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# 論文の要約

平成２５年度博士論文
Hyperbolic Volume, Fibered Commensurability,
and Exceptional Surgeries, Theory v.s. Computation

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# Outline of the paper

A hyperbolic 3-manifold is a Riemannian manifold of constant sectional curvature $-1$. We mainly consider hyperbolic 3-manifolds of finite volume. By the Mostow-Prasad rigidity theorem, two such manifolds are isometric if and only if they are homeomorphic. Hence we may study hyperbolic geometry via topology and vice versa. Further, thanks to the Mostow-Prasad rigidity, computers can be very useful tools for the study of hyperbolic 3-manifolds. In this outline, we explain what we study in the thesis. Here, we lay stress especially on how we used computer.

## 1 Theoretical study

The first main topic of this thesis is the hyperbolic volumes, which is one of the most fundamental geometric invariants of hyperbolic manifolds. We also consider fibered commensurability, a notion introduced in [5].

### 1.1 On volume preserving moves on graphs with parabolic meridians [16].

We discuss cutting operation on hyperbolic 3-manifolds along thrice punctured spheres. If we cut a hyperbolic link complement along thrice punctured spheres, we get a complement of hyperbolic graph with parabolic meridians. By using this relationship between links and graphs, we give a method to compute the best possible upper bounds for volumes of hyperbolic link complements in terms of the twist numbers. Here, twist number is defined for a link diagram as the number of twists, a maximal collection of bigon regions. We compute approximate values of the best possible upper bounds for the case where twist numbers are less than 10. To compute those values we used two programs called plantri [3] and Orb [9]. We first enumerate planar graphs by plantri and compute the (approximate) volumes of them by Orb. Those (approximate) values of upper bounds are better than the known bounds due to Agol-Thurston [14].

### 1.2 On the number of hyperbolic 3-manifolds of a given volume (joint work with Craig Hodgson) [10].

We discuss the number $N(v)$ of hyperbolic 3-manifolds of volume $v$. By a work of Jørgensen and Thurston (c.f. [8]), the number $N(v)$ is known to be finite for all $v \in \mathbb{R}$. We first investigate censuses on SnapPea and computed (approximate) volumes. Then we drew graphs (see figure 1). Those graphs suggest that

- $\sup_{w \leq v} N(w)$ grows exponentially as $v$ goes to infinity, and
- there are infinitely many $v$ with $N(v) = 1$.

We show that the experiments above are actually the case.

**Theorem 1.** *For each $n \geq 3$, there are at least $2^n/(2n)$ different hyperbolic link com-*

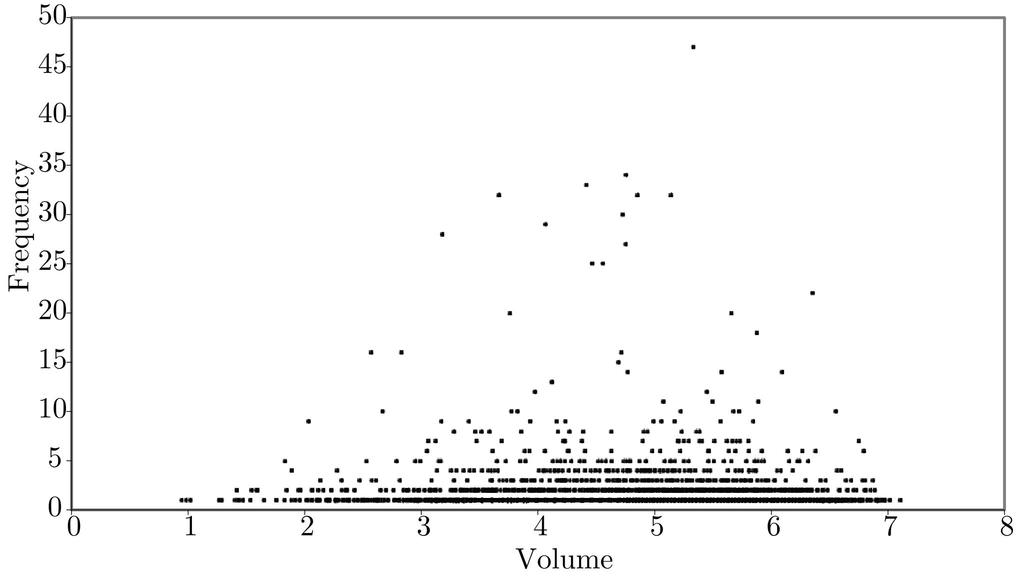


Fig. 1 Frequencies of volumes of hyperbolic 3-manifolds in the closed and cusped censuses.

*plements of volume* $4nV_8$.

**Theorem 2.** *There is an infinite sequence of hyperbolic Dehn fillings* $M(a_i, b_i)$ *on the figure eight knot complement* $M = m004$ *such that the manifolds* $M(a_i, b_i)$ *are determined by their volumes, amongst all finite volume orientable hyperbolic 3-manifolds.*

Here, "m004" is the SnapPea's notation of the figure eight knot complement.

## 1.3 Equidecomposability, volume formulae and orthospectra (joint work with Greg McShane)[18].

We compare two formulae by Bridgeman-Kahn [2] and Calegari [4] that compute the volume of a given compact hyperbolic $n$-manifold $M$ with totally geodesic boundary by using the data called *orthospectrum.* The orthospectrum of $M$ is the set of length of orthogeodesics, geodesics perpendicular to the boundary $\partial M$ at both ends. The two identities are derived by means of different decompositions. They defined decompositions of unit tangent bundle $T_1(M)$ such that each piece corresponds to an orthogeodesic. Let us denote by $\mathcal{B}(l)$ (resp. $\mathcal{C}(l)$) the piece defined by Bridgeman-Kahn (resp. Calegari) that corresponds to an orthogeodesic of length $l$. They also showed that the geometry of each piece is determined by the length of corresponding orthogeodesic. Since $\mathrm{vol}_{2n-1}(T_1(M)) = \mathrm{vol}_n(M)\mathrm{vol}_{n-1}(\mathbb{S}^{n-1})$ ($\mathbb{S}^m$ is the unit sphere of dimension $m$), the sum of the volumes of $\mathcal{B}(l)$ or $\mathcal{C}(l)$, where $l$ runs over the orthospectrum, is equal to $\mathrm{vol}_n(M)$ times $\mathrm{vol}_{n-1}(\mathbb{S}^{n-1})$, a function of $n-1$. Here, $\mathrm{vol}_m$ denotes the volume of an $m$-dimensional orientable manifold. Thus by associating the volume of $\mathcal{B}(l)$ or $\mathcal{C}(l)$ to each element $l$ in the orthospectrum of $M$, we can compute the volume of $M$. Since both formulae hold for every such hyperbolic manifolds, it is natural to ask if the volume of $\mathcal{B}(l)$ and $\mathcal{C}(l)$ coincide or not. For $n = 2$, Calegari showed that $\mathrm{vol}_3(\mathcal{B}(l)) = \mathrm{vol}_3(\mathcal{C}(l))$ [4]. Further, for $n = 3$, we numerically computed the volumes of $\mathcal{B}(l)$ and $\mathcal{C}(l)$, and observed that they are very close for many $l \in \mathbb{R}$. In §3, we verify that the volumes of $\mathcal{B}(l)$ and $\mathcal{C}(l)$ coincide for any $n \geq 2$.

**Theorem 3.** *For all $n \geq 2$,*

$$\mathrm{vol}_{2n-1}(\mathcal{B}(l)) = \mathrm{vol}_{2n-1}(\mathcal{C}(l)).$$

Both Bridgeman-Kahn and Calegari derived formulae for the volume of $\mathcal{B}(l)$ and $\mathcal{C}(l)$, which involve integrals and are complicated. We derived a simple formula of the volume of $\mathcal{C}(l)$ (and hence, $\mathcal{B}(l)$) for the case where $n = 3$.

**Theorem 4.**

$$\mathrm{vol}_5(\mathcal{C}(l)) = \frac{2\pi(l+1)}{e^{2l}-1}.$$

### 1.4 On commensurability of fibrations on a hyperbolic 3-manifold [17].

We consider fibered commensurability of fibrations on a hyperbolic 3-manifold. The notion of fibered commensurability is introduced by Calegari-Sun-Wang [5]. Note that for a given fibration on an orientable 3-manifold, we have an associated pair $(F, \phi)$ of the fibre surface $F$ and the monodromy $\phi : F \to F$. Roughly speaking, two fibrations (on possibly distinct manifolds) corresponding to $(F_1, \phi)$ and $(F_2, \phi)$ are said to be commensurable if there is a finite covering $\widetilde{F}$ of both $F_1$ and $F_2$, and the lifts $\widetilde{\phi_i}$ of $\phi_i$ $(i = 1, 2)$ satisfying $\widetilde{\phi_1}^{k_1} = \widetilde{\phi_2}^{k_2}$ for some $k_1, k_2 \in \mathbb{Z}\backslash\{0\}$. Two fibrations on a 3-manifold $M$ are said to be symmetric if there is a homeomorphism $\varphi : M \to M$ that maps one to the other. We give a necessary condition for manifolds to have non-symmetric but commensurable fibrations.

**Theorem 5.** *Suppose that $M$ is a hyperbolic 3-manifold that does not have hidden symmetries. Then, any pair of fibrations of $M$ is either symmetric or non-commensurable, but not both.*

Here, a hidden symmetry of $M$ is an element of the commensurator of $\Gamma$ which is not in the normalizer of $\Gamma$, where $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is the image of a holonomy representation of $\pi_1(M)$. On the other hand, we also show that there exist manifolds with many commensurable fibrations.

**Theorem 6.** *For any $n \in \mathbb{N}$, there exists a hyperbolic 3-manifolds with at least $n$ mutually non-symmetric but commensurable fibrations.*

## 2 Computational study

In the theoretical part, we mostly used computer to have questions that quite likely to be true, and verified them theoretically. In most of those questions, infinitely many values or manifolds are involved, and hence we cannot prove them naively by using computer. On the other hand, once we have a theory that shows "finiteness", then computers can be powerful tools to prove a theorem. In the second part of this thesis, we develop computer programs that perform rigorous proofs.

## 2.1 Verified computations for hyperbolic 3-manifolds (joint work with Neil Hoffman, Kazuhiro Ichihara, Masahide Kashiwagi, Shin'ichi Oishi, and Akitoshi Takayasu)[13].

We developed the package *hikmot* that enables us to prove the hyperbolicity of a given triangulated 3-manifold. The program hikmot can fail verifying the hyperbolicity even though a given manifold is actually hyperbolic, and hence it can NOT prove the non-hyperbolicity. However, hikmot usually works and in fact we can prove hyperbolicity of all manifolds in several censuses on SnapPy [7].

**Theorem 7.** *All the manifolds in OrientableCuspedCensus [6] are hyperbolic.*

**Theorem 8.** *All the manifolds in OrientableClosedCensus [11] are hyperbolic.*

Since to prove the hyperbolicity, we deal with equations with complex variables, so-called gluing equations, the notion of *interval arithmetic* plays a key role. On computer, we cannot deal with *all* real numbers or complex numbers. We usually use floating point arithmetic, that is an approximate computation. Therefore, in principle, it cannot prove any mathematical theorem concerning real values or complex values. The interval arithmetic has been introduced in order to solve this problem [20, 21, 19]. Instead of dealing with approximate value, in interval arithmetic, we compute intervals that contain rigorous values. In principle, this cannot prove any equality, however, it can prove inequalities. To solve a gluing equation, we use Newton's method. If we apply Newton's method by using floating point arithmetic, then we can only get approximate solutions and more critically, this cannot verify the convergence of Newton's method. There are several theorems that give sufficient conditions, which are inequalities, for the convergence of Newton's method. Putting those theorems and interval arithmetic together, we can prove that there is a solution for a gluing equation and hence the hyperbolicity of a manifold.

## 2.2 Exceptional surgeries on alternating knots (joint work with Kazuhiro Ichihara)[12].

We use hikmot together with a python code based on the codes in [15] to classify all the exceptional surgeries along alternating knots. Thurston [22] has proved that for a given hyperbolic knot, all but finitely many surgeries along the knot give hyperbolic manifolds. We call those finitely many surgeries that give non-hyperbolic manifolds exceptional. The codes in [15] are based on SnapPea. The code first applies so-called the 6-theorem due to Agol [1] and Lackenby [14]. The 6-theorem shows that all surgery slopes of geometric lengths greater than 6 give rise to hyperbolic manifolds by Dehn surgery. Then by using SnapPea, the code tries to find hyperbolic metrics on manifolds obtained by Dehn surgery along slopes of geometric length less than or equal to 6. We modified the codes [15] so that each computation is done by interval arithmetic. (To apply the 6-theorem we need to verify an inequality). For slopes of geometric length less than or equal to 6, we use hikmot to prove the hyperbolicity of the surgered manifolds. The codes might return slopes which are non-exceptional, however, never miss slopes of exceptional surgeries. Thanks to several known works to classify the exceptional surgeries along alternating knots, it suffices to prove that about 30000 links with a condition on the slopes for certain components do not admit exceptional surgeries. By applying our codes to those links, we see that none of such links admit any exceptional surgeries that satisfy the condition. In average, it takes about one hour to verify non-existence of exceptional surgeries for each link. Since 30,000 hours $\approx$ 3.4 years, we used TSUBAME, the supercomputer of Tokyo Tech. Thus with a big aid of computer, we have a classification of the exceptional surgeries along the alternating knots.

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