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PAPER

Minimal Forbidden Minors for the Family of Graphs with Proper-Path-Width at Most Two

Atsushi TAKAHASHI†, Shuichi UENO†, *and* Yoji KAJITANI†, *Members*

SUMMARY The family $\mathcal{P}_k$ of graphs with proper-path-width at most k is minor-closed. It is known that the number of minimal forbidden minors for a minor-closed family of graphs is finite, but we have few such families for which all the minimal forbidden minors are listed. Although the minimal acyclic forbidden minors are characterized for $\mathcal{P}_k$, all the minimal forbidden minors are known only for $\mathcal{P}_1$. This paper lists 36 minimal forbidden minors for $\mathcal{P}_2$, and shows that there exist no other minimal forbidden minors for $\mathcal{P}_2$.

key words: *proper-path-width, path-width, minor-closed family, minimal forbidden minor, VLSI layout*

1. Introduction

A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A family $\mathcal{F}$ of graphs is said to be *minor-closed* if the following condition holds: If $G \in \mathcal{F}$ and H is a minor of G then $H \in \mathcal{F}$. A graph G is a *minimal forbidden minor* for a minor-closed family $\mathcal{F}$ of graphs if $G \notin \mathcal{F}$ and any proper minor of G is in $\mathcal{F}$. $\mathcal{F}$ is characterized by the minimal forbidden minors for $\mathcal{F}$. It is proved by Robertson and Seymour [14] that every minor-closed family of graphs has a finite number of minimal forbidden minors. They also proved that the problem of deciding if a fixed graph is a minor of an input graph can be solved in polynomial time [13]. It follows that the problem of testing membership for any minor-closed family $\mathcal{F}$ of graphs can be solved in polynomial time provided that we know all the minimal forbidden minors for $\mathcal{F}$. As many important problems can be reduced to the problem, it is important to find all the minimal forbidden minors for minor-closed families of graphs. Although it is known that there is no general method to find all the minimal forbidden minors for any minor-closed family of graphs [5], [6], a special method could be applied to each minor-closed family of graphs. In fact, the minimal forbidden minors are known for some minor-closed families of graphs [1], [2], [4], [7], [8], [20], [21]. This paper lists the minimal forbidden minors for the family of graphs with proper-path-width at most two.

The family $\mathcal{F}_k$ ($\mathcal{P}_k$) of graphs with (proper-)path-width at most k is minor-closed. $\mathcal{F}_k$ and $\mathcal{P}_k$ are known to have applications to VLSI layout, linguistics, and games on graphs [9], [11], [17]. Thus, it is important to list the minimal forbidden minors for $\mathcal{F}_k$ and $\mathcal{P}_k$. In [16] the authors characterize the minimal acyclic forbidden minors for $\mathcal{F}_k$ and $\mathcal{P}_k$ ($k \geq 1$). (Partial results can be found in [9], [15].) Although Kinnersley and Langston [8] list all 110 minimal forbidden minors for $\mathcal{F}_2$, it is open to list all minimal forbidden minors for $\mathcal{F}_k$ ($k \geq 3$) and $\mathcal{P}_k$ ($k \geq 2$). In this paper, we list all 36 minimal forbidden minors for $\mathcal{P}_2$. Our proof contains many general methods that would be useful to characterize minimal forbidden minors for $\mathcal{P}_k$ ($k \geq 3$).

Robertson and Seymour [13] proved that there exists an $\mathcal{O}(n^3)$ time algorithm to decide if a given n-vertex graph is in a minor-closed family of graphs. The time complexity is reduced to $\mathcal{O}(n \log^2 n)$ if the family does not contain all planar graphs. This is obtained by combining the results in [13] and [10]. Moreover, the time complexity is reduced to $\mathcal{O}(n)$ if the family avoids a $2 \times k$ grid graph and a circus-graph [3]. It follows from our results that we have an $\mathcal{O}(n \log^2 n)$ time membership test algorithm for $\mathcal{P}_2$, since we have listed all the minimal forbidden minors for $\mathcal{P}_2$ and $\mathcal{P}_2$ does not contain all planar graphs. Notice that $\mathcal{P}_2$ contains a $2 \times k$ grid graph for any k. It should be noticed that our algorithm is the first explicit membership test algorithm for $\mathcal{P}_2$, although we believe that there exists a more efficient and practical algorithm for $\mathcal{P}_2$ which does not rely on a minor test algorithm.

The paper is organized as follows. In Sect. 2, we give the definition of the proper-path-width of graphs, and show a necessary condition for a graph to be a minimal forbidden minor for $\mathcal{P}_k$ ($k \geq 1$). We introduce two equivalence relations on graphs in Sect. 3. Using these equivalence relations we show, in Sect. 4, 36 minimal forbidden minors for $\mathcal{P}_2$. In Sect. 5, we show some necessary conditions for minimal forbidden minors with cut-vertices or bridges for $\mathcal{P}_k$ ($k \geq 2$). In Sect. 6, we show that the 36 graphs characterize $\mathcal{P}_2$. Sect. 7 concludes the paper.

2. Preliminaries and a General Result

Graphs we consider are finite undirected, but may have loops and multiple edges unless otherwise specified. Let G be a graph, and $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The proper-path-width

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is defined as follows.

Definition 1: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a sequence of subsets of $V(G)$. The *width* of $\mathcal{X}$ is $\max_{1 \leq i \leq r} |X_i| - 1$. $\mathcal{X}$ is called a *proper-path-decomposition* of G if the following conditions are satisfied:

(i) For any distinct i and j, $X_i \not\subseteq X_j$;

(ii) $\bigcup_{i=1}^{r} X_i = V(G)$;

(iii) For any edge $(u, v) \in E(G)$, there exists an i such that $u, v \in X_i$;

(iv) For all a, b, and c with $1 \leqq a \leqq b \leqq c \leqq r$, $X_a \cap X_c \subseteq X_b$;

(v) For all a, b, and c with $1 \leqq a < b < c \leqq r$, $|X_a \cap X_c| \leq |X_b| - 2$.

The *proper-path-width* of G, denoted by $ppw(G)$, is the minimum width over all proper-path-decompositions of G.

For any positive integer k, the family of graphs with proper-path-width at most k, denoted by $\mathcal{P}_k$, is minor-closed [16]. Let $\Omega(\mathcal{P}_k)$ be the set of all minimal forbidden minors for $\mathcal{P}_k$.

A sequence $\mathcal{X}$ of subsets of $V(G)$ is called a *path-decomposition* of G if $\mathcal{X}$ satisfies (i), (ii), (iii), and (iv). The *path-width* of G, denoted by $pw(G)$, is the minimum width over all path-decompositions of G. For any positive integer k, the family of graphs with path-width at most k, denoted by $\mathcal{F}_k$, is minor-closed [12]. The relations between proper-path-width and path-width are mentioned in [16]–[19].

A graph obtained from graphs H_1, H_2, and H_3 by the following construction is called a *star-composition* of H_1, H_2, and H_3: (i) choose a vertex $v_i \in V(H_i)$ for $i = 1, 2$, and 3; (ii) let v be a new vertex not in $V(H_1), V(H_2)$, or $V(H_3)$; (iii) connect v to v_i by an edge (v, v_i) for $i = 1, 2$, and 3. A *delta-composition* of graphs H_1, H_2, and H_3 is a graph obtained from H_1, H_2, and H_3 by the following construction: (i) choose a vertex $v_i \in V(H_i)$ for $i = 1, 2$, and 3; (ii) connect v_1 to v_2, v_2 to v_3, and v_3 to v_1 by edges (v_1, v_2), (v_2, v_3), and (v_3, v_1), respectively. The following three theorems are proved by the authors.

Theorem 1 (Takahashi et al. [16]): For any positive integer k, if H_1, H_2, and H_3 are (not necessarily distinct) graphs in $\Omega(\mathcal{P}_k)$ then any star-composition of H_1, H_2, and H_3 is in $\Omega(\mathcal{P}_{k+1})$.

Theorem 2 (Takahashi et al. [16]): For any positive integer k, if H_1, H_2, and H_3 are (not necessarily distinct) graphs in $\Omega(\mathcal{P}_k)$ then any delta-composition of H_1, H_2, and H_3 is in $\Omega(\mathcal{P}_{k+1})$.

Theorem 3 (Takahashi et al. [16]): For any positive integer k, a tree T is in $\Omega(\mathcal{P}_{k+1})$ if and only if T is a star-composition of three (not necessarily distinct) trees in $\Omega(\mathcal{P}_k)$.

By these theorems, we can prove that the maximum number of vertices of a graph in $\Omega(\mathcal{P}_k)$ is at least $(3^{k+1} - 1)/2$, and $|\Omega(\mathcal{P}_k)|$ is at least $(k!)^2$. We counted the number of minimal acyclic forbidden minors for $\mathcal{P}_k$ $(k \leqq 4)$. For example, the number of minimal acyclic forbidden minors for $\mathcal{P}_4$ is 2875919312080.

The sequence of subsets of $V(G)$ obtained by concatenating sequences $\mathcal{X}_1$ and $\mathcal{X}_2$ of subsets of $V(G)$ is denoted by $(\mathcal{X}_1, \mathcal{X}_2)$. For a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of subsets of $V(G)$ and a set $S \subseteq V(G)$, we denote $(X_1 \cup S, X_2 \cup S, \ldots, X_r \cup S)$ by $\mathcal{X} \cup S$, and $(X_1 \cap S, X_2 \cap S, \ldots, X_r \cap S)$ by $\mathcal{X} \cap S$ for simplicity.

In the rest of this section, we show that a necessary condition for a graph to be a minimal forbidden minor for $\mathcal{P}_k$.

Lemma 1: Let G be a graph and G' be an underlying simple graph of G. Then $ppw(G) = ppw(G')$.

Proof: Since G' is a minor of G and a proper-path-decomposition of G' is also a proper-path-decomposition of G, $ppw(G) = ppw(G')$. □

A proper-path-decomposition with width k is called a k-proper-path-decomposition. A k-proper-path-decomposition $(X_1, X_2, \ldots, X_r)$ is said to be *full* if $|X_i| = k + 1$ $(1 \leqq i \leqq r)$ and $|X_j \cap X_{j+1}| = k$ $(1 \leqq j \leqq r - 1)$.

Lemma 2 (Takahashi et al. [17]): For any graph G with $ppw(G) = k$, there exists a full k-proper-path-decomposition of G.

For any vertex $v \in V(G)$, $d_G(v)$ is the number of vertices adjacent to v. If an edge (u, v) is contracted from a graph, the vertex obtained by identifying u and v is denoted by u or v.

Lemma 3: Let G be a graph satisfying the following: there exist vertices $u, v \in V(G)$ such that $(u, v) \in E(G)$, $d_G(u) = 2$, and $d_G(v) = 1$. (See Fig. 1.) If G' is a graph obtained from G by contracting edge (u, v), then $ppw(G) = ppw(G')$.

Proof: Since G' is a minor of G, $ppw(G) \geq ppw(G')$. We will show that $ppw(G) \leqq ppw(G')$. Assume that $ppw(G') = k$. Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a k-proper-path-decomposition of G', and w be a unique vertex adjacent to u in G'. By Lemma 2, we assume that $\mathcal{X}$ is full. By Definition 1(iii), there exists an integer a such that $\{u, w\} \subseteq X_a$. If $\{u, w\} \subseteq X_1$, then

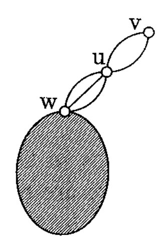


Fig. 1 An example of graph G such that $(u, v) \in E(G)$, $d_G(u) = 2$, and $d_G(v) = 1$.

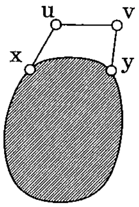


Fig. 2 An example of graph G such that $(u, v) \in E(G)$ and $d_G(u) = d_G(v) = 2$.

$((X_1 - \{w\}) \cup \{v\}, \mathcal{X})$ is a k-proper-path-decomposition of G, and $ppw(G) \leq k$. Notice that $(X_r, X_{r-1}, \ldots, X_1)$ is also a k-proper-path-decomposition of G. Thus, we assume that $\{u, w\} \subseteqq X_a$ $(1 < a < r)$. Suppose that $u \notin X_{a-1} \cup X_{a+1}$. Since $\mathcal{X}$ is full, $X_{a-1} \cap X_a = X_a \cap X_{a+1} = X_a - \{u\}$. However, we have $|X_a| \geqq |X_{a-1} \cap X_{a+1}| + 2 = k + 2$ by Definition 1(v), contradicting to the assumption that the width of $\mathcal{X}$ is k. Thus we assume, without loss of generality, that $u \in X_{a+1}$. For any integer i $(a + 1 \leqq i \leqq r)$, let $X'_i = (X_i - \{u\}) \cup \{v\}$ if $u \in X_i$, $X'_i = X_i$ otherwise. Then $(X_1, \ldots, X_a, (X_a \cap X_{a+1}) \cup \{v\}, X'_{a+1}, \ldots, X'_r)$ is a k-proper-path-decomposition of G. Thus $ppw(G) \leqq ppw(G')$. □

Lemma 4: Let G be a graph satisfying the following: there exist vertices $u, v \in V(G)$ such that $(u, v) \in E(G)$, $d_G(u) = d_G(v) = 2$, and u and v have no common adjacent vertex. (See Fig. 2.) If G' is a graph obtained from G by contracting edge (u, v), then $ppw(G) = ppw(G')$.

Proof: Since G' is a minor of G, $ppw(G) \geqq ppw(G')$. We will show that $ppw(G) \leqq ppw(G')$. Assume that $ppw(G') = k$. Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a full k-proper-path-decomposition of G', and x and y be vertices adjacent to u in G'. If $\{u, x, y\} \subseteqq X_1$, then $((X_1 - \{x\}) \cup \{v\}, \mathcal{X})$ is a k-proper-path-decomposition of G, and $ppw(G) \leqq k$. If $\{u, x, y\} \subseteqq X_i$ $(1 < i < r)$ then we can assume that $\{u, y\} \subseteqq X_{i+1}$ since $\mathcal{X}$ is full. Thus we assume that there exist distinct integers a and b such that $\{u, x\} \subseteqq X_a$, $\{u, y\} \subseteqq X_b$ by Definition 1(iii) $(1 \leqq a < b \leqq r)$. For any integer i $(a + 1 \leqq i \leqq r)$, let $X'_i = (X_i - \{u\}) \cup \{v\}$ if $u \in X_i$, $X'_i = X_i$ otherwise. Then $(X_1, \ldots, X_a, (X_a \cap X_{a+1}) \cup \{v\}, X'_{a+1}, \ldots, X'_r)$ is a k-proper-path-decomposition of G. Thus $ppw(G) \leqq ppw(G')$. □

Notice that a minimal forbidden minor for $\mathcal{P}_k$ is connected, since the proper-path-width of a graph G is the maximum proper-path-width over all connected components of G. From Lemmas 1, 3, and 4, we have the following theorem.

Theorem 4: If a graph G is in $\Omega(\mathcal{P}_k)$ then G is connected and simple, and there are no adjacent vertices u and v such that $d_G(v) \leqq 2$, $d_G(u) \leqq 2$, and they have no common adjacent vertex.

3. Equivalence Relations and Some General Results

In this section, we introduce two equivalence relations. These equivalence relations are very useful for listing minimal forbidden minors for $\mathcal{P}_k$ $(k \geqq 1)$.

Lemma 5: If there exists a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of subsets of $V(G)$ with width k satisfying Definition 1(ii), (iii), and (iv), and $|X_a \cap X_b \cap X_c| \leqq k-1$ for any X_a, X_b, and X_c such that each one is not a subset of the others $(1 \leqq a < b < c \leqq r)$, then $ppw(G) \leqq k$.

We omit the proof of this lemma since a similar lemma is proved in [17].

Lemma 6: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a proper-path-decomposition of G, and H be a subgraph of G with proper-path-width k. Let $X'_i = X_i \cap V(H)$. Then there exists an integer i $(1 \leqq i \leqq r)$ such that either (a) $|X'_i| \geqq k+1$, or (b) $|X'_i| = k$ and $|X'_{i-1} \cap X'_{i+1}| \geqq k-1$.

Proof: For otherwise, $\mathcal{X} \cap V(H)$ is a sequence with width $k - 1$ satisfying the condition of Lemma 5, contradicting to the assumption that $ppw(H) = k$. □

Lemma 7: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a proper-path-decomposition of G and H be a connected subgraph of G. Let $X'_i = X_i \cap V(H)$ $(1 \leqq i \leqq r)$. If $X'_a \neq \emptyset$ and $X'_b \neq \emptyset$ $(1 \leqq a \leqq b \leqq r)$, then $X'_i \neq \emptyset$ for any i $(a \leqq i \leqq b)$. Moreover, if $|X'_i| = 1$ for some i $(a < i < b)$, then $X'_i = X'_{i-1} \cap X'_{i+1}$.

Proof: Suppose that $X'_c = \emptyset$ for some c $(a < c < b)$. Each vertex of H appears in consecutive X_i's. Thus, if $P = \bigcup_{i=1}^{c-1} X'_i$ and $Q = \bigcup_{i=c+1}^{r} X'_i$, then $P \cap Q = \emptyset$. Since $V(H)$ is partitioned into P and Q, and H is connected, there exist $u \in P$ and $v \in Q$ such that $(u, v) \in E(H)$. However, $\{u, v\} \nsubseteqq X_i$ for any i $(1 \leqq i \leqq r)$, contradicting to Definition 1(iii).

Assume that $|X'_c| = 1$ for some c $(a < c < b)$. Let $v \in X'_c$. If $v \notin X'_{c-1}$ then $V(H)$ is partitioned into $\bigcup_{i=1}^{c-1} X'_i$ and $\bigcup_{i=c}^{r} X'_i$. This is contradicting to Definition 1(iii). Thus $v \in X'_{c-1}$. Similarly, $v \in X'_{c+1}$. □

$G \backslash S$ is the graph obtained from G by deleting $S \subseteqq V(G)$.

Lemma 8: Let G be a connected graph and v be a vertex of G. Let H_i be a connected graph and u_i be a vertex of H_i $(1 \leqq i \leqq 2)$. Define that G_i be the graph obtained from G and H_i by identifying v and u_i $(1 \leqq i \leqq 2)$. If $H_1, H_2 \in \Omega(\mathcal{P}_k)$ then $ppw(G_1) = ppw(G_2)$.

Proof: Assume that $ppw(G_1) = h$. Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a h-proper-path-decomposition of G_1. Let a and b be integers such that $v \in X_a \cap X_b$ and $v \notin X_{a-1} \cup X_{b+1}$ $(a \leqq b)$. Let $X_i^H = X_i \cap V(H_1)$ and $X_i^G = X_i - V(H_1)$ for any i $(1 \leqq i \leqq r)$. From Lemma 6, there exists an integer i such that either (a) $|X_i^H| \geqq k + 2$ $(1 \leqq i \leqq r)$, or (b) $|X_i^H| = k + 1$ and $|X_{i-1}^H \cap X_{i+1}^H| \geqq k$ $(1 < i < r)$. Let c be such an integer. Let $X'_i = X_i^G \cup \{v\}$ if $\min\{a, c\} \leqq i \leqq \max\{b, c\}$, $X'_i = X_i^G$ otherwise. Since H_1 is connected, $|X'_i| \leqq h + 1$ by Lemma 7 $(1 \leqq i \leqq r)$. Since

$H_2 \in \Omega(\mathcal{P}_k)$, $ppw(H_2\backslash\{u_2\}) = k$. Let $\mathcal{X}^*$ be a k-proper-path-decomposition of $H_2\backslash\{u_2\}$.

If $|X_c^H| \geqq k+2$ then $(X_1', \ldots, X_{c-1}', X_c' \cup \mathcal{X}^*, X_{c+1}', \ldots, X_r')$ is a sequence with width h satisfying the conditions of Lemma 5, and $ppw(G_2) \leqq h$. Notice that $|X_c'| = |X_c| - |X_c^H| + 1 \leqq h - k$. Thus we assume that $|X_c^H| = k+1$ and $|X_c^G| \leqq h-k$. If $|X_{c-1}^G \cap X_c^G| = |X_c^G \cap X_{c+1}^G| = h-k$ then $X_{c-1}^G \cap X_{c+1}^G = X_c^G$. We have that $|X_{c-1} \cap X_{c+1}| = |X_{c-1}^G \cap X_{c+1}^G| + |X_{c-1}^H \cap X_{c+1}^H| = (h-k)+k$. However, we have $|X_c| \geqq h+2$ by Definition 1(v), contradicting to the assumption that the width of $\mathcal{X}$ is h. Thus, we assume, without loss of generality, $|X_c^G \cap X_{c+1}^G| \leqq h-k-1$. Then, $(X_1', \ldots, X_c', (X_c' \cap X_{c+1}') \cup \{v\} \cup \mathcal{X}^*, X_{c+1}', \ldots, X_r')$ is a sequence with width h satisfying the conditions of Lemma 5, and $ppw(G_2) \leqq h$.

Therefore, we have $ppw(G_2) \leqq h = ppw(G_1)$. Similarly, we can prove that $ppw(G_1) \leqq ppw(G_2)$, and we have $ppw(G_1) = ppw(G_2)$. □

The following definition plays an important role to characterize $\Omega(\mathcal{P}_k)$.

Definition 2: Let H be a graph and x be a vertex of H. $R_1^1(H,x)$ is the graph obtained from H by adding vertices u, v, and w and edges (x,u), (x,v), and (x,w). $R_2^1(H,x)$ is the graph obtained from $R_1^1(H,x)$ by adding edge (u,w) and deleting v. $R_3^1(H,x)$ is the graph obtained from $R_1^1(H,x)$ by adding edges (u,v) and (v,w) and deleting edges (x,u) and (x,w). Graphs G_1 and G_2 are said to be *semi-1-equivalent*, denoted by $G_1 \overset{1}{\cong} G_2$, if there exist a graph H and a vertex $x \in V(H)$ such that $G_1 = R_i^1(H,x)$ $(1 \leqq i \leqq 3)$ and $G_2 = R_j^1(H,x)$ $(1 \leqq j \leqq 3)$. Graphs G_1 and G_i are said to be *1-equivalent* if there exists a sequence of graphs $G_1, G_2, \ldots, G_i$ $(i \geq 1)$ such that $G_1 \overset{1}{\cong} G_2 \overset{1}{\cong} \cdots \overset{1}{\cong} G_i$.

It is easy to see that the 1-equivalence is an equivalence relation on the graphs. An example of 1-equivalent graphs is shown in Fig. 3.

Theorem 5: If $G \in \Omega(\mathcal{P}_k)$ then a graph 1-equivalent to G is in $\Omega(\mathcal{P}_k)$.

Proof: Let H be a graph and $x \in V(H)$. Let $R_i = R_i^1(H,x)$ $(1 \leqq i \leqq 3)$. Assume that $R_a \in \Omega(\mathcal{P}_k)$ for some a $(1 \leqq a \leqq 3)$. To prove the lemma, it is sufficient to show that $R_i \in \Omega(\mathcal{P}_k)$ for any i $(1 \leqq i \leqq 3)$.

Let $H_i = R_i\backslash(V(H) - \{x\})$ $(1 \leqq i \leqq 3)$. Since H_1, H_2, $H_3 \in \Omega(\mathcal{P}_1)$, $ppw(R_1) = ppw(R_2) = ppw(R_3) = k+1$ by Lemma 8. Thus it is sufficient to show that R_i is minimal for any i, that is, the proper-path-width of a graph obtained from R_i by deleting or contracting an edge is at most k. Let G_i' be a graph obtained from R_i by deleting or contracting an edge $e \in E(H)$. We have $ppw(G_a') = ppw(G_i') = k$ for any i $(1 \leqq i \leqq 3)$ by Lemma 8. Let G_i'' be a graph obtained from R_i by deleting or contracting an edge $e \notin E(H)$. Proper-path-widths of connected components of G_i'' not containing x are at most one $(\leqq k)$. A connected component of G_i'' containing x is isomorphic to either graphs G_1^*, G_2^*, G_3^*, or G_4^* obtained from R_2 by deleting edge (x,u), contracting edge (x,u), deleting edge (u,w), or deleting vertices u and w, respectively (See Fig. 4). Since both G_3^* and G_4^* are proper minors of R_a, $ppw(G_3^*) \leqq k$ and $ppw(G_4^*) \leqq k$. Since the underlying simple graph of G_2^* is a proper minor of G_3^*, $ppw(G_2^*) \leqq ppw(G_3^*) \leqq k$ by Lemma 1. By Lemma 3, we have $ppw(G_1^*) = ppw(G_2^*)$. Thus $ppw(G_i'') \leqq k$ for any i $(1 \leqq i \leqq 3)$. Hence we have $R_i \in \Omega(\mathcal{P}_k)$ $(1 \leqq i \leqq 3)$. □

The following definition also plays an important role to characterize $\Omega(\mathcal{P}_k)$.

Definition 3: Let H be a graph, and x and y be non-adjacent vertices of H. $R_1^2(H,x,y)$ is the graph obtained from H by adding vertices u and v and edges (x,u), (y,u), and (u,v). $R_2^2(H,x,y)$ is the graph obtained from $R_1^2(H,x,y)$ by adding edge (x,y) and deleting v. Graphs G_1 and G_2 are said to be semi-2-equivalent, denoted by $G_1 \overset{2}{\cong} G_2$, if there exist a graph H and non-adjacent vertices $x \in V(H)$ and $y \in V(H)$ such that $G_1 = R_i^2(H,x,y)$ $(i = 1,2)$, $G_2 = R_j^2(H,x,y)$ $(j = 1,2)$. Graphs G_1 and G_i are said to be 2-equivalent if there exists a sequence of graphs $G_1, G_2, \ldots, G_i$ $(i \geq 1)$ such that $G_1 \overset{2}{\cong} G_2 \overset{2}{\cong} \cdots \overset{2}{\cong} G_i$.

It is easy to see that the 2-equivalence is an equivalence relation on the graphs. An example of 2-equivalent graphs is shown in Fig. 5.

Lemma 9: If a graph G is 2-equivalent to G' then $ppw(G) = ppw(G')$.

Proof: Let H be a graph and $x, y \in V(H)$ be non-adjacent vertices. Let $R_i = R_i^2(H,x,y)$ $(1 \leqq i \leqq 2)$. To prove the lemma, it is sufficient to show that $ppw(R_1) =$

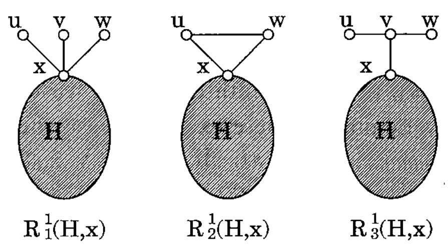


Fig. 3 1-equivalent graphs.

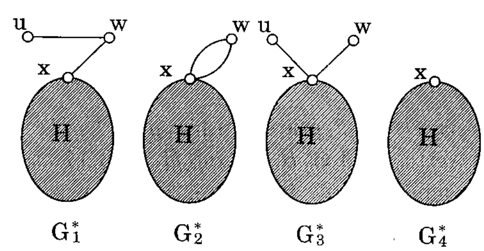
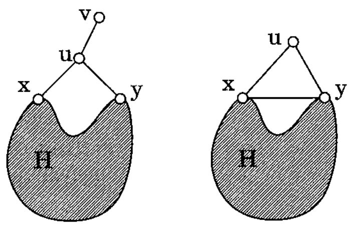
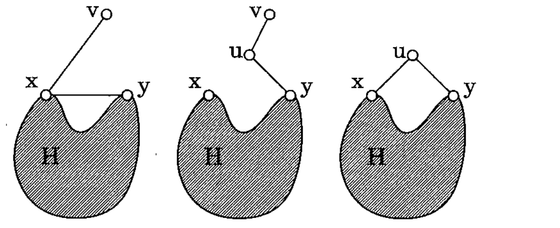


Fig. 4 Connected components of G_i'' containing x.



$R^2_1(H,x,y)$ $R^2_1(H,x,y)$

Fig. 5 2-equivalent graphs.



G_1^* G_2^* G_3^*

Fig. 6 Connected components of G_1'' containing x.

$ppw(R_2)$.

First, assume that $ppw(R_1) = k$. Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a full k-proper-path-decomposition of R_1. Let $X_i' = X_i - \{u,v\}$. If $\{u,x,y\} \subseteqq X_i$ for some i $(1 \leqq i \leqq r)$, $\mathcal{X}$ is a k-proper-path-decomposition of a graph Q obtained from R_1 by adding edge (x,y). Since R_2 is a minor of Q, $ppw(R_2) \leqq k$. Thus we assume that $\{u,x,y\} \not\subseteqq X_i$ for any i $(1 \leqq i \leqq r)$. Suppose that $\{u,x\} \in X_{a-1} \cap X_a$ and $\{u,y\} \in X_b$ for some a and b $(a < b)$. Notice that $|X'_{a-1} \cap X'_a| \leqq k-1$. Since $(X'_1, \ldots, X'_{a-1}, (X'_{a-1} \cap X'_a) \cup \{y,u\}, X'_a \cup \{y\}, \ldots, X'_{b-1} \cup \{y\}, X'_b, \ldots, X'_r)$ is a sequence satisfying the conditions of Lemma 5, $ppw(R_2) \leqq k$. Thus we assume that $\{u,x\} \subseteqq X_a$, $\{u,y\} \subseteqq X_b$, $x \notin X_{a+1}$, $y \notin X_{b-1}$, and $u \notin X_{a-1} \cup X_{b+1}$ for some a and b $(1 \leqq a < b \leqq r)$. There exists an integer c such that $\{u,v\} \subseteqq X_c$ $(a \leqq c \leqq b)$. Moreover, since $\mathcal{X}$ is full, we assume that $\{u,v\} \subseteqq X_c \cap X_{c+1}$ for some c $(a \leqq c < b)$. Notice that $|X'_c \cap X'_{c+1}| \leqq k-2$. Since $(X'_1, \ldots, X'_a, X'_{a+1} \cup \{x\}, \ldots, X'_c \cup \{x\}, (X'_c \cap X'_{c+1}) \cup \{x,y,u\}, X'_{c+1} \cup \{y\}, \ldots, X'_{b-1} \cup \{y\}, X'_b, \ldots, X'_r)$ is a sequence satisfying the conditions of Lemma 5, $ppw(R_2) \leqq k$. Thus we have that $ppw(R_2) \leqq ppw(R_1)$.

Next, assume that $ppw(R_2) = k$ and let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a full k-proper-path-decomposition of R_2. There exists an integer a such that $\{u,x,y\} \subseteqq X_a$. If $\{u,x,y\} \subseteqq X_1$, then $((X_1 - \{x\}) \cup \{v\}, \mathcal{X})$ is a k-proper-path-decomposition of R_1, and $ppw(R_1) \leqq k$. If $\{u,x,y\} \subseteqq X_a$ $(1 < a < r)$ then we assume without loss of generality that $\{u,x\} \subseteqq X_{a+1}$ since $\mathcal{X}$ is full. For any integer i $(a+1 \leqq i \leqq r)$, let $X'_i = (X_i - \{u\}) \cup \{v\}$ if $u \in X_i$, $X'_i = X_i$ otherwise. Then $(X_1, \ldots, X_a, (X_a \cap X_{a+1}) \cup \{v\}, X'_{a+1}, \ldots, X'_r)$ is a k-proper-path-decomposition of R_1. Thus $ppw(R_1) \leqq ppw(R_2)$.

Therefore we have $ppw(R_1) = ppw(R_2)$. □

Theorem 6: If $G \in \Omega(\mathcal{P}_k)$ then a graph 2-equivalent to G is in $\Omega(\mathcal{P}_k)$.

Proof: Let H be a graph and $x, y \in V(H)$ be non-adjacent vertices. Let $R_i = R_i^2(H,x,y)$ $(1 \leqq i \leqq 2)$. It is sufficient to show that $R_1 \in \Omega(\mathcal{P}_k)$ if and only if $R_2 \in \Omega(\mathcal{P}_k)$.

First, assume that $R_1 \in \Omega(\mathcal{P}_k)$. By Lemma 9, we have $ppw(R_2) = ppw(R_1) = k+1$. It is sufficient to show that R_2 is minimal, that is, the proper-path-width of a graph obtained from R_2 by deleting or contracting an edge is at most k. Let G'_1 and G'_2 be graphs obtained from R_1 and R_2 by deleting or contracting an edge $e \in E(H)$, respectively. We have $ppw(G'_2) = ppw(G'_1) = k$ by Lemma 9. Let G''_2 be a graph obtained from R_2 by deleting or contracting an edge $e \in \{(x,u), (y,u), (x,y)\}$. Since either G''_2 or the underlying simple graph of G''_2 is a proper minor of R_1, $ppw(G''_2) \leqq k$. Thus we have $R_2 \in \Omega(\mathcal{P}_k)$.

Next, assume that $R_2 \in \Omega(\mathcal{P}_k)$. Similar to argument above, we have $ppw(R_1) = ppw(R_2) = k+1$, and the proper-path-width of a graph obtained from R_1 by deleting or contracting an edge $e \in E(H)$ is k. Let G''_1 be a graph obtained from R_1 by deleting or contracting an edge $e \in \{(x,u), (y,u), (u,v)\}$. Proper-path-widths of connected components of G''_1 not containing x is at most one. A connected component of G''_1 containing x is isomorphic to either of graphs G_1^*, G_2^*, or G_3^* obtained from R_1 by contracting (x,u), deleting (x,u), or contracting (u,v), respectively (See Fig. 6). Let G_4^* be a graph obtained from G_2^* by contracting edge (u,v). We have $ppw(G_2^*) = ppw(G_4^*)$ by Lemma 3. Since both G_1^*, G_3^*, and G_4^* are proper minors of R_2, proper-path-widths of these graphs are at most k. Then the proper-path-width of G''_1 is at most k. Thus we have $R_1 \in \Omega(\mathcal{P}_k)$. □

4. Minimal Forbidden Minors for $\mathcal{P}_2$

In this section, we enumerate minimal forbidden minors for $\mathcal{P}_2$.

Lemma 10: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a proper-path-decomposition of a graph G, and $G_1, G_2, \ldots, G_s$, H be disjoint connected subgraphs of G. Let $X'_i = X_i \cap V(H)$ $(1 \leqq i \leqq r)$. Let a be an integer such that either $|X'_a| \geqq k+1$ or $|X'_a| = k$ and $|X'_{a-1} \cap X'_{a+1}| \geqq k-1$. If $|X_{a-1} \cap V(G_i)| \neq \emptyset$ and $|X_{a+1} \cap V(G_i)| \neq \emptyset$ for any i $(1 \leqq i \leqq s)$, then the width of $\mathcal{X}$ is at least $k+s$.

Proof: By Lemma 7, there exists at least one vertex in $X_a \cap V(G_i)$ $(1 \leqq i \leqq s)$. If either $|X'_a| \geqq k+1$ or $|X_a \cap V(G_i)| \geqq 2$ for some i, then $|X_a| \geqq |X'_a| + |X_a \cap V(G_1)| + \cdots + |X_a \cap V(G_s)| \geqq k+s+1$, and the width of $\mathcal{X}$ is at least $k+s$. Thus we assume that $|X'_a| = k$ and $|X_a \cap V(G_i)| = 1$ for any i. Let v_i be a vertex in

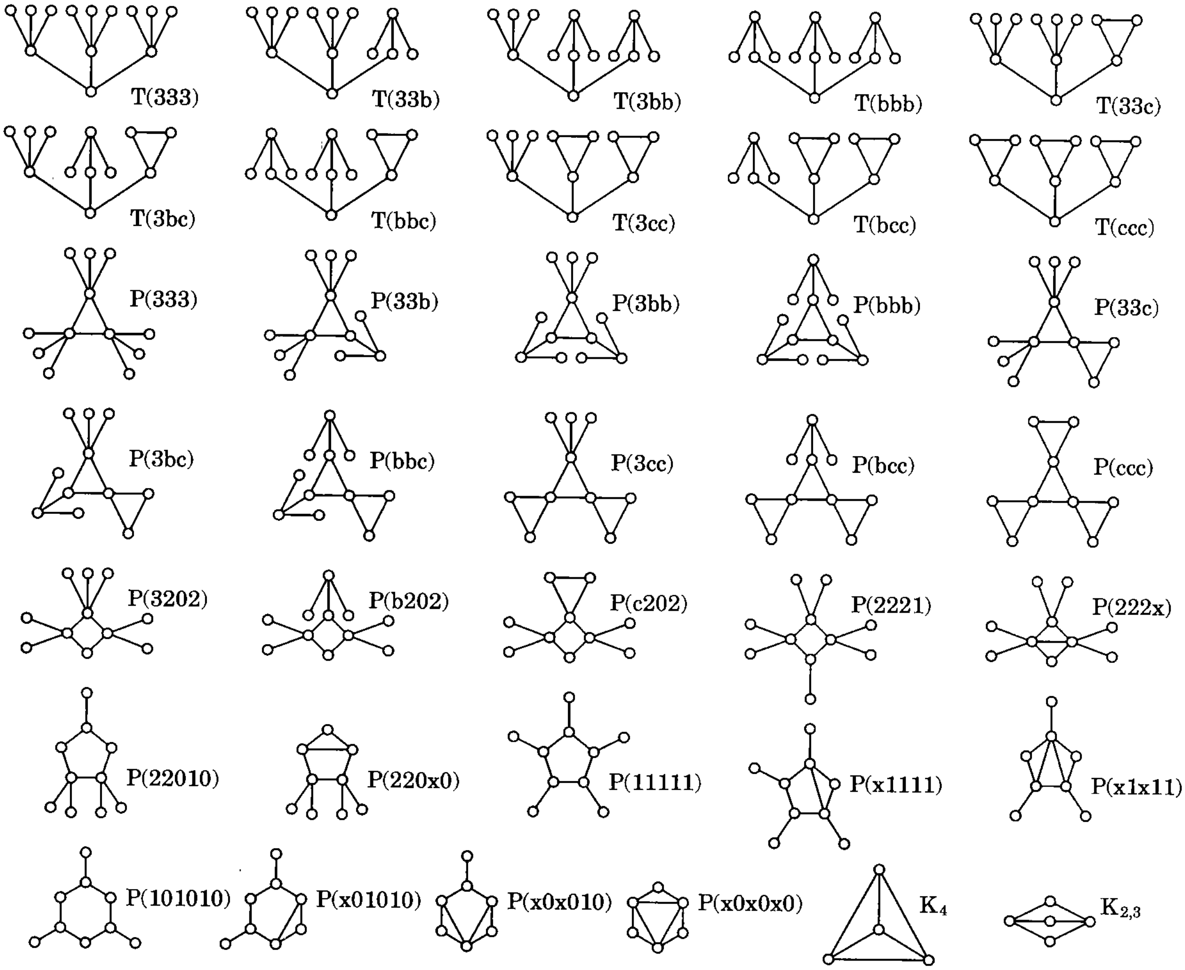


Fig. 7 Minimal forbidden minors for $\mathcal{P}_2$.

$X_a \cap V(G_i)$ $(1 \leqq i \leqq s)$. By Lemma 7, $v_i \in X_{a-1} \cap X_{a+1}$. Since $X_{a-1} \cap X_{a+1} \supseteq (X'_{a-1} \cap X'_{a+1}) \cup \{v_1, v_2, \dots, v_s\}$, we have that $|X_a| \geqq |X_{a-1} \cap X_{a+1}| + 2 \geqq |(X'_{a-1} \cap X'_{a+1})| + |\{v_1, v_2, \dots, v_s\}| + 2 = k + s + 1$ by Definition 1(v). Thus the width of $\mathcal{X}$ is at least $k + s$. □

Let $P(a_0, a_1, \dots, a_{n-1})$ be the graph obtained from a cycle with vertices $\{v_0, v_1, \dots, v_{n-1}\}$ and edges $\{(v_i, v_{(i+1) \bmod n}) | 0 \leqq i \leqq n-1\}$ by adding a_i vertices and connecting them with v_i by edges for all i $(0 \leqq i \leqq n-1)$.

Theorem 7: A graph isomorphic to either $P(333)$, $P(3202)$, $P(2221)$, $P(22010)$, $P(11111)$, $P(101010)$, K_4, or $K_{2,3}$ (see Fig. 7) is a minimal forbidden minor for $\mathcal{P}_2$.

Proof: Let G be a graph isomorphic to $P(11111)$. Let $u_0, u_1, \dots, u_4$ be vertices of G such that $(u_i, u_{(i+1) \bmod 5}) \in E(G)$, and $v_0, v_1, \dots, v_4$ be vertices of G such that $(v_i, u_i) \in E(G)$ $(0 \leqq i \leqq 4)$. Let $\mathcal{X} = (X_1, X_2, \dots, X_r)$ be a proper-path-decomposition of G. Let s_i be an integer such that $v_i, u_i \in X_{s_i}$ $(0 \leqq i \leqq 4)$. Let $t_0, t_1, \dots, t_4$ be a permutation of 0, 1, 2, 3, and 4 such that $s_{t_0} \leqq s_{t_1} \leqq \cdots \leqq s_{t_4}$. Without loss of generality, we assume that $t_2 = 0$ and (t_0, t_1) is a combination of either $(1,2)$, $(1,3)$, or $(1,4)$. Assume first that (t_0, t_1) is a combination of $(1,2)$. Let H, G_1 and G_2 be induced subgraphs of G on $\{v_0\}$, $\{v_1, v_4, u_0, u_1, u_4\}$ and $\{v_2, v_3, u_2, u_3\}$, respectively. Notice that H, G_1 and G_2 are disjoint connected subgraphs of G. If $s_1 < s_0 < s_4$ and $s_2 < s_0 < s_3$, then $|X_{s_0}| \geqq 4$ by Lemma 10, and the width of $\mathcal{X}$ is at least 3. Otherwise $s_0 = s_i$ for some i $(1 \leqq i \leqq 4)$ and $\{v_0, u_0, v_i, u_i\} \subseteqq X_{s_0}$, again, $|X_{s_0}| \geqq 4$ and the width of $\mathcal{X}$ is at least 3. Assume next that (t_0, t_1) is a combination of either $(1,3)$ or $(1,4)$. Let H, G_1 and G_2 be induced subgraphs of G on $\{v_0\}$, $\{v_1, v_2, u_1, u_2\}$ and $\{v_3, v_4, u_3, u_4\}$, respectively. Similarly we have the width of $\mathcal{X}$ is at least 3. Thus the width of $\mathcal{X}$ is at least 3 and we have $ppw(G) \geqq 3$. It is easy to see that there exists a 3-proper-path-decomposition of G, and there exists a 2-proper-path-decomposition for each proper minor of G. Thus G is a minimal forbidden minor for $\mathcal{P}_2$.

Although we can prove all the other cases similarly, we omit the proof for the sake of space. □

Let $P(a_0, a_1, \ldots, a_{n-1})$ be a graph such that $a_i \leqq 3$ $(0 \leqq i \leqq n-1)$. A graph 1-equivalent to $P(a_0, a_1, \ldots, a_{n-1})$ is denoted by $P(z_0, z_1, \ldots, z_{n-1})$ where z_i is a_i, b, or c if $a_i = 3$, $z_i = a_i$ otherwise $(1 \leqq i \leqq n-1)$. A graph 2-equivalent to $P(a_0, a_1, \ldots, a_{n-1})$ is denoted by $P(z_0, z_1, \ldots, z_{n-1})$ where z_i is a_i or x if $a_i = 1$, $z_i = a_i$ otherwise $(1 \leqq i \leqq n-1)$.

Corollary 1: 36 Graphs shown in Fig. 7 are minimal forbidden minors for $\mathcal{P}_2$.

Proof: The eight graphs stated in Theorem 7 are minimal forbidden minors for $\mathcal{P}_2$. There exist new eleven (seven) graphs 1-equivalent (2-equivalent) to the graphs stated in Theorem 7. These graphs are minimal forbidden minors for $\mathcal{P}_2$ by Theorems 5 and 6. Finally, ten graphs are minimal forbidden minors for $\mathcal{P}_2$ by Theorem 1. □

Notice that any minimal forbidden minor for $\mathcal{P}_2$ stated in Theorem 2 is isomorphic to a graph 1-equivalent to $P(333)$.

5. Properties of Graphs with Cut-Vertices or Bridges

In this section, we show necessary conditions for a graph obtained from two disjoint graphs with proper-path-width k by either identifying some vertex or adding an edge to be a minimal forbidden minor for $\mathcal{P}_k$ $(k \geqq 1)$. A vertex v is called a cut-vertex of a connected graph G if $G \backslash \{v\}$ has at least two connected components. An edge (u, v) is called a bridge of a connected graph G if the graph obtained from G by deleting (u, v) has at least two connected components.

Lemma 11: Let G be a connected graph with a cut-vertex v. Let H be a union of connected components of $G \backslash \{v\}$, H_1 be the induced subgraph of G on $V(H) \cup \{v\}$, and H_2 be the induced subgraph of G on $V(G) - V(H)$. If $ppw(G) = ppw(H_1) = ppw(H_2) = k$ and there exists at most one connected component of H with proper-path-width k, then there exists a k-proper-path-decomposition $(X_1, X_2, \ldots, X_r)$ of H_1 such that $v \in X_1$.

Proof: Assume that the proper-path-width of each connected component of H is at most $k-1$. Since $ppw(H) = k-1$, there exists a $(k-1)$-proper-path-decomposition $\mathcal{X}$ of H. Then $(\mathcal{X} \cup \{v\})$ is a k-proper-path-decomposition of H_1 satisfying the condition of this lemma. Thus we assume that there exists a connected component of H with proper-path-width k.

Let H' be a connected component of H with proper-path-width k and G' be the induced subgraph of G on $V(H') \cup V(H_2)$. Since H_2 is a minor of G' and G' is a minor of G, we have $ppw(G') = k$. Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a k-proper-path-decomposition of G'. From Lemma 6, there exists an integer c such that either (a) $|X_c \cap V(H_2)| \geqq k+1$ $(1 \leqq c \leqq r)$, or (b) $|X_c \cap V(H_2)| = k$ and $|X_{c-1} \cap X_{c+1} \cap V(H_2)| \geqq k-1$ $(1 < c < r)$. Suppose that there exist integers a and b such that $X_a \cap V(H') \neq \emptyset$, $X_b \cap V(H') \neq \emptyset$ $(a < c < b)$. Since H' is connected, $|X_c \cap V(H')| \geqq 1$ by Lemma 7. If $|X_c \cap V(H_2)| \geqq k+1$ then $|X_c| \geqq |X_c \cap V(H_2)| + |X_c \cap V(H')| \geqq k+2$, contradicting to that the width of $\mathcal{X}$ is k. Thus we suppose that $|X_c \cap V(H_2)| = k$ and $|X_c \cap V(H')| = 1$. However, we have $|X_{c-1} \cap X_{c+1}| \geqq k$ since $|X_{c-1} \cap X_{c+1} \cap V(H')| \geqq 1$ by Lemma 7 and $|X_{c-1} \cap X_{c+1} \cap V(H_2)| \geqq k-1$. Again this is contradicting to that the width of $\mathcal{X}$ is k since $|X_c| \geqq |X_{c-1} \cap X_{c+1}| + 2 \geqq k+2$. Thus, without loss of generality, we assume that $\bigcup_{i=1}^{c-1} X_i \cap V(H') = \emptyset$. Let d be the integer such that $v \in X_d$ and $v \notin X_{d+1}$. Notice that $c \leqq d$ since there exists a vertex $v' \in V(H')$ such that $(v, v') \in E(G')$. Let $X'_i = (X_i \cap V(H')) \cup \{v\}$ for any integer i $(c \leqq i \leqq d)$, and $X'_i = X_i \cap V(H')$ for any integer i $(d < i)$. Notice that $|X'_i| \leqq k+1$ since H_2 is connected.

Let H^* be the induced subgraph of H_1 on $V(H') \cup \{v\}$. Then $\mathcal{X}' = (X'_c, X'_{c+1}, \ldots, X'_r)$ is a sequence of subsets of $V(H^*)$ with width k satisfying the conditions of Lemma 5. Since the proper-path-width of $H_1 \backslash V(H^*)$ is at most $k-1$, there exists a $(k-1)$-proper-path-decomposition $\mathcal{X}''$ of $H_1 \backslash V(H_1^*)$. Then $(\mathcal{X}'' \cup \{v\}, \mathcal{X}')$ is a sequence of subsets of $V(H_1)$ with width k satisfying the conditions of Lemma 5. It is not difficult to see that we can obtain a k-proper-path-decomposition of H_1 satisfying the condition of this lemma from the sequence $(\mathcal{X}'' \cup \{v\}, \mathcal{X}')$. □

Theorem 8: Let G be a connected graph with a cut-vertex v. Let H be a union of connected components of $G \backslash \{v\}$, H_1 be the induced subgraph of G on $V(H) \cup \{v\}$, and H_2 be the induced subgraph of G on $V(G) - V(H)$. If $ppw(H_1) = ppw(H_2) = k$ and $G \in \Omega(\mathcal{P}_k)$, then either G is a star-composition of (not necessarily distinct) graphs in $\Omega(\mathcal{P}_{k-1})$, or $H_1 \in \Omega(\mathcal{P}_{k-1})$, or $H_2 \in \Omega(\mathcal{P}_{k-1})$.

Proof: Assume contrary that G is not a star-composition of three graphs in $\Omega(\mathcal{P}_{k-1})$, $H_1 \notin \Omega(\mathcal{P}_{k-1})$, and $H_2 \notin \Omega(\mathcal{P}_{k-1})$. Notice that $G \backslash \{v\}$ has at most two connected components with proper-path-width k, for otherwise, G is not minimal.

Assume first that $H_1 \backslash \{v\}$ and $H_2 \backslash \{v\}$ have at most one connected component with proper-path-width k, respectively. Let H' be a proper minor of H_2 with proper-path-width k. Let G' be the minor of G such that $V(G') = V(H_1) \cup V(H')$ and $E(G') = E(H_1) \cup E(H')$. Since $G \in \Omega(\mathcal{P}_k)$, we have $ppw(G') = k$. By Lemma 11, there exists a k-proper-path-decomposition $\mathcal{X}^1 = (X_1^1, X_2^1, \ldots, X_{r_1}^1)$ of H_1 such that $v \in X_{r_1}^1$. Similarly, there exists a k-proper-path-decomposition $\mathcal{X}^2 = (X_1^2, X_2^2, \ldots, X_{r_2}^2)$ of H_2 such that $v \in X_1^2$. Then $(\mathcal{X}^1, \mathcal{X}^2)$ is a k-proper-path-decomposition of G, contradicting to that $ppw(G) \geqq k+1$.

Assume next that $H_1 \backslash \{v\}$ has exactly two connected components with proper-path-width k. Let R_1 and R_2 be connected components of $H_1 \backslash \{v\}$ with proper-path-width k, $H_a = H_1 \backslash V(R_2)$, and H_b be the induced subgraph of G on $V(H_2) \cup V(R_2)$. Note that H_a and H_b are connected, $V(H_a) \cap V(H_b) = \{v\}$, and $E(H_a) \cup E(H_b) = E(G)$. Since $H_a \backslash \{v\}$ and $H_b \backslash \{v\}$ have exactly one connected components with proper-path-width k, respectively, $ppw(H_a) = ppw(H_b) = k$, and H_a and H_b are not contained in $\Omega(\mathcal{P}_{k-1})$. Thus, by similar argument, there exists a k-proper-path-decomposition of G, contradicting to that $ppw(G) \geqq k+1$. □

Theorem 9: Let G be a connected graph with a bridge (u, v). Let H_1 be the connected component containing u of the graph obtained from G by deleting (u, v), H_2 be the other component. If $ppw(H_1) = ppw(H_2) = k$ and $G \in \Omega(\mathcal{P}_k)$, then G is a star-composition of (not necessarily distinct) graphs in $\Omega(\mathcal{P}_{k-1})$.

Proof: Assume contrary that G is not a star-composition of three graphs in $\Omega(\mathcal{P}_{k-1})$. Since G is minimal, $H_1 \backslash \{u\}$ and $H_2 \backslash \{v\}$ have at most one connected component with proper-path-width at most k, respectively. Let G' be the graph obtained from G by contracting edge (u, v). Then $ppw(G') = k$, and graphs G', H_1, and H_2 satisfy the condition of Lemma 11. Thus there exist k-proper-path-decompositions $\mathcal{X}^1 = (X_1^1, X_2^1, \ldots, X_{r_1}^1)$ of H_1 and $\mathcal{X}^2 = (X_1^2, X_2^2, \ldots, X_{r_2}^2)$ of H_2 such that $u \in X_{r_1}^1$ and $v \in X_1^2$. Then $(\mathcal{X}^1, \{u, v\}, \mathcal{X}^2)$ is a k-proper-path-decomposition of G, contradicting to that $ppw(G) = k+1$. □

6. Only 36 Graphs

In this section, we show that a graph G is a minimal forbidden minor for $\mathcal{P}_2$ if and only if G is isomorphic to one of 36 graphs shown in Fig. 7.

A *block* of a graph G is a nontrivial connected subgraph of G, which contains no cut-vertices and which is maximal with respect to that property. A block with at least three vertices is called a *cyclic block*. Let G be a graph with a cyclic block B. For any vertex v of a cyclic block B in a graph G, $L(v)$ is the connected component of $G - E(B)$ containing v and $N(v)$ is the number of vertices of $L(v) \backslash \{v\}$ if $L(v) \backslash \{v\}$ consists of isolate vertices, infinite otherwise.

Lemma 12: If a graph G is in $\Omega(\mathcal{P}_2)$, then G is connected and simple, and satisfys one of the following conditions: (1) G is a star-composition of (not necessarily distinct) three graphs in $\Omega(\mathcal{P}_1)$; or (2) G has exactly one cyclic block B, and $N(v) \leqq 3$ for any vertex $v \in V(B)$; or (3) G is isomorphic to a graph 1-equivalent to a graph satisfying (2).

Proof: By Theorem 4, a minimal forbidden minor for $\mathcal{P}_k$ is connected and simple.

Let G be a graph in $\Omega(\mathcal{P}_2)$. Assume that G is not a star-composition of (not necessarily distinct) three graphs in $\Omega(\mathcal{P}_1)$. If G has no cyclic block then G is acyclic, contradicting to Theorem 3. Thus we assume that G has at least one cyclic block.

Assume that there exist distinct cyclic blocks B_1 and B_2 of G. Note that $ppw(B_1) = ppw(B_2) = 2$. If $V(B_1) \cap V(B_2) = \emptyset$ then there exists a bridge (u, v) of G such that the graph obtained from G by deleting (u, v) has two connected components H_1 and H_2, and B_1 and B_2 are subgraphs of H_1 and H_2, respectively. Notice also that $ppw(H_1) = ppw(H_2) = 2$. However, this is contradicting to Theorem 9. Thus $V(B_1) \cap V(B_2) \neq \emptyset$. Let $v \in V(B_1) \cap V(B_2)$. Let H_1 be the connected component of $G \backslash (V(B_2) - \{v\})$ such that B_1 is a subgraph of H_1, and $H_2 = G \backslash (V(H_1) - \{v\})$. Notice that $V(H_1) \cap V(H_2) = \{v\}$, $E(H_1) \cup E(H_2) = E(G)$, and $ppw(H_1) = ppw(H_2) = 2$. By Theorem 8, $H_1 \in \Omega(\mathcal{P}_1)$ or $H_2 \in \Omega(\mathcal{P}_1)$. Without loss of generality, we assume that $H_1 \in \Omega(\mathcal{P}_1)$. Since H_1 has a cyclic block, H_1 is isomorphic to a complete graph $K_3 \in \Omega(\mathcal{P}_1)$. Then G is isomorphic to a graph 1-equivalent to the minimal forbidden minor for $\mathcal{P}_2$ such that the number of connected components is one less than that of G.

Thus we assume, without loss of generality, G has exactly one cyclic block. Let B be the cyclic block of G and w be a vertex in $V(B)$. If $ppw(L(w)) \geqq 2$ then either $G \backslash (V(L(w)) - \{w\})$ or $L(w)$ is in $\Omega(\mathcal{P}_1)$ by Theorem 8. If $G \backslash (V(L(w)) - \{w\}) \in \Omega(\mathcal{P}_1)$ then G is 1-equivalent to a tree. Thus G is a star-composition of three graphs in $\Omega(\mathcal{P}_1)$, a contradiction. Thus $L(w) \in \Omega(\mathcal{P}_1)$. Hence $N(w) = 3$ or G is 1-equivalent to a graph with $N(w) = 3$. If $ppw(L(w)) = 1$, then $d_G(u) \leqq 2$ for any vertex $u \in L(w)$. Since $d_G(u) = 1$ for any vertex $u \in L(w) - \{w\}$ by Theorem 4, $N(w) \leqq 2$. Thus $N(v) \leqq 3$ for any vertex $v \in V(B)$, or G is isomorphic to a graph 1-equivalent to a graph with $N(v) \leqq 3$ for any vertex $v \in V(B)$. □

Since it is easy to characterize the graphs in $\Omega(\mathcal{P}_2)$ obtained by star-compositions of (not necessarily distinct) three graphs in $\Omega(\mathcal{P}_1)$, we consider, in the following, graphs with exactly one cyclic block. A graph is said to be *outer planar* if it can be embedded in the plane so that its edges intersect only at their ends and its vertices lie on the boundary on the exterior region. In the following, we call inner region of outer planar graph, simply, region.

Lemma 13: For any minimal forbidden minor G for $\mathcal{P}_2$, G is not outer planar if and only if G is isomorphic to either K_4 or $K_{2,3}$.

Proof: It is well-known that the class of all outer planar graphs is minor-closed, and K_4 and $K_{2,3}$ are the minimal forbidden minors for the class. Thus any non-outer planar graph G in $\Omega(\mathcal{P}_2)$ is either K_4 or $K_{2,3}$ by Corollary 1. □

To show that a graph $G \in \Omega(\mathcal{P}_2)$ is isomorphic to one of 36 graphs shown in Fig. 7, it is sufficient to consider graphs with one outer planar cyclic block. Let

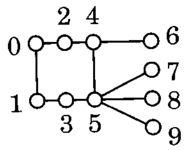


Fig. 8 $P(310000)$.

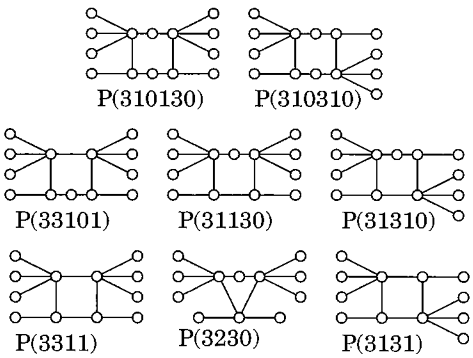


Fig. 9 Graphs with proper-path-widths two.

$P(310000)$ be the graph shown in Fig. 8.

Lemma 14: For any minor G of $P(310000)$, there exists a 2-proper-path-decomposition $(X_1, X_2, \ldots, X_r)$ of G such that $0, 1 \in X_1$.

Proof: $(\{0,1,2\}, \{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,6\}, \{5,6,7\}, \{5,7,8\}, \{5,8,9\})$ is a 2-proper-path-decomposition of $P(310000)$, and satisfys the condition of this lemma. Thus there exists such a proper-path-decomposition for any minor of $P(310000)$. □

Corollary 2: The proper-path-widths of $P(310310)$, $P(310130)$, $P(33101)$, $P(31310)$, $P(31130)$, $P(3311)$, $P(3230)$, $P(3131)$ (shown in Fig. 9) are two.

First, we consider the case that there exists exactly one region in outer planar cyclic block.

Lemma 15: Let G be a simple connected outer planar graph such that G has an outer planar cyclic block B with one region and $N(v) \leqq 3$ for any vertex $v \in V(B)$. If $G \in \Omega(\mathcal{P}_2)$ then G is isomorphic to either $P(333)$, $P(3202)$, $P(2221)$, $P(22010)$, $P(11111)$, or $P(101010)$.

Proof: Assume that G is not isomorphic to either $P(333)$, $P(3202)$, $P(2221)$, $P(22010)$, $P(11111)$, or $P(101010)$. We show that $G \notin \Omega(\mathcal{P}_2)$. To complete the proof, it is sufficient to show that either $ppw(G) = 2$ or there exists a proper minor of G that contained in $\Omega(\mathcal{P}_2)$.

In the following, we consider several cases according to the number of vertices in B.

1. $|V(B)| = 3$.

   Since G is a proper minor of $P(333) \in \Omega(\mathcal{P}_2)$, $ppw(G) = 2$.

2. $|V(B)| = 4$.

   Assume that a vertex $v_1 \in V(B)$ such that $N(v_1) = 3$ is adjacent to a vertex $v_2 \in V(B)$ such that $N(v_2) = 3$. If there exists a vertex $v \in V(B) - \{v_1, v_2\}$ such that $N(v) \geqq 2$, then $P(3202) \in \Omega(\mathcal{P}_2)$ is a proper minor of G, otherwise $ppw(G) = 2$ since G is a minor of $P(3311)$. Thus v_1 is not adjacent to v_2 if $N(v_1) = N(v_2) = 3$. If there exists a vertex $v \in V(B)$ such that $N(v) = 0$, then $ppw(G) = 2$ since G is either a minor of $P(3230)$ or a proper minor of $P(3202) \in \Omega(\mathcal{P}_2)$. Thus there exists no vertex $v \in V(B)$ such that $N(v) = 0$. If there exist at least three vertices v such that $N(v) \geqq 2$, then $P(2221) \in \Omega(\mathcal{P}_2)$ is a proper minor of G, otherwise $ppw(G) = 2$ since G is a minor of either $P(3311)$ or $P(3131)$.

3. $|V(B)| = 5$.

   Assume that a vertex $v_1 \in V(B)$ such that $N(v_1) \geqq 2$ is adjacent to a vertex $v_2 \in V(B)$ such that $N(v_2) \geqq 2$. Let v be a vertex in $V(B)$ such that v is adjacent to neither v_1 nor v_2. If $N(v) \geqq 1$, then $P(22010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus $N(v) = 0$, and $N(s) \geqq 1$ and $N(t) \geqq 1$ for the other two vertices s and t of B by Theorem 4. However, if $N(s) = N(t) = 1$ then $ppw(G) = 2$ since G is a minor of $P(33101)$, otherwise $P(2221) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus v_1 is not adjacent to v_2 if $N(v_1) \geqq 2$ and $N(v_2) \geqq 2$. If $N(v) \geqq 1$ for any vertex $v \in V(B)$, $P(11111) \in \Omega(\mathcal{P}_2)$ is a proper minor of G, otherwise $ppw(G) = 2$ since G is a minor of either $P(31310)$ or $P(31130)$.

4. $|V(B)| = 6$.

   If there exists at most one vertex v in $V(B)$ such that $N(v) = 0$, then $P(11111) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus we assume that there exist at least two vertices v such that $N(v) = 0$. Let v_1 and v_2 be vertices in $V(B)$ such that $N(v_1) = N(v_2) = 0$. Notice that v_1 is not adjacent to v_2 by Theorem 4. If there exists a vertex adjacent to both v_1 and v_2, then $P(101010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G by Theorem 4. Thus there exists no vertex adjacent to both v_1 and v_2. Hence $N(v) \geqq 1$ for each vertex in $V(B) - \{v_1, v_2\}$. If a vertex $v_3 \in V(B)$ such that $N(v_3) \geqq 2$ is adjacent to a vertex $v_4 \in V(B)$ such that $N(v_4) \geqq 2$, then $P(22010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G, otherwise $ppw(G) = 2$ since G is a minor of either $P(310310)$ or $P(310130)$.

5. $|V(B)| > 6$.

   Assume that $|V(B)| = n$ $(n \geqq 7)$. There exists at least $\lceil n/2 \rceil$ vertices v such that $N(v) \geqq 1$ by Theorem 4. If there exist at least five vertices v such that $N(v) \geqq 1$, then $P(11111) \in \Omega(\mathcal{P}_2)$ is a minor of G. Thus $n < 9$. In case when $n = 7, 8$, there exist at least four vertices v such that $N(v) \geqq 1$ and $P(101010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G.

Thus $G \notin \Omega(\mathcal{P}_2)$ and we complete the proof. $\square$

Next, we consider the case that there exist more than one region in outer planar cyclic block.

Lemma 16: Let G be a simple connected outer planar graph such that G has an outer planar cyclic block B with more than one region and $N(v) \leqq 3$ for any vertex $v \in V(B)$. If $G \in \Omega(\mathcal{P}_2)$ then G is isomorphic to either $P(222x)$, $P(220x0)$, $P(x1111)$, $P(x1x11)$, $P(x01010)$, $P(x0x010)$, or $P(x0x0x0)$.

Proof: Assume that G is not isomorphic to either $P(222x)$, $P(220x0)$, $P(x1111)$, $P(x1x11)$, $P(x01010)$, $P(x0x010)$, or $P(x0x0x0)$.

Distinct regions F and F' are said to be adjacent if there exists an edge $e \in E(F) \cap E(F')$. Suppose that some region in G is adjacent to more than two regions in G. Since $P(x0x0x0) \in \Omega(\mathcal{P}_2)$ is a proper minor of G, $G \notin \Omega(\mathcal{P}_2)$. Thus we assume that each region in G is adjacent to at most two regions in G. Let $F_1, F_2, \ldots, F_f$ $(f \geqq 2)$ be regions in G such that F_i and F_{i+1} are adjacent to each other $(1 \leqq i \leqq f-1)$. Notice that $|E(F_i) \cap E(F_{i+1})| = 1$ for any i $(1 \leqq i \leqq f-1)$ since G is outer planar. Let $(u, u') \in E(F_1) \cap E(F_2)$, and $u = u_0, u_1, u_2, \ldots, u_m, u_{m+1} = u'$ be vertices in F_1 such that $(u_i, u_{i+1}) \in E(B)$ $(0 \leqq i \leqq m, m \geqq 1)$. Similarly, let $(v, v') \in E(F_{f-1}) \cap E(F_f)$, and $v = v_0, v_1, v_2, \ldots, v_{m'}, v_{m'+1} = v'$ be vertices in F_f such that $(v_i, v_{i+1}) \in E(B)$ $(0 \leqq i \leqq m', m' \geqq 1)$.

Let $V_1 = V(F_1)$ if $u' \neq v'$, $V_1 = V(F_1) - \{u'\}$ otherwise. Let G_1 be the induced subgraph of G on $V(F_1) \cup \bigcup_{w \in V_1} V(L(w))$.

Claim 1: If $u \neq v$ or $N(u) = 0$ then G_1 is a minor of $P(310000)$.

Proof: Let $a = 1$ if $N(u) = 0$, $a = 0$ otherwise. Let $b = m+1$ if $u' \neq v'$, $b = m$ otherwise. Notice that G_1 is the induced subgraph of G on $V(F_1) \cup \bigcup_{a \leqq i \leqq b} V(L(u_i))$.

Assume that $N(u_i) \geqq 1$ and $N(u_{i+j}) \geqq 1$ for some i and j $(a \leqq i \leqq b-2, 2 \leqq j \leqq b-i)$. Then $P(x01010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G such that (v, v') corresponds to an inner edge of $P(x01010)$. If $N(u_i) \geqq 2$ and $N(u_j) \geqq 2$ for some i and j $(a \leqq i < j \leqq b)$ then $P(220x0) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus G_1 is a minor of $P(310000)$, since no vertices w with $N(w) = 0$ are adjacent to each other by Theorem 4. $\square$

Let $V_2 = V(F_f)$ if $u \neq v$, $V_2 = V(F_f) - \{v\}$ otherwise. Let G_2 be the induced subgraph of G on $V(F_f) \cup \bigcup_{w \in V_2} V(L(w))$. By a similar argument as above, if $u' \neq v'$ or $N(v') = 0$ then G_2 is a minor of $P(310000)$.

Let $V_3 = V(F_2) \cup V(F_3) \cup \cdots \cup V(F_{f-1})$. If there exists a vertex $w \in V_3 - \{u, u', v, v'\}$ such that $N(w) \geqq 1$, then $G \notin \Omega(\mathcal{P}_2)$ since $P(x0x010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus we assume that $N(w) = 0$ for each vertex $w \in V_3 - \{u, u', v, v'\}$. Let G_3 be the induced subgraph of G on V_3. Notice that $V(G) = V(G_1) \cup V(G_2) \cup V(G_3)$. We claim the following.

Claim 2: If both G_1 and G_2 are minors of $P(310000)$, then $G \notin \Omega(\mathcal{P}_2)$.

Proof: Since G_1 is a minor of $P(310000)$, there exists a 2-proper-path-decomposition $\mathcal{X}_1 = (X_1^1, X_2^1, \ldots, X_{r_1}^1)$ of G_1 such that $u, u' \in X_{r_1}^1$ by Lemma 14. Similarly, there exists a 2-proper-path-decomposition $\mathcal{X}_2 = (X_1^2, X_2^2, \ldots, X_{r_2}^2)$ such that $v, v' \in X_1^2$. Since $N(w) = 0$ for each vertex $w \in V_3 - \{u, u', v, v'\}$, it is not difficult to see that there exists a 2-proper-path-decomposition $\mathcal{X}_3 = (X_1^3, X_2^3, \ldots, X_{r_3}^3)$ of G_3 such that $u, u' \in X_1^3$ and $v, v' \in X_{r_3}^3$. Then $(\mathcal{X}_1, \mathcal{X}_3, \mathcal{X}_2)$ is a 2-proper-path-decomposition of G, and $G \notin \Omega(\mathcal{P}_2)$. $\square$

Thus by Claims 1 and 2, without loss of generality, we assume that $u = v$ and $N(u) > 0$. In the following, we consider two cases.

Case 1: Assume that $u' \neq v'$ or $N(u') = 0$. If $N(u_i) \geqq 1$ and $N(v_j) \geqq 1$ for some i and j $(2 \leqq i \leqq m+1, 2 \leqq j \leqq m'+1)$ then either $P(x1x11) \in \Omega(\mathcal{P}_2)$ or $P(101010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus, we have either $N(u_i) = 0$ for any i $(2 \leqq i \leqq m+1)$ or $N(v_j) = 0$ for any j $(2 \leqq j \leqq m'+1)$. Without loss of generality, we assume that $N(u_i) = 0$ for any i $(2 \leqq i \leqq m+1)$. Assume that $N(v_j) \geqq 1$ for some j $(2 \leqq j \leqq m'+1)$. If $N(u) \geqq 2$ and $N(u_1) \geqq 2$, $P(22010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus, either $N(u) = 1$ or $N(u_1) \leqq 1$, and G_1 is a minor of $P(310000)$. Since G_2 is a minor of $P(310000)$ by Claim 1, $G \notin \Omega(\mathcal{P}_2)$ by Claim 2. Thus, $N(v_j) = 0$ for any j $(2 \leqq j \leqq m'+1)$. There exist at most three vertices w such that $N(w) > 0$ in $V_1 \cup V_2$.

Assume that $N(u) \geqq 3$. If $N(u_1) \geqq 2$ and $N(v_1) \geqq 2$ then $P(3202) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus, without loss of generality, we assume that $N(u_1) \leqq 1$ and G_1 is a minor of $P(310000)$. Since G_2 is a minor of $P(310000)$ by Claim 1, $G \notin \Omega(\mathcal{P}_2)$ by Claim 2. Assume that $N(u) \leqq 2$. Let x and y be the vertices in $L(u) \backslash \{u\}$. Let G_a be the induced subgraph of G on $V(F_1) \cup \{x\}$, and G_b be the induced subgraph of G on $V(F_f) \cup \{y\}$. Since both G_a and G_b are minors of $P(310000)$, there exist a 2-proper-path-decomposition $\mathcal{X}_a = (X_1^a, X_2^a, \ldots, X_{r_a}^a)$ of G_a such that $u, u' \in X_{r_a}^a$, and a 2-proper-path-decomposition $\mathcal{X}_b = (X_1^b, X_2^b, \ldots, X_{r_b}^b)$ of G_b such that $v, v' \in X_1^b$. Then $(\mathcal{X}_a, \mathcal{X}_3, \mathcal{X}_b)$ is a 2-proper-path-decomposition of G, and $G \notin \Omega(\mathcal{P}_2)$.

Case 2: Assume that $u' = v'$ and $N(u') > 0$. If there exist at least two vertices u_i $(1 \leqq i \leqq m)$ such that $N(u_i) \geqq 1$, then $P(x1111) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus there exists at most one vertex u_i such that $N(u_i) \geqq 1$ for some i $(1 \leqq i \leqq m)$. We have $m \leqq 3$ by Theorem 4. However, if $m = 3$ then $P(101010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G since $N(u_2) \geqq 1$. Thus we have $m \leqq 2$. Similarly, there exists at most one vertex v_i such that $N(v_i) \geqq 1$ for some i $(1 \leqq i \leqq m')$, and $m' \leqq 2$. Without loss of generality, we assume that $m' \leqq m \leqq 2$.

Assume that $m = 2$. Without loss of generality, we assume that $N(u_1) \geq 1$. If $N(u) \geq 2$ and $N(u_1) \geq 2$, then $P(22010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus, G_1 is a minor of $P(310000)$. If $m' = 2$ and $N(v_1) \geq 1$ then $P(101010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus, we have either $m' = 2$ and $N(v_2) \geq 1$, or $m' = 1$. If $N(v') \geq 2$ and $N(v_{m'}) \geq 2$ then $P(22010) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus G_2 is a minor of $P(310000)$ and $G \notin \Omega(\mathcal{P}_2)$ by Claim 2.

Assume that $m = m' = 1$. Without loss of generality, we assume that $N(u) \geq N(u')$. If $N(u) = 1$ then both G_1 and G_2 are minors of $P(310000)$, since $N(u') = 1$. Thus we assume that $N(u) \geq 2$. If $N(u_1) \geq 2$ and $N(v_1) \geq 2$ then $P(2221) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus, without loss of generality, we assume that $N(u_1) \leq 1$. Then G_1 is a minor of $P(310000)$. If $N(u') \geq 2$ and $N(v_1) \geq 2$ then $P(222x) \in \Omega(\mathcal{P}_2)$ is a proper minor of G. Thus G_2 is a minor of $P(310000)$ and $G \notin \Omega(\mathcal{P}_2)$ by Claim 2.

Thus $G \notin \Omega(\mathcal{P}_2)$ and we complete the proof. □

Theorem 10: A graph G is contained in $\Omega(\mathcal{P}_2)$ if and only if G is isomorphic to one of 36 graphs shown in Fig. 7.

Proof: Since $\Omega(\mathcal{P}_1) = \{K_3, K_{1,3}\}$, the number of graphs in $\Omega(\mathcal{P}_2)$ obtained by star-compositions of (not necessarily distinct) three graphs in $\Omega(\mathcal{P}_1)$ is ten. A graph in $\Omega(\mathcal{P}_2)$ with a cyclic block B and $N(v) \leq 3$ for any vertex $v \in V(B)$ is isomorphic to one of the fifteen graphs shown in Lemmas 13, 15, and 16. The number of graphs 1-equivalent to these graphs are eleven. Thus by Lemma 12, the number of graphs in $\Omega(\mathcal{P}_2)$ is 36, and the graphs shown in Fig. 7 are the minimal forbidden minors for $\mathcal{P}_2$. □

7. Concluding Remarks

In this paper, we characterize $\mathcal{P}_2$ by the minimal forbidden minors for $\mathcal{P}_2$. For applications in linguistics, it would be of definite interest to know the structure of graphs with path-width at most 6 as mentioned by Kornai and Tuza in [9]. Although it is known that the number of minimal forbidden minors for $\mathcal{P}_k$ $(k \geq 3)$ is stupendous, we might be able to characterize all the minimal forbidden minors by simple compositions like as the minimal acyclic forbidden minors. In this sense, some general properties of minimal forbidden minors for $\mathcal{P}_k$ shown in this paper give us a clue to characterize not only the minimal forbidden minors for $\mathcal{P}_k$ but also the minimal forbidden minors for $\mathcal{F}_k$ $(k \geq 3)$.

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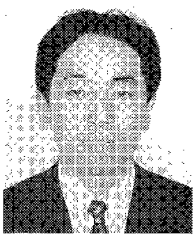

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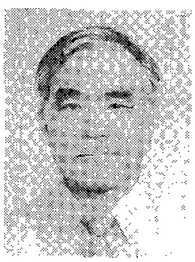

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