論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	Sampled-Parameter Feedback Control for Dynamical Systems with Stochastic Parameters
著者(和文)	AHMET CETINKAYA
Author(English)	AHMET CETINKAYA
出典(和文)	学位:博士(学術), 学位授与機関:東京工業大学, 報告番号:甲第9580号, 授与年月日:2014年3月26日, 学位の種別:課程博士, 審査員:早川　朋久,木村　康治,井村　順一,山北　昌毅,石井　秀明
Citation(English)	Degree:Doctor (Academic), Conferring organization: Tokyo Institute of Technology, Report number:甲第9580号, Conferred date:2014/3/26, Degree Type:Course doctor, Examiner:,,,,
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Sampled-Parameter Feedback Control for Dynamical Systems with Stochastic Parameters

by

Ahmet Cetinkaya

Submitted to Department of Mechanical and Environmental Informatics
in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

Tokyo Institute of Technology
February 2014

Table of Contents

List of Figures

Summary

Control of complex dynamical systems is an important and active research area. Dynamical models of various complex real life systems from biology, physics, finance, and engineering fields incorporate *randomly varying parameters*. These parameters often describe the state of *external environment* in which the system under consideration operates, and hence they may not be directly measurable or may not be observed as frequently as the state of the system itself. Therefore, when a control problem for a complex dynamical system with stochastic parameters is explored, one needs to take into account the fact that system parameters may *not* be available for control purposes at all time instants.

In this thesis we address feedback control problem for *linear stochastic parameter-varying systems,* under the assumption that the controller has access *only* to *sampled information* of parameters.

We first develop *sampled-parameter* feedback control frameworks for linear systems with a stochastic parameter that takes values from a finite set. Each value of the parameter represents a *mode* of the dynamical system. The overall system in this case is called a "switched system", since the operation mode of the dynamics *switches* when the parameter changes its value. Switched stochastic systems accurately characterize complex processes that are subject to dynamical changes due to sudden environmental variations. We investigate continuous-time switched systems with a randomly varying mode signal. This mode signal is assumed to be observed (sampled) periodically. In this case, information about the operation mode of the switched system is available for control purposes only at periodic mode observation instants. We propose a feedback control law that achieves stabilization of the system states by using only *periodically observed (sampled) mode information*. We then direct our attention to a more complex feedback control problem for

the case where the sampled mode information is subject to *time delay* before it becomes available for control purposes. This time delay may emanate from communication delays between the mode sampling mechanism and the controller or computational delays in mode detection. We propose stabilizing control laws that depend only on *delayed and sampled* version of the mode signal.

In addition to continous-time switched stochastic systems, we also explore the feedback control problem for discrete-time switched stochastic systems. We consider the case where the mode of the switched system is periodically observed. We develop a stabilizing feedback control framework that incorporates *sampled-mode-dependent* and *time-varying* feedback gains, which allow stabilization despite the uncertainty of the operation mode between consecutive mode observation instants. We utilize the *periodicity* induced in the system dynamics due to *periodic mode observations*, and employ discrete-time Floquet theory to obtain necessary and sufficient conditions for the stabilization of the system states. Furthermore, we address the case where mode information obtained through periodic observations is *imprecise*. *Imprecise mode information* characterizes the situation where some of the modes are indistinguishable by the mode detector. Specifically, in this situation, the modes of the switched system are divided into a number of groups, and the controller periodically receives information of the *group* that contains the active operation mode. For this case, we develop a feedback control law that guarantees stabilization by using only the group information rather than a precise information of the active mode.

Next, we address feedback control problem for continuous-time and discrete-time switched stochastic systems for the case where the mode of the switched system is observed at *random* time instants. For the continuous-time case, we develop a stabilizing control law under the assumption that the lengths of intervals between mode sampling instants are exponentially distributed independent random variables. For this particular case, we observe that the closed-loop system under our proposed sampled-mode control law can be modeled as a switched linear stochastic system with a mode signal that is defined to be a bivariate stochastic process composed of the actual mode signal and its sampled version. On the other hand, for the discrete-time case we do not assume a particular structure for the distribution of the lengths of intervals between the time instants at which mode is sampled. We observe that this characterization encapsulates periodic mode observations as a special case. Our investigation for the discrete-time case is predicated on the analysis of a sequence-valued process that encapsulates the stochastic nature of the evolution of active operation mode between mode observation instants.

Parameters of certain dynamical system models from engineering field evolve in multidimensional spaces composed of a continuum of points. Hence, dynamical systems with such kind of parameters can not be characterized as switched systems. In the last part of this thesis, we explore sampled-parameter feedback control of discrete-time dynamical systems with stochastic parameters that are defined on *multidimensional spaces*. Furthermore, we investigate a special class of linear parameter-varying systems where the system matrix depends *affinely* on the entries of the stochastic parameter vector. For this class of parameter-varying systems, we show that stabilization can be achieved by using a control law with a feedback gain that is an *affine function* of the entries of the *sampled parameter vector*.

All sampled-parameter feedback control frameworks that we develop in this thesis have *guaranteed stabilization properties*. Specifically, we obtain conditions under which our proposed control laws guarantee that the system states converge to the zero solution for all *possible* trajectories of the parameters.

Chapter 1

Introduction

1.1 Dynamical Systems with Stochastic Parameters

The framework developed for dynamical systems with time-varying parameters has been indispensable in modeling complex real life processes and has found applications in various fields such as aeronautics, energy, and automotive systems as well as bio-informatics and finance [1–3]. Specifically, parameters of many dynamical systems vary in a stochastic fashion. For example, the dynamical model of a flight control system includes the time-varying parameter *airspeed*, which is modeled as a stochastic process [4, 5]. Moreover, for power systems, *load profile* is a time-varying parameter of the dynamical model and characterized by a stochastic process in several studies (see [6, 7] and the references therein). Furthermore, researchers of population dynamics also use dynamical models that incorporate randomly varying parameters [8–13]. Note that *growth rate*, which is a parameter of the population model of a species, changes randomly within time depending on variations in environment such as increase or decrease in food resources. Dynamical models with stochastic parameters have also been adopted in finance. For instance, models that describe stock prices often include randomly varying parameters [14–17]. Specifically, *volatility*, which is a parameter of stock price models, is subject to random variations due to market trends that are influenced by external social, economical, and political changes.

There is a common feature between dynamical system models used in different application fields. Randomly varying parameters of these dynamical system models describe the state of *external environment*, and hence they may not be directly measurable or may not be observed as frequently as the state of the system itself. Therefore, when a control problem for dynamical systems with stochastic parameters is considered, one must take into account the fact that perfect knowledge of the parameters may not be available for control purposes at all time instants. Motivated by this point, in this thesis, we address the control problem for the case where *only sampled information* of parameters is available for control purposes. Our goal is to develop *sampled-parameter control frameworks* that are effective for controlling linear dynamical systems when parameters are not observed exactly, instantaneously, or as frequently as the system state.

1.2 Sampled-Parameter Feedback Control

Control of complex dynamical systems with time-varying parameters is an important and active research area. For many practical applications, it is desirable that the system state is stabilized around a set point by a controller despite the effect of varying parameters. Feedback control of linear systems with varying parameters has been explored in many studies (e.g. [18–40]). The state-space description of continuous-time linear dynamical systems with time-varying parameters is given by

$$\dot{x}(t) = A(\xi(t))x(t) + B(\xi(t))u(t), \quad t \geq 0, \tag{1.1}$$

where $x(\cdot)$ and $u(\cdot)$ respectively denote the state and control input vectors; furthermore, $A(\cdot)$ and $B(\cdot)$ are state and input matrices that depend on the parameter $\xi(\cdot)$.

In the literature, the control input $u(\cdot)$ is often designed based on the assumption that the parameter $\xi(\cdot)$ can be measured at all times instants $t \geq 0$ (see [18–26, 41, 42]). For example, many researchers have investigated stabilization of the state of the dynamical system (1.1) towards the origin with a control law of the form $u(t) \triangleq K(\xi(t))x(t)$. Note that in this case, $K(\cdot)$, which denotes the feedback gain, depends directly on the parameter $\xi(\cdot)$. On the other hand, in [43], a controller is designed for the case where only certain components of the parameter vector $\xi(\cdot)$ is measurable. Furthermore, in [43–45], researchers propose a feedback control law that depends on a noisy measurement of the parameter $\xi(\cdot)$. Note that all abovementioned control laws require continuously available information concerning the parameter. In the case where parameter information is not continuously available, these suggested control laws would not be applicable. The same issue arises also in the discrete-time case, where the dynamical model is described by the difference equation

$$x(k+1) = A(\xi(k))x(k) + B(\xi(k))u(k), \quad k \in \mathbb{N}_0. \tag{1.2}$$

Note that the parameter $\xi(\cdot)$ may not be available for control purposes at all time instants $k \in \mathbb{N}_0$.

In the case where the parameter $\xi(\cdot)$ is not observed at all time steps, we need alternative control frameworks that do not require knowledge of the parameter at all times. To deal with such cases, one can use feedback control laws that are independent of the parameter (e.g., [27–31, 37–40]). In this case, the control law takes the form $u(k) = Kx(k)$ for the discrete-time system, and $u(t) = Kx(t)$ for the continous-time system, where K is a fixed constant gain matrix that is independent of the parameter. However, in this problem setting, finding a constant feedback gain K that achieves stabilization despite the uncertainty of the time-varying parameter $\xi(\cdot)$ is a difficult problem. On the other hand, if the parameter can be observed (sampled) at certain time instants (even if rarely), this sampled parameter information can be utilized in the control framework.

In this thesis, we consider the stabilization problem under the assumption that the controller has access only to *sampled parameter information*. We explore the case where the parameter is observed periodically, as well as the case where the information about the parameter is obtained at random time instants. Furthermore, we address *sampled-parameter* feedback control problem for both linear systems with a stochastic parameter that takes values from a finite set and linear systems with a parameter that evolves in $\mathbb{R}^l$.

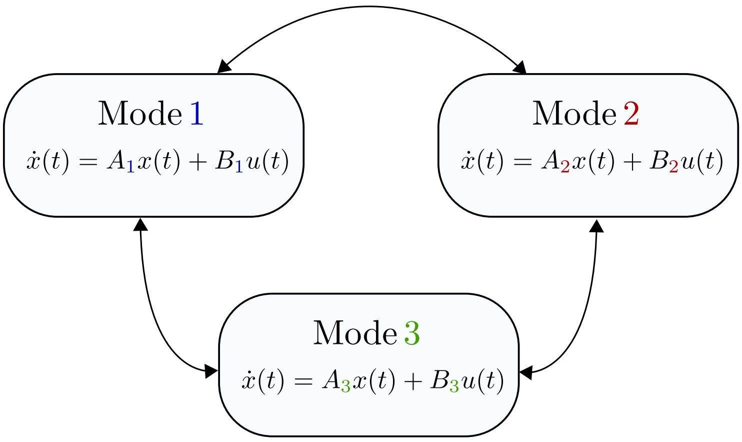

Figure 1.1: Modes of a continuous-time switched stochastic system

1.3 Sampled-Mode Feedback Control of Switched Stochastic Systems

In the case where the stochastic parameter of a dynamical system takes values from a finite set, each value of the parameter represents a *mode* of the dynamical system. The mode of the dynamics *switches* when the parameter changes its value. The overall system in this case is called a "switched system". Fig. 1.1 shows possible transitions (switches) between the modes of a switched stochastic system with 3 modes. Note that the dynamics of the ith mode is characterized by state and input matrices A_i and B_i.

Mode signal of a switched stochastic system characterizes the random transitions between the modes of the switched system and it is modeled as a finite-state stochastic process. In the literature, the mode signal is often modeled as a time-homogeneous Markov chain (e.g., [15, 46–53]). Note that the value of the mode signal determines the index of the active subsystem (mode) that will govern the dynamics until the next mode switching instant.

Feedback control problem for switched stochastic systems has been investigated in many studies (e.g., [4, 14, 41, 42, 49, 51, 54–66] and the references therein). Most of the control frameworks developed for switched stochastic systems require the availability of information on the active operation mode at all times. Note that for numerous applications the active mode describes the operating conditions of a physical process and is driven by external incidents of stochastic nature. The active mode, hence, may not be directly measurable and it may not be available for control purposes at all time instants during the course of operation. When the controller does not have access to any mode information, for achieving stabilization one can resort to adaptive control frameworks [67–69] or mode-independent control laws [63, 70–72]. Furthermore, an estimate of the mode signal can also be employed for control purposes [73, 74]. It is also mentioned in [75, 76] that mode information may be recovered from system state observations. However, mode information recovery is difficult when there is noise in the dynamics. On the other hand, if mode information can be observed (sampled) at certain time instants, this sampled information can be utilized in the control framework.

One of the main goals of this thesis is to explore the feedback control problem for switched stochastic systems under sampled mode information. In this regard, we first address the case where the mode is sampled periodically. However, periodic mode sampling (observation) is not always possible. Note that there are certain situations where mode information can only be obtained at *random* time instants. For example the mode may be sampled periodically; however, due to random losses in the communication between mode sampling mechanism and the controller, some of the mode samples do not reach the controller. Furthermore, in some applications, the active operation mode has to be detected, but the detected mode information would not be always accurate. In this case each mode detection has a confidence level. Mode information with low confidence is discarded. As a result, depending on the confidence level of detection, the controller may or may not receive the mode information at a particular mode detection instant. In this thesis, we also address the feedback control problem for such cases where the information of the mode signal is randomly available to the controller.

Note that feedback control problem setting with sampled mode information is appropriate for applications where the mode cannot be observed as frequently as the system state. On the other hand, sampled mode information may also be subject to *delays*. Specifically, each sampled mode data may become available to the controller after a delay. Addressing this problem is crucial, because in practical applications there may be delays in mode detection. In this thesis, we address this problem and propose a control law that depends only on the *sampled and delayed* mode information.

In the literature, switched stochastic systems have also been used for modeling fault-tolerant control systems (see [77, 78]). Fault-tolerant systems are composed of a normal operation mode and a number of faulty modes. Faulty modes are associated with failures of different components of a process. Furthermore, failures are often detected through diagnostic tests, which may fail to identify the exact type of the failure. When a failure is detected, the controller receives the information that there was a failure; however, the exact information of the type of the faulty mode may not be available for control purposes. In this thesis, we develop control frameworks that can deal with such situations. Specifically, we investigate feedback control of switched stochastic systems for the case where mode information obtained through observations is not precise. In order to model *imprecision* of mode information, we divide the modes of the switched system into a number of groups, and consider the case where the controller periodically receives information of the group that contains the active mode. We propose a control law that depends only on the periodically available mode group information, rather than the exact information of the mode.

1.4 Sampled-Parameter Feedback Control of Dynamical Systems with Parameters Defined on $\mathbb{R}^l$

In the switched system framework, the stochastic parameter of the system takes its values from a finite set. On the other hand, certain complex real life processes from biology, mechanical engineering, and finance incorporate randomly varying parameters that takes values from sets with uncountably many elements (see [79–81]; and the references therein). One of the main goals of this thesis is to address feedback control of dynamical systems with stochastic parameters that evolve in a multidimensional state space. Specifically, we develop a stabilizing control framework for discrete-time linear dynamical systems for the case where the system parameter is observed (sampled) periodically.

The analysis for systems with stochastic parameters that evolve in multidimensional spaces with uncountably many elements is more complicated than the analysis for switched stochastic systems. In our analysis we rely on *stationarity* and *ergodicity* properties of a stochastic process that represents the sequences of values that the system parameter takes between consecutive observation instants.

In many studies that deal with linear dynamical systems with time-varying parameters, researchers embrace models with affine parameter-dependence (e.g., [2, 82, 83]). In such models system matrices are affine functions of the entries of the parameter vector. In the literature, researchers often employ control laws that depend on perfect information of the parameter at all time instants. In this thesis, we explore linear parameter-varying systems where the state matrix is an affine function of the entries of the parameter vector. We show that stabilization for this class of parameter-varying systems can be achieved through a control law with a feedback gain that is an affine function of the entries of the sampled parameter vector.

1.5 Outline of the Thesis

We introduce the notation in Chapter 2, where we also present several definitions and some key results concerning continuous-time and discrete-time stochastic processes. Furthermore, the definitions of the stochastic stability notions “almost sure asymptotic stability” and “second moment asymptotic stability” are provided also in Chapter 2.

In Chapter 3, feedback stabilization of continuous-time switched linear stochastic dynamical systems is explored. The mode signal, which characterizes the switching between subsystems, is modeled as a continuous-time Markov chain. We propose a feedback control law that depends only on the uniformly (periodically) sampled mode information rather than the actual mode signal. We analyze the probabilistic dynamics of the sampled mode information, and develop a form of strong law of large numbers to show the *almost sure asymptotic stability* of the closed-loop system under the proposed control law.

In Chapters 4 and 5, we consider the feedback control problem for the case where the mode of a continuous-time switched system is periodically sampled at discrete time instants and obtained sampled mode information is subject to *time delays*. In Chapter 4 we analyze the stability of the closed-loop switched stochastic system under our proposed control law with a piecewise-constant feedback gain that depends on *delayed and sampled* version of the mode signal. The results presented in Chapter 4 are based on our probabilistic analysis of a bivariate stochastic process that is composed of the actual mode signal and its delayed sampled version. Next, in Chapter 5 we propose a new control framework that relies on a *probability-based feedback gain scheduling scheme* that utilizes the available delayed sampled mode data as well as a priori information concerning the probabilistic dynamics of the mode signal. Specifically, the feedback-gain scheduling method is based on selecting the gain associated with the mode that has the highest conditional probability of being active given the most recent sampled and delayed mode data.

In Chapter 6, *second-moment asymptotic stabilization* of a discrete-time switched stochastic system is investigated. Active operation mode of the switched system is assumed to be only periodically observed (sampled). We develop a stabilizing feedback control framework that incorporates sampled-mode-dependent time-varying feedback gains, which allow stabilization despite the uncertainty of the active operation mode between consecutive mode observation instants. We employ discrete-time Floquet theory to obtain necessary and sufficient conditions for second-moment asymptotic stabilization of the zero solution. Furthermore, we use Lyapunov-like functions with periodic coefficients to obtain alternative stabilization conditions, which we then employ for designing feedback gains.

In Chapter 7, we propose a feedback control law for discrete-time switched stochastic systems that depends only on the periodically obtained *imprecise* mode information. Specifically, the modes of the switched system are assumed to be divided into a number of groups, and the periodically available mode information indicates only the group that contains the active mode. We obtain sufficient conditions for second moment asymptotic stability of the closed-loop system under our proposed control law which depends only on mode group information.

In Chapters 8 and 9, we investigate feedback control of continuous- and discrete-time switched stochastic systems for the case where the mode of the switched system is observed at *random* time instants. Specifically, in Chapter 8, we explore almost sure asymptotic stabilization problem of continuous-time switched linear stochastic dynamical systems, for which the mode signal is modeled as a Markov chain. Intervals between the mode sampling time instants are assumed to be exponentially distributed random variables. We show that the *bivariate process* composed of the *actual mode signa*l and its *sampled version* is a finite-state continuous-time Markov chain due to the exponential distribution property of the mode sampling intervals. Based on this result, we obtain sufficient conditions under which our proposed control law achieves almost sure asymptotic stabilization. Next, in Chapter 9, feedback control of a discrete-time switched stochastic system is explored for the case where the active operation mode is observed only at *random* time instants. A stabilizing control law that utilizes the information obtained through mode observations is proposed. We analyze the probabilistic dynamics of a *sequence-valued stochastic process* that captures the key properties of the evolution of active mode between mode observation instants. We then use the results of our analysis to obtain sufficient conditions under which our proposed control law guarantees almost sure asymptotic stabilization.

In Chapter 10, feedback stabilization of a discrete-time linear stochastic parameter-varying system is explored. The parameter of the system is modeled as a discrete-time *stationary* and *ergodic Markov process* on $\mathbb{R}^l$. We develop a stabilizing control framework for the case where the system parameter is observed (sampled) periodically. We obtain sufficient conditions under which almost sure asymptotic stabilization of the closed-loop stochastic parameter-varying system is guaranteed by our proposed control law, which depends only on the sampled version of the system parameter. Furthermore, we explore a special class of linear parameter-varying systems where the state matrix is an *affine function* of the entries of the *parameter vector*. We show that stabilization for this class of parameter-varying systems can be achieved through a control law with a feedback gain that is an *affine function* of the entries of the *sampled parameter vector*.

Note that in each chapter we present illustrative numerical examples to demonstrate the efficacy of our results on the sampled-parameter feedback control problem for dynamical systems with stochastic parameters.

Finally, we give concluding remarks and provide a discussion on future extensions in Chapter 11.

Chapter 2

Mathematical Preliminaries

In this chapter, we first introduce the notation used in the thesis, we then present several definitions and some key results concerning continuous-time and discrete-time stochastic processes. Furthermore, we provide the definitions of several stochastic stability notions that are used throughout the following chapters.

2.1 Notation

We denote positive and nonnegative integers by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. Moreover, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices. We write $(\cdot)^{\mathrm{T}}$ for transpose, $\|\cdot\|$ for the Euclidean vector norm, and $\otimes$ for Kronecker product. Furthermore, we use $\lfloor \cdot \rfloor$ to denote the largest integer that is less than or equal to its real argument, $\mathrm{tr}(\cdot)$ for trace of a matrix, I_n for the identity matrix of dimension n, and $\lambda_{\min}(H)$ (resp., $\lambda_{\max}(H)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix H. We represent a finite-length sequence of ordered elements $q_1, q_2, \ldots, q_n$ by $q = (q_1, q_2, \ldots, q_n)$. Furthermore, the length (number of elements) of the sequence q is denoted by $|q|$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $V(x) > 0$, $x \neq 0$, and $V(0) = 0$. We use ∇V to denote the vector of the first order spatial derivatives of a twice continuously differentiable scalar-valued function V, that is, $\nabla V = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right]$, and we use $\nabla(\nabla V)$ to denote the matrix of the second-order spatial derivatives of V, that is,

$$\nabla(\nabla V) = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{bmatrix}.$$

Now, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We use the notation $\mathbb{E}[\cdot]$ to denote the expectation. Conditional expectation of a random variable $x : \Omega \to \mathbb{R}$ given an event $G \in \mathcal{F}$ (with $\mathbb{P}[G] > 0$) is denoted by

$$\mathbb{E}[x|G] \triangleq \frac{1}{\mathbb{P}[G]} \int_G x(\omega)\mathbb{P}(\mathrm{d}\omega). \tag{2.1}$$

Furthermore, conditional expectation of a random variable $x : \Omega \to \mathbb{R}$ given a σ-algebra $\mathcal{H}$ is defined to be the $\mathcal{H}$-measurable unique random variable $\mathbb{E}[x|\mathcal{H}]$ such that

$$\int_A \mathbb{E}[x|\mathcal{H}]\mathbb{P}(\mathrm{d}\omega) = \int_A x(\omega)\mathbb{P}(\mathrm{d}\omega), \quad A \in \mathcal{H}.$$

We use $\mathbb{1}_{[\cdot]} : \Omega \to \{0, 1\}$ to denote the indicator function, defined by

$$\mathbb{1}_{[G]}(\omega) = \begin{cases} 1, & \omega \in G, \\ 0, & \omega \notin G, \end{cases} \quad G \in \mathcal{F}. \tag{2.2}$$

2.2 Continuous-time Stochastic Processes

A continuous-time stochastic process is a collection of random variables $x_t : \Omega \to \mathbb{R}^n$ parametrized by the time variable $t \in [0, \infty)$ (see [84–86]). Furthermore, a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined to be a family of σ-algebras such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad 0 \leq s < t.$$

A stochastic process $\{x_t \in \mathbb{R}^n\}_{t\geq 0}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if the random variable $x_t : \Omega \to \mathbb{R}^n$ is $\mathcal{F}_t$-measurable, that is,

$$\{\omega \in \Omega : \, x_t(\omega) \in B\} \in \mathcal{F}_t, \quad t \geq 0,$$

for all Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra associated with $\mathbb{R}^n$.

In the following we provide definitions of continuous-time Markov chains and Poisson processes.

2.2.1 Continuous-Time Finite-State Markov Chains

A finite-state, continuous-time Markov chain $\{r(t) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{t\geq 0}$ with $r(0) = r_0 \in \mathcal{M}$, is an $\mathcal{F}_t$-adapted, piecewise-constant and right-continuous stochastic process characterized by a generator matrix $Q \in \mathbb{R}^{M\times M}$. The generator matrix $Q \in \mathbb{R}^{M\times M}$ determines the transition rates between each pair of states $i, j \in \mathcal{M}$ such that

$$\mathbb{P}[r(t+\Delta t) = j | r(t) = i] = \begin{cases} q_{i,j}\Delta t + o(\Delta t), & i \neq j, \\ 1 + q_{i,j}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $q_{i,j}$ denotes the (i,j)th element of the matrix Q. Note that $q_{i,j} \geq 0$, $i \neq j$, and $q_{i,i} = -\sum_{j\neq i} q_{i,j}$, $i \in \mathcal{M}$. Furthermore, transition probabilities for each pair of states $i, j \in \mathcal{M}$ are given by

$$\mathbb{P}[r(t+\tau) = j | r(t) = i] = p_{i,j}(\tau), \quad t, \tau \geq 0, \tag{2.3}$$

where $p_{i,j}(\tau)$ denotes the (i,j)th element of the matrix $e^{Q\tau}$. A Markov chain is called "irreducible" if it is possible to reach from any state to another state with one or more transitions. For all finite-state, irreducible, continuous-time Markov chains there exists a unique stationary probability distribution $\pi \triangleq [\pi_1, \ldots, \pi_M]^{\mathrm{T}} \in \mathbb{R}^M$ such that $\pi^{\mathrm{T}} Q = 0$, $\pi_i > 0$, $i \in \mathcal{M}$, and $\sum_{i\in\mathcal{M}} \pi_i = 1$ [86, 87].

In Chapters 3–5, and 8, the mode signal, which manages the transition between subsystems (modes) of a switched stochastic continuous-time dynamical system, is modeled as a finite-state continuous-time Markov chain.

2.2.2 Continuous-Time Poisson Processes

A continuous-time Poisson process is a stochastic process that counts the number of occurrences of some events. Mathematically, it is defined to be the $\mathcal{F}_t$-adapted stochastic process $\{N(t) \in \mathbb{N}_0\}_{t\geq 0}$ with $N(0) = 0$, where $N(t)$ denotes the number of events that occur in the time interval $(0, t]$. Probability of the occurrence of an event in a short time interval $(t, t + \Delta t]$ is given by

$$\mathbb{P}[N(t + \Delta t) = k + 1 \,|\, N(t) = k] = \lambda\Delta t + o(\Delta t), \quad k \in \mathbb{N}_0, \tag{2.4}$$

where $\lambda > 0$ denotes the intensity of occurrences. Length of intervals between consecutive events are distributed by the exponential distribution with parameter λ. A Poisson process has "stationary and independent increments". "Independent increments" property suggests that occurrences of events in non-overlapping intervals are independent. Moreover, as a result of "stationary increments" property, the number of events in any time interval is distributed with Poisson distribution depending only on the length of the interval. For Poisson processes, the probability of occurrences of more than one event at a time is zero. Additionally, only finite number of events occur in finite time intervals, almost surely.

Note that in Chapter 8, we employ a Poisson process to characterize the occurrences of random mode observations.

2.3 Discrete-Time Stochastic Processes

A discrete-time stochastic process is a collection of random variables $x_k : \Omega \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, a filtration in the discrete-time setting is defined to be a collection of σ-algebras $\{\mathcal{F}_k\}_{k\in\mathbb{N}_0}$ such that

$$\mathcal{F}_j \subset \mathcal{F}_k \subset \mathcal{F}, \quad j \leq k, \quad j, k \in \mathbb{N}_0.$$

A discrete-time stochastic process $\{x_k \in \mathbb{R}^n\}_{k\in\mathbb{N}_0}$ is called $\mathcal{F}_k$-adapted if the random variable $x_k : \Omega \to \mathbb{R}^n$ is $\mathcal{F}_k$-measurable, that is,

$$\{\omega \in \Omega : x_k(\omega) \in B\} \in \mathcal{F}_k, \quad k \in \mathbb{N}_0,$$

for all Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$ [88].

In what follows, we first define discrete-time finite-state Markov chains, and then we explain Markov chains on countable spaces. We follow with a definition of discrete-time renewal processes, and characterize Markov processes on $\mathbb{R}^l$. Finally, we explain notions of stationarity and ergodicity for discrete-time stochastic processes.

2.3.1 Discrete-Time Finite-State Markov Chains

A finite-state, discrete-time Markov chain $\{r(k) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{k\in\mathbb{N}_0}$ is an $\mathcal{F}_k$-adapted stochastic process characterized by an initial distribution $\nu : \mathcal{M} \to [0, 1]$ and a transition probability matrix $P \in \mathbb{R}^{M\times M}$ such that

$$\mathbb{P}[r(0) = i] = \nu_i, \quad i \in \mathcal{M}, \tag{2.5}$$

$$\mathbb{P}[r(k+1) = j | r(k) = i] = p_{i,j}, \quad i, j \in \mathcal{M}, \quad k \in \mathbb{N}_0, \tag{2.6}$$

where $p_{i,j} \in [0, 1]$ denotes the (i, j)th entry of the matrix P. Note that $\sum_{i\in\mathcal{M}} \nu_i = 1$ and $\sum_{j\in\mathcal{M}} p_{i,j} = 1$, $i \in \mathcal{M}$.

In Chapters 6, 7, and 9, we employ a time-homogeneous discrete-time Markov chain to model the active operation mode of a switched stochastic discrete-time dynamical system.

2.3.2 Discrete-Time Markov Chains on Countable State Spaces

A time-homogeneous, discrete-time Markov chain defined on a countable state space $\mathcal{S}$ is an $\mathcal{F}_k$-adapted stochastic process $\{s(k) \in \mathcal{S}\}_{k\in\mathbb{N}_0}$ characterized by an initial distribution $\lambda : \mathcal{S} \to [0, 1]$ and transition probabilities $\rho_{i,j} \in [0, 1]$, $i, j \in \mathcal{S}$, such that

$$\mathbb{P}[s(0) = i] = \lambda_i, \quad i \in \mathcal{S}, \tag{2.7}$$

$$\mathbb{P}[s(k+1) = j | s(k) = i] = \rho_{i,j}, \quad i, j \in \mathcal{S}, \quad k \in \mathbb{N}_0. \tag{2.8}$$

Note that $\sum_{i\in\mathcal{S}}\lambda_i = 1$ and $\sum_{j\in\mathcal{S}}\rho_{i,j} = 1$, $i \in \mathcal{S}$.

A discrete-time Markov chain $\{s(k) \in \mathcal{S}\}_{k\in\mathbb{N}_0}$ is called *aperiodic* if for every $i \in \mathcal{S}$, there exists $n \in \mathbb{N}$ such that for all $\bar{n} \geq n$, $\mathbb{P}[s(k+\bar{n}) = i|s(k) = i] > 0$, $k \in \mathbb{N}_0$. Note that states of aperiodic Markov chains are revisited aperiodically.

We call a discrete-time Markov chain $\{s(k) \in \mathcal{S}\}_{k\in\mathbb{N}_0}$ *irreducible* if for every $i, j \in \mathcal{S}$, there exists $n \in \mathbb{N}$ such that $\mathbb{P}[s(k+n) = j|s(k) = i] > 0$, $k \in \mathbb{N}_0$. In other words, for irreducible Markov chains it is possible to reach to any state from another state in finite transitions.

A distribution $\phi : \mathcal{S} \to [0,1] : j \mapsto \phi_j$ is called *invariant distribution* of the Markov chain $\{s(k) \in \mathcal{S}\}_{k\in\mathbb{N}_0}$ if $\phi_j = \sum_{i\in\mathcal{S}}\phi_i\rho_{i,j}$, $j \in \mathcal{S}$. Now, suppose $\{s(k) \in \mathcal{S}\}_{k\in\mathbb{N}_0}$ is an irreducible discrete-time Markov chain with the invariant distribution $\phi : \mathcal{S} \to [0,1]$. The strong law of large numbers (also called ergodic theorem; see [86, 87, 89]) for discrete-time Markov chains states that for any $\xi_i \in \mathbb{R}$, $i \in \mathcal{S}$, such that $\sum_{i\in\mathcal{S}}\phi_i|\xi_i| < \infty$, it follows that $\mathbb{P}[\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\xi_{s(k)} = \sum_{i\in\mathcal{S}}\phi_i\xi_i] = 1$.

In Chapter 9, we employ a time-homogeneous, discrete-time, *finite-state* Markov chain for modeling the mode transitions of a switched stochastic system. We consider the case where mode is observed at random time instants. Furthermore, the sequences of modes between random mode observation instants are characterized through a *countable-state* discrete-time Markov chain. Therefore, the ergodic theorem for countable-state Markov chains is crucial for developing our main results in Sections 9.2 and 9.3.

2.3.3 Discrete-Time Renewal Processes

A discrete-time renewal process $\{N(k) \in \mathbb{N}_0\}_{k\in\mathbb{N}_0}$ with initial value $N(0) = 0$ is an $\mathcal{F}_k$-adapted stochastic counting process defined by

$$N(k) = \sum_{i\in\mathbb{N}} \mathbb{1}_{[t_i \leq k]}, \tag{2.9}$$

where $t_i \in \mathbb{N}_0$, $i \in \mathbb{N}_0$, are random time instants such that $t_0 = 0$ and $\tau_i \triangleq t_i - t_{i-1} \in \mathbb{N}$, $i \in \mathbb{N}$, are identically distributed independent random variables with finite expectation (i.e., $\mathbb{E}[\tau_i] < \infty$, $i \in \mathbb{N}$). Note that τ_i, $i \in \mathbb{N}$, denote the lengths of intervals between time instants t_i, $i \in \mathbb{N}_0$. Furthermore, we use $\mu : \mathbb{N} \to [0,1]$ to denote the common distribution of the random variables τ_i, $i \in \mathbb{N}$, such that

$$\mathbb{P}[\tau_i = \tau] = \mu_\tau, \quad \tau \in \mathbb{N}, \quad i \in \mathbb{N}, \tag{2.10}$$

where $\mu_\tau \in [0,1]$. Note that $\sum_{\tau\in\mathbb{N}} \mu_\tau = 1$. Now, let $\hat{\tau} \triangleq \sum_{\tau\in\mathbb{N}} \tau\mu_\tau = \mathbb{E}[\tau_1](= \mathbb{E}[\tau_i]$, $i \in \mathbb{N})$. It follows as a consequence of strong law of large numbers for renewal processes (see [86]) that $\lim_{k\to\infty} \frac{N(k)}{k} = \frac{1}{\hat{\tau}}$.

Note that in Section 9.2, we employ a renewal process to characterize the occurrences of random mode observations.

2.3.4 Discrete-Time Markov Processes on $\mathbb{R}^l$

A time-homogeneous, discrete-time Markov process defined on state space $\mathbb{R}^l$ is a stochastic process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ characterized by an initial distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0,1]$ and transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0,1]$ such that

$$\mathbb{P}[\xi(0) \in S] = \nu(S), \tag{2.11}$$

$$\mathbb{P}[\xi(k+1) \in S | \xi(k) = s] = P(s,S), \tag{2.12}$$

for all $s \in \mathbb{R}^l$, $S \in \mathcal{B}(\mathbb{R}^l)$, $k \in \mathbb{N}_0$. Note that for each $s \in \mathbb{R}^l$, $P(s,\cdot) : \mathcal{B}(\mathbb{R}^l) \to [0,1]$ is a probability measure on the measurable space $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$; furthermore, for each $S \in \mathcal{B}(\mathbb{R}^l)$, $P(\cdot, S) : \mathbb{R}^l \to [0,1]$ is a measurable function on multidimensional space $\mathbb{R}^l$ (see [89,90]).

We define i-step transition probability functions $P^{(i)} : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0,1]$ by

$$P^{(0)}(s,S) \triangleq \begin{cases} 1, & \text{if } s \in S, \\ 0, & \text{otherwise}, \end{cases} \tag{2.13}$$

$$P^{(n+1)}(s,S) \triangleq \int_{\mathbb{R}^l} P^{(n)}(\bar{s},S)P(s,\mathrm{d}\bar{s}), \quad n \in \mathbb{N}. \tag{2.14}$$

Note that $P^{(1)}(s,S) = P(s,S)$, $s \in \mathbb{R}^l$, $S \in \mathcal{B}(\mathbb{R}^l)$. For a given time $k \in \mathbb{N}_0$ and step size $i \in \mathbb{N}_0$, $P^{(i)}(s,S)$ denotes the conditional probability that the Markov process will take a value inside the set $S \in \mathcal{B}(\mathbb{R}^l)$ at time $k+i$, given that it had the value $s \in \mathbb{R}^l$ at time k, that is

$$\mathbb{P}[\xi(k+i) \in S | \xi(k) = s] = P^{(i)}(s, S), \quad k, i \in \mathbb{N}_0. \tag{2.15}$$

A probability measure $\pi : \mathcal{B}(\mathbb{R}^l) \to [0,1]$ is called a *stationary distribution* of Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ if

$$\int_{\mathbb{R}^l} P(s, S)\pi(\mathrm{d}s) = \pi(S), \quad S \in \mathcal{B}(\mathbb{R}^l). \tag{2.16}$$

A Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is called *aperiodic* if there is no integer $d \geq 2$ and non-empty subsets $S_i \subseteq \mathbb{R}^l$, $i \in \{1, 2, \ldots, d\}$, such that $S_i \cap S_j = \emptyset$, $i \neq j$, $P(s, S_{i+1}) = 1$, $s \in S_i$, $i \in \{1, 2, \ldots, d-1\}$ and $P(s, S_1) = 1$, $s \in S_d$ (see [91]).

In Section 10.3, we employ an aperiodic Markov process defined on $\mathbb{R}^l$ to characterize the parameter of a discrete-time linear stochastic parameter-varying dynamical system.

2.3.5 Stationarity and Ergodicity of Discrete-Time Stochastic Processes

In this section we first give the definition of *stationarity*, and then we explain *measure preserving transformations* and *ergodic* stochastic processes.

A discrete-time stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is called *stationary* if for every $n \in \mathbb{N}$,

$$\begin{aligned} &\mathbb{P}[\zeta(i) \in S_1, \zeta(i+1) \in S_2, \ldots, \zeta(i+n-1) \in S_n] \\ &\quad = \mathbb{P}[\zeta(j) \in S_1, \zeta(j+1) \in S_2, \ldots, \zeta(j+n-1) \in S_n], \end{aligned} \tag{2.17}$$

for all $S_k \in \mathcal{B}(\mathbb{R}^l)$, $k \in \{1, 2, \ldots, n\}$, and $i, j \in \mathbb{N}_0$. Note that for a stationary stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$, the joint distribution of random variables $\zeta(k), \zeta(k+1), \ldots, \zeta(k+n)$ is the same for all $k \in \mathbb{N}_0$, in other words the joint distribution does not change over time [90, 92]. It is important to note that a time-homogeneous discrete-time Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ characterized with the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0,1]$ and the initial distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0,1]$ is stationary if the initial distribution $\nu(\cdot)$ is also a stationary distribution, that is,

$$\int_{\mathbb{R}^l} P(s,S)\nu(\mathrm{d}s) = \nu(S), \quad S \in \mathcal{B}(\mathbb{R}^l). \tag{2.18}$$

Now consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A measurable function $T : \Omega \to \Omega$ is called a *measure preserving transformation* if

$$\mathbb{P}[T^{-1}(F)] = \mathbb{P}[F], \quad F \in \mathcal{F},$$

where

$$T^{-1}(F) \triangleq \{\omega \in \Omega : T(w) \in F\}, \quad F \in \mathcal{F}. \tag{2.19}$$

Note that every stationary stochastic process is associated with a measure preserving transformation [90,92]. We define the measure preserving transformation associated with the stationary stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ in the following way. First, let $\Omega \triangleq \left(\mathbb{R}^l\right)^{\mathbb{N}_0}$ denote the space that includes all infinite-sequences of $\mathbb{R}^l$-valued vectors, and let $\mathcal{F} \triangleq \mathcal{B}\left(\left(\mathbb{R}^l\right)^{\mathbb{N}_0}\right)$ denote the product σ-algebra (see [90, 92]). Furthermore, let $\mathbb{P}$ be the probability measure induced by $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$. Note that all sequences of the form $\omega \triangleq \{\omega(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ are included in Ω; moreover, $\mathcal{F}$ includes all sets of the form $\{\omega \in \Omega : \omega(i) \in S_1, \omega(i+1) \in S_2, \ldots \omega(i+n-1) \in S_n\}$, for all $S_k \in \mathcal{B}(\mathbb{R}^l)$, $k \in \{1, 2, \ldots, n\}$, and $i \in \mathbb{N}_0$. For a fixed $\omega \in \Omega$, the stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is given by $\zeta(k) = \omega(k), k \in \mathbb{N}_0$. Now, we define $T_\zeta : \Omega \to \Omega$ by

$$T_\zeta(\{\omega(k)\}_{k\in\mathbb{N}_0}) \triangleq \{\omega(k+1)\}_{k\in\mathbb{N}_0}, \quad \omega \in \Omega. \tag{2.20}$$

Note that $T_\zeta : \Omega \to \Omega$ shifts the sequence $\omega \in \Omega$. The stationarity of the stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ implies that the function $T_\zeta : \Omega \to \Omega$ is a measure preserving transformation [90]. For the measure preserving transformation $T_\zeta : \Omega \to \Omega$, we define $T_\xi^i : \Omega \to \Omega$, by $T_\zeta^0(\omega) = \omega$ and $T_\zeta^{i+1}(\omega) = T_\zeta(T_\zeta^i(\omega))$, $i \in \mathbb{N}_0$.

Consider the stationary stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ and the associated measure preserving transformation $T_\zeta : \Omega \to \Omega$ defined in (2.20). The stationary stochastic process

$\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is called *ergodic* if $\mathbb{P}[F] = 0$ or $\mathbb{P}[F] = 1$ for all $F \in \mathcal{F}$ such that $T_\zeta^{-1}(F) = F$.

Now let $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ be a stationary and ergodic stochastic process. Furthermore, let $f : \mathbb{R}^l \to \mathbb{R}$ be a measurable function such that $\mathbb{E}[|f(\zeta(0))|] < \infty$. *Ergodic Theorem* [90, 92] states that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\zeta(k)) = \mathbb{E}[f(\zeta(0))]$, almost surely.

Stationarity and ergodicity notions are crucial for obtaining the main results of Chapter 10.

2.4 Stochastic Stability Definitions

In the literature, researchers employ various stability notions for analyzing stochastic dynamical systems. In this section, we give definitions of *almost sure asymptotic stability* and *second-moment asymptotic stability,* which we adopt in our study.

2.4.1 Almost Sure Asymptotic Stability

In Chapters 3–5, and 8, we investigate continuous-time stochastic dynamical systems, where the state variable is characterized by the stochastic process $\{x(t) \in \mathbb{R}^n\}_{t\geq 0}$. The zero solution $x(t) \equiv 0$ of a continuous-time stochastic dynamical system is called asymptotically stable almost surely if

$$\mathbb{P}[\lim_{t\to\infty} \|x(t)\| = 0] = 1. \tag{2.21}$$

Furthermore, in Chapters 9 and 10, we investigate almost sure asymptotic stability of discrete-time stochastic dynamical systems, for which the state variable is given by discrete-time stochastic process $\{x(k) \in \mathbb{R}^n\}_{k\in\mathbb{N}_0}$. The zero solution $x(k) \equiv 0$ of a discrete-time stochastic dynamical system is called asymptotically stable almost surely if

$$\mathbb{P}[\lim_{k\to\infty} \|x(k)\| = 0] = 1. \tag{2.22}$$

Note that almost sure asymptotic stability notion is also called "asymptotic stability with probability one" [14].

2.4.2 Second-Moment Asymptotic Stability

In Chapters 6 and 7, we investigate second-moment asymptotic stability of discrete-time stochastic dynamical systems. The zero solution $x(k) \equiv 0$ of a discrete-time stochastic system is called second-moment asymptotically stable if

$$\lim_{k \to \infty} \mathbb{E}[\|x(k)\|^2] = 0. \tag{2.23}$$

Chapter 3

Feedback Control of Continuous-Time Switched Linear Stochastic Systems Using Uniformly Sampled Mode Information

3.1 Introduction

Stabilization of stochastic hybrid systems have been investigated by many researchers. Particularly, feedback control of Markov jump systems have attracted considerable attention. Markov jump systems are composed of deterministic subsystems (modes). Transitions between these subsystems are characterized by a stochastic mode signal, which is modeled as a finite-state Markov chain. Feedback control of Markov jump linear systems have been discussed in [57]; output feedback stabilization of Markov jump systems have been investigated in [4] and [63]; moreover, stabilization problem under the effect of delays is explored in [62] and [64]. In addition to Markov jump systems, researchers have also explored more general "switching diffusion processes", which introduce stochasticity also in the subsystem dynamics. Several results regarding the stabilization of switching diffusion processes are provided in [13, 14, 54], and [49].

In the literature concerning the stabilization of switched stochastic systems, researchers often employ feedback control laws that require perfect knowledge of the mode signal. These suggested control laws are not suitable when mode information is not available or only available at certain instants. It is important to address the stabilization problem under limited mode information. In this regard, when there is no mode information available at all, under some conditions, stabilization can be achieved using a fixed feedback control law that is independent of the mode [63, 71]. Furthermore, an estimate of the mode signal can also be employed for control purposes [73, 74]. On the other hand, if mode information can be observed at certain time instants (even if rarely), this information can be utilized in the control framework.

In this chapter, we consider the stabilization problem for the case where mode information is sampled uniformly, that is, the mode sampling instants are equally spaced by a constant interval. Specifically, we consider a switched linear system composed of stochastic subsystems which include Brownian motion in their dynamics. The mode signal of the switched system is modeled as a time-homogeneous, finite-state Markov chain. We propose a control law that depends only on the uniformly sampled mode information. In this case, the closed-loop system under the proposed control law cannot be transformed into another switched linear stochastic system with a mode signal that is a time-homogeneous Markov chain. As a consequence, a new approach is needed to analyze stability of the closed-loop system. We first obtain a representation of the mode signal from the available samples employing the "sample and hold" technique. We investigate the probabilistic dynamics of this sampled version of the mode signal. Next, we derive and employ a type of strong law of large numbers for a bivariate process composed of the actual and the sampled mode signal to show that our proposed control law guarantees almost sure stability of the zero solution.

The contents of the chapter are as follows. In Section 3.5, we give the mathematical model for continuous-time switched linear stochastic dynamical systems, then we investigate the feedback control problem for these systems under uniformly sampled mode information and obtain some sufficient conditions of almost sure asymptotic stabilization. In Section 3.4, we explore the sampled-mode output feedback control problem. Furthermore, in Section 3.5, we investigate the sampled-mode stabilization problem for switched linear stochastic dynamical systems with multiplicative noise. We then present illustrative numerical examples in Section 3.6 to demonstrate the efficacy of our results. Finally, we conclude in Section 3.7.

3.2 Feedback Control of Switched Linear Stochastic Systems Using Periodically Sampled Mode Data

In this section, we investigate feedback control of switched linear stochastic systems that are composed of a number of *deterministic* subsystems and a stochastic mode signal, which characterizes the transition between the subsystems. We develop a control framework for the case where the mode signal of the switched system is sampled (observed) only at equally spaced discrete time instants. We start with the mathematical model for the switched stochastic dynamical system that we investigate.

3.2.1 Mathematical Model for Continuous-Time Switched Linear Stochastic Systems

We consider the continuous-time switched linear stochastic dynamical system with $M \in \mathbb{N}$ number of subsystems (modes) described by

$$\dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}u(t), \tag{3.1}$$

with initial conditions $x(0) = x_0$ and $r(0) = r_0$, where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and $A_i, \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$, are subsystem matrices. Transitions between the modes are characterized by the piecewise-constant $\mathcal{F}_t$-adapted mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$, which is assumed to be an continuous-time irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M\times M}$ with the stationary probability distribution $\pi \in \mathbb{R}^M$ (see Section 2.2.1 for the definition of continuous-time irreducible Markov chains).

3.2.2 Feedback Control Problem Under Periodic Mode Observations

We investigate feedback stabilization of the linear stochastic dynamical system (3.1) under the assumption that only a periodically-sampled version the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is available for control purposes. Specifically, we assume that the mode signal is sampled (observed) at time instants $0, \tau, 2\tau$,3τ,. . ., where $\tau > 0$ denotes the constant m*ode sampling period*. Our goal is to design a stabilizing feedback control law that depends only on the

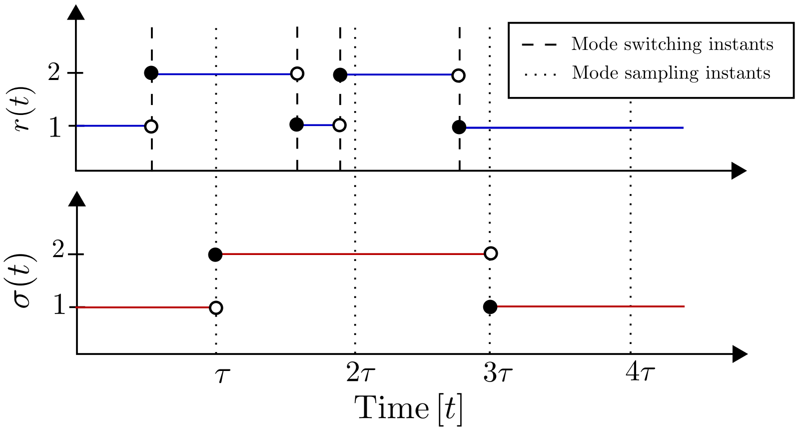

Figure 3.1: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time

sampled mode information.

First, by employing the "sample and hold" technique we obtain a representation of the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ using only the available mode samples $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$. We define this sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ by

$$\sigma(t) \triangleq r(k\tau), \quad t \in [k\tau, (k+1)\tau), \, k \in \mathbb{N}_0. \tag{3.2}$$

Furthermore, in order to achieve almost sure stabilization using only the sampled mode information, we consider the control law of the form

$$u(t) = K_{\sigma(t)}x(t). \tag{3.3}$$

In what follows we show that under certain conditions, the control law (3.3) guarantees the stability of the zero solution $x(t) \equiv 0$ of the closed-loop switched linear system (3.1). To this end, we first explore the relation between the actual mode signal $r(\cdot)$ and its sampled version $\sigma(\cdot)$. We then present some key results that are necessary for investigating the stabilization problem.

The sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ is a piecewise-constant stochastic process. It may be subject to jumps at the time instants $k\tau$, $k \in \mathbb{N}$, only when there is a mode switch in the time interval $((k-1)\tau, k\tau]$. In Fig. 3.1 we present sample paths of both the actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ of a switched system (3.1) with $M = 2$ modes. Note that both the sampling period $\tau > 0$ and frequency of the occurrences of mode transitions have an influence on how closely the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ resembles the actual mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$. For example, when mode samples are obtained relatively frequently compared to the occurrences of mode switches, the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ is likely to be a good representation of the actual mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$.

We denote the obtained mode samples by the sequence $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$, which is a discrete-time Markov chain with state transition probabilities given by

$$\mathbb{P}[r((k+1)\tau) = j | r(k\tau) = i] = p_{i,j}(\tau), \tag{3.4}$$

where $p_{i,j}(\tau)$ is the (i, j)th element of the transition matrix $e^{Q\tau}$. Since $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is irreducible, $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is also an irreducible Markov chain. Furthermore, $\pi \in \mathbb{R}^M$ is also the stationary probability distribution for the discrete-time Markov chain $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ [87].

The following lemma is concerned with finite-state, irreducible Markov chains and crucial for developing the main results of this chapter.

Lemma 3.1. Suppose $\{r(t) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{t\geq 0}$ is a finite-state, irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M\times M}$. Then for any $\phi_l \in \mathbb{R}$, $l \in \mathcal{M}$, $\tau > 0$, and $k \in \mathbb{N}$ such that $\mathbb{P}[r(k\tau) = i, r((k+1)\tau) = j] > 0$, it follows that

$$\begin{aligned}&\mathbb{E}[\int_{k\tau}^{(k+1)\tau} \phi_{r(s)}\mathrm{d}s | r(k\tau) = i, r((k+1)\tau) = j]\\ &\quad = \frac{1}{p_{i,j}(\tau)} \int_0^{\tau} \sum_{l\in\mathcal{M}} \phi_l p_{l,j}(\tau - s) p_{i,l}(s)\mathrm{d}s.\end{aligned} \tag{3.5}$$

Proof. First, let $F_l(t) \triangleq \{\omega \in \Omega : r_t(\omega) = l\}$, $t \geq 0$, $l \in \mathcal{M}$ and $G \triangleq F_i(k\tau) \cap F_j((k+$

$1)\tau)$. By the definition of conditional expectation given by (2.1), we have

$$\begin{aligned}
&\mathbb{E}[\int_{k\tau}^{(k+1)\tau} \phi_{r(s)} \mathrm{d}s | r(k\tau) = i, r((k+1)\tau) = j] \\
&= \mathbb{E}[\int_{k\tau}^{(k+1)\tau} \phi_{r(s)} \mathrm{d}s | G] \\
&= \frac{1}{\mathbb{P}[G]} \int_G \int_{k\tau}^{(k+1)\tau} \phi_{r(s)} \mathrm{d}s \mathbb{P}(\mathrm{d}\omega) \\
&= \frac{1}{\mathbb{P}[G]} \int_G \int_0^{\tau} \phi_{r(s'+k\tau)} \mathrm{d}s' \mathbb{P}(\mathrm{d}\omega) \\
&= \frac{1}{\mathbb{P}[G]} \int_G \int_0^{\tau} \sum_{l\in\mathcal{M}} \phi_l \mathbb{1}_{[r(s'+k\tau)=l]} \mathrm{d}s' \mathbb{P}(\mathrm{d}\omega),
\end{aligned} \tag{3.6}$$

where we also used the substitution $s' = s - k\tau$. By employing Fubini's Theorem [93], we change the order of integrals in (3.6) to obtain

$$\begin{aligned}
&\mathbb{E}[\int_{k\tau}^{(k+1)\tau} \phi_{r(s)} \mathrm{d}s | r(k\tau) = i, r((k+1)\tau) = j] \\
&= \frac{1}{\mathbb{P}[G]} \int_0^{\tau} \int_G \sum_{l\in\mathcal{M}} \phi_l \mathbb{1}_{[r(s'+k\tau)=l]} \mathbb{P}(\mathrm{d}\omega) \mathrm{d}s' \\
&= \frac{1}{\mathbb{P}[G]} \int_0^{\tau} \sum_{l\in\mathcal{M}} \phi_l \int_G \mathbb{1}_{[r(s'+k\tau)=l]} \mathbb{P}(\mathrm{d}\omega) \mathrm{d}s' \\
&= \frac{1}{\mathbb{P}[G]} \int_0^{\tau} \sum_{l\in\mathcal{M}} \phi_l \mathbb{P}[G \cap F_l(s'+k\tau)] \mathrm{d}s'.
\end{aligned} \tag{3.7}$$

Furthermore, it follows from (2.3) that

$$\begin{aligned}
\frac{\mathbb{P}[G \cap F_l(s'+k\tau)]}{\mathbb{P}[G]} &= \frac{\mathbb{P}[F_i(k\tau) \cap F_j((k+1)\tau) \cap F_l(s'+k\tau)]}{\mathbb{P}[F_i(k\tau) \cap F_j((k+1)\tau)]} \\
&= \frac{\mathbb{P}[F_j((k+1)\tau) | F_l(s'+k\tau)] \mathbb{P}[F_l(s'+k\tau) | F_i(k\tau)]}{\mathbb{P}[F_j((k+1)\tau) | F_i(k\tau)]} \\
&= \frac{p_{l,j}(\tau - s') p_{i,l}(s')}{p_{i,j}(\tau)}.
\end{aligned} \tag{3.8}$$

By substituting (3.8) to (3.7), we obtain (3.5), which completes the proof. $\square$

Next, in Lemma 3.2 we present a form of strong law of large numbers for the bivariate stochastic process $\{(r(t), \sigma(t)) \in \mathcal{M} \times \mathcal{M}\}_{t\geq 0}$. This result is then utilized for developing the main results below in Sections 3.3 and 3.5.

Lemma 3.2. Suppose $\{r(t) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{t\geq 0}$ is a finite-state, irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M\times M}$ with stationary probability distributions $\pi \in \mathbb{R}^M$; and $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ defined in (3.2) is the sampled version of $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ for a given sampling period $\tau > 0$. Then for any $\gamma_{i,j} \in \mathbb{R}$, $i, j \in \mathcal{M}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s = \frac{1}{\tau} \mathrm{tr}(\Pi \int_0^\tau e^{Qs} \mathrm{d}s\Gamma), \tag{3.9}$$

almost surely, where $\Pi \in \mathbb{R}^{M\times M}$ is the diagonal matrix with the diagonal elements $\pi_1, \pi_2, \ldots, \pi_M$, and $\Gamma \in \mathbb{R}^{M\times M}$ is the matrix with the elements $\gamma_{i,j}$, $i, j \in \mathcal{M}$.

Proof. First, we divide the interval $[0, t]$ into sub-intervals as

$$[0, t] = [0, \tau) \cup [\tau, 2\tau) \cup \ldots \cup [N(t)\tau, t], \quad t \geq 0, \tag{3.10}$$

where $N(t) \triangleq \lfloor t/\tau \rfloor + 1$, $t \geq 0$, denotes the number of mode samples obtained in the interval $[0, t]$. Consequently, we evaluate the integral over the interval $[0, t]$ in (3.9) by summing the integrals over each of the sub-intervals given in (3.10), that is,

$$\int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s = \sum_{k=0}^{N(t)-1} \int_{k\tau}^{(k+1)\tau} \gamma_{r(s),\sigma(s)} \mathrm{d}s + \int_{N(t)\tau}^{t} \gamma_{r(s),\sigma(s)} \mathrm{d}s. \tag{3.11}$$

Now, we define

$$N^{i,j}(t) \triangleq \sum_{k=1}^{N(t)} \mathbb{1}_{[r((k-1)\tau))=i, r(k\tau)=j]}, \; t \geq 0, \tag{3.12}$$

which denotes the number of times consecutive mode samples take the values i and j, respectively, in the interval $[0, t]$. Note that

$$\sum_{i,j\in\mathcal{M}} N^{i,j}(t) = N(t). \tag{3.13}$$

Furthermore, for each pair of modes $i, j \in \mathcal{M}$, we define the sequence of indices $\{k_n^{i,j} \in \mathbb{N}_0\}_{n\in\mathbb{N}}$ by

$$k_1^{i,j} = \min\{k \in \mathbb{N}_0 : r(k\tau) = i, r((k + 1)\tau) = j\}, \tag{3.14}$$

$$k_n^{i,j} = \min\{k > k_{n-1}^{i,j} : r(k\tau) = i, r((k + 1)\tau) = j\}, \tag{3.15}$$

for $n > 1$. Note that $r(k_n^{i,j}\tau) = i$ and $r((k_n^{i,j} + 1)\tau) = j$, $n \in \mathbb{N}$, $i, j \in \mathcal{M}$. As a result, for a given pair of modes $i, j \in \mathcal{M}$, $[k_n^{i,j}\tau, (k_n^{i,j} + 1)\tau)$ denotes the nth time interval between mode sampling instances for which the consecutive mode samples are i and j, respectively.

It follows from (3.11)-(3.15) that

$$\int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s = \sum_{i,j\in\mathcal{M}} \sum_{n=1}^{N^{i,j}(t)} \int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau} \gamma_{r(s),i} \mathrm{d}s + \int_{N(t)\tau}^{t} \gamma_{r(s),\sigma(s)} \mathrm{d}s, \tag{3.16}$$

where we also used $\gamma_{r(s),\sigma(s)} = \gamma_{r(s),i}$, $s \in [k_n^{i,j}\tau, (k_n^{i,j} + 1)\tau)$. Consequently, we calculate the limit in (3.9) as

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s \\ &\quad = \lim_{t\to\infty} \frac{1}{t} \Big(\sum_{i,j\in\mathcal{M}} \sum_{n=1}^{N^{i,j}(t)} \int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau} \gamma_{r(s),i} \mathrm{d}s + \int_{N(t)\tau}^{t} \gamma_{r(s),\sigma(s)} \mathrm{d}s \Big) \end{aligned} \tag{3.17}$$

Since $\int_{N(t)\tau}^{t} \gamma_{r(s),\sigma(s)} \mathrm{d}s \le |\max_{i,j\in\mathcal{M}} \gamma_{i,j}|\tau$, $t \ge 0$,

$$\lim_{t\to\infty} \frac{1}{t} \int_{N(t)\tau}^{t} \gamma_{r(s),\sigma(s)} \mathrm{d}s = 0. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s \\ &= \lim_{t\to\infty} \frac{1}{t} \sum_{i,j\in\mathcal{M}} \sum_{n=1}^{N^{i,j}(t)} \int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau} \gamma_{r(s),i} \mathrm{d}s \\ &= \lim_{t\to\infty} \frac{N(t)}{t} \sum_{i,j\in\mathcal{M}} \Big(\frac{N^{i,j}(t)}{N(t)} \frac{1}{N^{i,j}(t)} \sum_{n=1}^{N^{i,j}(t)} \int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau} \gamma_{r(s),i} \mathrm{d}s \Big) \\ &= \lim_{t\to\infty} \frac{N(t)}{t} \sum_{i,j\in\mathcal{M}} \Big(\lim_{t\to\infty} \frac{N^{i,j}(t)}{N(t)} \lim_{t\to\infty} \frac{1}{N^{i,j}(t)} \sum_{n=1}^{N^{i,j}(t)} \int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau} \gamma_{r(s),i} \mathrm{d}s \Big) \end{aligned} \tag{3.19}$$

In the following, we calculate the three limit terms on the right hand side of (3.19). First of all, by the definition of $N(t)$ we have

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\tau}. \tag{3.20}$$

Second, in order to evaluate $\lim_{t\to\infty}\frac{N^{i,j}(t)}{N(t)}$ in (3.19), we focus on the probabilistic dynamics of the sequence of mode samples $\{r(k\tau)\in\mathcal{M}\}_{k\in\mathbb{N}}$. Specifically, the sequence $\{r(k\tau)\in\mathcal{M}\}_{k\in\mathbb{N}}$ is a discrete-time Markov chain with transition probabilities given in (3.4). In addition, for the discrete-time Markov chain $\{r(k\tau)\in\mathcal{M}\}_{k\in\mathbb{N}}$, $N_{i,j}(t)$ and $N(t)$ respectively denote the number of state transitions from i to j, and the total number of state transitions. It follows by the strong law of large numbers for discrete-time Markov chains [86, 87] that

$$\lim_{t\to\infty}\frac{N^{i,j}(t)}{N(t)}=\pi_i p_{i,j}(\tau). \tag{3.21}$$

Third, we also employ strong law of large numbers to evaluate the last limit expression $\lim_{t\to\infty}\frac{1}{N^{i,j}(t)}\sum_{n=1}^{N^{i,j}(t)}\int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau}\gamma_{r(s),i}\mathrm{d}s$ in (3.19). Note that for given pair of modes $i,j\in\mathcal{M}$, the integrals $\int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau}\gamma_{r(s),i}\mathrm{d}s$, $n=\{1,\ldots,N^{i,j}(t)\}$, in (3.19), are $\mathbb{R}$-valued i.i.d. random variables, that is, for the Borel sets $B\in\mathcal{B}(\mathbb{R})$, $\psi(B)\triangleq\mathbb{P}[\int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau}\gamma_{r(s),i}\mathrm{d}s\in B]$ induces a probability measure on the measurable space $(\mathbb{R},\mathcal{B})$ independent of $n\in\mathbb{N}$. Moreover, by using Lemma 3.1, we obtain

$$\begin{aligned}
&\mathbb{E}[\int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau}\gamma_{r(s),i}\mathrm{d}s]\\
&=\sum_{k=0}^{\infty}\mathbb{E}[\int_{k\tau}^{(k+1)\tau}\gamma_{r(s),i}\mathrm{d}s|k_n^{i,j}=k]\mathbb{P}[k_n^{i,j}=k]\\
&=\sum_{k=0}^{\infty}\mathbb{E}[\int_{k\tau}^{(k+1)\tau}\gamma_{r(s),i}\mathrm{d}s|r(k\tau)=i,r((k+1)\tau)=j]\mathbb{P}[k_n^{i,j}=k]\\
&=\frac{1}{p_{i,j}(\tau)}\int_0^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,i}p_{l,j}(\tau-s)p_{i,l}(s)\mathrm{d}s\sum_{k=0}^{\infty}\mathbb{P}[k_n^{i,j}=k]\\
&=\frac{1}{p_{i,j}(\tau)}\int_0^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,i}p_{l,j}(\tau-s)p_{i,l}(s)\mathrm{d}s,
\end{aligned}\tag{3.22}$$

for $n\in\mathbb{N}$. Note also that $N^{i,j}(t)$ approaches infinity as $t\to\infty$, almost surely. Thus, the strong law of large numbers can be employed to obtain

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{N^{i,j}(t)}\sum_{n=1}^{N^{i,j}(t)}\int_{k_n^{i,j}\tau}^{(k_n^{i,j}+1)\tau}\gamma_{r(s),i}\mathrm{d}s&=\mathbb{E}[\int_{k_1^{i,j}\tau}^{(k_1^{i,j}+1)\tau}\gamma_{r(s),i}\mathrm{d}s]\\
&=\frac{1}{p_{i,j}(\tau)}\int_0^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,i}p_{l,j}(\tau-s)p_{i,l}(s)\mathrm{d}s,
\end{aligned}\tag{3.23}$$

Finally, we substitute (3.20), (3.21), and (3.23) into (3.19), and arrive at

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t \gamma_{r(s),\sigma(s)}\mathrm{d}s &= \frac{1}{\tau}\sum_{i,j\in\mathcal{M}}\pi_i\int_0^\tau\sum_{l\in\mathcal{M}}\gamma_{l,i}p_{l,j}(\tau-s)p_{i,l}(s)\mathrm{d}s\\ &= \frac{1}{\tau}\sum_{i\in\mathcal{M}}\pi_i\int_0^\tau\sum_{l\in\mathcal{M}}\gamma_{l,i}\sum_{j\in\mathcal{M}}p_{l,j}(\tau-s)p_{i,l}(s)\mathrm{d}s \end{aligned}\tag{3.24}$$

Moreover, since $\sum_{j\in\mathcal{M}}p_{l,j}(t)=1$, $t\geq 0$, $l\in\mathcal{M}$, it follows from (3.24) that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t \gamma_{r(s),\sigma(s)}\mathrm{d}s &= \frac{1}{\tau}\sum_{i\in\mathcal{M}}\pi_i\int_0^\tau\sum_{l\in\mathcal{M}}\gamma_{l,i}p_{i,l}(s)\mathrm{d}s\\ &= \frac{1}{\tau}\mathrm{tr}(\Pi\int_0^\tau e^{Qs}\mathrm{d}s\Gamma),\end{aligned}\tag{3.25}$$

which completes the proof. □

3.3 Sufficient Conditions for Almost Sure Asymptotic Stabilization

By utilizing the strong law of large numbers developed in Lemma 3.2 and employing a quadratic Lyapunov-like function, we now obtain sufficient conditions for the almost sure asymptotic stability of the closed-loop system (3.1), (3.3) under uniformly (periodically) sampled mode information.

Theorem 3.1. Consider the continuous-time switched linear stochastic control system (3.1), (3.3) with mode sampling period $\tau > 0$. If there exist $P > 0$ and scalars $\gamma_{i,j}\in\mathbb{R}$, $i,j\in\mathcal{M}$, such that

$$0 \geq (A_i+B_iK_j)^{\mathrm{T}}P + P(A_i+B_iK_j) - \gamma_{i,j}P, \quad i,j\in\mathcal{M}, \tag{3.26}$$

$$\mathrm{tr}(\Pi\int_0^\tau e^{Qs}\mathrm{d}s\Gamma) < 0, \tag{3.27}$$

where $\Pi\in\mathbb{R}^{M\times M}$ is the diagonal matrix with the diagonal elements $\pi_1,\pi_2,\ldots,\pi_M$, and $\Gamma\in\mathbb{R}^{M\times M}$ is the matrix with (i,j)th elements given by $\gamma_{i,j}$, then the zero solution $x(t)\equiv 0$ of the closed-loop control system (3.1), (3.3) is asymptotically stable almost surely.

Proof. First, consider the quadratic, positive-definite function $V(x)\triangleq x^{\mathrm{T}}Px$. It follows from (3.1) and (3.3) that

$$\dot{V}(x(t)) = x^{\mathrm{T}}(t)\big((A_{r(t)} + B_{r(t)}K_{\sigma(t)})^{\mathrm{T}}P + P(A_{r(t)} + B_{r(t)}K_{\sigma(t)})\big)x(t), \quad t \geq 0. \tag{3.28}$$

Now let

$$\alpha \triangleq \frac{\min_{i,j\in\mathcal{M}} \lambda_{\min}((A_i + B_iK_j)^{\mathrm{T}}P + P(A_i + B_iK_j))}{\lambda_{\max}(P)}. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$\dot{V}(x(t)) \geq \alpha V(x(t)), \quad t \geq 0. \tag{3.30}$$

Therefore,

$$V(x(t)) \geq e^{\alpha t}V(x(0)), \quad t \geq 0, \tag{3.31}$$

which proves that for nonzero values of the initial state $x(0) = x_0$, $V(x(t)) > 0$, $t \geq 0$. Now consider $\ln V(x(t))$, which is well defined for $t \geq 0$, since $V(x(t)) > 0$. It follows from (3.28) that

$$\begin{aligned}
&\frac{\mathrm{d}\ln V(x(t))}{\mathrm{d}t} \\
&\quad = \frac{1}{V(x(t))}\dot{V}(x(t)) \\
&\quad = \frac{1}{x^{\mathrm{T}}(t)Px(t)}x^{\mathrm{T}}(t)\big((A_{r(t)} + B_{r(t)}K_{\sigma(t)})^{\mathrm{T}}P + P(A_{r(t)} + B_{r(t)}K_{\sigma(t)})\big)x(t), \quad t \geq 0.
\end{aligned} \tag{3.32}$$

Now integrating (3.32) over the interval $[0, t]$ yields

$$\begin{aligned}
&\ln V(x(t)) \\
&\quad = \ln V(x(0)) \\
&\qquad + \int_0^t \frac{1}{x^{\mathrm{T}}(s)Px(s)}x^{\mathrm{T}}(s)\big((A_{r(s)} + B_{r(s)}K_{\sigma(s)})^{\mathrm{T}}P + P(A_{r(s)} + B_{r(s)}K_{\sigma(s)})\big)x(s)\mathrm{d}s.
\end{aligned} \tag{3.33}$$

It then follows from (3.26) that

$$\ln V(x(t)) \leq \ln V(x(0)) + \int_0^t \frac{1}{x^{\mathrm{T}}(s)Px(s)} \gamma_{r(s),\sigma(s)} x^{\mathrm{T}}(s)Px(s)\mathrm{d}s$$
$$= \ln V(x(0)) + \int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s, \quad t \geq 0. \tag{3.34}$$

By the strong law of large numbers (presented in Lemma 3.2),

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s = \frac{1}{\tau} \mathrm{tr}(\Pi \int_0^\tau e^{Qs} \mathrm{d}s \Gamma), \tag{3.35}$$

almost surely. By using (3.27), (3.34), and (3.35), we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \ln V(x(t)) \leq \frac{1}{\tau} \mathrm{tr}(\Pi \int_0^\tau e^{Qs} \mathrm{d}s \Gamma)$$
$$< 0. \tag{3.36}$$

It then follows that $\lim_{t\to\infty} \ln V(x(t)) = -\infty$, almost surely, and hence,

$$\mathbb{P}[\lim_{t\to\infty} V(x(t)) = 0] = 1, \tag{3.37}$$

which implies that the zero solution is asymptotically stable almost surely. □

Theorem 3.1 provides sufficient conditions under which the proposed control law 3.3 guarantees almost sure stabilization of the zero solution of the switched linear stochastic dynamical system (3.1). Note that the conditions of Theorem 3.1 depend not only on subsystem dynamics but also on the probabilistic dynamics of the mode signal as well as the mode sampling period $\tau > 0$.

Remark 3.1. Note that conditions (3.26), (3.27) can be used to assess stability of the closed-loop system (3.1), (3.3) when the gain matrices K_i, $i \in \mathcal{M}$, are already known. On the other hand, in practice, we often need to employ numerical methods for finding gain matrices so that the proposed control law (3.3) with those gains achieves almost sure asymptotic stabilization. In this regard, it is important to note that conditions (3.26), (3.27) are also well suited for finding *stabilizing feedback gain matrices* K_i, $i \in \mathcal{M}$. Specifically, note that for given $P > 0$, the inequalities (3.26), (3.27) are linear in K_i, $i \in \mathcal{M}$, and $\gamma_{i,j}$, $i,j \in \mathcal{M}$. Thus, numerical tools for *linear matrix inequalities* (see [94–96]) can be used for finding feedback gain matrices K_i, $i \in \mathcal{M}$, and scalars $\gamma_{i,j}$, $i,j \in \mathcal{M}$, that satisfy conditions (3.26), (3.27).

Remark 3.2. Note that the condition (3.72) of Theorem 3.1 depends not only on the transition rates between modes $q_{i,j}$, $i,j \in \mathcal{M}$, but also the sampling period $\tau > 0$. For a given mode sampling period $\tau > 0$, checking the condition (3.72) requires evaluation of the integral $\int_0^\tau e^{Qs}\mathrm{d}s$. A wide range of numerical integration algorithms can be used to calculate this integral accurately.

3.4 Sampled-Mode Output Feedback Control Problem

In the previous section we considered sampled-mode state feedback control problem. In this section we extend our results for the output feedback control problem. Specifically, we consider the dynamical system (3.1) together with

$$y(t) = C_{r(t)}x(t), \tag{3.38}$$

where $y(t) \in \mathbb{R}^l$ is denotes the output of the system and $C_i \in \mathbb{R}^{l\times n}$, $i \in \mathcal{M}$, are output matrices for each mode. In the output feedback control problem setting, the controller has access to output information rather than the system state. We propose an observer-based control framework which requires only output $y(t)$ and sampled mode information $\sigma(t)$. Specifically, we estimate the state with an observer and use the estimated state in our feedback control framework. To this end, we propose the observer

$$\dot{\hat{x}}(t) = A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - C_{\sigma(t)}\hat{x}(t)), \tag{3.39}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of state and $L_i \in \mathbb{R}^{n\times l}$, $i \in \mathcal{M}$, are observer gains. Furthermore, we consider the control law

$$u(t) = K_{\sigma(t)}\hat{x}(t), \quad t \geq 0. \tag{3.40}$$

Note that in our proposed observer (3.39) and control law (3.40), only sampled mode signal $\sigma(t)$ is required rather than the actual mode signal $r(t)$.

Let $e(t) \triangleq x(t) - \hat{x}(t)$. Note that $e(t) \in \mathbb{R}^n$ denotes the state estimation error at time $t \geq 0$. It follows from (3.1), (3.38), (3.39), and (3.40) that

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
&= A_{r(t)}x(t) + B_{r(t)}u(t) - \big(A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - C_{\sigma(t)}\hat{x}(t))\big) \\
&= A_{r(t)}x(t) + B_{r(t)}K_{\sigma(t)}\hat{x}(t) - A_{\sigma(t)}\hat{x}(t) - B_{\sigma(t)}K_{\sigma(t)}\hat{x}(t) \\
&\quad - L_{\sigma(t)}(C_{r(t)}x(t) - C_{\sigma(t)}\hat{x}(t)) \\
&= A_{r(t)}x(t) + B_{r(t)}K_{\sigma(t)}(x(t) - e(t)) - A_{\sigma(t)}(x(t) - e(t)) \\
&\quad - B_{\sigma(t)}K_{\sigma(t)}(x(t) - e(t)) - L_{\sigma(t)}(C_{r(t)}x(t) - C_{\sigma(t)}(x(t) - e(t))) \\
&= \big((A_{r(t)} - A_{\sigma(t)}) + (B_{r(t)} - B_{\sigma(t)})K_{\sigma(t)} - L_{\sigma(t)}(C_{r(t)} - C_{\sigma(t)})\big)x(t) \\
&\quad + \big(A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)} - (B_{r(t)} - B_{\sigma(t)})K_{\sigma(t)}\big)e(t). \qquad (3.41)
\end{aligned}
$$

Now, define $\bar{A}_{i,j}^{1,1}$, $\bar{A}_{i,j}^{1,2}$, $\bar{A}_{i,j}^{2,1}$, $\bar{A}_{i,j}^{2,2}$, $i, j \in \mathcal{M}$, by

$$\bar{A}_{i,j}^{1,1} \triangleq A_i + B_iK_j, \qquad (3.42)$$

$$\bar{A}_{i,j}^{1,2} \triangleq -B_iK_j, \qquad (3.43)$$

$$\bar{A}_{i,j}^{2,1} \triangleq (A_i - A_j) + (B_i - B_j)K_j - L_j(C_i - C_j), \qquad (3.44)$$

$$\bar{A}_{i,j}^{2,2} \triangleq A_j - L_jC_j - (B_i - B_j)K_j, \quad i, j \in \mathcal{M}. \qquad (3.45)$$

It then follows from (3.1) and (3.41) that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{r(t),\sigma(t)}^{1,1} & \bar{A}_{r(t),\sigma(t)}^{1,2} \\ \bar{A}_{r(t),\sigma(t)}^{2,1} & \bar{A}_{r(t),\sigma(t)}^{2,2} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \qquad (3.46)$$

In the following we obtain sufficient conditions under which the zero solution $x(t) \equiv 0$, $e(t) \equiv 0$ of (3.46) is asymptotically stable almost surely. Note that under these conditions both the state and estimation error converges to zero almost surely. We follow the approach that we used in the previous section. Specifically, we utilize the strong law of large numbers developed in Lemma 3.2 and employ a quadratic Lyapunov-like function.

In this section the Lyapunov-like function has the form $V(x, e) = x^{\mathrm{T}}P_1x + e^{\mathrm{T}}P_2e$, where both $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ are positive-definite matcies.

Theorem 3.2. Consider the continuous-time switched linear stochastic control system (3.1), (3.38), (3.39), and (3.40) with mode sampling period $\tau > 0$. If there exist $P_1 > 0$, $P_2 > 0$, and scalars $\gamma_{i,j} \in \mathbb{R}$, $i, j \in \mathcal{M}$, such that

$$0 \geq \begin{bmatrix} \bar{A}_{i,j}^{1,1\mathrm{T}}P_1 + P_1\bar{A}_{i,j}^{1,1} - \gamma_{i,j}P_1 & P_1\bar{A}_{i,j}^{1,2} + \bar{A}_{i,j}^{2,1\mathrm{T}}P_2 \\ \bar{A}_{i,j}^{1,2\mathrm{T}}P_1 + P_2\bar{A}_{i,j}^{2,1} & \bar{A}_{i,j}^{2,2\mathrm{T}}P_2 + P_2\bar{A}_{i,j}^{2,2} - \gamma_{i,j}P_2 \end{bmatrix}, \quad i, j \in \mathcal{M}, \tag{3.47}$$

$$\mathrm{tr}(\Pi \int_0^\tau e^{Qs}\mathrm{d}s\Gamma) < 0, \tag{3.48}$$

where $\bar{A}_{i,j}^{1,1}$, $\bar{A}_{i,j}^{1,2}$, $\bar{A}_{i,j}^{2,1}$, $\bar{A}_{i,j}^{2,2}$, $i, j \in \mathcal{M}$, are defined by (3.42)–(3.45), $\Pi \in \mathbb{R}^{M\times M}$ is the diagonal matrix with the diagonal elements $\pi_1, \pi_2, \ldots, \pi_M$, and $\Gamma \in \mathbb{R}^{M\times M}$ is the matrix with (i, j)th elements given by $\gamma_{i,j}$, then the zero solution $x(t) \equiv 0$, $e(t) \equiv 0$ of the closed-loop control system (3.1), (3.38), (3.39), and (3.40) is asymptotically stable almost surely.

Proof. First, consider the quadratic, positive-definite function $V(x, e) \triangleq x^{\mathrm{T}}P_1x + e^{\mathrm{T}}P_2e$. It follows from (3.1), (3.38), (3.39), and (3.40) that

$$\begin{aligned} \dot{V}(x(t), e(t)) = {} & x^{\mathrm{T}}(t)\big((A_{r(t)} + B_{r(t)}K_{\sigma(t)})^{\mathrm{T}}P_1 + P_1(A_{r(t)} + B_{r(t)}K_{\sigma(t)})\big)x(t) \\ & - 2x^{\mathrm{T}}(t)P_1B_{r(t)}K_{\sigma(t)}e(t) \\ & + e^{\mathrm{T}}(t)\big((A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)} - (B_{r(t)} - B_{\sigma(t)})K_{\sigma(t)})^{\mathrm{T}}P_2 \\ & + P_2(A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)} - (B_{r(t)} - B_{\sigma(t)})K_{\sigma(t)})\big)e(t) \\ & + 2e^{\mathrm{T}}(t)P_2\big((A_{r(t)} - A_{\sigma(t)}) - L_{\sigma(t)}(C_{r(t)} - C_{\sigma(t)}) \\ & + (B_{r(t)} - B_{\sigma(t)})K_{\sigma(t)}\big)x(t). \end{aligned} \tag{3.49}$$

Now by using (3.42)–(3.45), we obtain

$$\dot{V}(x(t), e(t)) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{A}_{i,j}^{1,1\mathrm{T}}P_1 + P_1\bar{A}_{i,j}^{1,1} & P_1\bar{A}_{i,j}^{1,2} + \bar{A}_{i,j}^{2,1\mathrm{T}}P_2 \\ \bar{A}_{i,j}^{1,2\mathrm{T}}P_1 + P_2\bar{A}_{i,j}^{2,1} & \bar{A}_{i,j}^{2,2\mathrm{T}}P_2 + P_2\bar{A}_{i,j}^{2,2} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \tag{3.50}$$

Hence, it follows from (3.47) that

$$\dot{V}(x(t), e(t)) \leq \gamma_{r(t),\sigma(t)}\big(x^{\mathrm{T}}(t)P_1x(t) + e^{\mathrm{T}}(t)P_2e(t)\big)$$
$$= \gamma_{r(t),\sigma(t)}V(x(t), e(t)), \quad t \geq 0. \tag{3.51}$$

Now by using a similar argument that we employ in the proof of Theorem 3.1, we can show that $V(x(t), e(t)) > 0$, $t \geq 0$. Now consider $\ln V(x(t), e(t))$, which is well defined since for $t \geq 0$, since $V(x(t), e(t)) > 0$. It follows that

$$\frac{\mathrm{d} \ln V(x(t), e(t))}{\mathrm{d}t} = \frac{1}{V(x(t), e(t))}\dot{V}(x(t), e(t)). \tag{3.52}$$

Furthermore after integrating (3.52) over the interval $[0, t]$ and using (3.51), we obtain

$$\ln V(x(t), e(t)) = \ln V(x(0), e(0)) + \int_0^t \frac{1}{V(x(s), e(s))}\dot{V}(x(s), e(s))\mathrm{d}s$$
$$\leq \ln V(x(0), e(0)) + \int_0^t \gamma_{r(s),\sigma(s)}\mathrm{d}s. \tag{3.53}$$

By the strong law of large numbers (presented in Lemma 3.2),

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \gamma_{r(s),\sigma(s)}\mathrm{d}s = \frac{1}{\tau}\mathrm{tr}(\Pi \int_0^\tau e^{Qs}\mathrm{d}s\Gamma), \tag{3.54}$$

almost surely. By using (3.48), (3.53), and (3.54), we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \ln V(x(t), e(t)) \leq \frac{1}{\tau}\mathrm{tr}(\Pi \int_0^\tau e^{Qs}\mathrm{d}s\Gamma)$$
$$< 0. \tag{3.55}$$

It then follows that $\lim_{t\to\infty} \ln V(x(t), e(t)) = -\infty$, almost surely, and hence,

$$\mathbb{P}[\lim_{t\to\infty} V(x(t), e(t)) = 0] = 1, \tag{3.56}$$

which implies that the zero solution $x(t) \equiv 0$, $e(t) \equiv 0$ is asymptotically stable almost surely. □

Theorem 3.2 provides sufficient conditions under which our observer-based sampled-mode feedback control framework guarantees convergence of state $x(t)$ and estimation error $e(t)$ to zero. It is important to note that given positive-definite matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, the inequalities (3.47)–(3.48) are linear in feedback and observer gains $K_i \in \mathbb{R}^{m\times n}$, $L_i \in \mathbb{R}^{n\times l}$, $i \in \mathcal{M}$. Hence, efficient numerical methods can be used to find these gains so that the conditions (3.47)–(3.48) are satisfied. Precisely, there are polynomial-time algorithms for checking feasibility of linear matrix inequalities. On the other hand, note that if the dynamical system has a nominal mode i, which is controllable and observable, $P_1, P_2 \in \mathbb{R}^{n\times n}$ can be heuristically assigned by solving algebraic Riccati equations $A_i^{\mathrm{T}} P_1 + P_1 A_i - P_1 B_i R_1^{-1} B_i^T P_1 + T_1 = 0$ and $A_i P_2 + P_2 A_i^{\mathrm{T}} + P_2 C_i^{\mathrm{T}} R_2^{-1} C_i P_2 + T_2 = 0$, where $R_1, R_2, T_1, T_2 \in \mathbb{R}^{n\times n}$ are known positive definite matrices.

In the next section, we consider the sampled-mode feedback control problem for switched linear stochastic systems with multiplicative noise.

3.5 Sampled-Mode Feedback Control of Switched Linear Stochastic Systems with Multiplicative Noise

In this section we consider the continuous-time switched stochastic dynamical system with $M \in \mathbb{N}$ modes given by

$$\mathrm{d}x(t) = A_{r(t)}x(t)\mathrm{d}t + B_{r(t)}u(t)\mathrm{d}t + D_{r(t)}x(t)\mathrm{d}W(t), \tag{3.57}$$

with the initial conditions $x(0) = x_0$ and $r(0) = r_0$, where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ respectively denote the state vector and the control input, $\{W(t) \in \mathbb{R}\}_{t\geq 0}$ is an $\mathcal{F}_t$-adapted Wiener process, A_i, $D_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$, are subsystem matrices. The mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is assumed to be an $\mathcal{F}_t$-adapted, irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M\times M}$ with the stationary probability distribution $\pi \in \mathbb{R}^M$. The Wiener process $\{W(t) \in \mathbb{R}\}_{t\geq 0}$ and the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ are assumed to be mutually independent stochastic processes.

Remark 3.3. Note that the switched stochastic system (3.1) discussed in Section 3.2 is a special case of the dynamical system (3.57) where $D_i = 0$, $i \in \mathcal{M}$. Note that the term $D_{r(t)}x(t)\mathrm{d}W(t)$ in (3.57) characterizes the effect of noise on system dynamics. This type of noise (called multiplicative noise) often characterizes stochastic disturbance on the system parameters of a dynamical system [97]. In engineering applications, state matrices of subsystems of the dynamical system (3.1) may be subject to disturbance. In such cases, the state matrix of the ith subsystem can be characterized by $A_i+D_i\eta(t)$, where $A_i, D_i \in \mathbb{R}^{n\times n}$ are constant matrices and $\eta(t)$ denotes *white noise*. Now note that white noise $\eta(t)$ is considered as the "informal time derivative" of Wiener process $W(t)$ (see [85,98]). Moreover, dynamical systems that involve white noise can be characterized as stochastic differential equations that incorporate Wiener processes. In the case where the state matrices of subsystems take the form $A_i + D_i\eta(t)$, by setting $\eta(t)\mathrm{d}t = \mathrm{d}W(t)$, we can characterize the overall dynamics of the switched system by (3.57). In this study for simplicity of exposition, we only consider the case with one-dimensional noise (characterized by the Wiener process $\{W(t) \in \mathbb{R}\}_{t\geq 0}$). It is important to note that our proposed framework can easily be extended to the more general case with multi-dimensional noise.

In what follows, we explore the state-feedback stabilization problem for the case where the mode signal is observed (sampled) periodically. To achieve stabilization, we employ the state feedback control law (3.3), which incorporates a feedback gain that depends only on the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$. In Theorem 3.4 below, we extend the results presented in Section 3.2, and obtain sufficient conditions for the almost sure asymptotic stability of the switched stochastic control system (3.57),(3.3).

Theorem 3.3. Consider the continuous-time switched linear stochastic control system (3.57), (3.3) with mode sampling period $\tau > 0$. If there exist $P > 0$ and scalars $\gamma_{i,j} \in \mathbb{R}$, $i \in \mathcal{M}$, such that

$$0 \geq (A_i + B_iK_j)^{\mathrm{T}}P + P(A_i + B_iK_j) + D_i^{\mathrm{T}}PD_i - \gamma_{i,j}P, \quad i,j \in \mathcal{M}, \tag{3.58}$$

$$\frac{1}{\tau}\mathrm{tr}(\Pi\int_0^\tau e^{Qs}\mathrm{d}s\Gamma) - \sum_{i\in\mathcal{M}}\pi_i\frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P + PD_i)}{2\lambda_{\max}^2(P)} < 0, \tag{3.59}$$

where $\Pi \in \mathbb{R}^{M\times M}$ is the diagonal matrix with the diagonal elements $\pi_1, \pi_2, \ldots, \pi_M$, and $\Gamma \in \mathbb{R}^{M\times M}$ is the matrix with the elements given by $\gamma_{i,j}$, $i, j \in \mathcal{M}$, then the zero solution $x(t) \equiv 0$ of the closed-loop system (3.57), (3.3) is asymptotically stable almost surely.

Proof. First, we define the quadratic, positive-definite function $V(x) \triangleq x^{\mathrm{T}}Px$. The closed-loop system (3.57) under the control law (3.3) is described by multi-dimensional Ito stochastic differential equations. Using Ito formula, we obtain

$$
\begin{aligned}
&\mathrm{d}V(x(t)) \\
&= \Big(\nabla V(x(t))(A_{r(t)} + B_{r(t)}K_{\sigma(t)})x(t) + \frac{1}{2}\mathrm{tr}\Big(D_{r(t)}x(t)x^{\mathrm{T}}(t)D_{r(t)}^{\mathrm{T}}\nabla\big(\nabla V(x(t))\big)\Big)\Big)\mathrm{d}t \\
&\quad + \nabla V(x(t))D_{r(t)}x(t)\mathrm{d}W(t) \\
&= x^{\mathrm{T}}(t)\Big((A_{r(t)} + B_{r(t)}K_{\sigma(t)})^{\mathrm{T}}P + P(A_{r(t)} + B_{r(t)}K_{\sigma(t)}) + D_{r(t)}^{\mathrm{T}}PD_{r(t)}\Big)x(t)\mathrm{d}t \\
&\quad + 2x^{\mathrm{T}}(t)PD_{r(t)}x(t)\mathrm{d}W(t). \qquad (3.60)
\end{aligned}
$$

It is shown in [14,51] that for nonlinear and time-dependent stochastic systems that satisfy local Lipschitz continuity and linear growth conditions in the state variable (see [98]), if the initial state is nonzero (i.e., $x_0 \neq 0$), then it follows that $x(t) \neq 0$, for all $t \geq 0$, almost surely. The same result holds for the state $x(t)$ of (3.57), (3.3), since the *linear* closed-loop system (3.57), (3.3) satisfies local Lipschitz continuity and linear growth conditions. Therefore, it is guaranteed by the positive-definiteness of $V(\cdot)$ that $V(x(t)) > 0$, $t \geq 0$. Now consider the function $\ln V(x(t))$, which is well-defined for all $t \geq 0$, since $V(x(t)) > 0$, $t \geq 0$. We use Ito formula once again to compute

$$
\begin{aligned}
&\mathrm{d}\ln V(x(t)) \\
&= \frac{1}{V(x(t))}x^{\mathrm{T}}(t)\Big((A_{r(t)} + B_{r(t)}K_{\sigma(t)})^{\mathrm{T}}P + P(A_{r(t)} + B_{r(t)}K_{\sigma(t)}) + D_{r(t)}^{\mathrm{T}}PD_{r(t)}\Big)x(t)\mathrm{d}t \\
&\quad - \frac{1}{2V^2(x(t))}(2x^{\mathrm{T}}(t)PD_{r(t)}x(t))^2\mathrm{d}t + \frac{1}{V(x(t))}2x^{\mathrm{T}}(t)PD_{r(t)}x(t)\mathrm{d}W(t). \qquad (3.61)
\end{aligned}
$$

Integrating (3.61) over the time interval $[0,\, t]$ yields

$$
\begin{aligned}
&\ln V(x(t)) \\
&= \ln V(x_0) + \int_0^t \frac{1}{V(x(s))}x^{\mathrm{T}}(s)\Big((A_{r(s)} + B_{r(s)}K_{\sigma(s)})^{\mathrm{T}}P + P(A_{r(s)} + B_{r(s)}K_{\sigma(s)}) \\
&\quad + D_{r(s)}^{\mathrm{T}}PD_{r(s)}\Big)x(s)\mathrm{d}s - \int_0^t \frac{1}{2V^2(x(s))}(2x^{\mathrm{T}}(s)PD_{r(s)}x(s))^2\mathrm{d}s + L(t), \qquad (3.62)
\end{aligned}
$$

where $L(t) \triangleq \int_0^t \frac{1}{V(x(s))} 2x^{\mathrm{T}}(s)PD_{r(s)}x(s)\mathrm{d}W(s)$. We note that

$$\begin{aligned} 2x^{\mathrm{T}}(s)PD_{r(s)}x(s) &= x^{\mathrm{T}}(s)(D_{r(s)}^{\mathrm{T}}P + PD_{r(s)})x(s) \\ &\geq \lambda_{\min}(D_{r(s)}^{\mathrm{T}}P + PD_{r(s)})x^{\mathrm{T}}(s)x(s) \\ &\geq \frac{\lambda_{\min}(D_{r(s)}^{\mathrm{T}}P + PD_{r(s)})}{\lambda_{\max}(P)} x^{\mathrm{T}}(s)Px(s). \end{aligned} \tag{3.63}$$

Furthermore, by (3.58), (3.73), (3.62), and (3.63),

$$\ln V(x(t)) \leq \ln V(x_0) + \int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s - \int_0^t \frac{\lambda_{\min}^2(D_{r(s)}^{\mathrm{T}}P + PD_{r(s)})}{2\lambda_{\max}^2(P)} \mathrm{d}s + L(t). \tag{3.64}$$

By the strong law of large numbers (Lemma 3.2),

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\sigma(s)} \mathrm{d}s = \frac{1}{\tau} \mathrm{tr}(\Pi \int_0^\tau e^{Qs} \mathrm{d}s \Gamma), \tag{3.65}$$

almost surely. Moreover, by the strong law of large numbers for continuous-time irreducible Markov chains [86, 87] we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t -\frac{\lambda_{\min}^2(D_{r(s)}^{\mathrm{T}}P + PD_{r(s)})}{2\lambda_{\max}^2(P)} \mathrm{d}s = -\sum_{i\in\mathcal{M}} \pi_i \frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P + PD_i)}{2\lambda_{\max}^2(P)}, \tag{3.66}$$

almost surely. In addition, note that the Ito integral $L(t)$ in inequality (3.62) is a local martingale with quadratic variation given by

$$\begin{aligned} [L]_t &= \int_0^t (\frac{1}{V(x(s))} 2x^{\mathrm{T}}(s)PD_{r(s)}x(s))^2 \mathrm{d}s \\ &= \int_0^t \frac{1}{V^2(x(s))} (2x^{\mathrm{T}}(s)PD_{r(s)}x(s))^2 \mathrm{d}s \\ &\leq \int_0^t \frac{1}{V^2(x(s))} (x^{\mathrm{T}}(s)(D_{r(s)}^{\mathrm{T}}P + PD_{r(s)})x(s))^2 \mathrm{d}s \\ &\leq \int_0^t \frac{\lambda_{\max}^2(D_{r(s)}^{\mathrm{T}}P + PD_{r(s)})}{\lambda_{\min}^2(P)} \mathrm{d}s \\ &\leq \frac{\max_{i\in\mathcal{M}} \lambda_{\max}^2(D_i^{\mathrm{T}}P + PD_i)}{\lambda_{\min}^2(P)} t. \end{aligned} \tag{3.67}$$

It follows from (3.67) that $\lim_{t\to\infty} \frac{1}{t}[L]_t < \infty$. Thus, we can employ the strong law of large numbers for local martingales [14, 49, 51] to show

$$\lim_{t\to\infty}\frac{1}{t}L(t)=0, \tag{3.68}$$

almost surely. Moreover, it follows from (3.59), (3.62), (3.65), (3.66), and (3.68) that

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\ln V(x(t)) &\leq \frac{1}{\tau}\mathrm{tr}(\Pi\int_0^\tau e^{Qs}\mathrm{d}s\Gamma)-\sum_{i\in\mathcal{M}}\pi_i\frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P+PD_i)}{2\lambda_{\max}^2(P)}\\ &<0.\end{aligned} \tag{3.69}$$

Finally, it follows that $\lim_{t\to\infty}\ln V(x(t))=-\infty$, almost surely, and hence,

$$\mathbb{P}[\lim_{t\to\infty}V(x(t))=0]=1, \tag{3.70}$$

which implies that the zero solution is asymptotically stable almost surely. □

Remark 3.4. Theorem 3.3 provides conditions that can be used to verify the almost sure asymptotic stability of the closed-loop system (3.57) under the control law (3.3), when the gain matrices $K_i\in\mathbb{R}^{m\times n}$, $i\in\mathcal{M}$, are known. Note that conditions (3.58) and (3.59) are obtained through a quadratic Lyapunov function approach. Specifically, we consider the Lyapunov function candidate $V(x(t))\triangleq x^T(t)Px(t)$, where $P\in\mathbb{R}^{n\times n}$ is a positive-definite matrix. In Theorem 3.4 below, we show that under certain conditions, almost sure asymptotic stability of the closed-loop system (3.57) is guaranteed by the control law (3.3) with the feedback gain given by $K_{\sigma(t)}=-B_{\sigma(t)}P$.

Theorem 3.4. Consider the continuous-time switched linear stochastic dynamical system (3.57) with mode sampling period $\tau>0$. If there exist $P>0$ and scalars $\zeta_i\in\mathbb{R}$, $i\in\mathcal{M}$, such that

$$0\geq A_i^{\mathrm{T}}P+PA_i+D_i^{\mathrm{T}}PD_i-2PB_iB_i^{\mathrm{T}}P-\zeta_iP,\ i\in\mathcal{M}, \tag{3.71}$$

$$\frac{1}{\tau}\mathrm{tr}(\Pi\int_0^\tau e^{Qs}\mathrm{d}s\Gamma)-\sum_{i\in\mathcal{M}}\pi_i\frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P+PD_i)}{2\lambda_{\max}^2(P)}<0, \tag{3.72}$$

where $\Pi\in\mathbb{R}^{M\times M}$ is the diagonal matrix with the diagonal elements $\pi_1,\pi_2,\ldots,\pi_M$, and

$\Gamma \in \mathbb{R}^{M \times M}$ is the matrix with (i, j)th elements given by

$$\gamma_{i,j} = \begin{cases} \zeta_j, & i = j, \\ \zeta_i + \frac{2\lambda_{\max}(PB_iB_i^{\mathrm{T}}P)}{\lambda_{\min}(P)} - \frac{\lambda_{\min}(P(B_jB_i^{\mathrm{T}}+B_iB_j^{\mathrm{T}})P)}{\lambda_{\max}(P)}, & i \neq j, \end{cases} \tag{3.73}$$

then the feedback control law (3.3) with the feedback gain matrix given by

$$K_{\sigma(t)} = -B_{\sigma(t)}^{\mathrm{T}} P \tag{3.74}$$

guarantees that the zero solution $x(t) \equiv 0$ of the closed-loop system (3.57) and (3.3) is asymptotically stable almost surely.

Proof. Note that 3.71 and 3.73 imply 3.58 with $K_i = -B_i^{\mathrm{T}} P$, $i \in \mathcal{M}$. Hence, the result follows from Theorem 3.3. □

Note that when the mode sampling period is very small, mode samples are obtained frequently; therefore, the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t \geq 0}$ is expected to resemble the actual mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$ closely. Hence, when ith mode is active, the feedback gain is likely to be K_i. Moreover, as the mode sampling period τ tends to zero, the problem at hand becomes a stabilization problem with full mode information. In this case the condition (3.72) takes a simpler form. Specifically, note that

$$\begin{aligned} \frac{1}{\tau}\mathrm{tr}(\Pi \int_0^\tau e^{Qs} \mathrm{d}s\Gamma) &= \frac{1}{\tau}\mathrm{tr}(\Pi \int_0^\tau \sum_{n=0}^{\infty} \frac{s^n Q^n}{n!} \mathrm{d}s\Gamma) \\ &= \frac{1}{\tau}\mathrm{tr}(\Pi \sum_{n=0}^{\infty} \frac{\tau^{n+1} Q^n}{(n+1)!}\Gamma) \\ &= \sum_{n=0}^{\infty} \frac{\tau^n}{(n+1)!}\mathrm{tr}(\Pi Q^n \Gamma). \end{aligned} \tag{3.75}$$

It follows that as the mode sampling period τ tends to zero, $\frac{1}{\tau}\mathrm{tr}(\Pi \int_0^\tau e^{Qs} \mathrm{d}s\Gamma)$ approaches to $\mathrm{tr}(\Pi\Gamma) = \sum_{i \in \mathcal{M}} \pi_i \zeta_i$. Consequently, the condition (3.72) reduces to

$$\sum_{i \in \mathcal{M}} \pi_i \big(\zeta_i - \frac{\lambda_{\min}^2(D_i^{\mathrm{T}} P + P D_i)}{2\lambda_{\max}^2(P)}\big) < 0. \tag{3.76}$$

Note that (3.71) and (3.76) are the conditions we provide in Chapter 8 for almost sure stabilization with a control law that depends on the perfect knowledge of the mode signal.

3.6 Illustrative Numerical Examples

In this section, we present numerical examples in order to illustrate the efficacy of our approach regarding the feedback control of a switched stochastic system using uniformly sampled mode information.

Example 3.1. In this example, we present a practical application of our proposed sampled-mode feedback control framework. In [4,5], a linear dynamical model of a helicopter flight control system is provided. This dynamical model incorporates the time-varying parameter *airspeed*, which is modeled as a stochastic process. Specifically, in [4, 5], researchers consider a switched linear stochastic system model of the form (3.1) with $M = 3$ modes which correspond to *nominal*, *low*, and *high* values of the airspeed. These modes are characterized by the subsystem matrices

$$A_1 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.42 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.0664 & -0.707 & 0.1198 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.4422 & 0.1761 \\ 0.9775 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.5047 & -0.707 & 2.546 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.4422 & 0.1761 \\ 5.112 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}.$$

Note that $(3, 2)$th and $(3, 4)$th entries of the state matrices A_i, $i \in \mathcal{M}$, are different for each mode. Furthermore, $(2, 1)$th entry of the input matrices B_i, $i \in \mathcal{M}$, are also mode-dependent. It is important to note that the states $x_1(\cdot)$, $x_2(\cdot)$, $x_3(\cdot)$, $x_4(\cdot)$ of the

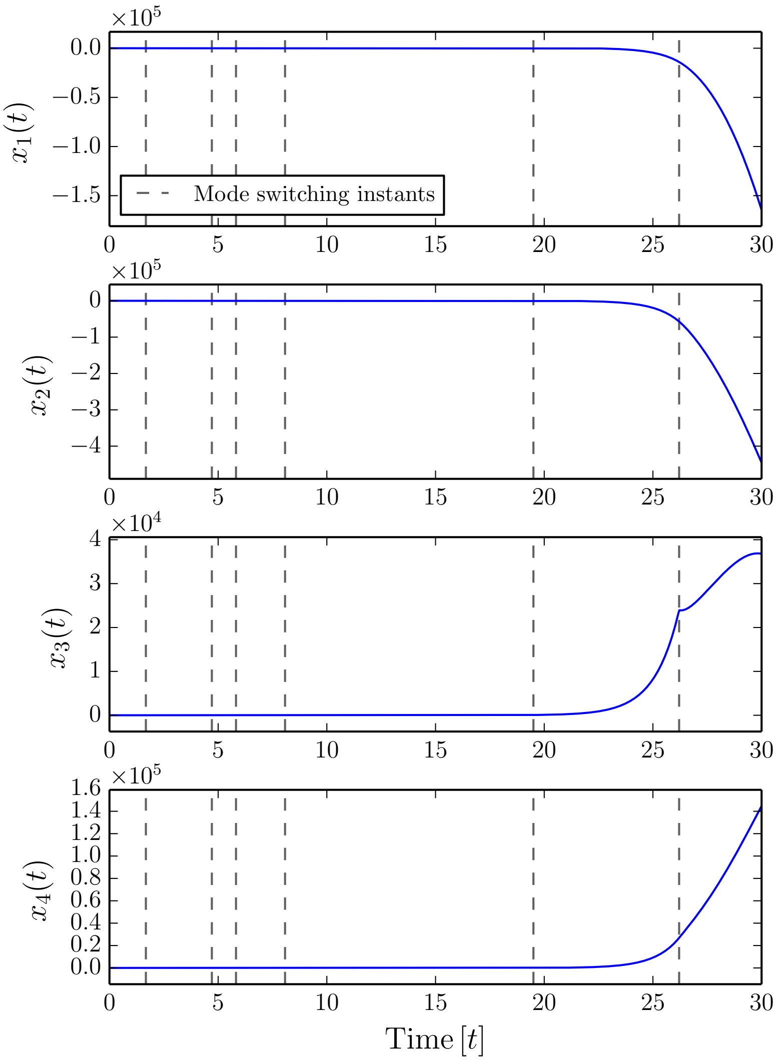

Figure 3.1: State trajectory of the uncontrolled system versus time

switched stochastic system (3.1) correspond respectively to longitudinal velocity, vertical velocity, pitch rate, and pitch angle of a helicopter. Moreover, control inputs $u_1(\cdot)$ and $u_2(\cdot)$ correspond respectively to collective and longitudinal cyclic commands of the helicopter.

The mode signal $\{r(t) \in \mathcal{M} \triangleq \{1, 2, 3\}\}_{t \geq 0}$ of the switched system is characterized by a continuous-time Markov chain with the generator matrix

$$Q = \begin{bmatrix} -0.0907 & 0.0671 & 0.0236 \\ 0.0671 & -0.0671 & 0 \\ 0.0236 & 0 & -0.0236 \end{bmatrix}, \tag{3.77}$$

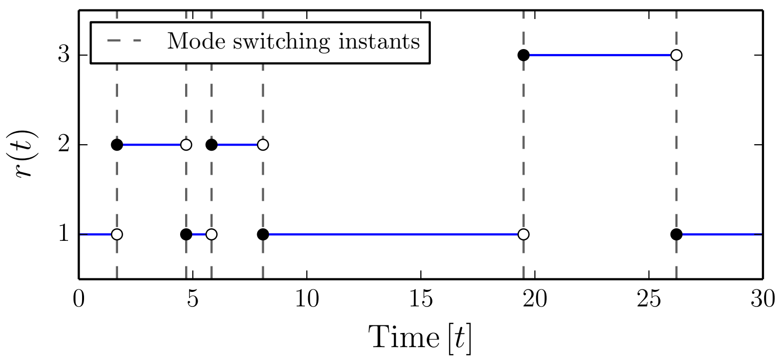

Figure 3.2: Mode signal versus time

with stationary probability distributions, $\pi_i = \frac{1}{3}$, $i \in \mathcal{M}$. Note that the mode signal is an irreducible Markov chain.

Figure 3.1 shows state trajectory of the uncontrolled system (3.1) (with $u(t) \equiv 0$) obtained with initial conditions $x(0) = [1,\ 1,\ 1,\ 1]^{\mathrm{T}}$ and $r(0) = 1$. Furthermore, Figure 3.2 shows the mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$. Note that the uncontrolled switched stochastic system clearly indicates unstable behavior, as the state trajectories diverge.

The hover condition of the helicopter is characterized by the zero solution $x(t) \equiv 0$ of the switched linear stochastic system (3.1). Note that stabilization of the zero solution of switched linear stochastic system (3.1) has been investigated in [4, 5] under the assumption that the mode information is continuously observable. In the remainder of this section, we show that feedback control of the helicopter described by the switched stochastic system (3.1) can be achieved by only using sampled mode information.

Now, consider the case where the mode signal is sampled periodically and hence available for control purposes only at the time instants $k\tau$, $k \in \mathbb{N}_0$, where $\tau = 1$ is the mode sampling period.

The conditions (3.26) and (3.27) of Theorem 3.1 are satisfied by matrices

$$P = \begin{bmatrix} 1.8873 & 0.0557 & 0.0179 & -0.8975 \\ 0.0557 & 0.1843 & 0.1205 & -0.0461 \\ 0.0179 & 0.1205 & 0.2425 & 0.0949 \\ -0.8975 & -0.0461 & 0.0949 & 1.3219 \end{bmatrix}, \quad (3.78)$$

$$K_1 = \begin{bmatrix} -0.6368 & 0.4777 & 0.6430 & 0.5966 \\ 0.2139 & 1.1880 & -0.5608 & -1.3941 \end{bmatrix}, \quad (3.79)$$

$$K_2 = \begin{bmatrix} -0.6414 & 1.0274 & 0.5375 & 0.1207 \\ 0.7746 & 1.1947 & -1.1604 & -2.0611 \end{bmatrix}, \quad (3.80)$$

$$K_3 = \begin{bmatrix} -0.7219 & -0.0531 & 0.8595 & 1.1602 \\ -0.6138 & 1.3216 & 0.1941 & -0.5091 \end{bmatrix}, \quad (3.81)$$

and scalars $\gamma_{1,1} = -0.1925$, $\gamma_{1,2} = 1.4185$, $\gamma_{1,3} = 1.4217$, $\gamma_{2,1} = 1.0957$, $\gamma_{2,2} = -0.2014$, $\gamma_{2,3} = 1.1928$, $\gamma_{3,1} = 1.1588$, $\gamma_{3,2} = 1.3642$, $\gamma_{3,3} = -0.3316$. It follows that the zero solution $x(t) \equiv 0$ of the system given by (3.1) under the control law (3.3) with feedback gains K_i, $i \in \mathcal{M}$, given by (3.79)–(3.81), is asymptotically stable almost surely. It is important to note that in order to obtain feedback gain matrices K_i, $i \in \mathcal{M}$, and scalars $\gamma_{i,j}$, $i, j \in \mathcal{M}$, that satisfies conditions of Theorem 3.1, we first set the positive-definite matrix P by solving algebraic Riccati equation (see [99]) for mode 1 (which corresponds to the subsystem associated with the nominal value of the airspeed parameter for the helicopter flight control system). Then we use numerical tools to find feasible solutions to matrix inequalities (3.26) and (3.27), which are linear in matrices K_i, $i \in \mathcal{M}$, and scalars $\gamma_{i,j}$, $i, j \in \mathcal{M}$, given the matrix P.

Figures 3.3 and 3.4 respectively show sample paths of $x(t)$ and $u(t)$ obtained with the initial conditions $x(0) = [1, 1, 1, 1]^{\mathrm{T}}$ and $r(0) = 1$. Furthermore, the actual mode signal $r(t)$ and its sampled version $\sigma(t)$ are shown in Figure 3.5. Note that for obtaining the state and control input trajectories, we used the sample path of the mode signal shown in Figure 3.2. Figure 3.3 indicates that the proposed sampled-mode control framework is effective for achieving convergence of the state trajectories to the origin (indicating hover flight condition of helicopter). Note that control input $u(\cdot)$ is subject to jumps at mode

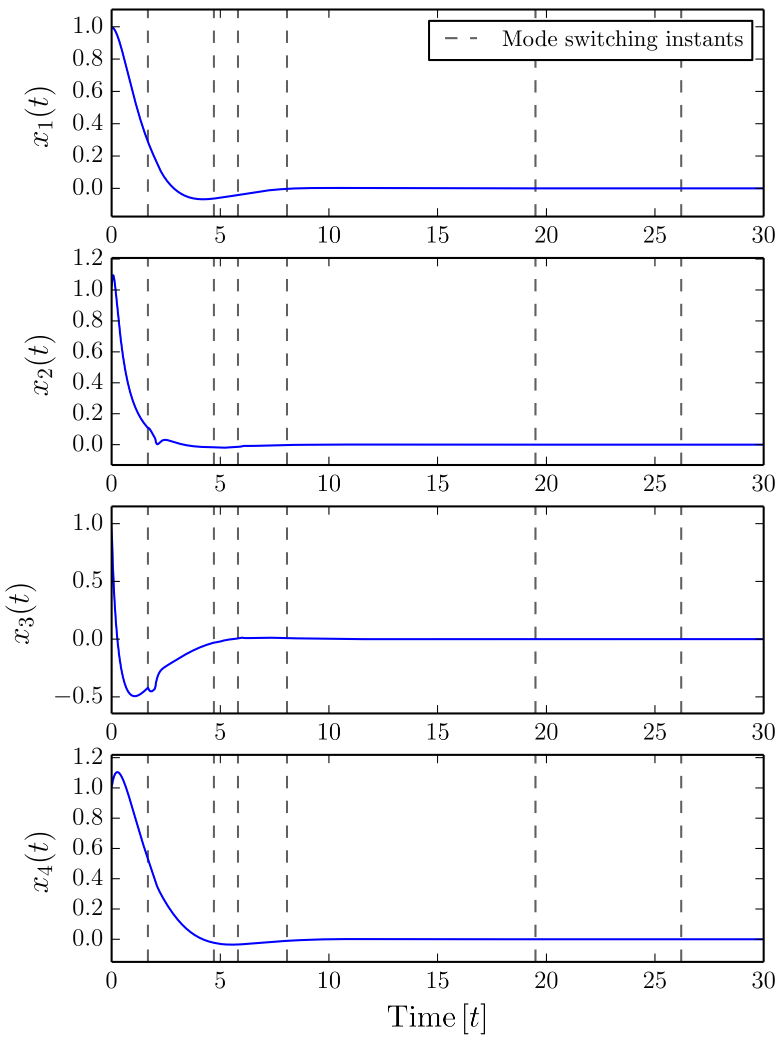

Figure 3.3: State trajectory versus time

sampling instants when sampled mode signal $\sigma(t)$ changes its value (see Figure 3.4). At mode sampling instants, the feedback gain, which depends on the sampled mode signal, is switched.

In this example the sampled mode signal is an accurate representation of the actual mode signal (see Figure 3.5), since the mode switches occur relatively rarely compared to the frequency of mode observations. In the following we consider the case where the mode switches occur more frequently. We show that the sampled-mode feedback stabilization can still be achieved for the sampling period $\tau = 1$. Specifically, we consider the case where the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ of the switched system is characterized by the

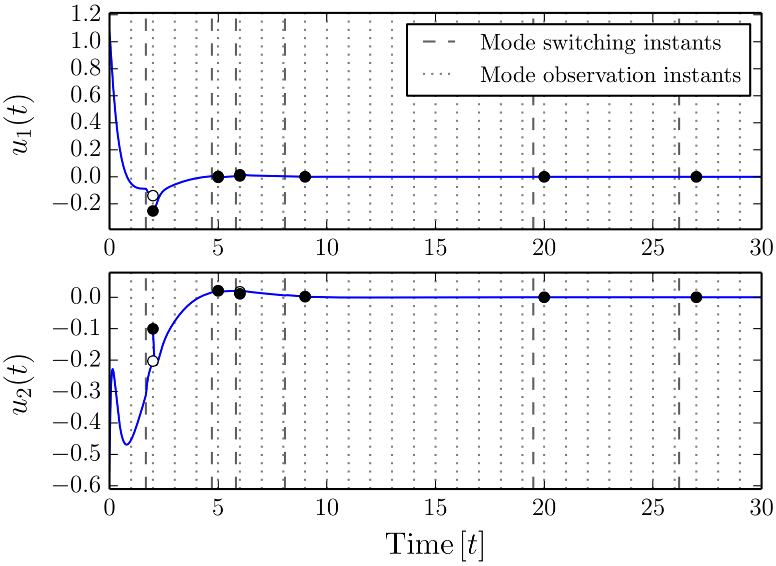

Figure 3.4: Control input versus time

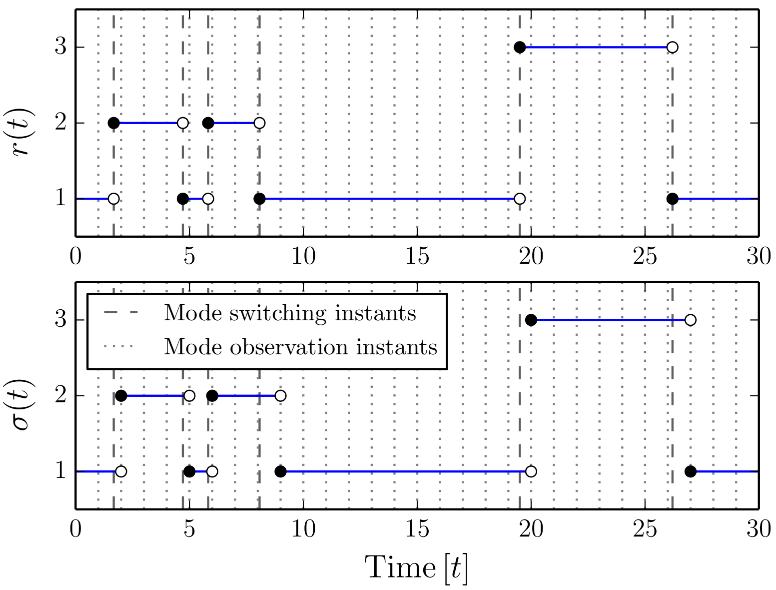

Figure 3.5: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time

generator matrix

$$Q = \begin{bmatrix} -1.814 & 1.342 & 0.472 \\ 1.342 & -1.342 & 0 \\ 0.472 & 0 & -0.472 \end{bmatrix}, \tag{3.82}$$

with stationary probability distributions, $\pi_i = \frac{1}{3}$, $i \in \mathcal{M}$. We assume that the mode signal is sampled periodically with period $\tau = 1$.

Note that the inequalities (3.26) and (3.27) are satisfied by matrices

$$P = \begin{bmatrix} 1.8873 & 0.0557 & 0.0179 & -0.8975 \\ 0.0557 & 0.1843 & 0.1205 & -0.0461 \\ 0.0179 & 0.1205 & 0.2425 & 0.0949 \\ -0.8975 & -0.0461 & 0.0949 & 1.3219 \end{bmatrix}, \tag{3.83}$$

$$K_1 = \begin{bmatrix} -0.6541 & 0.5125 & 0.5533 & 0.5181 \\ 0.3139 & 1.0222 & -0.6048 & -1.5197 \end{bmatrix}, \tag{3.84}$$

$$K_2 = \begin{bmatrix} -0.6736 & 0.7789 & 0.4721 & 0.2981 \\ 0.6171 & 0.9992 & -0.9348 & -1.8636 \end{bmatrix}, \tag{3.85}$$

$$K_3 = \begin{bmatrix} -0.7589 & 0.0683 & 0.7029 & 1.0019 \\ -0.3119 & 1.0889 & -0.0242 & -0.8238 \end{bmatrix}, \tag{3.86}$$

and scalars $\gamma_{1,1} = -0.0966$, $\gamma_{1,2} = 0.293$, $\gamma_{1,3} = 0.5988$, $\gamma_{2,1} = 0.1076$, $\gamma_{2,2} = -0.255$, $\gamma_{2,3} = 0.5915$, $\gamma_{3,1} = 0.4215$, $\gamma_{3,2} = 0.6753$, $\gamma_{3,3} = -0.4522$. It follows from Theorem 3.1 that the zero solution $x(t) \equiv 0$ of the system given by (3.1) under the control law (3.3) with feedback gains K_i, $i \in \mathcal{M}$, given by (3.84)–(3.86), is asymptotically stable almost surely. Note that the feedback gains (3.84)–(3.86), obtained for the case with generator matrix (3.82) are different from the feedback gains (given in (3.79)–(3.81)) that we obtained for the generator matrix given by (3.77).

Figures 3.6 and 3.7 respectively show sample paths of state $x(t)$ and control input $u(t)$ obtained with the initial conditions $x(0) = [1,\, 1,\, 1,\, 1]^{\mathrm{T}}$ and $r(0) = 1$. Moreover, actual mode signal $r(t)$ and its sampled version $\sigma(t)$ are shown in Figure 3.8.

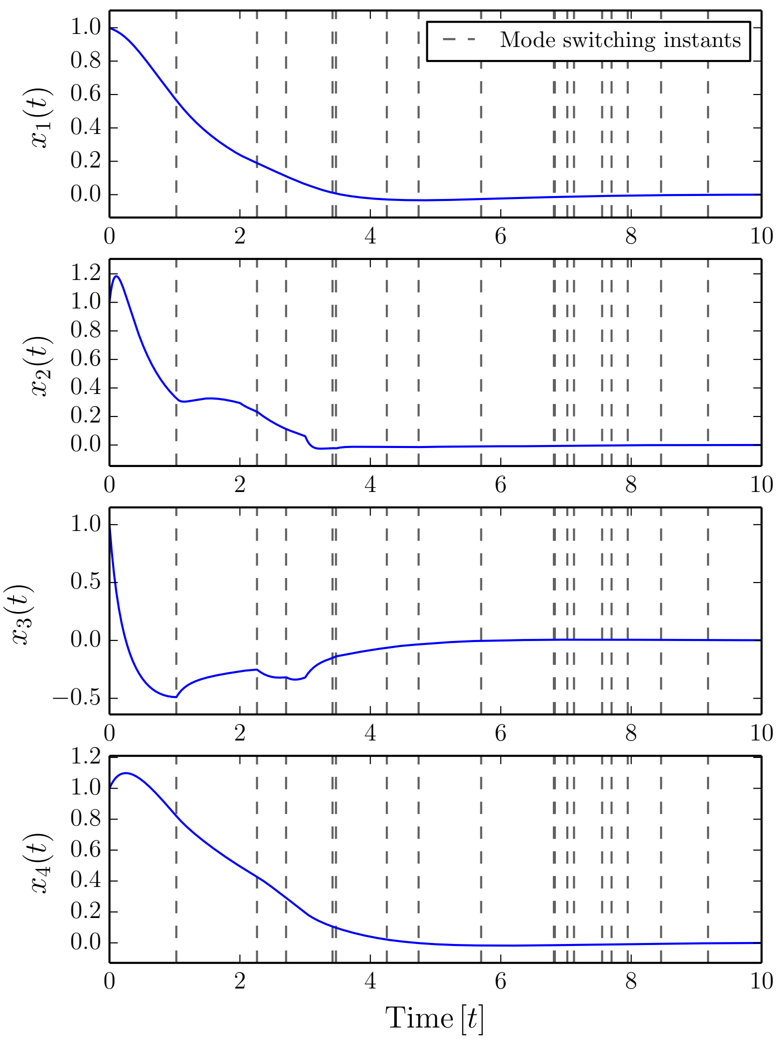

Figure 3.6: State trajectory versus time

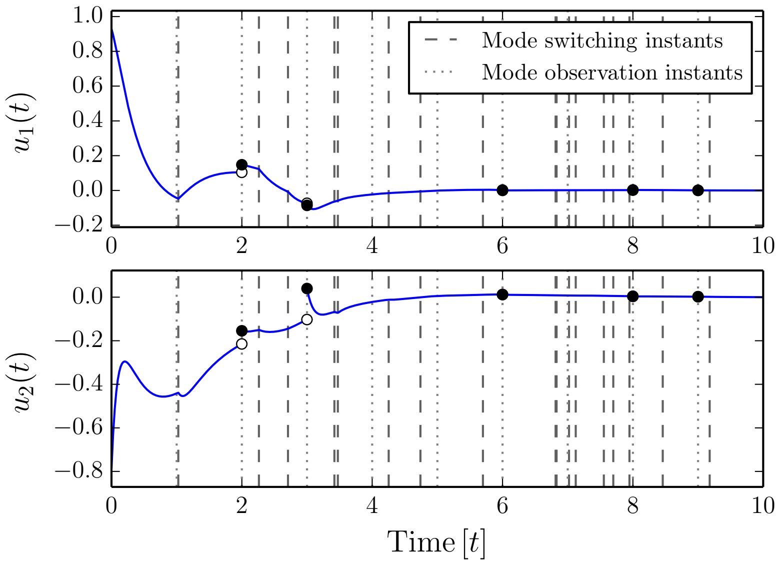

Figure 3.7: Control input versus time

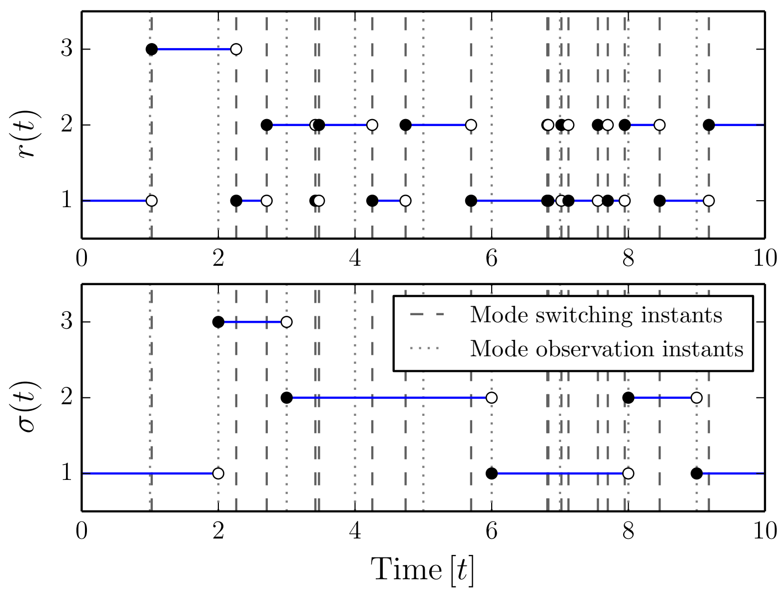

Figure 3.8: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time

Example 3.2. In this example, we consider the switched linear stochastic dynamical system (3.57) with $M = 3$ modes characterized by the subsystem matrices

$$A_1 = \begin{bmatrix} 0.5 & 1 \\ -4 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & -6 \\ 1 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and $D_1 = D_2 = D_3 = I_2$. The mode signal $\{r(t) \in \mathcal{M} \triangleq \{1, 2, 3\}\}_{t \geq 0}$ of the switched system is assumed to be a Markov chain with the generator matrix

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{3.87}$$

with stationary probability distributions, $\pi_i = \frac{1}{3}$, $i \in \mathcal{M}$. Furthermore, the mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$ is assumed to be sampled periodically and hence available only at the time instants $k\tau$, $k \in \mathbb{N}_0$, where $\tau = 0.1$ is the mode sampling period.

The conditions (3.71) and (3.72) of Theorem 3.4 are satisfied by the positive-definite matrix $P = I_2$ and the scalars $\zeta_1 = -1$, $\zeta_2 = -1.4$, $\zeta_3 = 2.5$. It follows that the zero solution $x(t) \equiv 0$ of the system given by (3.57) under the control law (3.3) with feedback gains $K_i = -B_i^{\mathrm{T}} P$, $i \in \mathcal{M}$, is asymptotically stable almost surely.

Figures 3.9 and 3.10 respectively show sample paths of $x(t)$ and $u(t)$ obtained with the initial conditions $x(0) = [1,\, 1]^{\mathrm{T}}$ and $r(0) = 1$. The piecewise-continuous control law (3.3) depends on the sampled mode signal information $\sigma(t)$. As a consequence, control profile is subject to jumps when sampled mode signal $\sigma(t)$ changes its value at mode sampling instants.

The quality of the representation of the actual mode signal by the sampled mode signal affects the stabilization performance. In this numerical example, mode samples are

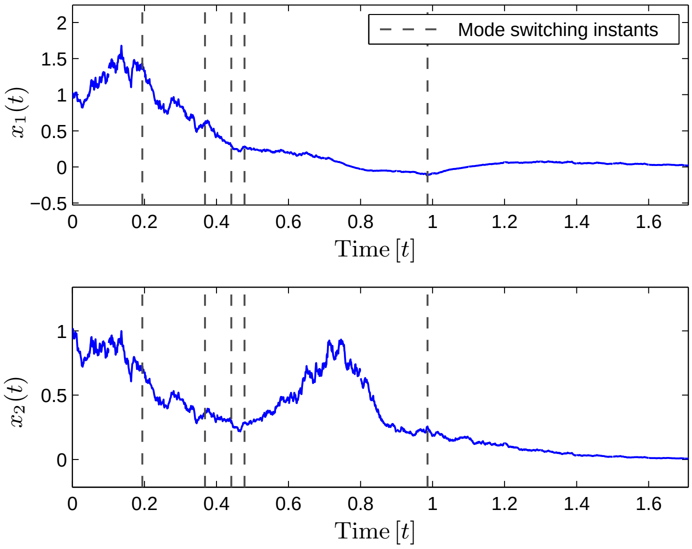

Figure 3.9: State trajectory versus time (mode sampling period $\tau = 0.1$)

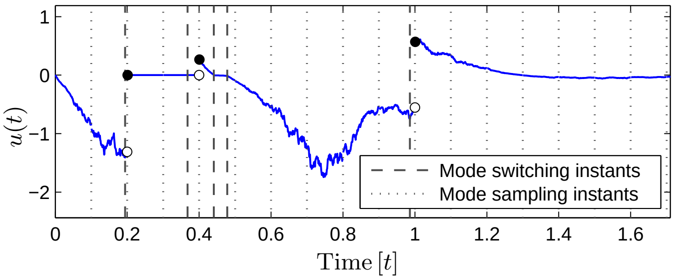

Figure 3.10: Control input versus time (mode sampling period $\tau = 0.1$)

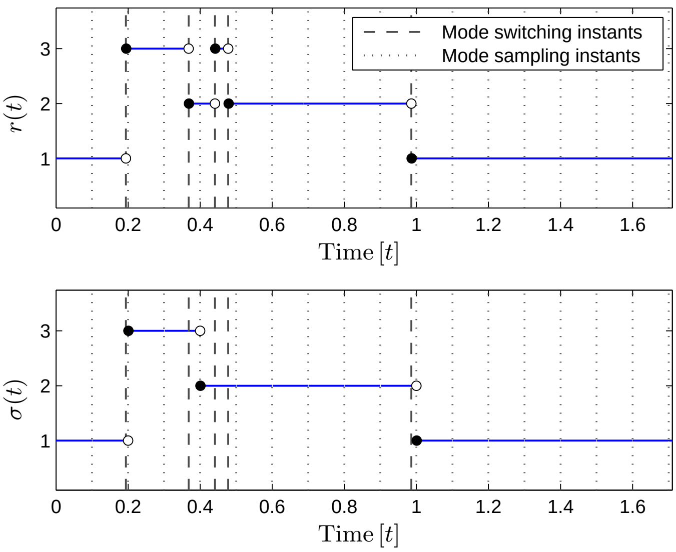

Figure 3.11: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time (mode sampling period $\tau = 0.1$)

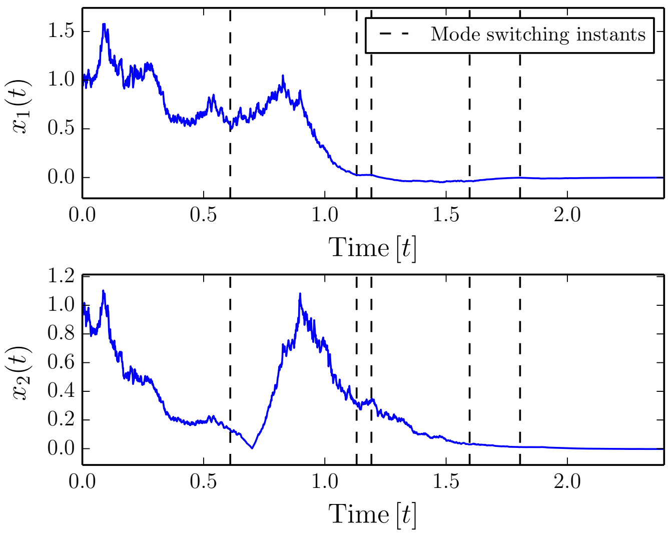

Figure 3.12: State trajectory versus time (mode sampling period $\tau = 0.1$)

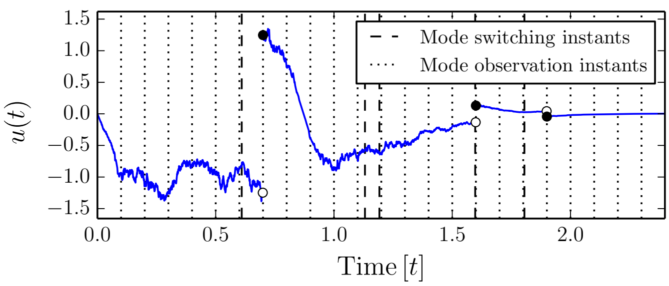

Figure 3.13: Control input versus time (mode sampling period $\tau = 0.1$)

obtained frequently compared to the occurrences of mode switches. As a consequence, the sampled mode signal $\sigma(t)$ is a good representation of the actual mode signal $r(t)$ (see Figure 3.11). Hence, the states converge to the origin (Figure 3.9).

Note that Theorem 3.4 guarantees that state trajectories $x(t)$ converges to the origin with probability one. Figures 3.12–3.14 show different sample paths of state, control input, actual mode signal, and sampled mode signal obtained with the same initial conditions ($x(0) = [1, 1]^{\mathrm{T}}$ and $r(0) = 1$) used for obtaining the sample paths shown in Figures 3.9–3.11. Figure 3.11 shows that for the sampling period $\tau = 0.1$, the sampled mode signal $\sigma(t)$ can be considered as a good representation of the actual mode signal $r(t)$, even though there are intervals where the actual mode signal differs from its sampled version. On the other hand, note that if we consider larger mode sampling periods, the sampled mode signal $\sigma(t)$ would no longer be an accurate representation of the actual mode of the

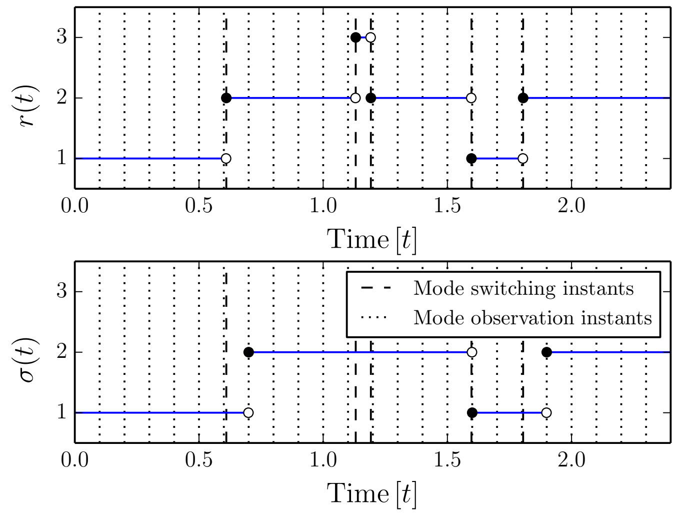

Figure 3.14: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time (mode sampling period $\tau = 0.1$)

switched system (3.57). In such cases, the stabilization performance of the control law (3.57) would deteriorate.

Now, we use the sample path of the actual mode signal $r(t)$ presented in Figure 3.14, and by considering a large mode sampling period ($\tau = 0.6$), we obtain the sampled mode signal $\sigma(t)$ (see Figure 3.17). Furthermore, in order to illustrate the deterioration of the stabilization performance for the large mode sampling period $\tau = 0.6$, we obtain sample paths of the state $x(t)$ and the control input $u(t)$ (see Figures 3.15 and 3.16). Note that when the mode signal is sampled very rarely, sampled mode signal may not provide a good information of the actual mode of the switched system. Therefore, control performance may be subject to deterioration.

3.7 Conclusion

A piecewise-continuous feedback control law that depends only on the uniformly sampled mode information has been proposed for continuous-time switched linear stochastic systems. In order to analyze the almost sure asymptotic stability of the closed loop system under the proposed control law, we first examined the relation between the sampled and the actual mode signal. Furthermore, we developed a kind of strong law of large numbers for the bivariate process comprising the sampled and the actual mode signal. We obtained sufficient conditions of almost sure asymptotic stabilization by using a quadratic

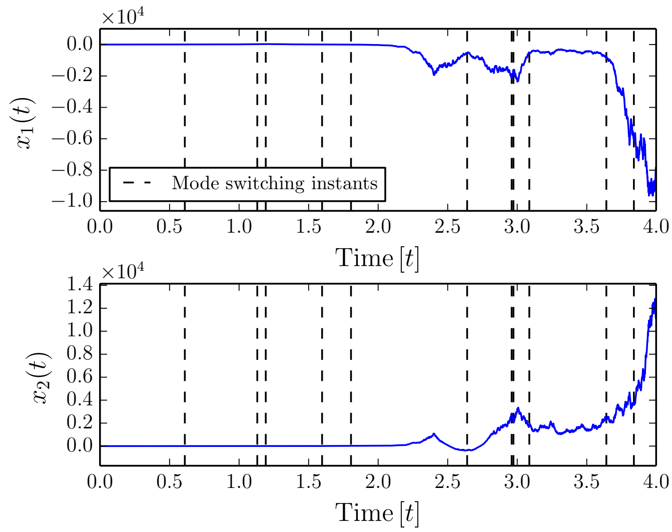

Figure 3.15: State trajectory versus time (mode sampling period $\tau = 0.6$)

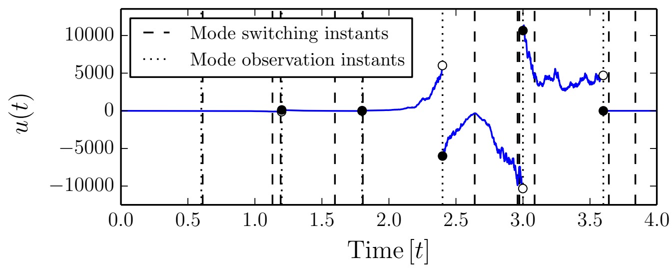

Figure 3.16: Control input versus time (mode sampling period $\tau = 0.6$)

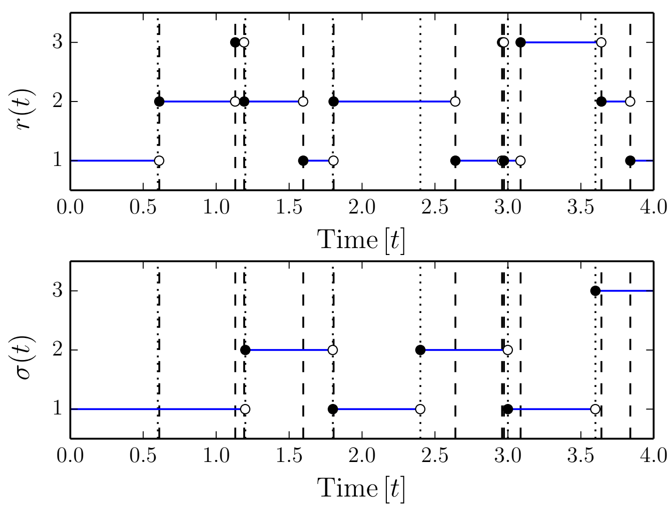

Figure 3.17: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time (mode sampling period $\tau = 0.6$)

Lyapunov-like function and taking advantage of the developed strong law of large numbers.

Note that in this chapter, we only proposed a control law for stabilizing continuous-time switched linear stochastic dynamical systems under sampled mode information. In Chapters 4 and 5 we extend the results of this chapter to the case where the sampled mode information is also subject to *time delay*.

Furthermore, note that in this chapter, we considered the case where the mode is sampled periodically. Hence, mode sampling instants considered in this chapter are deterministic. In Chapter 8, we will address the case where the mode signal is sampled at time instants that are separated by exponentially distributed independent random time intervals.

Chapter 4

Stabilization of Switched Linear Stochastic Systems Under Delayed Discrete Mode Observations

4.1 Introduction

Most of the feedback control laws documented in the literature of stochastic hybrid systems require perfect knowledge of the mode signal. As a consequence, when the mode of the switched system is not available for control purposes or only sampled mode information is available, these control laws cannot be used for stabilization. It is therefore important to develop feedback control frameworks for the limited mode information case. In the previous chapter, we explored a feedback control problem for switched stochastic systems for the case where the mode information is observed (sampled) only at discrete time instants. Specifically, we proposed a feedback control law for stabilizing continuous-time switched linear stochastic dynamical systems under periodically sampled mode information. In this chapter, we extend our results presented in Chapter 3 to the case where the sampled mode information is subject to *time delay*.

A feedback control problem for stochastic hybrid systems under the effect of delays is explored in several studies. Specifically, stabilization with delayed state feedback has been investigated in [59, 62, 64, 100–103] and stabilization of discrete-time Markov jump systems over communication networks with delays has been discussed in [104, 105]. Furthermore, an optimal controller is obtained in [106] for stabilizing discrete-time linear Markov jump systems under "one time-step" delayed mode observations.

In this chapter, we explore the feedback control problem for continuous-time switched stochastic systems under sampled and delayed mode information. These systems are composed of linear stochastic subsystems, which include Brownian motion in their dynamics. A continuous-time, finite-state Markov chain is employed for modeling the mode signal of the switched system. We focus on the case where the mode signal of the switched system is observed (sampled) only at equally spaced discrete time instants and the obtained mode samples are available to the controller only after a time delay. The mode information time delay can capture communication delays between the mode sampling mechanism and the controller. On the other hand, computational delays in mode detection might as well be modeled by the delayed mode observations. For example, mode information delays may correspond to failure-detection delays for a fault tolerant control system with normal/faulty modes and a "fault detection and isolation scheme" explored in [77, 78]. We propose a piecewise-continuous control law that depends only on the delayed version of the sampled mode signal rather than the actual mode signal. We employ a quadratic Lyapunov-like function to obtain sufficient conditions under which our proposed control law guarantees almost sure asymptotic stability of the switched stochastic system.

The contents of this chapter are as follows. We explain the feedback control problem for switched stochastic systems under sampled and delayed mode information in Section 4.2. We then obtain sufficient conditions under which our proposed control law guarantees almost sure asymptotic stability in Section 4.3. In Section 4.4, we give an illustrative numerical example. Finally, we provide the conclusion in Section 4.5.

4.2 Feedback Control Problem Under Delayed Sampled Mode Information

Consider the continuous-time switched stochastic system given by (3.57). In the following, we investigate the feedback control problem for the case where the mode signal of the switched system is sampled and the sampled mode information is subject to time delay. Specifically, the mode signal is assumed to be observed (sampled) periodically with period

$\tau > 0$. Moreover, the obtained mode samples are assumed to be available to the controller after a constant time delay $T_{\mathrm{D}} > 0$.

We denote the available mode samples by the sequence $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$. By employing the "sample and hold" technique we obtain the sampled version of the mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ defined by

$$\sigma(t) \triangleq r(k\tau), \quad t \in [k\tau, (k+1)\tau), \quad k \in \mathbb{N}_0. \tag{4.1}$$

In Chapter 3, we had considered a control law of the form $u(t) = K_{\sigma(t)}x(t)$, which depends only on the sampled mode information. In this chapter, each mode sample data is assumed to be subject to delay $T_{\mathrm{D}} > 0$, and hence only a delayed version of the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ is available for control purposes. As a result, the control law presented in Chapter 3 cannot be directly employed. We assume that the initial mode is known to the controller and propose a new control law of the form

$$u(t) = \begin{cases} K_{r_0}x(t), & 0 \leq t < T_{\mathrm{D}}, \\ K_{\sigma(t-T_{\mathrm{D}})}x(t), & t \geq T_{\mathrm{D}}, \end{cases} \tag{4.2}$$

which depends only on the delayed version of the sampled mode signal. Henceforth, our main objective is to obtain sufficient conditions of almost sure asymptotic stability of the closed-loop system (3.57) under our proposed control law (4.2).

In the remainder of this section, we first discuss the relation between the actual mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$, the discrete mode sample sequence $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$, and the time delayed version of the sampled mode signal $\{\sigma(t - T_{\mathrm{D}}) \in \mathcal{M}\}_{t\geq T_{\mathrm{D}}}$. We then extend our results presented in Chapter 3 for the time-delay case and obtain a strong law of large numbers for the continuous-time bivariate stochastic process $\{(r(t), \sigma(t - T_{\mathrm{D}})) \in \mathcal{M} \times \mathcal{M}\}_{t\geq T_{\mathrm{D}}}$.

Note that the sequence of mode samples $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$, is a discrete-time Markov chain with state transition probabilities given by

$$\mathbb{P}[r((k+1)\tau) = j | r(k\tau) = i] = p_{i,j}(\tau), \tag{4.3}$$

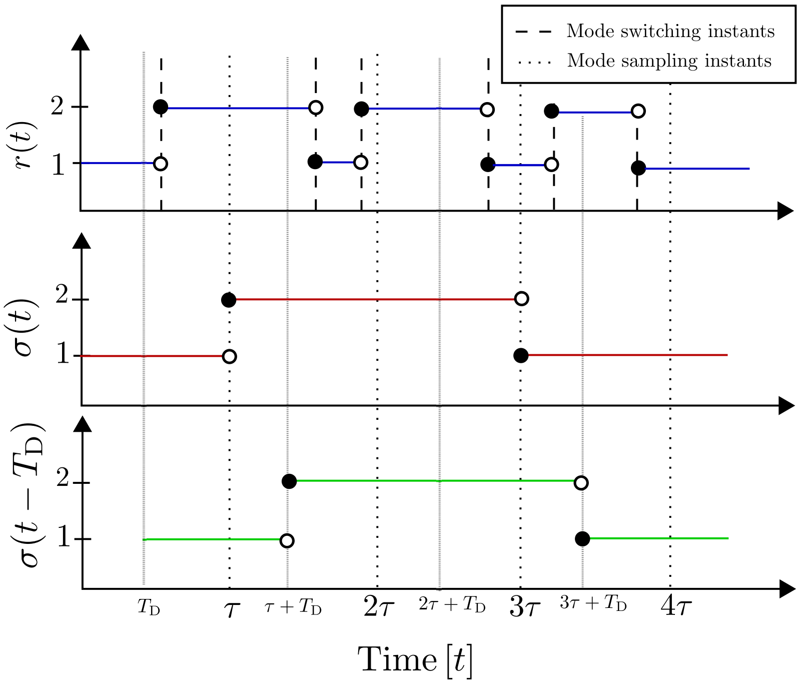

Figure 4.1: Actual mode signal $r(t)$, the sampled mode signal $\sigma(t)$, and the delayed version of the sampled mode signal $\sigma(t-T_{\mathrm{D}})$ versus time

where $p_{i,j}(\tau)$ represents the (i, j)th entry of the transition matrix $e^{Q\tau}$. Note that $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is an irreducible Markov chain, since the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is irreducible. Moreover, the stationary probability distribution for $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is given by $\pi \in \mathbb{R}^M$, which is also the stationary probability distribution for $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ [87].

The delayed sampled mode signal $\{\sigma(t - T_{\mathrm{D}}) \in \mathcal{M}\}_{t\geq T_{\mathrm{D}}}$ is a piecewise-constant stochastic process that depends on the actual mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$. Note that the frequency of the occurrences of mode transitions, mode sampling period $\tau > 0$, and sampled mode information delay $T_{\mathrm{D}} > 0$ affect how accurately the actual mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is represented by the delayed sampled version $\{\sigma(t - T_{\mathrm{D}}) \in \mathcal{M}\}_{t\geq T_{\mathrm{D}}}$. Figure 4.1 shows sample paths of $r(t)$, $\sigma(t)$, and $\sigma(t - T_{\mathrm{D}})$ of a switched system (3.57) with $M = 2$ modes.

Our goal now is to analyze the long run average of a piecewise-constant function that depends on the actual mode signal $r(t)$ as well as its sampled and delayed version $\sigma(t - T_{\mathrm{D}})$. This analysis is important for obtaining sufficient stability conditions for the closed-loop system (3.57), (4.2) through a Lyapunov-like approach. The following lemma presents a preliminary result that is necessary for our analysis.

Lemma 4.1. Let $\{r(t) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{t\geq 0}$ be a finite-state, irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M\times M}$. Then for any $\phi_l \in \mathbb{R}$, $l \in \mathcal{M}$, $t_1, t_2 \in [0, \infty)$ and $G \in \mathcal{F}$ such that $t_1 \leq t_2$ and $\mathbb{P}[G] > 0$, it follows that

$$\mathbb{E}[\int_{t_1}^{t_2} \phi_{r(s)}\mathrm{d}s|G] = \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}} \phi_l \mathbb{P}[G \cap F_l(s)]\mathrm{d}s, \tag{4.4}$$

where $F_l(t) \triangleq \{\omega \in \Omega : r_t(\omega) = l\}$, $t \geq 0$.

Proof. By using the definition of conditional expectation given in (2.1), we obtain

$$\begin{aligned}\mathbb{E}[\int_{t_1}^{t_2} \phi_{r(s)}\mathrm{d}s|G] &= \frac{1}{\mathbb{P}[G]}\int_G\int_{t_1}^{t_2} \phi_{r(s)}\mathrm{d}s\mathbb{P}(\mathrm{d}\omega)\\ &= \frac{1}{\mathbb{P}[G]}\int_G\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}} \phi_l \mathbb{1}_{[F_l(s)]}(\omega)\mathrm{d}s\mathbb{P}(\mathrm{d}\omega). \end{aligned}\tag{4.5}$$

Moreover, we employ Fubini's Theorem [93] to change the order of integrals in (4.5). It follows that

$$\begin{aligned}\mathbb{E}[\int_{t_1}^{t_2} \phi_{r(s)}\mathrm{d}s|G] &= \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\int_G\sum_{l\in\mathcal{M}} \phi_l \mathbb{1}_{[F_l(s)]}(\omega)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}s\\ &= \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}} \phi_l \int_G \mathbb{1}_{[F_l(s)]}(\omega)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}s\\ &= \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}} \phi_l \mathbb{P}[G \cap F_l(s)]\mathrm{d}s, \end{aligned}\tag{4.6}$$

which completes the proof. □

Now, by using the result presented in Lemma 4.1, we obtain a strong law of large numbers for the bivariate stochastic process $\{(r(t), \sigma(t - T_{\mathrm{D}}))\}_{t\geq T_{\mathrm{D}}}$ in Lemma 4.2.

Lemma 4.2. Let $\{r(t) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{t\geq 0}$ be an irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M\times M}$ with stationary probability distribution $\pi \in \mathbb{R}^M$. Moreover, let $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ defined in (4.1) be the sampled version of $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ for a given sampling period $\tau > 0$, and let $T_{\mathrm{D}} > 0$ be the time-delay constant. Then, for any $\gamma_{i,j} \in \mathbb{R}$, $i, j \in \mathcal{M}$,

$$\lim_{t\to\infty}\frac{1}{t}\int_{T_{\mathrm{D}}}^{t} \gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s = \frac{1}{\tau}\mathrm{tr}(\Pi\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}} e^{Qs}\mathrm{d}s\Gamma), \tag{4.7}$$

almost surely, where $\Pi \in \mathbb{R}^{M\times M}$ denotes the diagonal matrix with the diagonal entries $\pi_1, \pi_2, \ldots, \pi_M$, and $\Gamma \in \mathbb{R}^{M\times M}$ denotes the matrix with the (i,j)th entries given by $\gamma_{i,j}$, $i,j \in \mathcal{M}$.

The delay-free version of the result presented in Lemma 4.2 was proved in Chapter 3. For proving Lemma 4.2, we employ a method similar to the one used in Chapter 3; however, some additional key steps are required due to the effect of time-delay.

Proof. First, let $\{N(t) \in \mathbb{N}_0\}_{t\geq T_\mathrm{D}}$ be the counting process defined by

$$N(t) = \max\{k \in \mathbb{N}_0 \,:\, k\tau + T_\mathrm{D} \leq t\}, \quad t \geq T_\mathrm{D}. \tag{4.8}$$

Note that $N(t)$ represents the number of mode samples obtained until time t. The integral on the left hand side of (4.7) can be computed as

$$\begin{aligned}\int_{T_\mathrm{D}}^{t} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s = &\sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_\mathrm{D}}^{k\tau+T_\mathrm{D}} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s \\ &+ \int_{N(t)\tau+T_\mathrm{D}}^{t} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s,\end{aligned} \tag{4.9}$$

for $t \geq T_\mathrm{D}$. We observe that

$$|\int_{N(t)\tau+T_\mathrm{D}}^{t} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s| \leq \max_{i,j\in\mathcal{M}} |\gamma_{i,j}|\tau, \tag{4.10}$$

for all $t \geq T_\mathrm{D}$, and hence

$$\lim_{t\to\infty} \frac{1}{t} \int_{N(t)\tau+T_\mathrm{D}}^{t} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s = 0. \tag{4.11}$$

Consequently, by using (4.9) and (4.11), we obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_{T_\mathrm{D}}^{t} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s = \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_\mathrm{D}}^{k\tau+T_\mathrm{D}} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s. \tag{4.12}$$

Next we consider two cases: the case where $T_\mathrm{D} \leq \tau$ and the case where $T_\mathrm{D} > \tau$. For both cases we evaluate the limit $\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_\mathrm{D}}^{k\tau+T_\mathrm{D}} \gamma_{r(s),\sigma(s-T_\mathrm{D})}\mathrm{d}s$ and show that the limit is given by the right hand side of (4.7) in both cases.

Case 1) We now consider the case where $T_{\mathrm{D}} \leq \tau$. Note that in this case the information delay is less than the mode sampling interval, and hence the kth sampled mode data $r(k\tau)$ becomes available for control purposes before time $(k+1)\tau$.

Now, let $\{N^{h,i,j}(t) \in \mathbb{N}_0\}_{t \geq T_{\mathrm{D}}}$ be the counting process defined by

$$N^{h,i,j}(t) = \sum_{k=1}^{N(t)} \mathbf{1}_{[r((k-1)\tau)=h, r(k\tau)=i, r((k+1)\tau)=j]}, \quad t \geq T_{\mathrm{D}}. \tag{4.13}$$

Note that for all $h, i, j \in \mathcal{M}$, the counting process $\{N^{h,i,j}(t) \in \mathbb{N}_0\}_{t \geq T_{\mathrm{D}}}$ is a stochastic process that depends on the mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$. Note also that

$$\sum_{h,i,j \in \mathcal{M}} N^{h,i,j}(t) = N(t), \quad t \geq T_{\mathrm{D}}. \tag{4.14}$$

Furthermore, for all $h, i, j \in \mathcal{M}$, let the sequence of indices $\{k_n^{h,i,j} \in \mathbb{N}\}_{n \in \mathbb{N}}$ be defined by

$$k_n^{h,i,j} = \min\{k \in \mathbb{N} : N^{h,i,j}(k\tau + T_{\mathrm{D}}) = n\},\ n \in \mathbb{N}. \tag{4.15}$$

Now, note that $r((k_n^{h,i,j}-1)\tau) = \sigma(k_n^{h,i,j}\tau - T_{\mathrm{D}}) = h$, $r(k_n^{h,i,j}\tau) = i$, and $r((k_n^{h,i,j}+1)\tau) = j$, $n \in \mathbb{N}$, $h, i, j \in \mathcal{M}$. Furthermore, $\sigma(s - T_{\mathrm{D}}) = r((k_n^{h,i,j} - 1)\tau) = h$, for $s \in [(k_n^{h,i,j} - 1)\tau + T_{\mathrm{D}}, k_n^{h,i,j}\tau + T_{\mathrm{D}})$. As a consequence, it follows from (4.13) and (4.15) that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}} \gamma_{r(s),\sigma(s-T_{\mathrm{D}})} \mathrm{d}s \\
&\quad = \lim_{t\to\infty} \frac{1}{t} \sum_{h,i,j\in\mathcal{M}} \sum_{n=1}^{N^{h,i,j}(t)} \int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}} \gamma_{r(s),\sigma(s-T_{\mathrm{D}})} \mathrm{d}s \\
&\quad = \lim_{t\to\infty} \frac{1}{t} \sum_{h,i,j\in\mathcal{M}} \sum_{n=1}^{N^{h,i,j}(t)} \int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}} \gamma_{r(s),h} \mathrm{d}s.
\end{aligned} \tag{4.16}$$

We multiply the integrals in the right hand side of (4.16) by $\frac{N(t)}{N(t)}\frac{N^{h,i,j}(t)}{N^{h,i,j}(t)}$ to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{h,i,j\in\mathcal{M}}\Big(\frac{N^{h,i,j}(t)}{N(t)}\frac{1}{N^{h,i,j}(t)}\sum_{n=1}^{N^{h,i,j}(t)}\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),h}\mathrm{d}s\Big)\\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{h,i,j\in\mathcal{M}}\Big(\lim_{t\to\infty}\frac{N^{h,i,j}(t)}{N(t)}\lim_{t\to\infty}\frac{1}{N^{h,i,j}(t)}\sum_{n=1}^{N^{h,i,j}(t)}\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),h}\mathrm{d}s\Big).
\end{aligned}\tag{4.17}$$

We start by computing $\lim_{t\to\infty}\frac{N(t)}{t}$. By the definition of $N(t)$ given in (4.8), we have

$$N(t)\tau+T_{\mathrm{D}}\le t\le (N(t)+1)\tau+T_{\mathrm{D}},\;\; t\ge T_{\mathrm{D}}.\tag{4.18}$$

Therefore,

$$\frac{t-\tau-T_{\mathrm{D}}}{\tau}\le N(t)\le\frac{t-T_{\mathrm{D}}}{\tau},\quad t\ge T_{\mathrm{D}}.\tag{4.19}$$

Since $\lim_{t\to\infty}\frac{1}{t}\frac{t-\tau-T_{\mathrm{D}}}{\tau}=\lim_{t\to\infty}\frac{1}{t}\frac{t-T_{\mathrm{D}}}{\tau}=\frac{1}{\tau}$, it follows from (4.19) that

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\tau}.\tag{4.20}$$

Next, we evaluate $\lim_{t\to\infty}\frac{N^{h,i,j}(t)}{N(t)}$ in (4.17). The counting process $N_{h,i,j}(t)$ denotes the number of time instants $k\in\{1,2,\ldots,N(t)\}$ such that $r((k-1)\tau)=h$, $r(k\tau)=i$, and $r((k+1)\tau)=j$. Furthermore, note that $N(t)=\sum_{h,i,j\in\mathcal{M}}N^{h,i,j}(t)$. By using the strong law of large numbers [86, 87] for the discrete-time Markov chain $\{r(k\tau)\in\mathcal{M}\}_{k\in\mathbb{N}_0}$, we obtain

$$\lim_{t\to\infty}\frac{N^{h,i,j}(t)}{N(t)}=\pi_h p_{h,i}(\tau)p_{i,j}(\tau),\tag{4.21}$$

where $\pi_h>0$ is the stationary probability distribution for state $h\in\mathcal{M}$ and $p_{h,i}(\tau)$ and $p_{i,j}(\tau)$ are transition probabilities characterized in (4.3).

As the third step, we will employ the strong law of large numbers for independent and identically distributed random variables in order to compute the limit

$$\lim_{t\to\infty}\frac{1}{N^{h,i,j}(t)}\sum_{n=1}^{N^{h,i,j}(t)}\int_{(k_n^{h,i,j}-1)\tau+T_{\rm D}}^{k_n^{h,i,j}\tau+T_{\rm D}}\gamma_{r(s),h}{\rm d}s$$

Note that

$$\int_{(k_n^{h,i,j}-1)\tau+T_{\rm D}}^{k_n^{h,i,j}\tau+T_{\rm D}}\gamma_{r(s),h}{\rm d}s=\int_{(k_n^{h,i,j}-1)\tau+T_{\rm D}}^{k_n^{h,i,j}\tau}\gamma_{r(s),h}{\rm d}s+\int_{k_n^{h,i,j}\tau}^{k_n^{h,i,j}\tau+T_{\rm D}}\gamma_{r(s),h}{\rm d}s. \tag{4.22}$$

Now let

$$y_n^{h,i,j}\triangleq\int_{(k_n^{h,i,j}-1)\tau+T_{\rm D}}^{k_n^{h,i,j}\tau}\gamma_{r(s),h}{\rm d}s,\quad n\in\mathbb{N},\quad h,i,j\in\mathcal{M}, \tag{4.23}$$

$$z_n^{h,i,j}\triangleq\int_{k_n^{h,i,j}\tau}^{k_n^{h,i,j}\tau+T_{\rm D}}\gamma_{r(s),h}{\rm d}s,\quad n\in\mathbb{N},\quad h,i,j\in\mathcal{M}. \tag{4.24}$$

Note that by definition (4.15), the mode signal takes the values h and i at time instants $(k_n^{h,i,j}-1)\tau$ and $k_n^{h,i,j}\tau$, respectively, for all $n\in\mathbb{N}$. The value of the mode signal during the interval $\big((k_n^{h,i,j}-1)\tau,k_n^{h,i,j}\tau\big)$ may differ for each $n\in\mathbb{N}$. However, the probability of the mode taking the value $l\in\mathcal{M}$ at time $(k_n^{h,i,j}-1)\tau+s$, where $s\in(0,\tau)$, does not depend on $n\in\mathbb{N}$. Hence, for given $h,i,j\in\mathcal{M}$, the random variables $y_n^{h,i,j}$, $n\in\mathbb{N}$, are independent and identically distributed. Similarly, for given $h,i,j\in\mathcal{M}$, the random variables $z_n^{h,i,j}$, $n\in\mathbb{N}$, are also independent and identically distributed. Now, we calculate $\mathbb{E}[y_n^{h,i,j}]$ and $\mathbb{E}[z_n^{h,i,j}]$. It follows from (4.23) that

$$\begin{aligned}\mathbb{E}[y_n^{h,i,j}]&=\sum_{k=1}^{\infty}\mathbb{E}[\int_{(k_n^{h,i,j}-1)\tau+T_{\rm D}}^{k_n^{h,i,j}\tau}\gamma_{r(s),h}{\rm d}s|k_n^{h,i,j}=k]\mathbb{P}[k_n^{h,i,j}=k]\\&=\sum_{k=1}^{\infty}\mathbb{E}[\int_{(k-1)\tau+T_{\rm D}}^{k\tau}\gamma_{r(s),h}{\rm d}s|G]\mathbb{P}[k_n^{h,i,j}=k],\end{aligned} \tag{4.25}$$

where

$$G\triangleq\{\omega\in\Omega:r_{(k-1)\tau}(\omega)=h,r_{k\tau}(\omega)=i,r_{(k+1)\tau}(\omega)=j\}. \tag{4.26}$$

We set $t_1\triangleq(k-1)\tau+T_{\rm D}$, $t_2\triangleq k\tau$, $\phi_{r(s)}\triangleq\gamma_{r(s),h}$, $s\in[t_1,t_2)$, and employ the result presented in Lemma 4.1 to obtain

$$\mathbb{E}[y_n^{h,i,j}] = \sum_{k=1}^{\infty} \frac{1}{\mathbb{P}[G]} \int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau} \sum_{l\in\mathcal{M}} \gamma_{l,h}\mathbb{P}[G\cap F_l(s)]\mathrm{d}s\mathbb{P}[k_n^{i,j}=k], \tag{4.27}$$

where $F_l(s) \triangleq \{\omega\in\Omega : r_s(\omega)=l\}$, $s\in[(k-1)\tau+T_{\mathrm{D}}, k\tau)$. Note that $G = F_h((k-1)\tau)\cap F_i(k\tau)\cap F_j((k+1)\tau)$ and hence

$$\begin{aligned}
\frac{\mathbb{P}[G\cap F_l(s)]}{\mathbb{P}[G]} &= \frac{\mathbb{P}[F_h((k-1)\tau)\cap F_i(k\tau)\cap F_j((k+1)\tau)\cap F_l(s)]}{F_h((k-1)\tau)\cap F_i(k\tau)\cap F_j((k+1)\tau)} \\
&= \frac{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]}{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]} \\
&= \frac{\mathbb{P}[F_i(k\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_h((k-1)\tau)]}{\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]} \\
&= \frac{p_{l,i}(k\tau-s)p_{h,l}(s-(k-1)\tau)}{p_{h,i}(\tau)}, \quad s\in[(k-1)\tau+T_{\mathrm{D}}, k\tau),
\end{aligned} \tag{4.28}$$

where $p_{h,i}(\tau)$ is given by (4.3). We substitute (4.28) into (4.27) and set $\tilde{s}\triangleq s-(k-1)\tau$ to arrive at

$$\begin{aligned}
\mathbb{E}[y_n^{h,i,j}] &= \sum_{k=1}^{\infty}\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,h}\frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)}\mathrm{d}\tilde{s}\mathbb{P}[k_n^{h,i,j}=k] \\
&= \int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,h}\frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)}\mathrm{d}\tilde{s}\sum_{k=1}^{\infty}\mathbb{P}[k_n^{h,i,j}=k] \\
&= \int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,h}\frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)}\mathrm{d}\tilde{s}, \quad n\in\mathbb{N}.
\end{aligned} \tag{4.29}$$

On the other hand it follows from (4.24) that

$$\begin{aligned}
\mathbb{E}[z_n^{h,i,j}] &= \sum_{k=1}^{\infty}\mathbb{E}[\int_{k_n^{h,i,j}\tau}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),h}\mathrm{d}s|k_n^{h,i,j}=k]\mathbb{P}[k_n^{h,i,j}=k] \\
&= \sum_{k=1}^{\infty}\mathbb{E}[\int_{k\tau}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),h}\mathrm{d}s|G]\mathbb{P}[k_n^{h,i,j}=k],
\end{aligned} \tag{4.30}$$

where G is given by (4.26). We now set $t_1\triangleq k\tau$, $t_2\triangleq k\tau+T_{\mathrm{D}}$, $\phi_{r(s)}\triangleq\gamma_{r(s),h}$, $s\in[t_1,t_2)$, and employ the result presented in Lemma 4.1 to obtain

$$\mathbb{E}[z_n^{h,i,j}] = \sum_{k=1}^{\infty}\frac{1}{\mathbb{P}[G]}\int_{k\tau}^{k\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,h}\mathbb{P}[G\cap F_l(s)]\mathrm{d}s\mathbb{P}[k_n^{h,i,j}=k]. \tag{4.31}$$

where $F_l(s) \triangleq \{\omega \in \Omega : r_s(\omega) = l\}$, $s \in [k\tau, k\tau + T_{\rm D})$. Now note that for $s \in [k\tau, k\tau + T_{\rm D})$,

$$\begin{aligned}\frac{\mathbb{P}[G \cap F_l(s)]}{\mathbb{P}[G]} &= \frac{\mathbb{P}[F_h((k-1)\tau) \cap F_i(k\tau) \cap F_j((k+1)\tau) \cap F_l(s)]}{F_h((k-1)\tau) \cap F_i(k\tau) \cap F_j((k+1)\tau)} \\ &= \frac{\mathbb{P}[F_j((k+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]}{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]} \\ &= \frac{\mathbb{P}[F_j((k+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i(k\tau)]}{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]} \\ &= \frac{p_{l,j}((k+1)\tau - s)p_{i,l}(s-k\tau)}{p_{i,j}(\tau)}. \end{aligned} \tag{4.32}$$

We now use (4.32) and set $\tilde{s} \triangleq s - (k-1)\tau$ in (4.31) to arrive at

$$\begin{aligned}\mathbb{E}[z_n^{h,i,j}] &= \sum_{k=1}^{\infty} \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,h} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s}\mathbb{P}[k_n^{h,i,j}=k] \\ &= \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,h} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s} \sum_{k=1}^{\infty} \mathbb{P}[k_n^{h,i,j}=k] \\ &= \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,h} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.33}$$

Note that $\lim_{t\to\infty} N^{h,i,j}(t) = \infty$, almost surely. Therefore, it follows from the strong law of large numbers and (4.29) and (4.33) that

$$\lim_{t\to\infty} \frac{1}{N^{h,i,j}(t)} \sum_{n=1}^{N^{h,i,j}(t)} y_n^{h,i,j} = \int_{T_{\rm D}}^{\tau} \sum_{l\in\mathcal{M}} \gamma_{l,h} \frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)} \mathrm{d}\tilde{s}, \tag{4.34}$$

$$\lim_{t\to\infty} \frac{1}{N^{h,i,j}(t)} \sum_{n=1}^{N^{h,i,j}(t)} z_n^{h,i,j} = \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,h} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s}. \tag{4.35}$$

Now it follows from (4.22)–(4.24), (4.34), and (4.35) that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{N^{h,i,j}(t)} \sum_{n=1}^{N^{h,i,j}(t)} \int_{(k_n^{h,i,j}-1)\tau+T_{\rm D}}^{k_n^{h,i,j}\tau+T_{\rm D}} \gamma_{r(s),h} \mathrm{d}s &= \int_{T_{\rm D}}^{\tau} \sum_{l\in\mathcal{M}} \gamma_{l,h} \frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)} \mathrm{d}\tilde{s} \\ &\quad + \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,h} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s}. \end{aligned} \tag{4.36}$$

As a final step, we substitute the limits evaluated in (4.20), (4.21), and (4.36) into (4.17), and obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{h,i}(\tau)p_{i,j}(\tau)\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,h}\frac{p_{l,i}(\tau-s)p_{h,l}(s)}{p_{h,i}(\tau)}\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{h,i}(\tau)p_{i,j}(\tau)\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,h}\frac{p_{l,j}(2\tau-s)p_{i,l}(s-\tau)}{p_{i,j}(\tau)}\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{i,j}(\tau)\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,h}p_{l,i}(\tau-s)p_{h,l}(s)\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{h,i}(\tau)\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,h}p_{l,j}(2\tau-s)p_{i,l}(s-\tau)\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,h}p_{h,l}(s)\sum_{i\in\mathcal{M}}p_{l,i}(\tau-s)\sum_{j\in\mathcal{M}}p_{i,j}(\tau)\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,h}\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau)\sum_{j\in\mathcal{M}}p_{l,j}(2\tau-s)\mathrm{d}s. \qquad (4.37)
\end{aligned}$$

Note that $\sum_{j\in\mathcal{M}}p_{i,j}(t)=1,\ t\geq 0$, for all $i\in\mathcal{M}$. We use this fact to obtain $\sum_{j\in\mathcal{M}}p_{i,j}(\tau)=1$, $\sum_{i\in\mathcal{M}}p_{l,i}(\tau-s)=1$, and $\sum_{j\in\mathcal{M}}p_{l,j}(2\tau-s)=1$ in (4.37). Furthermore, note that $\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau)=p_{h,l}(s)$, for all $h,l\in\mathcal{M}$. Therefore, it follows from (4.37) that

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s &=\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,h}p_{h,l}(s)\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,h}p_{h,l}(s)\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,h}p_{h,l}(s)\mathrm{d}s\\
&=\frac{1}{\tau}\mathrm{tr}(\Pi\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}e^{Qs}\mathrm{d}s\Gamma). \qquad (4.38)
\end{aligned}$$

Case 2) We now consider the case where $T_{\mathrm{D}}>\tau$ and compute $\lim_{t\to\infty}\frac{1}{t}\int_{T_{\mathrm{D}}}^{t}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s$. Note that in this case, the information delay is larger than the mode sampling interval, and hence the kth sampled mode data $r(k\tau)$ becomes available for control purposes after time $(k+1)\tau$. Let

$$\underline{k}\triangleq\max\{k\in\mathbb{N}:k\tau\leq T_{\mathrm{D}}\}. \qquad (4.39)$$

Note that the kth sampled mode data $r(k\tau)$ becomes available for control purposes before time $(k+\underline{k}+1)\tau$. Now, for given $g,h,i,j\in\mathcal{M}$, let $\{N^{g,h,i,j}(t)\in\mathbb{N}_0\}_{t\geq T_{\mathrm{D}}}$ be the counting process defined by

$$N^{g,h,i,j}(t)=\sum_{k=1}^{N(t)}\mathbf{1}_{[r((k-1)\tau)=g,r((k-1+\underline{k})\tau)=h,r((k+\underline{k})\tau)=i,r((k+\underline{k}+1)\tau)=j]},\quad t\geq T_{\mathrm{D}}. \tag{4.40}$$

Note that for all $g,h,i,j\in\mathcal{M}$, the counting process $\{N^{g,h,i,j}(t)\in\mathbb{N}_0\}_{t\geq T_{\mathrm{D}}}$ is a stochastic process that depends on the mode signal $\{r(t)\in\mathcal{M}\}_{t\geq 0}$. Furthermore, note that

$$\sum_{g,h,i,j\in\mathcal{M}}N^{g,h,i,j}(t)=N(t),\quad t\geq T_{\mathrm{D}}. \tag{4.41}$$

Now, for all $g,h,i,j\in\mathcal{M}$, we define the sequence of indices $\{k_n^{g,h,i,j}\in\mathbb{N}\}_{n\in\mathbb{N}}$ by

$$k_n^{g,h,i,j}=\min\{k\in\mathbb{N}:N^{g,h,i,j}(k\tau+T_{\mathrm{D}})=n\},\quad n\in\mathbb{N}. \tag{4.42}$$

Now, note that $r((k_n^{g,h,i,j}-1)\tau)=\sigma((k_n^{g,h,i,j}-1+\underline{k})\tau-T_{\mathrm{D}})=g$, $r((k_n^{g,h,i,j}-1+\underline{k})\tau)=h$, $r((k_n^{g,h,i,j}+\underline{k})\tau)=i$, and $r((k_n^{g,h,i,j}+\underline{k}+1)\tau)=j$, $n\in\mathbb{N}$, $g,h,i,j\in\mathcal{M}$. Furthermore, note that $\sigma(s-T_{\mathrm{D}})=r((k_n^{g,h,i,j}-1)\tau)=g$, for $s\in[(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}},k_n^{g,h,i,j}\tau+T_{\mathrm{D}})$. As a consequence, it follows from (4.12)–(4.42) that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}&\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s\\&=\lim_{t\to\infty}\frac{1}{t}\sum_{g,h,i,j\in\mathcal{M}}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s\\&=\lim_{t\to\infty}\frac{1}{t}\sum_{g,h,i,j\in\mathcal{M}}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),g}\mathrm{d}s.\end{aligned} \tag{4.43}$$

We now multiply the integral in the right hand side of (4.43) by $\frac{N(t)}{N(t)}\frac{N^{g,h,i,j}(t)}{N^{g,h,i,j}(t)}$ to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s \\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{g,h,i,j\in\mathcal{M}}\Big(\frac{N^{g,h,i,j}(t)}{N(t)}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),g}\mathrm{d}s\Big) \\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{g,h,i,j\in\mathcal{M}}\Big(\lim_{t\to\infty}\frac{N^{g,h,i,j}(t)}{N(t)}\lim_{t\to\infty}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),g}\mathrm{d}s\Big).
\end{aligned} \tag{4.44}$$

We now evaluate $\lim_{t\to\infty}\frac{N^{g,h,i,j}(t)}{N(t)}$ in (4.44). The counting process $N_{g,h,i,j}(t)$ denotes the number of time instants $k\in\{1,2,\ldots,N(t)\}$ such that $r((k-1)\tau)=g$, $r((k-1+\underline{k})\tau)=h$, $r((k+\underline{k})\tau)=i$, and $r((k+\underline{k}+1)\tau)=j$. Note that $N(t)=\sum_{g,h,i,j\in\mathcal{M}}N^{g,h,i,j}(t)$. We use the strong law of large numbers [86,87] for discrete-time Markov chain $\{r(k\tau)\in\mathcal{M}\}_{k\in\mathbb{N}_0}$ and obtain

$$\lim_{t\to\infty}\frac{N^{g,h,i,j}(t)}{N(t)}=\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)p_{i,j}(\tau), \tag{4.45}$$

where $\pi_g>0$ is the stationary probability distribution for state $g\in\mathcal{M}$. Moreover, $p_{h,i}(\underline{k}\tau)$, $p_{h,i}(\tau)$ and $p_{i,j}(\tau)$ are transition probabilities characterized in (4.3).

Next, our goal is to compute the limit

$$\lim_{t\to\infty}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),g}\mathrm{d}s. \tag{4.46}$$

Note that by the definition (4.39)

$$\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),g}\mathrm{d}s=\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{(k_n^{g,h,i,j}+\underline{k})\tau}\gamma_{r(s),g}\mathrm{d}s+\int_{(k_n^{g,h,i,j}+\underline{k})\tau}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),g}\mathrm{d}s. \tag{4.47}$$

Now let

$$y_n^{g,h,i,j} \triangleq \int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{(k_n^{g,h,i,j}+\underline{k})\tau} \gamma_{r(s),g} {\rm d}s, \quad n \in \mathbb{N}, \quad g,h,i,j \in \mathcal{M}, \tag{4.48}$$

$$z_n^{g,h,i,j} \triangleq \int_{(k_n^{g,h,i,j}+\underline{k})\tau}^{k_n^{g,h,i,j}\tau+T_{\rm D}} \gamma_{r(s),g} {\rm d}s, \quad n \in \mathbb{N}, \quad g,h,i,j \in \mathcal{M}. \tag{4.49}$$

Note that by definition (4.42), the mode signal takes the values g and i at time instants $(k_n^{g,h,i,j} - 1 + \underline{k})\tau$ and $(k_n^{g,h,i,j} + \bar{k})\tau$, respectively, for all $n \in \mathbb{N}$. The value of the mode signal during the interval $\big((k_n^{g,h,i,j} - 1 + \underline{k})\tau, (k_n^{g,h,i,j} + \bar{k})\tau\big)$ may differ for each $n \in \mathbb{N}$. However, the probability of the mode taking the value $l \in \mathcal{M}$ at time $(k_n^{g,h,i,j} - 1 + \underline{k})\tau + s$, where $s \in (0, \tau)$, does not depend on $n \in \mathbb{N}$. Note that the integration in (4.48) is over the interval $[(k_n^{g,h,i,j} - 1)\tau + T_{\rm D}, (k_n^{g,h,i,j} + \bar{k})\tau)$, where $(k_n^{g,h,i,j} - 1)\tau + T_{\rm D} > (k_n^{g,h,i,j} - 1 + \underline{k})\tau$. Hence, for given $g,h,i,j \in \mathcal{M}$, the random variables $y_n^{g,h,i,j}$, $n \in \mathbb{N}$, are independent and distributed identically. A similar argument can be used to show that for given $g,h,i,j \in \mathcal{M}$, the random variables $z_n^{g,h,i,j}$, $n \in \mathbb{N}$, are also independent and distributed identically. Now, it follows from (4.48) that

$$\begin{aligned}\mathbb{E}[y_n^{g,h,i,j}] &= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{(k_n^{g,h,i,j}+\underline{k})\tau} \gamma_{r(s),g} {\rm d}s | k_n^{g,h,i,j} = k] \mathbb{P}[k_n^{g,h,i,j} = k] \\ &= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k-1)\tau+T_{\rm D}}^{(k+\underline{k})\tau} \gamma_{r(s),g} {\rm d}s | G] \mathbb{P}[k_n^{g,h,i,j} = k],\end{aligned} \tag{4.50}$$

where

$$G \triangleq \{\omega \in \Omega : r_{(k-1)\tau}(\omega) = g, r_{(k-1+\underline{k})\tau}(\omega) = h, r_{(k+\underline{k})\tau}(\omega) = i, r_{(k+\underline{k}+1)\tau}(\omega) = j\}. \tag{4.51}$$

We set $t_1 \triangleq (k-1)\tau + T_{\rm D}$, $t_2 \triangleq (k+\underline{k})\tau$, $\phi_{r(s)} \triangleq \gamma_{r(s),g}$, $s \in [t_1, t_2)$, and employ the result presented in Lemma 4.1 to obtain

$$\mathbb{E}[y_n^{g,h,i,j}] = \sum_{k=1}^{\infty} \frac{1}{\mathbb{P}[G]} \int_{(k-1)\tau+T_{\rm D}}^{(k+\underline{k})\tau} \sum_{l \in \mathcal{M}} \gamma_{l,g} \mathbb{P}[G \cap F_l(s)] {\rm d}s \mathbb{P}[k_n^{g,h,i,j} = k], \tag{4.52}$$

where $F_l(s) \triangleq \{\omega \in \Omega : r_s(\omega) = l\}$, $s \in [(k-1)\tau + T_{\rm D}, (k+\underline{k})\tau)$. Note that

$$G = F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau), \tag{4.53}$$

and hence

$$
\begin{aligned}
&\frac{\mathbb{P}[G \cap F_l(s)]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau) \cap F_l(s)]}{F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau)} \\
&= \frac{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]\mathbb{P}[F_i((k+\underline{k})\tau)|F_l(s)]}{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]} \\
&\quad \cdot \frac{\mathbb{P}[F_l(s)|F_h((k-1+\underline{k})\tau)]\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]}{\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]} \\
&= \frac{\mathbb{P}[F_i((k+\underline{k})\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_h((k-1+\underline{k})\tau)]}{\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]} \\
&= \frac{p_{l,i}((k+\underline{k})\tau - s)p_{h,l}(s-(k-1+\underline{k})\tau)}{p_{h,i}(\tau)}, \quad s \in [(k-1)\tau + T_{\mathrm{D}}, (k+\underline{k})\tau), \qquad (4.54)
\end{aligned}
$$

where $p_{h,i}(\tau)$ is given by (4.3). We substitute (4.54) into (4.52) and set $\tilde{s} \triangleq s-(k-1)\tau$ to arrive at

$$
\begin{aligned}
\mathbb{E}[y_n^{g,h,i,j}] &= \sum_{k=1}^{\infty} \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,i}((\underline{k}+1)\tau - \tilde{s})p_{h,l}(\tilde{s}-\underline{k}\tau)}{p_{h,i}(\tau)} \mathrm{d}\tilde{s}\mathbb{P}[k_n^{g,h,i,j}=k] \\
&= \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,i}((\underline{k}+1)\tau - \tilde{s})p_{h,l}(\tilde{s}-\underline{k}\tau)}{p_{h,i}(\tau)} \mathrm{d}\tilde{s} \sum_{k=1}^{\infty} \mathbb{P}[k_n^{g,h,i,j}=k] \\
&= \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,i}((\underline{k}+1)\tau - \tilde{s})p_{h,l}(\tilde{s}-\underline{k}\tau)}{p_{h,i}(\tau)} \mathrm{d}\tilde{s}, \quad n \in \mathbb{N}. \qquad (4.55)
\end{aligned}
$$

On the other hand it follows from (4.49) that

$$
\begin{aligned}
\mathbb{E}[z_n^{g,h,i,j}] &= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k_n^{g,h,i,j}+\underline{k})\tau}^{k_n^{g,h,i,j}\tau + T_{\mathrm{D}}} \gamma_{r(s),g} \mathrm{d}s | k_n^{g,h,i,j}=k] \mathbb{P}[k_n^{g,h,i,j}=k] \\
&= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k+\underline{k})\tau}^{k\tau + T_{\mathrm{D}}} \gamma_{r(s),g} \mathrm{d}s | G] \mathbb{P}[k_n^{g,h,i,j}=k], \qquad (4.56)
\end{aligned}
$$

where G is given by (4.51). We now set $t_1 \triangleq (k+\underline{k})\tau$, $t_2 \triangleq k\tau + T_{\mathrm{D}}$, $\phi_{r(s)} \triangleq \gamma_{r(s),g}$, $s \in [t_1, t_2)$, and employ the result presented in Lemma 4.1 to obtain

$$
\mathbb{E}[z_n^{g,h,i,j}] = \sum_{k=1}^{\infty} \frac{1}{\mathbb{P}[G]} \int_{(k+\underline{k})\tau}^{k\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} \mathbb{P}[G \cap F_l(s)] \mathrm{d}s \mathbb{P}[k_n^{g,h,i,j}=k]. \qquad (4.57)
$$

where $F_l(s) \triangleq \{\omega \in \Omega : r_s(\omega) = l\}$, $s \in [(k+\underline{k})\tau, k\tau + T_{\mathrm{D}})$. Note that

$$\begin{aligned}
&\frac{\mathbb{P}[G \cap F_l(s)]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau) \cap F_l(s)]}{F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau)} \\
&= \frac{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i((k+\underline{k})\tau)]}{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]} \\
&\quad \cdot \frac{\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]}{\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]} \\
&= \frac{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i((k+\underline{k})\tau)]}{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]} \\
&= \frac{p_{l,j}((k+\underline{k}+1)\tau - s)p_{i,l}(s-(k+\underline{k})\tau)}{p_{i,j}(\tau)}, \quad s \in [(k+\underline{k})\tau, k\tau + T_{\mathrm{D}}). \qquad (4.58)
\end{aligned}$$

We now use (4.58) and set $\tilde{s} \triangleq s - (k-1)\tau$ in (4.31) to arrive at

$$\begin{aligned}
\mathbb{E}[z_n^{g,h,i,j}] &= \sum_{k=1}^{\infty} \int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,j}((\underline{k}+2)\tau - \tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s}\,\mathbb{P}[k_n^{g,h,i,j} = k] \\
&= \int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,j}((\underline{k}+2)\tau - \tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s} \sum_{k=1}^{\infty} \mathbb{P}[k_n^{g,h,i,j} = k] \\
&= \int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,j}((\underline{k}+2)\tau - \tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s}, \quad n \in \mathbb{N}. \qquad (4.59)
\end{aligned}$$

Note that $\lim_{t\to\infty} N^{g,h,i,j}(t) = \infty$, almost surely. Consequently, it follows from the strong law of large numbers and (4.29) and (4.59) that

$$\lim_{t\to\infty} \frac{1}{N^{g,h,i,j}(t)} \sum_{n=1}^{N^{g,h,i,j}(t)} y_n^{g,h,i,j} = \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,i}((\underline{k}+1)\tau - \tilde{s})p_{h,l}(\tilde{s}-\underline{k}\tau)}{p_{h,i}(\tau)} \mathrm{d}\tilde{s}, \qquad (4.60)$$

$$\lim_{t\to\infty} \frac{1}{N^{g,h,i,j}(t)} \sum_{n=1}^{N^{g,h,i,j}(t)} z_n^{g,h,i,j} = \int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} \frac{p_{l,j}((\underline{k}+2)\tau - \tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)} \mathrm{d}\tilde{s}. \qquad (4.61)$$

It then follows from (4.22)–(4.24), (4.60), and (4.61) that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{k_n^{g,h,i,j}\tau+T_{\rm D}}\gamma_{r(s),g}{\rm d}s\\
&\quad=\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,g}\frac{p_{l,i}((\underline{k}+1)\tau-\tilde{s})p_{h,l}(\tilde{s}-\underline{k}\tau)}{p_{h,i}(\tau)}{\rm d}\tilde{s}\\
&\quad\quad+\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,g}\frac{p_{l,j}((\underline{k}+2)\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}{\rm d}\tilde{s}.
\end{aligned}\tag{4.62}$$

Finally, we substitute the limits evaluated in (4.20), (4.45), and (4.62) into (4.44), and obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\rm D}}^{k\tau+T_{\rm D}}\gamma_{r(s),\sigma(s-T_{\rm D})}{\rm d}s\\
&=\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)p_{i,j}(\tau)\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,g}\frac{p_{l,i}((\underline{k}+1)\tau-s)p_{h,l}(s-\underline{k}\tau)}{p_{h,i}(\tau)}{\rm d}s\\
&\quad+\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)p_{i,j}(\tau)\\
&\qquad\qquad\cdot\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,g}\frac{p_{l,j}((\underline{k}+2)\tau-s)p_{i,l}(s-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}{\rm d}s\\
&=\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{i,j}(\tau)\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,g}p_{l,i}((\underline{k}+1)\tau-s)p_{h,l}(s-\underline{k}\tau){\rm d}s\\
&\quad+\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,g}p_{l,j}((\underline{k}+2)\tau-s)p_{i,l}(s-\tau-\underline{k}\tau){\rm d}s\\
&=\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\Big(\sum_{l\in\mathcal{M}}\gamma_{l,g}\\
&\quad\cdot\sum_{h\in\mathcal{M}}p_{g,h}(\underline{k}\tau)p_{h,l}(s-\underline{k}\tau)\sum_{i\in\mathcal{M}}p_{l,i}((\underline{k}+1)\tau-s)\sum_{j\in\mathcal{M}}p_{i,j}(\tau)\Big){\rm d}s\\
&\quad+\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\Big(\sum_{l\in\mathcal{M}}\gamma_{l,g}\sum_{h\in\mathcal{M}}p_{g,h}(\underline{k}\tau)\\
&\quad\cdot\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau-\underline{k}\tau)\sum_{j\in\mathcal{M}}p_{l,j}((\underline{k}+2)\tau-s)\Big){\rm d}s.
\end{aligned}\tag{4.63}$$

By using the fact that $\sum_{j\in\mathcal{M}}p_{i,j}(t)=1$, $t\geq 0$, for all $i\in\mathcal{M}$, we obtain $\sum_{j\in\mathcal{M}}p_{i,j}(\tau)=1$, $\sum_{i\in\mathcal{M}}p_{l,i}((\underline{k}+1)\tau-s)=1$, and $\sum_{j\in\mathcal{M}}p_{l,j}((\underline{k}+2)\tau-s)=1$ in (4.63). Note also that $\sum_{h\in\mathcal{M}}p_{g,h}(\underline{k}\tau)p_{h,l}(s-\underline{k}\tau)=p_{g,l}(s)$, for all $g,l\in\mathcal{M}$. Moreover, $\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau-$

$\underline{k}\tau) = p_{h,l}(s - \underline{k}\tau)$, for all $h, l \in \mathcal{M}$. Therefore, it follows from (4.63) that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}} \gamma_{r(s),\sigma(s-T_{\mathrm{D}})} \mathrm{d}s \\
&= \frac{1}{\tau} \sum_{g\in\mathcal{M}} \pi_g \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,g} p_{g,l}(s) \mathrm{d}s \\
&\quad + \frac{1}{\tau} \sum_{g\in\mathcal{M}} \pi_g \int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} \sum_{h\in\mathcal{M}} p_{g,h}(\underline{k}\tau) p_{h,l}(s-\underline{k}\tau) \mathrm{d}s. \qquad (4.64)
\end{aligned}$$

Now note that $\sum_{h\in\mathcal{M}} p_{g,h}(\underline{k}\tau)p_{h,l}(s-\underline{k}\tau) = p_{g,l}(s)$, for all $g, l \in \mathcal{M}$. As a consequence,

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}} \gamma_{r(s),\sigma(s-T_{\mathrm{D}})} \mathrm{d}s &= \frac{1}{\tau} \sum_{g\in\mathcal{M}} \pi_g \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,g} p_{g,l}(s) \mathrm{d}s \\
&\quad + \frac{1}{\tau} \sum_{g\in\mathcal{M}} \pi_g \int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} p_{g,l}(s) \mathrm{d}s \\
&= \frac{1}{\tau} \sum_{g\in\mathcal{M}} \pi_g \int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}} \sum_{l\in\mathcal{M}} \gamma_{l,g} p_{g,l}(s) \mathrm{d}s \\
&= \frac{1}{\tau} \mathrm{tr}(\Pi \int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}} e^{Qs} \mathrm{d}s \Gamma). \qquad (4.65)
\end{aligned}$$

We evaluated the limit $\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}} \gamma_{r(s),\sigma(s-T_{\mathrm{D}})} \mathrm{d}s$ for cases where $T_{\mathrm{D}} \leq \tau$ and $T_{\mathrm{D}} > \tau$. It is shown in (4.38) and (4.65) that for both cases the limit is given by $\frac{1}{\tau}\mathrm{tr}(\Pi \int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}} e^{Qs} \mathrm{d}s \Gamma)$. It then follows from (4.12) that for all $\tau > 0$ and $T_{\mathrm{D}} > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \int_{T_{\mathrm{D}}}^{t} \gamma_{r(s),\sigma(s-T_{\mathrm{D}})} \mathrm{d}s = \frac{1}{\tau} \mathrm{tr}(\Pi \int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}} e^{Qs} \mathrm{d}s \Gamma), \qquad (4.66)$$

which completes the proof. □

4.3 Sufficient Conditions for Almost Sure Asymptotic Stabilization

In this section, we employ a quadratic Lyapunov approach similar to the one used in Chapter 3 and utilize the strong law of large numbers developed in Lemma 4.2 to obtain sufficient conditions of almost sure asymptotic stabilization under sampled and delayed mode information.

Theorem 4.1. Consider the switched linear stochastic system (3.57) with mode sampling period $\tau > 0$ and sampled mode information constant time delay $T_{\rm D} > 0$. If there exist $P > 0$ and scalars $\zeta_i \in \mathbb{R}$, $i \in \mathcal{M}$, such that

$$0 \geq A_i^{\rm T} P + PA_i + D_i^{\rm T} PD_i - 2PB_iB_i^{\rm T} P - \zeta_i P, \tag{4.67}$$

for $i \in \mathcal{M}$, and

$$\frac{1}{\tau}{\rm tr}(\Pi \int_{T_{\rm D}}^{T_{\rm D}+\tau} e^{Qs}{\rm d}s\Gamma) - \sum_{i\in\mathcal{M}} \pi_i \frac{\lambda_{\rm min}^2(D_i^{\rm T} P + PD_i)}{2\lambda_{\rm max}^2(P)} < 0, \tag{4.68}$$

where $\Pi \in \mathbb{R}^{M\times M}$ denotes the diagonal matrix with the diagonal entries π_1,π_2,. . .,π_M, and $\Gamma \in \mathbb{R}^{M\times M}$ denotes the matrix with the (i, j)th entries given by

$$\gamma_{i,j} = \begin{cases} \zeta_j, & i = j, \\ \zeta_i + \frac{2\lambda_{\rm max}(PB_iB_i^{\rm T}P)}{\lambda_{\rm min}(P)} - \frac{\lambda_{\rm min}(P(B_jB_i^{\rm T}+B_iB_j^{\rm T})P)}{\lambda_{\rm max}(P)}, & i \neq j, \end{cases} \tag{4.69}$$

then the feedback control law (4.2) with the feedback gain matrices given by

$$K_i = -B_i^{\rm T} P, \quad i \in \mathcal{M}, \tag{4.70}$$

guarantees that the zero solution $x(t) \equiv 0$ of the closed-loop system (3.57) and (4.2) is asymptotically stable almost surely.

Proof. We can describe the closed-loop system (3.57) under the control law (4.2) by the multi-dimensional Ito stochastic differential equation

$$ {\rm d}x(t) = (A_{r(t)} + B_{r(t)}K_{\rho(t)})x(t){\rm d}t + D_{r(t)}x(t){\rm d}W(t), \quad t \geq 0, \tag{4.71}$$

where

$$\rho(t) \triangleq \begin{cases} r_0, & 0 \leq t < T_{\rm D}, \\ \sigma(t - T_{\rm D}), & t \geq T_{\rm D}. \end{cases} \tag{4.72}$$

Now, let $V(x) \triangleq x^{\rm T}Px$ and consider the function $\ln V(x(t))$. Note that $V(\cdot)$ is a positive-definite function; therefore, $\ln V(x(t))$ is well-defined for non-zero values of the state. It follows from Ito formula and (4.70) that

$$\begin{aligned}\ln V(x(t)) = \ln V(x_0) + \int_0^t \frac{1}{V(x(s))} x^{\mathrm{T}}(s)\Big(A_{r(s)}^{\mathrm{T}} P + P A_{r(s)} \\ - P B_{\rho(s)} B_{r(s)}^{\mathrm{T}} P - P B_{r(s)} B_{\rho(s)}^{\mathrm{T}} P + D_{r(s)}^{\mathrm{T}} P D_{r(s)}\Big) x(s) \mathrm{d}s \\ - \int_0^t \frac{1}{2V^2(x(s))} (2x^{\mathrm{T}}(s) P D_{r(s)} x(s))^2 \mathrm{d}s + L(t),\end{aligned} \tag{4.73}$$

where $L(t) \triangleq \int_0^t \frac{1}{V(x(s))} 2x^{\mathrm{T}}(s) P D_{r(s)} x(s) \mathrm{d}W(s)$. We observe that

$$\begin{aligned}2x^{\mathrm{T}}(s) P D_{r(s)} x(s) &= x^{\mathrm{T}}(s)(D_{r(s)}^{\mathrm{T}} P + P D_{r(s)}) x(s) \\ &\geq \frac{\lambda_{\min}(D_{r(s)}^{\mathrm{T}} P + P D_{r(s)})}{\lambda_{\max}(P)} x^{\mathrm{T}}(s) P x(s).\end{aligned} \tag{4.74}$$

By (4.67), (4.69), (4.73), and (4.74), we arrive at

$$\ln V(x(t)) \leq \ln V(x_0) + \int_0^t \gamma_{r(s),\rho(s)} \mathrm{d}s - \int_0^t \frac{\lambda_{\min}^2(D_{r(s)}^{\mathrm{T}} P + P D_{r(s)})}{2\lambda_{\max}^2(P)} \mathrm{d}s + L(t). \tag{4.75}$$

Note that $\rho(s) = r_0$, $s \in [0, T_{\mathrm{D}})$, and consequently

$$|\int_0^{T_{\mathrm{D}}} \gamma_{r(s),\rho(s)} \mathrm{d}s| \leq \max_{i \in \mathcal{M}} |\gamma_{i,r_0}| T_{\mathrm{D}}.$$

Therefore,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^{T_{\mathrm{D}}} \gamma_{r(s),\rho(s)} \mathrm{d}s = 0. \tag{4.76}$$

Furthermore, it follows from (4.72), (4.76) and the strong law of large numbers presented in Lemma 4.2 that

$$\begin{aligned}\lim_{t \to \infty} \frac{1}{t} \int_0^t \gamma_{r(s),\rho(s)} \mathrm{d}s &= \lim_{t \to \infty} \frac{1}{t} \int_{T_{\mathrm{D}}}^t \gamma_{r(s),\rho(s - T_{\mathrm{D}})} \mathrm{d}s \\ &= \frac{1}{\tau} \mathrm{tr}(\Pi \int_{T_{\mathrm{D}}}^{\tau + T_{\mathrm{D}}} e^{Qs} \mathrm{d}s \Gamma),\end{aligned} \tag{4.77}$$

almost surely. In addition, by the strong law of large numbers for continuous-time, finite-state, irreducible Markov chains [86, 87]

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t -\frac{\lambda_{\min}^2(D_{r(s)}^{\mathrm{T}}P+PD_{r(s)})}{2\lambda_{\max}^2(P)}\mathrm{d}s = -\sum_{i\in I}\pi_i\frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P+PD_i)}{2\lambda_{\max}^2(P)}, \tag{4.78}$$

almost surely. Furthermore, note that the Ito integral $L(t)$ in inequality (4.73) is a local martingale with quadratic variation given by

$$\begin{aligned}[L]_t &= \int_0^t \Big(\frac{1}{V(x(s))}2x^{\mathrm{T}}(s)PD_{r(s)}x(s)\Big)^2\mathrm{d}s \\ &= \int_0^t \frac{1}{V^2(x(s))}(2x^{\mathrm{T}}(s)PD_{r(s)}x(s))^2\mathrm{d}s \\ &\leq \int_0^t \frac{1}{V^2(x(s))}(x^{\mathrm{T}}(s)(D_{r(s)}^{\mathrm{T}}P+PD_{r(s)})x(s))^2\mathrm{d}s \\ &\leq \int_0^t \frac{\lambda_{\max}^2(D_{r(s)}^{\mathrm{T}}P+PD_{r(s)})}{\lambda_{\min}^2(P)}\mathrm{d}s \\ &\leq \frac{\max_{i\in I}\lambda_{\max}^2(D_i^{\mathrm{T}}P+PD_i)}{\lambda_{\min}^2(P)}t. \end{aligned} \tag{4.79}$$

It follows from (4.79) that $\lim_{t\to\infty}\frac{1}{t}[L]_t < \infty$. Hence, by using the same approach presented in [14,49,51], we can employ the strong law of large numbers for local martingales to show

$$\lim_{t\to\infty}\frac{1}{t}L(t) = 0, \tag{4.80}$$

almost surely. Moreover, by using (4.68), (4.75), (4.77), (4.78), and (4.80), we arrive at

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\ln V(x(t)) &\leq \frac{1}{\tau}\mathrm{tr}(\Pi\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}} e^{Qs}\mathrm{d}s\Gamma) - \sum_{i\in\mathcal{M}}\pi_i\frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P+PD_i)}{2\lambda_{\max}^2(P)} \\ &< 0. \end{aligned} \tag{4.81}$$

As a consequence of (4.81), $\lim_{t\to\infty}\ln V(x(t)) = -\infty$, almost surely; moreover,

$$\mathbb{P}[\lim_{t\to\infty}V(x(t)) = 0] = 1. \tag{4.82}$$

Therefore, the zero solution $x(t)\equiv 0$ of the closed-loop system (3.57), (4.2) is asymptotically stable, almost surely. □

Note that the stabilization conditions (4.67) and (4.68) depend not only on the subsystem dynamics, but also on the probabilistic dynamics of the mode signal, as well as the mode sampling period $\tau > 0$ and the sampled mode information delay $T_{\mathrm{D}} > 0$.

Remark 4.1. When time delay for the sampled mode information T_{D} tends to 0, the conditions of Theorem 4.1 reduces to the stabilization conditions presented in Theorem 3.4 for the case where time delay for sampled mode information is not present.

Remark 4.2. Note that under the conditions (4.67) and (4.68), our proposed control law (4.2) guarantees almost sure stabilization even if the mode samples are subject to different time delays that are upper-bounded by a constant $T_{\mathrm{D}} \in (0, \tau]$. Specifically, consider the case where the kth mode sample data $r(k\tau)$ is subject to time delay $T_k > 0$, $k \in \mathbb{N}_0$. In this case, the sequence $\{k\tau + T_k\}_{k \in \mathbb{N}_0}$ denotes the time instants at which the obtained mode samples become available to the controller. If there exists a constant $T_{\mathrm{D}} \in (0, \tau]$ such that $T_k \in (0, T_{\mathrm{D}}]$, $k \in \mathbb{N}_0$, then the mode samples become available to the controller *in order*, that is,

$$k_1\tau + T_{k_1} \leq k_2\tau + T_{k_2}, \quad k_1 \leq k_2, \quad k_1, k_2 \in \mathbb{N}_0. \tag{4.83}$$

Furthermore,

$$k\tau + T_k \leq k\tau + T_{\mathrm{D}}, \quad k \in \mathbb{N}_0, \tag{4.84}$$

which implies that the controller has the sampled mode information $r(k\tau)$ at time $k\tau + T_{\mathrm{D}}$, and hence the proposed control law (4.2) can still be employed for stabilizing the switched linear stochastic dynamical system (3.57).

4.4 Illustrative Numerical Example

In this section, we demonstrate the efficacy of our approach concerned with the stabilization of switched linear stochastic dynamical systems under sampled and delayed mode information. Specifically, we consider the switched linear stochastic system (3.57) composed of $M = 3$ subsystems characterized by the subsystem matrices

$$A_1 = \begin{bmatrix} 1 & -5 \\ 1 & 1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1.6 \\ 1.6 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 0.75 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0.5 & 0.5 \\ -4.75 & 0.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

and $D_1 = D_2 = D_3 = I_2$. The mode signal $\{r(t) \in \mathcal{M} \triangleq \{1,2,3\}\}_{t\geq 0}$ of the switched system is assumed to be a Markov chain with the generator matrix

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{4.85}$$

with stationary probability distributions, $\pi_i = \frac{1}{3}$, $i \in \mathcal{M}$. Furthermore, the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is assumed to be sampled periodically at the time instants $k\tau$, $k \in \mathbb{N}_0$, where $\tau = 0.1$ is the mode sampling period. Furthermore, the obtained mode samples are assumed to be available to the controller after a constant time delay $T_{\mathrm{D}} = 0.03$.

Note that the positive-definite matrix $P = I_2$ and the scalars $\zeta_1 = -0.35$, $\zeta_2 = 2.75$, $\zeta_3 = -2.25$ satisfy (4.67) and (4.68). Therefore, it follows from Theorem 4.1 that the zero solution $x(t) \equiv 0$ of the switched stochastic system given by (3.57) under the proposed control law (4.2) is asymptotically stable almost surely.

Figures 4.1 and 4.2 respectively show the sample paths of $x(t)$ and $u(t)$ obtained with the initial conditions $x(0) = [1,\, 1]^{\mathrm{T}}$ and $r(0) = 1$. The sampled mode signal $\sigma(t)$ may change its value at mode sampling time instants denoted by the sequence $\{k\tau\}_{k\in\mathbb{N}}$. Since, the piecewise-continuous control law (4.2) depends on the delayed sampled mode signal $\sigma(t - T_{\mathrm{D}})$, the control input trajectory is subject to jumps at time instants denoted by $\{k\tau + T_{\mathrm{D}}\}_{k\in\mathbb{N}}$.

Note that the feedback control performance is directly related to the quality of the rep-

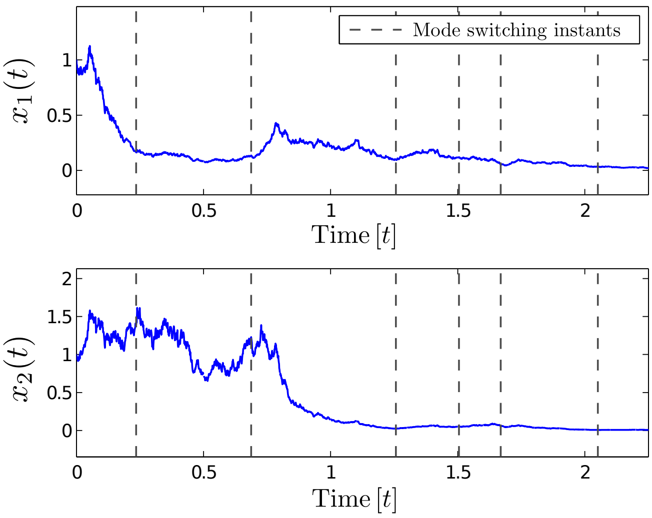

Figure 4.1: State trajectory versus time

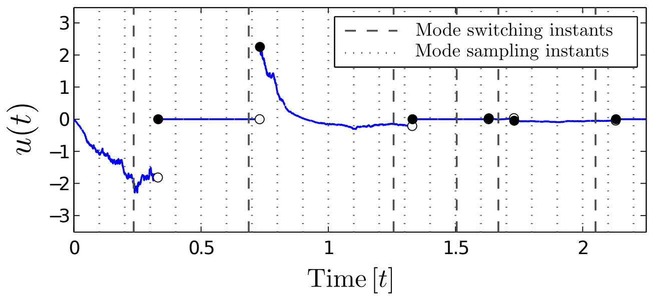

Figure 4.2: Control input versus time

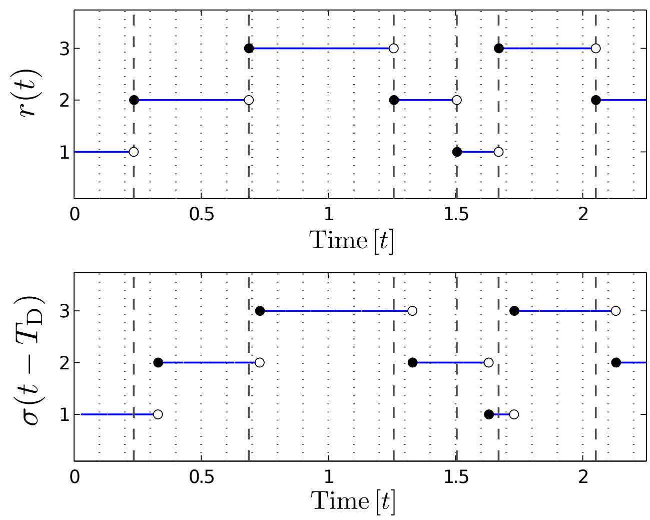

Figure 4.3: Actual mode signal $r(t)$ and the delayed version of the sampled mode signal $\sigma(t - T_{\mathrm{D}})$ versus time

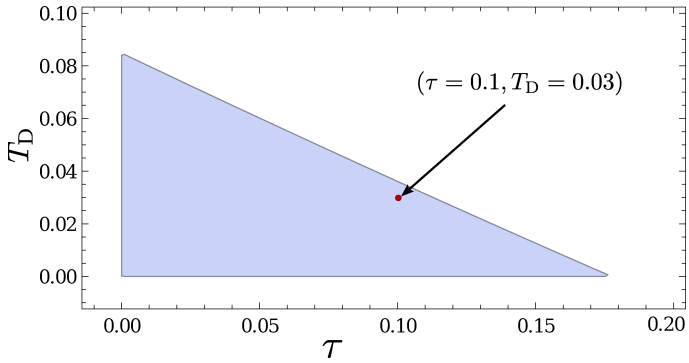

Figure 4.4: Stabilization region with respect to τ and T_{D}

resentation of the actual mode signal by the sampled and delayed version that is available to the controller. Owing to frequent mode sampling and small mode information time delay, the delayed sampled mode signal $\sigma(t - T_{\mathrm{D}})$ of this numerical example is a good representation of the actual mode signal $r(t)$ (see Figure 4.3).

Stabilization performance depends on both the mode sampling period $\tau > 0$ and the mode sample information delay $T_{\mathrm{D}} > 0$. Figure 4.4 shows the numerically obtained values of the constants τ and T_{D} that satisfy the condition (4.68) of Theorem 4.1 for the positive-definite matrix $P = I_2$ and the scalars $\zeta_1 = -0.35$, $\zeta_2 = 2.75$, $\zeta_3 = -2.25$. The dark region represents the set of values $(\tau, T_{\mathrm{D}}) \in (0, \infty) \times (0, \infty)$, for which the stabilization is guaranteed by the control law (4.2) according to Theorem 4.1. Note that the sample paths of $x(t)$, $u(t)$, $r(t)$ and $\sigma(t - T_{\mathrm{D}})$ shown in Figures 4.1–4.3 are obtained for the values $\tau = 0.1$, $T_{\mathrm{D}} = 0.03$, which corresponds to a point in the stabilization region shown in Figure 4.4.

4.5 Conclusion

In this chapter, stabilization of switched linear stochastic systems has been investigated. A feedback control framework has been developed for the case where the mode of the switched system is periodically sampled and the obtained mode samples are available to the controller only after a time delay. We employed a quadratic Lyapunov-like function for obtaining sufficient conditions under which our proposed control framework guarantees almost sure asymptotic stabilization.

The control law proposed in this chapter incorporates a feedback gain that depends on the sampled and delayed version of the mode signal. Note that this feedback gain remains constant between consecutive time instants at which sampled mode data becomes available. In Chapter 5, we propose a new feedback gain scheduling mechanism for selecting the feedback gain associated with the mode that has the highest conditional probability of being active given the available sampled and delayed mode data.

Chapter 5

Probability-Based Feedback Gain Scheduling for Stabilizing Switched Linear Stochastic Systems Under Delayed Sampled Mode Information

5.1 Introduction

In Chapter 4, we investigated the stabilization of a continuous-time switched linear stochastic system for the case where the mode information is observed only at discrete time instants and the observed mode information is available to the controller only after a delay. In this chapter we develop a *new* control framework for the same problem setting. Specifically, to guarantee feedback stabilization of continuous-time switched linear stochastic dynamical systems under delayed sampled mode information, in Chapter 4, we proposed a piecewise-continuous linear state feedback control law. The feedback gain of the control law presented in Chapter 4 is switched between the gains associated with each mode of the system depending on the delayed sampled mode signal. This approach is based on picking the feedback gain associated with the mode that was active at the most recent mode sampling instant. Furthermore, the same feedback gain is maintained until the next mode sample data becomes available (see Chapter 4). Note that when the mode is sampled rarely and the mode information delay is large, the delayed sampled mode signal will not be an accurate representation of the actual mode signal, and hence the control law presented in Chapter 4 may fail to stabilize the switched system. In this chapter, in order to relax the requirements on the mode sampling period and the mode information time delay, we develop a new control framework with probability-based feedback gain scheduling scheme.

In the literature, probabilistic gain schedulers have been employed in different problem settings. For example, in [107], probability-based gain schedulers are used for designing filters for a discrete-time system with missing output measurements. Furthermore, a stochastic scheduling scheme is proposed in [108], where a Markov chain is used for the scheduling between finite number of controllers to stabilize a discrete-time linear system.

Our proposed scheduling method uses the available delayed sampled mode data of a continuous-time switched stochastic system to identify the conditional probability distribution of the modes at any given time. In our proposed control framework, the gain scheduler selects the feedback gain associated with the mode that has the highest probability of being active. We employ a quadratic Lyapunov function approach to obtain sufficient conditions under which our proposed control framework with the probability-based gain scheduling scheme guarantees asymptotic stabilization.

The contents of this chapter are organized as follows. In Section 5.2, we provide a preliminary result concerning continuous-time Markov chains. In Section 5.3, we introduce the feedback control problem for switched stochastic systems under delayed sampled mode information; furthermore, in Section 5.4 we explain our proposed control framework based on a probabilistic feedback gain scheduling scheme. We obtain sufficient conditions of almost sure stabilization in Section 5.5. In Section 5.6, we provide a numerical example. Finally, we conclude the chapter in Section 5.7.

5.2 Mathematical Preliminaries

We provide a key result concerned with continuous-time finite-state Markov chains in Lemma 5.1 below, which we employ for obtaining the main results provided in Section 5.3.

Lemma 5.1. Let $\{r(t) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{t \geq 0}$ be a finite-state irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M \times M}$. Let $\phi_l : [0, \infty) \to \mathbb{R}$, $l \in \mathcal{M}$, be bounded piecewise-constant functions. Then for any $t_1, t_2 \in [0, \infty)$ and $G \in \mathcal{F}$ such that

$t_1 \le t_2$ and $\mathbb{P}[G] > 0$, it follows that

$$\mathbb{E}[\int_{t_1}^{t_2} \phi_{r(s)}(s)\mathrm{d}s|G] = \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}}\phi_l(s)\mathbb{P}[G\cap F_l(s)]\mathrm{d}s, \tag{5.1}$$

where $F_l(t) \triangleq \{\omega \in \Omega : r_t(\omega) = l\}$, $t \ge 0$.

Proof. By using the definition of conditional expectation given in (2.1), we obtain

$$\begin{aligned}\mathbb{E}[\int_{t_1}^{t_2} \phi_{r(s)}\mathrm{d}s|G] &= \frac{1}{\mathbb{P}[G]}\int_G\int_{t_1}^{t_2}\phi_{r(s)}(s)\mathrm{d}s\mathbb{P}(\mathrm{d}\omega)\\ &= \frac{1}{\mathbb{P}[G]}\int_G\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}}\phi_l(s)\mathbb{1}_{[F_l(s)]}(\omega)\mathrm{d}s\mathbb{P}(\mathrm{d}\omega).\end{aligned} \tag{5.2}$$

Moreover, we employ Fubini's Theorem [93] to change the order of integrals in (5.2). It follows that

$$\begin{aligned}\mathbb{E}[\int_{t_1}^{t_2} \phi_{r(s)}\mathrm{d}s|G] &= \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\int_G\sum_{l\in\mathcal{M}}\phi_l(s)\mathbb{1}_{[F_l(s)]}(\omega)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}s\\ &= \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}}\phi_l(s)\int_G\mathbb{1}_{[F_l(s)]}(\omega)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}s\\ &= \frac{1}{\mathbb{P}[G]}\int_{t_1}^{t_2}\sum_{l\in\mathcal{M}}\phi_l(s)\mathbb{P}[G\cap F_l(s)]\mathrm{d}s,\end{aligned} \tag{5.3}$$

which completes the proof. □

Note that Lemma 5.1 is a generalized version of Lemma 4.1 in Chapter 4, in the sense that it allows the integrand $\phi_{r(\cdot)}(\cdot)$ in the integral of the left hand side of (5.1) to not only depend on the Markov chain $r(\cdot)$ but also depend directly on time.

5.3 Feedback Control Problem for Switched Linear Stochastic Systems Under Sampled and Delayed Mode Information

In this section, we explain the stabilization problem under sampled and delayed mode information for switched linear stochastic systems. Specifically, we consider the continuous-time switched stochastic dynamical system given by (3.57). We assume that the mode signal of the switched system is observed (sampled) periodically and the obtained sampled mode data is subject to a constant time delay. We denote the mode sampling period by $\tau > 0$ and the mode information time delay by $T_{\rm D} > 0$.

We denote the obtained mode samples by the sequence $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$, which forms a discrete-time Markov chain with state transition probabilities given by

$$\mathbb{P}[r((k+1)\tau) = j | r(k\tau) = i] = p_{i,j}(\tau), \tag{5.4}$$

where $p_{i,j}(\tau)$ represents the (i,j)th entry of the transition matrix $e^{Q\tau}$. It follows from the irreducibility of the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ that $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is also irreducible. Furthermore, the stationary probability distribution for the discrete-time Markov chain $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is given by $\pi \in \mathbb{R}^M$, which is also the stationary probability distribution for $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ [87].

We use the "sample and hold" technique and define the sampled version of the mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ by

$$\sigma(t) \triangleq r(k\tau), \quad t \in [k\tau, (k+1)\tau),\, k \in \mathbb{N}_0. \tag{5.5}$$

As each mode sample data is subject to time delay $T_{\rm D} > 0$, only a delayed version of the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ is available for control purposes. We denote the delayed sampled mode signal by the stochastic process $\{\sigma(t - T_{\rm D}) \in \mathcal{M}\}_{t\geq T_{\rm D}}$.

In Chapter 4, under the assumption that the initial mode r_0 is known to the controller, we had proposed the stabilizing feedback control law (4.2). The control law (4.2) depends only on the delayed sampled mode signal $\{\sigma(t - T_{\rm D}) \in \mathcal{M}\}_{t\geq T_{\rm D}}$, which switches the feedback gain of the controller between the gains $K_1, K_2, \ldots, K_M$ associated with the modes of the switched stochastic system (3.57).

The delayed sampled mode signal $\{\sigma(t - T_{\rm D}) \in \mathcal{M}\}_{t\geq T_{\rm D}}$ and hence the feedback gain remain constant in the time intervals $[k\tau + T_{\rm D}, (k+1)\tau + T_{\rm D})$, $k \in \mathbb{N}_0$. Moreover, the feedback gain in (4.2) is switched at the time instant $k\tau + T_{\rm D}$, only if the consecutive mode samples $r((k-1)\tau) \in \mathcal{M}$ and $r(k\tau) \in \mathcal{M}$ are different. Stabilization performance of the feedback control law (4.2) is directly related to the quality of the representation of the actual mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ by the sampled and delayed version $\{\sigma(t - T_{\rm D}) \in \mathcal{M}\}_{t\geq T_{\rm D}}$.

Note that the frequency of the mode transitions, the mode sampling period $\tau > 0$, as well as the sampled mode information delay $T_{\rm D} > 0$ affect how accurately the actual mode signal is represented by its time delayed sampled version. If the mode is sampled very frequently and the mode information time delay is small, then the delayed sampled mode signal $\{\sigma(t - T_{\rm D}) \in \mathcal{M}\}_{t \geq T_{\rm D}}$ is likely to resemble the actual mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$ accurately. In this case, the feedback gain is likely to be K_i when mode i is active. On the other hand, if the mode is observed rarely and the mode information delay is large, then the delayed sampled mode signal $\{\sigma(t - T_{\rm D}) \in \mathcal{M}\}_{t \geq T_{\rm D}}$ is likely to be a poor representation of the actual mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$. Consequently, depending also on the subsystem dynamics and the mode switching frequency, the control performance may deteriorate. In order to overcome this issue associated with the control law (4.2), in the following section we will propose a new probability-based scheme to schedule the switching between the feedback gains $K_1, K_2, \ldots, K_M$, in the time intervals $[k\tau + T_{\rm D}, (k+1)\tau + T_{\rm D})$, $k \in \mathbb{N}_0$.

5.4 Probability-Based Feedback Gain Scheduling

In this section, we develop a probabilistic feedback gain scheduling framework that is based on selecting the feedback gain associated with the mode that has the highest probability of being active.

The mode signal $\{r(t) \in \mathcal{M}\}_{t \geq 0}$ of the switched stochastic system (3.57) may change its value in the time intervals between mode sampling instants. However, the trajectory of the mode signal in the time intervals $(k\tau, (k+1)\tau)$, $k \in \mathbb{N}_0$, is not available to the controller. Nevertheless, the available delayed sampled mode data and the a priori information concerning the probabilistic dynamics of the mode signal can be utilized to compute the probability distribution regarding the active mode in those intervals.

The conditional probability of the mode signal taking the value $j \in \mathcal{M}$ at time $t_2 \geq 0$ given that it had the value $i \in \mathcal{M}$ at an earlier time $t_1 \in [0, t_2]$, is given by

$$\mathbb{P}[r(t_2) = j | r(t_1) = i] = p_{i,j}(t_2 - t_1), \tag{5.6}$$

where $p_{i,j}(t)$, again, represents the (i, j)th entry of the matrix e^{Qt}, $t \geq 0$. Note that

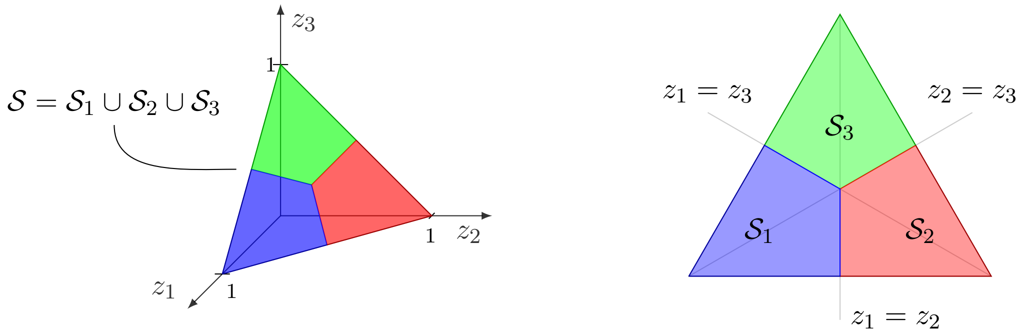

Figure 5.1: The structure and the partitions of the set $\mathcal{S}$

$\sum_{j\in\mathcal{M}} p_{i,j}(t) = 1$, $t \geq 0$, $i \in \mathcal{M}$.

We define the *conditional probability distributions* $p_i : [0, \infty) \to \mathbb{R}^M, i \in \mathcal{M}$, by

$$p_i(t) \triangleq [p_{i,1}(t), p_{i,2}(t), \ldots, p_{i,M}(t)]^{\mathrm{T}}, \ t \geq 0, i \in \mathcal{M}. \tag{5.7}$$

It follows that $p_i(t_2 - t_1) \in \mathcal{M}$ represents the *conditional probability distribution* regarding the active mode at time $t_2 \geq 0$ given that the mode signal had the value $i \in \mathcal{M}$ at time $t_1 \in [0, t_2]$.

Now let $\mathcal{S} \triangleq \{z \in \mathbb{R}^M : \sum_{i\in\mathcal{M}} z_i = 1; z_i \geq 0, i \in \mathcal{M}\}$. Note that the trajectories of the conditional probability distributions $p_i(\cdot), i \in \mathcal{M}$, move on the set $\mathcal{S}$, that is, $p_i(t) \in \mathcal{S}$, $t \geq 0, i \in \mathcal{M}$. We partition the set $\mathcal{S}$ into M subsets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_M$ given by

$$\mathcal{S}_i \triangleq \{z \in \mathcal{S} : z_i > z_j, j \in \{1, \ldots, i-1\}; z_i \geq z_j, j \in \{i+1, \ldots, M\}\}, i \in \mathcal{M}. \tag{5.8}$$

Note that $\mathcal{S} = \bigcup_{i\in\mathcal{M}} \mathcal{S}_i$ and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, $i \neq j$. For example, when $M = 3$, the trajectories of the conditional probability distributions $p_i(\cdot)$, $i \in \{1, 2, 3\}$, move on a triangular surface characterized by the set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ (see Figure 5.1). Note that the trajectories of the conditional distributions converge to $\pi \in \mathbb{R}^M$, which denotes the stationary distribution for the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$. For example, Figure 5.2 shows the conditional probability distribution $p_3(\cdot)$ which starts in the subset S_3 and moves towards the stationary distribution $\pi \in \mathbb{R}^M$, which is in the subset S_1.

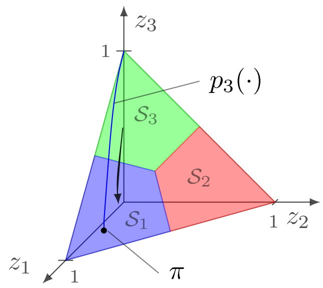

Figure 5.2: Trajectory of the conditional probability distribution $p_3(\cdot)$ on the set $\mathcal{S}$

Now we construct the feedback gain switching (scheduling) signal $\{\rho(t) \in \mathcal{M}\}_{t\geq 0}$ by

$$\rho(t) \triangleq \begin{cases} \eta(r_0, t), & \text{if } t \in [0, T_{\mathrm{D}}), \\ \eta(r(k\tau), t - k\tau), & \text{if } t \in \mathcal{T}_k, \quad k \in \mathbb{N}_0, \end{cases} \tag{5.9}$$

where $\mathcal{T}_k \triangleq [k\tau + T_{\mathrm{D}}, (k+1)\tau + T_{\mathrm{D}}), k \in \mathbb{N}_0$, and

$$\eta(i, \cdot) \triangleq \begin{cases} 1, & \text{if } p_i(\cdot) \in \mathcal{S}_1, \\ \quad \vdots & \\ M, & \text{if } p_i(\cdot) \in \mathcal{S}_M, \end{cases} \qquad i \in \mathcal{M}. \tag{5.10}$$

For a given mode $i \in \mathcal{M}$, $\eta(i, \cdot)$ is a piecewise-constant, deterministic function of time. The trajectory of $\eta(i, \cdot)$ depends only on the generator matrix $Q \in \mathbb{R}^{M \times M}$ of the mode signal, which is assumed to be known. Furthermore, $\eta(i, t_2 - t_1)$ denotes the index of the mode that has the highest conditional probability of being active at time $t_2 \geq 0$ given that the ith mode was active at an earlier time $t_1 \in [0, t_2]$. Note that the information of an obtained mode sample $r(k\tau)$ become available to the controller at the time instant $k\tau + T_{\mathrm{D}}$. This mode sample is used for computing $\eta(r(k\tau), t - k\tau)$ in (5.9), which represents the index of the mode with the highest conditional probability at time $t \in [k\tau + T_{\mathrm{D}}, (k+1)\tau + T_{\mathrm{D}})$ given the mode information $r(k\tau)$. Hence, for a given time $t \geq 0$, $\rho(t)$ denotes the index of the mode that has the highest conditional probability of being active at time t given the most recent sampled mode information.

Note that the conditional probability distributions $p_i(\cdot)$, $i \in \mathcal{M}$, cross the boundaries between the sets $\mathcal{S}_i$, $i \in \mathcal{M}$, finitely many times in every finite time interval. As a conse-

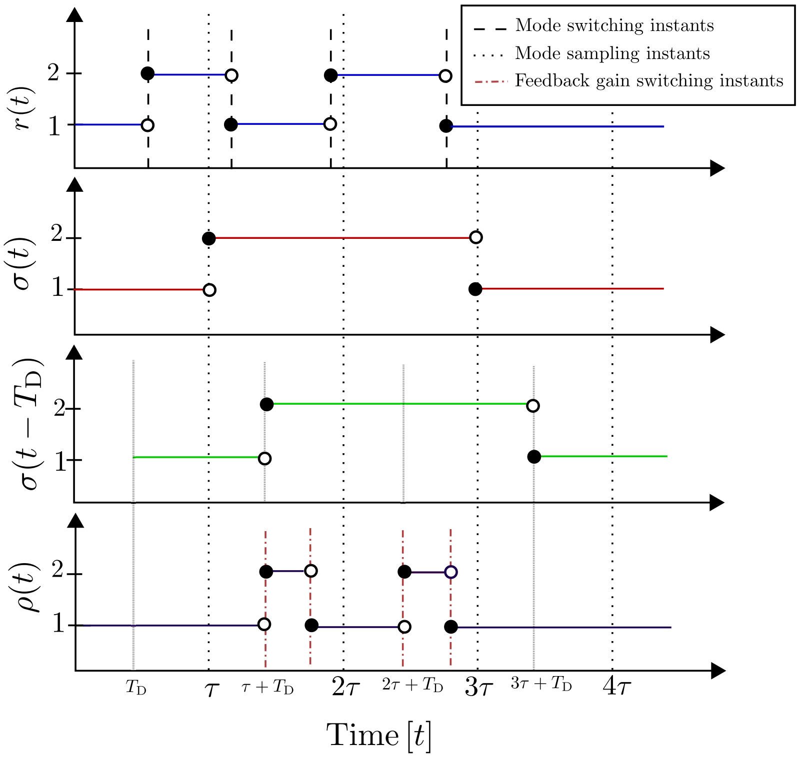

Figure 5.3: Actual mode signal $r(t)$, the sampled mode signal $\sigma(t)$, the delayed version of the sampled mode signal $\sigma(t - T_{\mathrm{D}})$, and the feedback gain switching signal $\rho(t)$ versus time

quence, $\{\rho(t) \in \mathcal{M}\}_{t\geq 0}$ changes its value only finite number of times in every finite time interval. For an example switched system (3.57) with $M = 2$ modes, Figure 5.3 shows sample paths of $r(t)$, $\sigma(t)$, $\sigma(t - T_{\rm D})$, and $\rho(t)$.

We employ $\{\rho(t) \in \mathcal{M}\}_{t\geq 0}$ for switching between the feedback gains $K_1, K_2, \ldots, K_M$ and propose a new control law of the form

$$u(t) = K_{\rho(t)} x(t), \quad t \geq 0. \tag{5.11}$$

Note that in the closed-loop system (3.57), (5.11), the active mode is denoted by $r(t)$, whereas the index of the active feedback gain is denoted by $\rho(t)$. In Lemma 5.2 below we present a strong law of large numbers for the bivariate stochastic process $\{(r(t), \rho(t)) \in \mathcal{M} \times \mathcal{M}\}_{t\geq 0}$, so that we obtain sufficient almost sure asymptotic stability conditions for the closed-loop system (3.57), (5.11) in Section 5.5.

Lemma 5.2. Let $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ be the irreducible Markov chain characterized by the generator matrix $Q \in \mathbb{R}^{M\times M}$. Furthermore, let $\{\rho(t) \in \mathcal{M}\}_{t\geq 0}$ be the stochastic process defined in (5.9). Then for any $\gamma_{i,j} \in \mathbb{R}$, $i, j \in \mathcal{M}$, it follows that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\rho(s)} {\rm d}s = \frac{1}{\tau} \sum_{i,j\in\mathcal{M}} \pi_i \int_{T_{\rm D}}^{\tau+T_{\rm D}} \gamma_{j,\eta(i,s)} p_{i,j}(s) {\rm d}s, \tag{5.12}$$

almost surely.

In order to prove Lemma 5.2, we use the method that we employed previously in Chapter 4 for proving Lemma 4.2.

Proof. We define the counting process $\{N(t) \in \mathbb{N}_0\}_{t\geq T_{\rm D}}$ by

$$N(t) = \max\{k \in \mathbb{N}_0 \,:\, k\tau + T_{\rm D} \leq t\}, \quad t \geq T_{\rm D}. \tag{5.13}$$

The number of mode samples obtained until time $t \geq T_{\rm D}$, is given by $N(t)$. It follows from

5.13 that

$$\int_0^t \gamma_{r(s),\rho(s)} \mathrm{d}s = \int_0^{T_\mathrm{D}} \gamma_{r(s),\rho(s)} \mathrm{d}s + \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_\mathrm{D}}^{k\tau+T_\mathrm{D}} \gamma_{r(s),\rho(s)} \mathrm{d}s$$
$$+ \int_{N(t)\tau+T_\mathrm{D}}^{t} \gamma_{r(s),\rho(s)} \mathrm{d}s, \quad t \geq 0. \tag{5.14}$$

Now note that

$$|\int_0^{T_\mathrm{D}} \gamma_{r(s),\rho(s)} \mathrm{d}s| \leq \max_{i,j\in\mathcal{M}} |\gamma_{i,j}| T_\mathrm{D}, \tag{5.15}$$
$$|\int_{N(t)\tau+T_\mathrm{D}}^{t} \gamma_{r(s),\rho(s)} \mathrm{d}s| \leq \max_{i,j\in\mathcal{M}} |\gamma_{i,j}| \tau, \quad t \geq T_\mathrm{D}. \tag{5.16}$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^{T_\mathrm{D}} \gamma_{r(s),\rho(s)} \mathrm{d}s = 0, \tag{5.17}$$
$$\lim_{t\to\infty} \frac{1}{t} \int_{N(t)\tau+T_\mathrm{D}}^{t} \gamma_{r(s),\rho(s)} \mathrm{d}s = 0. \tag{5.18}$$

By using (5.14), (5.17), and 5.18 we obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\rho(s)} \mathrm{d}s = \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_\mathrm{D}}^{k\tau+T_\mathrm{D}} \gamma_{r(s),\rho(s)} \mathrm{d}s. \tag{5.19}$$

We now consider the following cases: the case where $T_\mathrm{D} \leq \tau$ and the case where $T_\mathrm{D} > \tau$. For each case we evaluate the limit $\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} \int_{(k-1)\tau+T_\mathrm{D}}^{k\tau+T_\mathrm{D}} \gamma_{r(s),\rho(s)} \mathrm{d}s$ and show that the limit is given by the right hand side of (5.12) in both of the cases.

Case 1) Consider the case where $T_\mathrm{D} \leq \tau$. It is important to note that in this case the information delay is less than the mode sampling interval, and hence the kth sampled mode data $r(k\tau)$ becomes available for control purposes before time $(k+1)\tau$.

Now, let $\{N^{h,i,j}(t) \in \mathbb{N}_0\}_{t\geq T_\mathrm{D}}$ be the counting process defined by

$$N^{h,i,j}(t) = \sum_{k=1}^{N(t)} \mathbb{1}_{[r((k-1)\tau)=h, r(k\tau)=i, r((k+1)\tau)=j]}, \quad t \geq T_\mathrm{D}. \tag{5.20}$$

Note that for all $h, i, j \in \mathcal{M}$, the counting process $\{N^{h,i,j}(t) \in \mathbb{N}_0\}_{t\geq T_\mathrm{D}}$ is a stochastic process that depends on the mode signal $\{r(t)\in\mathcal{M}\}_{t\geq 0}$. Note also that

$$\sum_{h,i,j\in\mathcal{M}} N^{h,i,j}(t)=N(t),\quad t\geq T_{\mathrm{D}}. \tag{5.21}$$

Furthermore, for all $h,i,j\in\mathcal{M}$, let the sequence of indices $\{k_n^{h,i,j}\in\mathbb{N}\}_{n\in\mathbb{N}}$ be defined by

$$k_n^{h,i,j}=\min\{k\in\mathbb{N}: N^{h,i,j}(k\tau+T_{\mathrm{D}})=n\},\ n\in\mathbb{N}. \tag{5.22}$$

Now, note that $r((k_n^{h,i,j}-1)\tau)=\sigma(k_n^{h,i,j}\tau-T_{\mathrm{D}})=h$, $r(k_n^{h,i,j}\tau)=i$, and $r((k_n^{h,i,j}+1)\tau)=j$, $n\in\mathbb{N}$, $h,i,j\in\mathcal{M}$. Furthermore, $\sigma(s-T_{\mathrm{D}})=r((k_n^{h,i,j}-1)\tau)=h$, for $s\in[(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}},k_n^{h,i,j}\tau+T_{\mathrm{D}})$. It follows from (5.9) that

$$\rho(s)=\eta(h,s-(k_n^{h,i,j}-1)\tau),\quad s\in[(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}},k_n^{h,i,j}\tau+T_{\mathrm{D}}). \tag{5.23}$$

It then follows from (5.20), (5.22), and (5.23) that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{1}{t}\sum_{h,i,j\in\mathcal{M}}\sum_{n=1}^{N^{h,i,j}(t)}\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{1}{t}\sum_{h,i,j\in\mathcal{M}}\sum_{n=1}^{N^{h,i,j}(t)}\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)}\mathrm{d}s.
\end{aligned} \tag{5.24}$$

We multiply the integrals in the right hand side of (5.24) by $\frac{N(t)}{N(t)}\frac{N^{h,i,j}(t)}{N^{h,i,j}(t)}$ to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{h,i,j\in\mathcal{M}}\Big(\frac{N^{h,i,j}(t)}{N(t)}\frac{1}{N^{h,i,j}(t)}\sum_{n=1}^{N^{h,i,j}(t)}\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)}\mathrm{d}s\Big)\\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{h,i,j\in\mathcal{M}}\Big(\lim_{t\to\infty}\frac{N^{h,i,j}(t)}{N(t)}\\
&\quad\cdot\lim_{t\to\infty}\frac{1}{N^{h,i,j}(t)}\sum_{n=1}^{N^{h,i,j}(t)}\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)}\mathrm{d}s\Big).
\end{aligned} \tag{5.25}$$

We start by computing $\lim_{t\to\infty} \frac{N(t)}{t}$. By the definition of $N(t)$ given in (5.13), we have

$$N(t)\tau + T_{\mathrm{D}} \le t \le (N(t)+1)\tau + T_{\mathrm{D}}, \;\; t \ge T_{\mathrm{D}}. \tag{5.26}$$

Therefore,

$$\frac{t-\tau-T_{\mathrm{D}}}{\tau} \le N(t) \le \frac{t-T_{\mathrm{D}}}{\tau}, \quad t \ge T_{\mathrm{D}}. \tag{5.27}$$

Since $\lim_{t\to\infty} \frac{1}{t}\frac{t-\tau-T_{\mathrm{D}}}{\tau} = \lim_{t\to\infty} \frac{1}{t}\frac{t-T_{\mathrm{D}}}{\tau} = \frac{1}{\tau}$, it follows from (5.27) that

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\tau}. \tag{5.28}$$

Next, we evaluate $\lim_{t\to\infty} \frac{N^{h,i,j}(t)}{N(t)}$ in (5.25). The counting process $N_{h,i,j}(t)$ denotes the number of time instants $k \in \{1,2,\ldots,N(t)\}$ such that $r((k-1)\tau) = h$, $r(k\tau) = i$, and $r((k+1)\tau) = j$. Furthermore, note that $N(t) = \sum_{h,i,j\in\mathcal{M}} N^{h,i,j}(t)$. By using the strong law of large numbers [86, 87] for the discrete-time Markov chain $\{r(k\tau) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$, we obtain

$$\lim_{t\to\infty} \frac{N^{h,i,j}(t)}{N(t)} = \pi_h p_{h,i}(\tau) p_{i,j}(\tau), \tag{5.29}$$

where $\pi_h > 0$ is the stationary probability distribution for state $h \in \mathcal{M}$ and $p_{h,i}(\tau)$ and $p_{i,j}(\tau)$ are transition probabilities characterized in (5.4).

As the third step, we will employ the strong law of large numbers for independent and identically distributed random variables in order to compute the limit

$$\lim_{t\to\infty} \frac{1}{N^{h,i,j}(t)} \sum_{n=1}^{N^{h,i,j}(t)} \int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)} \mathrm{d}s.$$

Note that

$$\begin{aligned} &\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)} \mathrm{d}s \\ &\quad = \int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)} \mathrm{d}s + \int_{k_n^{h,i,j}\tau}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)} \mathrm{d}s. \end{aligned} \tag{5.30}$$

Now let

$$y_n^{h,i,j} \triangleq \int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)}\mathrm{d}s, \quad n \in \mathbb{N}, \quad h,i,j \in \mathcal{M}, \tag{5.31}$$

$$z_n^{h,i,j} \triangleq \int_{k_n^{h,i,j}\tau}^{k_n^{h,i,j}\tau+T_{\mathrm{D}}} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)}\mathrm{d}s, \quad n \in \mathbb{N}, \quad h,i,j \in \mathcal{M}. \tag{5.32}$$

Note that by definition (5.22), the mode signal takes the values h and i at time instants $(k_n^{h,i,j}-1)\tau$ and $k_n^{h,i,j}\tau$, respectively, for all $n \in \mathbb{N}$. The value of the mode signal during the interval $\big((k_n^{h,i,j}-1)\tau, k_n^{h,i,j}\tau\big)$ may differ for each $n \in \mathbb{N}$. However, the probability of the mode taking the value $l \in \mathcal{M}$ at time $(k_n^{h,i,j}-1)\tau + s$, where $s \in (0,\tau)$, does not depend on $n \in \mathbb{N}$. Hence, for given $h,i,j \in \mathcal{M}$, the random variables $y_n^{h,i,j}$, $n \in \mathbb{N}$, are independent and identically distributed. Similarly, for given $h,i,j \in \mathcal{M}$, the random variables $z_n^{h,i,j}$, $n \in \mathbb{N}$, are also independent and identically distributed. Now, we calculate $\mathbb{E}[y_n^{h,i,j}]$ and $\mathbb{E}[z_n^{h,i,j}]$. It follows from (5.31) that

$$\begin{aligned}\mathbb{E}[y_n^{h,i,j}] &= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k_n^{h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{h,i,j}\tau} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)}\mathrm{d}s | k_n^{h,i,j} = k]\mathbb{P}[k_n^{h,i,j} = k] \\ &= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau} \gamma_{r(s),\eta(h,s-(k-1)\tau)}\mathrm{d}s | G]\mathbb{P}[k_n^{h,i,j} = k],\end{aligned} \tag{5.33}$$

where

$$G \triangleq \{\omega \in \Omega : r_{(k-1)\tau}(\omega) = h, r_{k\tau}(\omega) = i, r_{(k+1)\tau}(\omega) = j\}. \tag{5.34}$$

We set $t_1 \triangleq (k-1)\tau + T_{\mathrm{D}}$, $t_2 \triangleq k\tau$, $\phi_{r(s)}(s) \triangleq \gamma_{r(s),\eta(h,s-(k-1)\tau)}$, $s \in [t_1, t_2)$, and employ the result presented in Lemma 5.1 to obtain

$$\mathbb{E}[y_n^{h,i,j}] = \sum_{k=1}^{\infty} \frac{1}{\mathbb{P}[G]} \int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau} \sum_{l \in \mathcal{M}} \gamma_{l,\eta(h,s-(k-1)\tau)} \mathbb{P}[G \cap F_l(s)]\mathrm{d}s\mathbb{P}[k_n^{i,j} = k], \tag{5.35}$$

where $F_l(s) \triangleq \{\omega \in \Omega : r_s(\omega) = l\}$, $s \in [(k-1)\tau + T_{\mathrm{D}}, k\tau)$. Note that $G = F_h((k-1)\tau) \cap$

$F_i(k\tau) \cap F_j((k+1)\tau)$ and hence

$$\begin{aligned}
\frac{\mathbb{P}[G \cap F_l(s)]}{\mathbb{P}[G]} &= \frac{\mathbb{P}[F_h((k-1)\tau) \cap F_i(k\tau) \cap F_j((k+1)\tau) \cap F_l(s)]}{F_h((k-1)\tau) \cap F_i(k\tau) \cap F_j((k+1)\tau)} \\
&= \frac{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]}{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]} \\
&= \frac{\mathbb{P}[F_i(k\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_h((k-1)\tau)]}{\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]} \\
&= \frac{p_{l,i}(k\tau - s)p_{h,l}(s-(k-1)\tau)}{p_{h,i}(\tau)}, \quad s \in [(k-1)\tau + T_\mathrm{D}, k\tau),
\end{aligned} \tag{5.36}$$

where $p_{h,i}(\tau)$ is given by (5.4). We substitute (5.36) into (5.35) and set $\tilde{s} \triangleq s-(k-1)\tau$ to arrive at

$$\begin{aligned}
\mathbb{E}[y_n^{h,i,j}] &= \sum_{k=1}^{\infty} \int_{T_\mathrm{D}}^{\tau} \sum_{l \in \mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)} \mathrm{d}\tilde{s} \mathbb{P}[k_n^{h,i,j} = k] \\
&= \int_{T_\mathrm{D}}^{\tau} \sum_{l \in \mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)} \mathrm{d}\tilde{s} \sum_{k=1}^{\infty} \mathbb{P}[k_n^{h,i,j} = k] \\
&= \int_{T_\mathrm{D}}^{\tau} \sum_{l \in \mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)} \mathrm{d}\tilde{s}, \quad n \in \mathbb{N}.
\end{aligned} \tag{5.37}$$

On the other hand it follows from (5.32) that

$$\begin{aligned}
\mathbb{E}[z_n^{h,i,j}] &= \sum_{k=1}^{\infty} \mathbb{E}[\int_{k_n^{h,i,j}\tau}^{k_n^{h,i,j}\tau + T_\mathrm{D}} \gamma_{r(s),\eta(h,s-(k_n^{h,i,j}-1)\tau)} \mathrm{d}s | k_n^{h,i,j} = k] \mathbb{P}[k_n^{h,i,j} = k] \\
&= \sum_{k=1}^{\infty} \mathbb{E}[\int_{k\tau}^{k\tau + T_\mathrm{D}} \gamma_{r(s),\eta(h,s-(k-1)\tau)} \mathrm{d}s | G] \mathbb{P}[k_n^{h,i,j} = k],
\end{aligned} \tag{5.38}$$

where G is given by (5.34). We now set $t_1 \triangleq k\tau$, $t_2 \triangleq k\tau + T_\mathrm{D}$, $\phi_{r(s)}(s) \triangleq \gamma_{r(s),\eta(h,s-(k-1)\tau)}$, $s \in [t_1, t_2)$, and employ the result presented in Lemma 5.1 to obtain

$$\mathbb{E}[z_n^{h,i,j}] = \sum_{k=1}^{\infty} \frac{1}{\mathbb{P}[G]} \int_{k\tau}^{k\tau + T_\mathrm{D}} \sum_{l \in \mathcal{M}} \gamma_{l,\eta(h,s-(k_n^{h,i,j}-1)\tau)} \mathbb{P}[G \cap F_l(s)] \mathrm{d}s \mathbb{P}[k_n^{h,i,j} = k]. \tag{5.39}$$

where $F_l(s) \triangleq \{\omega \in \Omega : r_s(\omega) = l\}$, $s \in [k\tau, k\tau + T_{\rm D})$. Now note that for $s \in [k\tau, k\tau + T_{\rm D})$,

$$\begin{aligned}\frac{\mathbb{P}[G \cap F_l(s)]}{\mathbb{P}[G]} &= \frac{\mathbb{P}[F_h((k-1)\tau) \cap F_i(k\tau) \cap F_j((k+1)\tau) \cap F_l(s)]}{F_h((k-1)\tau) \cap F_i(k\tau) \cap F_j((k+1)\tau)} \\ &= \frac{\mathbb{P}[F_j((k+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]}{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]\mathbb{P}[F_i(k\tau)|F_h((k-1)\tau)]\mathbb{P}[F_h((k-1)\tau)]} \\ &= \frac{\mathbb{P}[F_j((k+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i(k\tau)]}{\mathbb{P}[F_j((k+1)\tau)|F_i(k\tau)]} \\ &= \frac{p_{l,j}((k+1)\tau - s)p_{i,l}(s-k\tau)}{p_{i,j}(\tau)}. \end{aligned} \tag{5.40}$$

We now use (5.40) and set $\tilde{s} \triangleq s - (k-1)\tau$ in (5.39) to arrive at

$$\begin{aligned}\mathbb{E}[z_n^{h,i,j}] &= \sum_{k=1}^{\infty} \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} {\rm d}\tilde{s}\,\mathbb{P}[k_n^{h,i,j} = k] \\ &= \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} {\rm d}\tilde{s} \sum_{k=1}^{\infty} \mathbb{P}[k_n^{h,i,j} = k] \\ &= \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} {\rm d}\tilde{s}, \quad n \in \mathbb{N}. \end{aligned} \tag{5.41}$$

Note that $\lim_{t\to\infty} N^{h,i,j}(t) = \infty$, almost surely. Therefore, it follows from the strong law of large numbers and (5.37) and (5.41) that

$$\lim_{t\to\infty} \frac{1}{N^{h,i,j}(t)} \sum_{n=1}^{N^{h,i,j}(t)} y_n^{h,i,j} = \int_{T_{\rm D}}^{\tau} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)} {\rm d}\tilde{s}, \tag{5.42}$$

$$\lim_{t\to\infty} \frac{1}{N^{h,i,j}(t)} \sum_{n=1}^{N^{h,i,j}(t)} z_n^{h,i,j} = \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} {\rm d}\tilde{s}. \tag{5.43}$$

Now it follows from (5.30)–(5.32), (5.42), and (5.43) that

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{N^{h,i,j}(t)} \sum_{n=1}^{N^{h,i,j}(t)} \int_{(k_n^{h,i,j}-1)\tau+T_{\rm D}}^{k_n^{h,i,j}\tau+T_{\rm D}} \gamma_{r(s),\rho(s)} {\rm d}s \\ &= \int_{T_{\rm D}}^{\tau} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,i}(\tau-\tilde{s})p_{h,l}(\tilde{s})}{p_{h,i}(\tau)} {\rm d}\tilde{s} + \int_{\tau}^{\tau+T_{\rm D}} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(h,\tilde{s})} \frac{p_{l,j}(2\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau)}{p_{i,j}(\tau)} {\rm d}\tilde{s}. \end{aligned} \tag{5.44}$$

As a final step, we substitute the limits evaluated in (5.28), (5.29), and (5.44) into (5.25), and obtain

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{h,i}(\tau)p_{i,j}(\tau)\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}\frac{p_{l,i}(\tau-s)p_{h,l}(s)}{p_{h,i}(\tau)}\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{h,i}(\tau)p_{i,j}(\tau)\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}\frac{p_{l,j}(2\tau-s)p_{i,l}(s-\tau)}{p_{i,j}(\tau)}\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{i,j}(\tau)\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}p_{l,i}(\tau-s)p_{h,l}(s)\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h,i,j\in\mathcal{M}}\pi_h p_{h,i}(\tau)\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}p_{l,j}(2\tau-s)p_{i,l}(s-\tau)\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}p_{h,l}(s)\sum_{i\in\mathcal{M}}p_{l,i}(\tau-s)\sum_{j\in\mathcal{M}}p_{i,j}(\tau)\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau)\sum_{j\in\mathcal{M}}p_{l,j}(2\tau-s)\mathrm{d}s. \qquad (5.45)
\end{aligned}
$$

Note that $\sum_{j\in\mathcal{M}}p_{i,j}(t)=1,\ t\geq 0$, for all $i\in\mathcal{M}$. We use this fact to obtain $\sum_{j\in\mathcal{M}}p_{i,j}(\tau)=1$, $\sum_{i\in\mathcal{M}}p_{l,i}(\tau-s)=1$, and $\sum_{j\in\mathcal{M}}p_{l,j}(2\tau-s)=1$ in (5.45). Furthermore, note that $\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau)=p_{h,l}(s)$, for all $h,l\in\mathcal{M}$. Therefore, it follows from (5.45) that

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s&=\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{T_{\mathrm{D}}}^{\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}p_{h,l}(s)\mathrm{d}s\\
&\quad+\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}p_{h,l}(s)\mathrm{d}s\\
&=\frac{1}{\tau}\sum_{h\in\mathcal{M}}\pi_h\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(h,s)}p_{h,l}(s)\mathrm{d}s. \qquad (5.46)
\end{aligned}
$$

Case 2) We now consider the case where $T_{\mathrm{D}}>\tau$ and compute the limit given by

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s. \qquad (5.47)$$

Note that in this case, the information delay is larger than the mode sampling interval, and hence the kth sampled mode data $r(k\tau)$ becomes available for control purposes after time $(k+1)\tau$. Let

$$\underline{k} \triangleq \max\{k \in \mathbb{N} : k\tau \leq T_{\mathrm{D}}\}. \tag{5.48}$$

Note that the kth sampled mode data $r(k\tau)$ becomes available for control purposes before time $(k+\underline{k}+1)\tau$. Now, for given $g,h,i,j \in \mathcal{M}$, let $\{N^{g,h,i,j}(t) \in \mathbb{N}_0\}_{t\geq T_{\mathrm{D}}}$ be the counting process defined by

$$N^{g,h,i,j}(t) = \sum_{k=1}^{N(t)} \mathbb{1}_{[r((k-1)\tau)=g, r((k-1+\underline{k})\tau)=h, r((k+\underline{k})\tau)=i, r((k+\underline{k}+1)\tau)=j]}, \quad t \geq T_{\mathrm{D}}. \tag{5.49}$$

Note that for all $g,h,i,j \in \mathcal{M}$, the counting process $\{N^{g,h,i,j}(t) \in \mathbb{N}_0\}_{t\geq T_{\mathrm{D}}}$ is a stochastic process that depends on the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$. Furthermore, note that

$$\sum_{g,h,i,j\in\mathcal{M}} N^{g,h,i,j}(t) = N(t), \quad t \geq T_{\mathrm{D}}. \tag{5.50}$$

Now, for all $g,h,i,j \in \mathcal{M}$, we define the sequence of indices $\{k_n^{g,h,i,j} \in \mathbb{N}\}_{n\in\mathbb{N}}$ by

$$k_n^{g,h,i,j} = \min\{k \in \mathbb{N} : N^{g,h,i,j}(k\tau + T_{\mathrm{D}}) = n\}, \quad n \in \mathbb{N}. \tag{5.51}$$

Now, note that $r((k_n^{g,h,i,j}-1)\tau) = \sigma((k_n^{g,h,i,j}-1+\underline{k})\tau - T_{\mathrm{D}}) = g$, $r((k_n^{g,h,i,j}-1+\underline{k})\tau) = h$, $r((k_n^{g,h,i,j}+\underline{k})\tau) = i$, and $r((k_n^{g,h,i,j}+\underline{k}+1)\tau) = j$, $n \in \mathbb{N}$, $g,h,i,j \in \mathcal{M}$. Furthermore, note that $\sigma(s - T_{\mathrm{D}}) = r((k_n^{g,h,i,j}-1)\tau) = g$, for $s \in [(k_n^{g,h,i,j}-1)\tau + T_{\mathrm{D}}, k_n^{g,h,i,j}\tau + T_{\mathrm{D}})$. It follows from (5.9) that

$$\rho(s) = \eta(g, s - (k_n^{g,h,i,j}-1)\tau), \quad s \in [(k_n^{g,h,i,j}-1)\tau + T_{\mathrm{D}}, k_n^{g,h,i,j}\tau + T_{\mathrm{D}}). \tag{5.52}$$

As a consequence, it follows from (5.19)–(5.51) that

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{1}{t}\sum_{g,h,i,j\in\mathcal{M}}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{1}{t}\sum_{g,h,i,j\in\mathcal{M}}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)}\mathrm{d}s. \qquad (5.53)
\end{aligned}
$$

We now multiply the integral in the right hand side of (5.53) by $\frac{N(t)}{N(t)}\frac{N^{g,h,i,j}(t)}{N^{g,h,i,j}(t)}$ to obtain

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s\\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{g,h,i,j\in\mathcal{M}}\Big(\frac{N^{g,h,i,j}(t)}{N(t)}\\
&\quad\cdot\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)}\mathrm{d}s\Big)\\
&=\lim_{t\to\infty}\frac{N(t)}{t}\sum_{g,h,i,j\in\mathcal{M}}\Big(\lim_{t\to\infty}\frac{N^{g,h,i,j}(t)}{N(t)}\\
&\quad\cdot\lim_{t\to\infty}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)}\mathrm{d}s\Big). \qquad (5.54)
\end{aligned}
$$

We now evaluate $\lim_{t\to\infty}\frac{N^{g,h,i,j}(t)}{N(t)}$ in (5.54). The counting process $N_{g,h,i,j}(t)$ denotes the number of time instants $k\in\{1,2,\ldots,N(t)\}$ such that $r((k-1)\tau)=g$, $r((k-1+\underline{k})\tau)=h$, $r((k+\underline{k})\tau)=i$, and $r((k+\underline{k}+1)\tau)=j$. Note that $N(t)=\sum_{g,h,i,j\in\mathcal{M}}N^{g,h,i,j}(t)$. We use the strong law of large numbers [86,87] for discrete-time Markov chain $\{r(k\tau)\in\mathcal{M}\}_{k\in\mathbb{N}_0}$ and obtain

$$
\lim_{t\to\infty}\frac{N^{g,h,i,j}(t)}{N(t)}=\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)p_{i,j}(\tau), \qquad (5.55)
$$

where $\pi_g>0$ is the stationary probability distribution for state $g\in\mathcal{M}$. Moreover, $p_{h,i}(\underline{k}\tau)$, $p_{h,i}(\tau)$ and $p_{i,j}(\tau)$ are transition probabilities characterized in (5.4).

Next, our goal is to compute the limit

$$\lim_{t\to\infty} \frac{1}{N^{g,h,i,j}(t)} \sum_{n=1}^{N^{g,h,i,j}(t)} \int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{k_n^{g,h,i,j}\tau+T_{\rm D}} \gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)} {\rm d}s. \tag{5.56}$$

Note that by the definition (5.48)

$$\begin{aligned}
&\int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{k_n^{g,h,i,j}\tau+T_{\rm D}} \gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)} {\rm d}s \\
&\quad = \int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{(k_n^{g,h,i,j}+\underline{k})\tau} \gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)} {\rm d}s + \int_{(k_n^{g,h,i,j}+\underline{k})\tau}^{k_n^{g,h,i,j}\tau+T_{\rm D}} \gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)} {\rm d}s.
\end{aligned} \tag{5.57}$$

Now let

$$y_n^{g,h,i,j} \triangleq \int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{(k_n^{g,h,i,j}+\underline{k})\tau} \gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)} {\rm d}s, \quad n \in \mathbb{N}, \quad g,h,i,j \in \mathcal{M}, \tag{5.58}$$

$$z_n^{g,h,i,j} \triangleq \int_{(k_n^{g,h,i,j}+\underline{k})\tau}^{k_n^{g,h,i,j}\tau+T_{\rm D}} \gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)} {\rm d}s, \quad n \in \mathbb{N}, \quad g,h,i,j \in \mathcal{M}. \tag{5.59}$$

Note that by definition (5.51), the mode signal takes the values g and i at time instants $(k_n^{g,h,i,j} - 1 + \underline{k})\tau$ and $(k_n^{g,h,i,j} + \bar{k})\tau$, respectively, for all $n \in \mathbb{N}$. The value of the mode signal during the interval $\big((k_n^{g,h,i,j} - 1 + \underline{k})\tau, (k_n^{g,h,i,j} + \bar{k})\tau\big)$ may differ for each $n \in \mathbb{N}$. However, the probability of the mode taking the value $l \in \mathcal{M}$ at time $(k_n^{g,h,i,j} - 1 + \underline{k})\tau + s$, where $s \in (0, \tau)$, does not depend on $n \in \mathbb{N}$. Note that the integration in (5.58) is over the interval $[(k_n^{g,h,i,j} - 1)\tau + T_{\rm D}, (k_n^{g,h,i,j} + \bar{k})\tau)$, where $(k_n^{g,h,i,j} - 1)\tau + T_{\rm D} > (k_n^{g,h,i,j} - 1 + \underline{k})\tau$. Hence, for given $g, h, i, j \in \mathcal{M}$, the random variables $y_n^{g,h,i,j}$, $n \in \mathbb{N}$, are independent and distributed identically. A similar argument can be used to show that for given $g, h, i, j \in \mathcal{M}$, the random variables $z_n^{g,h,i,j}$, $n \in \mathbb{N}$, are also independent and distributed identically. Now, it follows from (5.58) that

$$\begin{aligned}
\mathbb{E}[y_n^{g,h,i,j}] &= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k_n^{g,h,i,j}-1)\tau+T_{\rm D}}^{(k_n^{g,h,i,j}+\underline{k})\tau} \gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)} {\rm d}s | k_n^{g,h,i,j} = k] \mathbb{P}[k_n^{g,h,i,j} = k] \\
&= \sum_{k=1}^{\infty} \mathbb{E}[\int_{(k-1)\tau+T_{\rm D}}^{(k+\underline{k})\tau} \gamma_{r(s),\eta(g,s-(k-1)\tau)} {\rm d}s | G] \mathbb{P}[k_n^{g,h,i,j} = k],
\end{aligned} \tag{5.60}$$

where

$$G \triangleq \{\omega \in \Omega : r_{(k-1)\tau}(\omega) = g, r_{(k-1+\underline{k})\tau}(\omega) = h, r_{(k+\underline{k})\tau}(\omega) = i, r_{(k+\underline{k}+1)\tau}(\omega) = j\}. \quad (5.61)$$

We set $t_1 \triangleq (k-1)\tau + T_{\mathrm{D}}$, $t_2 \triangleq (k+\underline{k})\tau$, $\phi_{r(s)}(s) \triangleq \gamma_{r(s),\eta(g,s-(k-1)\tau)}$, $s \in [t_1, t_2)$, and employ the result presented in Lemma 5.1 to obtain

$$\mathbb{E}[y_n^{g,h,i,j}] = \sum_{k=1}^{\infty} \frac{1}{\mathbb{P}[G]} \int_{(k-1)\tau+T_{\mathrm{D}}}^{(k+\underline{k})\tau} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(g,s-(k-1)\tau)} \mathbb{P}[G \cap F_l(s)] \mathrm{d}s \mathbb{P}[k_n^{g,h,i,j} = k], \quad (5.62)$$

where $F_l(s) \triangleq \{\omega \in \Omega : r_s(\omega) = l\}$, $s \in [(k-1)\tau + T_{\mathrm{D}}, (k+\underline{k})\tau)$. Note that

$$G = F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau), \quad (5.63)$$

and hence

$$\begin{aligned}
&\frac{\mathbb{P}[G \cap F_l(s)]}{\mathbb{P}[G]} \\
&= \frac{\mathbb{P}[F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau) \cap F_l(s)]}{F_g((k-1)\tau) \cap F_h((k-1+\underline{k})\tau) \cap F_i((k+\underline{k})\tau) \cap F_j((k+\underline{k}+1)\tau)} \\
&= \frac{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]\mathbb{P}[F_i((k+\underline{k})\tau)|F_l(s)]}{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]} \\
&\quad \cdot \frac{\mathbb{P}[F_l(s)|F_h((k-1+\underline{k})\tau)]\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]}{\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]} \\
&= \frac{\mathbb{P}[F_i((k+\underline{k})\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_h((k-1+\underline{k})\tau)]}{\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]} \\
&= \frac{p_{l,i}((k+\underline{k})\tau - s)p_{h,l}(s-(k-1+\underline{k})\tau)}{p_{h,i}(\tau)}, \quad s \in [(k-1)\tau + T_{\mathrm{D}}, (k+\underline{k})\tau), \quad (5.64)
\end{aligned}$$

where $p_{h,i}(\tau)$ is given by (5.4). We substitute (5.64) into (5.62) and set $\tilde{s} \triangleq s - (k-1)\tau$ to arrive at

$$\begin{aligned}
\mathbb{E}[y_n^{g,h,i,j}] &= \sum_{k=1}^{\infty} \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(g,\tilde{s})} \frac{p_{l,i}((\underline{k}+1)\tau - \tilde{s})p_{h,l}(\tilde{s} - \underline{k}\tau)}{p_{h,i}(\tau)} \mathrm{d}\tilde{s} \mathbb{P}[k_n^{g,h,i,j} = k] \\
&= \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(g,\tilde{s})} \frac{p_{l,i}((\underline{k}+1)\tau - \tilde{s})p_{h,l}(\tilde{s} - \underline{k}\tau)}{p_{h,i}(\tau)} \mathrm{d}\tilde{s} \sum_{k=1}^{\infty} \mathbb{P}[k_n^{g,h,i,j} = k] \\
&= \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau} \sum_{l\in\mathcal{M}} \gamma_{l,\eta(g,\tilde{s})} \frac{p_{l,i}((\underline{k}+1)\tau - \tilde{s})p_{h,l}(\tilde{s} - \underline{k}\tau)}{p_{h,i}(\tau)} \mathrm{d}\tilde{s}, \quad n \in \mathbb{N}. \quad (5.65)
\end{aligned}$$

On the other hand it follows from (5.59) that

$$\begin{aligned}\mathbb{E}[z_n^{g,h,i,j}] &= \sum_{k=1}^{\infty}\mathbb{E}[\int_{(k_n^{g,h,i,j}+\underline{k})\tau}^{k_n^{g,h,i,j}\tau+T_{\rm D}}\gamma_{r(s),\eta(g,s-(k_n^{g,h,i,j}-1)\tau)}{\rm d}s|k_n^{g,h,i,j}=k]\mathbb{P}[k_n^{g,h,i,j}=k]\\ &= \sum_{k=1}^{\infty}\mathbb{E}[\int_{(k+\underline{k})\tau}^{k\tau+T_{\rm D}}\gamma_{r(s),\eta(g,s-(k-1)\tau)}{\rm d}s|G]\mathbb{P}[k_n^{g,h,i,j}=k],\end{aligned}\tag{5.66}$$

where G is given by (5.61). We now set $t_1 \triangleq (k+\underline{k})\tau$, $t_2 \triangleq k\tau + T_{\rm D}$, $\phi_{r(s)}(s) \triangleq \gamma_{r(s),\eta(g,s-(k-1)\tau)}$, $s \in [t_1, t_2)$, and employ the result presented in Lemma 5.1 to obtain

$$\mathbb{E}[z_n^{g,h,i,j}] = \sum_{k=1}^{\infty}\frac{1}{\mathbb{P}[G]}\int_{(k+\underline{k})\tau}^{k\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s-(k-1)\tau)}\mathbb{P}[G\cap F_l(s)]{\rm d}s\mathbb{P}[k_n^{g,h,i,j}=k]. \tag{5.67}$$

where $F_l(s) \triangleq \{\omega\in\Omega : r_s(\omega) = l\}$, $s\in[(k+\underline{k})\tau, k\tau+T_{\rm D})$. Note that

$$\begin{aligned}&\frac{\mathbb{P}[G\cap F_l(s)]}{\mathbb{P}[G]}\\ &= \frac{\mathbb{P}[F_g((k-1)\tau)\cap F_h((k-1+\underline{k})\tau)\cap F_i((k+\underline{k})\tau)\cap F_j((k+\underline{k}+1)\tau)\cap F_l(s)]}{F_g((k-1)\tau)\cap F_h((k-1+\underline{k})\tau)\cap F_i((k+\underline{k})\tau)\cap F_j((k+\underline{k}+1)\tau)}\\ &= \frac{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i((k+\underline{k})\tau)]}{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]}\\ &\quad\cdot\frac{\mathbb{P}[F_i((k+\underline{k})\tau)|F_h((k-1+\underline{k})\tau)]\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]}{\mathbb{P}[F_h((k-1+\underline{k})\tau)|F_g((k-1)\tau)]\mathbb{P}[F_g((k-1)\tau)]}\\ &= \frac{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_l(s)]\mathbb{P}[F_l(s)|F_i((k+\underline{k})\tau)]}{\mathbb{P}[F_j((k+\underline{k}+1)\tau)|F_i((k+\underline{k})\tau)]}\\ &= \frac{p_{l,j}((k+\underline{k}+1)\tau-s)p_{i,l}(s-(k+\underline{k})\tau)}{p_{i,j}(\tau)},\quad s\in[(k+\underline{k})\tau, k\tau+T_{\rm D}).\end{aligned}\tag{5.68}$$

We now use (5.68) and set $\tilde{s}\triangleq s-(k-1)\tau$ in (5.39) to arrive at

$$\begin{aligned}\mathbb{E}[z_n^{g,h,i,j}] &= \sum_{k=1}^{\infty}\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,\tilde{s})}\frac{p_{l,j}((\underline{k}+2)\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}{\rm d}\tilde{s}\mathbb{P}[k_n^{g,h,i,j}=k]\\ &= \int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,\tilde{s})}\frac{p_{l,j}((\underline{k}+2)\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}{\rm d}\tilde{s}\sum_{k=1}^{\infty}\mathbb{P}[k_n^{g,h,i,j}=k]\\ &= \int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,\tilde{s})}\frac{p_{l,j}((\underline{k}+2)\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}{\rm d}\tilde{s},\quad n\in\mathbb{N}.\end{aligned}\tag{5.69}$$

Note that $\lim_{t\to\infty}N^{g,h,i,j}(t)=\infty$, almost surely. Consequently, it follows from the strong law of large numbers and (5.37) and (5.69) that

$$\lim_{t\to\infty}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)} y_n^{g,h,i,j} = \int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,\tilde{s})}\frac{p_{l,i}((\underline{k}+1)\tau-\tilde{s})p_{h,l}(\tilde{s}-\underline{k}\tau)}{p_{h,i}(\tau)}\mathrm{d}\tilde{s}, \tag{5.70}$$

$$\lim_{t\to\infty}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)} z_n^{g,h,i,j} = \int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,\tilde{s})}\frac{p_{l,j}((\underline{k}+2)\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}\mathrm{d}\tilde{s}. \tag{5.71}$$

It then follows from (5.30)–(5.32), (5.70), and (5.71) that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{N^{g,h,i,j}(t)}\sum_{n=1}^{N^{g,h,i,j}(t)}\int_{(k_n^{g,h,i,j}-1)\tau+T_{\mathrm{D}}}^{k_n^{g,h,i,j}\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s\\
&=\int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,\tilde{s})}\frac{p_{l,i}((\underline{k}+1)\tau-\tilde{s})p_{h,l}(\tilde{s}-\underline{k}\tau)}{p_{h,i}(\tau)}\mathrm{d}\tilde{s}\\
&\quad+\int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,\tilde{s})}\frac{p_{l,j}((\underline{k}+2)\tau-\tilde{s})p_{i,l}(\tilde{s}-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}\mathrm{d}\tilde{s}.
\end{aligned} \tag{5.72}$$

Finally, we substitute the limits evaluated in (5.28), (5.55), and (5.72) into (5.54), and obtain

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\rm D}}^{k\tau+T_{\rm D}}\gamma_{r(s),\rho(s)}{\rm d}s$$
$$=\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)p_{i,j}(\tau)\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}\frac{p_{l,i}((\underline{k}+1)\tau-s)p_{h,l}(s-\underline{k}\tau)}{p_{h,i}(\tau)}{\rm d}s$$
$$+\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)p_{i,j}(\tau)$$
$$\cdot\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}\frac{p_{l,j}((\underline{k}+2)\tau-s)p_{i,l}(s-\tau-\underline{k}\tau)}{p_{i,j}(\tau)}{\rm d}s$$
$$=\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{i,j}(\tau)\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}p_{l,i}((\underline{k}+1)\tau-s)p_{h,l}(s-\underline{k}\tau){\rm d}s$$
$$+\frac{1}{\tau}\sum_{g,h,i,j\in\mathcal{M}}\pi_g p_{g,h}(\underline{k}\tau)p_{h,i}(\tau)\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}p_{l,j}((\underline{k}+2)\tau-s)p_{i,l}(s-\tau-\underline{k}\tau){\rm d}s$$
$$=\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\Big(\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}$$
$$\cdot\sum_{h\in\mathcal{M}}p_{g,h}(\underline{k}\tau)p_{h,l}(s-\underline{k}\tau)\sum_{i\in\mathcal{M}}p_{l,i}((\underline{k}+1)\tau-s)\sum_{j\in\mathcal{M}}p_{i,j}(\tau)\Big){\rm d}s$$
$$+\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\Big(\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}\sum_{h\in\mathcal{M}}p_{g,h}(\underline{k}\tau)$$
$$\cdot\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau-\underline{k}\tau)\sum_{j\in\mathcal{M}}p_{l,j}((\underline{k}+2)\tau-s)\Big){\rm d}s. \tag{5.73}$$

By using the fact that $\sum_{j\in\mathcal{M}}p_{i,j}(t)=1$, $t\geq 0$, for all $i\in\mathcal{M}$, we obtain $\sum_{j\in\mathcal{M}}p_{i,j}(\tau)=1$, $\sum_{i\in\mathcal{M}}p_{l,i}((\underline{k}+1)\tau-s)=1$, and $\sum_{j\in\mathcal{M}}p_{l,j}((\underline{k}+2)\tau-s)=1$ in (5.73). Note also that $\sum_{h\in\mathcal{M}}p_{g,h}(\underline{k}\tau)p_{h,l}(s-\underline{k}\tau)=p_{g,l}(s)$, for all $g,l\in\mathcal{M}$. Moreover, $\sum_{i\in\mathcal{M}}p_{h,i}(\tau)p_{i,l}(s-\tau-\underline{k}\tau)=p_{h,l}(s-\underline{k}\tau)$, for all $h,l\in\mathcal{M}$. Therefore, it follows from (5.73) that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\rm D}}^{k\tau+T_{\rm D}}\gamma_{r(s),\rho(s)}{\rm d}s$$
$$=\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{T_{\rm D}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}p_{g,l}(s){\rm d}s$$
$$+\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{(\underline{k}+1)\tau}^{\tau+T_{\rm D}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}\sum_{h\in\mathcal{M}}p_{g,h}(\underline{k}\tau)p_{h,l}(s-\underline{k}\tau){\rm d}s. \tag{5.74}$$

Now note that $\sum_{h\in\mathcal{M}} p_{g,h}(\underline{k}\tau)p_{h,l}(s-\underline{k}\tau) = p_{g,l}(s)$, for all $g, l \in \mathcal{M}$. As a consequence,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s &= \frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{T_{\mathrm{D}}}^{(\underline{k}+1)\tau}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}p_{g,l}(s)\mathrm{d}s\\ &\quad+\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{(\underline{k}+1)\tau}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}p_{g,l}(s)\mathrm{d}s\\ &=\frac{1}{\tau}\sum_{g\in\mathcal{M}}\pi_g\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}\sum_{l\in\mathcal{M}}\gamma_{l,\eta(g,s)}p_{g,l}(s)\mathrm{d}s. \end{aligned}\tag{5.75}$$

We evaluated the limit $\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{N(t)}\int_{(k-1)\tau+T_{\mathrm{D}}}^{k\tau+T_{\mathrm{D}}}\gamma_{r(s),\rho(s)}\mathrm{d}s$ for cases where $T_{\mathrm{D}}\leq\tau$ and $T_{\mathrm{D}}>\tau$. By changing the variables for sums in (5.46) and (5.75), we can show for both cases that the limit is given by $\frac{1}{\tau}\sum_{i\in\mathcal{M}}\pi_i\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}\sum_{j\in\mathcal{M}}\gamma_{j,\eta(i,s)}p_{i,j}(s)\mathrm{d}s$. Consequently, it follows from (5.19) that for all $\tau>0$ and $T_{\mathrm{D}}>0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t\gamma_{r(s),\sigma(s-T_{\mathrm{D}})}\mathrm{d}s &= \frac{1}{\tau}\sum_{i\in\mathcal{M}}\pi_i\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}\sum_{j\in\mathcal{M}}\gamma_{j,\eta(i,s)}p_{i,j}(s)\mathrm{d}s\\ &=\frac{1}{\tau}\sum_{i,j\in\mathcal{M}}\pi_i\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}\gamma_{j,\eta(i,s)}p_{i,j}(s)\mathrm{d}s,\end{aligned}\tag{5.76}$$

which completes the proof. □

Lemma 5.2 provides a strong law of large numbers for the bivariate stochastic process $\{(r(t),\rho(t))\}_{t\geq0}$. The result presented in Lemma 5.2 is crucial for obtaining the main results of this study. Specifically, the integral expression obtained in (5.12) for the long-run average of the piecewise-constant stochastic process $\{\gamma_{r(t),\rho(t)}\}_{t\geq0}$ will be used in Section 5.5.

5.5 Sufficient Conditions of Almost Sure Asymptotic Stabilization

In this section we provide sufficient conditions under which our proposed control law (5.11) achieves almost sure asymptotic stabilization of the switched linear stochastic system (3.57).

Theorem 5.1. Consider the switched linear stochastic system (3.57) with mode sampling period $\tau>0$ and sampled mode information constant time delay $T_{\mathrm{D}}>0$. If there exist

$P > 0$ and scalars $\zeta_i \in \mathbb{R}$, $i \in \mathcal{M}$, such that

$$0 \geq A_i^{\mathrm{T}} P + P A_i + D_i^{\mathrm{T}} P D_i - 2 P B_i B_i^{\mathrm{T}} P - \zeta_i P, \tag{5.77}$$

for $i \in \mathcal{M}$, and

$$\frac{1}{\tau} \sum_{i,j \in \mathcal{M}} \pi_i \int_{T_{\mathrm{D}}}^{\tau + T_{\mathrm{D}}} \gamma_{j,\eta(i,s)} p_{i,j}(s) \mathrm{d}s - \sum_{i \in \mathcal{M}} \pi_i \frac{\lambda_{\min}^2 (D_i^{\mathrm{T}} P + P D_i)}{2 \lambda_{\max}^2 (P)} < 0, \tag{5.78}$$

where

$$\gamma_{i,j} \triangleq \begin{cases} \zeta_j, & \text{if } i = j, \\ \zeta_i + \frac{2\lambda_{\max}(P B_i B_i^{\mathrm{T}} P)}{\lambda_{\min}(P)} - \frac{\lambda_{\min}(P(B_j B_i^{\mathrm{T}} + B_i B_j^{\mathrm{T}})P)}{\lambda_{\max}(P)}, & \text{if } i \neq j, \end{cases} \tag{5.79}$$

then the feedback control law (5.11) with the feedback gain matrices given by

$$K_i = -B_i^{\mathrm{T}} P, \quad i \in \mathcal{M}, \tag{5.80}$$

guarantees that the zero solution $x(t) \equiv 0$ of the closed-loop system (3.57) and (5.11) is asymptotically stable almost surely.

For proving Theorem 5.1 we employ the strong law of large numbers developed in Lemma 5.2, and utilize a quadratic Lyapunov approach similar to the one used in Chapters 3 and 4.

Proof. The closed-loop system (3.57) under the control law (5.11) is given by the multi-dimensional Ito stochastic differential equation

$$\mathrm{d}x(t) = (A_{r(t)} + B_{r(t)} K_{\rho(t)}) x(t) \mathrm{d}t + D_{r(t)} x(t) \mathrm{d}W(t),$$

for $t \geq 0$. First, we define $V(x) \triangleq x^{\mathrm{T}} P x$ and consider the function $\ln V(x)$. Since $V(\cdot)$ is a positive-definite function, $\ln V(x)$ is well-defined for non-zero values of the state. Using

Ito formula and (5.80), we obtain

$$\begin{aligned}
&\ln V(x(t)) \\
&\quad = \ln V(x_0) + \int_0^t \frac{1}{V(x(s))} x^{\mathrm{T}}(s)\Big(A_{r(s)}^{\mathrm{T}} P + P A_{r(s)} - P B_{\rho(s)} B_{r(s)}^{\mathrm{T}} P - P B_{r(s)} B_{\rho(s)}^{\mathrm{T}} P \\
&\qquad + D_{r(s)}^{\mathrm{T}} P D_{r(s)}\Big) x(s) \mathrm{d}s - \int_0^t \frac{1}{2V^2(x(s))} (2x^{\mathrm{T}}(s) P D_{r(s)} x(s))^2 \mathrm{d}s + L(t), \qquad (5.81)
\end{aligned}$$

where $L(t) \triangleq \int_0^t \frac{1}{V(x(s))} 2x^{\mathrm{T}}(s) P D_{r(s)} x(s) \mathrm{d}W(s)$. Noting that

$$2x^{\mathrm{T}}(s) P D_{r(s)} x(s) \geq \frac{\lambda_{\min}(D_{r(s)}^{\mathrm{T}} P + P D_{r(s)})}{\lambda_{\max}(P)} x^{\mathrm{T}}(s) P x(s), \qquad (5.82)$$

it follows from (5.77), (5.79), and (5.81) that

$$\ln V(x(t)) \leq \ln V(x_0) + \int_0^t \gamma_{r(s),\rho(s)} \mathrm{d}s - \int_0^t \frac{\lambda_{\min}^2(D_{r(s)}^{\mathrm{T}} P + P D_{r(s)})}{2\lambda_{\max}^2(P)} \mathrm{d}s + L(t). \qquad (5.83)$$

We now apply the strong law of large numbers presented in Lemma 5.2 to obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \gamma_{r(s),\rho(s)} \mathrm{d}s = \frac{1}{\tau} \sum_{i,j\in\mathcal{M}} \pi_i \int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}} \gamma_{j,\eta(i,s)} p_{i,j}(s) \mathrm{d}s, \qquad (5.84)$$

almost surely. In addition, by the strong law of large numbers for continuous-time, finite-state, irreducible Markov chains [86, 87],

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t -\frac{\lambda_{\min}^2(D_{r(s)}^{\mathrm{T}} P + P D_{r(s)})}{2\lambda_{\max}^2(P)} \mathrm{d}s = -\sum_{i\in\mathcal{M}} \pi_i \frac{\lambda_{\min}^2(D_i^{\mathrm{T}} P + P D_i)}{2\lambda_{\max}^2(P)}, \qquad (5.85)$$

almost surely. Furthermore, note that the Ito integral $L(t)$ in inequality (5.81) is a local martingale with quadratic variation given by

$$\begin{aligned}
[L]_t &= \int_0^t \Big(\frac{1}{V(x(s))} 2x^{\mathrm{T}}(s) P D_{r(s)} x(s)\Big)^2 \mathrm{d}s \\
&\leq \int_0^t \frac{\lambda_{\max}^2(D_{r(s)}^{\mathrm{T}} P + P D_{r(s)})}{\lambda_{\min}^2(P)} \mathrm{d}s \\
&\leq \frac{\max_{i\in\mathcal{M}} \lambda_{\max}^2(D_i^{\mathrm{T}} P + P D_i)}{\lambda_{\min}^2(P)} t, \qquad (5.86)
\end{aligned}$$

so that $\lim_{t\to\infty} \frac{1}{t}[L]_t < \infty$. Hence, by employing the same approach presented in [14, 49, 51], it follows from the strong law of large numbers for local martingales that

$$\lim_{t\to\infty}\frac{1}{t}L(t)=0, \tag{5.87}$$

almost surely. Finally, using (5.78), (5.83), (5.84), (5.85), and (5.87), we arrive at

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\ln V(x(t)) &\leq \frac{1}{\tau}\sum_{i,j\in\mathcal{M}}\pi_i\int_{T_{\mathrm{D}}}^{\tau+T_{\mathrm{D}}}\gamma_{j,\eta(i,s)}p_{i,j}(s)\mathrm{d}s-\sum_{i\in\mathcal{M}}\pi_i\frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P+PD_i)}{2\lambda_{\max}^2(P)}\\ &< 0.\end{aligned} \tag{5.88}$$

Hence, $\lim_{t\to\infty}\ln V(x(t))=-\infty$ almost surely; moreover, $\mathbb{P}[\lim_{t\to\infty}V(x(t))=0]=1$. Therefore, the zero solution $x(t)\equiv 0$ of the closed-loop system (3.57), (5.11) is asymptotically stable almost surely. □

The conditions of Theorem 5.1 reflect the effect of the mode sampling period $\tau>0$ and the sampled mode information delay $T_{\mathrm{D}}>0$ on the stabilization. Note that the subsystem dynamics as well as the mode switching frequency also affect the stabilization.

Sufficient conditions of almost sure asymptotic stabilization presented in Theorem 5.1 are obtained through a quadratic Lyapunov function approach. Specifically, we consider the Lyapunov function candidate $V(x(t))\triangleq x^T(t)Px(t)$. The condition (5.77) guarantees an upper-bound on the stochastic Lyapunov derivative. Namely, under the condition (5.77), we have

$$\begin{aligned}\mathrm{L}V(x(t)) &\triangleq x^T(t)(A_{r(t)}^{\mathrm{T}}P+PA_{r(t)}+D_{r(t)}^{\mathrm{T}}PD_{r(t)}-PB_{\rho(t)}B_{r(t)}^{\mathrm{T}}P-PB_{r(t)}B_{\rho(t)}^{\mathrm{T}}P)x(t)\\ &\leq \gamma_{r(t),\rho(t)}V(x(t)),\quad t\geq 0,\end{aligned} \tag{5.89}$$

where $\gamma_{i,j}\in\mathbb{R}$, $i,j\in\mathcal{M}$, are given in (5.79). On the other hand, the condition (5.78) characterizes the requirement on the long run average of $\gamma_{r(t),\rho(t)}$, which we obtain through the strong law of large numbers presented in Lemma 5.2. Note that we do not require $\gamma_{i,j}<0$, for all $i,j\in\mathcal{M}$. In fact, as long as the condition (5.78) is satisfied, almost sure asymptotic stability of the closed-loop system is guaranteed even if $\gamma_{i,j}>0$ for some $i,j\in\mathcal{M}$.

The control framework developed in this study is based on a probability-based feedback gain scheduling. When a mode sample data becomes available to the controller after a delay, the feedback gain is switched according to the conditional probability distribution of the active mode given the available sampled mode data. This method is in contrast with the approach presented in Chapter 4, where the same feedback gain is maintained until the next mode sample data becomes available. The effect of the proposed feedback gain scheduling on the stabilization is characterized in the condition (5.78) by $\eta(i,\cdot)$, $i \in \mathcal{M}$. Note that $\eta(i,\cdot)$, $i \in \mathcal{M}$, (defined in (5.10)) are deterministic functions of time that can be computed easily for a given generator matrix $Q \in \mathbb{R}^{M\times M}$.

In the development of our control framework above, we have considered the case where the mode information delay is constant. However, in certain applications, the sampled mode information delay may vary for every mode sample. It is important to note that our proposed control law (5.11) can still be employed for almost sure asymptotic stabilization even if each mode sample is subject to a different time delay. Let $T_k > 0$ denote the time delay after which the kth mode sample data $r(k\tau)$ become available for control purposes. If there exists a positive constant $T_{\mathrm{D}} \leq \tau$ such that $T_k \leq T_{\mathrm{D}}$, $k \in \mathbb{N}_0$, sampled mode data reach the controller *in order*; furthermore, the controller has access to the mode sample $r(k\tau)$ at time $k\tau + T_{\mathrm{D}}$. In this case, as long as the conditions (5.77) and (5.78) are satisfied for the time delay upper-bound constant T_{D}, stabilization of the zero solution is guaranteed by our proposed control law (5.11).

5.6 Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of our proposed framework. Specifically, consider the 2-dimensional continuous-time switched linear stochastic system (3.57) composed of $M = 3$ modes characterized by the subsystem matrices

$$A_1 = \begin{bmatrix} 0.5 & 0.1 \\ -5 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.25 & 0 \\ 0.25 & 0.25 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 1 & -5.5 \\ 0.75 & 0.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

and $D_1 = D_2 = D_3 = I_2$. The mode signal $\{r(t) \in \mathcal{M} \triangleq \{1,2,3\}\}_{t\geq 0}$ of the switched system is assumed to be an irreducible Markov chain with the generator matrix

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -5 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \tag{5.90}$$

and the stationary probability distribution given by $\pi_1 = \frac{5}{7}$, $\pi_2 = \pi_3 = \frac{1}{7}$. Moreover, the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is assumed to be sampled periodically with the mode sampling period $\tau = 0.5$. In addition, the obtained mode samples are assumed to be available to controller after a constant time delay $T_{\mathrm{D}} = 0.4$.

Note that the positive-definite matrix $P = I_2$ and the scalars $\zeta_1 = -2.9$, $\zeta_2 = 1.75$, $\zeta_3 = -2.2$ satisfy the conditions (5.77) and (5.78). As a consequence, it follows that the control law (5.11) guarantees almost sure asymptotic stabilization of the zero solution $x(t) \equiv 0$ of the switched stochastic system (3.57).

The sample paths of $x(t)$ and $u(t)$ obtained with the initial conditions $x(0) = [1,\ 1]^{\mathrm{T}}$ and $r(0) = 3$ are shown in Figures 5.1 and 5.2, respectively. Moreover, Figure 5.3 shows the sample paths of the actual mode signal $r(t)$ and the feedback gain switching signal $\rho(t)$. Note that the control input trajectory is subject to jumps at feedback gain switching instants, as the feedback gain switching signal $\rho(t)$ changes its value.

Both the mode sampling period $\tau > 0$ and the mode sample information delay $T_{\mathrm{D}} > 0$ have effects on the stabilization. The whole dark region in Figure 5.4 shows the numerically obtained values of the constants τ and T_{D} that satisfy the condition (5.78) of

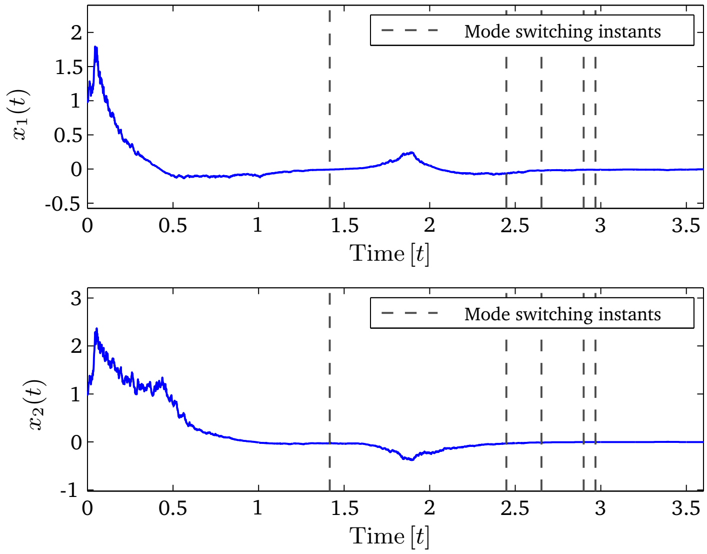

Figure 5.1: State trajectory versus time

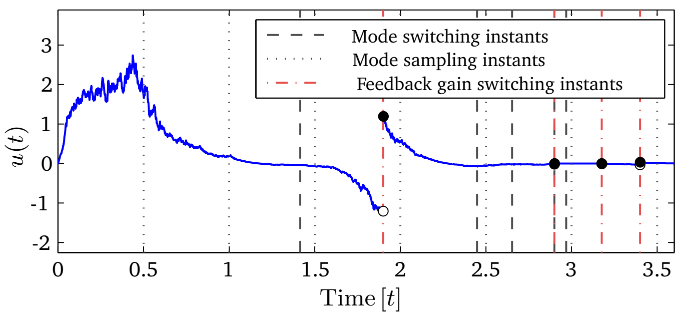

Figure 5.2: Control input versus time

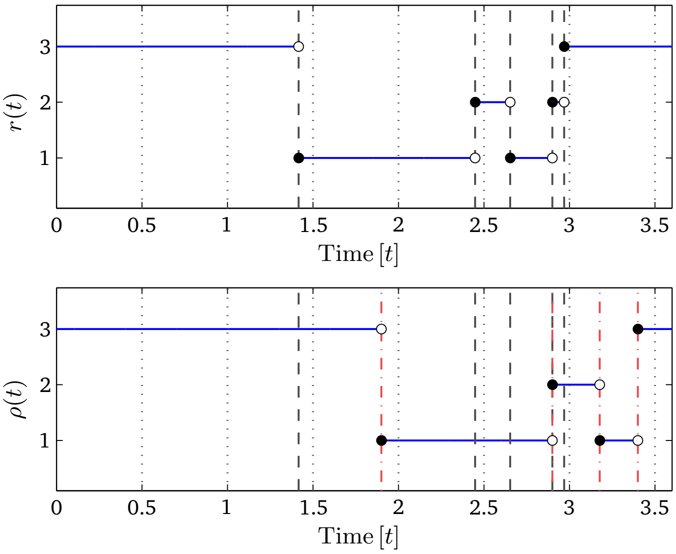

Figure 5.3: Actual mode signal $r(t)$ and the feedback gain switching signal $\rho(t)$ versus time

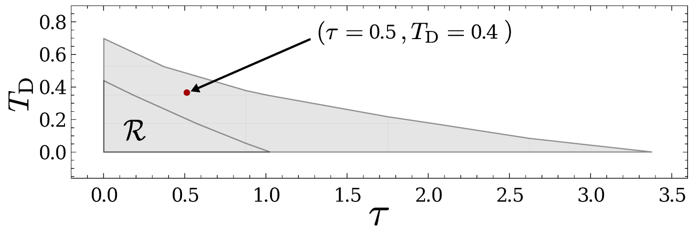

Figure 5.4: Stabilization region with respect to τ and $T_{\rm D}$

Theorem 5.1 for the positive-definite matrix $P = I_2$ and the scalars $\zeta_1 = -2.9$, $\zeta_2 = 1.75$, $\zeta_3 = -2.2$. Hence, our proposed control law (5.11) based on the feedback gain scheduling framework developed in Section 5.4 is guaranteed to achieve almost sure asymptotic stabilization for any values of τ and $T_{\rm D}$ selected from the whole dark region. On the other hand, the smaller region denoted by $\mathcal{R}$ in Figure 5.4 represents the values of τ and $T_{\rm D}$ such that the control law given in (4.2) guarantees the stabilization according to Theorem 4.1 in Chapter 4. This indicates that in comparison to the control law (4.2), our new control law (5.11) that relies on the probability-based feedback gain scheduling framework offer more relaxed stabilization conditions with respect to the mode sampling period $\tau > 0$ and the mode sample information delay $T_{\rm D} > 0$ for this example.

The sample paths of $x(t)$, $u(t)$, $r(t)$ and $\rho(t)$ shown in Figures 5.1–5.3 are obtained for the case where the mode sampling period is $\tau = 0.5$ and the mode sample information delay is $T_{\mathrm{D}} = 0.4$. Note that the point $(\tau = 0.5, T_{\mathrm{D}} = 0.4)$ lies only in the stabilization region associated with the control law (5.11) that relies on the probability-based feedback gain scheduling framework (see Figure 5.4).

5.7 Conclusion

Feedback control of switched linear stochastic systems under sampled and delayed mode information have been investigated. A new feedback gain scheduling method has been developed. This method is based on selecting the feedback gain associated with the mode that has the highest conditional probability of being active given the available sampled and delayed mode data. Sufficient conditions of almost sure asymptotic stabilization under the proposed control law have been obtained by employing a quadratic Lyapunov approach.

In Chapters 3–5, sampled-mode stabilization of continuous-time switched stochastic systems was explored. In Chapters 6, 7, and 9 below, we direct our attention to *discrete-time* switched stochastic systems.

Chapter 6

Sampled-Mode-Dependent Time-Varying Control Strategy for Stabilizing Discrete-Time Switched Stochastic Systems

6.1 Introduction

In Chapters 3–5, we explored feedback control of *continuous-time* switched stochastic systems that incorporate a continuous-time mode signal. Under the assumption that the mode of the switched system can be periodically observed, we proposed stabilizing feedback control laws that utilize the available sampled mode information.

In this chapter, our goal is to investigate the feedback control problem for *discrete-time* switched linear stochastic systems. Discrete-time switched linear stochastic systems are composed of a number of deterministic subsystems that are described by difference equations. The transitions between the subsystems (modes) of a discrete-time switched stochastic system is managed by a discrete-time stochastic mode signal. In this chapter, we use a finite-state discrete-time Markov chain (see Section 2.3.2) to model the mode signal. We consider the feedback control problem for the case where the mode signal is observed periodically. Therefore, the feedback control problem we explore in this chapter can be considered as a discrete-time analogue of the problem discussed in Chapter 3. Note that the feedback gain of the control law developed in Chapter 3, is set to a constant gain associated with the last observed mode. Furthermore, the same constant feedback gain is maintained between consecutive mode observation instants. The control law developed in Chapter 3, guarantee stabilization when the sampled mode information is an accurate representation of the actual operation mode, which is the case when transitions between modes occur rarely, and the operation mode of the system is frequently observed. In this chapter, we develop a *time-varying* control strategy that guarantees second-moment asymptotic stabilization of a discrete-time switched linear stochastic dynamical system. In our proposed control law we utilize sampled-mode-dependent feedback gains that *vary* during the intervals between consecutive mode observation instants. Note that these feedback gains can be designed for each time step to effectively compensate the uncertainty of the operation mode during large mode observation intervals. Therefore, our present control framework allows us to relax the tight requirements characterized in Chapter 3 on the mode observation period. Note that the control framework that we propose in this chapter and the control framework that we developed in Chapter 5 are similar in the sense that feedback gains in both frameworks are allowed to *vary* between the time instants at which consecutive mode observations become available for control purposes.

In this chapter, to obtain conditions under which our proposed control framework guarantees stabilization, we first investigate the properties of a bivariate process composed of the actual mode and its sampled version. We also observe that the dynamics that govern the evolution of the state covariance is periodic due to periodic mode observations. We then apply the discrete-time analogue of Floquet theory (see [109–111]), to obtain necessary and sufficient conditions of second-moment asymptotic stability of the zero solution. Furthermore, by employing Lyapunov-like functions with periodic coefficients, we also obtain alternative stabilization conditions, which we use for designing feedback gains.

The contents of this chapter are as follows. In Section 6.2, we propose our feedback control framework for stabilizing discrete-time switched stochastic systems with periodic mode observations. Then in Section 6.3, we present conditions under which our proposed control law guarantees second-moment asymptotic stabilization. In Section 6.4, we present a numerical example. Finally, in Section 6.5, we conclude the chapter.

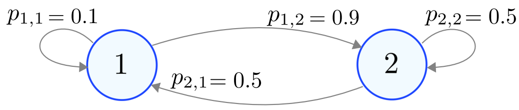

Figure 6.1: Mode transition diagram for $\{r(k) \in \mathcal{M} \triangleq \{1, 2\}\}_{k \in \mathbb{N}_0}$

6.2 Feedback Control of Switched Stochastic Systems with Periodically Observed Active Operation Mode

In this section, we propose a feedback control framework for stabilizing a switched stochastic system by using only the periodically obtained mode information.

6.2.1 Mathematical Model of Discrete-Time Switched Linear Stochastic Systems

We consider the discrete-time switched linear stochastic dynamical system with $M \in \mathbb{N}$ number of modes given by

$$x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k), \quad k \in \mathbb{N}_0, \tag{6.1}$$

with the initial condition $x(0) = x_0$, where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ respectively denote the state vector and the control input; furthermore, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $i \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$, are the subsystem matrices. The mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ is assumed to be an $\mathcal{F}_k$-adapted, M-state discrete-time Markov chain characterized by the initial mode distribution, $\nu : \mathcal{M} \to [0, 1]$ and the transition probability matrix $P \in \mathbb{R}^{M \times M}$ with entries $p_{i,j} \in \mathcal{M}$, $i, j \in \mathcal{M}$ (see Section 2.3.2 for the definition and properties of discrete-time Markov chains). Let $p_{i,j}^{(l)}$ denote the (i, j)th entry of the matrix P^l. Note that $p_{i,j}^{(l)} \in [0, 1]$, $i, j \in \mathcal{M}$, characterize l-step transition probabilities between the modes of the switched system, that is,

$$p_{i,j}^{(l)} = \mathbb{P}[r(k+l) = j | r(k) = i], \; l \in \mathbb{N}_0, \; i, j \in \mathcal{M}, \tag{6.2}$$

with $p_{i,i}^{(0)} = 1$, $i \in \mathcal{M}$, $p_{i,j}^{(0)} = 0$, $i \neq j$. Moreover, $p_{i,j}^{(1)} = p_{i,j}$, $i, j \in \mathcal{M}$.

We use transition diagrams to graphically represent possible transitions between the operation modes of a switched system. A mode transition diagram for a switched system

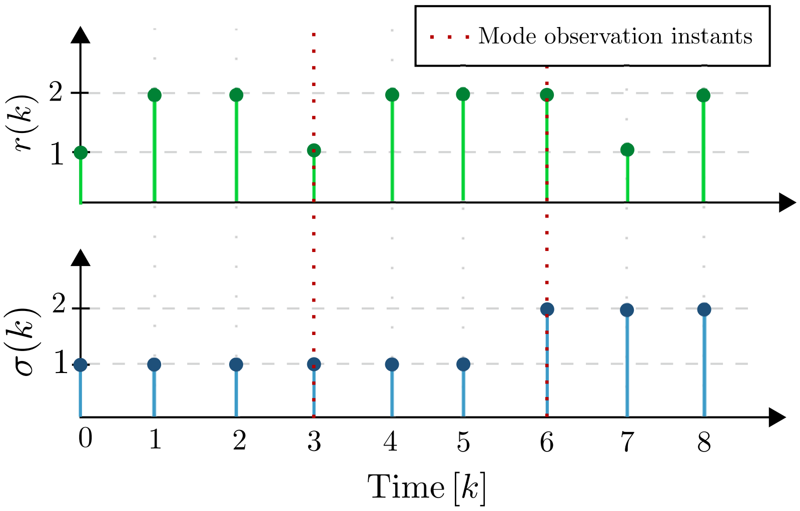

Figure 6.2: Actual mode signal $r(k)$ and its sampled version $\sigma(k)$

with $M = 2$ modes is shown in Figure 6.1. The labels on the directed edges indicate probability of associated transitions. For example, Figure 6.1 indicates that probability of transition from mode 1 to mode 2 is given by $p_{1,2} = 0.9$.

6.2.2 Control under Periodic Observations of the Active Operation Mode

Our goal in this section is to investigate a feedback control problem for the case where operation mode of the switched stochastic system (6.1) is observed (sampled) periodically at time instants $0, \tau, 2\tau, \ldots$, where $\tau \in \mathbb{N}$ denotes the mode observation period.

The sampled mode information that is available to the controller can be represented by the discrete-time stochastic process $\{\sigma(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ defined by

$$\sigma(k) = r(n\tau),\ k \in \{n\tau, n\tau + 1, \ldots (n+1)\tau - 1\}, \tag{6.3}$$

for $n \in \mathbb{N}_0$.

In Figure 6.2, we show sample paths of the actual mode $r(\cdot)$ and the corresponding sampled mode $\sigma(\cdot)$ for a switched stochastic system with $M = 2$ modes. In this example, active mode is observed at every $\tau = 3$ steps. Note that at mode observation instants, the operation mode of the switched system is known by the controller with certainty. However, at the other time instants, the actual mode signal may change its value according to transition probabilities $p_{i,j}$, $i, j \in \mathcal{M}$, and hence, sampled mode may differ from the actual mode.

Remark 6.1. Note that when the operation mode rarely switches and the mode observations are frequent, the sampled mode $\sigma(\cdot)$ is likely to be an accurate representation of the actual mode $r(\cdot)$ of the switched system. Therefore, stabilization can be achieved by a feedback control law of the form

$$u(k) = K_{\sigma(k)}x(k), \quad k \in \mathbb{N}_0, \tag{6.4}$$

where $K_i \in \mathbb{R}^{m \times n}$ is a constant gain matrix designed for mode $i \in \mathcal{M}$. In Chapter 3, we employed continuous-time version of the control law (6.4), for stabilizing continuous-time switched stochastic systems.

It is important to note that when the mode observation period $\tau \in \mathbb{N}$ is large, the sampled mode is likely to be a poor representation of the actual mode, and therefore, the control law (6.4) may not suffice to stabilize the switched stochastic system. To illustrate this issue, we consider l-step transition probabilities $p_{i,j}^{(l)} \in [0,1]$, $i, j \in \mathcal{M} \triangleq \{1,2\}$, for a switched system with $M = 2$ modes. Operation mode of the switched system is assumed to switch randomly according to transition probabilities given by the transition diagram in Figure 6.1. Note that given the n-th sampled mode information $\sigma(n\tau) = r(n\tau)$, the probability of mode j being active at time $n\tau+l$, is given by the l-step transition probability $p_{\sigma(n\tau),j}^{(l)}$. For example, consider the case where $\sigma(n\tau) = r(n\tau) = 1$, that is, at time $k = n\tau$, the switched system is in mode 1. In this case $p_{1,1}^{(l)}$ and $p_{1,2}^{(l)}$, which are shown in Figure 6.3, respectively denote probabilities of mode 1 and mode 2 being active $l \in \mathbb{N}_0$ steps after the mode observation instant $n\tau$. As shown in Figure 6.3, it is likely that mode 2 will be active 1 step after the mode observation instant $n\tau$. Note that this information is not taken into account in the control law (6.4), and the feedback gain is set to K_1 until the next mode observation instant $(n+1)\tau$. Therefore, the control performance may be poor, as the feedback gain is kept constant at K_1, when mode 2 is likely to be active.

In order to overcome the above-mentioned issue with the stabilization problem for the case of large mode observation periods, we propose a sampled-mode-dependent time-

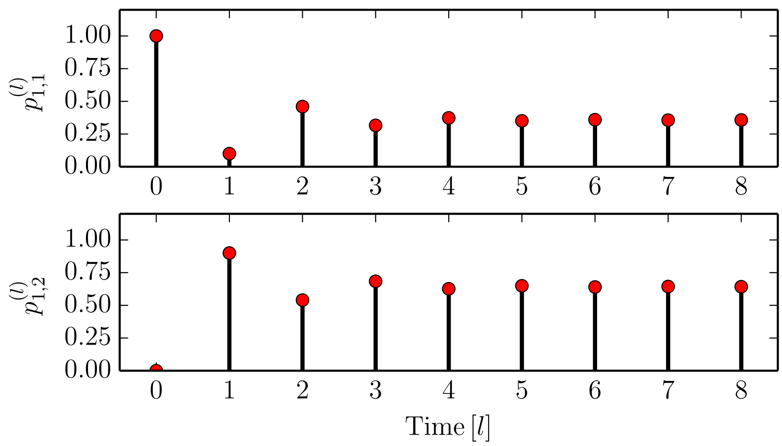

Figure 6.3: Evolution of conditional mode transition probabilities over time

varying feedback control strategy. Specifically, we consider the control law

$$u(k) = K_{\sigma(k)}(k)x(k), \quad k \in \mathbb{N}_0, \tag{6.5}$$

where $K_i(\cdot) : \mathbb{N}_0 \to \mathbb{R}^{m\times n}$, $i \in \mathcal{M}$, are τ-periodic matrix functions, that is, $K_i(k+\tau) = K_i(k)$, $k \in \mathbb{N}_0$, $i \in \mathcal{M}$. Note that with this new control framework, feedback gains $K_i(0), K_i(1), \ldots, K_i(\tau-1)$, can be designed effectively by utilizing the l-step conditional transition probabilities $p_{i,j}^{(l)}$, $j \in \mathcal{M}$.

6.3 Conditions for Second-Moment Asymptotic Stabilization

In this section, we obtain conditions under which the control law (6.5) guarantees second-moment asymptotic stabilization of the switched linear stochastic system (6.1).

Note that the closed-loop system dynamics (6.1), (6.5) can be treated as a switched stochastic system on its own with a bivariate mode signal $\{(r(k), \sigma(k)) \in \mathcal{M}\times\mathcal{M}\}_{k\in\mathbb{N}_0}$. In the following, we present some key results on the probabilistic dynamics of the bivariate process $\{(r(k), \sigma(k)) \in \mathcal{M}\times\mathcal{M}\}_{k\in\mathbb{N}_0}$.

It is important to note that for given $\hat{i}, \hat{j}, i, j \in \mathcal{M}$, the conditional probability $\mathbb{P}[\big(r(k+1), \sigma(k+1)\big) = (\hat{i}, \hat{j}) | \big(r(k), \sigma(k)\big) = (i, j)]$ cannot be unambiguously defined in the case where $\mathbb{P}[\big(r(k), \sigma(k)\big) = (i, j)] = 0$ (Borel's paradox, [93]). For example, at mode observation instants $k \in \{0, \tau, 2\tau, \ldots\}$, we have $\mathbb{P}[\big(r(k), \sigma(k)\big) = (i, j)] = 0$, when $i \neq j$. Furthermore, depending on mode transition probabilities and the initial mode distribution, there may be other time instants k such that $\mathbb{P}[\big(r(k),\sigma(k)\big)=(i,j)]=0$.

In Lemma 6.1 below, we show that under certain conditions on the mode transition probabilities $p_{i,j}\in[0,1]$, $i,j\in\mathcal{M}$, and the initial mode distribution $\nu:\mathcal{M}\to[0,1]$, there exist τ-periodic functions $\gamma_{(i,j),(\hat{i},\hat{j})}:\mathbb{N}_0\to[0,1]$, $i,j,\hat{i},\hat{j}\in\mathcal{M}$, such that

$$\mathbb{P}[\big(r(k+1),\sigma(k+1)\big)=(\hat{i},\hat{j})|\big(r(k),\sigma(k)\big)]=\gamma_{(r(k),\sigma(k)),(\hat{i},\hat{j})}(k),\quad k\in\mathbb{N}_0. \tag{6.6}$$

Note that the τ-periodicity of the functions $\gamma_{(i,j),(\hat{i},\hat{j})}(\cdot)$, $i,j,\hat{i},\hat{j}\in\mathcal{M}$, and subsystem matrices $A_i+B_iK_j(\cdot)$, $i,j\in\mathcal{M}$, is crucial for obtaining our main stability results.

Lemma 6.1. Let $\gamma_{(i,j),(\hat{i},\hat{j})}:\mathbb{N}_0\to[0,1]$, $i,j,\hat{i},\hat{j}\in\mathcal{M}$, be τ-periodic functions defined by

$$\gamma_{(i,j),(\hat{i},\hat{j})}(l\tau+k)=\begin{cases}p_{i,\hat{i}}, & \text{if } \hat{i}=\hat{j} \text{ and } p_{j,i}^{(k)}>0,\\ 0, & \text{otherwise,}\end{cases} \tag{6.7}$$

for $k=\tau-1$, $l\in\mathbb{N}_0$, and

$$\gamma_{(i,j),(\hat{i},\hat{j})}(l\tau+k)=\begin{cases}p_{i,\hat{i}}, & \text{if } j=\hat{j} \text{ and } p_{j,i}^{(k)}>0,\\ 0, & \text{otherwise,}\end{cases} \tag{6.8}$$

for $k\in\{0,1,\ldots,\tau-2\}$, $l\in\mathbb{N}_0$. If $p_{i,j}>0$, $i,j\in\mathcal{M}$, and $\nu_i>0$, $i\in\mathcal{M}$, then it follows that for $i,j,\hat{i},\hat{j}\in\mathcal{M}$, and $k\in\mathbb{N}_0$ such that $\mathbb{P}[r(k)=i,\sigma(k)=j]=0$,

$$\gamma_{(i,j),(\hat{i},\hat{j})}(k)=0, \tag{6.9}$$

moreover,

$$\mathbb{P}[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}|\big(r(k),\sigma(k)\big)]=\gamma_{(r(k),\sigma(k)),(\hat{i},\hat{j})}(k),\quad k\in\mathbb{N}_0. \tag{6.10}$$

Proof. First, let $\{\rho(k)\in\mathcal{M}\times\mathcal{M}\}_{k\in\mathbb{N}_0}$ be a bivariate process defined by

$$\rho(k)\triangleq(r(k),\sigma(k)),\quad k\in\mathbb{N}_0. \tag{6.11}$$

Furthermore, for all $k \in \{0, 1, \ldots, \tau - 1\}$ define

$$\mathcal{N}(k) \triangleq \{(i, j) \in \mathcal{M} \times \mathcal{M} : \mathbb{P}[\rho(k) = (i, j)] > 0\}. \tag{6.12}$$

Let, $\overline{\mathcal{N}}(k) \subset \mathcal{M} \times \mathcal{M}$ denote the complement of the set $\mathcal{N}(k)$, that is,

$$\overline{\mathcal{N}}(k) \triangleq (\mathcal{M} \times \mathcal{M}) \setminus \mathcal{N}(k). \tag{6.13}$$

Now, note that

$$\begin{aligned}\mathbb{P}[\rho(l\tau + k) = (i, j)] &= \mathbb{P}[r(l\tau + k) = i, \sigma(l\tau) = j] \\ &= \mathbb{P}[r(l\tau + k) = i | \sigma(l\tau) = j]\mathbb{P}[\sigma(l\tau) = j] \\ &= p_{j,i}^{(k)} \mathbb{P}[\sigma(l\tau) = j],\end{aligned} \tag{6.14}$$

for $k \in \{0, 1, \ldots, \tau - 1\}$, $l \in \mathbb{N}_0$. By the assumption that $p_{i,j} \in (0, 1)$, $i, j \in \mathcal{M}$, and $\nu_i \in (0, 1)$, $i \in \mathcal{M}$, we obtain $\mathbb{P}[\sigma(l\tau) = j] > 0$, $l \in \mathbb{N}_0$. Therefore, by (6.14), $\mathbb{P}[\rho(l\tau + k) = (i, j)] > 0$ if and only if $p_{j,i}^{(k)} > 0$, and hence, it follows from (6.12) that

$$\mathcal{N}(k) = \{(i, j) \in \mathcal{M} \times \mathcal{M} : p_{j,i}^{(k)} > 0\}. \tag{6.15}$$

By the definition of conditional probability of an event given a random variable [93, 112], it follows that

$$\begin{aligned}&\mathbb{P}[\rho(l\tau + k + 1) = (\hat{i}, \hat{j}) | \rho(l\tau + k)] \\ &\quad = \sum_{(i,j) \in \mathcal{N}(k)} \mathbb{P}[\rho(l\tau + k + 1) = (\hat{i}, \hat{j}) | \rho(l\tau + k) = (i, j)] \mathbb{1}_{[\rho(l\tau+k)=(i,j)]},\end{aligned} \tag{6.16}$$

for $k \in \{0, 1, \ldots, \tau - 1\}$, $l \in \mathbb{N}_0$. Note that $r(l\tau + k + 1) = \sigma(l\tau + k + 1)$ and $\mathbb{P}[r(l\tau + k + 1) = \hat{i} | r(l\tau + k) = i] = p_{i,\hat{i}}$ for $k = \tau - 1$, $l \in \mathbb{N}_0$, because at time instants $0, \tau, 2\tau, \ldots$, the mode is sampled. Hence, by (6.7) and (6.16), for $k = \tau - 1$ and $l \in \mathbb{N}_0$,

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{N}(k)} & \mathbb{P}[\rho(l\tau+k+1) = (\hat{\imath},\hat{\jmath})|\rho(l\tau+k) = (i,j)]\mathbb{1}_{[\rho(l\tau+k)=(i,j)]} \\
&= \begin{cases} \sum_{(i,j)\in\mathcal{N}(k)} p_{i,\hat{\imath}}\mathbb{1}_{[\rho(l\tau+k)=(i,j)]}, & \text{if } \hat{\imath} = \hat{\jmath}, \\ 0, & \text{otherwise}, \end{cases} \\
&= \sum_{(i,j)\in\mathcal{N}(k)} \gamma_{(i,j),(\hat{\imath},\hat{\jmath})}(l\tau+k)\mathbb{1}_{[\rho(l\tau+k)=(i,j)]}. \qquad (6.17)
\end{aligned}$$

Furthermore, since $\sigma(l\tau+k+1) = \sigma(l\tau+k)$, $k \in \{0, 1, \ldots, \tau-2\}$, $l \in \mathbb{N}_0$, it follows from (6.8) and (6.16) that for $k \in \{0, 1, \ldots, \tau-2\}$, $l \in \mathbb{N}_0$.

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{N}(k)} & \mathbb{P}[\rho(l\tau+k+1) = (\hat{\imath},\hat{\jmath})|\rho(l\tau+k) = (i,j)]\mathbb{1}_{[\rho(l\tau+k)=(i,j)]} \\
&= \begin{cases} \sum_{(i,j)\in\mathcal{N}(k)} p_{i,\hat{\imath}}\mathbb{1}_{[\rho(l\tau+k)=(i,j)]}, & \text{if } j = \hat{\jmath}, \\ 0, & \text{otherwise}, \end{cases} \\
&= \sum_{(i,j)\in\mathcal{N}(k)} \gamma_{(i,j),(\hat{\imath},\hat{\jmath})}(l\tau+k)\mathbb{1}_{[\rho(l\tau+k)=(i,j)]}. \qquad (6.18)
\end{aligned}$$

Next, note that (6.7), (6.8), and (6.15) imply (6.9), that is, $\gamma_{(i,j),(\hat{\imath},\hat{\jmath})}(l\tau+k) = 0$ for $(i,j) \in \overline{\mathcal{N}}(k)$, $k \in \{0, 1, \ldots, \tau-1\}$, $l \in \mathbb{N}_0$. Consequently,

$$\sum_{(i,j)\in\overline{\mathcal{N}}(k)} \gamma_{(i,j),(\hat{\imath},\hat{\jmath})}(k)\mathbb{1}_{[\rho(k)=(i,j)]} = 0, \quad k \in \mathbb{N}_0. \qquad (6.19)$$

Since $\mathcal{N}(k) \cup \overline{\mathcal{N}}(k) = \mathcal{M} \times \mathcal{M}$, it follows from (6.16)–(6.19) that

$$\begin{aligned}
\mathbb{P}[r(k+1) = \hat{\imath}, \sigma(k+1) = \hat{\jmath}|\big(r(k),\sigma(k)\big)] &= \mathbb{P}[\rho(k+1) = (\hat{\imath},\hat{\jmath})|\rho(k)] \\
&= \sum_{(i,j)\in\mathcal{M}\times\mathcal{M}} \gamma_{(i,j),(\hat{\imath},\hat{\jmath})}(k)\mathbb{1}_{[\rho(k)=(i,j)]} \\
&= \sum_{i,j\in\mathcal{M}} \gamma_{(i,j),(\hat{\imath},\hat{\jmath})}(k)\mathbb{1}_{[r(k)=i,\sigma(k)=j]} \\
&= \gamma_{(r(k),\sigma(k)),(\hat{\imath},\hat{\jmath})}(k), \quad k \in \mathbb{N}_0, \qquad (6.20)
\end{aligned}$$

which completes the proof. □

Remark 6.2. Note that $\gamma_{(i,j),(\hat{i},\hat{j})}(k)$, $i,j,\hat{i},\hat{j} \in \mathcal{M}$, defined in (6.7), (6.8) cannot be considered as transition probabilities between the states of the bivariate process $\{(r(k),\sigma(k)) \in \mathcal{M}\times\mathcal{M}\}_{k\in\mathbb{N}_0}$. Specifically, Lemma 6.1 shows that $\gamma_{(i,j),(\hat{i},\hat{j})}(k)$ can *only* be considered as the transition probability from state (i,j) to state $(\hat{i},\hat{j})$ of the bivariate process $\{(r(k),\sigma(k)) \in \mathcal{M}\times\mathcal{M}\}$, if $\mathbb{P}[r(k)=i,\sigma(k)=j] \neq 0$. On the other hand, when $\mathbb{P}[r(k)=i,\sigma(k)=j]=0$, we have $\gamma_{(i,j),(\hat{i},\hat{j})}(k)=0$, $\hat{i},\hat{j}\in\mathcal{M}$.

In the following, we utilize the result presented in Lemma 6.1 to obtain necessary and sufficient conditions for second-moment asymptotic stability of a class of switched linear stochastic control systems (6.1), (6.5) with nonzero mode transition probabilities and random initial mode. Specifically, we consider the case where the mode signal is characterized by transition probabilities and initial distribution that satisfy $p_{i,j}>0$, $i,j\in\mathcal{M}$, and $v_i>0$, $i\in\mathcal{M}$, respectively.

Theorem 6.1. Consider the switched linear stochastic control system (6.1), (6.5) with a mode signal characterized by transition probabilities and initial distribution that satisfy $p_{i,j}>0$, $i,j\in\mathcal{M}$, and $\nu_i>0$, $i\in\mathcal{M}$, respectively. Let $\Lambda:\mathbb{N}_0\to\mathbb{R}^{M^2n^2\times M^2n^2}$ be a τ-periodic matrix function given in block matrix form as

$$\Lambda(k) \triangleq \begin{bmatrix} \Lambda_{1,1}(k) & \cdots & \Lambda_{1,M^2}(k) \\ \vdots & \ddots & \vdots \\ \Lambda_{M^2,1}(k) & \cdots & \Lambda_{M^2,M^2}(k) \end{bmatrix}, \quad k\in\mathbb{N}_0, \tag{6.21}$$

where $\Lambda_{(\hat{i}-1)M+\hat{j},(i-1)M+j}(k) \triangleq \gamma_{(i,j),(\hat{i},\hat{j})}(k)\big(A_i+B_iK_j(k)\big)\otimes\big(A_i+B_iK_j(k)\big)$ for $i,j,\hat{i},\hat{j}\in\mathcal{M}$ and $\gamma_{(i,j),(\hat{i},\hat{j})}(k)$ is given by (6.7) and (6.8). Furthermore, let

$$\Phi \triangleq \Lambda(\tau-1)\Lambda(\tau-2)\cdots\Lambda(1)\Lambda(0). \tag{6.22}$$

Then the zero solution $x(k)\equiv 0$ of the closed-loop system (6.1) and (6.5) is second-moment asymptotically stable if and only if all eigenvalues of the matrix $\Phi\in\mathbb{R}^{M^2n^2\times M^2n^2}$ are inside the unit circle of the complex plane.

The proof of Theorem 6.1 is based on stability analysis for a linear periodic system, the states of which are closely related to the second moment $\mathbb{E}[\|x(k)\|^2]$. Specifically, by using an approach similar to the one employed in [58, 113, 114], we analyze the dynamics that govern the evolution of the state covariance (given in vector form by $\mathbb{E}[x(k) \otimes x(k)]$). In our analysis, the dynamics of the state covariance is affected by both the actual mode and its sampled version.

Proof. First, define

$$y(k) \triangleq \mathbb{E}[\bar{r}(k) \otimes \bar{\sigma}(k) \otimes x(k) \otimes x(k)], \tag{6.23}$$

where $\bar{r}(\cdot) : \mathbb{N}_0 \to \mathbb{R}^M$, $\bar{\sigma}(\cdot) : \mathbb{N}_0 \to \mathbb{R}^M$ are vector-functions given by

$$\bar{r}_i(k) \triangleq \mathbb{1}_{[r(k)=i]}, \quad i \in \mathcal{M}, \quad k \in \mathbb{N}_0, \tag{6.24}$$

$$\bar{\sigma}_i(k) \triangleq \mathbb{1}_{[\sigma(k)=i]}, \quad i \in \mathcal{M}, \quad k \in \mathbb{N}_0. \tag{6.25}$$

Our goal is to obtain a difference equation that characterizes the evolution of $y(k)$. Now, let $\bar{A}_{i,j} : \mathbb{N}_0 \to \mathbb{R}^{n^2 \times n^2}$, $i, j \in \mathcal{M}$, be matrix functions defined by

$$\bar{A}_{i,j}(k) \triangleq \big(A_i + B_i K_j(k)\big) \otimes \big(A_i + B_i K_j(k)\big),\ k \in \mathbb{N}_0. \tag{6.26}$$

Note that $\bar{A}_{i,j}(\cdot)$, $i, j \in \mathcal{M}$, are τ-periodic matrix functions, because $K_j(\cdot)$, $j \in \mathcal{M}$, are τ-periodic. Now, for each pair of modes $\hat{i}, \hat{j} \in \mathcal{M}$,

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} x(k+1) \otimes x(k+1)] \\
&\quad = \mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} \big(A_{r(k)} + B_{r(k)} K_{\sigma(k)}(k)\big) x(k) \otimes \big(A_{r(k)} + B_{r(k)} K_{\sigma(k)}(k)\big) x(k)] \\
&\quad = \mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} \bar{A}_{r(k),\sigma(k)}(k) \big(x(k) \otimes x(k)\big)] \\
&\quad = \mathbb{E}[\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} \bar{A}_{r(k),\sigma(k)}(k) \big(x(k) \otimes x(k)\big) | \mathcal{H}_k]],\ k \in \mathbb{N}_0,
\end{aligned} \tag{6.27}$$

where $\mathcal{H}_k$ denotes the σ-algebra generated by the random variables $x(k)$, $r(k)$, and $\sigma(k)$. It follows from (6.27) that

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} x(k+1) \otimes x(k+1)] \\
&\quad = \mathbb{E}[\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} | \mathcal{H}_k] \bar{A}_{r(k),\sigma(k)}(k) \big(x(k) \otimes x(k)\big)],\ \hat{i}, \hat{j} \in \mathcal{M},\ k \in \mathbb{N}_0.
\end{aligned} \tag{6.28}$$

Now, note that $r(k+1) = \sigma(k+1)$, $k \in \{n\tau - 1 : n \in \mathbb{N}_0\}$. Since $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ is a Markov process, the random variables $r(k+1)$ and $\sigma(k+1)$ are independent of the random variable $x(k)$ given $r(k)$ for $k \in \{n\tau - 1 : n \in \mathbb{N}_0)$. On the other hand $\sigma(k+1) = \sigma(k)$ for $k \in \mathbb{N}_0 \setminus \{n\tau - 1 : n \in \mathbb{N}_0)$. Therefore, for $k \in \mathbb{N}_0 \setminus \{n\tau - 1 : n \in \mathbb{N}_0)$, $r(k+1)$ and $\sigma(k+1)$ are independent of the random variable $x(k)$ given $r(k)$ and $\sigma(k)$. Consequently,

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} x(k+1) \otimes x(k+1)] \\ &\quad = \mathbb{E}[\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} | \big(r(k), \sigma(k)\big)] \bar{A}_{r(k),\sigma(k)}(k) \big(x(k) \otimes x(k)\big)], \quad \hat{i}, \hat{j} \in \mathcal{M}, \quad (6.29) \end{aligned}$$

for $k \in \mathbb{N}_0$. Now by Lemma 6.1,

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]} x(k+1) \otimes x(k+1)] \\ &\quad = \mathbb{E}[\gamma_{(r(k),\sigma(k)),(\hat{i},\hat{j})}(k) \bar{A}_{r(k),\sigma(k)}(k) \big(x(k) \otimes x(k)\big)] \\ &\quad = \sum_{i,j} \gamma_{(i,j),(\hat{i},\hat{j})}(k) \bar{A}_{i,j}(k) \mathbb{E}[\mathbb{1}_{[r(k)=i,\sigma(k)=j]} x(k) \otimes x(k)], \quad \hat{i}, \hat{j} \in \mathcal{M}, \quad k \in \mathbb{N}_0. \quad (6.30) \end{aligned}$$

It then follows from (6.23) and (6.30) that

$$y(k+1) = \Lambda(k) y(k), \quad k \in \mathbb{N}_0, \quad (6.31)$$

where $\Lambda(\cdot) : \mathbb{N}_0 \to \mathbb{R}^{M^2 n^2 \times M^2 n^2}$ defined in (6.21) is a τ-periodic matrix function. Note that

$$y((l+1)\tau) = \Phi y(l\tau), \quad l \in \mathbb{N}_0, \quad (6.32)$$

where $\Phi \in \mathbb{R}^{M^2 n^2 \times M^2 n^2}$ defined in (6.22) is in fact the monodromy matrix associated with the discrete-time deterministic linear periodic system (6.31). Based on discrete-time version of Floquet theory [111], we study asymptotic behavior of the solutions of (6.31) through the associated monodromy matrix Φ. Specifically, $\lim_{k\to\infty} y(k) = 0$, if and only if all eigenvalues of the monodromy matrix Φ are inside the unit circle of the complex plane. Note that for $x \in \mathbb{R}^n$, $\bar{r}, \bar{\sigma} \in \mathbb{R}^M$, such that $\bar{r} \neq \bar{\sigma}$, we have $\Lambda(0)\mathbb{E}[\bar{r} \otimes \bar{\sigma} \otimes x \otimes x] = 0$, and hence $\Phi\mathbb{E}[\bar{r} \otimes \bar{\sigma} \otimes x \otimes x] = 0$. On the other hand, for $x \in \mathbb{R}^n$, $\bar{r}, \bar{\sigma} \in \mathbb{R}^M$, such that $\bar{r} = \bar{\sigma}$, $\lim_{k\to\infty} \Phi^k \mathbb{E}[\bar{r} \otimes \bar{\sigma} \otimes x \otimes x] = 0$ if all eigenvalues of the monodromy matrix Φ are inside the unit circle of the complex plane.

Now, as a consequence of (6.23), the zero solution is second-moment asymptotically stable (i.e., $\lim_{k\to\infty}\mathbb{E}[\|x(k)\|^2]=0$) if and only if $\lim_{k\to\infty}y(k)=0$. Therefore, we have $\lim_{k\to\infty}\mathbb{E}[\|x(k)\|^2]=0$ if and only if all eigenvalues of the monodromy matrix Φ are inside the unit circle of the complex plane. □

Theorem 6.1 shows that the stability of the zero solution of the closed-loop switched stochastic control system (6.1), (6.5) can be deduced through the eigenvalues of the matrix $\Phi\in\mathbb{R}^{M^2n^2\times M^2n^2}$ (given in (6.22)), which depends not only on the mode transition probabilities and subsystem dynamics but also on the mode observation period $\tau\in\mathbb{N}$.

Next, we present alternative stabilization conditions by considering quadratic Lyapunov function with periodic coefficients that depend on both the actual mode signal $r(\cdot)$ and its sampled version $\sigma(\cdot)$.

Theorem 6.2. Consider the switched linear stochastic control system (6.1), (6.5) with a mode signal characterized by transition probabilities and initial distribution that satisfy $p_{i,j}>0$, $i,j\in\mathcal{M}$, and $\nu_i>0$, $i\in\mathcal{M}$, respectively. If there exist τ-periodic matrix functions $R_{i,j}(\cdot),Q_{i,j}(\cdot)>0$, $i,j\in\mathcal{M}$, such that

$$\begin{aligned}0=&\sum_{\tilde{i},\tilde{j}\in\mathcal{M}}\gamma_{(i,j),(\tilde{i},\tilde{j})}(k)\hat{A}^{\mathrm{T}}_{i,j}(k)R_{\tilde{i},\tilde{j}}(k+1)\hat{A}_{i,j}(k)\\&-R_{i,j}(k)+Q_{i,j}(k),\quad k\in\{0,1,\ldots\tau-1\},\quad i,j\in\mathcal{M},\end{aligned}\tag{6.33}$$

where $\hat{A}_{i,j}(k)\triangleq A_i+B_iK_j(k)$, and $\gamma_{(i,j),(\tilde{i},\tilde{j})}(k)$ is defined by (6.7), (6.8), then the zero solution $x(k)\equiv 0$ of the closed-loop system (6.1), (6.5) is second-moment asymptotically stable.

Proof. First, let $V(x(k),k)\triangleq x^{\mathrm{T}}(k)R_{r(k),\sigma(k)}(k)x(k)$, $k\in\mathbb{N}_0$. We now show that the Lyapunov function candidate decreases in expectation. It follows from (6.1) and (6.5) that

$$\begin{aligned}&\mathbb{E}[V(x(k+1),k+1)]\\&\quad=\mathbb{E}[x^{\mathrm{T}}(k)\hat{A}^{\mathrm{T}}_{r(k),\sigma(k)}(k)R_{r(k+1),\sigma(k+1)}(k+1)\hat{A}_{r(k),\sigma(k)}(k)x(k)],\\&\quad=\mathbb{E}[\mathbb{E}[x^{\mathrm{T}}(k)\hat{A}^{\mathrm{T}}_{r(k),\sigma(k)}(k)R_{r(k+1),\sigma(k+1)}(k+1)\hat{A}_{r(k),\sigma(k)}(k)x(k)|\mathcal{H}_k]],\end{aligned}\tag{6.34}$$

where $\mathcal{H}_k$ denotes the σ-algebra generated by the random variables $x(k)$, $r(k)$, and $\sigma(k)$. Note that the random variables $x(k)$, $r(k)$, and $\sigma(k)$ are $\mathcal{H}_k$-measurable, and hence, it follows that

$$\begin{aligned}&\mathbb{E}[V(x(k+1),k+1)]\\&\quad=\mathbb{E}[x^{\mathrm{T}}(k)\hat{A}^{\mathrm{T}}_{r(k),\sigma(k)}(k)\mathbb{E}[R_{r(k+1),\sigma(k+1)}(k+1)|\mathcal{H}_k]\hat{A}_{r(k),\sigma(k)}(k)x(k)].\end{aligned}\tag{6.35}$$

Note also that $\mathbb{E}[(r(k+1),\sigma(k+1))|\mathcal{H}_k]=\mathbb{E}[(r(k+1),\sigma(k+1))|(r(k),\sigma(k))]$. Consequently, it follows from Lemma 6.1 that

$$\begin{aligned}\mathbb{E}[R_{r(k+1),\sigma(k+1)}(k+1)|\mathcal{H}_k]&=\mathbb{E}[R_{r(k+1),\sigma(k+1)}(k+1)|\big(r(k),\sigma(k)\big)]\\&=\sum_{\hat{i},\hat{j}\in\mathcal{M}}\mathbb{E}[\mathbb{1}_{[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}]}|\big(r(k),\sigma(k)\big)]R_{\hat{i},\hat{j}}(k+1)\\&=\sum_{\hat{i},\hat{j}\in\mathcal{M}}\mathbb{P}[r(k+1)=\hat{i},\sigma(k+1)=\hat{j}|\big(r(k),\sigma(k)\big)]R_{\hat{i},\hat{j}}(k+1)\\&=\sum_{\hat{i},\hat{j}\in\mathcal{M}}\gamma_{(r(k),\sigma(k)),(\hat{i},\hat{j})}R_{\hat{i},\hat{j}}(k+1),\quad k\in\mathbb{N}_0.\end{aligned}\tag{6.36}$$

Now, by substituting (6.36) into (6.35), we obtain

$$\begin{aligned}&\mathbb{E}[V(x(k+1),k+1)]\\&\quad=\mathbb{E}[x^{\mathrm{T}}(k)\big(\sum_{\hat{i},\hat{j}\in\mathcal{M}}\gamma_{(r(k),\sigma(k)),(\hat{i},\hat{j})}\hat{A}^{\mathrm{T}}_{r(k),\sigma(k)}(k)R_{\hat{i},\hat{j}}(k+1)\hat{A}_{r(k),\sigma(k)}(k)\big)x(k)].\end{aligned}\tag{6.37}$$

Moreover, as a consequence of (6.33),

$$\mathbb{E}[V(x(k+1),k+1)]-\mathbb{E}[V(x(k),k)]=-\mathbb{E}[x^{\mathrm{T}}(k)Q_{r(k),\sigma(k)}(k)x(k)].\tag{6.38}$$

It follows from the positive definiteness of the matrices $Q_{r(k),\sigma(k)}(k)$, $k\in\mathbb{N}_0$, that the sequence $\{\mathbb{E}[V(x(k),k)]\}_{k\in\mathbb{N}_0}$ is monotone decreasing with respect to time k. Furthermore, $V(x(k),k)$ is nonnegative for all $k\in\mathbb{N}_0$, and hence, $\mathbb{E}[V(x(k),k)]\geq 0$, $k\in\mathbb{N}_0$. It follows from monotone convergence theorem that

$$\lim_{k\to\infty}\mathbb{E}[V(x(k),k)]=0.\tag{6.39}$$

Now note that

$$\mathbb{E}[\|x(k)\|^2] \leq \underline{r}\mathbb{E}[V(x(k),k)], \quad k \in \mathbb{N}_0, \tag{6.40}$$

where $\underline{r} \triangleq \min\{\lambda_{\min}(R_{i,j}(k)) \ : \ i,j \in \mathcal{M}, \ k \in \{0,\ldots,\tau-1\}\}$. Since $R_{i,j}(k) > 0$, $k \in \{0,\ldots,\tau-1\}$, we have $\underline{r} > 0$. Thus, it follows from (6.39) and (6.40) that $\mathbb{E}[\|x(k)\|^2] \to 0$ as $k \to \infty$, which completes the proof. □

Remark 6.3. Note that Theorem 6.2 requires the initial mode to be randomly distributed, that is, $\nu_i > 0$, $i \in \mathcal{M}$. This requirement is relaxed in Theorem 6.3 below. Specifically, the result presented in Theorem 6.3 can also be used for assessing second-moment asymptotic stability of the switched stochastic control system (6.1), (6.5) with a deterministic initial mode r_0, such that $\nu_{r_0} = 1$, $\nu_i = 0$, $i \neq r_0$.

Theorem 6.3. Consider the switched linear stochastic control system (6.1), (6.5) with a mode signal characterized by transition probabilities that satisfy $p_{i,j} > 0$, $i,j \in \mathcal{M}$. If there exist τ-periodic matrix functions $R_{i,j}(\cdot), Q_{i,j}(\cdot) > 0$, $i,j \in \mathcal{M}$, such that (6.33) holds, then the zero solution $x(k) \equiv 0$ of the closed-loop system (6.1), (6.5) is second-moment asymptotically stable.

Proof. Let $\tilde{x}(k) \triangleq x(k+\tau)$, $\tilde{r}(k) \triangleq r(k+\tau)$, $\tilde{\sigma}(k) \triangleq \sigma(k+\tau)$, $k \in \mathbb{N}_0$. Since $K_i(\cdot)$, $i \in \mathcal{M}$, are τ-periodic, it follows from (6.1) and (6.5) that

$$\tilde{x}(k+1) = A_{\tilde{r}(k)}\tilde{x}(k) + B_{\tilde{r}(k)}\tilde{u}(k), \quad k \in \mathbb{N}_0, \tag{6.41}$$

$$\tilde{u}(k) \triangleq K_{\tilde{\sigma}(k)}(k)\tilde{x}(k). \tag{6.42}$$

Note that the random variable $\tilde{r}(0)$ denotes the initial mode of the switched stochastic control system (6.41), (6.42). Furthermore, $\tilde{\nu}_i \triangleq \mathbb{P}[\tilde{r}(0) = i] = p^{(\tau)}_{r(0),i} > 0$, $i \in \mathcal{M}$, because $p_{i,j} \in (0,1), i,j \in \mathcal{M}$. Now it follows from Theorem 6.2 that for any initial state $\tilde{x}(0) = \tilde{x}_0 \in \mathbb{R}^n$, the zero solution $x(k) \equiv 0$ of the switched stochastic control system (6.41), (6.42) is second-moment asymptotically stable, that is, $\lim_{k\to\infty}\mathbb{E}[\|\tilde{x}(k)\|^2] = 0$. As a direct consequence of the definition of $\tilde{x}(k)$, we obtain $\lim_{k\to\infty}\mathbb{E}[\|x(k)\|^2] = 0$, which completes the proof. □

The second-moment asymptotic stability of the switched linear stochastic system (6.1) under the control law (6.5) can be analyzed through the results presented in Theorems 6.1, 6.2, and 6.3, when the feedback gain matrices $K_i(k)$, $k \in \{0, 1, \ldots, \tau-1\}$, $i \in \mathcal{M}$, are already known. On the other hand, we often need to *find* feedback gains such that the proposed control law (6.5) with those gains achieves second-moment asymptotic stabilization. In Corollary 6.1 below, we present sufficient stabilization conditions, which are well suited for finding stabilizing feedback gains through numerical methods.

Corollary 6.1. Consider the switched linear stochastic system (6.1) with a mode signal characterized by transition probabilities that satisfy $p_{i,j} > 0$, $i, j \in \mathcal{M}$. If there exist τ-periodic matrix functions $\tilde{S}_j(\cdot) > 0$, $L_j(\cdot) \in \mathbb{R}^{m \times n}$, $j \in \mathcal{M}$, such that

$$0 > \sum_{\tilde{i},\tilde{j} \in \mathcal{M}} \gamma_{(i,j),(\tilde{i},\tilde{j})}(k) \tilde{A}_{i,j}^{\mathrm{T}}(k) \tilde{S}_{\tilde{j}}^{-1}(k+1) \tilde{A}_{i,j}(k) - \tilde{S}_j(k), \tag{6.43}$$

for all $k \in \{0, 1, \ldots \tau - 1\}$, and $i, j \in \mathcal{M}$, where $\tilde{A}_{i,j}(k) \triangleq A_i \tilde{S}_j(k) + B_i L_j(k)$, and $\gamma_{(i,j),(\tilde{i},\tilde{j})}(k)$ is defined by (6.7), (6.8), then the feedback control law (6.5) with the feedback gain matrix $K_{\sigma(k)}(k) \triangleq L_{\sigma(k)}(k) \tilde{S}_{\sigma(k)}^{-1}(k)$ guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (6.1), (6.5) is second-moment asymptotically stable.

Proof. The result is a direct consequence of Theorem 6.3 with τ-periodic matrix functions $R_{i,j}(\cdot) > 0$ and $Q_{i,j}(\cdot) > 0$, $i, j \in \mathcal{M}$, given by

$$R_{i,j}(k) = \tilde{S}_j^{-1}(k), \quad i, j \in \mathcal{M}, \tag{6.44}$$

$$Q_{i,j}(k) = \tilde{S}_j^{-1}(k) - \sum_{\tilde{i},\tilde{j} \in \mathcal{M}} \gamma_{(i,j),(\tilde{i},\tilde{j})}(k) \hat{A}_{i,j}^{\mathrm{T}}(k) \tilde{S}_{\tilde{j}}^{-1}(k+1) \hat{A}_{i,j}(k), \quad i, j \in \mathcal{M}, \tag{6.45}$$

where $\hat{A}_{i,j}(k) \triangleq A_i + B_i K_j(k)$, $k \in \{0, 1, \ldots \tau - 1\}$. □

Corollary 6.1 shows that if (6.43) can be verified for τ-periodic matrix functions $\tilde{S}_j(\cdot) > 0$, $L_j(\cdot) \in \mathbb{R}^{m \times n}$, for each mode $j \in \mathcal{M}$, then second-moment asymptotic stabilization of the zero solution is guaranteed under the control law (6.5) with feedback gain matrix $K_{\sigma(k)}(k) \triangleq L_{\sigma(k)}(k) \tilde{S}_{\sigma(k)}^{-1}(k)$.

Remark 6.4. For numerical verification of condition (6.43) in Corollary 6.1, we employ a numerical technique that involves *linear matrix inequalities* (see [94,95]). Specifically, we use Schur complements (see [115]) to transform condition (6.43) into matrix inequalities given by

$$0 < \begin{bmatrix} \tilde{S}_j(k) & T_{i,j}^{1\mathrm{T}}(k) & \cdots & T_{i,j}^{M^2\mathrm{T}}(k) \\ T_{i,j}^{1}(k) & U^1(k+1) & & 0 \\ \vdots & & \ddots & \\ T_{i,j}^{M^2}(k) & 0 & & U^{M^2}(k+1) \end{bmatrix}, \tag{6.46}$$

for $i, j \in \mathcal{M}$, $k \in \{0, 1, \ldots \tau - 1\}$, where

$$U^{(\hat{i}-1)M+\hat{j}}(k) \triangleq \tilde{S}_{\hat{j}}(k), \tag{6.47}$$

$$T_{i,j}^{(\hat{i}-1)M+\hat{j}}(k) \triangleq \sqrt{\gamma_{(i,j),(\hat{i},\hat{j})}(k)}\tilde{A}_{i,j}(k), \hat{i}, \hat{j} \in \mathcal{M}, \tag{6.48}$$

with $\tilde{A}_{i,j}(k) \triangleq A_i\tilde{S}_j(k) + B_iL_j(k)$. Note that the matrix inequalities (6.46) are *linear* in $\tilde{S}_j(\cdot) > 0$, $L_j(\cdot) \in \mathbb{R}^{m\times n}$, $j \in \mathcal{M}$. Furthermore, note that since $\tilde{S}_j(\cdot)$, $j \in \mathcal{M}$, are τ-periodic matrix functions, $U^{(\hat{i}-1)M+\hat{j}}(\tau) = U^{(\hat{i}-1)M+\hat{j}}(0)$, $\hat{i}, \hat{j} \in \mathcal{M}$. In this study, we use numerical methods to search for positive-definite matrices $\tilde{S}_j(k) > 0$, $k \in \{0, 1, \ldots, \tau - 1\}$, $i, j \in \mathcal{M}$, and matrices $L_j(k) \in \mathbb{R}^{m\times n}$, $k \in \{0, 1, \ldots, \tau - 1\}$, $i, j \in \mathcal{M}$, that satisfy the linear matrix inequalities (6.46).

Remark 6.5. Investigation of the applicability of our results to large-scale systems is important. To this end, we note that in large-scale systems both the state size $n \in \mathbb{N}$ and the number of modes $M \in \mathbb{N}$ may be large. Depending on how large these values are, our numerical methods characterized through linear matrix inequalities in Remark 6.4 may require long computation time. It is crucial to investigate computational complexity of these numerical methods. Note that as explained in Remark 6.4, our numerical method is based on checking feasible solutions to linear matrix inequalities. Linear matrix inequalities can be accurately solved in an efficient manner [116, 117]. Specifically, the worst-case computational complexity of the method presented in [116] is given as $O(ab^3)$ where $a \in \mathbb{N}$ denotes the number of total row size of the linear matrix inequalities and $b \in \mathbb{N}$ denotes the number of decision variables. Now, note that the row size of the linear matrix inequality (6.46) is given by nM^2; furthermore, there are τM^2 of these inequalities. On the other hand, since we look for positive-definite matrices, we consider linear matrix inequlities of the form $\tilde{S}_j(k) > 0$, $j \in \mathcal{M}$, $k \in \{1, 2, \ldots, \tau\}$, which have total row size $\tau n M$. Hence, $a = \tau M^2 n M^2 + \tau n M = \tau n M^4 + \tau n M$. On the other hand, decision variables are entries of $\tilde{S}_j(k) > 0$, $L_j(k) \in \mathbb{R}^{m \times n}$, $j \in \mathcal{M}$, $k \in \{1, 2, \ldots, \tau\}$. Therefore, $b = \tau n^2 M + \tau m n M$, where $m \in \mathbb{N}$ is the control input size. Assuming $m \leq n$, the worst-case computational complexity of our problem is then given by $O(ab^3) = O(\tau^4 n^7 M^7)$. This shows that the computatioal complexity grows similarly for state size $n \in \mathbb{N}$ and the number of modes $M \in \mathbb{N}$.

6.4 Illustrative Numerical Example

We now demonstrate our results with an illustrative numerical example. Specifically, consider the switched linear stochastic system (6.1) with $M = 2$ modes described by the subsystems matrices

$$A_1 = \begin{bmatrix} 0.01 & 1 \\ 1.5 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.4 & 1.2 \\ 1 & 0.01 \end{bmatrix},$$

$B_1 = [0,\, 1]^{\mathrm{T}}$, and $B_2 = [-1,\, 0]^{\mathrm{T}}$. The mode signal $\{r(k) \in \mathcal{M} \triangleq \{1, 2\}\}_{k \in \mathbb{N}_0}$ of the switched system is assumed to be a discrete-time Markov chain characterized by the transition probabilities $p_{1,1} = 0.1$, $p_{1,2} = 0.9$ and $p_{2,1} = p_{2,2} = 0.5$ (see Figure 6.1 for the corresponding transition diagram).

The mode signal of the switched system is assumed to be sampled at every 5 time steps, that is, $\tau = 5$. Note that positive-definite τ-periodic matrix functions $\tilde{S}_j(\cdot) > 0$,

$j \in \mathcal{M} \triangleq \{1, 2\}$, with values

$$\tilde{S}_1(0) = \begin{bmatrix} 15.062 & 0.395 \\ 0.395 & 2.621 \end{bmatrix}, \tilde{S}_2(0) = \begin{bmatrix} 3.633 & 0.961 \\ 0.961 & 45.827 \end{bmatrix},$$
$$\tilde{S}_1(1) = \begin{bmatrix} 2.662 & -2.754 \\ -2.754 & 3.03 \end{bmatrix}, \tilde{S}_2(1) = \begin{bmatrix} 1.849 & -2.241 \\ -2.241 & 3.701 \end{bmatrix},$$
$$\tilde{S}_1(2) = \begin{bmatrix} 3.069 & -2.073 \\ -2.073 & 3.459 \end{bmatrix}, \tilde{S}_2(2) = \begin{bmatrix} 3.7 & -2.683 \\ -2.683 & 3.111 \end{bmatrix},$$
$$\tilde{S}_1(3) = \begin{bmatrix} 3.46 & -1.97 \\ -1.97 & 4.766 \end{bmatrix}, \tilde{S}_2(3) = \begin{bmatrix} 3.155 & -1.461 \\ -1.461 & 4.841 \end{bmatrix},$$
$$\tilde{S}_1(4) = \begin{bmatrix} 4.805 & 0.242 \\ 0.242 & 3.865 \end{bmatrix}, \tilde{S}_2(4) = \begin{bmatrix} 4.873 & 0.094 \\ 0.094 & 3.847 \end{bmatrix},$$

and τ-periodic row vectors functions $L_j(\cdot) : \mathbb{N}_0 \to \mathbb{R}^{1\times 2}$, $j \in \mathcal{M} \triangleq \{1, 2\}$, with values

$$L_1(0) = \begin{bmatrix} -23.082 & -2.869 \end{bmatrix}, L_2(0) = \begin{bmatrix} 4.812 & 56.236 \end{bmatrix},$$
$$L_1(1) = \begin{bmatrix} -1.502 & 2.035 \end{bmatrix}, L_2(1) = \begin{bmatrix} -0.701 & 1.058 \end{bmatrix},$$
$$L_1(2) = \begin{bmatrix} -1.084 & 1.472 \end{bmatrix}, L_2(2) = \begin{bmatrix} -2.153 & 2.799 \end{bmatrix},$$
$$L_1(3) = \begin{bmatrix} -2.307 & 3.631 \end{bmatrix}, L_2(3) = \begin{bmatrix} -1.714 & 2.737 \end{bmatrix},$$
$$L_1(4) = \begin{bmatrix} 1.817 & 0.358 \end{bmatrix}, L_2(4) = \begin{bmatrix} 1.754 & 0.58 \end{bmatrix},$$

satisfy (6.43). Therefore, it follows from Corollary 6.1 that the proposed control law (6.5) with sampled-mode-dependent τ-periodic feedback gains $K_i(k) \triangleq L_i(k)\tilde{S}_i^{-1}(k)$, $i \in \mathcal{M} \triangleq \{1, 2\}$, $k \in \mathbb{N}_0$, guarantees second-moment asymptotic stability of the closed-loop system (6.1), (6.5).

Sample paths of the state $x(k)$ and the control input $u(k)$ (obtained with initial conditions $x(0) = [1,\ -1]^{\mathrm{T}}$ and $r(0) = 1$) are shown in Figures 6.1 and 6.2. Moreover, Figure 6.3 shows sample paths of the actual mode $r(k)$ and its sampled version $\sigma(k)$.

We can see in Figure 6.3 that the sampled version of the mode signal $\sigma(k)$ is not a

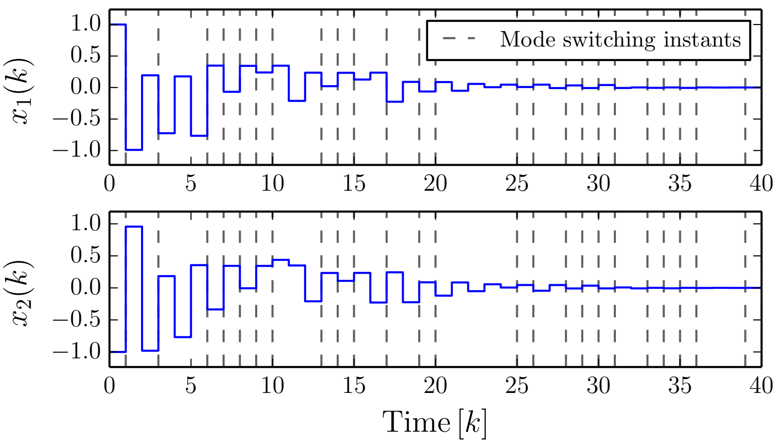

Figure 6.1: State trajectory versus time

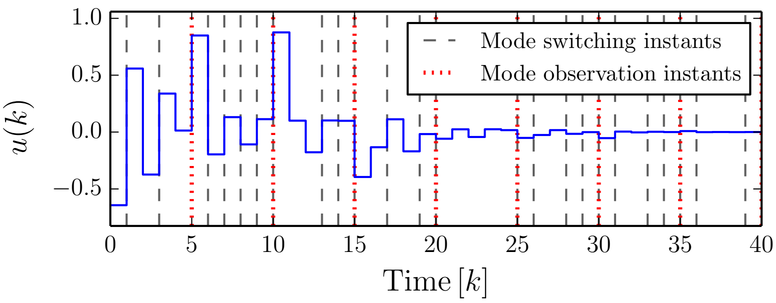

Figure 6.2: Control input versus time

good representation of the actual mode signal $r(k)$ due to frequent mode switches and rare mode observations. However, our new *sampled-mode-dependent time-varying* feedback control strategy characterized in (6.5) takes possible mode transitions between mode observation time instants into account. As it is indicated in Figures 6.1–6.3, the proposed control law achieves stabilization of the state.

Furthermore, we obtain 5000 sample paths of the state trajectory $x(k)$ and estimate the second moment $\mathbb{E}[\|x(k)\|^2]$ of the state by

$$\mathbb{E}[\|x(k)\|^2] \approx \frac{1}{5000}\sum_{i=1}^{5000}\|x(k)\|^2_{(i)}, \quad k \in \mathbb{N}_0, \tag{6.49}$$

where $\|x(k)\|_{(i)}$ denotes the norm of state $x(k)$ of the ith sample path at time k. Figure 6.4 shows the numerically approximated second moment, which converges to 0.

Note that at mode observation instants, the active operation mode of the switched system is known by the controller with certainty; however, between mode observation

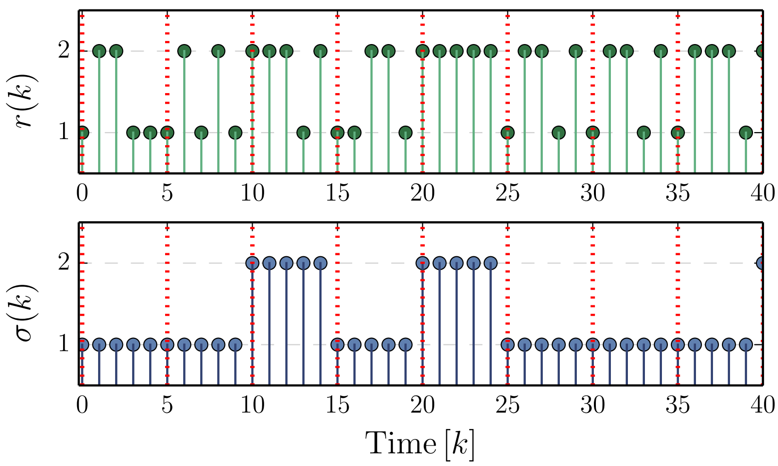

Figure 6.3: Actual mode signal $r(k)$ and sampled mode signal $\sigma(k)$

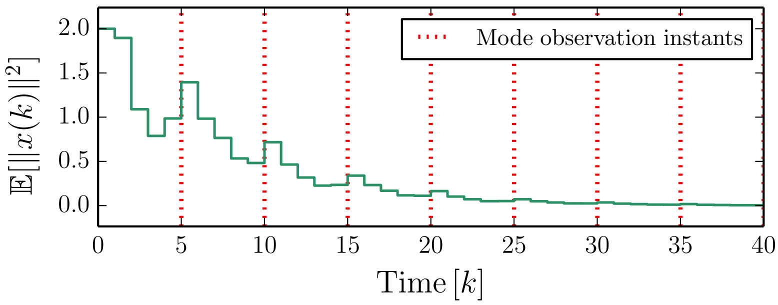

Figure 6.4: Second moment trajectory versus time

instants only the probability distribution of the possibly active mode is known. Hence, the control performance may start to deteriorate over time as the certainty of the mode information decreases, until the next mode observation instant, where perfect operation mode information becomes available once again. Note that although the second moment of the state increases for a few time steps before each mode sampling instant, the controller is effective enough to compensate the uncertainty on the mode information so that $\mathbb{E}[\|x(k)\|^2]$ takes a lower value at each mode observation instant, and eventually it converges to 0 (see Figure 6.4).

6.5 Conclusion

In this section, we investigated second-moment asymptotic stabilization of discrete-time switched linear stochastic systems. Specifically, we developed a control law that guarantees stabilization even when only periodically observed version of the active operation mode of the switched system is available for control purposes. Our proposed control law, which incorporates sampled-mode-dependent time-varying feedback gains, is effective for compensating the uncertainty on the information about the operation mode of the system between mode observation instants. We utilized the periodicity induced in the closed-loop system dynamics due to periodic mode observations, and employed discrete-time Floquet theory to obtain necessary and sufficient conditions for second-moment asymptotic stabilization of the zero solution. Furthermore, we used Lyapunov-like functions with periodic coefficients to obtain alternative stabilization conditions, which we then employed for designing feedback gains.

Chapter 7

Stabilizing Discrete-Time Switched Linear Stochastic Systems Using Periodically Available Imprecise Mode Information

7.1 Introduction

In Chapters 3–5, we considered stabilization problems for continuous-time switched stochastic systems under sampled mode information. Furthermore, in Chapter 6, we explored the sampled-mode feedback control problem for the discrete-time case.

In Chapters 3 and 6, the mode signal of a switched stochastic system is assumed to be periodically observed. Note that the control frameworks developed in Chapters 3 and 6 guarantee stabilization under the assumption that *perfect information* of the operation mode of the switched system is obtained at mode observation instants. In contrast to the problem setting in Chapters 3 and 6, in this chapter we assume that the mode information obtained through the observations is *not precise*. In other words, the controller does not receive perfect mode information at mode observation instants. Specifically, we assume that modes of the switched system are divided into a number of groups, and the controller periodically receives information of the group that contains the active mode. In summary, in our new framework, the control law depends only on periodically available *imprecise* mode information, rather than the exact information of the mode.

In the literature, stabilization problem under imprecise mode information has been previously studied in [118, 119] for the case where the mode information is available at all time instants. Specifically, H_2-control of discrete-time switched systems with imprecise mode information is explored in [118]. Furthermore, stabilization conditions are obtained in [119] for continuous-time switched systems under continuously available imprecise mode information. In this chapter we investigate the case where the imprecise mode information is only available periodically.

Note that the imprecise mode information characterizes the case where some of the modes are indistinguishable by the mode detector. For example, for a fault tolerant control system, the fault detector detects a failure, but the type of the failure may not be exactly known. Thus, the control system has only imprecise information of the failure. There are other studies that deal with feedback stabilization problems using imperfect mode information. For example, in [73, 74, 120] the authors propose stabilizing control laws that depend on estimates of the mode signal. The difference between mode estimates and imprecise mode information is that mode estimates may lack accuracy; however, imprecise mode information lacks exactness, although it is accurate.

This chapter is organized as follows. In Section 7.2, we introduce the feedback control problem for discrete-time switched stochastic systems under periodically available imprecise mode information; furthermore, we obtain sufficient conditions under which our proposed control law achieves second-moment asymptotic stabilization of the zero solution. We present an illustrative numerical example in Section 7.3. Finally, we conclude the chapter in Section 7.4.

7.2 State Feedback Control of Switched Stochastic Systems Using Periodically Available Imprecise Mode Information

Consider the discrete-time switched stochastic dynamical system with $M \in \mathbb{N}$ modes given by (6.1) with the initial conditions $x(0) = x_0$ and $r(0) = r_0$. Hence, the initial distribution of the mode signal $\{r(k) \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}\}_{k \in \mathbb{N}_0}$ is given by $\nu : \mathcal{M} \to [0, 1]$ such that $\nu_{r_0} = 1$ and $\nu_i = 0$, $i \neq r_0$.

A mode transition diagram for the switched system (6.1) with $M = 4$ modes is shown

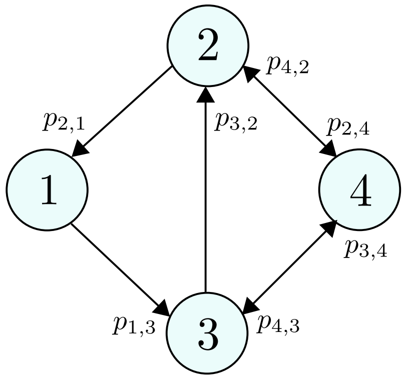

Figure 7.1: Mode transition diagram for $\{r(k) \in \mathcal{M} \triangleq \{1, 2, 3, 4\}\}_{k \in \mathbb{N}_0}$

in Figure 7.1. The labels on the directed edges indicate probability of associated transitions. In Figure 7.1, we only show the edges that correspond to transitions with nonzero probabilities.

7.2.1 Periodic Mode Observations

For the switched stochastic system (6.1), we study the state feedback stabilization problem for the case where mode information is observed periodically at time instants $0, \tau, 2\tau, \ldots$, where $\tau \in \mathbb{N}$ denotes the mode observation period. In this chapter, we specifically consider the case where $\tau \geq 2$ such that at certain time instants no mode information will be available for control purposes.

When the observations at time instants $0, \tau, 2\tau, \ldots$, provide perfect knowledge of the active mode, the sampled mode information that is available to the controller can be represented by the discrete-time stochastic process $\{\sigma(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ defined by

$$\sigma(k) = r(n\tau), \quad k \in \{n\tau, n\tau + 1, \ldots (n+1)\tau - 1\}, \quad n \in \mathbb{N}_0. \tag{7.1}$$

Note that when the observation period τ is small, and the mode switches occur rarely, the sampled mode signal $\{\sigma(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ is likely to be a good representation of the actual mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$. Hence, the stabilization can be achieved by a control law of the form

$$u(k) = K_{\sigma(k)} x(k), \quad k \in \mathbb{N}_0. \tag{7.2}$$

In Chapter 3, in order to stabilize continuous-time switched stochastic systems, we proposed the continuous-time version of the feedback control law (7.2) that depends only on the sampled mode signal, rather than the actual mode signal. In this chapter we extend our results to the case of *imprecise* mode information.

7.2.2 Imprecise Mode Information

In this section we provide mathematical definition of the imprecise mode information.

We assume that the index set $\mathcal{M}$ is divided into a number of nonempty subsets $\mathcal{M}_i \subset \mathcal{M}, i \in \{1, \ldots, N\}$, where $N \leq M$, such that $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset, i \neq j$, and $\cup_{i \in \{1,\ldots,N\}} \mathcal{M}_i = \mathcal{M}$. We call the $\mathcal{M}_i \subset \mathcal{M}, i \in \{1, \ldots, N\}$, the mode groups, since each subset $\mathcal{M}_i$ represents a group of modes. Now let $\mathcal{N} \triangleq \{1, \ldots, N\}$ and define $\eta : \mathcal{M} \to \mathcal{N}$ by

$$\eta(i) = j, \quad i \in \mathcal{M}_j, \;\; j \in \mathcal{N}. \tag{7.3}$$

Note that the function $\eta(\cdot)$ maps each mode into its respective group.

In this chapter, we assume that periodically available imprecise mode information indicates only the group that contains the active mode. For example if mode i is active at a mode observation time instant $n\tau$, the only information that is available for control purposes is $\eta(i)$, which indicates the mode group $\mathcal{M}_{\eta(i)}$ that contains mode i. Note that this information lacks precision when $\mathcal{M}_{\eta(i)}$ also contains modes other than mode i. In this case, the controller has the imprecise information that one of the modes contained in $\mathcal{M}_{\eta(i)}$ is active at the mode observation time instant $n\tau$. A suitable measure for the precision of the periodically available mode information would be the scalar $\mu \triangleq \frac{N}{M}$. For example, if μ is close to 1, it means that the periodically available mode information has high precision.

We consider the case where the imprecise mode information is only available periodically at mode observation time instants $0, \tau, 2\tau, \ldots$. We denote the available mode information by the imprecise mode information signal $\{\eta(\sigma(k)) \in \mathcal{N}\}_{k \in \mathbb{N}_0}$, and propose the control law given by

$$u(k) = K_{\eta(\sigma(k))} x(k), \quad k \in \mathbb{N}_0. \tag{7.4}$$

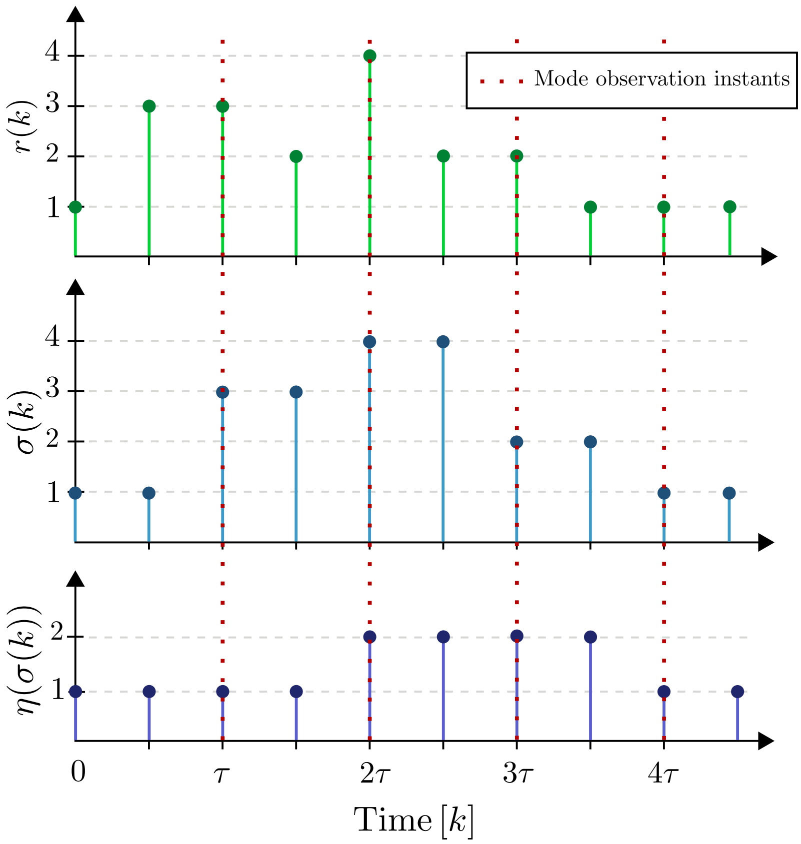

Figure 7.2: Actual mode signal $r(k)$, sampled mode signal $\sigma(k)$, and imprecise mode information signal $\eta(\sigma(k))$

In the next section, we obtain sufficient conditions of stabilization of the switched stochastic system (6.1) under the control law (7.4).

It is important to note that both the sampled mode signal $\{\sigma(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ and the imprecise mode information signal $\{\eta(\sigma(k)) \in \mathcal{N}\}_{k\in\mathbb{N}_0}$ are stochastic processes that depend on the actual mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$. In Figure 7.2, we show sample paths of $r(k)$, $\sigma(k)$, and $\eta(\sigma(k))$ for a discrete-time switched stochastic system with $M = 4$ modes. For this switched system, we assume that the index set $\mathcal{M} \triangleq \{1, 2, 3, 4\}$ is divided into subsets $\mathcal{M}_1 \triangleq \{1, 3\}$ and $\mathcal{M}_2 \triangleq \{2, 4\}$ (see Figure 7.3). Therefore, when the mode is observed at time instants 0, τ, 2τ,. . ., the imprecise mode information signal $\eta(\sigma(k))$ takes the values either 1 or 2, which indicate the group of the active mode. For example, at time $k = 3\tau$, mode 2 is active, that is $r(3\tau) = 2$. Consequently, $\sigma(3\tau) = 2$, and $\eta(\sigma(3\tau)) = 2$. Furthermore, note that neither $\sigma(k)$ nor $\eta(\sigma(k))$ indicate any information about the mode switches between two consecutive mode observation instants.

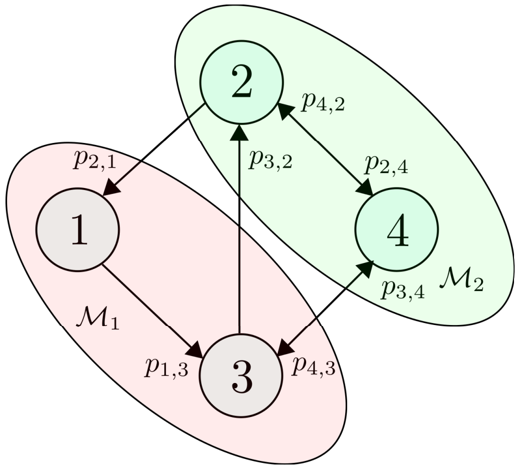

Figure 7.3: Groups of modes characterized by $\mathcal{M}_1 \triangleq \{1,3\}$ and $\mathcal{M}_2 \triangleq \{2,4\}$

Remark 7.1. The case of $\mathcal{N} = \mathcal{M}$ corresponds to the situation when the mode observations provide perfect knowledge of the active mode. In this particular case, the problem turns into the stabilization problem under sampled mode information.

Remark 7.2. Note that the situation where no information is available can also be characterized within the imprecise mode information framework. Specifically, consider the case where all modes are collected in a single group $\mathcal{M}_1 = \mathcal{M}$. Hence, $\mathcal{N} = \{1\}$, and $\eta(\sigma(k)) = 1$, $k \in \mathbb{N}_0$. In this case, stabilization has to be achieved by the control law (7.4) with the fixed feedback gain matrix $K_1 \in \mathbb{R}^{m\times n}$.

7.2.3 Sufficient Conditions for Second-Moment Asymptotic Stabilization

In this section we present sufficient conditions under which our proposed control law (7.4) guarantees the second-moment asymptotic stability (see Section 2.4.2) of the zero solution $x(k) \equiv 0$ of the closed-loop system (6.1), (7.4).

Theorem 7.1. Consider the switched linear stochastic system (6.1) with control input (7.4), which depends on imprecise mode information that is available periodically at time instants $0, \tau, 2\tau, \ldots$, where $\tau \geq 2$. If there exist $\tilde{P}_i > 0$, $i \in \mathcal{N}$, $L_i \in \mathbb{R}^{m\times n}$, $i \in \mathcal{N}$, and scalars $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, such that

$$0 \geq (A_i\tilde{P}_{\eta(i)} + B_iL_{\eta(i)})^{\mathrm{T}}\tilde{P}^{-1}_{\eta(i)}(A_i\tilde{P}_{\eta(i)} + B_iL_{\eta(i)}) - \alpha\tilde{P}_{\eta(i)}, \quad i \in \mathcal{M}, \tag{7.5}$$

$$0 \geq (A_i\tilde{P}_{\eta(j)} + B_iL_{\eta(j)})^{\mathrm{T}}\tilde{P}^{-1}_{\eta(j)}(A_i\tilde{P}_{\eta(j)} + B_iL_{\eta(j)}) - \beta\tilde{P}_{\eta(j)}, \quad i,j \in \mathcal{M}, \tag{7.6}$$

$$0 \geq \sum_{j\in\mathcal{M}} p_{i,j}(A_i\tilde{P}_{\eta(l)} + B_iL_{\eta(l)})^{\mathrm{T}}\tilde{P}^{-1}_{\eta(j)}(A_i\tilde{P}_{\eta(l)} + B_iL_{\eta(l)}) - \gamma\tilde{P}_{\eta(l)}, \quad i,l \in \mathcal{M}, \tag{7.7}$$

and

$$\alpha\beta^{\tau-2}\gamma < 1, \tag{7.8}$$

then the feedback control law (7.4) with the feedback gain matrix

$$K_{\eta(\sigma(k))} = L_{\eta(\sigma(k))}\tilde{P}^{-1}_{\eta(\sigma(k))}, \tag{7.9}$$

guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (6.1) and (7.4) is second-moment asymptotically stable.

The proof of Theorem 7.1 is based on showing the asymptotic convergence of the expectation of a quadratic Lyapunov function candidate.

Proof. First, we define the positive-definite matrices $P_i \triangleq \tilde{P}^{-1}_{\eta(i)}$, $i \in \mathcal{M}$, and the positive-definite function $V(x(k),k) \triangleq x^{\mathrm{T}}(k)P_{\sigma(k)}x(k)$, $k \in \mathbb{N}_0$. Our initial goal is to show that $\mathbb{E}[V(x(n\tau), n\tau)] \to 0$ as $n \to \infty$.

Note that

$$\begin{aligned}\mathbb{E}[V(x(k),k)] &= \sum_{i,j\in\mathcal{M}} \mathbb{E}[V(x(k),k)\mathbb{1}_{[r(k)=i,\sigma(k)=j]}] \\ &= \sum_{i,j\in\mathcal{M}} \mathbb{E}[x^{\mathrm{T}}(k)P_jx(k)\mathbb{1}_{[r(k)=i,\sigma(k)=j]}], \quad k \in \mathbb{N}_0.\end{aligned} \tag{7.10}$$

Now let $\tilde{A}_{i,j} \triangleq A_i + B_iK_j$, $i \in \mathcal{M}$, $j \in \mathcal{N}$. Hence, $\tilde{A}_{r(k),\eta(\sigma(k))}$ denotes the closed-loop state matrix such that $x(k+1) = \tilde{A}_{r(k),\eta(\sigma(k))}x(k)$, $k \in \mathbb{N}_0$. It follows that

$$\begin{aligned}\mathbb{E}[V(x(k+1),k+1)] &= \sum_{i,j\in\mathcal{M}} \mathbb{E}[x^{\mathrm{T}}(k+1)P_j x(k+1)\mathbb{1}_{[r(k+1)=i,\sigma(k+1)=j]}] \\ &= \sum_{i,j\in\mathcal{M}} \mathbb{E}[\mathbb{E}[x^{\mathrm{T}}(k+1)P_j x(k+1)\mathbb{1}_{[r(k+1)=i,\sigma(k+1)=j]}|\mathcal{F}_k]] \\ &= \sum_{i,j\in\mathcal{M}} \mathbb{E}[\mathbb{E}[x^{\mathrm{T}}(k)\tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))}P_j \\ &\quad\cdot \tilde{A}_{r(k),\eta(\sigma(k))}x(k)\mathbb{1}_{[r(k+1)=i,\sigma(k+1)=j]}|\mathcal{F}_k]], \quad k\in\mathbb{N}_0. \end{aligned} \tag{7.11}$$

Since the random variables $x(k)$, $r(k)$, $\sigma(k)$ are all $\mathcal{F}_k$-measurable, we have

$$\begin{aligned}\mathbb{E}[V(x(k+1),k+1)] &= \sum_{i,j\in\mathcal{M}} \mathbb{E}[x^{\mathrm{T}}(k)\tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))}P_j\tilde{A}_{r(k),\eta(\sigma(k))}x(k) \\ &\quad\cdot \mathbb{E}[\mathbb{1}_{[r(k+1)=i,\sigma(k+1)=j]}|\mathcal{F}_k]], \quad k\in\mathbb{N}_0. \end{aligned} \tag{7.12}$$

Note that $\sigma(k+1) = r(k+1)$ for $k = n\tau - 1$, $n \in \mathbb{N}$. It follows from (7.12) that

$$\begin{aligned}\mathbb{E}[V(x(k+1),k+1)] &= \sum_{j\in\mathcal{M}} \mathbb{E}[x^{\mathrm{T}}(k)\tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))}P_j\tilde{A}_{r(k),\eta(\sigma(k))}x(k)\mathbb{E}[\mathbb{1}_{[r(k+1)=j]}|\mathcal{F}_k]] \\ &= \sum_{j\in\mathcal{M}} \mathbb{E}[x^{\mathrm{T}}(k)\tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))}P_j\tilde{A}_{r(k),\eta(\sigma(k))}x(k)p_{r(k),j}] \\ &= \mathbb{E}[x^{\mathrm{T}}(k)\big(\sum_{j\in\mathcal{M}} \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))}P_j\tilde{A}_{r(k),\eta(\sigma(k))}p_{r(k),j}\big)x(k)], \end{aligned} \tag{7.13}$$

for $k = n\tau - 1$, $n \in \mathbb{N}$.

Now set $L_{\eta(j)} = K_{\eta(j)}P_j^{-1}$, $j \in \mathcal{M}$. It follows from the definitions $P_i \triangleq \tilde{P}^{-1}_{\eta(i)}$, $i \in \mathcal{M}$, $\tilde{A}_{i,j} \triangleq A_i + B_iK_j$, $i \in \mathcal{M}$, $j \in \mathcal{N}$, and the condition (7.7) that

$$\sum_{j\in\mathcal{M}} P^{-1}_{\sigma(k)}\tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))}P_j\tilde{A}_{r(k),\eta(\sigma(k))}p_{r(k),j}P^{-1}_{\sigma(k)} \le \gamma P^{-1}_{\sigma(k)}, \quad k = n\tau-1,\ n\in\mathbb{N}. \tag{7.14}$$

By pre- and post-multiplying both sides of (7.14) with $P_{\sigma(k)}$,

$$\sum_{j\in\mathcal{M}} \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))}P_j\tilde{A}_{r(k),\eta(\sigma(k))}p_{r(k),j} \le \gamma P_{\sigma(k)}, \tag{7.15}$$

for $k = n\tau - 1$, $n \in \mathbb{N}$. Furthermore, by using (7.13) and (7.15), we obtain

$$\begin{aligned}\mathbb{E}[V(x(k+1), k+1)] &\leq \gamma \mathbb{E}[x^{\mathrm{T}}(k) P_{\sigma(k)} x(k)] \\ &= \gamma \mathbb{E}[V(x(k), k)], \end{aligned} \tag{7.16}$$

for $k = n\tau - 1$, $n \in \mathbb{N}$. Note that $\sigma(k+1) = \sigma(k)$, and hence $\sigma(k+1)$ is $\mathcal{F}_k$-measurable, for $k \in \{(n-1)\tau, (n-1)\tau + 1, \ldots, n\tau - 2\}$, $n \in \mathbb{N}$. It follows from (7.12) that

$$\begin{aligned}\mathbb{E}[V(x(k+1), k+1)] &= \sum_{i \in \mathcal{M}} \mathbb{E}[x^{\mathrm{T}}(k) \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} x(k) \mathbb{E}[\mathbb{1}_{[r(k+1)=i]} | \mathcal{F}_k]] \\ &= \sum_{i \in \mathcal{M}} \mathbb{E}[x^{\mathrm{T}}(k) \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} x(k) p_{r(k),i}] \\ &= \mathbb{E}[x^{\mathrm{T}}(k) \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} x(k) \big(\sum_{i \in \mathcal{M}} p_{r(k),i} \big)] \\ &= \mathbb{E}[x^{\mathrm{T}}(k) \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} x(k)], \end{aligned} \tag{7.17}$$

for $k \in \{(n-1)\tau, \ldots, n\tau - 2\}$, $n \in \mathbb{N}$. Note that by the condition (7.6), for $n \in \mathbb{N}$,

$$P^{-1}_{\sigma(k)} \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} P^{-1}_{\sigma(k)} \leq \beta P^{-1}_{\sigma(k)}, \; k \in \{(n-1)\tau + 1, \ldots, n\tau - 2\}. \tag{7.18}$$

We now pre- and post-multiply both sides of (7.18) with $P_{\sigma(k)}$, to obtain

$$\tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} \leq \beta P_{\sigma(k)}, \quad k \in \{(n-1)\tau + 1, \ldots, n\tau - 2\}, \quad n \in \mathbb{N}. \tag{7.19}$$

Hence, by (7.17),

$$\begin{aligned}\mathbb{E}[V(x(k+1), k+1)] &\leq \beta \mathbb{E}[x^{\mathrm{T}}(k) P_{\sigma(k)} x(k)] \\ &= \beta \mathbb{E}[V(x(k), k)], \quad k \in \{(n-1)\tau + 1, \ldots, n\tau - 2\}, \quad n \in \mathbb{N}. \end{aligned} \tag{7.20}$$

Furthermore, since $r(k) = \sigma(k)$, for $k = n\tau$, $n \in \mathbb{N}_0$, it follows form condition (7.5) that

$$P^{-1}_{\sigma(k)} \tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} P^{-1}_{\sigma(k)} \leq \alpha P^{-1}_{\sigma(k)}, \tag{7.21}$$

for $k = (n-1)\tau$, $n \in \mathbb{N}$. After pre- and post-multiplying both sides of (7.21) with $P_{\sigma(k)}$, we get

$$\tilde{A}^{\mathrm{T}}_{r(k),\eta(\sigma(k))} P_{\sigma(k)} \tilde{A}_{r(k),\eta(\sigma(k))} \leq \alpha P_{\sigma(k)}, \quad k = (n-1)\tau, \quad n \in \mathbb{N}. \tag{7.22}$$

It then follows that

$$\begin{aligned} \mathbb{E}[V(x(k+1), k+1)] &\leq \alpha \mathbb{E}[x^{\mathrm{T}}(k) P_{\sigma(k)} x(k)] \\ &= \alpha \mathbb{E}[V(x(k), k)], \quad k = (n-1)\tau, \quad n \in \mathbb{N}. \end{aligned} \tag{7.23}$$

Finally, by using (7.16), (7.20), and (7.23), we obtain

$$\begin{aligned} \mathbb{E}[V(x((n+1)\tau), (n+1)\tau)] &\leq \gamma \mathbb{E}[V(x((n+1)\tau - 1), (n+1)\tau - 1)] \\ &\leq \beta^{\tau-2} \gamma \mathbb{E}[V(x(n\tau + 1), n\tau + 1)] \\ &\leq \alpha \beta^{\tau-2} \gamma \mathbb{E}[V(x(n\tau), n\tau)], \quad n \in \mathbb{N}_0. \end{aligned} \tag{7.24}$$

As a consequence of (7.8) and (7.24), $\mathbb{E}[V(x(n\tau), n\tau)] \to 0$ as $n \to \infty$. Moreover, since $\mathbb{E}[\|x(n\tau)\|^2] \leq \frac{\mathbb{E}[V(x(n\tau), n\tau)]}{\min_{j \in \mathcal{M}} \lambda_{\min}(P_j)}$, it follows that

$$\lim_{n \to \infty} \mathbb{E}[\|x(n\tau)\|^2] = 0. \tag{7.25}$$

Now let $c \triangleq \max_{i,j \in \mathcal{M}} \lambda_{\max}(\tilde{A}^{\mathrm{T}}_{i,\eta(j)} \tilde{A}_{i,\eta(j)})$. Note that $\mathbb{E}[\|x(k+1)\|^2] \leq c\mathbb{E}[\|x(k)\|^2]$, $k \in \mathbb{N}_0$. It follows from (7.25) that for every $\epsilon > 0$, there exists $\hat{N} \in \mathbb{N}$ such that $\mathbb{E}[\|x(n\tau)\|^2] < \epsilon c^{\tau}$, for $n > \hat{N}$. Consequently, $\mathbb{E}[\|x(k)\|^2] < \epsilon$, for $k > \hat{N}\tau$. Therefore, $\mathbb{E}[\|x(k)\|^2] \to 0$ as $k \to \infty$, which completes the proof. □

Theorem 7.1 provides sufficient conditions under which our proposed feedback control law (7.4) guarantees second-moment asymptotic stabilization of the zero solution. Note that feedback control performance is directly related to subsystem dynamics, mode switching frequency, and mode observation period τ. The condition (7.8) of Theorem 7.1 indicates the effect of the mode observation period τ on the stability of the closed-loop system.

Note that the nonnegative scalars α, β, and γ characterize upper bounds on the growth of the expectation of a positive-definite function $V(x(k),k) \triangleq x^{\mathrm{T}}(k)P_{\sigma(k)}x(k)$ with $P_{\sigma(k)} \triangleq \tilde{P}^{-1}_{\eta(\sigma(k))}$. Specifically, the scalar $\alpha \geq 0$ characterizes the growth right after an imprecise mode information becomes available to the controller. Strictly speaking, $\mathbb{E}[V(x(n\tau+1),n\tau+1)] \leq \alpha\mathbb{E}[V(x(n\tau),n\tau)]$, $n \in \mathbb{N}_0$. The scalar $\gamma \geq 0$ characterizes the growth right before the mode observation instants, that is, $\mathbb{E}[V(x(n\tau),n\tau)] \leq \gamma\mathbb{E}[V(x(n\tau-1),n\tau-1)]$, $n \in \mathbb{N}$. On the other hand, the nonnegative scalar β characterizes the growth between two consecutive mode observation instants when the available mode information may no longer be an accurate representation of the active mode. As a consequence, β has to satisfy the condition (7.6) for all pairs of modes $i,j \in \mathcal{M}$. Hence, when the dynamics of the subsystems are significantly different from each other, the condition (7.6) can only be satisfied for $\beta \geq 1$. In this case, the mode observation period τ has to be sufficiently small so that the condition (7.8) can be satisfied.

In addition, the effect of the precision of the mode information is also reflected in conditions (7.5)–(7.7) through the function $\eta(\cdot)$, which maps individual modes into their respective groups. In order to check whether the proposed control law (7.4) guarantees stabilization or not, we have to find positive-definite matrices $\tilde{P}_i > 0$, $i \in \mathcal{N}$, such that conditions (7.5)–(7.7) hold. It is important to note that these conditions are harder to satisfy when the mode information is very imprecise, as one has to satisfy conditions (7.5)–(7.7) simultaneously with fewer variables, $\tilde{P}_i > 0$, $L_i \in \mathbb{R}^{m\times n}$, $i \in \mathcal{N}$.

Although, it is not easily seen from the conditions (7.5)–(7.8), in general, one can compensate the imprecision of the observed mode information by observing the mode more frequently. Note that this method would not work for the case $\mathcal{N} = \{1\}$, as mode observations would not provide any distinctive mode information regardless of the mode observation period.

Remark 7.3. When $\mathcal{N} = \{1\}$, the conditions (7.5)–(7.7) reduce, respectively, to

$$0 \geq (A_i\tilde{P}_1 + B_iL_1)^{\mathrm{T}}\tilde{P}_1^{-1}(A_i\tilde{P}_1 + B_iL_1) - \alpha\tilde{P}_1, \tag{7.26}$$

$$0 \geq (A_i\tilde{P}_1 + B_iL_1)^{\mathrm{T}}\tilde{P}_1^{-1}(A_i\tilde{P}_1 + B_iL_1) - \beta\tilde{P}_1, \tag{7.27}$$

$$0 \geq (A_i\tilde{P}_1 + B_iL_1)^{\mathrm{T}}\tilde{P}_1^{-1}(A_i\tilde{P}_1 + B_iL_1) - \gamma\tilde{P}_1, \tag{7.28}$$

for $i \in \mathcal{M}$. After setting $\alpha = \beta = \gamma$, the final condition (7.8) becomes $\alpha^\tau < 1$, which is satisfied only when $\alpha < 1$. It is natural that when no distinctive mode information is available through observations ($\mathcal{N} = \{1\}$), stabilization conditions do not depend on the mode observation period τ.

Remark 7.4. When $\tau = 2$, the conditions of Theorem 7.1 take a simpler form. Note that the condition (7.6) can always be satisfied with $\beta = \frac{\max_{i,j\in\mathcal{M}} \lambda_{\max}(C_{i,\eta(j)})}{\min_{j\in\mathcal{M}} \lambda_{\min}(\tilde{P}_{\eta(j)})}$, where $C_{i,\eta(j)} \triangleq (A_i\tilde{P}_{\eta(j)} + B_iL_{\eta(j)})^{\mathrm{T}}\tilde{P}_{\eta(j)}^{-1}(A_i\tilde{P}_{\eta(j)} + B_iL_{\eta(j)})$. Thus, for the case where $\tau = 2$, the stabilization conditions reduce to the inequalities (7.5), (7.7), and $\alpha\gamma < 1$.

Remark 7.5. Note that the inequalities (7.5)–(7.8) are not linear in $L_i, \tilde{P}_i, i \in \mathcal{N}$, $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$, due to the terms $\alpha\tilde{P}_{\eta(i)}$, $\beta\tilde{P}_{\eta(j)}$, $\gamma\tilde{P}_{\eta(l)}$, and $\alpha\beta^{\tau-2}\gamma$. However, given the scalars $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$ that satisfy (7.8), the conditions (7.5)–(7.7) can be transformed into matrix inequalities that are linear in $L_i, \tilde{P}_i, i \in \mathcal{N}$, using Schur complements [115]. Specifically, let $\hat{A}_{i,\eta(j)} \triangleq (A_i\tilde{P}_{\eta(j)} + B_iL_{\eta(j)})$, $i, j \in \mathcal{M}$, and define

$$Q_i \triangleq \begin{bmatrix} \alpha\tilde{P}_{\eta(i)} & \hat{A}_{i,\eta(i)}^{\mathrm{T}} \\ \hat{A}_{i,\eta(i)} & \tilde{P}_{\eta(i)} \end{bmatrix}, \quad i \in \mathcal{M}, \tag{7.29}$$

$$R_{i,j} \triangleq \begin{bmatrix} \beta\tilde{P}_{\eta(j)} & \hat{A}_{i,\eta(j)}^{\mathrm{T}} \\ \hat{A}_{i,\eta(j)} & \tilde{P}_{\eta(j)} \end{bmatrix}, \quad i, j \in \mathcal{M}, \tag{7.30}$$

$$S_{i,l} \triangleq \begin{bmatrix} \gamma\tilde{P}_{\eta(l)} & T_{i,l}^{1\mathrm{T}} & \cdots & T_{i,l}^{M\mathrm{T}} \\ T_{i,l}^{1} & \tilde{P}_{\eta(1)} & & 0 \\ \vdots & & \ddots & \\ T_{i,l}^{M} & 0 & & \tilde{P}_{\eta(M)} \end{bmatrix}, \quad i, l \in \mathcal{M}, \tag{7.31}$$

where $T_{i,l}^{j} \triangleq \sqrt{p_{i,j}}\hat{A}_{i,\eta(l)}$, $i, j, l \in \mathcal{M}$. It follows that the conditions (7.5)–(7.7) are equivalent to the inequalities $Q_i \geq 0$, $i \in \mathcal{M}$, $R_{i,j} \geq 0$, $i, j \in \mathcal{M}$, and $S_{i,l} \geq 0$, $i, l \in \mathcal{M}$, which are linear in $L_i \in \mathbb{R}^{m\times n}, \tilde{P}_i > 0, i \in \mathcal{N}$, given $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$. Hence, we can iterate over a set of the values of $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$ that satisfy (7.8) and look for feasible solutions to the linear matrix inequalities $Q_i \geq 0$, $i \in \mathcal{M}$, $R_{i,j} \geq 0$, $i, j \in \mathcal{M}$, and $S_{i,l} \geq 0$, $i, l \in \mathcal{M}$. In Section 7.3, we employ this approach and find values for $L_i \in \mathbb{R}^{m\times n}, \tilde{P}_i > 0, i \in \mathcal{N}$, $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$ that satisfy (7.5)–(7.8) for a given discrete-time switched linear system.

7.3 Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of our main results presented in Section 7.2. Consider the 2-dimensional discrete-time switched linear stochastic dynamical system with $M = 3$ modes described by the subsystems matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1.6 & -0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & -1 \\ -1.45 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -0.9 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 1 \\ 1.5 & 0.3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0.9 \end{bmatrix},$$

The mode signal $\{r(k) \in \mathcal{M} \triangleq \{1,2,3\}\}_{k\in\mathbb{N}_0}$ of the discrete-time switched stochastic system is assumed to be a time-homogeneous Markov chain characterized by the transition probabilities $p_{i,i} = 0.6$, $i \in \mathcal{M}$, $p_{i,j} = 0.2$, $i \neq j$, $i, j \in \mathcal{M}$. The controller is assumed to have access to imprecise information concerning the mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$, at time instants $0, \tau, 2\tau, \ldots$, where $\tau = 5$.

Modes of the switched system are assumed to be divided into $N = 2$ groups, $\mathcal{M}_1 \triangleq \{1,3\}$ and $\mathcal{M}_2 \triangleq \{2\}$. Periodically available mode information is characterized by the imprecise mode information signal $\{\eta(\sigma(k)) \in \{1,2\}\}_{k\in\mathbb{N}_0}$. Hence, the modes 1 and 3 are indistinguishable by the controller.

Note that the positive-definite matrices,

$$\tilde{P}_1 = \begin{bmatrix} 4.5221 & -0.2061 \\ -0.2061 & 0.2448 \end{bmatrix}, \tag{7.32}$$
$$\tilde{P}_2 = \begin{bmatrix} 3.6290 & -0.1609 \\ -0.1609 & 0.2492 \end{bmatrix}, \tag{7.33}$$

the row vectors $L_1 = [-7.3576 \;\; 0.2998]$, $L_2 = [-6.0399 \;\; 0.4816]$, and scalars $\alpha = 0.1$, $\beta = 1.45$, $\gamma = 1.5$, satisfy the conditions (7.5)–(7.8). It follows from Theorem 7.1 that the control law (7.4) with the feedback gain matrices $K_1 = L_1\tilde{P}_1^{-1} = [-1.6339 \;\; -0.1510]$

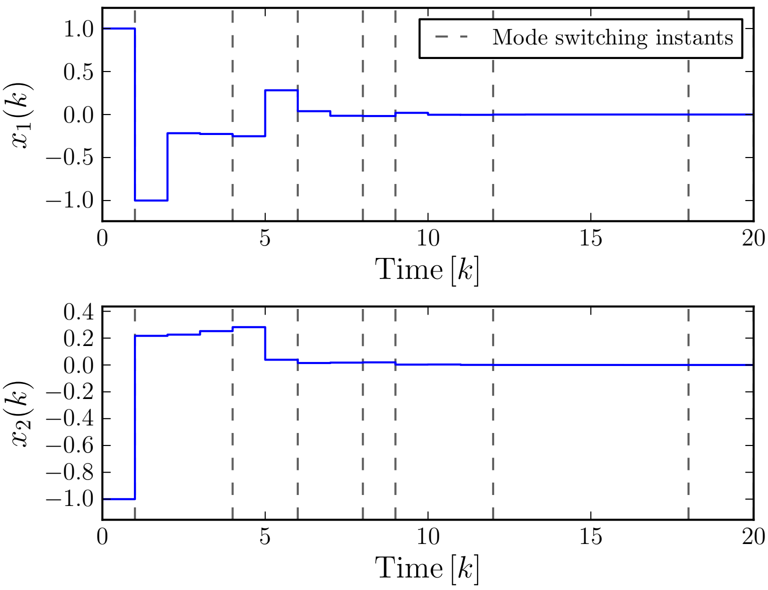

Figure 7.1: State trajectory versus time

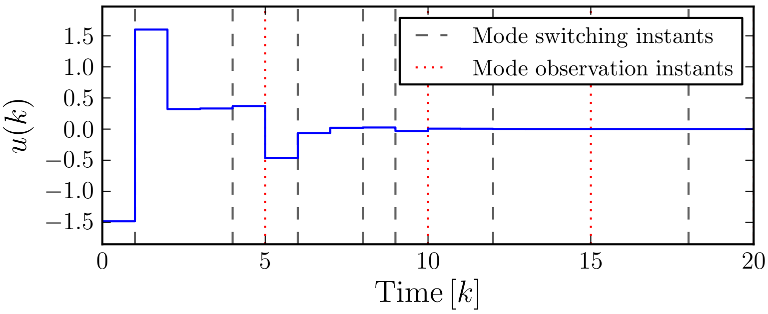

Figure 7.2: Control input versus time

and $K_2 = L_2 \tilde{P}_2^{-1} = [-1.6252 \;\; 0.8837]$, guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (6.1), (7.4) is second-moment asymptotically stable.

Sample paths of the state and control input, $x(k)$ and $u(k)$, obtained with the initial conditions $x(0) = [1, \, -1]^{\mathrm{T}}$ and $r(0) = 1$ are presented in Figs. 7.1 and 7.2, respectively. Furthermore, sample paths of the actual mode signal $r(k)$, sampled version of the mode signal $\sigma(k)$, and the imprecise mode information signal $\eta(\sigma(k))$ are shown in Figure 7.3.

Note that the states $1, 2 \in \mathcal{N}$ of the imprecise mode information signal $\eta(\sigma(k))$, correspond to the mode groups $\mathcal{M}_1$ and $\mathcal{M}_2$. For example, at time instant $k = 5$, imprecise mode information signal $\eta(\sigma(k))$ takes the value 1, which corresponds to the group $\mathcal{M}_1 = \{1, 3\}$ (Figure 7.3). Hence, at time $k = 5$, information obtained through mode observation is not precise. Specifically, given the information, the active mode can either

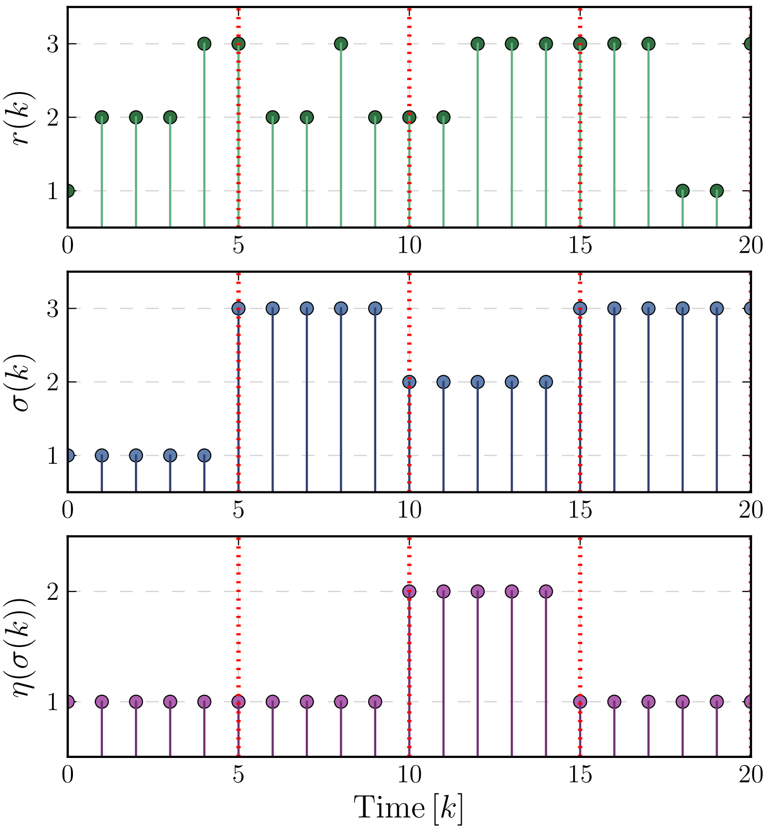

Figure 7.3: Actual mode signal $r(k)$, sampled mode signal $\sigma(k)$, and imprecise mode information signal $\eta(\sigma(k))$

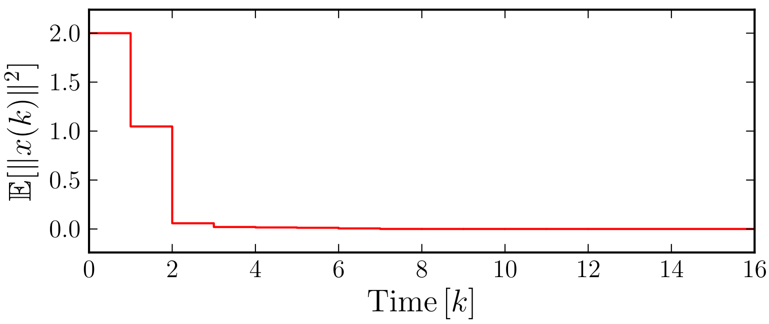

Figure 7.4: Second moment trajectory versus time

be 1 or 3. Note also that in the case $\eta(\sigma(n\tau)) = 2$, for some $n \in \mathbb{N}$, information that is available for control purposes is precise, since the group $\mathcal{M}_2 = \{2\}$ contains only mode 2.

In order to demonstrate the convergence of the second moment of the state, we obtain 1000 sample paths of the state trajectory $x(k)$ (with the initial conditions $x(0) = [1, -1]^{\mathrm{T}}$ and $r(0) = 1$) to estimate $\mathbb{E}[\|x(k)\|^2]$ by

$$\mathbb{E}[\|x(k)\|^2] \approx \frac{1}{1000} \sum_{i=1}^{1000} \|x(k)\|^2_{(i)}, \quad k \in \mathbb{N}_0, \tag{7.34}$$

where $\|x(k)\|_{(i)}$ denotes the state norm for the ith sample path at time k. Figure 7.4 shows the numerically approximated second moment $\mathbb{E}[\|x(k)\|^2]$, which converges to 0.

7.4 Conclusion

We proposed a state feedback control framework for stabilizing discrete-time switched linear stochastic systems. We considered the case where the controller has access to mode information only at certain time instants. Furthermore, the exact modes which are active at those instants are unknown to the controller. The controller is assumed to have only the information of a set of modes one of which is guaranteed to be active. We obtained sufficient conditions of second-moment asymptotic stabilization under the assumption that the imprecise mode information is available periodically.

Note that in Chapters 3–7, we explored the stabilization problem for switched stochastic systems for the case where the mode of the switched system is sampled periodically at deterministic time instants. In Chapters 8 and 9 below, we will address the case where the mode signal is sampled at random time instants.

Chapter 8

Sampled-Mode Stabilization of Switched Linear Stochastic Dynamical Systems With Exponentially Distributed Random Mode Sampling Intervals

8.1 Introduction

In most of the studies that deal with stabilization of switched stochastic systems, proposed control laws depend on full information of the mode signal of the switched system. As a result, these control laws may not be appropriate when the mode information is sampled and only available at sampling instants. In Chapters 3–7 we considered the feedback control problem for the case where the mode is sampled periodically. Hence, mode sampling instants considered in Chapters 3–7 are deterministic.

In this chapter, we investigate stabilization of continuous-time switched linear stochastic systems for the case where the mode signal is sampled at *random* time instants. Specifically, the intervals between mode sampling instants are assumed to be exponentially distributed independent random variables. First, we provide stability analysis for a continuous-time switched linear stochastic dynamical system without control input. The mode signal, which manages the transition between the subsystems of the switched system, is modeled as a finite-state continuous-time Markov chain (see Section 2.2.1). Based on our stability analysis, we first propose a stabilizing control law that depends on the actual mode signal. Next, we consider the case where the mode signal information is sampled and hence available only at *random* time instants. By using "sample and hold" technique, we construct the sampled version of the mode signal. We then propose a control law that depends only on the sampled mode signal rather than the actual mode signal. We observe that the closed-loop control system under our proposed control law can be characterized as a switched linear stochastic system with a mode signal defined to be a bivariate stochastic process composed of the actual mode signal and its sampled version. Due to the fact that the time intervals between mode sampling instants are exponentially distributed, the bivariate process composed of the actual mode signal and its sampled version turns out to be a finite-state continuous-time Markov chain. Based on our stability analysis for switched linear stochastic dynamical systems, we obtain sufficient conditions under which the proposed control law achieves almost sure asymptotic stabilization (see Section 2.4.1). Note that the closed-loop system under the control law that we propose resembles a fault tolerant control system with normal/faulty modes and a "fault detection and isolation scheme" which is explored in [77] and [78]. In this sense, investigation of the stability of this closed-loop system is also important due to possible applications in the field of fault-tolerant control systems as well.

This chapter is organized as follows. In Section 8.2, we present the mathematical model for continuous-time switched linear stochastic dynamical systems, and provide sufficient conditions of stability. Furthermore we propose a stabilizing control law that depends on the mode signal. We investigate feedback stabilization of switched linear stochastic systems under randomly *sampled* mode information in Section 8.3. A numerical example is provided in Section 8.4 to demonstrate the utility of our results. Finally, we conclude the chapter in Section 8.5.

8.2 Stability and Stabilization of Switched Linear Stochastic Dynamical Systems

In this section, we first provide the mathematical model for switched linear stochastic dynamical systems. We obtain sufficient conditions of almost sure asymptotic stability.

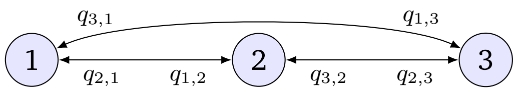

Figure 8.1: Transition diagram of a 3-state Markov chain

Then, we consider switched linear stochastic dynamical systems with control input. Based on our stability analysis, we propose a piecewise-continuous control strategy that achieves stabilization of the zero solution of continuous-time switched linear stochastic dynamical systems.

8.2.1 Sufficient Conditions of Almost Sure Asymptotic Stability

Consider the continuous-time switched linear stochastic dynamical system given by

$$\mathrm{d}x(t) = A_{r(t)}x(t)\mathrm{d}t + D_{r(t)}x(t)\mathrm{d}W(t), \quad t \geq 0, \tag{8.1}$$

with initial conditions $x(0) = x_0$ and $r(0) = r_0$, where $\{x(t)\}_{t\geq 0}$ is the $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted state vector, $\{W(t)\}_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t$-adapted Wiener process, $A_i, D_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$, are subsystem matrices. The dynamical system (8.1) is assumed to have $M \geq 1$ number of subsystems (modes). Transition between the modes is characterized by the piecewise constant $\mathcal{F}_t$-adapted mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$, which is assumed to be an irreducible Markov chain with generator matrix $Q \in \mathbb{R}^{M\times M}$ and with a stationary probability distribution $\pi \in \mathbb{R}^M$. We assume that the Wiener process $\{W(t) \in \mathbb{R}\}_{t\geq 0}$ and the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ are mutually independent stochastic processes.

Figure 8.1 shows the mode transition diagram for a switched system with $M = 3$ modes. Nodes in the figure represent the states of the modes of the switched system, arrowed edges represent a possible transition between the modes in the direction of the arrows, and the labels on the edges indicate the transition rates between the paired modes.

Stability of the dynamical system given by (8.1) can be analyzed using a quadratic Lyapunov-like function.

Theorem 8.1. Consider the switched linear stochastic system given by (8.1). If there exist

$P > 0$ and scalars $\zeta_i \in \mathbb{R}$, $i \in I$, such that

$$0 \geq A_i^{\mathrm{T}} P + P A_i + D_i^{\mathrm{T}} P D_i - \zeta_i P, \quad i \in \mathcal{M}, \tag{8.2}$$

$$\sum_{i \in \mathcal{M}} \pi_i \Big(\zeta_i - \frac{\lambda_{\min}^2(D_i^{\mathrm{T}} P + P D_i)}{2\lambda_{\max}^2(P)}\Big) < 0, \tag{8.3}$$

then the zero solution $x(t) \equiv 0$ of the system described by (8.1) is asymptotically stable almost surely.

Proof. We start by defining the quadratic, positive-definite function $V(x) \triangleq x^{\mathrm{T}} P x$. All modes of the switched system (8.1) are described by multi-dimensional Ito stochastic differential equations. We can employ Ito formula to obtain

$$\begin{aligned}
\mathrm{d}V(x(t)) &= \Big(\nabla V(x(t)) A_{r(t)} x(t) + \frac{1}{2}\mathrm{tr}\Big(D_{r(t)} x(t) x^{\mathrm{T}} D_{r(t)}^{\mathrm{T}} \nabla\big(\nabla V(x(t))\big)\Big)\Big)\mathrm{d}t \\
&\quad + \nabla V(x(t)) D_{r(t)} x(t) \mathrm{d}W(t) \\
&= x^{\mathrm{T}}(t)(A_{r(t)}^{\mathrm{T}} P + P A_{r(t)} + D_{r(t)}^{\mathrm{T}} P D_{r(t)}) x(t) \mathrm{d}t + 2x^{\mathrm{T}}(t) P D_{r(t)} x(t) \mathrm{d}W(t)
\end{aligned} \tag{8.4}$$

which determines the evolution of $V(x(t))$, between consequent switching instants, when the ith mode is active. Now consider the function $\ln V(x(t))$, which is well-defined for non-zero values of the state, since $V(\cdot)$ is a positive-definite function. We use Ito formula once again to compute

$$\begin{aligned}
\mathrm{d}\ln V(x(t)) &= \frac{1}{V(x(t))} x^{\mathrm{T}}(t)(A_{r(t)}^{\mathrm{T}} P + P A_{r(t)} + D_{r(t)}^{\mathrm{T}} P D_{r(t)}) x(t) \mathrm{d}t \\
&\quad - \frac{1}{2V^2(x(t))} \|2x^{\mathrm{T}}(t) P D_{r(t)} x(t)\|^2 \mathrm{d}t \\
&\quad + \frac{1}{V(x(t))} 2x^{\mathrm{T}}(t) P D_{r(t)} x(t) \mathrm{d}W(t).
\end{aligned} \tag{8.5}$$

We integrate (8.5) over the time interval $[0,\, t]$ to obtain

$$\begin{aligned}
\ln V(x(t)) = \ln V(x_0) &+ \int_0^t \frac{x^{\mathrm{T}}(\tau)(A_{r(\tau)}^{\mathrm{T}} P + P A_{r(\tau)} + D_{r(\tau)}^{\mathrm{T}} P D_{r(\tau)}) x(\tau)}{V(x(\tau))} \mathrm{d}\tau \\
&- \int_0^t \frac{1}{2V^2(x(\tau))} \|2x^{\mathrm{T}}(\tau) P D_{r(\tau)} x(\tau)\|^2 \mathrm{d}\tau \\
&+ \int_0^t \frac{1}{V(x(\tau))} 2x^{\mathrm{T}}(\tau) P D_{r(\tau)} x(\tau) \mathrm{d}W(\tau).
\end{aligned} \tag{8.6}$$

Note that

$$
\begin{aligned}
2x^{\mathrm{T}}(\tau)PD_{r(\tau)}x(t) &= x^{\mathrm{T}}(\tau)(D_{r(\tau)}^{\mathrm{T}}P + PD_{r(\tau)})x(\tau) \\
&\geq \lambda_{\min}(D_{r(\tau)}^{\mathrm{T}}P + PD_{r(\tau)})x^{\mathrm{T}}(\tau)x(\tau) \\
&\geq \frac{\lambda_{\min}(D_{r(\tau)}^{\mathrm{T}}P + PD_{r(\tau)})}{\lambda_{\max}(P)}x^{\mathrm{T}}(\tau)Px(\tau). \qquad (8.7)
\end{aligned}
$$

It follows from (8.2), (8.6) and (8.7) that

$$
\begin{aligned}
\ln V(x(t)) \leq \ln V(x_0) &+ \int_0^t (\zeta_{r(\tau)} - \frac{\lambda_{\min}^2(D_{r(\tau)}^{\mathrm{T}}P + PD_{r(\tau)})}{2\lambda_{\max}^2(P)})\mathrm{d}\tau \\
&+ \int_0^t \frac{1}{V(x(\tau))}2x^{\mathrm{T}}(\tau)PD_{r(\tau)}x(\tau)\mathrm{d}W(\tau). \qquad (8.8)
\end{aligned}
$$

By the strong law of large numbers for irreducible Markov chains [87] we have

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t (\zeta_{r(\tau)} - \frac{\lambda_{\min}^2(D_{r(\tau)}^{\mathrm{T}}P + PD_{r(\tau)})}{2\lambda_{\max}^2(P)})\mathrm{d}\tau = \sum_{i\in I}\pi_i(\zeta_i - \frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P + PD_i)}{2\lambda_{\max}^2(P)}), \qquad (8.9)
$$

almost surely. Furthermore, the Ito integral in inequality (8.8),

$$
L(t) = \int_0^t \frac{1}{V(x(\tau))}2x^{\mathrm{T}}(\tau)PD_{r(\tau)}x(\tau)\mathrm{d}W(\tau) \qquad (8.10)
$$

is a local martingale with quadratic variation

$$
\begin{aligned}
[L]_t &= \int_0^t \big(\frac{1}{V(x(\tau))}2x^{\mathrm{T}}(\tau)PD_{r(\tau)}x(\tau)\big)^2\mathrm{d}\tau \\
&= \int_0^t \frac{1}{V^2(x(\tau))}\big(2x^{\mathrm{T}}(\tau)PD_{r(\tau)}x(\tau)\big)^2\mathrm{d}\tau \\
&\leq \int_0^t \frac{1}{V^2(x(\tau))}\big(x^{\mathrm{T}}(\tau)(D_{r(\tau)}^{\mathrm{T}}P + PD_{r(\tau)})x(\tau)\big)^2\mathrm{d}\tau \\
&\leq \int_0^t \frac{\lambda_{\max}^2(D_{r(\tau)}^{\mathrm{T}}P + PD_{r(\tau)})}{\lambda_{\min}^2(P)}\mathrm{d}\tau \\
&\leq \frac{\max_{i\in\mathcal{M}}\lambda_{\max}^2(D_i^{\mathrm{T}}P + PD_i)}{\lambda_{\min}^2(P)}t \qquad (8.11)
\end{aligned}
$$

Consequently, $\lim_{t\to\infty}\frac{1}{t}[L]_t < \infty$. Thus, by using the same approach presented in [49,51], we can employ the strong law of large numbers for local martingales [14] to show

$$\lim_{t\to\infty}\frac{1}{t}L(t)=0, \tag{8.12}$$

almost surely. Moreover, it follows from (8.8), (8.9), and (8.12) that

$$\limsup_{t\to\infty}\frac{1}{t}\ln V(x(t))\leq\sum_{i\in\mathcal{M}}\pi_i(\zeta_i-\frac{\lambda_{\min}^2(D_i^{\mathrm{T}}P+PD_i)}{2\lambda_{\max}^2(P)}). \tag{8.13}$$

Finally, by (8.3),

$$\mathrm{P}[\lim_{t\to\infty}V(x(t))=0]=1, \tag{8.14}$$

which implies almost sure asymptotic stability of the zero solution. □

We employ the stability result presented in Theorem 8.1 for investigating almost sure feedback stabilization problem in the following sections.

8.2.2 Feedback Stabilization with Continuously Observed Mode

In this section, we develop a stabilizing control law for switched linear stochastic dynamical systems. Consider the continuous-time switched linear stochastic system with control input given by (3.57). The stabilization problem we discuss in this section is to design a feedback control law which guarantees the almost sure asymptotic stability of the zero solution $x(t)\equiv 0$. By assuming that information on the mode signal $\{r(t)\in\mathcal{M}\}_{t\geq 0}$ is available to the controller for $t\geq 0$, we propose a control law of the form $u(t)=K_{r(t)}x(t)$, where $K_i\in\mathbb{R}^{m\times n}$ denotes the state feedback gain for the ith mode. Note that the feedback matrix is switched when there is a mode transition. As a result, control input $u(\cdot)$ may have discontinuities at mode switching instants, which we denote by the sequence $\{t_1, t_2, \ldots\}$.

Corollary 8.1. Consider the continuous-time switched linear stochastic dynamical system given by (3.57). If there exist $P>0$ and scalars $\zeta_i\in\mathbb{R}$, $i\in\mathcal{M}$, such that

$$0\geq A_i^{\mathrm{T}}P+PA_i+D_i^{\mathrm{T}}PD_i-2PB_iB_i^{\mathrm{T}}P-\zeta_iP,\;\; i\in\mathcal{M}, \tag{8.15}$$

and (8.3) are satisfied, then the feedback control law

$$u(t) = -B_{r(t)}^{\mathrm{T}} P x(t) \tag{8.16}$$

guarantees that the zero solution $x(t) \equiv 0$ of the switched stochastic system (3.57) is asymptotically stable almost surely.

Proof. The result is a direct consequence of Theorem 8.1 with A_i replaced by $A_i - B_i B_i^{\mathrm{T}} P$, $i \in \mathcal{M}$. $\square$

The proposed control law (8.16) is a function of the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$, and hence cannot be used for stabilization when the mode information is available only at certain time instants or when it is not available at all. For the case where the mode signal information is not available, one can seek a control law of the form

$$u(t) = Kx(t), \tag{8.17}$$

which does not depend on the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$. On the other hand, when mode signal is sampled and only available at certain time instants, sampled mode information can also be employed in the control law.

8.3 Feedback Stabilization Under Sampled Mode Information with Exponentially Distributed Random Sampling Intervals

In this section we explore feedback stabilization problem for the case where the mode signal information $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ of the switched linear stochastic system (3.57) is available only at certain time instants, which we denote by the sequence $\{t_0 = 0, t_1, t_2, \ldots\}$. We assume that the length of time intervals between these instants are independent random variables that are distributed by exponential distribution with parameter $\lambda > 0$. As a result, these time instants correspond to occurrences of events of a Poisson process $\{N(t) \in \mathbb{N}_0\}_{t\geq 0}$ with the parameter $\lambda > 0$ (see Section 2.2.2 for explanation of the properties of Poisson processes). We call λ the mode sampling intensity parameter.

The elements of the sequence $\{t_0,\, t_1,\, t_2,\, \ldots\}$ are characterized by

$$t_k \triangleq \inf\{t\,:\, N(t) \geq k\}, \quad k \in \mathbb{N}_0. \tag{8.18}$$

Note that when the mode sampling intensity λ is small, the length of the time intervals $(t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, are likely to be large; therefore, the mode signal information is expected to be rarely available.

By employing the "sample and hold" technique we construct the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ of the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ by using only the available mode samples $\{r(t_0), r(t_1), r(t_2), \ldots\}$ as

$$\sigma(t) \triangleq r(t_{N(t)}), \quad t \geq 0. \tag{8.19}$$

At time instants $\{t_0, t_1, t_2, \ldots\}$, the sampled mode signal is equal to the actual mode signal of the switched system, that is, $\sigma(t_k) = r(t_k)$, $k \in \mathbb{N}_0$. Furthermore, the sampled mode signal may be discontinuous at the time instant t_k, $k \in \mathbb{N}$, if a mode switch occurs in the time interval $(t_{k-1}, t_k]$. Figure 8.1 shows a sample path of the actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ of a switched system (3.57) with $M = 3$ modes. Note that when the mode sampling intensity parameter λ is sufficiently large, mode signal information samples will be frequently available; therefore, $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ is likely to be a good representation of the mode signal.

Now, we show that under certain conditions, the zero solution of the switched linear system (3.57) can be stabilized by a controller that depends only on the sampled information of the mode signal rather than the actual mode signal. Specifically, we consider the control law of the form

$$u(t) = K_{\sigma(t)} x(t). \tag{8.20}$$

The closed-loop system (3.57) under the control law (8.20) is given by

$$\mathrm{d}x(t) = (A_{r(t)} + B_{r(t)} K_{\sigma(t)}) x(t)\mathrm{d}t + D_{r(t)} x(t)\mathrm{d}W(t). \tag{8.21}$$

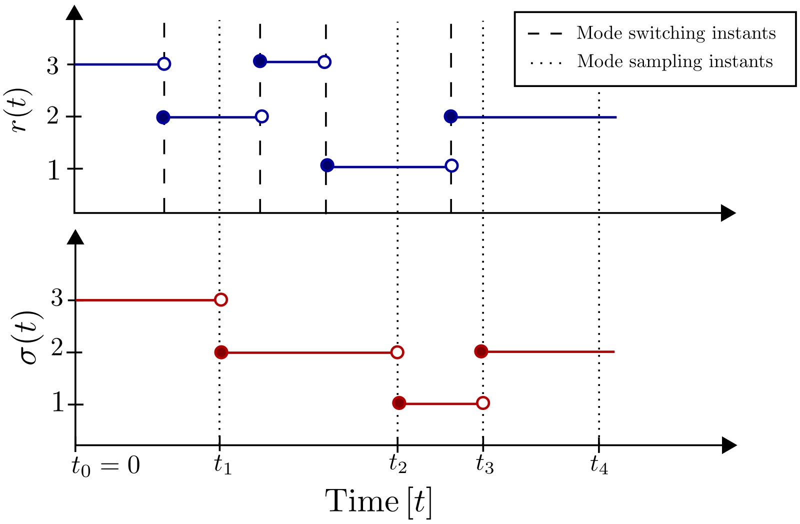

Figure 8.1: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time

We now verify that the closed-loop system (8.21) can be expressed as a switched linear stochastic dynamical system described by (8.1). For finite values of the mode sampling intensity parameter λ, the sampled mode signal is imperfect, that is, the actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ may take different values when $t \neq t_k$, $k \in \mathbb{N}_0$. We define the bivariate stochastic process

$$\{\hat{r}(t)\}_{t\geq 0} \triangleq \{(r(t), \sigma(t))\}_{t\geq 0}. \tag{8.22}$$

Under the assumption that the Poisson process $\{N(t) \in \mathbb{N}_0\}_{t\geq 0}$ and the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ are independent stochastic processes, for any $i, j, k, l \in \mathcal{M}$,

$$\begin{aligned} &\mathbb{P}[\hat{r}(t+\Delta t) = (j,l) \,|\, \hat{r}(t) = (i,k)] \\ &= \begin{cases} q_{i,j}\Delta t + o(\Delta t), & i \neq j,\ k = l, \\ 1 + q_{i,i}\Delta t + o(\Delta t), & i = j = k = l, \\ \lambda\Delta t + o(\Delta t), & i = j,\ k \neq l,\ i \neq k, \\ 1 + q_{i,i}\Delta t - \lambda\Delta t + o(\Delta t), & i = j,\ k = l,\ i \neq k, \\ o(\Delta t), & \text{otherwise}. \end{cases} \end{aligned} \tag{8.23}$$

It follows that the bivariate stochastic process $\{\hat{r}(t)\}_{t\geq 0}$ is a Markov chain with M^2 states given by $\{(1,1), (2,1), \ldots, (M,1), (1,2), (2,2), \ldots, (M,2), \ldots, (1,M), (1,M), \ldots, (M,M)\}$.

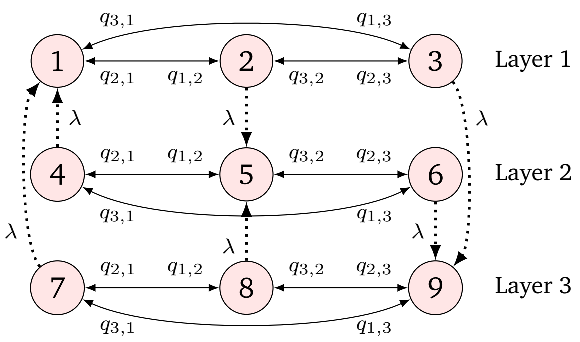

Figure 8.2: Transition diagram of a Markov chain of 9 states with a special structure for $M = 3$

We enumerate the states in this order as $\hat{\mathcal{M}} \triangleq \{1, 2, \ldots, M^2\}$. Furthermore, the generator of the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}}\}_{t \geq 0}$ is given by

$$\hat{Q} = \begin{bmatrix} T^1 & \lambda J_{\mathrm{M}}^2 & \cdots & \lambda J_M^M \\ \lambda J_M^1 & T^2 & \cdots & \lambda J_M^M \\ \vdots & \vdots & \ddots & \vdots \\ \lambda J_M^1 & \lambda J_M^2 & \cdots & T^M \end{bmatrix}, \tag{8.24}$$

where $T^i = Q - \lambda I_M + \lambda J_M^i$, $i \in \mathcal{M}$, and $J_M^i \in \mathbb{R}^{M \times M}$ denotes the matrix with the (i, i) entry being 1 and the rest of the entries being zero.

The Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}} = \{1, 2, \ldots, M^2\}\}_{t \geq 0}$ can be represented by a transition diagram with a special graph structure of M^2 nodes (Figure 8.2). In this graph structure, the nodes are placed in M layers. Nodes in the ith layer are numbered as $\{(i-1)M + 1, (i-1)M + 2, \ldots, (i-1)M + M\}$. Graph structure of each separate layer resembles the transition diagram of the Markov chain $\{r(t) \in \mathcal{M}\}_{t \geq 0}$. For example, an arrowed edge directed from the $((i-1)M + j)$th node to the $((i-1)M + k)$th node represents a possible transition from the state j to the state k of the Markov chain $\{r(t) \in I\}_{t \geq 0}$. On the other hand, between two distinct layers i and j in the graph structure of the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}}\}_{t \geq 0}$, there exist two directed edges: one from the $((i-1)M + j)$th node in the ith layer to the $((j-1)M + j)$th node in the jth layer, and another one from the $((j-1)M + i)$th node in the jth layer to the $((i-1)M + j)$th node in the ith layer. The directed edge from the ith layer to the jth layer represents a possible change in the state of the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t \geq 0}$ from i to j.

Since the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is irreducible, there exists a directed path between each pair of nodes within each layer of the transition diagram of the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}}\}_{t\geq 0}$. Furthermore, there exists a directed edge from each layer to another layer. It follows that there exists a directed path from each node to another node in the transition diagram of the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}}\}_{t\geq 0}$. We conclude that the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}}\}_{t\geq 0}$ is also irreducible. Consequently, there exists a unique stationary probability distribution $\hat{\pi} \in \mathbb{R}^{M^2}$ such that $\hat{\pi}^{\mathrm{T}}\hat{Q} = 0$, $\hat{\pi}_i > 0$, $i \in \hat{\mathcal{M}}$, and $\sum_{i\in\hat{\mathcal{M}}} \hat{\pi}_i = \sum_{i\in\mathcal{M}} \sum_{j\in\mathcal{M}} \hat{\pi}_{(i-1)M+j} = 1$

The Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}}\}_{t\geq 0}$ is irreducible; therefore, we can express the closed-loop system (8.21) as a comparison system which is a switched linear stochastic dynamical system of M^2 modes described by (8.1) with subsystem matrices $A_{(i-1)M+j}$ replaced by $A_j - B_j K_i$, and $D_{(i-1)M+j}$ replaced by D_j, for $i, j \in \mathcal{M}$. The transition between the modes of this comparison system is represented by the transition diagram of the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}}\}_{t\geq 0}$ with M layers.

We now state our main result on the almost sure asymptotic stabilization of the switched stochastic dynamical system (3.57) under sampled mode information. The result is based on the stability analysis for the comparison system (8.1) stated in Theorem 8.1.

Theorem 8.2. Consider the continuous-time switched linear stochastic dynamical system given by (3.57). If there exist $P > 0$ and scalars $\zeta_i \in \mathbb{R}$, $i \in \mathcal{M}$, such that (8.15) and

$$\sum_{i\in\mathcal{M}} \sum_{j\in\mathcal{M}} \hat{\pi}_{(i-1)M+j} \Big(\beta_{i,j} - \frac{\lambda_{\min}^2(D_j^{\mathrm{T}} P + P D_j)}{2\lambda_{\max}^2(P)} \Big) < 0, \tag{8.25}$$

where

$$\beta_{i,j} = \begin{cases} \zeta_j, & i = j, \\ \zeta_j + \frac{2\lambda_{\max}(P B_j B_j^{\mathrm{T}} P)}{\lambda_{\min}(P)} - \frac{\lambda_{\min}(P(B_j B_i^{\mathrm{T}} + B_i B_j^{\mathrm{T}})P)}{\lambda_{\max}(P)}, & i \neq j, \end{cases} \tag{8.26}$$

and $\hat{\pi} \in \mathbb{R}^{M^2}$ is the unique stationary distribution of the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}} = \{1, 2, \ldots, M^2\}\}_{t\geq 0}$ characterized by the generator matrix $\hat{Q}$ given in (8.24), then the feedback control law (8.20) with the feedback gain matrix given by

$$K_{\sigma(t)} = -B_{\sigma(t)}^{\mathrm{T}} P, \tag{8.27}$$

guarantees that the zero solution $x(t) \equiv 0$ of the closed-loop system (3.57) and (8.20) is asymptotically stable almost surely.

Proof. The closed-loop system under the control law (8.27) can be expressed as a comparison system which is a switched linear stochastic system given by (8.1). The comparison system is composed of M^2 modes described by the subsystem matrices $A_{(i-1)M+j}$ replaced by $A_j - B_j B_i^{\mathrm{T}} P$, and $D_{(i-1)M+j}$ replaced by D_j, for $i,\ j \in \mathcal{M}$. The mode signal of the comparison system is the Markov chain $\{\hat{r}(t) \in \hat{\mathcal{M}} = \{1, 2, \ldots, M^2\}\}_{t\geq 0}$ characterized by the generator matrix $\hat{Q}$ given in (8.24). We set the initial conditions of the comparison system as $x(0) = x_0$ and $r(0) = \hat{r}(0)$. Almost sure asymptotic stability of the zero solution of the comparison system (8.1) implies almost sure asymptotic stability of the zero solution of the system (3.57). Thus the result follows from Theorem 8.1 with $\zeta_{(i-1)M+j}$ replaced by $\beta_{i,j}$, for $i, j \in \mathcal{M}$. □

The transition rates $q_{i,j},\ i, j \in \mathcal{M}$, as well as the mode sampling intensity λ affect the stability conditions of the closed-loop system under the control law (8.27). Note that the stationary distribution $\hat{\pi} \in \mathbb{R}^{M^2}$ also depends on the values of both $q_{i,j},\ i, j \in \mathcal{M}$, and λ. Therefore, the condition (8.25), which involves the stationary distribution $\hat{\pi} \in \mathbb{R}^{M^2}$, is satisfied only for certain values of $q_{i,j},\ i, j \in \mathcal{M}$, and λ.

When the transition rates $q_{i,j},\ i, j \in \mathcal{M}$, are large, switchings between the modes of the system (3.57) are likely to be frequent. In this case, if the mode sampling intensity λ is very small, then the stationary probability distributions associated with the states $\{(i-1)M+j\ :\ i, j \in \mathcal{M},\ i \neq j\}$ are high. Furthermore, the sampled mode signal $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ is expected to differ from the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$. On the contrary, when the mode switchings are statistically rare and the mode sampling intensity λ is sufficiently large, the stationary probability distributions associated with the states $\{(i-1)M+i\ :\ i \in \mathcal{M}\}$ are high. Moreover, $\{\sigma(t) \in \mathcal{M}\}_{t\geq 0}$ is expected to be a good representation of the mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$.

8.4 Illustrative Numerical Example

In this section, we present a numerical example to demonstrate the efficacy of our approach. Specifically, we consider the switched linear stochastic dynamical system (3.57) with $M = 3$ modes described by the subsystem matrices given by

$$A_1 = \begin{bmatrix} 2 & -2 \\ 3 & 1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 2 & 1 \\ -0.5 & 3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 & 0.3 \\ 0.2 & 1 \end{bmatrix},$$

and $D_1 = D_2 = D_3 = I_2$. The mode signal $\{r(t) \in \mathcal{M} \triangleq \{1, 2, 3\}\}_{t\geq 0}$ of the system is assumed to be a 3-state Markov chain characterized by the generator matrix

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}. \tag{8.28}$$

The mode signal $\{r(t) \in \mathcal{M}\}_{t\geq 0}$ is assumed to be available only at certain time instants. Furthermore, intervals between these time instants are assumed to be distributed independently by exponential distribution with the parameter $\lambda = 5$.

The bivariate stochastic process $\{\hat{r}(t) \in \hat{\mathcal{M}} \triangleq \{1, 2, \ldots, 9\}\}_{t\geq 0}$ defined in (8.22) is a Markov chain with the unique invariant distribution given by

$$\hat{\pi}_{(i-1)M+j} = \begin{cases} \frac{1}{4}, & i, j \in \mathcal{M},\ i = j, \\ \frac{1}{24}, & i, j \in \mathcal{M},\ i \neq j. \end{cases}$$

Note that the positive-definite matrix $P = 5I_2$ and the scalars $\zeta_1 = -4.3$, $\zeta_2 = 5$, $\zeta_3 = -3.3$ satisfy the conditions (8.15) and (8.25). Therefore, it follows from Theorem 8.2 that the control law (8.20) guarantees almost sure asymptotic stability of the zero solution

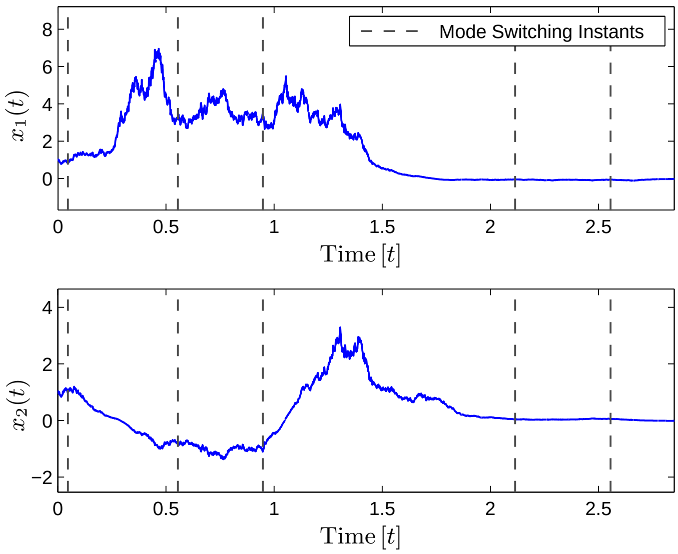

Figure 8.1: State trajectory versus time (mode sampling intensity parameter $\lambda = 5$)

$x(t) \equiv 0$ of the system given by (3.57).

With initial conditions $x(0) = [1,\ 1]^{\mathrm{T}}$ and $r(0) = 1$, Figures 8.1 and 8.2 show sample paths of $x(t)$ and $u(t)$, respectively.

The piecewise-continuous control law (8.20) depends on the sampled mode signal information $\sigma(t)$. As a consequence, control profile is subject to jumps when $\sigma(t)$ changes its value at mode sampling instants. Note that both the mode sampling intensity and the frequency of mode switches directly affect the quality of the representation of the actual mode signal by the sampled mode signal. In this example, the sampling intensity $\lambda = 5$ is relatively high compared to the frequency of mode switches; consequently, the sampled mode signal $\sigma(t)$ closely matches the actual mode signal $r(t)$ (see Figure 8.3).

When the mode sampling intensity parameter λ is small, the mode signal is sampled statistically rarely, and hence the performance of the approximation is expected to be poor. As a consequence, the control performance may also deteriorate. In order to demonstrate the effect of a relatively small mode sampling intensity parameter, we set $\lambda = 1$ and obtain sample paths of the state $x(t)$, the control input $u(t)$, as well as the actual mode signal $r(t)$ and its sampled version $\sigma(t)$, which we present in Figures 8.4–8.6 respectively.

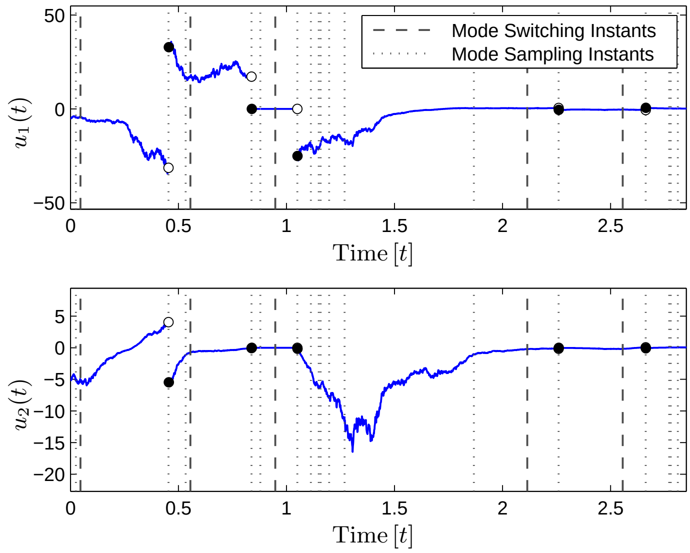

Figure 8.2: Control input versus time (mode sampling intensity parameter $\lambda = 5$)

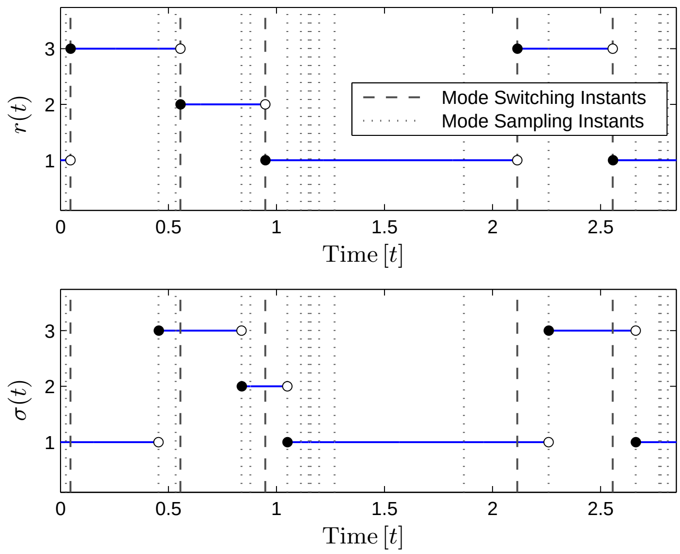

Figure 8.3: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time (mode sampling intensity parameter $\lambda = 5$)

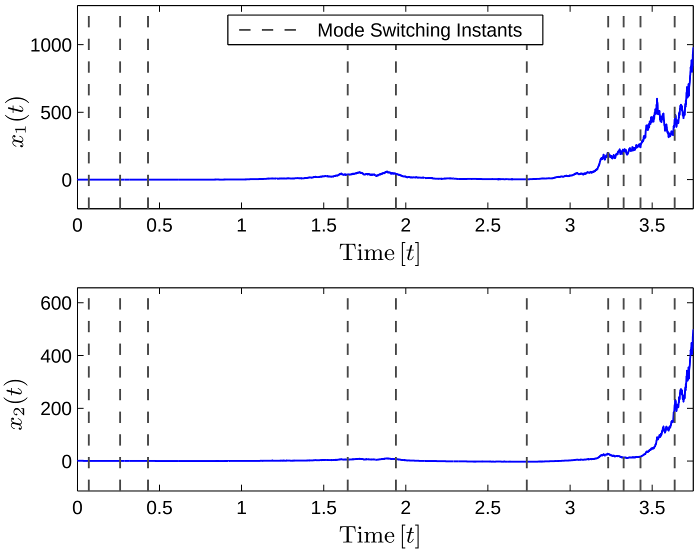

Figure 8.4: State trajectory versus time (mode sampling intensity parameter $\lambda = 1$)

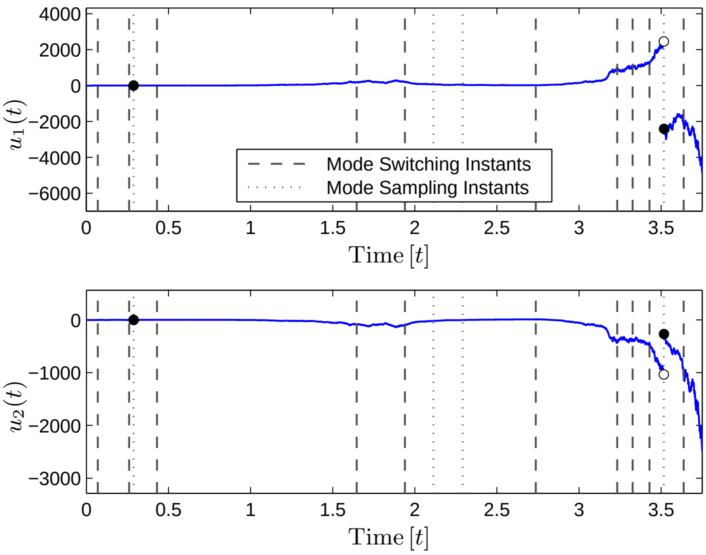

Figure 8.5: Control input versus time (mode sampling intensity parameter $\lambda = 1$)

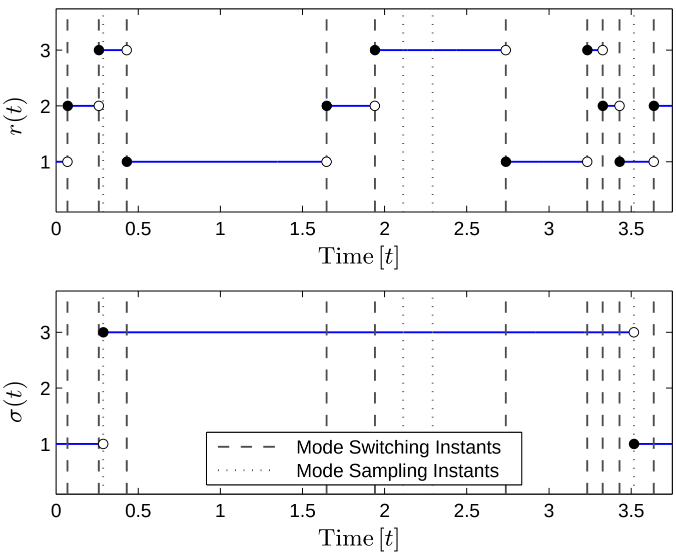

Figure 8.6: Actual mode signal $r(t)$ and the sampled mode signal $\sigma(t)$ versus time (mode sampling intensity parameter $\lambda = 1$)

8.5 Conclusion

The stability of continuous-time switched linear stochastic systems was investigated. A quadratic Lyapunov-like function has been employed for obtaining sufficient almost sure asymptotic stability conditions. Moreover, feedback stabilization of the zero solution under sampled mode information was explored. The intervals between mode sampling time instants are assumed to be exponentially distributed random variables. We proposed a piecewise-continuous control law that guarantees almost sure asymptotic stability of the zero solution. The proposed control law depends only on the sampled mode signal which is constructed from the available mode samples by using "sample and hold" technique.

In Chapter 9 below, we investigate stabilization of a discrete-time switched stochastic system under the assumption that the active operation mode of the switched system is available for control purposes at *random* time instants. In this sense the control problem we consider in Chapter 9 is closely related to the feedback control problem that we have explored in this chapter.

Chapter 9

Feedback Control of Discrete-Time Switched Stochastic Systems Using Randomly Available Active Mode Information

9.1 Introduction

In Chapters 6 and 7 we investigated the stabilization of discrete-time switched stochastic systems for the case where only *sampled* mode information is available for control purposes. Under the assumption that the active mode can be *periodically* observed, we proposed stabilizing feedback control frameworks that utilize the available mode information.

In this chapter our goal is to explore feedback stabilization of discrete-time switched stochastic systems for the case where the active operation mode, which is modeled as a finite-state discrete-time Markov chain (see Sections 2.3.1 and 2.3.2), is observed only at *random* time instants. Specifically, we assume that the length of intervals between consecutive mode observation instants are identically distributed independent random variables. We employ a renewal process (see Section 2.3.3) to characterize the occurrences of random mode observations. This characterization allows us to also explore periodic mode observations, which are discussed in Chapters 3, 6, and 7, as a special case.

We propose a linear feedback control law with a piecewise-constant gain matrix that is switched depending on the value of a randomly sampled version of the mode signal. In order to investigate the evolution of the active mode together with its randomly sampled version, we construct a stochastic process that represents sequences of values the mode takes between random mode observation instants. This sequence-valued stochastic process turns out to be a countable-state Markov chain defined over a set that is composed of all possible mode sequences of finite length. We first analyze the probabilistic dynamics of this sequence-valued Markov chain. Then based on our analysis, we obtain sufficient stabilization conditions for the closed-loop switched stochastic system under our proposed control framework. These stabilization conditions let us assess whether the closed-loop system is stable for a given probability distribution for the length of intervals between consecutive mode observation instants. As this probability distribution is not assumed to have a certain structure, the result presented in this chapter can also be considered as a generalization of the result provided in Chapter 8, where stabilization problem is discussed in continuous time and the random intervals between mode sampling instants are specifically assumed to be exponentially distributed. In this chapter we also explore the case where perfect information regarding the probability distribution for the length of intervals between consecutive mode observation instants is not available. For this problem setting, we present alternative sufficient stabilization conditions which can be used for verifying stability even if the distribution is not exactly known.

The contents of this chapter are as follows. In Section 9.2, we propose our feedback control framework for stabilizing discrete-time switched stochastic systems under randomly available mode information. Then in Section 9.3, we present sufficient conditions under which our proposed control law guarantees almost sure asymptotic stabilization. In Section 9.4, we demonstrate the efficacy of our results with two illustrative numerical examples. Finally, in Section 9.5 we conclude the chapter.

9.2 Stabilizing Discrete-Time Switched Stochastic Systems with Randomly Available Mode Information

In this section, we propose a feedback control framework for stabilizing a switched stochastic system by using only the randomly available mode information. Specifically, we con-

Figure 9.1: Mode transition diagram for $\{r(k) \in \mathcal{M} \triangleq \{1, 2\}\}_{k\in\mathbb{N}_0}$

sider the discrete-time switched linear stochastic dynamical system with $M \in \mathbb{N}$ number of modes given by (6.1) with the initial conditions $x(0) = x_0$ and $r(0) = r_0$. Hence, the initial distribution of the mode signal $\{r(k) \in \mathcal{M} \triangleq \{1, 2, , \ldots, M\}\}_{k\in\mathbb{N}_0}$ is given by $\nu : \mathcal{M} \to [0, 1]$ such that $\nu_{r_0} = 1$ and $\nu_i = 0$, $i \neq r_0$.

We use the matrix $P \in \mathbb{R}^{M\times M}$ to characterize probability of transitions between the modes of the switched system. Specifically, $p_{i,j} \in [0, 1]$, which is the (i, j)th entry of the matrix P, denotes the probability of a transition from mode i to mode j. Furthermore, we use $p^{(l)}_{i,j}$ to denote (i, j)th entry of the matrix P^l. Note that $p^{(l)}_{i,j} \in [0, 1]$ is in fact the l-step transition probability from mode i to mode j, that is,

$$p^{(l)}_{i,j} \triangleq \mathbb{P}[r(k + l) = j|r(k) = i], \ l \in \mathbb{N}_0, \ i, j \in \mathcal{M}, \tag{9.1}$$

with $p^{(0)}_{i,i} = 1$, $i \in \mathcal{M}$, $p^{(0)}_{i,j} = 0$, $i \neq j$. Furthermore, $p^{(1)}_{i,j} = p_{i,j}$, $i, j \in \mathcal{M}$. Mode signal can be represented using a transition diagram, which shows possible transitions between the operation modes of the switched system. Mode transition diagram for a switched system with two modes is shown in Figure 9.1.

In this chapter, we assume that the mode signal is an aperiodic, irreducible Markov chain and has the invariant distribution $\pi : \mathcal{M} \to [0, 1]$.

9.2.1 Feedback Control Under Randomly Observed Mode Information

In this chapter, active mode of the switched stochastic system (6.1) is assumed to be observed only at random time instants, which we denote by $t_i \in \mathbb{N}_0$, $i \in \mathbb{N}_0$. We assume that $t_0 = 0$ and $\tau_i \triangleq t_i - t_{i-1} \in \mathbb{N}$, $i \in \mathbb{N}$, are independent random variables that are distributed according to a common distribution $\mu : \mathbb{N} \to [0, 1]$ for all $i \in \mathbb{N}$ such that $\hat{\tau} \triangleq \sum_{\tau\in\mathbb{N}} \tau\mu_\tau < \infty$. In this problem setting, the initial mode information r_0 is assumed to be available to the controller, and a renewal process $\{N(k) \in \mathbb{N}_0\}_{k\in\mathbb{N}_0}$ is employed for counting the number of mode observations that are obtained after the initial time. We assume that the renewal process $\{N(k) \in \mathbb{N}_0\}_{k\in\mathbb{N}_0}$ and the mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ are mutually independent.

Following our approach in Chapter 3, we employ a linear feedback control law with a ‘piecewise-constant’ feedback gain matrix that depends only on the obtained mode information. Specifically, we consider the control law

$$u(k) = K_{\sigma(k)}x(k), \quad k \in \mathbb{N}_0, \tag{9.2}$$

where $\{\sigma(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is the sampled version of the mode signal defined by

$$\sigma(k) \triangleq r(t_{N(k)}), \quad k \in \mathbb{N}_0. \tag{9.3}$$

Note that the sampled mode signal $\{\sigma(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ acts as a switching mechanism for the linear feedback gain, which remains constant between two consecutive mode observation instants, that is, $K_{\sigma(k)} = K_{r(t_i)}$ for $k \in [t_i, t_{i+1})$.

Between two consecutive mode observation instants, the feedback gain $K_{\sigma(\cdot)}$ stays constant, whereas the active mode $r(\cdot)$ of the dynamical system (6.1) may change its value. Stabilization performance under the control law (9.2) hence depends not only on the length of the intervals between random mode observation instants, but also on how the active mode switches during the intervals.

In Figure 9.2, we show sample paths of the active mode signal $r(\cdot)$ and its sampled version $\sigma(\cdot)$ for a switched stochastic system with $M = 2$ modes. In this example, active mode is observed at time instants $t_0 = 0$, $t_1 = 2$, $t_2 = 5$, $t_3 = 6$, $t_4 = 8$, $\ldots$. Note that at mode observation instants actual mode signal $r(\cdot)$ and its sampled version $\sigma(\cdot)$ have the same value. However, at the other time instants, sampled mode signal may differ from the actual mode, since between mode observation instants, the actual mode signal may change its value.

In order to investigate the evolution of the active mode between consecutive mode observation instants, we construct a new stochastic process $\{s(i)\}_{i\in\mathbb{N}_0}$ that takes values from a countable set of mode sequences of variable length. Specifically, we define the

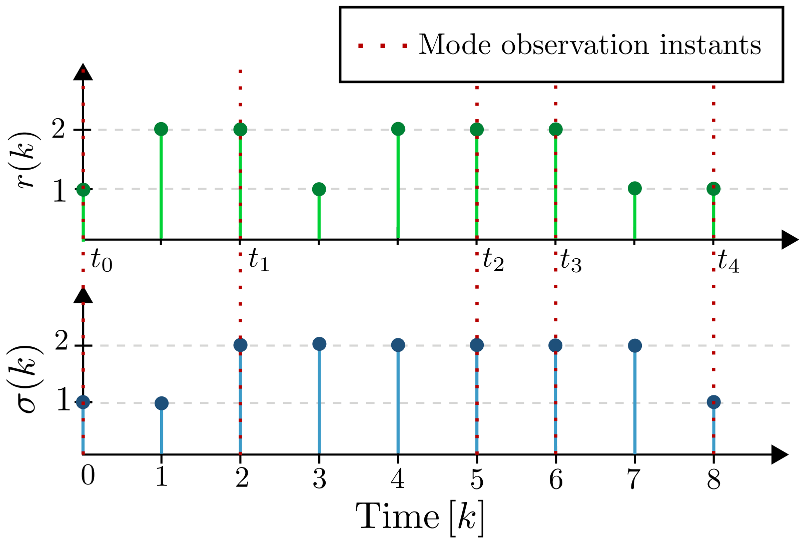

Figure 9.2: Actual mode signal $r(k)$ and its sampled version $\sigma(k)$

sequence-valued stochastic process $\{s(i)\}_{i\in\mathbb{N}_0}$ by

$$s(i) \triangleq \big(r(t_i), r(t_i+1), \ldots, r(t_{i+1}-1)\big), \ i \in \mathbb{N}_0, \tag{9.4}$$

with t_i, $i \in \mathbb{N}_0$, being the random mode observation instants. By the definition given in (9.4), $s(i)$ represents the sequence of values that the active mode $r(\cdot)$ takes between the mode observation instants t_i and t_{i+1}. Hence, $s_n(i)$, which denotes the nth element of the sequence $s(i)$, represents the value of the active mode $r(\cdot)$ at time t_i+n-1. Furthermore, the value of the sampled mode signal $\sigma(\cdot)$ between time instants t_i and t_{i+1} is represented by $s_1(i) = r(t_i)$. Note that the active mode is observed and becomes available for control purposes only at time instants t_i, $i \in \mathbb{N}_0$. Thus, the controller has access only to the observed mode data $\sigma(t_i) = r(t_i)$, $i \in \mathbb{N}_0$, which correspond to the first elements of the sequences $s(i)$, $i \in \mathbb{N}_0$.

For the sample paths of active mode signal $r(\cdot)$ and its sampled version $\sigma(\cdot)$ shown in Figure 9.2, mode sequences between mode observation instants $t_0 = 0$, $t_1 = 2$, $t_2 = 5$, $t_3 = 6$, $t_4 = 8$, are given as $s(0) = (1,2)$, $s(1) = (2,1,2)$, $s(2) = (2)$, $s(3) = (2,1)$. The key property of the stochastic process $\{s(i)\}_{i\in\mathbb{N}_0}$ is that, a given mode sequence $s(i)$ indicates full information of the active mode as well as the information the controller has during the time interval between consecutive mode observation instants t_i and t_{i+1}.

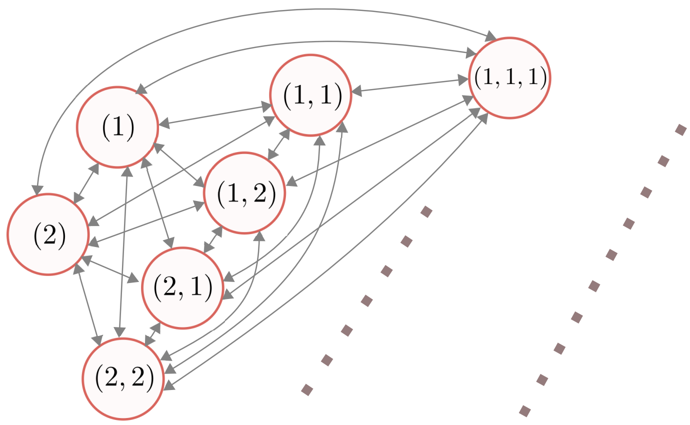

Figure 9.3: Transition diagram of the sequence-valued discrete-time countable-state Markov chain $\{s(i) \in \mathcal{S} \triangleq \{(1), (2), (1,1), \ldots\}\}_{i\in\mathbb{N}_0}$ over the set of mode sequences of variable length

In what follows, we explain the probabilistic dynamics of the stochastic process $\{s(i)\}_{i\in\mathbb{N}_0}$ and provide key results that we will use in Section 9.3 for analyzing stability of the closed-loop switched stochastic control system (6.1), (9.2).

9.2.2 Probabilistic Dynamics of Mode Sequences

The possible values of sequence that the stochastic process $\{s(i)\}_{i\in\mathbb{N}_0}$ may take are characterized by the set

$$\mathcal{S} \triangleq \{(q_1, q_2, \ldots, q_\tau) : p_{q_n, q_{n+1}} > 0,\ n \in \{1, \ldots, \tau - 1\}; \\ q_n \in \mathcal{M},\ n \in \{1, \ldots, \tau\};\ \mu_\tau > 0\}. \tag{9.5}$$

Note that the sequence-valued stochastic process $\{s(i)\}_{i\in\mathbb{N}_0}$ is a discrete-time Markov chain on the countable state space represented by $\mathcal{S}$, which contains all possible mode sequences for all possible lengths of intervals between consecutive mode observation instants. For example, consider the case where the switched system (6.1) has two modes. Furthermore, suppose that $\mu_\tau > 0$ for all $\tau \in \mathbb{N}$. In other words, lengths of intervals between mode observation instants may take any positive integer value. In this case, the state space $\mathcal{S} = \{(1), (2), (1,1), (1,2), \ldots\}$ contains all finite-length mode sequences composed of elements from $\mathcal{M} = \{1, 2\}$. See Figure 9.3 for the transition diagram of countable-state Markov chain $\{s(i) \in \mathcal{S}\}_{i\in\mathbb{N}_0}$ of this example.

It is important to note that if the set $\{\tau \in \mathbb{N} : \mu_\tau > 0\}$ has finite number of elements, then set $\mathcal{S}$ will also contain finite number of sequences. In other words, if the lengths of intervals between mode observation instants have finite number of possible values, then the number of possible sequences is also finite. For example, consider the case where the operation mode of the switched system, which takes values from the index set $\mathcal{M} = \{1, 2\}$, is observed periodically with period 2, that is, $\mu_2 = 1$. In this case, $\mathcal{S} = \{(1,1), (1,2), (2,1), (2,2)\}$ (see Figure 9.4).

We now characterize the initial distribution and the state-transition probabilities of the discrete-time Markov chain $\{s(i) \in \mathcal{S}\}_{i\in\mathbb{N}_0}$ as functions of the initial distribution and the state-transition probabilities of the mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$. Specifically, the initial distribution $\lambda : \mathcal{S} \to [0,1]$ of the discrete-time Markov chain $\{s(i) \in \mathcal{S}\}_{i\in\mathbb{N}_0}$ is given by

$$\begin{aligned}\lambda_q &= \mathbb{P}[s(0) = q] \\ &= \mathbb{P}[t_1 = |q|, r(0) = q_1, \ldots, r(|q|-1) = q_{|q|}] \\ &= \mathbb{P}[t_1 = |q| \bigm| r(0) = q_1, \ldots, r(|q|-1) = q_{|q|}] \\ &\quad \cdot \mathbb{P}[r(0) = q_1, \ldots, r(|q|-1) = q_{|q|}], \ q \in \mathcal{S}. \end{aligned} \tag{9.6}$$

Since the mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ and the mode observation counting process $\{N(k) \in \mathbb{N}_0\}_{k\in\mathbb{N}_0}$ are mutually independent, mode transitions and mode observations occur independently. Hence, $t_1 = \tau_1$ is independent of $r(n)$ for every $n \in \mathbb{N}_0$. As a consequence,

$$\begin{aligned}\lambda_q &= \mathbb{P}[t_1 = |q|]\,\mathbb{P}[r(0) = q_1, \ldots, r(|q|-1) = q_{|q|}] \\ &= \mathbb{P}[t_1 = |q|]\,\mathbb{P}[r(0) = q_1] \\ &\quad \cdot \prod_{n=1}^{|q|-1} \mathbb{P}[r(n) = q_{n+1} | r(n-1) = q_n] \\ &= \begin{cases} \mu_{|q|} \prod_{n=1}^{|q|-1} p_{q_n, q_{n+1}}, & \text{if } q_1 = r_0, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{9.7}$$

for $q \in \mathcal{S}$. Note that $s_1(0)$, which is the first element of the first mode sequence $s(0)$, is equal to the initial mode r_0.

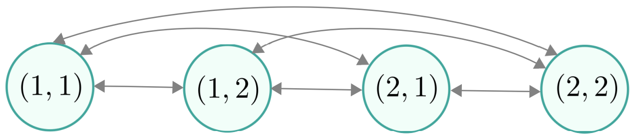

Figure 9.4: Transition diagram of the sequence-valued discrete-time Markov chain $\{s(i) \in \mathcal{S} \triangleq \{(1,1),(1,2),(2,1),(2,2)\}\}_{i\in\mathbb{N}_0}$

Probability of a transition from a mode sequence $q \in \mathcal{S}$ to another mode sequence $\bar{q} \in \mathcal{S}$ is given by

$$\begin{aligned}\rho_{q,\bar{q}} &= \mathbb{P}[s(i+1) = \bar{q} | s(i) = q],\\ &= \mathbb{P}\big[\tau_{i+1} = |\bar{q}|, r(t_{i+1}) = \bar{q}_1, \ldots,\\ &\qquad r(t_{i+1} + |\bar{q}| - 1) = \bar{q}_{|\bar{q}|} \,\big|\, \tau_i = |q|,\\ &\qquad r(t_i) = q_1, \ldots, r(t_i + |q| - 1) = q_{|q|}\big],\end{aligned} \tag{9.8}$$

for $i \in \mathbb{N}_0$. Note that τ_{i+1} is independent of the random variables $r(n)$, $n \in \mathbb{N}_0$, and τ_i. Furthermore, given $r(t_i + \tau_i - 1)$, the random variable $r(t_{i+1})$ is conditionally independent of $r(t_i), \ldots, r(t_i + \tau_i - 2)$, and τ_i. It follows that

$$\begin{aligned}\rho_{q,\bar{q}} &= \mathbb{P}\big[\tau_{i+1} = |\bar{q}|, r(t_{i+1}) = \bar{q}_1, \ldots,\\ &\qquad r(t_{i+1} + |\bar{q}| - 1) = \bar{q}_{|\bar{q}|} \,\big|\, r(t_i + |q| - 1) = q_{|q|}\big]\\ &= \mathbb{P}[r(t_{i+1}) = \bar{q}_1 \,|\, r(t_i + |q| - 1) = q_{|q|}]\mathbb{P}[\tau_{i+1} = |\bar{q}|]\\ &\quad \cdot \prod_{n=1}^{|\bar{q}|-1} \mathbb{P}[r(t_{i+1} + n) = \bar{q}_{n+1} | r(t_{i+1} + n - 1) = \bar{q}_n]\\ &= p_{q_{|q|},\bar{q}_1} \;\; \mu_{|\bar{q}|} \prod_{n=1}^{|\bar{q}|-1} p_{\bar{q}_n,\bar{q}_{n+1}}, \quad i \in \mathbb{N}_0.\end{aligned} \tag{9.9}$$

Note that $\mu_{|\bar{q}|}$ in (9.9) represents the probability that length of the interval between two mode observation instants is equal to the length of the sequence $\bar{q}$, whereas $p_{q_{|q|},\bar{q}_1} \in [0, 1]$ represents the transition probability from the mode represented by the last element of sequence q, to the mode represented by the first element of the sequence $\bar{q}$. Furthermore, the expression $\prod_{n=1}^{|\bar{q}|-1} p_{\bar{q}_n,\bar{q}_{n+1}}$ denotes the joint probability that the active mode takes the values denoted by the elements of the sequence $\bar{q}$ until the next mode observation instant.

Since the mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is aperiodic and irreducible, mode sequences may start with any of the possible modes indicated by the index set $\mathcal{M} = \{1, \ldots, M\}$. Furthermore, it is possible to reach from any mode sequence to another mode sequence in a finite number of mode observations. Hence, the discrete-time Markov chain $\{s(i) \in \mathcal{S}\}_{i\in\mathbb{N}_0}$ is irreducible. In Lemma 9.1 below, we provide the invariant distribution for the countable-state discrete-time Markov chain $\{s(i) \in \mathcal{S}\}_{i\in\mathbb{N}_0}$. The invariant distribution for the case where $\mathcal{S}$ contains only sequences of fixed length $T \in \mathbb{N}$ is provided in [86]. In Lemma 9.1, we consider the more general case where $\mathcal{S}$ may contain countably infinite number of sequences of all possible lengths.

Lemma 9.1. Discrete-time Markov chain $\{s(i) \in \mathcal{S}\}_{i\in\mathbb{N}_0}$ has the invariant distribution $\phi : \mathcal{S} \to [0, 1] : q \mapsto \phi_q$ given by

$$\phi_q \triangleq \pi_{q_1}\mu_{|q|} \prod_{n=1}^{|q|-1} p_{q_n,q_{n+1}}, \quad q \in \mathcal{S}, \tag{9.10}$$

where $\pi : \mathcal{M} \to [0, 1]$ and $p_{i,j}$, $i, j \in \mathcal{M}$, respectively denote the invariant distribution and transition probabilities of the finite-state Markov chain $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$.

Proof. We prove this result by showing that $\phi_{\bar{q}} = \sum_{q\in\mathcal{S}} \phi_q \rho_{q,\bar{q}}$, for all $\bar{q} \in \mathcal{S}$. It follows from (9.9) and (9.10) that

$$\sum_{q\in\mathcal{S}} \phi_q\rho_{q,\bar{q}} = \Big(\sum_{q\in\mathcal{S}} \pi_{q_1}\mu_{|q|}\Big(\prod_{n=1}^{|q|-1} p_{q_n,q_{n+1}}\Big)p_{q_{|q|},\bar{q}_1}\Big)\mu_{|\bar{q}|} \prod_{n=1}^{|\bar{q}|-1} p_{\bar{q}_n,\bar{q}_{n+1}}, \ \bar{q} \in \mathcal{S}. \tag{9.11}$$

Now let $\mathcal{S}_\tau \triangleq \{q \in \mathcal{S} : |q| = \tau\}$, $\tau \in \mathbb{N}$. Note that the set $\mathcal{S}_\tau$ contains all mode sequences of length τ. We rewrite the sum in (9.11) to obtain

$$\begin{aligned}\sum_{q\in\mathcal{S}} \pi_{q_1}\mu_{|q|}\Big(\prod_{n=1}^{|q|-1} p_{q_n,q_{n+1}}\Big)p_{q_{|q|},\bar{q}_1} &= \sum_{\tau\in\mathbb{N}} \mu_\tau \sum_{q\in\mathcal{S}_\tau} \pi_{q_1}\Big(\prod_{n=1}^{\tau-1} p_{q_n,q_{n+1}}\Big)p_{q_\tau,\bar{q}_1} \\ &= \sum_{\tau\in\mathbb{N}} \mu_\tau \sum_{q_\tau\in\mathcal{M}} \cdots \sum_{q_1\in\mathcal{M}} \pi_{q_1}\Big(\prod_{n=1}^{\tau-1} p_{q_n,q_{n+1}}\Big)p_{q_\tau,\bar{q}_1}. \end{aligned} \tag{9.12}$$

Note that since $\pi : \mathcal{M} \to [0, 1]$ is the invariant distribution of the finite-state Markov chain $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$, it follows that $\sum_{i\in\mathcal{M}} \pi_i p_{i,j} = \pi_j$, $i, j \in \mathcal{M}$. Thus, we have $\sum_{q_n\in\mathcal{M}} \pi_{q_n} p_{q_n,q_{n+1}} = \pi_{q_{n+1}}$, $n \in \{1, \ldots, \tau - 1\}$, and $\sum_{q_\tau\in\mathcal{M}} \pi_{q_\tau} p_{q_\tau,\bar{q}_1} = \pi_{\bar{q}_1}$. As a result, from (9.12) we obtain

$$\sum_{q\in\mathcal{S}}\pi_{q_1}\mu_{|q|}\Big(\prod_{n=1}^{|q|-1}p_{q_n,q_{n+1}}\Big)p_{q_{|q|},\bar{q}_1}=\sum_{\tau\in\mathbb{N}}\mu_\tau\pi_{\bar{q}_1}$$
$$=\pi_{\bar{q}_1}. \tag{9.13}$$

Finally, substituting (9.13) into (9.11) yields

$$\sum_{q\in\mathcal{S}}\phi_q\rho_{q,\bar{q}}=\pi_{\bar{q}_1}\mu_{|\bar{q}|}\prod_{n=1}^{|\bar{q}|-1}p_{\bar{q}_n,\bar{q}_{n+1}}$$
$$=\phi_{\bar{q}},\quad \bar{q}\in\mathcal{S}, \tag{9.14}$$

which completes the proof. □

We have now established that the countable-state Markov chain $\{s(k)\in\mathcal{S}\}_{k\in\mathbb{N}_0}$ is irreducible and has the invariant distribution presented in Lemma 9.1. In the next section, we use the ergodic theorem provided in Section 2.3.2 for $\{s(k)\in\mathcal{S}\}_{k\in\mathbb{N}_0}$ to obtain the main results of this chapter.

9.3 Sufficient Conditions for Almost Sure Asymptotic Stabilization

In this section, we employ the results presented in Section 9.2 to obtain sufficient conditions for almost sure asymptotic stabilization of the closed-loop system (6.1) under the control law (9.2).

Theorem 9.1. Consider the switched linear stochastic system (6.1). If there exist matrices $\tilde{R}>0$, $L_i\in\mathbb{R}^{m\times n}$, $i\in\mathcal{M}$, and scalars $\zeta_{i,j}\in(0,\infty)$, $i,j\in\mathcal{M}$, such that

$$0\geq(A_i\tilde{R}+B_iL_j)^{\mathrm{T}}\tilde{R}^{-1}(A_i\tilde{R}+B_iL_j)-\zeta_{i,j}\tilde{R},\quad i,j\in\mathcal{M}, \tag{9.15}$$

$$\sum_{\tau\in\mathbb{N}}\mu_\tau\sum_{l=1}^{\tau}\sum_{i,j\in\mathcal{M}}\pi_i p_{i,j}^{(l-1)}\ln\zeta_{j,i}<0, \tag{9.16}$$

then the linear feedback control law (9.2) with the feedback gain matrix

$$K_{\sigma(k)} = L_{\sigma(k)}\tilde{R}^{-1}, \tag{9.17}$$

guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (6.1) and (9.2) is asymptotically stable almost surely.

Proof. First, we define $V(x) \triangleq x^{\mathrm{T}}Rx$, where $R \triangleq \tilde{R}^{-1}$. It follows from (6.1) and (9.2) that

$$V(x(k+1)) = x^{\mathrm{T}}(k)(A_{r(k)} + B_{r(k)}K_{\sigma(k)})^{\mathrm{T}}R(A_{r(k)} + B_{r(k)}K_{\sigma(k)})x(k),\ k \in \mathbb{N}_0. \tag{9.18}$$

We set $L_j = K_jR^{-1}$, $j \in \mathcal{M}$, and use (9.15) and (9.18) to obtain

$$\begin{aligned} V(x(k+1)) &\leq \zeta_{r(k),\sigma(k)}V(x(k)) \\ &\leq \eta(k)V(x(0)), \quad k \in \mathbb{N}_0, \end{aligned} \tag{9.19}$$

where $\eta(k) \triangleq \prod_{n=0}^{k} \zeta_{r(n),\sigma(n)}$, $k \in \mathbb{N}$. We will first show that $\eta(k) \to 0$ almost surely as $k \to \infty$. Note that $\eta(k) > 0$, $k \in \mathbb{N}_0$. Then, it follows that

$$\ln\eta(k) = \sum_{n=0}^{k} \ln\zeta_{r(n),\sigma(n)}. \tag{9.20}$$

By using the definitions of stochastic processes $\{N(k) \in \mathbb{N}_0\}_{k\in\mathbb{N}_0}$ and $\{s(i) \in \mathcal{S}\}_{i\in\mathbb{N}_0}$, we obtain

$$\begin{aligned} \ln\eta(k) &= \sum_{n=0}^{t_{N(k)}-1} \ln\zeta_{r(n),\sigma(n)} + \sum_{n=t_{N(k)}}^{k} \ln\zeta_{r(n),\sigma(n)} \\ &= \sum_{i=0}^{N(k)-1} \xi_{s(i)} + \sum_{n=t_{N(k)}}^{k} \ln\zeta_{r(n),\sigma(n)}, \end{aligned} \tag{9.21}$$

where $\xi_q \triangleq \sum_{n=1}^{|q|} \ln \zeta_{q_n,q_1}$, $q \in \mathcal{S}$.

Next, in order to evaluate $\lim_{k\to\infty} \frac{1}{k}\ln\eta(k)$, note that $\lim_{k\to\infty} \frac{1}{k} \sum_{n=t_{N(k)}}^{k} \ln\zeta_{r(n),\sigma(n)} =$

0. Consequently,

$$\lim_{k\to\infty}\frac{1}{k}\ln\eta(k) = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{N(k)-1}\xi_{s(i)}$$
$$= \lim_{k\to\infty}\frac{N(k)}{k}\frac{1}{N(k)}\sum_{i=0}^{N(k)-1}\xi_{s(i)}. \tag{9.22}$$

It follows as a consequence of the strong law of large numbers for renewal processes (Section 2.3.3) that $\lim_{k\to\infty}\frac{N(k)}{k}=\frac{1}{\hat{\tau}}$, where $\hat{\tau}=\sum_{\tau\in\mathbb{N}}\tau\mu_\tau$. Furthermore, by the ergodic theorem for countable-state Markov chains (Section 2.3.2), it follows that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\xi_{s(i)} = \sum_{q\in\mathcal{S}}\phi_q\xi_q$. Using the invariant distribution $\phi:\mathcal{S}\to[0,1]$ given by (9.10), we get

$$\lim_{k\to\infty}\frac{1}{k}\ln\eta(k) = \frac{1}{\hat{\tau}}\sum_{q\in\mathcal{S}}\big(\pi_{q_1}\mu_{|q|}\prod_{n=1}^{|q|-1}p_{q_n,q_{n+1}}\big)\sum_{m=1}^{|q|}\ln\zeta_{q_m,q_1}. \tag{9.23}$$

Now let $\mathcal{S}_\tau \triangleq \{q\in\mathcal{S} : |q|=\tau\}$, $\tau\in\mathbb{N}$. Note that the set $\mathcal{S}_\tau$ contains all mode sequences of length τ. It follows from (9.23) that

$$\lim_{k\to\infty}\frac{1}{k}\ln\eta(k) = \frac{1}{\hat{\tau}}\sum_{\tau\in\mathbb{N}}\sum_{q\in\mathcal{S}_\tau}\big(\pi_{q_1}\mu_{|q|}\prod_{n=1}^{|q|-1}p_{q_n,q_{n+1}}\big)\sum_{m=1}^{|q|}\ln\zeta_{q_m,q_1}$$
$$= \frac{1}{\hat{\tau}}\sum_{\tau\in\mathbb{N}}\mu_\tau\sum_{q\in\mathcal{S}_\tau}\pi_{q_1}(\prod_{n=1}^{\tau-1}p_{q_n,q_{n+1}})\sum_{m=1}^{\tau}\ln\zeta_{q_m,q_1}$$
$$= \frac{1}{\hat{\tau}}\sum_{\tau\in\mathbb{N}}\mu_\tau\sum_{m=1}^{\tau}\sum_{q\in\mathcal{S}_\tau}\pi_{q_1}(\prod_{n=1}^{\tau-1}p_{q_n,q_{n+1}})\ln\zeta_{q_m,q_1}. \tag{9.24}$$

Furthermore, let $\mathcal{S}_{\tau,l}^{i,j} \triangleq \{q\in\mathcal{S}_\tau : q_1=i, q_l=j\}$, $i,j\in\mathcal{M}$, $l\in\{1,2,\ldots,\tau-1\}$. The set $\mathcal{S}_{\tau,l}^{i,j}$ contains all mode sequences of length τ that have $i\in\mathcal{M}$ and $j\in\mathcal{M}$ as the 1st and the lth elements, respectively. We use (9.1) to obtain

$$\sum_{q\in\mathcal{S}_\tau}\pi_{q_1}(\prod_{n=1}^{\tau-1}p_{q_n,q_{n+1}})\ln\zeta_{q_l,q_1} = \sum_{i,j\in\mathcal{M}}\sum_{q\in\mathcal{S}_{\tau,l}^{i,j}}\pi_{q_1}(\prod_{n=1}^{\tau-1}p_{q_n,q_{n+1}})\ln\zeta_{q_l,q_1}$$
$$= \sum_{i,j\in\mathcal{M}}\pi_i\ln\zeta_{j,i}\sum_{q\in\mathcal{S}_{\tau,l}^{i,j}}(\prod_{n=1}^{\tau-1}p_{q_n,q_{n+1}})$$
$$= \sum_{i,j\in\mathcal{M}}\pi_i\ln\zeta_{j,i}p_{i,j}^{(l-1)}. \tag{9.25}$$

Substituting (9.25) into (9.24) yields

$$\lim_{k\to\infty}\frac{1}{k}\ln\eta(k) = \frac{1}{\hat{\tau}}\sum_{\tau\in\mathbb{N}}\mu_\tau\sum_{l=1}^{\tau}\sum_{i,j\in\mathcal{M}}\pi_i p_{i,j}^{(l-1)}\ln\zeta_{j,i}. \tag{9.26}$$

Now, since $\hat{\tau} = \sum_{\tau\in\mathbb{N}}\tau\mu_\tau < \infty$, as a result of (9.16), we have

$$\lim_{k\to\infty}\frac{1}{k}\ln\eta(k) < 0. \tag{9.27}$$

Thus, $\lim_{k\to\infty}\ln\eta(k) = -\infty$ almost surely; furthermore, $\mathbb{P}[\lim_{k\to\infty}\eta(k) = 0] = 1$. By (9.19) we have $V(x(k+1)) \le \eta(k)V(x(0))$, $k\in\mathbb{N}$, and therefore,

$$\mathbb{P}[\lim_{k\to\infty}V(x(k)) = 0] = 1, \tag{9.28}$$

which implies that the zero solution of the closed-loop system (6.1), (9.2) is asymptotically stable almost surely. □

Theorem 9.1 provides sufficient conditions for almost sure asymptotic stability of the closed-loop system (6.1) and (9.2). Conditions (9.15) and (9.16) of Theorem 9.1 indicate dependence of stabilization performance on subsystem dynamics, mode transition probabilities, and random mode observations. Specifically, $\zeta_{i,j}$ in (9.15) characterizes an upper bound on the growth of a Lyapunov-like function, when the switched system evolves according to dynamics of the ith subsystem and the jth feedback gain. Furthermore, the effect of mode transitions on the stabilization is reflected in (9.15) through the limiting distribution $\pi : \mathcal{M} \to [0,1]$ as well as l-step transition probabilities $p_{i,j}^{(l)}$, $i,j\in\mathcal{M}$. Finally, the effect of random mode observations is indicated in condition (9.15) by $\mu : \mathbb{N}\to[0,1]$, which represents the distribution of the lengths of intervals between consecutive mode observation instants.

Remark 9.1. It is important to investigate conservativeness of the obtained stabilization conditions. To this end, first note that we analyze the stability of the closed-loop system through the Lyapunov-like function $V(x) \triangleq x^{\mathrm{T}}Rx$ with $R = \tilde{R}^{-1}$, where $\tilde{R}$ is a positive-definite matrix that satisfy (9.15). The scalar $\zeta_{i,j}\in(0,\infty)$ in (9.15) characterizes an upper bound on the growth of the Lyapunov-like function, when the switched system evolves according to dynamics of the ith subsystem and the jth feedback gain. Note that if $\zeta_{i,j} \in (0,1)$ for all $i,j \in \mathcal{M}$, it is guaranteed that the Lyapunov-like function will decrease at each time step. However, we do not require $\zeta_{i,j} \in (0,1)$ for all $i,j \in \mathcal{M}$. There may be pairs $i,j \in \mathcal{M}$ such that $\zeta_{i,j} > 1$, hence Lyapunov-like function $V(\cdot)$ may grow when ith subsystem and the jth feedback gain is active. As long as $\zeta_{i,j}$, $i,j \in \mathcal{M}$, satisfy (9.16) the Lyapunov-like is guaranteed to converge to zero in the long-run (even if it may grow at certain instants). Note that eventhough the conditions (9.15), (9.16) allow unstable subsystem-feedback gain pairs, some conservativeness may still arise due the characterization with single Lyapunov-like function. This conservatism can be reduced with an alternative approach with multiple Lyapunov-like functions assigned for each subsystem-feedback gain pairs (see Chapter 6).

Remark 9.2. In order to verify conditions (9.15) and (9.16) of Theorem 9.1, we take an approach similar to the one presented in Chapters 6 and 7. Specifically, we use Schur complements (see [115]) to transform condition (9.15) into the matrix inequalities

$$0 \leq \begin{bmatrix} \zeta_{i,j}\tilde{R} & \hat{A}_{i,j}^{\mathrm{T}} \\ \hat{A}_{i,j} & \tilde{R} \end{bmatrix}, \quad i,j \in \mathcal{M}, \tag{9.29}$$

where $\hat{A}_{i,j} \triangleq (A_i\tilde{R} + B_iL_j)$, $i,j \in \mathcal{M}$. Note that the inequalities (9.29) are linear in $\tilde{R}$ and L_i, $i \in \mathcal{M}$. In our numerical method, we iterate over a set of the values of $\zeta_{i,j}$, $i,j \in \mathcal{M}$, that satisfy (9.16) and at each iteration we look for feasible solutions to the linear matrix inequalities (9.29). In Section 9.4 below, we employ this method and find values for matrices $\tilde{R} \in \mathbb{R}^{n\times n}, L_i \in \mathbb{R}^{m\times n}, i \in \mathcal{M}$, and scalars $\zeta_{i,j} \in (0,\infty), i,j \in \mathcal{M}$ that satisfy (9.15), (9.16) for a given discrete-time switched linear system. It is important to note that the scalars $\zeta_{i,j} \in (0,\infty), i,j \in \mathcal{M}$, that satisfy (9.16) form an unbounded set. Note that this set is smaller than the entire nonnegative orthant in $\mathbb{R}^{M^2}$. However, we still need to reduce the search space of $\zeta_{i,j}, i,j \in \mathcal{M}$. To this end, first note that it is harder to find feasible solutions to linear matrix inequalities given by (9.29) when the scalars $\zeta_{i,j}, i,j \in \mathcal{M}$, are close to zero. Note also that if there exist a feasible solution to (9.29) for certain values of $\zeta_{i,j}, i,j \in \mathcal{M}$, then it is guaranteed that feasible solutions to (9.29) exist also for *larger* values of $\zeta_{i,j}, i,j \in \mathcal{M}$. Therefore, we can restrict our search space and iterate over large values of $\zeta_{i,j}, i, j \in \mathcal{M}$, that satisfy (9.16), and check feasible solutions to (9.29). Specifically, we only iterate over $\zeta_{i,j}, i, j \in \mathcal{M}$, that is close to the search space's boundary identified by $\sum_{\tau \in \mathbb{N}} \mu_\tau \sum_{l=1}^{\tau} \sum_{i,j \in \mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} = 0$. Now note that in order for (9.16) to be satisfied, there must exist at least a pair $i, j \in \mathcal{M}$ such that $\zeta_{i,j} < 1$. Since the scalar $\zeta_{i,j}$ represens the stability/instability margin for the dynamics characterized by the ith subsystem and the jth feedback gain, we expect $\zeta_{i,i} < 1$ for stabilizable modes $i \in \mathcal{M}$. This further reduces the search space for our numerical method.

Remark 9.3. Note that conditions (9.15) and (9.16) presented in Theorem 9.1 can also be used for determining almost sure asymptotic stability of the switched stochastic control system (6.1), (9.2) with periodically observed mode information. The renewal process characterization presented in this chapter in fact encompasses periodic mode observations (explored previously in Chapters 6 and 7) as a special case. Specifically, suppose that the mode observation instants are given by $t_i = iT$, $i \in \mathbb{N}_0$, where $T \in \mathbb{N}$ denotes the mode observation period. Our present framework allows us to characterize periodic mode observations by setting the distribution $\mu : \mathbb{N} \to [0, 1]$ such that $\mu_T = 1$ and $\mu_\tau = 0$, $\tau \neq T$. Note that condition (9.16) of Theorem 9.1 for this case reduces to

$$\sum_{l=1}^{T} \sum_{i,j \in \mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} < 0. \tag{9.30}$$

Furthermore, if the controller has perfect mode information at all time instants ($T = 1$, hence $\sigma(k) = r(k)$, $k \in \mathbb{N}_0$), condition (9.16) takes even a simpler form given by the inequality

$$\sum_{i \in \mathcal{M}} \pi_i \ln \zeta_{i,i} < 0. \tag{9.31}$$

Remark 9.4. Condition (9.16) of Theorem 9.1 has a simpler form also for the case where the length of intervals between consecutive mode observation instants are uniformly distributed over the set $\{\tau_{\mathrm{L}}, \tau_{\mathrm{L}} + 1, \ldots, \tau_{\mathrm{H}}\}$ with $\tau_{\mathrm{L}}, \tau_{\mathrm{H}} \in \mathbb{N}$ such that $\tau_{\mathrm{L}} \leq \tau_{\mathrm{H}}$. In this case

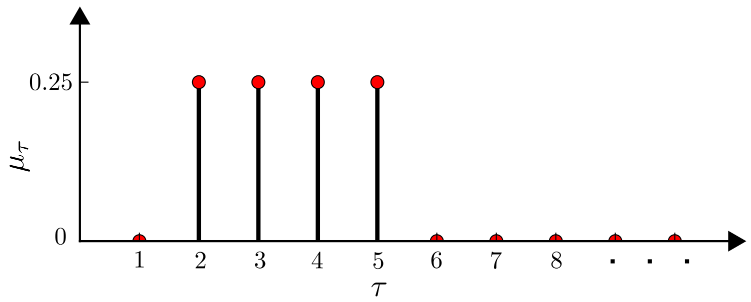

Figure 9.1: Uniform distribution given by (9.32) with $\tau_{\rm L} = 2$ and $\tau_{\rm H} = 5$ for the length of intervals between consecutive mode observation instants

the distribution $\mu : \mathbb{N} \to [0, 1]$ is given by

$$\mu_\tau \triangleq \begin{cases} \frac{1}{\tau_{\rm H} - \tau_{\rm L} + 1}, & \text{if } \tau \in \{\tau_{\rm L}, \tau_{\rm L} + 1, \ldots, \tau_{\rm H}\}, \\ 0, & \text{otherwise.} \end{cases} \quad (9.32)$$

Figure 9.1 shows the distribution (9.32) for an example case with $\tau_{\rm L} = 2$ and $\tau_{\rm H} = 5$.

With (9.32), condition (9.16) of Theorem 9.1 reduces to the inequality

$$\sum_{\tau=\tau_{\rm L}}^{\tau_{\rm H}} \sum_{l=1}^{\tau} \sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} < 0. \quad (9.33)$$

Remark 9.5. Note that our probabilistic characterization of mode observation instants also allows us to explore the feedback control problem under missing mode samples. Specifically, consider the case where the mode is sampled at all time instants; however, some of the mode samples are lost during communication between mode sampling mechanism and the controller. Suppose that the controller receives a sampled mode data at each time step $k \in \mathbb{N}$ with probability $\theta \in (0, 1)$. In other words, the mode data is lost with probability $1 - \theta$. We investigate this problem by setting

$$\mu_\tau \triangleq (1 - \theta)^{\tau - 1}\theta, \quad \tau \in \mathbb{N}. \quad (9.34)$$

Figure 9.2 shows the geometric distribution (9.34) with $\theta = 0.3$.

It turns out that for $\mu_\tau : \mathbb{N} \to [0, 1]$ given by (9.34), the left-hand side of condition (9.16) has a closed-form expression. Note that by changing the order of summations and

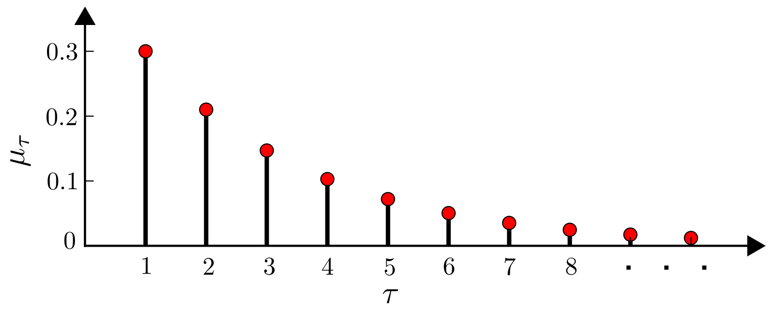

Figure 9.2: Geometric distribution given by (9.34) with $\theta = 0.3$ for the length of intervals between consecutive mode observation instants

using (9.34), we can rewrite the left-hand side of (9.16) as

$$\begin{aligned}\sum_{\tau\in\mathbb{N}} \mu_\tau \sum_{l=1}^{\tau} \sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} &= \sum_{i,j\in\mathcal{M}} \pi_i \ln \zeta_{j,i} \sum_{\tau\in\mathbb{N}} \mu_\tau \sum_{l=1}^{\tau} p_{i,j}^{(l-1)} \\ &= \sum_{i,j\in\mathcal{M}} \pi_i \ln \zeta_{j,i} \sum_{l=1}^{\infty} p_{i,j}^{(l-1)} \sum_{\tau=l}^{\infty} \mu_\tau \\ &= \sum_{i,j\in\mathcal{M}} \pi_i \ln \zeta_{j,i} \sum_{l=1}^{\infty} p_{i,j}^{(l-1)} (1 - \sum_{\tau=1}^{l-1} \mu_\tau) \\ &= \sum_{i,j\in\mathcal{M}} \pi_i \ln \zeta_{j,i} \sum_{l=1}^{\infty} p_{i,j}^{(l-1)} \big(1 - \sum_{\tau=1}^{l-1} (1-\theta)^{\tau-1}\theta\big). \qquad (9.35)\end{aligned}$$

Note that $\big(1 - \sum_{\tau=1}^{l-1}(1-\theta)^{\tau-1}\theta\big) = \big(1 - \theta\frac{1-(1-\theta)^{l-1}}{1-(1-\theta)}\big) = (1-\theta)^{l-1}$. Therefore,

$$\sum_{\tau\in\mathbb{N}} \mu_\tau \sum_{l=1}^{\tau} \sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} = \sum_{i,j\in\mathcal{M}} \pi_i \ln \zeta_{j,i} \sum_{l=1}^{\infty} p_{i,j}^{(l-1)} (1-\theta)^{l-1}. \qquad (9.36)$$

Let $Z \triangleq \sum_{l=1}^{\infty} P^{l-1}(1-\theta)^{l-1}$, where $P \in \mathbb{R}^{M\times M}$ denotes the transition probability matrix for the mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$. Note that the infinite sum in the definition of Z converges, because the eigenvalues of the matrix $(1-\theta)P$ are strictly inside the unit circle of the complex plane. By using the formula for geometric series of matrices [115], we obtain

$$Z = \big(I - (1-\theta)P\big)^{-1}. \qquad (9.37)$$

Furthermore, it follows from (9.36) that

$$\sum_{\tau\in\mathbb{N}}\mu_\tau\sum_{l=1}^{\tau}\sum_{i,j\in\mathcal{M}}\pi_i p_{i,j}^{(l-1)}\ln\zeta_{j,i}=\sum_{i,j\in\mathcal{M}}\pi_i\ln\zeta_{j,i}z_{i,j}, \tag{9.38}$$

and therefore, when $\mu:\mathbb{N}\to[0,1]$ is given by (9.34), condition (9.16) of Theorem 9.1 takes the form

$$\sum_{i,j\in\mathcal{M}}\pi_i\ln\zeta_{j,i}z_{i,j}<0, \tag{9.39}$$

where $z_{i,j}$ is the (i,j)th entry of the matrix Z given by (9.37).

Remark 9.6. Note that in order to check condition (9.16) of Theorem 9.1, one needs to have perfect information regarding the distribution $\mu:\mathbb{N}\to[0,1]$, according to which the lengths of intervals between consecutive mode observation instants are distributed. In Theorem 9.2 below, we present alternative sufficient stabilization conditions, which do not require exact knowledge of $\mu:\mathbb{N}\to[0,1]$. Specifically, we consider the case where the mode observation instants t_i, $i\in\mathbb{N}_0$, satisfy

$$\mathbb{P}[t_{i+1}-t_i\le\bar{\tau}]=1,\quad i\in\mathbb{N}_0, \tag{9.40}$$

where $\bar{\tau}\in\mathbb{N}$ is a known constant. Note that in this case time instants of consecutive mode observations are assumed to be at most $\bar{\tau}\in\mathbb{N}$ steps apart.

Theorem 9.2. Consider the switched linear stochastic system (6.1). Suppose that the mode-transition probability matrix $P\in\mathbb{R}^{M\times M}$ possesses only positive real eigenvalues. If there exist matrices $\tilde{R}>0$, $L_i\in\mathbb{R}^{m\times n}$, $i\in\mathcal{M}$, and scalars $\bar{\tau}\in\mathbb{N}$, $\zeta_{i,j}\in(0,\infty)$, $i,j\in\mathcal{M}$, such that (9.15), (9.40),

$$0\le\zeta_{j,i}-\zeta_{i,i},\quad i,j\in\mathcal{M}, \tag{9.41}$$

$$\sum_{l=1}^{\bar{\tau}}\sum_{i,j\in\mathcal{M}}\pi_i p_{i,j}^{(l-1)}\ln\zeta_{j,i}<0, \tag{9.42}$$

hold, then the linear feedback control law (9.2) with the feedback gain matrix (9.17) guarantees that the zero solution $x(k)\equiv 0$ of the closed-loop system is asymptotically stable almost surely.

Proof. The mode signal $\{r(k) \in \mathcal{M}\}_{k\in\mathbb{N}_0}$ is an irreducible and aperiodic Markov chain; therefore, the invariant distribution $\pi : \mathcal{M} \to [0,1]$ is also the limiting distribution [87]. Thus, for all $i, j \in \mathcal{M}$ and $k \in \mathbb{N}_0$,

$$\lim_{l\to\infty} p_{i,j}^{(l)} = \lim_{l\to\infty} \mathbb{P}[r(k+l) = j | r(k) = i] = \pi_j. \tag{9.43}$$

Now, let $p_i^{(l)} \in [0,1]^{1\times M}$,$i \in \mathcal{M}$, denote the row vector with the jth element given by the l-step transition probability $p_{i,j}^{(l)}$. Note that $p_i^{(\cdot)}$ is the unique solution of the difference equation

$$p_i^{(l+1)} = p_i^{(l)} P, \quad l \in \mathbb{N}_0, \tag{9.44}$$

with the initial condition $p_{i,i}^{(0)} = 1$ and $p_{i,j}^{(0)} = 0$, $i \neq j$, $j \in \mathcal{M}$. Since all the eigenvalues of the mode-transition probability matrix $P \in \mathbb{R}^{M\times M}$ are positive real numbers, the solution $p_i^{(\cdot)}$ of the difference equation (9.44) does not comprise any oscillatory components, and l-step transition probabilities $p_{i,j}^{(l)}$, $i, j \in \mathcal{M}$, converge towards their limiting values *monotonically*, that is,

$$p_{i,i}^{(l+1)} \le p_{i,i}^{(l)}, \quad i \in \mathcal{M},\ l \in \mathbb{N}_0, \tag{9.45}$$

$$p_{i,j}^{(l+1)} \ge p_{i,j}^{(l)}, \quad i \neq j,\ \ i, j \in \mathcal{M},\ \ l \in \mathbb{N}_0. \tag{9.46}$$

Now note that for all $i, j \in \mathcal{M}$, and $\tau \in \mathbb{N}$,

$$\frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} = \frac{1}{\tau+1}\Big(\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} + \frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(l-1)}\Big). \tag{9.47}$$

By (9.46), we have $p_{i,j}^{(l-1)} \le p_{i,j}^{\tau}$, $l \in \{1, 2, \ldots, \tau\}$, $i, j \in \mathcal{M}$, $i \neq j$. Hence, it follows from

(9.47) that

$$\frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} \leq \frac{1}{\tau+1}\Big(\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} + \frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(\tau)}\Big)$$
$$= \frac{1}{\tau+1}\Big(\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} + p_{i,j}^{(\tau)}\Big)$$
$$= \frac{1}{\tau+1}\sum_{l=1}^{\tau+1} p_{i,j}^{(l-1)}, \quad \tau \in \mathbb{N},\ \ i \neq j. \tag{9.48}$$

As a consequence, for all $\tau \leq \bar{\tau}$ it follows that

$$\frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} \leq \frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}} p_{i,j}^{(l-1)}, \quad i \neq j,\ \ i,j \in \mathcal{M}. \tag{9.49}$$

Next, we show that (9.40)–(9.42) together with (9.49) imply (9.16). First, let

$$\kappa_{\tau,\bar{\tau}}^{i,j} \triangleq \frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} - \frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}} p_{i,j}^{(l-1)}, \quad i,j \in \mathcal{M}. \tag{9.50}$$

It follows that

$$\frac{1}{\tau}\sum_{l=1}^{\tau}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} = \sum_{i,j\in\mathcal{M}} \pi_i \ln \zeta_{j,i} \frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(l-1)}$$
$$= \sum_{i,j\in\mathcal{M}} \pi_i \ln \zeta_{j,i}\kappa_{\tau,\bar{\tau}}^{i,j} + \frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i}$$
$$= \sum_{i\in\mathcal{M}} \pi_i \ln \zeta_{i,i}\kappa_{\tau,\bar{\tau}}^{i,i} + \sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{M}, j\neq i} \pi_i \ln \zeta_{j,i}\kappa_{\tau,\bar{\tau}}^{i,j}$$
$$+ \frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i}. \tag{9.51}$$

Note that by (9.49), we have $\kappa_{\tau,\bar{\tau}}^{i,j} \leq 0$, $\tau \leq \bar{\tau}$, $i \neq j$. It follows from (9.41) that, for $\tau \leq \bar{\tau}$,

$$\ln \zeta_{j,i}\kappa_{\tau,\bar{\tau}}^{i,j} \leq \ln \zeta_{i,i}\kappa_{\tau,\bar{\tau}}^{i,j}, \quad i \neq j,\ i,j \in \mathcal{M}. \tag{9.52}$$

Furthermore, since $\sum_{j\in\mathcal{M}} p_{i,j}^{(l)} = 1$, $l \in \mathbb{N}_0$, $i \in \mathcal{M}$, we have

$$\begin{aligned}
\sum_{j\in\mathcal{M}} \kappa_{\tau,\bar{\tau}}^{i,j} &= \sum_{j\in\mathcal{M}} \frac{1}{\tau}\sum_{l=1}^{\tau} p_{i,j}^{(l-1)} - \sum_{j\in\mathcal{M}} \frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}} p_{i,j}^{(l-1)} \\
&= \frac{1}{\tau}\sum_{l=1}^{\tau}\sum_{j\in\mathcal{M}} p_{i,j}^{(l-1)} - \frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}}\sum_{j\in\mathcal{M}} p_{i,j}^{(l-1)} \\
&= \frac{\tau}{\tau} - \frac{\bar{\tau}}{\bar{\tau}} \\
&= 0, \quad i \in \mathcal{M}.
\end{aligned} \tag{9.53}$$

We use (9.51)–(9.53) to obtain

$$\begin{aligned}
\frac{1}{\tau}\sum_{l=1}^{\tau}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l)} \ln\zeta_{j,i} &\leq \sum_{i\in\mathcal{M}} \pi_i \ln\zeta_{i,i}\kappa_{\tau,\bar{\tau}}^{i,j} + \sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{M}, j\neq i} \pi_i \ln\zeta_{i,i}\kappa_{\tau,\bar{\tau}}^{i,j} \\
&\quad + \sum_{i,j\in\mathcal{M}} \pi_i \ln\zeta_{j,i}\frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}} p_{i,j}^{(l-1)} \\
&= \sum_{i\in\mathcal{M}} \pi_i \ln\zeta_{i,i}\sum_{j\in\mathcal{M}}\kappa_{\tau,\bar{\tau}}^{i,j} + \sum_{i,j\in\mathcal{M}} \pi_i \ln\zeta_{j,i}\frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}} p_{i,j}^{(l-1)} \\
&= \frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln\zeta_{j,i}, \quad \tau \leq \bar{\tau}.
\end{aligned} \tag{9.54}$$

Finally, by (9.40) and (9.54), it follows that

$$\begin{aligned}
\sum_{\tau\in\mathbb{N}} \mu_\tau \sum_{l=1}^{\tau}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln\zeta_{j,i} &= \sum_{\tau\in\mathbb{N}} \mu_\tau \tau\big(\frac{1}{\tau}\sum_{l=1}^{\tau}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln\zeta_{j,i}\big) \\
&\leq \sum_{\tau\in\mathbb{N}} \mu_\tau \tau\big(\frac{1}{\bar{\tau}}\sum_{l=1}^{\bar{\tau}}\sum_{i,j\in\mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln\zeta_{j,i}\big).
\end{aligned} \tag{9.55}$$

Note that (9.42) and (9.55) imply (9.16). Hence, the result follows from Theorem 9.1. □

Conditions of Theorem 9.2 can be utilized for assessing stability of a switched stochastic control system, even if exact knowledge of the distribution $\mu : \mathbb{N} \to [0,1]$ is not available. Note that the requirement on the knowledge of $\mu : \mathbb{N} \to [0,1]$ is relaxed in Theorem 9.2 by imposing other conditions on the mode-transition probability matrix $P \in \mathbb{R}^{M\times M}$ and the scalars $\zeta_{i,j} \in (0,\infty)$, $i,j \in \mathcal{M}$.

In the following we provide an illustrative discussion on differences between Theorems 9.1 and 9.2.

Note that we can use Theorem 9.1 to find feedback gains that guarantee stabilization of the closed-loop system for the case where mode is observed at every $\bar{\tau} \in \mathbb{N}$ steps. Note also that these feedback gains do not necessarily guarantee stabilization if mode is observed more frequently. For example, for the case where the length of mode intervals are given by $\tau_i = \bar{\tau} - 1$, $i \in \mathbb{N}$, the stabilizing feedback gains obtained for the case where mode is observed at every $\bar{\tau} \in \mathbb{N}$ steps, may make the state diverge. In order to demonstrate this issue, we first obtain a set of sufficient conditions for *instability* of the closed-loop system.

Theorem 9.3. Consider the switched linear stochastic control system (6.1), (9.2) with the feedback gain matrices $K_i \in \mathbb{R}^{m \times n}$, $i \in \mathcal{M}$. If there exist a matrix $R > 0$, and scalars $\zeta_{i,j} \in (0, \infty)$, $i, j \in \mathcal{M}$, such that

$$0 \leq (A_i + B_i K_j)^{\mathrm{T}} R (A_i + B_i K_j) - \zeta_{i,j} R, \quad i, j \in \mathcal{M}, \tag{9.56}$$

$$\sum_{\tau \in \mathbb{N}} \mu_\tau \sum_{l=1}^{\tau} \sum_{i,j \in \mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} > 0, \tag{9.57}$$

then $\lim_{k \to \infty} \|x(k)\|^2 = \infty$, almost surely.

Proof. It follows from (6.1) and (9.2) that

$$V(x(k+1)) = x^{\mathrm{T}}(k)(A_{r(k)} + B_{r(k)} K_{\sigma(k)})^{\mathrm{T}} R (A_{r(k)} + B_{r(k)} K_{\sigma(k)}) x(k),\ k \in \mathbb{N}_0. \tag{9.58}$$

We use (9.56) and (9.58) to obtain

$$\begin{aligned} V(x(k+1)) &\geq \zeta_{r(k),\sigma(k)} V(x(k)) \\ &\geq \eta(k) V(x(0)), \quad k \in \mathbb{N}_0, \end{aligned} \tag{9.59}$$

where $\eta(k) \triangleq \left(\prod_{n=0}^{k} \zeta_{r(n),\sigma(n)}\right)$, $k \in \mathbb{N}$. By using the same approach that we employed in proof of Theorem 9.1, we show that

$$\lim_{k \to \infty} \frac{1}{k} \ln \eta(k) = \frac{1}{\hat{\tau}} \sum_{\tau \in \mathbb{N}} \mu_\tau \sum_{l=1}^{\tau} \sum_{i,j \in \mathcal{M}} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i}. \tag{9.60}$$

where $\hat{\tau} = \sum_{\tau\in\mathbb{N}} \tau\mu_\tau < \infty$. Now as a result of (9.57), we have

$$\lim_{k\to\infty} \frac{1}{k}\ln\eta(k) > 0. \tag{9.61}$$

Thus, $\lim_{k\to\infty} \ln\eta(k) = \infty$ almost surely; furthermore, $\mathbb{P}[\lim_{k\to\infty} \eta(k) = \infty] = 1$. By (9.59) we have $V(x(k+1)) \geq \eta(k)V(x(0))$, $k \in \mathbb{N}$, and therefore,

$$\mathbb{P}[\lim_{k\to\infty} V(x(k)) = \infty] = 1, \tag{9.62}$$

which implies that $\lim_{k\to\infty} \|x(k)\|^2 = \infty$, almost surely. □

Now, consider the switched linear *scalar* stochastic system (6.1), (9.2) with $M = 2$ modes described by $A_1 = A_2 = 1.1$, $B_1 = 1$, $B_2 = -1$. The mode signal $\{r(k) \in \mathcal{M} \triangleq \{1, 2\}\}_{k\in\mathbb{N}_0}$ is assumed to be characterized by the mode transition probability matrix given by

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}. \tag{9.63}$$

Mode is assumed to be periodically observed at every 2 steps. Periodic observations for this case can be characterized by setting $\mu_2 = 1$, and $\mu_\tau = 0$, $\tau \neq 2$. Note that $\tilde{R} = 1$ and $L_1 = 1$, $L_2 = -1$, and scalars $\zeta_{1,1} = \zeta_{2,2} = 4.41$, $\zeta_{1,2} = \zeta_{2,1} = 0.01$, satisfy the conditions (9.15), (9.16) of Theorem 9.1. Therefore, the proposed feedback control law (9.2) with the feedback gains $K_1 = L_1\tilde{R}^{-1} = 1$ and $K_2 = L_2\tilde{R}^{-1} = -1$, guarantees almost sure stabilization of the zero solution of the closed-loop system. On the other hand, now suppose that mode is observed at every step $k \in \mathbb{N}_0$. Periodic observations for this case can be characterized by setting $\mu_1 = 1$, and $\mu_\tau = 0$, $\tau \neq 1$. Furthermore, suppose that the control law has the feedback gains $K_1 = 1$ and $K_2 = -1$. For the case where mode is observed at every time step, $R = 1$, and scalars $\zeta_{1,1} = \zeta_{2,2} = 4.41$, $\zeta_{1,2} = \zeta_{2,1} = 0.01$ satisfy conditions of Theorem 9.3. Hence, the state diverges almost surely, that is, $\mathbb{P}[\lim_{k\to\infty} \|x(k)\|^2 = \infty] = 1$. Note that the proposed control law (9.16) with gains $K_1 = 1$ and $K_2 = -1$ *stabilize* the system when mode is observed at every 2 steps, and *destabilize* the system when mode is observed at every step. This issue arrieses due to the fact that feedback gain K_1 works better for mode 2, and feedback gain K_2 works better for mode 1. When mode is observed at every step, feedback gain K_1 is always used with mode 1 and feedback gain K_2 is always used with mode 2, since $r(k) = \sigma(k)$, $k \in \mathbb{N}_0$. On the other hand, when mode is observed at every 2 steps, for some time steps k, $r(k) \neq \sigma(k)$. Hence, when mode is observed at every 2 steps, occasionaly feedback gain is set to K_1 when mode 2 is active, and occasionaly feedback gain is set to K_2 when mode 1 is active. This results in stabilization of the system.

It is important to note that Theorem 9.2 provides a form of monotonicity with respect to frequency of mode observations that Theorem 9.1 does not provide. Specifically, consider feedback gains $K_i = L_i \tilde{R}^{-1}$, $i \in \mathcal{M}$, with $\tilde{R} \in \mathbb{R}^{n \times n}$ and $L_i \in \mathbb{R}^{m \times n}$, $i \in \mathcal{M}$, that satisfy conditions of Theorem 9.2 for constant $\bar{\tau} \in \mathbb{N}$. It follows that these feedback gains guarantee stabilization if mode observation intervals satisfy $\tau_i \leq \bar{\tau}$, $i \in \mathbb{N}$. Hence, feedback gains obtained with Theorem 9.2 for periodic mode observation interval $\bar{\tau} \in \mathbb{N}$ are guaranteed to achieve stabilization also for the case where mode is observed more frequently, that is, mode observations intervals are smaller than $\bar{\tau} \in \mathbb{N}$.

9.3.1 Stabilization Conditions for Reducible Mode Signal

Theorems 9.1 and 9.2 provide stabilization conditions for the case where the mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ is an irreducible Markov chain (see Sections 2.3.1 and 2.3.2). In this section, our goal is to extend our framework to the case where the mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ is not necessarily irreducible. In order to deal with this problem setting, we first give definitions of *communicating set* (also called communicating class) and *closed communicating set* for discrete-time Markov chains [87].

A mode $i \in \mathcal{M}$ is said to *communicate* with mode $j \in \mathcal{M}$ if there exist $n_1, n_2 \in \mathbb{N}_0$ such that $\mathbb{P}[s(k+n_1) = j | s(k) = i] > 0$, $\mathbb{P}[s(k+n_2) = i | s(k) = j] > 0$, $k \in \mathbb{N}_0$. Note that mode i communicates with mode j if there is a directed path from node i to node j in the transition diagram of the mode signal. Note also that every mode communicates with itself. A set of modes $\mathcal{C} \subset \mathcal{M}$ is called a *communicating set* if mode i communicates with mode j for all $i, j \in \mathcal{C}$, and mode i does not communicate with mode j for all $i \in \mathcal{C}$ and $j \in \mathcal{M} \setminus \mathcal{C}$. Note that $\mathcal{M}$ (the set of all modes) is partitioned to a number of disjoint communicating sets. A communicating set $\mathcal{C}$ is called *closed communicating set* if for all

$i \in \mathcal{C}$ and $j \in \mathcal{M} \setminus \mathcal{C}$, $\mathbb{P}[s(k+n) = j | s(k) = i] = 0$, $k, n \in \mathbb{N}_0$. Note that a transition from a mode belonging to a closed communicating set to another mode outside of the closed communicating set is not possible. In other words, there is no path to outside from a closed communicating set in the transition diagram of the mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$.

Note that there is at least one closed communicating set in $\mathcal{M}$, since $\mathcal{M}$ has finitely many modes [87]. If the mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ is irreducible, then $\mathcal{M}$ is a closed communicating set itself. Now, note that if the initial mode r_0 belongs to a closed communicating set $\mathcal{C} \subset \mathcal{M}$, then the mode does not leave outside set $\mathcal{C}$, that is $r(k) \in \mathcal{C}$, $k \in \mathbb{N}_0$. In that case, mode signal is given by $\{r(k) \in \mathcal{C}\}_{k \in \mathbb{N}_0}$. The stationary distributions $\pi_i \in [0,1], i \in \mathcal{C}$, in this case, satisfy $\pi_j = \sum_{i \in \mathcal{C}} \pi_i p_{i,j}$, $j \in \mathcal{C}$. For checking the stability of the closed-loop system, we can use Theorem 9.1 with $\mathcal{M}$ replaced with $\mathcal{C}$. On the other hand, if r_0 does not belong to a closed communicating set, then the mode $r(\cdot)$ will reach a closed communicating set in finite time with probability one. Now let $\mathcal{C}_h \subset \mathcal{M}$, $h \in \{1, 2, \ldots, c\}$, denote all closed communicating sets such that there exist $n_h \in \mathbb{N}_0$, $h \in \{1, 2, \ldots, c\}$, with $\mathbb{P}[r(n_h) \in \mathcal{C}_h | r(0) = r_0] > 0$, $h \in \{1, 2, \ldots, c\}$, where $c \in \{1, 2, \ldots, M\}$ is the total number of such sets. Note that in the transition diagram of the mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$, $\mathcal{C}_h \subset \mathcal{M}$, $h \in \{1, 2, \ldots, c\}$, correspond to all closed communicating sets that has a directed path from the initial mode r_0. Furthermore, note that after the mode signal reaches one of the closed communicating sets $\mathcal{C}_h \subset \mathcal{M}$, $h \in \{1, 2, \ldots, c\}$, in finite time, it stays in that set. Now let $\pi_i \in [0,1], i \in \mathcal{C}_h$, $h \in \{1, 2, \ldots, c\}$, be distributions such that $\pi_j = \sum_{i \in \mathcal{C}_h} \pi_i p_{i,j}$, $j \in \mathcal{C}_h$, $h \in \{1, 2, \ldots, c\}$. The distributions $\pi_i \in [0,1], i \in \mathcal{C}_h$, denote the stationary probabilities for modes after mode signal reaches in set $\mathcal{C}_h$.

Using the definitions provided above, we now present sufficient stabilization conditions for the case where the mode signal $\{r(k) \in \mathcal{M}\}_{k \in \mathbb{N}_0}$ is not necessarily irreducible.

Corollary 9.1. Consider the switched linear stochastic system (6.1). Let $\mathcal{C}_h \subset \mathcal{M}$, $h \in \{1, 2, \ldots, c\}$, denote closed communicating sets with the property that there exist $n_h \in \mathbb{N}_0$ such that $\mathbb{P}[r(n_h) \in \mathcal{C}_h | r(0) = r_0] > 0$, $h \in \{1, 2, \ldots, c\}$, where $c \in \{1, 2, \ldots, M\}$ is the total number of such sets. If for all $h \in \{1, 2, \ldots, c\}$, there exist matrices $\tilde{R}_h > 0$,

$L_i \in \mathbb{R}^{m\times n}$, $i \in \mathcal{C}_h$, and scalars $\zeta_{i,j} \in (0,\infty)$, $i,j \in \mathcal{C}_h$, such that

$$0 \geq (A_i\tilde{R}_h + B_iL_j)^{\mathrm{T}}\tilde{R}_h^{-1}(A_i\tilde{R}_h + B_iL_j) - \zeta_{i,j}\tilde{R}_h, \quad i,j \in \mathcal{C}_h, \tag{9.64}$$

$$\sum_{\tau\in\mathbb{N}} \mu_\tau \sum_{l=1}^{\tau} \sum_{i,j\in\mathcal{C}_h} \pi_i p_{i,j}^{(l-1)} \ln \zeta_{j,i} < 0, \tag{9.65}$$

then the control law (9.2) with the feedback gain matrices

$$K_i = \begin{cases} L_i\tilde{R}_h^{-1}, & \text{if } i \in \mathcal{C}_h, \\ 0, & \text{otherwise}, \end{cases} \tag{9.66}$$

guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (6.1) and (9.2) is asymptotically stable almost surely.

Proof. First, we define

$$t_{\mathrm{C}} \triangleq \inf\{k \in \mathbb{N}_0 : r(k) \in \bigcup_{h\in\{1,2,\ldots,c\}} \mathcal{C}_h\}. \tag{9.67}$$

Note that $t_{\mathrm{C}} \in \mathbb{N}_0$ denotes the time instant at which the mode reaches one of the closed communicating sets $\mathcal{C}_h \subset \mathcal{M}$, $h \in \{1,2,\ldots,c\}$. Now let $r_{\mathrm{C}} \triangleq r(t_{\mathrm{C}})$, $x_{\mathrm{C}} \triangleq x(t_{\mathrm{C}})$. Futhermore, let $h_{\mathrm{C}} \in \{1,2,\ldots,c\}$ denote index of the closed communicating set that includes the mode r_{C}. Note that $r(k) \in \mathcal{C}_{h_{\mathrm{C}}}$ for $k \geq t_{\mathrm{C}}$. Hence, after time t_{C}, the dynamics characterized by (6.1) can be considered as a switched linear stochastic system with mode signal $\{r(k) \in \mathcal{C}_{h_{\mathrm{C}}}\}_{k\geq t_{\mathrm{C}}}$. The initial state and mode of this switched system is given by $x_{\mathrm{C}} \in \mathbb{R}^n$ and $r_{\mathrm{C}} \in \mathcal{C}_{h_{\mathrm{C}}}$, respectively. Now, note that (9.64) and (9.65) imply (9.15) and (9.16) with $\mathcal{M}$ replaced by $\mathcal{C}_{h_{\mathrm{C}}}$. Hence, the result follows from Theorem 9.1. □

9.4 Illustrative Numerical Examples

In this section we provide numerical examples to demonstrate the results presented in this chapter.

Example 9.1. Consider the switched linear stochastic system (6.1) with $M = 2$ modes described by the subsystems matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1.6 & -0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -0.5 & 1.4 \end{bmatrix},$$

$B_1 = [0,\ 1]^{\mathrm{T}}$, and $B_2 = [0,\ -1]^{\mathrm{T}}$. The mode signal $\{r(k) \in \mathcal{M} \triangleq \{1,2\}\}_{k\in\mathbb{N}_0}$ of the switched system is assumed to be an aperiodic and irreducible Markov chain characterized by the transition probabilities $p_{1,2} = p_{2,1} = 0.3$ and $p_{1,1} = p_{2,2} = 0.7$. The invariant distribution for $\{r(k) \in \mathcal{M} \triangleq \{1,2\}\}_{k\in\mathbb{N}_0}$ is given by $\pi_1 = \pi_2 = 0.5$.

Moreover, $\mu : \mathbb{N} \to [0,1]$, according to which the lengths of intervals between consecutive mode observation instants are distributed, is assumed to be given by $\mu_\tau = (1-\theta)^\tau \theta$, $\tau \in \mathbb{N}$, with $\theta = 0.3$. In this case, at each time step $k \in \mathbb{N}$, the mode may be observed with probability $\theta = 0.3$ (see Remark 9.5).

Note that

$$\tilde{R} = \begin{bmatrix} 3.0143 & -0.1485 \\ -0.1485 & 1.5280 \end{bmatrix}, \tag{9.68}$$

$$L_1 = [-3.5326\ \ 0.9608], \tag{9.69}$$

$$L_2 = [-3.0029\ \ 1.8284], \tag{9.70}$$

and the scalars $\zeta_{1,1} = 0.7$, $\zeta_{1,2} = 1.8$, $\zeta_{2,1} = 2$, and $\zeta_{2,2} = 0.8$ satisfy (9.15) and (9.16). Now, it follows from Theorem 9.1 that the proposed control law (9.2) with feedback gain matrices

$$K_1 = L_1 \tilde{R}^{-1} = [-1.1465\ \ 0.5174], \tag{9.71}$$

$$K_2 = L_2 \tilde{R}^{-1} = [-0.9718\ \ 1.1021], \tag{9.72}$$

guarantees almost sure asymptotic stability of the closed-loop system (6.1), (9.2).

Sample paths of the state $x(k)$ and the control input $u(k)$ (obtained with initial conditions $x(0) = [1,\ -1]^{\mathrm{T}}$ and $r(0) = 1$) are shown in Figures 9.1 and 9.2. Furthermore, Figure 9.3 shows a sample path of the actual mode signal $r(k)$ and its sampled version $\sigma(k)$. Figures 9.1–9.3 indicate that our proposed control framework guarantees stabiliza-

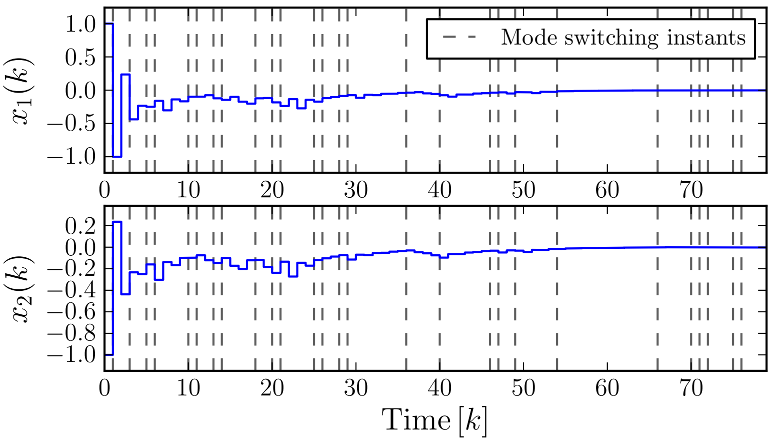

Figure 9.1: State trajectory versus time

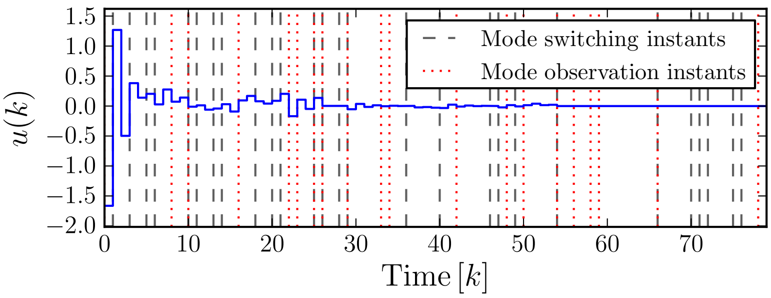

Figure 9.2: Control input versus time

tion even for the case where operation mode of the switched system is observed only at random time instants.

The control law (9.2) with feedback gain matrices (9.71) and (9.72) guarantee stabilization of the closed-loop system with random mode observations characterized by distribution $\mu_\tau = (1-\theta)^{\tau-1}\theta$ with $\theta = 0.3$. Note that for each time step, θ represents the probability of mode information being available for control purposes. In order to investigate conservativeness of our results, we search all values of parameter θ for which the control law (9.2) with feedback gains (9.71) and (9.72) achieve stabilization. To this end, first, we search values of θ such that there exist a positive-definite matrix $\tilde{R}$, and scalars $\zeta_{i,j}$, $i,j \in \mathcal{M}$ that satisfy conditions (9.15) and (9.16) of Theorem 9.1 with $L_1 = K_1\tilde{R}$ and $L_2 = K_2\tilde{R}$, where K_1 and K_2 are given by (9.71) and (9.72). We find that for parameter values $\theta \in [0.2, 1]$, conditions (9.15) and (9.16) are satisfied. Hence Theorem 9.1

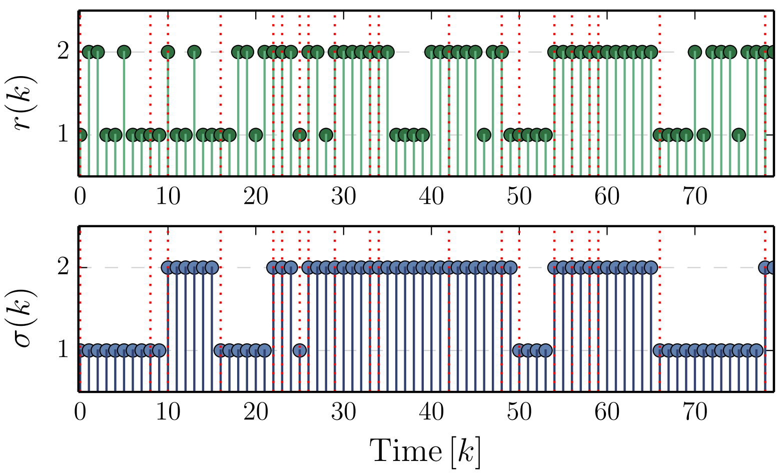

Figure 9.3: Actual mode signal $r(k)$ and sampled mode signal $\sigma(k)$

guarantees stabilization for the case where parameter θ is inside the range $[0.2, 1]$. On the other hand, through repetitive numerical simulations we observe that the states of the closed-loop system converge to the origin in fact for a larger range of parameter values ($\theta \in [0.12, 1]$), which indicate some conservativeness in the conditions of Theorem 9.1 (see Remark 9.1).

Example 9.2. Consider the switched linear stochastic system (6.1) with $M = 3$ modes described by the subsystems matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1.5 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0.5 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 \\ 1.1 & 1.2 \end{bmatrix},$$

$B_1 = [0,\ 1]^{\mathrm{T}}$, $B_2 = [0,\ 0.2]^{\mathrm{T}}$, and $B_3 = [0,\ 0.7]^{\mathrm{T}}$. The mode signal $\{r(k) \in \mathcal{M} \triangleq \{1, 2, 3\}\}_{k \in \mathbb{N}_0}$ of the switched system is assumed to be an aperiodic and irreducible Markov chain characterized by the transition matrix

$$P \triangleq \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}.$$

The invariant distribution for $\{r(k) \in \mathcal{M} \triangleq \{1, 2, 3\}\}_{k \in \mathbb{N}_0}$ is given by $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$. Furthermore, note that the transition matrix P possesses positive real eigenvalues 0.4

(with algebraic multiplicity 2) and 1.

The lengths of intervals between consecutive mode observation instants are assumed to be uniformly distributed over the set $\{2,3,4,5\}$ (see Remark 9.4). In other words, the distribution $\mu : \mathbb{N} \to [0,1]$ is assumed to be given by (9.32) with $\tau_{\rm L} = 2$ and $\tau_{\rm H} = 5$. In this example, we will demonstrate Theorem 9.2, which can be utilized for assessing closed-loop system stability for the case where the exact knowledge of $\mu : \mathbb{N} \to [0,1]$ is not available. Note that for this example the mode observation instants t_i, $i \in \mathbb{N}_0$, satisfy (9.40) with $\bar{\tau} = 5$.

Furthermore, note that

$$\tilde{R} = \begin{bmatrix} 2.6465 & -0.7851 \\ -0.7851 & 1.2568 \end{bmatrix}, \tag{9.73}$$

$$L_1 = [-3.5858 \quad 0.1413], \tag{9.74}$$

$$L_2 = [-4.7066 \quad -0.3329], \tag{9.75}$$

$$L_3 = [-3.2532 \quad -0.3601], \tag{9.76}$$

and the scalars $\zeta_{1,1} = 0.6$, $\zeta_{1,2} = 1.7$, $\zeta_{1,3} = 1.5$, $\zeta_{2,1} = 1.6$, $\zeta_{2,2} = 0.7$, $\zeta_{2,3} = 2$, $\zeta_{3,1} = 2$, $\zeta_{3,2} = 2$, and $\zeta_{3,3} = 0.5$ satisfy (9.15), (9.41), and (9.42). Therefore, it follows from Theorem 9.2 that the proposed control law (9.2) with feedback gain matrices

$$K_1 = L_1\tilde{R}^{-1} = [-1.6222 \quad -0.9009], \tag{9.77}$$

$$K_2 = L_2\tilde{R}^{-1} = [-2.2794 \quad -1.6888], \tag{9.78}$$

$$K_3 = L_3\tilde{R}^{-1} = [-1.6132 \quad -1.2942], \tag{9.79}$$

guarantees almost sure asymptotic stability of the closed-loop system (6.1), (9.2).

Figures 9.4 and 9.5 respectively show sample paths of the state $x(k)$ and the control input $u(k)$ obtained with initial conditions $x(0) = [1,\ -1]^{\rm T}$ and $r(0) = 1$. Furthermore, a sample path of the actual mode signal $r(k)$ and its sampled version $\sigma(k)$ are shown in Figure 9.6. As it is indicated in Figures 9.4–9.6, the proposed control framework (9.2) achieves asymptotic stabilization of the zero solution. It is important to note that the feedback gains K_1, K_2, and K_3 are designed by utilizing Theorem 9.2 without using in-

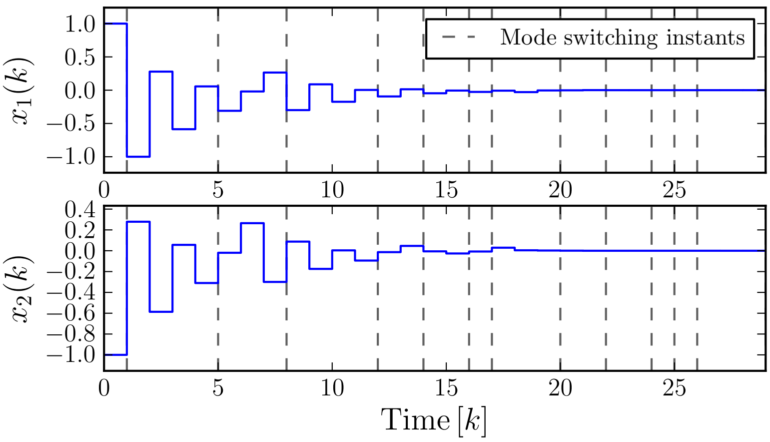

Figure 9.4: State trajectory versus time

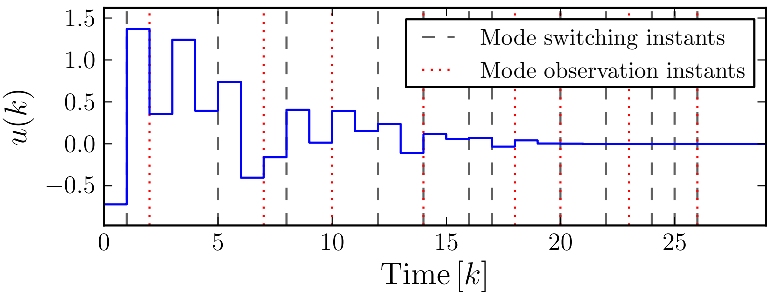

Figure 9.5: Control input versus time

formation on the distribution $\mu : \mathbb{N} \to [0, 1]$, according to which the lengths of intervals between consecutive mode observation instants are distributed. Note that Theorem 9.2 requires only the knowledge of an upper-bounding constant $\bar{\tau} \in \mathbb{N}$ for the length of intervals between consecutive mode observation instants, instead of the exact knowledge of $\mu : \mathbb{N} \to [0, 1]$.

9.5 Conclusion

We proposed a feedback control framework for stochastic stabilization of discrete-time switched linear stochastic systems under randomly available mode information. In this problem setting, information on the active operation mode of the switched system is assumed to be available for control purposes only at random time instants. We presented a probabilistic analysis concerning a sequence-valued stochastic process that captures the

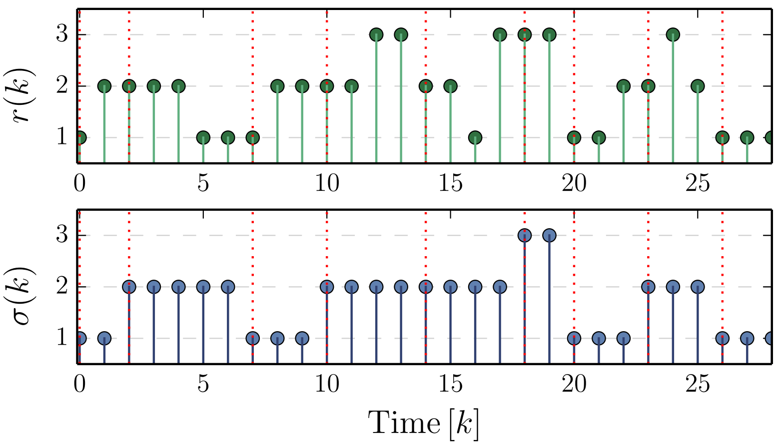

Figure 9.6: Actual mode signal $r(k)$ and sampled mode signal $\sigma(k)$

evolution of active operation mode between mode observation instants. We then used the results of this analysis to obtain sufficient almost sure asymptotic stability conditions for the zero solution of the closed-loop system. These conditions can be used to verify almost sure asymptotic stability of the closed-loop system for the case where stochastic properties of mode observation instants are fully known. Furthermore, we characterized a numerical method (based on linear matrix inequalities) for finding feedback gain matrices so that the proposed control law with those gains achieves almost sure asymptotic stabilization. Note that we also explored the case where exact knowledge of the stochastic properties of mode observation instants is not available. For this case, we presented a set of alternative stabilization conditions.

Chapter 10

Sampled-Parameter Feedback Control of Discrete-time Linear Stochastic Parameter-Varying Systems

10.1 Introduction

Feedback control of dynamical systems with stochastic parameters have been explored in several studies [4, 51, 55–60]. Most of the documented control frameworks for stochastic parameter-varying systems require the availability of parameter information at all time instants. Note that the parameters of a system usually describe the state of external environment, and may not be directly measurable or may not be observed as frequently as the state of the system itself. Hence, it is important to investigate the control problem for the case where the parameters are not available for control purposes at all time instants.

In Chapters 3–9, we investigated stabilization problem for Markov jump systems for the case where the controller has access only to *sampled information of the system mode*, which is modeled by a finite-state Markov chain.

In this chapter we explore feedback control of discrete-time linear stochastic *parameter-varying* systems under *sampled parameter information*. Specifically, the parameter of the system is modeled as a discrete-time aperiodic, stationary, and ergodic Markov process defined on $\mathbb{R}^l$ (see Section 2.3.4). We assume that this parameter is observed (sampled) periodically. In order to achieve stabilization, we develop a control framework that depends *only* on the *sampled version of the parameter*. We obtain sufficient conditions of almost sure asymptotic stabilization of the closed-loop system by utilizing the stationarity and ergodicity properties of a stochastic process that represents the sequences of values that the system parameter takes between consecutive observation instants. We then explore a special class of linear parameter-varying systems where the state matrix is an affine function of the entries of the parameter vector. Note that linear parameter-varying systems with affine parameter dependence has been previosly studied by many researchers (see [2, 82, 83], and the references therein). Our goal in this chapter is to show that stabilization for this class of parameter-varying systems can be achieved through a control law with a feedback gain that is an affine function of the entries of the sampled parameter vector.

This chapter is organized as follows. In Section 10.2, we provide a key result concerning discrete-time Markov processes on $\mathbb{R}^l$. In Section 10.3, we present the mathematical model for discrete-time linear stochastic parameter-varying systems and explain the feedback control problem under periodically sampled parameter information. We obtain sufficient conditions under which our proposed control law guarantees almost sure asymptotic stabilization in Section 10.4. Then, in Section 10.5 we discuss almost sure asymptotic stabilization problem for linear parameter-varying systems with affine parameter dependence. In Section 10.6, we present an illustrative numerical example. Finally, in Section 10.7 we conclude the chapter.

10.2 Mathematical Preliminaries

In this section, we present a key result that is necessary for developing the main results in the following sections.

In Section 10.3 below, we consider a discrete-time linear stochastic parameter-varying dynamical system. The parameter of the dynamical system is modeled as an aperiodic, *stationary,* and *ergodic* discrete-time Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$. Note that Section 2.3.4 provides a detailed characterization of discrete-Time Markov processes on $\mathbb{R}^l$; furthermore, Section 2.3.5 provides the definitions of *stationarity* and *ergodicity* notions for discrete-time stochastic processes. We investigate the stabilization problem for the case where the parameter process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is observed (sampled) at every $\tau \in \mathbb{N}$ time steps. The sequences of values that the parameter $\xi(\cdot)$ takes between consecutive observation instants are characterized through the stochastic process $\{\hat{\xi}(n) \in \underbrace{\mathbb{R}^l \times \mathbb{R}^l \times \cdots \times \mathbb{R}^l}_{\tau \text{ terms}}\}_{n\in\mathbb{N}_0}$ defined by

$$\hat{\xi}(n) \triangleq \big(\xi(n\tau), \xi(n\tau+1), \ldots, \xi((n+1)\tau-1)\big), \tag{10.1}$$

for $n \in \mathbb{N}_0$. Our main results presented in Section 10.3 rely on Lemma 10.1 below, where we show that the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ defined in (10.1) is also *stationary* and *ergodic*.

Lemma 10.1. Suppose that $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is a discrete-time aperiodic, stationary, and ergodic Markov process. Then the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ that is defined in (10.1) for a given $\tau \in \mathbb{N}$ is stationary and ergodic.

Proof. We first show that $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is stationary. For all $S_{k,m} \in \mathcal{B}(\mathbb{R}^l)$, $m \in \{1, 2, \ldots, \tau\}$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, and $i \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathbb{P}[\hat{\xi}(i) \in \hat{S}_1, \hat{\xi}(i+1) \in \hat{S}_2, \ldots \hat{\xi}(i+n-1) \in \hat{S}_n] \\
&= \mathbb{P}[\xi(i\tau) \in S_{1,1}, \ldots, \xi((i+1)\tau-1) \in S_{1,\tau}, \\
&\quad \xi((i+1)\tau) \in S_{2,1}, \ldots, \xi((i+2)\tau-1) \in S_{2,\tau}, \ldots, \\
&\quad \xi((i+n-1)\tau) \in S_{n,1}, \ldots, \xi((i+n)\tau-1) \in S_{n,\tau}],
\end{aligned} \tag{10.2}$$

where $\hat{S}_k \triangleq S_{k,1} \times S_{k,2} \times \cdots \times S_{k,\tau}$, $k \in \{1, 2, \ldots, n\}$. The stationarity of $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ implies that for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}[\xi(i\tau) \in S_{1,1}, \ldots, \xi((i+n)\tau-1) \in S_{n,\tau}] \\
&\quad = \mathbb{P}[\xi(j\tau) \in S_{1,1}, \ldots, \xi((j+n)\tau-1) \in S_{n,\tau}],
\end{aligned} \tag{10.3}$$

for all $S_{k,m} \in \mathcal{B}(\mathbb{R}^l)$, $m \in \{1, 2, \ldots, \tau\}$, $k \in \{1, 2, \ldots, n\}$, and $i, j \in \mathbb{N}_0$. Now it follows from (10.2), (10.3), and the definition of $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ given in (10.1) that for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}[\hat{\xi}(i) \in \hat{S}_1, \hat{\xi}(i+1) \in \hat{S}_2, \ldots \hat{\xi}(i+n-1) \in \hat{S}_n] \\
&\quad = \mathbb{P}[\hat{\xi}(j) \in \hat{S}_1, \hat{\xi}(j+1) \in \hat{S}_2, \ldots \hat{\xi}(j+n-1) \in \hat{S}_n],
\end{aligned} \tag{10.4}$$

for all $S_{k,m} \in \mathcal{B}(\mathbb{R}^l)$, $m \in \{1, 2, \ldots, \tau\}$, $k \in \{1, 2, \ldots, n\}$, and $i, j \in \mathbb{N}_0$, which shows that $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is stationary.

Next, we show that $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is an ergodic stochastic process. Now let $\Omega \triangleq \big(\mathbb{R}^l\big)^{\mathbb{N}_0}$ denote the space that includes all infinite-sequences of $\mathbb{R}^l$-valued vectors, and let $\mathcal{F} \triangleq \mathcal{B}\big(\big(\mathbb{R}^l\big)^{\mathbb{N}_0}\big)$ denote the product σ-algebra. Furthermore, let $\mathbb{P}$ be the probability measure induced by $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$. For a fixed $\omega \in \Omega$, the stochastic process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is given by $\xi(k) = \omega(k), k \in \mathbb{N}_0$. The measure preserving transformation $T_\xi : \Omega \to \Omega$ associated with the stationary process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is given by

$$T_\xi(\{\omega(k)\}_{k\in\mathbb{N}_0}) \triangleq \{\omega(k+1)\}_{k\in\mathbb{N}_0}, \quad \omega \in \Omega. \tag{10.5}$$

Now, for a fixed $\omega \in \Omega$, the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is given by $\hat{\xi}(n) = \big(\omega(n\tau), \omega(n\tau+1), \ldots, \omega((n+1)\tau - 1)\big), n \in \mathbb{N}_0$. Furthermore, the measure preserving transformation $T_{\hat{\xi}} : \Omega \to \Omega$ associated with the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is given by

$$T_{\hat{\xi}}(\{\omega(k)\}_{k\in\mathbb{N}_0}) \triangleq \{\omega(k+\tau)\}_{k\in\mathbb{N}_0}, \quad \omega \in \Omega. \tag{10.6}$$

Moreover, note that $T_{\hat{\xi}}(\omega) = T_\xi^\tau(\omega), \omega \in \Omega$. Hence, for all $F \in \mathcal{F}$,

$$\begin{aligned}
T_{\hat{\xi}}^{-1}(F) &= \{\omega \in \Omega : T_{\hat{\xi}}(\omega) \in F\} \\
&= \{\omega \in \Omega : T_\xi^\tau(\omega) \in F\}.
\end{aligned} \tag{10.7}$$

Note that for the case of $\tau = 1$, $T_{\hat{\xi}}(\omega) = T_\xi(\omega), \omega \in \Omega$, and hence ergodicity of $\{\xi(k)\}_{k\in\mathbb{N}_0}$ implies ergodicity of $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$. Now, consider the case where $\tau \geq 2$. Let $F \in \mathcal{F}$ be a set such that $\{\omega \in \Omega : T_\xi^\tau(\omega) \in F\} = F$. Then in the case where $\{\omega \in \Omega : T_\xi(\omega) \in F\} = F$, we have $\mathbb{P}[F] = 0$ or $\mathbb{P}[F] = 1$. Otherwise, aperiodicity of $\{\xi(k)\}_{k\in\mathbb{N}_0}$ implies that $\mathbb{P}[F] \in \{0, 1\}$. Hence, for all $F \in \mathcal{F}$ such that $T_{\hat{\xi}}^{-1}(F) = F$, we obtain $\mathbb{P}[F] = 0$ or $\mathbb{P}[F] = 1$, which shows that the stationary stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is ergodic. □

10.3 Sampled-Parameter Feedback Control of Discrete-Time Linear Stochastic Parameter-Varying Systems

In this section, we first provide the mathematical model for a discrete-time linear stochastic parameter-varying system. Then we explain the feedback control problem under periodically observed (sampled) parameter information and present our proposed sampled-parameter control framework for stabilizing discrete-time linear stochastic parameter-varying systems.

10.3.1 Mathematical Model

We consider the discrete-time linear stochastic dynamical system given by

$$x(k+1) = A(\xi(k))x(k) + B(\xi(k))u(k),\ k \in \mathbb{N}_0, \tag{10.8}$$

with the initial condition $x(0) = x_0$, where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input. Furthermore, $A : \mathbb{R}^l \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^l \to \mathbb{R}^{n\times m}$ denote the parameter-dependent system matrices. The parameter denoted by $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is assumed to be an aperiodic, stationary, and ergodic Markov process characterized by the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ and the *initial stationary* distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$. Note that

$$\mathbb{P}[\xi(0) \in S] = \nu(S), \quad S \in \mathcal{B}(\mathbb{R}^l), \tag{10.9}$$

$$\int_{\mathbb{R}^l} P(s, S)\nu(\mathrm{d}s) = \nu(S), \quad S \in \mathcal{B}(\mathbb{R}^l). \tag{10.10}$$

Note that a class of *switched* stochastic systems can be modeled as stochastic parameter-varying systems of the form (10.8). For instance, the discrete-time switched linear stochastic system discussed in Chapters 6, 7, and 9 is a special case of the dynamical system (10.8), where $\{\xi(k)\}_{k\in\mathbb{N}_0}$ is modeled as an aperiodic and irreducible finite-state Markov chain (see Remark 10.1). Note that in Chapters 6, 7, and 9, the parameter $\xi(\cdot)$ indicates

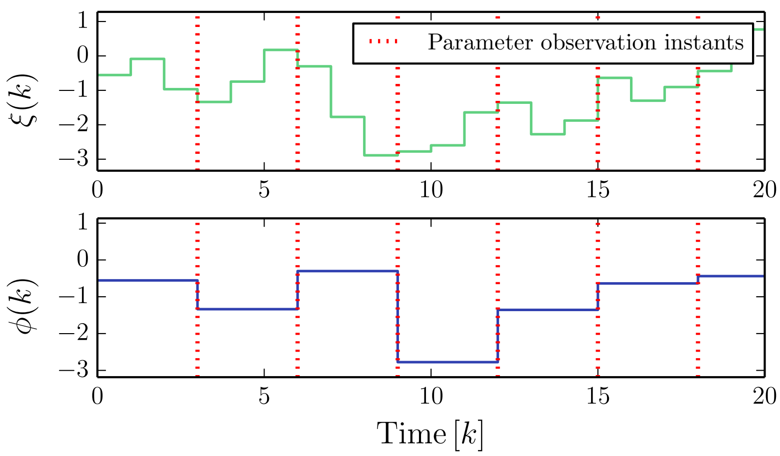

Figure 10.1: Actual parameter $\xi(k)$ and its sampled version $\phi(k)$

the active mode (subsystem) that governs the overall dynamics of a switched system. Furthermore, note that linear systems with stationary and ergodic *autoregressive* parameters can also be characterized through (10.8), since vector autoregressions are Markov processes (see [89]). In Section 10.6, we present an illustrative discussion on the almost sure asymptotic stabilization of a stochastic parameter-varying system with stationary and ergodic autoregressive parameters.

10.3.2 Control Under Periodic Parameter Observations

In this chapter, we investigate feedback stabilization of the linear parameter-varying dynamical system (10.8) under the assumption that only a periodically-sampled version of the parameter process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is available for control purposes. Specifically, we assume that the parameter $\xi(\cdot)$ is observed (sampled) periodically at time instants $0, \tau, 2\tau, \ldots$, where $\tau \in \mathbb{N}$ denotes the parameter observation period. The sampled parameter information that is available to the controller is characterized through the stochastic process $\{\phi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ defined by

$$\phi(k) = \xi(n\tau),\ k \in \{n\tau, n\tau + 1, \ldots (n+1)\tau - 1\}, \tag{10.11}$$

for $n \in \mathbb{N}_0$.

In order to achieve stabilization of the dynamical system (10.8), we propose the control law

$$u(k) = K(\phi(k))x(k), \quad k \in \mathbb{N}_0, \tag{10.12}$$

where $K : \mathbb{R}^l \to \mathbb{R}^{m\times n}$ denotes the sampled-parameter-dependent feedback gain. Note that the control law (10.12) requires *only* sampled parameter information.

Figure 10.1 shows sample paths of a parameter process $\{\xi(k) \in \mathbb{R}\}_{k\in\mathbb{N}_0}$ (modeled as an autoregressive process), and its sampled version $\{\phi(k) \in \mathbb{R}\}_{k\in\mathbb{N}_0}$. In this example, the parameter $\xi(\cdot)$ is observed (sampled) at every $\tau = 3$ steps. At these parameter observation instants, actual parameter and its sampled version share the same value. However, at other time instants, actual parameter may differ from its sampled version, since the parameter may change its value between the observation instants. Hence, the perfect knowledge of the actual parameter is available to the controller only at the parameter observation instants. In the following, we obtain sufficient conditions for stabilization of the parameter-varying system (10.8) under the proposed control law (10.12) that depends *only* on the sampled parameter information.

Note that the system dynamics in (10.8) depend on the actual parameter $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$, whereas the feedback gain of the control law (10.12) depends on the sampled version of the parameter, $\{\phi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$. In the following lemma we present an ergodic theorem for the coupled stochastic process $\{(\xi(t), \phi(k)) \in \mathbb{R}^l \times \mathbb{R}^l\}_{k\in\mathbb{N}_0}$, which is composed of the original parameter process and its sampled version. Note that the result provided in Lemma 10.2 below is crucial for developing the main results of this chapter presented in Theorems 10.1, 10.2, and Corollary 10.1.

Lemma 10.2. Suppose $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is an aperiodic, stationary, and ergodic Markov process characterized by the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ and the *initial stationary* distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$. Furthermore, let $\{\phi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ defined as in (10.11) be the periodically sampled version of $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ for a given sampling period $\tau \in \mathbb{N}$. Then for any Borel measurable function $\gamma : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$, it follows that

$$\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) = \frac{1}{\tau} \sum_{i=0}^{\tau-1} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \gamma(\bar{\xi}, \bar{\phi}) P^{(i)}(\bar{\phi}, \mathrm{d}\bar{\xi}) \nu(\mathrm{d}\bar{\phi}), \tag{10.13}$$

almost surely.

Proof. Let $\{\hat{\xi}(n) \in \underbrace{\mathbb{R}^l \times \mathbb{R}^l \times \cdots \times \mathbb{R}^l}_{\tau \text{ terms}}\}_{n\in\mathbb{N}_0}$ be the stochastic process defined in (10.1). Note that $\hat{\xi}(n)$ denotes the sequence of values that the parameter $\xi(\cdot)$ takes between consecutive observation instants $n\tau$ and $(n+1)\tau$. Furthermore, let $N(k) \triangleq \lfloor k/\tau \rfloor$, $k \in \mathbb{N}_0$. The number of mode samples obtained upto time $k \in \mathbb{N}_0$ is given by $N(k)+1$. Note that, for all $\bar{n} \in \mathbb{N}$ such that $\bar{n} > \tau$, we have

$$\sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) = \sum_{n=0}^{N(\bar{n})-1} \sum_{i=0}^{\tau-1} \gamma(\xi(n\tau+i), \phi(n\tau+i)) + \sum_{k=N(\bar{n})\tau}^{\bar{n}-1} \gamma(\xi(k), \phi(k)). \quad (10.14)$$

Since $\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=N(\bar{n})\tau}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) = 0$, it follows from (10.14) that

$$\begin{aligned}
\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) &= \lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{n=0}^{N(\bar{n})-1} \sum_{i=0}^{\tau-1} \gamma(\xi(n\tau+i), \phi(n\tau+i)) \\
&= \lim_{\bar{n}\to\infty} \frac{N(\bar{n})}{\bar{n}} \frac{1}{N(\bar{n})} \sum_{n=0}^{N(\bar{n})-1} \sum_{k=0}^{\tau-1} \gamma(\xi(n\tau+i), \phi(n\tau+i)) \\
&= \lim_{\bar{n}\to\infty} \frac{N(\bar{n})}{\bar{n}} \frac{1}{N(\bar{n})} \sum_{n=0}^{N(\bar{n})-1} \hat{\gamma}(\hat{\xi}(n)), \quad (10.15)
\end{aligned}$$

where $\hat{\gamma}(\hat{\xi}(n)) \triangleq \sum_{k=0}^{\tau-1} \gamma(\xi(n\tau+k), \phi(n\tau+k))$. Now, by using the definition of $N(\cdot)$, we obtain $\lim_{\bar{n}\to\infty} \frac{N(\bar{n})}{\bar{n}} = \frac{1}{\tau}$. Furthermore, it follows from Lemma 10.1 that the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is stationary and ergodic. Thus, by the ergodic theorem for stationary and ergodic stochastic processes, we obtain $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \bar{\gamma}(\hat{\xi}(n)) = \mathbb{E}[\hat{\gamma}(\hat{\xi}(0))]$. Therefore,

$$\begin{aligned}
\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) &= \frac{1}{\tau} \mathbb{E}[\hat{\gamma}(\hat{\xi}(0))] \\
&= \frac{1}{\tau} \mathbb{E}[\sum_{i=0}^{\tau-1} \gamma(\xi(i), \phi(i))] \\
&= \frac{1}{\tau} \sum_{i=0}^{\tau-1} \mathbb{E}[\gamma(\xi(i), \phi(i))]. \quad (10.16)
\end{aligned}$$

Note that since the value of sampled parameter process $\phi(\cdot)$ does not change between parameter observation instants, we have $\phi(i) = \phi(0) = \xi(0)$, $i \in \{0, 1, \ldots, \tau-1\}$. It then follows that

$$\lim_{\bar{n}\to\infty}\frac{1}{\bar{n}}\sum_{k=0}^{\bar{n}-1}\gamma(\xi(k),\phi(k)) = \frac{1}{\tau}\sum_{i=0}^{\tau-1}\mathbb{E}[\gamma(\xi(i),\xi(0))]. \tag{10.17}$$

Now by using the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0,1]$ and the initial stationary distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0,1]$, we obtain

$$\begin{aligned}\lim_{\bar{n}\to\infty}\frac{1}{\bar{n}}\sum_{k=0}^{\bar{n}-1}\gamma(\xi(k),\phi(k)) &= \frac{1}{\tau}\sum_{i=0}^{\tau-1}\int_{\mathbb{R}^l}\int_{\mathbb{R}^l}\gamma(\bar{\xi},\bar{\phi})\mathbb{P}[\xi(i)\in \mathrm{d}\bar{\xi},\xi(0)\in \mathrm{d}\bar{\phi}]\\ &= \frac{1}{\tau}\sum_{i=0}^{\tau-1}\int_{\mathbb{R}^l}\int_{\mathbb{R}^l}\gamma(\bar{\xi},\bar{\phi})P^{(i)}(\bar{\phi},\mathrm{d}\bar{\xi})\nu(\mathrm{d}\bar{\phi}), \end{aligned}\tag{10.18}$$

which completes the proof. □

10.4 Sufficient Conditions for Almost Sure Asymptotic Stabilization

In this section, we utilize the result presented in Lemma 10.2 and obtain sufficient almost sure asymptotic stabilization conditions for the closed-loop stochastic parameter-varying system (10.8), (10.12).

Theorem 10.1. Consider the linear parameter-varying control system (10.8), (10.12). If there exist a matrix $R > 0$ and a measurable function $\lambda : \mathbb{R}^l \times \mathbb{R}^l \to (0,\infty)$ such that

$$0 \geq (A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}}R(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) - \lambda(\bar{\xi},\bar{\phi})R, \quad \bar{\xi},\bar{\phi}\in\mathbb{R}^l, \tag{10.19}$$

$$\sum_{i=0}^{\tau-1}\int_{\mathbb{R}^l}\int_{\mathbb{R}^l}\ln(\lambda(\bar{\xi},\bar{\phi}))P^{(i)}(\bar{\phi},\mathrm{d}\bar{\xi})\nu(\mathrm{d}\bar{\phi}) < 0, \tag{10.20}$$

then the zero solution $x(k) \equiv 0$ of the closed-loop system (10.8), (10.12) is asymptotically stable almost surely.

Proof. First, let $V : \mathbb{R}^n \to [0,\infty)$ be the positive-definite function defined by $V(x) \triangleq$

$x^{\mathrm{T}}Rx$. It follows from (10.8) and (10.12) that for $k \in \mathbb{N}_0$,

$$V(x(k+1)) = x^{\mathrm{T}}(k)\big(A(\xi(k)) + B(\xi(k))K(\phi(k))\big)^{\mathrm{T}} R\big(A(\xi(k)) + B(\xi(k))K(\phi(k))\big)x(k). \tag{10.21}$$

We now use (10.19), (10.21) and definition of $V(\cdot)$ to obtain

$$\begin{aligned} V(x(k+1)) &\leq \lambda(\xi(k), \phi(k))V(x(k)) \\ &\leq \theta(k)V(x(0)), \quad k \in \mathbb{N}_0, \end{aligned} \tag{10.22}$$

where $\theta : \mathbb{N}_0 \to (0, \infty)$ is given by

$$\theta(k) \triangleq \prod_{n=0}^{k} \lambda(\xi(n), \phi(n)), \quad k \in \mathbb{N}_0. \tag{10.23}$$

Now, it follows from (10.23) that

$$\ln(\theta(k)) = \sum_{n=0}^{k} \ln(\lambda(\xi(n), \phi(n))), \quad k \in \mathbb{N}_0. \tag{10.24}$$

Furthermore, as a consequence of Lemma 10.2,

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{k} \ln(\theta(k)) &= \lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k} \ln(\lambda(\xi(n), \phi(n))) \\ &= \frac{1}{\tau} \sum_{i=0}^{\tau-1} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \ln(\lambda(\bar{\xi}, \bar{\phi}))P^{(i)}(\bar{\phi}, \mathrm{d}\bar{\xi})\nu(\mathrm{d}\bar{\phi}). \end{aligned} \tag{10.25}$$

It then follows from (10.20) and (10.25) that

$$\lim_{k\to\infty} \frac{1}{k} \ln(\theta(k)) < 0, \tag{10.26}$$

almost surely. Hence, $\lim_{k\to\infty} \ln \theta(k) = -\infty$, almost surely, and therefore,

$$\mathbb{P}[\lim_{k\to\infty} \theta(k) = 0] = 1. \tag{10.27}$$

Now, as a result of (10.22) and (10.27), $\mathbb{P}[\lim_{k\to\infty} V(x(k)) = 0] = 1$, which implies that the zero solution $x(k) \equiv 0$ of the closed-loop system (10.8), (10.12) is asymptotically stable almost surely. $\square$

Theorem 10.1 provides sufficient conditions for almost sure asymptotic stability of the zero solution of the closed-loop system (10.8) under the control law (10.12). Conditions (10.19) and (10.20) of Theorem 10.1 reflect that the stabilization performance depend not only on the system dynamics but also on the probabilistic dynamics of parameter transitions as well as the parameter observation period $\tau \in \mathbb{N}$.

Remark 10.1. Note that the parameter-varying dynamical system model defined by (10.8) includes switched linear stochastic system models (explored in Chapters 6, 7, and 9) as a special case. Consequently, conditions (10.19) and (10.20) allow us to also assess almost sure asymptotic stability of closed-loop switched linear stochastic systems. Specifically, let the parameter process $\{\xi(k) \in \mathbb{R}\}_{k\in\mathbb{N}_0}$ be characterized through the transition probability function $P : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ and the *initial stationary* distribution $\nu : \mathcal{B}(\mathbb{R}) \to [0,1]$ given by

$$P(\bar{\phi}, \{\bar{\xi}\}) = p_{\bar{\phi},\bar{\xi}}, \quad \bar{\phi}, \bar{\xi} \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}, \tag{10.28}$$

$$\nu(\{\bar{\phi}\}) = \nu_{\bar{\phi}}, \quad \bar{\phi}, \in \mathcal{M}, \tag{10.29}$$

where $p_{\bar{\phi},\bar{\xi}} \in [0,1]$, $\nu_{\bar{\phi}} \in [0,1]$, $\bar{\phi}, \bar{\xi} \in \mathcal{M}$, are scalars that satisfy $\sum_{\bar{\xi}\in\mathcal{M}} p_{\bar{\phi},\bar{\xi}} = 1$, $\bar{\phi} \in \mathcal{M}$, $\sum_{\bar{\phi}\in\mathcal{M}} \nu_{\bar{\phi}} = 1$, and $\sum_{\bar{\phi}\in\mathcal{M}} \nu_{\bar{\phi}} p_{\bar{\phi},\bar{\xi}} = \nu_{\bar{\xi}}$, $\bar{\xi} \in \mathcal{M}$. With this characterization, the parameter-varying system (10.8) represents a switched linear stochastic system composed of $M \in \mathbb{N}$ number of subsystems (modes); moreover, the parameter process $\{\xi(k)\}_{k\in\mathbb{N}_0}$ corresponds to the mode signal that is modeled as an aperiodic and irreducible Markov chain with transition probabilities $p_{\bar{\phi},\bar{\xi}} \in [0,1]$, $\bar{\phi}, \bar{\xi} \in \mathcal{M}$, and *initial stationary* distributions $\nu_{\bar{\phi}} \in [0,1]$, $\bar{\phi} \in \mathcal{M}$. Furthermore, the condition (10.20) for this case reduces to

$$\sum_{i=0}^{\tau-1} \sum_{\bar{\xi}\in\mathcal{M}} \sum_{\bar{\phi}\in\mathcal{M}} \ln(\lambda(\bar{\xi}, \bar{\phi})) p^{(i)}_{\bar{\phi},\bar{\xi}} \nu_{\bar{\phi}} < 0, \tag{10.30}$$

where $p^{(i)}_{\bar{\phi},\bar{\xi}} \in [0,1]$, $\bar{\phi}, \bar{\xi} \in \mathcal{M}$, denote i-step mode transition probabilities of the switched system.

In the next section, we explore the sampled-parameter control problem for a linear parameter-varying system with a state matrix that depend affinely on the stochastic parameter $\{\xi(k)\}_{k\in\mathbb{N}_0}$.

10.5 Stabilization of Linear Parameter-Varying Systems with Affine Parameter Dependence

We now consider a special case of the parameter-varying dynamical system (10.8) where the state matrix $A(\cdot)$ is defined as an affine function of the entries of the parameter vector $\xi(\cdot) \in \mathbb{R}^l$; moreover, the input matrix $B(\cdot)$ is defined as a constant matrix. Specifically, we consider the linear parameter-varying system (10.8) with

$$A(\bar{\xi}) \triangleq \big(\bar{A}_0 + \sum_{i=1}^{l} \bar{\xi}_i \bar{A}_i\big), \tag{10.31}$$

$$B(\bar{\xi}) \triangleq \bar{B}, \quad \bar{\xi} \in \mathbb{R}^l, \tag{10.32}$$

where $\bar{A}_i \in \mathbb{R}^{m\times n}$, $i \in \{0, 1, \ldots, l\}$, and $\bar{B} \in \mathbb{R}^{n\times m}$ are constant matrices. In order to achieve stabilization of the zero solution of dynamical system (10.8) with state and input matrices given by (10.31) and (10.32), we employ the control law (10.12) with the sampled-parameter-dependent feedback gain function

$$K(\bar{\phi}) \triangleq \bar{K}_0 + \sum_{i=1}^{l} \bar{\phi}_i \bar{K}_i, \quad \bar{\phi} \in \mathbb{R}^l, \tag{10.33}$$

where $\bar{K}_i \in \mathbb{R}^{m\times n}$, $i \in \{0, 1, \ldots, l\}$, are constant matrices. Note that the feedback gain (10.33) is an affine function of the entries of the sampled parameter vector $\phi(\cdot)$.

In Theorem 10.2 below, we present sufficient conditions under which the proposed control law (10.12) with the feedback gain (10.33) guarantees almost sure asymptotic stabilization of the linear stochastic parameter-varying system (10.8) with state and input matrices given by (10.31) and (10.32).

Theorem 10.2. Consider the linear parameter-varying system (10.8) with state and input matrices given by (10.31) and (10.32). If there exist a matrix $R > 0$ and scalars $\alpha_i \in$

$(0, \infty), i \in \{1, 2, \ldots, l\}$, $\beta_i \in (0, \infty)$, $i \in \{0, 1, \ldots, l\}$, such that

$$0 \geq \bar{A}_i^{\mathrm{T}} R \bar{A}_i - \alpha_i R, \quad i \in \{1, \ldots, l\}, \tag{10.34}$$

$$0 \geq (\bar{A}_i + \bar{B}\bar{K}_i)^{\mathrm{T}} R (\bar{A}_i + \bar{B}\bar{K}_i) - \beta_i R, \quad i \in \{0, 1, \ldots, l\}, \tag{10.35}$$

and (10.20) hold with $\lambda : \mathbb{R}^l \times \mathbb{R}^l \to (0, \infty)$ given by

$$\lambda(\bar{\xi}, \bar{\phi}) \triangleq (2l+1)\big(\beta_0^2 + \sum_{i=1}^{l} \big((\bar{\xi}_i - \bar{\phi}_i)^2 \alpha_i^2 + \bar{\phi}_i^2 \beta_i^2\big)\big), \tag{10.36}$$

then the control law (10.12) with the feedback gain (10.33) guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (10.8), (10.12) is asymptotically stable almost surely.

Proof. Using the definitions (10.31), (10.32), we obtain

$$\begin{aligned}
&(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}} R (A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) \\
&\quad = \big(\bar{A}_0 + \sum_{i=1}^{l} \bar{\xi}_i \bar{A}_i + \bar{B}(\bar{K}_0 + \sum_{i=1}^{l} \bar{\phi}_i \bar{K}_i)\big)^{\mathrm{T}} R \big(\bar{A}_0 + \sum_{i=1}^{l} \bar{\xi}_i \bar{A}_i + \bar{B}(\bar{K}_0 + \sum_{i=1}^{l} \bar{\phi}_i \bar{K}_i)\big) \\
&\quad = \big(\bar{A}_0 + \bar{B}\bar{K}_0 + \sum_{i=1}^{l} (\bar{\xi}_i \bar{A}_i + \bar{\phi}_i \bar{B}\bar{K}_i)\big)^{\mathrm{T}} R \big(\bar{A}_0 + \bar{B}\bar{K}_0 + \sum_{i=1}^{l} (\bar{\xi}_i \bar{A}_i + \bar{\phi}_i \bar{B}\bar{K}_i)\big).
\end{aligned} \tag{10.37}$$

Note that for $i \in \{1, 2, \ldots, l\}$,

$$\begin{aligned}
\bar{\xi}_i \bar{A}_i + \bar{\phi}_i \bar{B}\bar{K}_i &= \bar{\xi}_i \bar{A}_i - \bar{\phi}_i \bar{A}_i + \bar{\phi}_i \bar{A}_i + \bar{\phi}_i \bar{B}\bar{K}_i \\
&= (\bar{\xi}_i - \bar{\phi}_i)\bar{A}_i + \bar{\phi}_i (\bar{A}_i + \bar{B}\bar{K}_i).
\end{aligned} \tag{10.38}$$

As a consequence of (10.37) and (10.38), we have

$$\begin{aligned}
&(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}} R (A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) \\
&\quad = \big(\bar{A}_0 + \bar{B}\bar{K}_0 + \sum_{i=1}^{l} (\bar{\xi}_i - \bar{\phi}_i)\bar{A}_i + \sum_{i=1}^{l} \bar{\phi}_i (\bar{A}_i + \bar{B}\bar{K}_i)\big)^{\mathrm{T}} \\
&\qquad \cdot R \big(\bar{A}_0 + \bar{B}\bar{K}_0 + \sum_{i=1}^{l} (\bar{\xi}_i - \bar{\phi}_i)\bar{A}_i + \sum_{i=1}^{l} \bar{\phi}_i (\bar{A}_i + \bar{B}\bar{K}_i)\big).
\end{aligned} \tag{10.39}$$

Now, let $C_i \in \mathbb{R}^{n\times n}$, $i \in \{0, 1, \ldots, 2l\}$, be defined by $C_0 \triangleq \bar{A}_0 + \bar{B}\bar{K}_0$, $C_i \triangleq (\bar{\xi}_i - \bar{\phi}_i)\bar{A}_i$, $C_{i+l} \triangleq \bar{\phi}_i(\bar{A}_i + \bar{B}\bar{K}_i)$, $i \in \{1, 2, \ldots, l\}$. Then, it follows from (10.39) that

$$\begin{aligned}
&(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}}R(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) \\
&\quad = \big(\sum_{i=0}^{2l} C_i\big)^{\mathrm{T}} R\big(\sum_{i=0}^{2l} C_i\big)^{\mathrm{T}} \\
&\quad = \sum_{i=0}^{2l}\sum_{j=i}^{2l}\big(C_i^{\mathrm{T}}RC_j + C_j^{\mathrm{T}}RC_i\big) - \sum_{i=0}^{2l} C_i^{\mathrm{T}}RC_i
\end{aligned} \tag{10.40}$$

Note that for any pair of matrices $C_i, C_j \in \mathbb{R}^{n\times n}$, we have $C_i^{\mathrm{T}}RC_i + C_j^{\mathrm{T}}RC_j - \big(C_i^{\mathrm{T}}RC_j + C_j^{\mathrm{T}}RC_i\big) = (C_i - C_j)^{\mathrm{T}}R(C_i - C_j) \geq 0$. Hence, (10.40) yields

$$\begin{aligned}
&(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}}R(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) \\
&\quad \leq \sum_{i=0}^{2l}\sum_{j=i}^{2l}\big(C_i^{\mathrm{T}}RC_i + C_j^{\mathrm{T}}RC_j\big) - \sum_{i=0}^{2l} C_i^{\mathrm{T}}RC_i \\
&\quad = \sum_{i=0}^{2l}\sum_{j=i}^{2l} C_i^{\mathrm{T}}RC_i + \sum_{i=0}^{2l}\sum_{j=i}^{2l} C_j^{\mathrm{T}}RC_j - \sum_{i=0}^{2l} C_i^{\mathrm{T}}RC_i.
\end{aligned} \tag{10.41}$$

Now note that

$$\sum_{i=0}^{2l}\sum_{j=i}^{2l} C_j^{\mathrm{T}}RC_j = \sum_{j=0}^{2l}\sum_{i=j}^{2l} C_i^{\mathrm{T}}RC_i = \sum_{i=0}^{2l}\sum_{j=0}^{i} C_i^{\mathrm{T}}RC_i. \tag{10.42}$$

Inserting (10.42) into (10.41) yields

$$\begin{aligned}
&(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}}R(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) \\
&\quad \leq \sum_{i=0}^{2l}\sum_{j=i}^{2l} C_i^{\mathrm{T}}RC_i + \sum_{i=0}^{2l}\sum_{j=0}^{i} C_i^{\mathrm{T}}RC_i - \sum_{i=0}^{2l} C_i^{\mathrm{T}}RC_i \\
&\quad = \sum_{i=0}^{2l}\sum_{j=0}^{2l} C_i^{\mathrm{T}}RC_i + \sum_{i=0}^{2l} C_i^{\mathrm{T}}RC_i - \sum_{i=0}^{2l} C_i^{\mathrm{T}}RC_i \\
&\quad = (2l+1)\sum_{i=0}^{2l} C_i^{\mathrm{T}}RC_i.
\end{aligned} \tag{10.43}$$

By using the definitions of $C_i \in \mathbb{R}^{n\times n}$, $i \in \{0, 1, \ldots, 2l\}$, together with (10.34)–(10.36) we obtain

$$\begin{aligned}
&(A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}} R (A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) \\
&\quad \le (2l+1)\big((\bar{A}_0 + \bar{B}\bar{K}_0)^{\mathrm{T}} R (\bar{A}_0 + \bar{B}\bar{K}_0) + \sum_{i=1}^{l} (\bar{\xi}_i - \bar{\phi}_i)^2 \bar{A}_i^{\mathrm{T}} R \bar{A}_i \\
&\qquad + \sum_{i=1}^{l} \bar{\phi}_i^2 (\bar{A}_i + \bar{B}\bar{K}_i)^{\mathrm{T}} R (\bar{A}_i + \bar{B}\bar{K}_i)\big) \\
&\quad = (2l+1)\big(\beta_0^2 R + \sum_{i=1}^{l} \big((\bar{\xi}_i - \bar{\phi}_i)^2 \alpha_i^2 + \bar{\phi}_i^2 \beta_i^2\big) R\big) \\
&\quad = \lambda(\bar{\xi}, \bar{\phi}) R, \qquad (10.44)
\end{aligned}$$

which implies (10.19). Hence, the result follows from Theorem 10.1. □

Note that the conditions presented in Theorem 10.2 can be used for assessing almost sure asymptotic stability of the closed-loop system (10.8), (10.12) with the system matrices (10.31), (10.32) when the gain matrices $\bar{K}_i \in \mathbb{R}^{m \times n}$, $i \in \{0, 1, \ldots, l\}$, for the control law (10.12), (10.33) are already known. In practice, we often need to employ numerical methods for finding gain matrices so that the proposed control law (10.12) with those gains achieves almost sure asymptotic stabilization. In Corollary 10.1 below, we present an alternative set of sufficient almost sure asymptotic stabilization conditions, which are well suited for finding *stabilizing gain matrices* $\bar{K}_i \in \mathbb{R}^{m \times n}$, $i \in \{0, 1, \ldots, l\}$, through numerical methods.

Corollary 10.1. Consider the linear parameter-varying system (10.8) with state and input matrices given by (10.31) and (10.32). If there exist matrices $\tilde{R} > 0$, $\bar{L}_i \in \mathbb{R}^{m \times n}$, $i \in \{0, 1, \ldots, l\}$, and scalars $\alpha_i \in (0, \infty), i \in \{1, 2, \ldots, l\}$, $\beta_i \in (0, \infty)$, $i \in \{0, 1, \ldots, l\}$, such that

$$0 \ge (\bar{A}_i \tilde{R})^{\mathrm{T}} \tilde{R}^{-1} (\bar{A}_i \tilde{R}) - \alpha_i \tilde{R}, \quad i \in \{1, 2, \ldots, l\}, \qquad (10.45)$$

$$0 \ge (\bar{A}_i \tilde{R} + \bar{B}\bar{L}_i)^{\mathrm{T}} \tilde{R}^{-1} (\bar{A}_i \tilde{R} + \bar{B}\bar{L}_i) - \beta_i \tilde{R}, \quad i \in \{0, 1, \ldots, l\}, \qquad (10.46)$$

and (10.20) hold with $\lambda : \mathbb{R}^l \times \mathbb{R}^l \to (0, \infty)$ given in (10.36), then the control law (10.12), (10.33) with gain matrices $\bar{K}_i = \bar{L}_i \tilde{R}^{-1}$, $i \in \{0, 1, \ldots, l\}$, guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (10.8), (10.12) is asymptotically stable almost surely.

Proof. The result is a direct consequence of Theorem 10.2 with $R = \tilde{R}^{-1}$. $\square$

Remark 10.2. We verify conditions (10.20), (10.45), and (10.46) of Corollary 10.1 by employing a numerical technique. Specifically, following the approach presented in Chapter 9, we transform conditions (10.45) and (10.46) into the matrix inequalities

$$0 \leq \begin{bmatrix} \alpha_i \tilde{R} & (\bar{A}_i \tilde{R})^{\mathrm{T}} \\ (\bar{A}_i \tilde{R}) & \tilde{R} \end{bmatrix}, \quad i \in \{1, \ldots, l\}, \tag{10.47}$$

$$0 \leq \begin{bmatrix} \beta_i \tilde{R} & (\bar{A}_i \tilde{R} + \bar{B}\bar{L}_i)^{\mathrm{T}} \\ (\bar{A}_i \tilde{R} + \bar{B}\bar{L}_i) & \tilde{R} \end{bmatrix}, i \in \{0, 1, \ldots, l\}, \tag{10.48}$$

by using Schur complements (see [115]). Note that the inequalities (10.47) and (10.48) are linear in $\tilde{R}$ and $\bar{L}_i$, $i \in \{0, 1, \ldots, l\}$. Our numerical method is based on iterating over a set of the values of $\alpha_i, i \in \{1, 2, \ldots, l\}$, and β_i, $i \in \{0, 1, \ldots, l\}$, that satisfy (10.20) with $\lambda(\cdot)$ calculated according to (10.36). At each iteration we look for feasible solutions to the linear matrix inequalities (10.47) and (10.48). We use this method in Section 10.6 below to find matrices $\tilde{R} > 0, \bar{L}_i \in \mathbb{R}^{m \times n}$, $i \in \{0, 1, \ldots, l\}$, and scalars $\alpha_i \in (0, \infty), i \in \{1, 2, \ldots, l\}$, and $\beta_i \in (0, \infty)$, $i \in \{0, 1, \ldots, l\}$, that satisfy (10.20), (10.45), and (10.46) for a given discrete-time linear stochastic parameter-varying system (10.8) with affine parameter dependence characterized in (10.31) and (10.32).

10.6 Illustrative Numerical Example

In this section, we present an illustrative numerical example to demonstrate the main results of this chapter. Specifically, we consider the linear parameter-varying stochastic system (10.8) with state and input matrices given by

$$A(\bar{\xi}) = \begin{bmatrix} 0.3 + 0.1\bar{\xi}_2 & 0.3 + 0.1(\bar{\xi}_1 + \bar{\xi}_2) \\ 4 + \bar{\xi}_1 & 1 + \bar{\xi}_2 \end{bmatrix}, \tag{10.49}$$

and $B(\bar{\xi}) = \bar{B} = [0,\, 1]^{\mathrm{T}}$, for all $\bar{\xi} \in \mathbb{R}^2$. Note that the state matrix $A(\cdot)$ given in (10.49) is an affine function of the form (10.30) with

$$\bar{A}_0 = \begin{bmatrix} 0.3 & 0.3 \\ 4 & 1 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 0 & 0.1 \\ 1 & 0 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 1 \end{bmatrix}. \tag{10.50}$$

The parameter $\{\xi(k) \in \mathbb{R}^2\}_{k\in\mathbb{N}_0}$ of the system is assumed to be a Markov process with autoregressive entries characterized by

$$\xi_1(k+1) = \rho_1\xi_1(k) + w_1(k+1), \tag{10.51}$$

$$\xi_2(k+1) = \rho_2\xi_2(k) + w_2(k+1), \tag{10.52}$$

where $\rho_1 = 0.8$, $\rho_2 = 0.4$; furthermore, $\{w_1(k) \in \mathbb{R}\}_{k\in\mathbb{N}}$ and $\{w_2(k) \in \mathbb{R}\}_{k\in\mathbb{N}}$ are mutually independent stochastic processes. We assume that the random variables $w_1(k)$, $k \in \mathbb{N}$, are independent and identically distributed by normal distribution $\mathcal{N}(0, \sigma_1^2 = 0.7)$ (i.e., mean value = 0, variance = 0.7). Similarly, the random variables $w_2(k)$, $k \in \mathbb{N}$, are assumed to be independent and identically distributed by normal distribution $\mathcal{N}(0, \sigma_2^2 = 0.3)$. The initial values $\xi_1(0)$ and $\xi_2(0)$ are assumed to be distributed by normal distributions $\mathcal{N}(0, \frac{\sigma_1^2}{1-\rho_1^2})$ and $\mathcal{N}(0, \frac{\sigma_2^2}{1-\rho_2^2})$, respectively. Note that $\{\xi(k) \in \mathbb{R}^2\}_{k\in\mathbb{N}_0}$ defined in (10.51) and (10.52) is an aperiodic, stationary, and ergodic Markov process characterized through the transition probability function $P : \mathbb{R}^2 \times \mathcal{B}(\mathbb{R}^2) \to [0,1]$ and the *initial stationary* distribution $\nu : \mathcal{B}(\mathbb{R}^2) \to [0,1]$ given by

$$P(\bar{\phi}, [a_1, b_1) \times [a_2, b_2)) = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f_1(\bar{\xi}_1 - \bar{\phi}_1) f_2(\bar{\xi}_2 - \bar{\phi}_2)\mathrm{d}\bar{\xi}_1\mathrm{d}\bar{\xi}_2, \tag{10.53}$$

$$\nu([a_1, b_1) \times [a_2, b_2)) = \int_{a_2}^{b_2}\int_{a_1}^{b_1} \tilde{f}_1(\bar{\phi}_1)\tilde{f}_2(\bar{\phi}_2)\mathrm{d}\bar{\phi}_1\mathrm{d}\bar{\phi}_2, \tag{10.54}$$

for $a_1, b_1, a_2, b_2 \in \mathbb{R}$ with $a_1 \leq b_1$ and $a_2 \leq b_2$, where $f_1 : \mathbb{R} \to [0,\infty)$, $f_2 : \mathbb{R} \to [0,\infty)$, $\tilde{f}_1 : \mathbb{R} \to [0,\infty)$, and $\tilde{f}_2 : \mathbb{R} \to [0,\infty)$ respectively denote the probability density functions for the normal distributions $\mathcal{N}(0, \sigma_1^2)$, $\mathcal{N}(0, \sigma_2^2)$, $\mathcal{N}(0, \frac{\sigma_1^2}{1-\rho_1^2})$, and $\mathcal{N}(0, \frac{\sigma_2^2}{1-\rho_2^2})$. Moreover, by using (10.53) we obtain the i-step transition probability function $P^{(i)} : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0,1]$

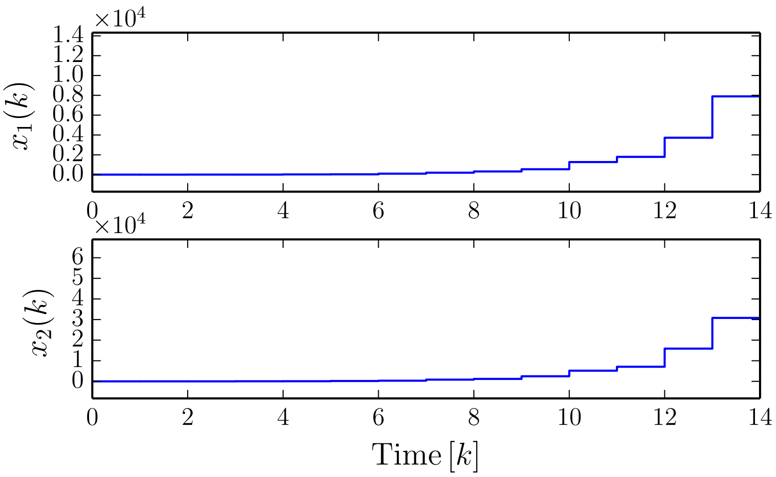

Figure 10.1: State trajectory of the uncontrolled system versus time

as

$$P^{(i)}(\bar{\phi}, [a_1, b_1) \times [a_2, b_2)) = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f_1^{(i)}(\bar{\xi}_1 - \rho_1^i \bar{\phi}_1) f_2^{(i)}(\bar{\xi}_2 - \rho_2^i \bar{\phi}_2) \mathrm{d}\bar{\xi}_1 \mathrm{d}\bar{\xi}_2, \tag{10.55}$$

for $a_1, b_1, a_2, b_2 \in \mathbb{R}$ with $a_1 \leq b_1$ and $a_2 \leq b_2$, where the functions $f_1^{(i)} : \mathbb{R} \to [0, \infty)$ and $f_2^{(i)} : \mathbb{R} \to [0, \infty)$ denote the probability density functions for the normal distributions $\mathcal{N}(0, \sum_{j=1}^{i} \rho_1^{j-1} \sigma_1^2)$ and $\mathcal{N}(0, \sum_{j=1}^{i} \rho_2^{j-1} \sigma_2^2)$ for a given step size $i \in \mathbb{N}$.

Figure 10.1 shows state trajectory of the uncontrolled system (10.8) (with $u(t) \equiv 0$). Note that the uncontrolled parameter-varying system clearly indicates unstable behavior. In the remainder of this section, we will show that stabilization of the system (10.8) can be achieved through our proposed control law (10.12) with the affine feedback gain (10.33) even if only a sampled version of the parameter is available for control purposes. Specifically, we assume that the parameter $\{\xi(k) \in \mathbb{R}^2\}_{k \in \mathbb{N}_0}$ is observed (sampled) at every $\tau = 3$ steps. Hence the controller receives information about the parameter only at the time instants $0, 3, 6, \ldots$.

Now, note that

$$\tilde{R} = \begin{bmatrix} 4.6524 & -2.1804 \\ -2.1804 & 4.1509 \end{bmatrix}, \tag{10.56}$$

$$\bar{L}_0 = [-16.7758 \quad 4.2915], \tag{10.57}$$

$$\bar{L}_1 = [-4.5515 \quad 1.9864], \tag{10.58}$$

$$\bar{L}_2 = [2.0641 \quad -4.2430], \tag{10.59}$$

and the scalars $\alpha_1 = \alpha_2 = 1.5$, $\beta_0 = 0.1$, $\beta_1 = \beta_2 = 0.01$ satisfy (10.45), (10.46), and (10.20). Therefore, it follows from Corollary 10.1 that the proposed sampled-parameter-dependent control law (10.12), (10.33) with gain matrices

$$\bar{K}_0 = \bar{L}_0\tilde{R}^{-1} = [-4., 1407 \quad -1.1412], \tag{10.60}$$

$$\bar{K}_1 = \bar{L}_1\tilde{R}^{-1} = [-1.0003 \quad -0.0469], \tag{10.61}$$

$$\bar{K}_2 = \bar{L}_2\tilde{R}^{-1} = [-0.047 \quad -1.0469], \tag{10.62}$$

guarantees almost sure asymptotic stabilization of the closed-loop system (10.8), (10.12).

Sample paths of the state $x(k)$ and the control input $u(k)$ obtained with initial condition $x(0) = [1, -1]^{\mathrm{T}}$ are shown in Figs. 10.2 and 10.3, respectively. Moreover, a sample path of the actual parameter $\xi(k)$ and its sampled version $\phi(k)$ are shown in Figure 10.4. Figures 10.2–10.4 indicate that our proposed control framework (10.12) which affinely depends on the sampled parameter $\phi(k)$ achieves asymptotic stabilization of the zero solution.

10.7 Conclusion

In this chapter, we investigated feedback control of discrete-time linear stochastic systems with time-varying parameters under sampled parameter information. Specifically, we considered the case where the parameter of the system is observed (sampled) periodically; furthermore, we proposed a control law that depends only on the sampled version of the parameter. We obtained sufficient conditions under which our control framework

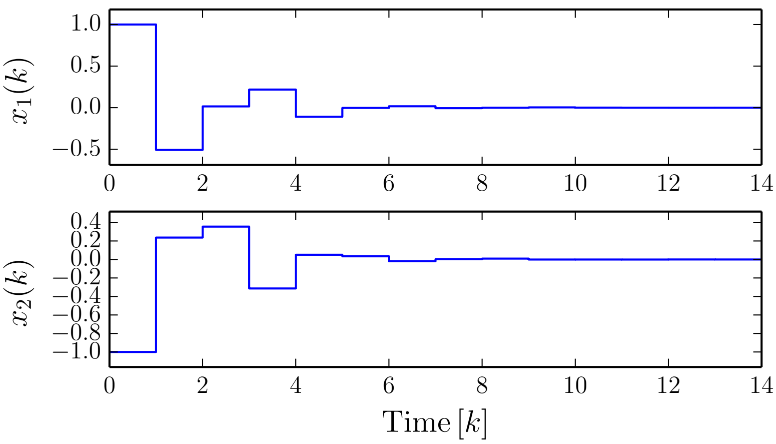

Figure 10.2: State trajectory versus time

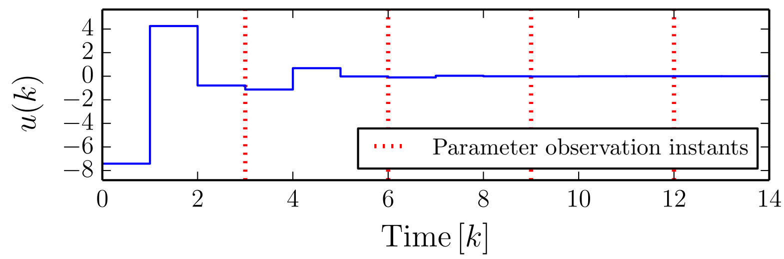

Figure 10.3: Control input versus time

guarantees almost sure asymptotic stabilization of the zero solution. With a numerical example, we presented an illustrative discussion on the almost sure asymptotic stabilization of a stochastic parameter-varying system with stationary and ergodic autoregressive parameters. The results obtained in this chapter suggest that our proposed control law successfully achieves almost sure asymptotic stabilization even if only a sampled version of the parameter is available for control purposes.

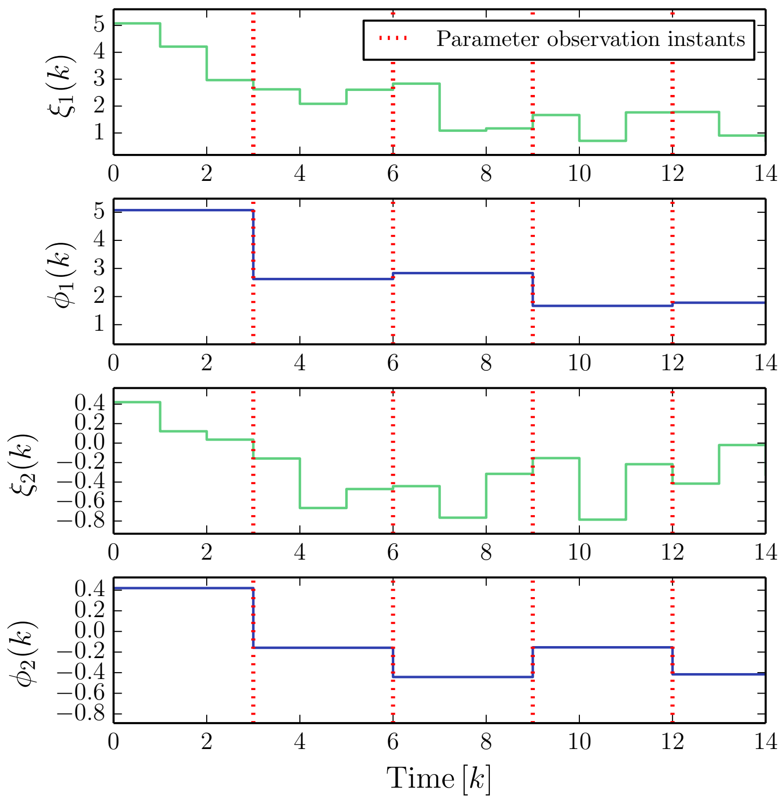

Figure 10.4: Actual parameter $\xi(k)$ and sampled parameter $\phi(k)$

Chapter 11

Concluding Remarks and Recommendations for Future Research

11.1 Conclusions

The goal of this thesis was to address sampled-parameter feedback control of dynamical systems with stochastic parameters. Specifically, we considered linear dynamical systems with time-varying parameters that are modeled by stochastic processes. These stochastic parameters describe the state of a randomly varying environment in which the dynamical system under consideration operates. We proposed control frameworks for the case where the parameter of the dynamical system is observed (sampled) only at certain time instants. Furthermore, we obtained conditions under which our proposed control frameworks guarantee convergence of the system state trajectories towards origin.

In this thesis, we explored the case where the parameter takes values from a finite set as well as the case where the parameter evolves in a space of uncountably many values. The case where the parameter of a dynamical system takes values from a finite set is characterized by switched stochastic system models. Switched stochastic systems are dynamical systems that are composed of a number of subsystems (modes). Each mode of the switched stochastic system describes the dynamics for a specific parameter value.

In Chapter 3, we investigated feedback control of continuous-time switched stochastic systems under the assumption that the mode of the switched system is sampled only at periodic time instants. In this problem setting, the information of the system state is continuously available for control purposes, however, the controller is assumed to have only *sampled* information of the mode signal. This problem setting is appropriate for applications where the perfect knowledge of the mode is not available for control purposes at all time instants. We developed a sampled-mode control framework in Chapter 3. This control framework incorporates a piecewise-constant linear feedback gain. It is important to note that the feedback gain in our control law depends only on the sampled mode signal. Specifically, feedback gain of our control law is switched periodically between a number of constant gains depending on the sampled mode information. We showed that under certain conditions, our proposed control law guarantees that the state trajectory of the closed-loop system converges to the origin almost surely. These sufficient stabilization conditions depend not only on the subsystem and mode transition dynamics but also on the mode sampling period. In order to obtain these conditions, we developed a form of strong law of large numbers (ergodic theorem) for a bivariate stochastic process composed of the actual mode and its sampled version. Using this key mathematical tool, we also showed that, our proposed control law can be used for stabilizing the system state even if the dynamics also include Wiener noise. Specifically, in Section 3.5, we considered a switched stochastic dynamical system where the dynamics of each mode is described by Ito-type stochastic differential equations which involve Wiener processes. We obtained sufficient stabilization conditions under which our proposed sampled-mode control framework achieves stabilization of the zero solution of the closed-loop system. The numerical examples, which we presented in Chapter 3, illustrate the efficacy of our sampled-mode control strategy in stabilizing continuous-time switched stochastic systems.

In Chapter 4, we explored sampled-mode feedback control of continuous-time switched stochastic systems under the effect of *mode information delays*. In this problem setting, the mode of the switched system is assumed to be periodically sampled, and each sampled mode data becomes available to the controller after a delay. This problem setting is important for applications, because in practice there may be delays in mode detection. In order to address this problem, we proposed a control law that depends only on the *sampled and delayed* mode information. We obtained some sufficient conditions under which the asymptotic stability of the state of the closed-loop switched stochastic system is guaranteed with our proposed control law.

We directed our attention to developing a new probability-based feedback gain scheduling mechanism in Chapter 5. Specifically, we investigated feedback control of continuous-time switched stochastic systems under delayed and sampled mode information (which was also explored in Chapter 4). Our probability-based feedback gain scheduling scheme utilizes the available delayed sampled mode data as well as a priori information concerning the probabilistic dynamics of the mode signal. In this scheme, the feedback gain of our controller is set to the gain associated with the mode that has the highest conditional probability of being active given the most recent sampled mode information. We demonstrated the utility of this new scheme through a numerical example. In our demonstration, we showed that probability-based feedback gain scheduling mechanism offers less conservative stabilization conditions with respect to the mode sampling period and the sampled mode information delay. In other words, with the probability-base feedback gain scheduling, stabilization can be guaranteed for larger values of the mode sampling period and the mode sample information delay compared to the case where we use the control law developed in Chapter 4.

In addition to continuous-time switched stochastic systems, we also explored feedback control of discrete-time switched stochastic systems. Specifically, in Chapter 6, we considered a discrete-time switched stochastic system which incorporates a stochastic mode signal that characterizes the switching between a number of deterministic subsystems that are described by difference equations. We investigated feedback control of discrete-time switched stochastic systems for the case where the mode is observed periodically. In order to achieve stabilization of the system state, we proposed a time-varying control strategy. Our proposed control law incorporates a feedback gain that depends not only on the sampled mode signal and but also on the time explicitly. Specifically, in this control strategy, a *time-varying*, *periodic* feedback gain is assigned to each mode of the switched system. When a sampled mode information becomes available to the controller, the gain associated with the sampled mode is set to be the feedback gain of the controller. Until the next mode sample becomes available, this time-varying gain is used. We obtained the dynamical equations that govern the evolution of the expectation of a stochastic process related to the covariance of the system state. We showed that these dynamical equations are *deterministic* and *show periodic behavior* due to periodicity of mode observations and periodicity of the feedback gains associated with the modes. By using discrete-time Floquet theory we obtained *necessary and sufficient* conditions for second-moment asymptotic stabilization of the zero solution. These conditions let us assess the stability of the closed-loop system for known feedback gains. However, for practical applications, we are required to find the feedback gains that we use in our control law. To this end, we also obtained alternative stabilization conditions that are better suited for finding feedback gains. In this regard, we provided efficient numerical methods to find stabilizing feedback gains. It is important to note that mode observation (sampling) period and the number of subsystems that we consider are finite. Hence, eventhough we consider a time-varying control strategy, the number of feedback gains we have to find is finite due to the *discrete-time* setting. Furthermore, our numerical methods can be used to efficiently determine these feedback gains.

In the literature, switched stochastic systems have been used for modeling fault-tolerant control systems which are composed of a normal operation mode and a number of faulty modes that are associated with failures of different components of the process. When there is a failure, information about this failure may not be instantaneously available to the controller. The failure is often detected through diagnostic tests. However, these tests may fail to identify the exact type of the failure. Hence, although the controller has the information that there was a failure and the system is in one of the faulty modes, it does not have the exact information of the mode. In order to develop control frameworks that can deal with such situations, in Chapter 7, we investigated the feedback control problem for a discrete-time switched stochastic system for the case where mode information obtained through the observations is not precise. We assumed that the modes of the switched system are divided into a number of groups, and the controller periodically receives information of the group that contains the active mode. We then proposed a control law that depends only on the periodically available imprecise mode information, rather than the exact information of the mode. We obtained a set of numerically verifiable conditions under which our proposed control law guarantees stabilization of the closed-loop system.

In practical applications, it would be ideal if the mode information of a switched system is available for control purposes at all time instants or at least periodically. However, there are certain cases where mode information is obtained at *random* time instants. This situation occurs for example when the mode is sampled at all time instants; however, some of the mode samples are randomly lost during communication between mode sampling mechanism and the controller. On the other hand, in some applications, the mode has to be detected, but the detected mode information may not be always accurate. In this case each mode detection has a confidence level. Mode information with low confidence is discarded. As a result, depending on the confidence level of detection, the controller may or may not receive the mode information at a particular mode detection instant. In order to deal with such cases where the information of the mode signal is randomly available to the controller, the methods that we developed in Chapters 8 and 9 can be employed. Note that in Chapter 8, we investigated sampled-mode feedback control problem for continuous-time switched stochastic systems for the case where the lengths of intervals between mode sampling time instants are exponentially distributed random variables. Futhermore, in Chapter 9, we proposed a sampled-mode feedback controller for stabilizing discrete-time switched stochastic systems. In Chapter 9, we did not assume a particular distribution for the length of intervals between mode observation (sampling) time instants. In fact, the sufficient stabilization conditions that we obtained in Chapter 9 can be used in various situations related to the nature mode observations. For example, these sufficient conditions can be used to assess stability of the closed-loop system under our proposed sampled-mode-dependent control law for cases such as uniformly or geometrically distributed mode observation intervals, or for the case where the mode is sampled periodically.

In Chapters 3–9, we investigated sampled-mode stabilization of switched stochastic control systems. Switched stochastic systems constitute an important class of dynamical systems with randomly varying parameters. Specifically, switched stochastic system models can be used for modeling processes with parameters that take values from a finite set. In the switched system framework, instantaneous change of the value of a parameter is modeled as a mode switch. On the other hand, for certain applications, we may also need to consider the case where the parameter of a process takes values from a space composed of a continuum of points. In this case, there are uncountably many values that parameter may take over time. In Chapter 10, we investigated feedback control of a dynamical system with stochastic parameters that evolve in a multidimensional space. We developed a stabilizing control framework for the case where the system parameter is observed (sampled) periodically. We obtained sufficient conditions under which almost sure asymptotic stabilization of the closed-loop stochastic parameter-varying system is guaranteed by our proposed control law. As a key step towards obtaining these sufficient conditions, we utilized the stationarity and ergodicity properties of a stochastic process that represents the sequences of values that the system parameter takes between consecutive observation instants. In Chapter 10, we also explored a special class of linear parameter-varying systems where the state matrix is an affine function of the entries of the parameter vector. We proved that stabilization for this class of parameter-varying systems can be achieved through a control law with a feedback gain that is an affine function of the entries of the sampled parameter vector. We presented a numerical example to illustrate the efficacy of our approach for stabilizing linear systems with autoregressive parameters.

In conclusion, in this thesis, we investigated feedback control of dynamical systems with parameters that evolve randomly. Specifically, we explored the case where the information of the system parameter is only available for control purposes at certain observation (sampling) instants. We proposed a range of different control methods to achieve stabilization of the system state by using only sampled mode information. We proved that our proposed control frameworks guarantee stabilization despite the uncertainty of the parameter between the observation instants. The sampled-parameter control frameworks that we developed are well-suited for controlling complex systems that work under the effect of stochastically varying environments, as the changes in the environmental conditions may not be observed exactly, instantaneously, or as frequently as the state of the system itself.

11.2 Recommendations for Future Research

In this thesis we worked on the sampled-parameter feedback control problem for linear systems with randomly varying parameters. We considered the case where the probabilistic dynamics that govern the evolution of the parameter is available *a priori*. Specifically, in Chapters 3–9, where we considered feedback control of switched stochastic systems, we assumed to have the knowledge of mode *transition rates* for the continuous-time case and mode *transition probabilities* for the discrete-time case. Furthermore, in Chapter 10, we investigated feedback control of a linear parameter-varying system with parameters that are Markov processes that evolve in $\mathbb{R}^l$ according to known *transition probability functions*. A future direction to thesis is the investigation of the case where the probabilistic dynamics of the parameters are unknown. In such cases one may develop adaptive control frameworks. A possible approach is to estimate the probabilistic dynamics of the parameter process by using the observed values of the parameter, and adjust the controller based on this estimation in an iterative fashion. A second approach would be considering a *direct* adaptive control framework. In this case, the feedback gain is directly adjusted based on the system state and parameter observations, since direct adaptive control does not require estimation of the probabilistic dynamics of the parameter.

For certain applications, we need to consider the case where the evolution of the parameters also depend on the system state. In such cases, when a switched stochastic system is considered, we can model the evolution of the mode through state-dependent transition rate or transition probability matrices. On the other hand, if the parameter takes values in $\mathbb{R}^l$, transition probability functions would be state-dependent. In this thesis, we considered the case where the evolution of the parameter is not state-dependent. As a future work, the results presented in this thesis can be extended to include the case where the parameter variation depends also on the state.

In Chapter 7, we considered the feedback control of a switched stochastic system for the case where only *sampled* and *imprecise* information of the mode signal is available to the controller. In that problem setting, the information of the mode was assumed to indicate only the *group* of modes that include the active mode. This group information is not an exact characterization of the active mode, although it *accurately* identifies the group of modes, one of which is guaranteed to be active. For many applications, mode information is provided to the controller by sensors, which may not have perfect accuracy. It is therefore important to address the feedback control problem for the case where the sampled mode information is *not accurate*. In this case, the information about the active mode may be wrong with a certain probability. Future research in this direction includes exploration of the relation between the accuracy of the mode information and the stabilization performance.

In this thesis, randomness is introduced in the dynamics through parameters. Specifically, we investigated feedback control problem for dynamical systems with parameters that evolve randomly. Note that in practice, randomness also appear as an input to the dynamical system. In this case the dynamics for the continuous-time case can be represented in the form

$$\dot{x}(t) = Ax(t) + Bu(t) + \xi(t), \quad t \geq 0, \tag{11.1}$$

where $\xi(\cdot)$ represents a randomly evolving input. This type of dynamics with random inputs are often seen in mechanical systems. In [121], response of a mechanical dynamical system to random excitation is analyzed. Furthermore, when a liquid contained in a tank is part of dynamical structure, the movement of liquid surface due to random excitation affects the overall dynamics. Investigation of liquid sloshing due to random excitation is conducted in [122, 123]. On the other hand, when the suspension system of a car is considered, road profile acts as a randomly evolving input [124, 125]. It is often the case, where the random input $\xi(\cdot)$ can not be measured as frequently and as precisely as the states $x(\cdot)$ of the system. Therefore, we are required to investigate the control problem for the case where only sampled information of the random input $\xi(\cdot)$ is available for control purposes.

References

[1] C. G. Cassandras and J. Lygeros, eds., *Stochastic Hybrid Systems*. Boca Raton: CRC Press, 2006.

[2] R. Tóth, *Modeling and identification of linear parameter-varying systems*, vol. 403 of *Lecture Notes in Control and Information Sciences*. Springer:Berlin, 2010.

[3] J. Mohammadpour and C. W. Scherer, eds., *Control of linear parameter varying systems with applications*. Springer:New York, 2012.

[4] D. P. de Farias, J. C. Geromel, J. B. R. do Val, and O. L. V. Costa, "Output feedback control of Markov jump linear systems in continuous-time," *IEEE Trans. Autom. Contr.*, vol. 45, pp. 944–949, 2000.

[5] S. Aberkane, J. C. Ponsart, M. Rodrigues, and D. Sauter, "Output feedback control of a class of stochastic hybrid systems," *Automatica*, vol. 44, pp. 1325–1332, 2008.

[6] V. Ugrinovskii and H. R. Pota, "Decentralized control of power systems via robust control of uncertain Markov jump parameter systems," *Int. J. Contr.*, vol. 78, no. 9, pp. 662–677, 2005.

[7] L. Li, V. A. Ugrinovskii, and R. Orsi, "Decentralized robust control of uncertain Markov jump parameter systems via output feedback," *Automatica*, vol. 43, no. 11, pp. 1932–1944, 2007.

[8] G. A. Becus, "Stochastic prey-predator relationships: A random evolution approach," *Bul. Math. Bio.*, vol. 41, pp. 543–554, 1979.

[9] Q. Luo and X. Mao, "Stochastic population dynamics under regime switching," *J. Math. Anal. Appl.*, vol. 334, pp. 69–84, 2007.

[10] C. Yuan, X. Mao, and J. Lygeros, "Stochastic hybrid delay population dynamics: Well-posed models and extinction," *J. Bio. Dyn.*, vol. 3, pp. 1–21, 2009.

[11] C. Zhu and G. Yin, "On hybrid competetive Lotka-Volterra ecosystems," *Nonl. Anal.*, vol. 71, pp. 1370–1379, 2009.

[12] C. Zhu and G. Yin, "On competitive Lotka-Volterra model in random environments," *J. Math. Anal. Appl.*, vol. 357, pp. 154–170, 2010.

[13] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*. Springer-Verlag: New York, 2010.

[14] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*. Imperial College Press: London, 2006.

[15] G. G. Yin, B. Zhang, and C. Zhu, "Practical stability and instability of regime-switching diffusions," *J. Contr. Theo. Appl.*, vol. 6, pp. 105–114, 2008.

[16] Q. Zhang, "Stock trading: An optimal selling rule," *SIAM J. Contr. Optim.*, vol. 40, pp. 64–87, 2001.

[17] N. Rohan and T. V. Ramanathan, "Geometric ergodicity of asymmetric volatility models with stochastic parameters," *Electron. J. Probab.*, vol. 18, pp. 1–12, 2013.

[18] P. Apkarian, P. Gahinet, and G. Becker, "Self-scheduled H_∞ control of linear parameter-varying systems: a design example," *Automatica*, vol. 31, no. 9, pp. 1251–1261, 1995.

[19] P. Apkarian and R. J. Adams, "Advanced gain-scheduling techniques for uncertain systems," *IEEE Trans. Contr. Sys. Tech.*, vol. 6, no. 1, pp. 21–32, 1998.

[20] D. J. Leith and W. E. Leithead, "Survey of gain-scheduling analysis and design," *International Journal of Control*, vol. 73, no. 11, pp. 1001–1025, 2000.

[21] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, 2000.

[22] F. Wang and V. Balakrishnan, "Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems," *IEEE Trans. Autom. Contr.*, vol. 47, no. 5, pp. 720–734, 2002.

[23] Y. Fujisaki, F. Dabbene, and R. Tempo, "Probabilistic design of LPV control systems," *Automatica*, vol. 39, no. 8, pp. 1323–1337, 2003.

[24] F. Wu and K. Dong, "Gain-scheduling control of LFT systems using parameter-dependent Lyapunov functions," *Automatica*, vol. 42, no. 1, pp. 39–50, 2006.

[25] P. Yan and H. Ozbay, "On switching H^{∞} controllers for a class of linear parameter varying systems," *Sys. Contr. Lettr.*, vol. 56, no. 7, pp. 504–511, 2007.

[26] E. Prempain and I. Postlethwaite, "L_2 and H_2 performance analysis and gain-scheduling synthesis for parameter-dependent systems," *Automatica*, vol. 44, no. 8, pp. 2081–2089, 2008.

[27] L. Benvenuti and L. Farina, "Constrained control for uncertain discrete-time linear systems," *Int. J. Rob. Nonl. Cont.*, vol. 8, no. 7, pp. 555–565, 1998.

[28] J. Geromel, P. D. Peres, and J. Bernussou, "On a convex parameter space method for linear control design of uncertain systems," *SIAM J. Contr. Optm.*, vol. 29, no. 2, pp. 381–402, 1991.

[29] F. Jabbari, "Output feedback controllers for systems with structured uncertainty," *IEEE Trans. Autom. Contr.*, vol. 42, no. 5, pp. 715–719, 1997.

[30] M. Rodrigues, D. Theilliol, S. Aberkane, and D. Sauter, "Fault tolerant control design for polytopic LPV systems," *Int. J. Appl. Math. Comput. Sci.*, vol. 17, no. 1, pp. 27–37, 2007.

[31] L. Xie, M. Fu, and C. E. de Souza, "H_{∞} control and quadratic stabilization of systems with parameter uncertainty via output feedback," *IEEE Trans. Autom. Contr.*, vol. 37, no. 8, pp. 1253–1256, 1992.

[32] F. Bruzelius, S. Pettersson, and C. Breitholtz, "Linear parameter-varying descriptions of nonlinear systems," in *Proc. Amer. Contr. Conf.*, vol. 2, pp. 1374–1379, IEEE, 2004.

[33] A. Casavola, D. Famularo, and G. Franze, "Predictive control of constrained nonlinear systems via LPV linear embeddings," *Int. J. Rob. Nonl. Cont.*, vol. 13, no. 3-4, pp. 281–294, 2003.

[34] D. J. Stilwell and W. J. Rugh, "Interpolation of observer state feedback controllers for gain scheduling," *Automatic Control, IEEE Transactions on*, vol. 44, no. 6, pp. 1225–1229, 1999.

[35] E. Yaz and R. E. Skelton, "Parametrization of all linear compensators for discrete-time stochastic parameter systems," *Automatica*, vol. 30, no. 6, pp. 945–955, 1994.

[36] A. Tiedemann and W. de Koning, "Statistical and stabilizability properties of equivalent discrete-time systems with stochastic parameters," *Int. J. Contr.*, vol. 40, no. 4, pp. 723–729, 1984.

[37] K. Fujimoto, S. Ogawa, Y. Ota, and M. Nakayama, "Optimal control of linear systems with stochastic parameters for variance suppression: The finite time horizon case," in *Proc. IFAC World Congr.*, pp. 12605–12610, 2011.

[38] K. Fujimoto, A. Satoh, and S. Fukunaga, "System identification based on variational Bayes method and the invariance under coordinate transformations," in *Proc. IEEE Conf. Dec. Contr. Eur. Contr. Conf.*, (Orlando, FL), pp. 3882–3888, 2011.

[39] E. Yaz, "Stabilization of discrete-time systems with stochastic parameters," *Sys. Contr. Lettr.*, vol. 5, no. 5, pp. 321–326, 1985.

[40] E. Yaz and A. Ray, "Grammian assignment for stochastic-parameter systems and their stabilization under randomly varying delays," in *Proc. IEEE Conf. Dec. Contr.*, vol. 3, pp. 2176–2181, 1994.

[41] M. Mariton and P. Bertrand, "Output feedback for a class of linear systems with stochastic jump parameters," *IEEE Trans. Autom. Contr.*, vol. 30, no. 9, pp. 898–900, 1985.

[42] O. L. Costa and E.-K. Boukas, "Necessary and sufficient condition for robust stability and stabilizability of continuous-time linear systems with Markovian jumps," *J. Optim. Theo. Appl.*, vol. 99, no. 2, pp. 359–379, 1998.

[43] I. Emre Kose and F. Jabbari, "Control of LPV systems with partly measured parameters," *IEEE Tran. Autom. Contr.*, vol. 44, no. 3, pp. 658–663, 1999.

[44] W. Heemels, J. Daafouz, and G. Millerioux, "Design of observer-based controllers for LPV systems with unknown parameters," in *Proc. IEEE Conf. Dec. Contr.*, (Shanghai, China), pp. 1836–1841, 2009.

[45] L. Jetto, V. Orsini, and R. Romagnoli, "An iterative LMI scheme for output regulation of continuous-time LPV systems with inexactly measured parameters," in *Proc. Med. Conf. Contr. Autom.*, (Barcelona, Spain), pp. 5–11, 2012.

[46] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, "Stochastic stability properties of jump linear systems," *IEEE Trans. Autom. Contr.*, vol. 37, pp. 38–53, 1992.

[47] E. K. Boukas and H. Yang, "Stability of stochastic systems with jumps," *Math. Prob. Eng.*, vol. 3, pp. 173–185, 1996.

[48] P. Bolzern, P. Colaneri, and G. D. Nicolao, "On almost sure stability of continoustime Markov jump linear systems," *Automatica*, vol. 42, pp. 983–988, 2006.

[49] C. Yuan and J. Lygeros, "On the exponential stability of switching diffusion process," *IEEE Trans. Autom. Contr.*, vol. 50, pp. 1422–1426, 2005.

[50] J. Luo, "Comparison principle and stability of Ito stochastic differential delay equations with Poisson jump and Markovian switching," *Nonl. Anal.*, vol. 64, pp. 253–262, 2006.

[51] X. Mao, G. G. Yin, and C. Yuan, "Stabilization and destabilization of hybrid systems of stochastic differential equations," *Automatica*, vol. 43, pp. 264–273, 2007.

[52] R. Z. Khasminskii, C. Zhu, and G. Yin, "Stability of regime-switching diffusions," *Stoc. Prc. Appl.*, vol. 117, pp. 1037–1051, 2007.

[53] G. Yin and F. Xi, “Stability of regime-switching jump diffusions,” *SIAM J. Contr. Optim.*, vol. 48, pp. 4525–4549, 2010.

[54] M. K. Ghosh, A. Arapostathis, and S. I. Marcus, “Optimal control of switching diffusions with application to flexible manufacturing systems,” *SIAM J. Contr. Optm.*, vol. 30, pp. 1–23, 1992.

[55] L. E. Ghaoui and M. A. Rami, “Robust state-feedback stabilization of jump linear systems via LMIs,” *Int. J. Robust Nonl. Contr.*, vol. 6, pp. 1015–1022, 1996.

[56] A. El Bouhtouri, D. Hinrichsen, and A. J. Pritchard, “H^{∞}-type control for discrete-time stochastic systems,” *Int. J. Rob. Nonl. Contr.*, vol. 9, no. 13, pp. 923–948, 1999.

[57] Y. Fang and K. A. Loparo, “Stabilization of continous-time jump linear systems,” *IEEE Trans. Autom. Contr.*, vol. 47, pp. 1590–1602, 2002.

[58] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*. Springer, 2004.

[59] S. Sathanantan, O.Adetona, C. Beane, and L. H. Keel, “Feedback stabilization of Markov jump linear system with time-varying delay,” *Stoc. Anal. App.*, vol. 26, pp. 577–594, 2008.

[60] J. C. Geromel, A. P. C. Goncalves, and A. R. Fioravanti, “Dynamic output feedback control of discrete-time Markov jump linear systems through linear matrix inequalities,” *SIAM J. Contr. Optm.*, vol. 48, no. 2, pp. 573–593, 2009.

[61] E. K. Boukas, *Stochastic switching systems: Analysis and design*. Springer, 2006.

[62] E. K. Boukas, “Delay-dependent stability and output feedback stabilisation of Markov jump system with time-delay,” *IEE Proc. Contr. Theo. Appl.*, vol. 149, pp. 379–386, 2002.

[63] E. K. Boukas, “Static output feedback control for stochastic hybrid systems: LMI approach,” *Automatica*, vol. 42, pp. 183–188, 2006.

[64] Z. Fei, H. Gao, and P. Shi, “New results on stabilization of Markovian jump systems with time delay,” *Automatica*, vol. 45, pp. 2300–2306, 2009.

[65] V. Dragan and T. Morozan, "Discrete-time Riccati type equations and the tracking problem," *ICIC Expr. Lett.*, vol. 2, pp. 109–116, 2008.

[66] A. P. C. Goncalves, A. R. Fioravanti, and J. C. Geromel, "Dynamic output feedback $\mathcal{H}_\infty$ control of discrete-time Markov jump linear systems through linear matrix inequalities," in *Proc. IEEE Conf. Dec. Contr.*, (Cancun, Mexico), pp. 4787–4792, 2008.

[67] K. Nassiri-Toussi and P. E. Caines, "On the adaptive stabilization and ergodic behaviour of stochastic jump-Markov systems via nonlinear filtering," in *Proc. IEEE Conf. Dec. Contr.*, (Brighton, England), pp. 1784–1785, 1991.

[68] P. E. Caines and J. F. Zhang, "Adaptive control for jump parameter systems via nonlinear filtering," in *Proc. IEEE Conf. Dec. Contr.*, (Tucson, AZ), pp. 699–704, 1992.

[69] B. Bercu, F. Dufour, and G. G. Yin, "Almost sure stabilization for feedback controls of regime-switching linear systems with a hidden Markov chain," *IEEE Trans. Autom. Contr.*, vol. 54, pp. 2114–2125, 2009.

[70] A. N. Vargas, J. B. do Val, and E. F. Costa, "Receding horizon control of Markov jump linear systems subject to noise and unobserved state chain," in *IEEE Conf. Dec. Contr.*, vol. 4, (Atlantis, Paradise Island, Bahamas), pp. 4381–4386, 2004.

[71] A. N. Vargas, W. Furloni, and J. B. R. do Val, "Constrained model predictive control of jump linear systems with noise and non-observed Markov state," in *Proc. Amer. Contr. Conf.*, (Minneapolis, MN), pp. 929–934, 2006.

[72] C. A. Silva and E. F. Costa, "An algorithm for the long run average cost problem for linear systems with non-observed markov jump parameters," in *Proc. Amer. Contr. Conf.*, (St. Louis, MO), pp. 4434–4439, 2009.

[73] M. D. Fragoso and O. L. V. Costa, "Mean square stabilizability of continuous-time linear systems with partial information on the Markovian jumping parameters," *Stoc. Anal. App.*, vol. 22, pp. 99–111, 2005.

[74] N. Xiao, L. Xie, and M. Fu, "Stabilization of Markov jump linear systems using quantized state feedback," *Automatica*, vol. 46, pp. 1696–1702, 2010.

[75] M. Baglietto, G. Battistelli, and P. Tesi, "Mode-observability conditions for linear and nonlinear systems," in *Proc. IEEE Conf. Dec. Contr.*, (Maui, HI), pp. 1941–1947, 2012.

[76] M. Baglietto, G. Battistelli, and P. Tesi, "Projection-based degree of distinguishability in switching linear systems," in *Proc. IEEE Conf. Dec. Contr.*, (Firenze, Italy), pp. 4048–4053, 2013.

[77] M. Mahmoud, J. Jiang, and Y. Zhang, "Analysis of the stochastic stability for fault tolerant control systems," in *Proc. IEEE Conf. Dec. Contr.*, (Phoenix, AZ), pp. 3188–3193, 1999.

[78] F. Tao and Q. Zhao, "Synthesis of active fault-tolerant control based on Markovian jump system models," *IET Contr. Theo. App.*, vol. 1, pp. 1160–1168, 2007.

[79] W. L. De Koning, "Optimal estimation of linear discrete-time systems with stochastic parameters," *Automatica*, vol. 20, no. 1, pp. 113–115, 1984.

[80] P. Bolzern, P. Colaneri, and G. D. Nicolao, "Almost sure stability of stochastic linear systems with ergodic parameters," *Eur. J. Contr.*, vol. 2, pp. 114–123, 2008.

[81] W.-C. Xie, *Dynamic stability of structures*. Cambridge University Press:New York, 2011.

[82] J. Daafouz and J. Bernussou, "Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties," *Sys. Contr. Lett.*, vol. 43, no. 5, pp. 355–359, 2001.

[83] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Trans. Autom. Contr.*, vol. 41, no. 3, pp. 436–442, 1996.

[84] Z. Brzeźniak and T. J. Zastawniak, *Basic Stochastic Processes: A Course Through Exercices*. Springer:London, 1999.

[85] B. Øksendal, *Stochastic Differential Equations*. Springer:New York, 2005.

[86] R. Serfozo, *Basics of Applied Stochastic Processes*. Springer-Verlag: New York, 2009.

[87] J. Norris, *Markov Chains*. Cambridge University Press: New York, 2009.

[88] D. Williams, *Probability with Martingales*. Cambridge University Press: Cambridge, 1991.

[89] R. Durrett, *Probability: Theory and Examples*. Cambridge University Press: New York, 2010.

[90] K. B. Athreya and S. N. Lahiri, *Measure Theory and Probability Theory*. Springer: New York, 2006.

[91] J. S. Rosenthal, *A First Look at Rigorous Probability Theory*. World Scientific Publishing: Singapore, 2011.

[92] A. Klenke, *Probability Theory: A Comprehensive Course*. Springer-Verlag: London, 2008.

[93] P. Billingsley, *Probability and Measure*. John Wiley and Sons:New Jersey, 2012.

[94] S. Boyd, V. Balakrishnan, E. Feron, and L. ElGhaoui, "Control system analysis and synthesis via linear matrix inequalities," in *Proc. Amer. Contr. Conf*, pp. 2147–2154, IEEE, 1993.

[95] S. P. Boyd, *Linear matrix inequalities in system and control theory*, vol. 15. Siam, 1994.

[96] L. El Ghaoui and S.-I. Niculescu, *Advances in linear matrix inequality methods in control*, vol. 2. Siam, 2000.

[97] Y. A. Phillis, "Controller design of systems with multiplicative noise," *IEEE Trans. Autom. Contr.*, vol. 30, no. 10, 1985.

[98] H.-H. Kuo, *Introduction to Stochastic Integration*. Springer:New York, 2006.

[99] F. W. Fairman, *Linear control theory: the state space approach*. Wiley:New York, 1998.

[100] W. Xie, G. Ma, C. Wang, and B. Feng, "Delay-dependent robust sampled-data stability for Markovian jump systems with time-delay," in *IEEE Conf. Ind. Elec. Appl.*, 2007.

[101] S. Sathananthan, C. Beane, and L. H. Keel, "Delay-dependent stability criteria for Markovian switching networks with time-varying delay," in *Proc. IFAC World Congr.*, (Seoul, Republic of Korea), pp. 6–11, 2008.

[102] S. He and F. Liu, "On delay-dependent stability of Markov jump systems with distributed time-delays," *Circ. Sys. Sign. Proc.*, vol. 30, pp. 323–337, 2011.

[103] H. Mukaidani, M. Unno, H. Xu, and V. Dragan, "Pareto-optimal solutions for Markov jump stochastic systems with delay," in *Proc. Amer. Contr. Conf.*, (Washington, DC), pp. 4660–4665, 2013.

[104] M. Liu, D. Ho, and Y. Niu, "Stabilization of Markovian jump linear system over networks with random communication delay," *Automatica*, vol. 45, pp. 416–421, 2009.

[105] J. Wang and H. Yang, "Output-feedback stabilization of Markovian jump linear systems over networks with time delays," *J. Int. Inf. Comp. Sci.*, vol. 13, pp. 2545–2552, 2011.

[106] I. Matei, N. C. Martins, and J. S. Baras, "Optimal linear quadratic regulator for Markovian jump linear systems in the presence of one time-step delayed mode observations," in *Proc. 17th World. Cong. IFAC*, (Seoul, Korea), pp. 8056–8061, 2008.

[107] G. Wei, Z. Wang, B. Shen, and M. Li, "Probability-dependent gain-scheduled filtering for stochastic systems with missing measurements," *IEEE Trans. Circ. Sys. II*, vol. 58, pp. 753–757, 2011.

[108] L. Greco, D. Fontanelli, and A. Bicchi, "Almost sure stability of anytime controllers via stochastic scheduling," in *Proc. IEEE Conf. Dec. Contr.*, (New Orleans, LA), pp. 5640–5645, 2007.

[109] J. Sreedhar and P. Van Dooren, "Discrete-time periodic systems: A Floquet approach," in *Proc. Conf. Inf. Sci. Sys.*, pp. 194–195, 1993.

[110] Y. Hayakawa and T. Jimbo, "Floquet transformations for discrete-time systems: Equivalence between periodic systems and time-invariant ones," in *IEEE Conf. Dec. Contr.*, (Cancun, Mexico), pp. 5140–5145, 2008.

[111] S. Bittanti and P. Colaneri, *Periodic Systems: Filtering and Control*. Springer-Verlag: London, 2009.

[112] E. Kopp, *From measures to Ito integrals*. Cambridge University Press: New York, 2011.

[113] M. Mariton, "Almost sure and moments stability of jump linear systems," *Sys. Contr. Lett.*, vol. 11, pp. 393–397, 1988.

[114] Y. Fang and K. A. Loparo, "Stochastic stability of jump linear systems," *IEEE Trans. Autom. Contr.*, vol. 47, pp. 1204–1208, 2002.

[115] D. Bernstein, *Matrix mathematics: Theory, Facts, and Formulas*. Princeton University Press: Princeton, 2009.

[116] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *LMI control toolbox-for use with Matlab*. The MATH Works, Inc:Natick, 1995.

[117] J. Lofberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *IEEE Int. Symp. Comp. Aided Contr. Sys. Design*, pp. 284–289, 2004.

[118] J. B. Do Val, J. C. Geromel, and A. P. C. Gonçalves, "The H2-control for jump linear systems: cluster observations of the Markov state," *Automatica*, vol. 38, no. 2, pp. 343–349, 2002.

[119] D. Li, D. Zhang, and H. Ji, "Stabilization of jump linear systems with partial observation of Markov mode," *Int. J. Pure Appl. Math.*, vol. 27, pp. 31–38, 2006.

[120] A. Sadon and D. Choukroun, "Fault-tolerant spacecraft magnetic attitude control," in *Advances in Aerospace Guidance, Navigation and Control*, pp. 741–760, Springer: Berlin Heidelberg, 2013.

[121] K. Kimura and M. Sakata, “Non-stationary responses of a non-symmetric non-linear system subjected to a wide class of random excitation,” *J. Snd. Vibr.*, vol. 76, no. 2, pp. 261–272, 1981.

[122] M. Sakata, K. Kimura, and M. Utsumi, “Non-stationary response of non-linear liquid motion in a cylindrical tank subjected to random base excitation,” *J. Snd. Vibr.*, vol. 94, no. 3, pp. 351–363, 1984.

[123] T. Ikeda and R. Ibrahim, “Nonlinear random responses of a structure parametrically coupled with liquid sloshing in a cylindrical tank,” *J. Snd. Vibr.*, vol. 284, no. 1, pp. 75–102, 2005.

[124] J. D. Robson and C. J. Dodds, “Stochastic road inputs and vehicle response,” *Vehicle Syst. Dyn.*, vol. 5, no. 1, pp. 1–13, 1976.

[125] J. Marzbanrad, G. Ahmadi, H. Zohoor, and Y. Hojjat, “Stochastic optimal preview control of vehicle suspension,” *J. Sound Vibr.*, vol. 275, no. 3, pp. 973–990, 2004.

List of Publications

[a] A. Cetinkaya and T. Hayakawa, "Sampled-parameter feedback control of discrete-time linear stochastic parameter-varying systems," in *Proc. IFAC World Congress*, 2014, to appear. (Chapter 10)

[b] A. Cetinkaya and T. Hayakawa, "Sampled-mode-dependent time-varying control strategy for stabilizing discrete-time switched stochastic systems," in *Proc. Amer. Contr. Conf.*, 2014, to appear. (Chapter 6)

[c] A. Cetinkaya and T. Hayakawa, "Discrete-time switched stochastic control systems with randomly observed operation mode," in *Proc. IEEE Conf. Dec. Contr.*, (Firenze, Italy), pp. 85-90, 2013. (Chapter 9)

[d] A. Cetinkaya and T. Hayakawa, "Stabilizing discrete-time switched linear stochastic systems using periodically available imprecise mode information," in *Proc. Amer. Contr. Conf.*, (Washington, DC, USA), pp. 3266-3271, 2013. (Chapter 7)

[e] A. Cetinkaya and T. Hayakawa, "Probability-based feedback gain scheduling for stabilizing switched linear stochastic systems under delayed sampled mode information," in *Proc. IEEE Conf. Dec. Contr.*, (Maui, HI), pp. 6843-6848, 2012. (Chapter 5)

[f] A. Cetinkaya and T. Hayakawa, "Stabilization of switched linear stochastic systems under delayed discrete mode observations," in *Proc. Int. Symp. Math. Theo. Netw. Sys.*, (Melbourne, Australia), 2012. (Chapter 4)

[g] A. Cetinkaya and T. Hayakawa, "Feedback control of switched stochastic systems using uniformly sampled mode information," in *Proc. Amer. Contr. Conf.*, (Montreal,

Canada), pp. 3778-3783, 2012. (Chapter 3)

[h] A. Cetinkaya and T. Hayakawa, “Stabilization of switched linear stochastic dynamical systems under limited mode information,” in *Proc. IEEE Conf. Dec. Contr.*, (Orlando, FL), pp. 8032-8037, 2011. (Chapter 8)

[i] A. Cetinkaya and T. Hayakawa, “Stability of switched stochastic dynamical systems driven by Brownian motion and Markov modulated compound Poisson process,” in *Proc. Amer. Contr. Conf.*, (San Francisco, CA), pp. 1458-1463, 2011. (Chapters 3 and 8)

[j] A. Cetinkaya, K. Kashima, and T. Hayakawa, “Stability and stabilization of switching stochastic differential equations subject to probabilistic state jumps,” in *Proc. IEEE Conf. Dec. Contr.*, (Atlanta, GA), pp. 2378-2383, 2010. (Chapters 6 and 7)

[k] A. Cetinkaya, K. Kashima, and T. Hayakawa, “Stability of stochastic systems with probabilistic mode switchings and state jumps,” in *Proc. Amer. Contr. Conf.*, (Baltimore, MD), pp. 4046-4051, 2010. (Chapters 6 and 7)