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Parallel and Modular Structures for FIR Digital Filters

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SUMMARY The scope of this paper is the realization of FIR digital filters with an emphasis on linear phase and maximally flat cases. The transfer functions of FIR digital filters are polynomials and polynomial evaluation algorithms can be utilized as realization schemes of these filters. In this paper we investigate the application of a class of polynomial evaluation algorithms called "recursive triangles" to the realization of FIR digital filters. The realization of an arbitrary transfer function using De Casteljau algorithm, a member of the recursive triangles used for evaluating Bernstein polynomials, is studied and it is shown that in some special and important cases it yields efficient modular structures. Realization of two dimensional filters based on Bernstein approximation is also considered. We also introduce recursive triangles for evaluating the power basis representation of polynomials and give a new multiplier-less maximally flat structure based on them. Finally, we generalize the structure further and show that Chebyshev polynomials can also be evaluated by the triangles. This is the triangular counterpart of the well-known Chebyshev structure. In general, the triangular structures yield highly modular digital filters that can be mapped to an array of concurrent processors resulting in high speed and efficient filtering specially for maximally flat transfer functions.
key words: *FIR digital filters, digital signal processing*

1. Introduction

The scope of this paper is the realization of FIR digital filters (DFs) with an emphasis on linear phase and maximally flat (MAXFLAT) cases. The realization of FIRDFs has been studied intensively by many researchers during the past few decades. Sensitivity, noise, complexity, modularity, and maximum possible on-line processing speed are among the most important features that specify one DF structure from another and make it suitable for a special application. When implementation on special purpose hardware for high-speed filtering is considered, the most desirable features are modularity, regularity, symmetry and potential for effective parallel processing on dedicated VLSI structures.

Since the transfer functions of FIRDFs are polynomials, polynomial evaluation algorithms can be utilized as realization schemes of these filters. Based on this idea Schüssler derived FIRDF structures from Lagrange and Newton interpolation formulas [1]. Chebyshev structure is another example where the recursive evaluation algorithm of Chebyshev polynomials is employed [2].

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In our earlier work we presented a novel structure based on De Casteljau algorithm which led us to multiplier-less realization of linear phase MAXFLAT FIRDFs [3]. This structure is highly modular and an attractive candidate for VLSI implementation. The De Casteljau algorithm is well-known in the field of computer-aided geometric design and is used for evaluating polynomials represented in Bernstein basis functions. This algorithm belongs to a family of recursive polynomial evaluation algorithms called "recursive triangles" [4] or simply triangles in this paper. Triangles for evaluating polynomials in Lagrange, Newton, Polya and power (monomial) bases are among other members of this family.

The purpose of this paper is to investigate the application of the above mentioned triangles to DF realizations. In Sect. 2, we study the realization of an arbitrary transfer function using the De Casteljau algorithm and show that in some special and important cases it yields efficient modular structures. We also present an idea for realizing two-dimensional (2-D) DFs that are obtained using Bernstein approximation. In Sect. 3, we introduce triangles for evaluating the power basis representation of polynomials and propose a new multiplier-less MAXFLAT structure based on them. In Sect. 4, we generalize the triangular structure further and show that Chebyshev polynomials can also be evaluated by the triangles. General properties of triangular structures are discussed in Sect. 5 and the paper is concluded in Sect. 6.

2. Bernstein Polynomials and De Casteljau Algorithm

Consider a real polynomial $f(x)$ of degree N expressed in its power form as

$$f(x) = \sum_{i=0}^{N} p_i \, x^i. \tag{1}$$

$f(x)$ can be converted to its Bernstein form

$$f(x) = \sum_{i=0}^{N} b_i \, B_i^{(N)}(x), \tag{2}$$

where $\{B_i^{(N)}(x)\}$ are Bernstein basis functions and $\{b_i\}$ are Bernstein coefficients. Bernstein basis functions on interval $[0, 1]$ become

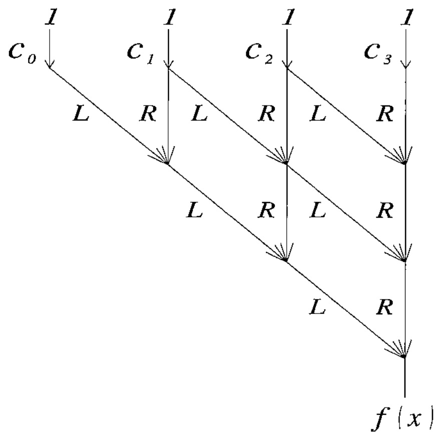

Fig. 1 Triangular evaluation of $f(x) = \sum_{i=0}^{3} c_i F_i(x)$; structure ∇_3.

$$B_i^{(N)}(x) = \binom{N}{i} (1-x)^{N-i} x^i$$

and the coefficients $\{b_i\}$ and $\{p_i\}$ are related as

$$b_i = \sum_{j=0}^{i} p_j \frac{\binom{i}{j}}{\binom{N}{j}}. \tag{3}$$

An efficient method for computing $\{b_i\}$ is proposed in [5]. In fact, Bernstein basis functions can be defined on an arbitrary interval but we restrict our discussion to $[0,1]$ without loss of generality. Unlike the power coefficients $\{p_i\}$, $\{b_i\}$ are geometrically meaningful and reflect the shape of the curve composed by $f(x)$ on $[0,1]$ to some extent. Here we list some of their important properties on this interval [6], [7].

1. Interpolation at boundaries: $f(0) = b_0$, $f(1) = b_N$.
2. Bounds on the range of $f(x)$: $\min_{i\in[0,N]} b_i \leqq f(x) \leqq \max_{i\in[0,N]} b_i$ for $x \in [0,1]$.
3. Derivatives at boundaries: The r-th order derivative of $f(x)$ at $x = 0$ ($x = 1$) depends only on $b_0, \ldots, b_r$ ($b_{N-r}, \ldots, b_N$).
4. Convex hull property: $f(x)$ lies inside the convex hull of its control polygon. The control polygon is the ordered sequence of points (x_i, y_i), where $x_i = i/N$, $y_i = b_i$ and $i = 0(1)N$ †.

Evaluation of (2) using De Casteljau algorithm can be performed by the following procedure.

$$\begin{aligned} b_i^{(0)} &\leftarrow b_i, \quad i = 0(1)N \\ b_i^{(j)} &\leftarrow (1-x)\, b_{i-1}^{(j-1)} + x\, b_i^{(j-1)}, \\ &\qquad j = 1(1)N, \quad i = j(1)N \\ f(x) &= b_N^{(N)}. \end{aligned} \tag{4}$$

Table 1 Definition of labels and coefficients.

Structure	L	R	c_i
De Casteljau	$1-x$	x	b_i
Power	1	x	p'_i
Chebyshev	x	$\sqrt{x^2-1}$	$\frac{1+(-1)^i}{2}$

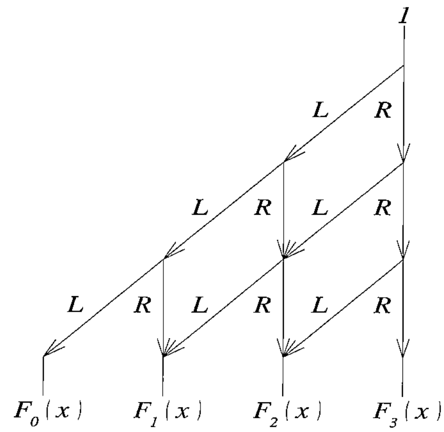

Fig. 2 Triangular evaluation of basis functions $F_i(x) = \binom{3}{i} L^{3-i} R^i$; structure Δ_3.

This is illustrated in Fig. 1 for $N = 3$, where L, R and $\{c_i\}$ are defined in Table 1. Note that the structure of Fig. 1 is the reversed Pascal's triangle with its parallel edges weighted by $R = x$ or $L = 1 - x$ for the De Casteljau algorithm. This is why the binomial coefficients can be realized together with the powers of x and $1 - x$ to form the Bernstein basis functions. We will denote this type of triangular structure, suitable for evaluating a polynomial of degree N, by ∇_N. For arbitrary L and R the result of evaluation through ∇_N is

$$f(x) = \sum_{i=0}^{N} c_i \binom{N}{i} L^{N-i} R^i.$$

Another scheme for evaluating (2) is computing the basis functions $\{B_i^{(N)}(x)\}$ first and performing an inner product between the coefficient and basis vectors afterward. For computing the basis functions we can use the triangle shown in Fig. 2, that is, Pascal's triangle with weighted edges. For the Bernstein basis functions we should, again, set $R = x$ and $L = 1 - x$ in this triangle. Triangles of this type, suitable for simultaneous evaluation of basis functions, will be denoted by Δ_N in this paper. Δ_N can be used for evaluating basis functions of the type

$$F_i(x) = \binom{N}{i} L^{N-i} R^i, \quad i = 0(1)N.$$

If two or more polynomials are to be evaluated simultaneously this scheme is useful.

† $i = i_1(i_2)i_3$ denotes $i = i_1, i_1 + i_2, i_1 + 2i_2, \ldots, i_3$.

2.1 Application to DF Realization

A straightforward way of utilizing the De Casteljau algorithm for FIRDF realization is converting the power form transfer function of an M-th order filter to its Bernstein form directly by setting $x = z^{-1}$ and computing the Bernstein coefficients using (3). A structure can be obtained easily by replacing x with a delay element in ∇_M and interpreting it as a signal flow graph. Using Δ_M would be an alternative with the difference that $M+1$ output signals of the triangle should be tapped by the Bernstein coefficients and added up to produce the output signal. This structure is the transposed form of ∇_M. It should be noted that the properties of coefficients $\{b_i\}$ specify the transfer function on the real interval $[0,1]$ not on the unit circle. That is, none of the properties stated above (1–4) holds for the frequency response of the filter in this case. Thus, the Bernstein coefficients are not geometrically meaningful with respect to the shape of the frequency response and we do not see any advantage in using this type of realization.

However, in a linear phase filter we can reflect the magnitude response characteristics of the filter to $\{b_i\}$ by utilizing the impulse response symmetry. As an example, consider an odd length, even symmetric transfer function $H(z)$ of order $2N$. Using the Chebyshev polynomials, the zero phase version of $H(z)$ can be written in the form

$$\begin{aligned} H_0(z) &= \sum_{i=0}^{N} h_i \, T_i\left(\frac{z+z^{-1}}{2}\right) \\ &= \sum_{i=0}^{N} t_i \left(\frac{z+z^{-1}}{2}\right)^i, \end{aligned} \tag{5}$$

where $T_i(x)$ denotes the i-th Chebyshev polynomial and coefficients t_i can be computed by expanding these polynomials. Using the transformation

$$x = \frac{1 - \frac{z+z^{-1}}{2}}{2}, \tag{6}$$

$H_0(z)$ can be transformed to

$$H_0(x) = \sum_{i=0}^{N} t'_i \, x^i,$$

a polynomial of degree N in the power basis. $H_0(x)$ can be expressed in its Bernstein form by setting $p_j = t'_j$ in (3) and computing the Bernstein coefficients $\{b_i\}$. Evaluation of $H_0(x)$ at $x \in [0,1]$ gives the magnitude response at $\omega \in [0,\pi]$, where $x = \sin^2(\omega/2)$. Hence, now, the Bernstein coefficients express the shape of the magnitude response.

The realization of the transformed transfer function, $H_0(x)$, using the De Casteljau algorithm is shown in Fig. 3, where we have put every two neighboring edges of ∇_N (L and R) in one block to enhance modularity. The structure consists of two parts. In the first part we have $N+1$ multiplier coefficients, $c_i = b_i$, receiving the input signal as their inputs. The second part is a symmetric and regular array of $N(N+1)/2$ multiplierless and identical building blocks. The multiplication operations on the input signal can be performed in parallel and the results are then sent to the triangular part. Using (4) and (6), the input-output relationship of the blocks becomes

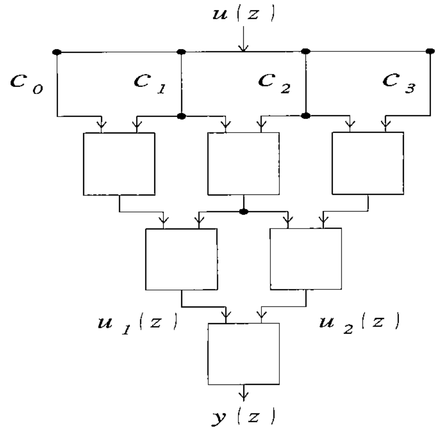

Fig. 3 Digital structure based on triangle ∇_3.

$$y(z) = \frac{1 + \frac{z+z^{-1}}{2}}{2}\, u_1(z) + \frac{1 - \frac{z+z^{-1}}{2}}{2}\, u_2(z),$$

where $u_1(z)$ and $u_2(z)$ correspond to $b_{i-1}^{(j-1)}$ and $b_i^{(j-1)}$ in (4), respectively. For a causal circuit we have

$$y(z) = \frac{(1+z^{-1})^2}{4}\, u_1(z) - \frac{(1-z^{-1})^2}{4}\, u_2(z). \tag{7}$$

This shows that no multiplication is needed in the second part of the structure. Interestingly, for the case $u_1(z) = u_2(z)$, i.e. a block with two identical input signals, the output becomes

$$y(z) = z^{-1}\, u_1(z) = z^{-1}\, u_2(z) \tag{8}$$

and the process reduces to a simple delay operation. In this case we may replace the block with a delay element and save the hardware cost.

By inserting proper latches in each building block we can obtain a systolic structure where all blocks are active in the same sampling interval. This enables us to process the signals through the array in a very high speed that is only dominated by the speed of addition operation. For an example of a systolic structure see [8]. In general, when the coefficients b_i are arbitrary real numbers the De Casteljau structure is expensive to implement. However, we will show that in some interesting situations, the cost can be reduced.

2.2 MAXFLAT Filters

These filters can be realized in a multiplier-less way utilizing the De Casteljau structure. In MAXFLAT highpass and lowpass filters, coefficients $b_i(=c_i)$ are either 0 or 1 [9] and therefore no multiplication is required for the whole triangular structure. In addition, considering that some of the blocks receive two equivalent inputs, simplification of the structure is also possible. Specifically, for a filter with degrees of flatness L and K at $\omega=0$ and $\omega=\pi$ respectively,† the total number of building blocks can be reduced to LK [3]; a reduction rate of at least $\frac{N-1}{2N}$. Ironically, the most expensive structure among all combinations of L and K for an odd N is a half-band structure where $L=K=\frac{N+1}{2}$. For an even N, half-band filters are not possible but the most expensive case occurs when $L=\frac{N}{2}$ or $K=\frac{N}{2}$.

Another interesting case is when the magnitude response of a linear phase, odd length, even symmetric filter is flat at $\omega=0$ $(x=0)$ and $\omega=\pi$ $(x=1)$ and an arbitrary approximation (minimax, least squares, etc.) elsewhere, i.e., $N>K+L-1$. Again, some of $b_i(=c_i)$ are 1 and 0 due to the Property 3 and we have a decrease in the number of multiplier coefficients and elimination of some blocks becomes possible.

2.3 Half-Band Filters

A half-band FIR filter is an odd length even symmetric filter of length $4P-1$ where $2P+1$ of its impulse responses are generally non-zero [10]. The canonic number of multipliers needed for realizing this filter is P and its zero phase transfer function satisfies the relation

$$H_0(z)+H_0(-z)=1.$$

This is equivalent to

$$H_0(x)+H_0(1-x)=1,$$

where $H_0(x)$ is the transformed polynomial of degree $N=2P-1$ and x is related to z as in (6). Using identities

$$B_i^{(N)}(x)=B_{N-i}^{(N)}(1-x),$$
$$\sum_{i=0}^{N} B_i^N(x)=1,$$

it turns out that the Bernstein coefficients of $H_0(x)$ should satisfy

$$b_{2P-1-i}+b_i=1 \quad \text{for} \quad i=0(1)(P-1).$$

Consequently, we need only P independent multiplier coefficients in the De Casteljau realization of a half-band filter.

2.4 Bernstein Polynomials as Approximants

Until now we assumed that the transfer function was obtained somehow and then converted to its Bernstein form. However, it is well-known that Bernstein polynomials can be used for uniform approximation [11] of a function $g(x)$, continuous on $[0,1]$, by setting

$$b_i=g(\frac{i}{N}), \quad i=0(1)N$$

in (2). This means that by using the De Casteljau structure we can easily obtain an approximation to an arbitrary shape. Although this approximation is not optimum in any sense, it may be useful for some applications.

The 2-D generalization of this may be more interesting. This is because designing a 2-D FIRDF with an arbitrary frequency response shape is more cumbersome, in general, than the 1-D case. A bivariate function, $g(x,y)$, continuous on $[0,1]\times[0,1]$ can be approximated uniformly through the bivariate polynomial

$$f(x,y)=\sum_{i=0}^{N_x}\sum_{j=0}^{N_y} g(\frac{i}{N_x},\frac{j}{N_y})\, B_i^{(N_x)}(x)\, B_j^{(N_y)}(y).$$

This polynomial can be mapped to the z domain using

$$x=\frac{1-\frac{z_1+z_1^{-1}}{2}}{2}, \quad y=\frac{1-\frac{z_2+z_2^{-1}}{2}}{2}.$$

As an example consider the design of a zero phase fan filter with ideal magnitude response of the form

$$I_0(\omega_1,\omega_2)=\begin{cases}1, & \arctan\left|\frac{\omega_2}{\omega_1}\right| \le \theta\\ 0, & \arctan\left|\frac{\omega_2}{\omega_1}\right| > \theta.\end{cases}$$

Considering the ideal magnitude response at the origin and the relationships $x=\sin^2\frac{\omega_1}{2}$ and $y=\sin^2\frac{\omega_2}{2}$, the Bernstein coefficients should be set as

$$b_{i,j}=g(\frac{i}{N_x},\frac{j}{N_y})=\begin{cases}1, & i=j=0\\ 1, & \arctan\frac{\arcsin\sqrt{\frac{j}{N_y}}}{\arcsin\sqrt{\frac{i}{N_x}}}\le\theta\\ 0, & \arctan\frac{\arcsin\sqrt{\frac{j}{N_y}}}{\arcsin\sqrt{\frac{i}{N_x}}}>\theta.\end{cases}$$

It is clear that the condition $\tan\theta>\frac{1}{N_y}$ can be used as a rule of thumb for obtaining a reasonable approximation. Figure 4 shows the magnitude response of the filter for $\theta=\frac{\pi}{7}$ and $N_x=N_y=10$.

Now we present a structure for realizing this class of filters. Let us define

$$b_j(x)=\sum_{i=0}^{N_x} g(\frac{i}{N_x},\frac{j}{N_y})\, B_i^{(N_x)}(x), \quad j=0(1)N_y.$$

We need all N_y+1 of these functions which should be used as the Bernstein coefficients in a ∇_{N_y} to yield

†With this definition we have $N=L+K-1$ in (5).

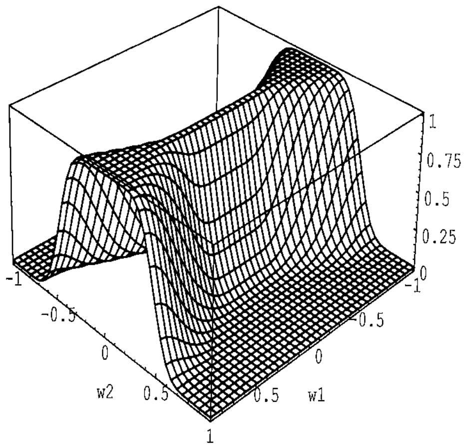

Fig. 4 Magnitude response of the approximated fan filter. ($w_{1,2} = \frac{\omega_{1,2}}{\pi}$)

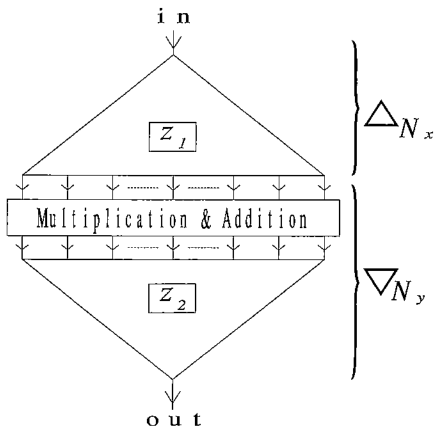

Fig. 5 Triangular realization of a 2-D DF.

$$f(x, y) = \sum_{j=0}^{N_y} b_j(x)\; B_j^{(N_y)}(y).$$

All of the Bernstein coefficients $\{b_j(x)\}$ can be realized using a Δ_{N_x} whose output signals are tapped by coefficients $g(\frac{i}{N_x}, \frac{j}{N_y})$ for $i = 0(1)N_x, j = 0(1)N_y$. The block diagram of the overall structure is shown in Fig. 5.

2.5 Degradation Due to Coefficient Quantization

Assume that the finite word-length representation of $\{b_i\}$ introduces errors $\{e_i\}$ and

$$-d_i \leqq e_i \leqq d_i, \qquad i = 0(1)N,$$

where $\{d_i\}$ depend on the quantization scheme. To estimate the deviation of $H_0(x)$ from its nominal value, it is desirable to find upper and lower bounds on

$$E_0(x) = Q_0(x) - H_0(x) = \sum_{i=0}^{N} e_i\, B_i^{(N)}(x),$$

where $Q_0(x)$ is the transfer function with quantized coefficients $q_i = b_i + e_i$. Since $\{e_i\}$ are Bernstein coefficients in the above Equation, their minimum and maximum values offer lower and upper bounds on $E_0(x)$, respectively. These bounds are sharp if and only if they are equal to e_0 or e_N [5].

3. Triangular Structures for Power Basis Representation

Assume that $f(x)$ is expressed in the power form (1) and let

$$p_i = \binom{N}{i}\, p_i', \quad i = 0(1)N.$$

In terms of p_i' we have

$$f(x) = \sum_{i=0}^{N} p_i' \binom{N}{i} x^i. \tag{9}$$

Now that we have binomial coefficients appeared explicitly, a simple recursive triangle based on Pascal's triangle can be used to evaluate (9). The general recursion for triangular evaluation of the power form is [4]

$$\begin{aligned} p_i^{(0)} &\leftarrow p_i' \ , \quad i = 0(1)N \\ p_i^{(j)} &\leftarrow p_{i-1}^{(j-1)} + x\, p_i^{(j-1)}, \\ &\qquad j = 1(1)N, \quad i = j(1)N \\ f(x) &= p_N^{(N)}. \end{aligned} \tag{10}$$

This can be illustrated by a ∇_N (cf. Fig. 1) with $L = 1$, $R = x$ and $c_i = p_i'$. Note that the boundary coefficients are $\{p_i'\}$, not $\{p_i\}$. For simultaneous evaluation of the power basis functions weighted by binomial coefficients, i.e., $\binom{N}{i} x^i$, see Fig. 2 (structure Δ_N).

A causal DF structure corresponding to ∇_N can be derived by using Fig. 3 with building blocks of the form

$$y(z) = z^{-1}\, u_1(z) - \frac{(1 - z^{-1})^2}{4}\, u_2(z), \tag{11}$$

assuming that x and z are related by (6). $u_1(z)$ and $u_2(z)$ correspond to $p_{i-1}^{(j-1)}$ and $p_i^{(j-1)}$ in (10), respectively. Again, the transposed form of this structure corresponds to a Δ_N whose outputs are tapped by $\{p_i'\}$ and then added up.

Another point to observe is that if $\{p_i\}$ are symmetric with respect to the middle coefficient $p_{\frac{N}{2}}$, this property will be preserved in $\{p_i'\}$ because of the symmetry of the binomial coefficients.

3.1 MAXFLAT Filters

Here we present a new multiplier-less structure based on the power basis triangle. Herrmann has shown that the transformed transfer function of a MAXFLAT filter can be written as [12]

$$H_0(x) = (1-x)^K \sum_{i=0}^{N-K} \binom{K+i-1}{i} x^i \qquad (12)$$

where x is defined as in (6).

Realization of the term $(1-x)^K$ is trivial. The summation term can be realized by observing that the term $\binom{K+i-1}{i} x^i$ is the i-th basis function of the $(K+i-1)$-th row of Δ_{K+i-1}. To realize all these terms simultaneously, a Δ_{N-1}, corresponding to $i = N-K$, is needed. Now, we can add up appropriate nodes and calculate the sum.

Let us show this for $N=5$ and $K=2$. From (12) we realize that only nodes with outputs of the form $\binom{1}{0}$, $\binom{2}{1} x$, $\binom{3}{2} x^2$ and $\binom{4}{3} x^3$ are needed from Δ_4. These nodes are specified by circles in Fig. 6. From this figure we can see that some nodes are unrelated to the circled nodes and need not be computed (the dotted part of Δ_4). We eliminate them to save the number of delays and adders. The causal version of the overall structure is depicted in Fig. 7. We may also obtain this structure by manipulating the signal flow graph of the structure

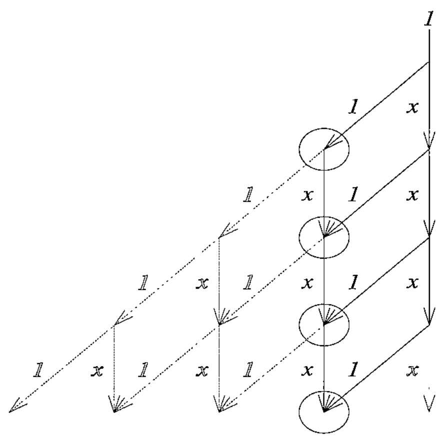

Fig. 6 Structure Δ_4 for power basis MAXFLAT. Only the circled nodes are needed for realizing the summation part of (12).

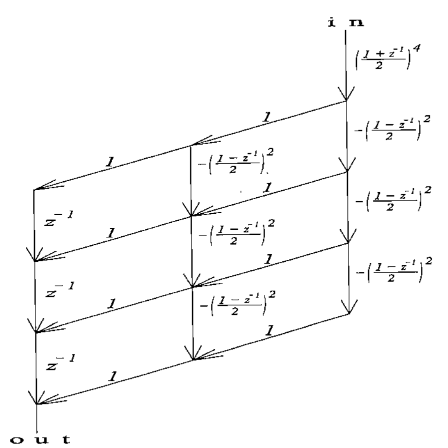

Fig. 7 New multiplier-less MAXFLAT structure based on Fig. 6 ($N=5$, $K=2$).

of [3] but our approach here is intended to demonstrate the application of power basis triangles in filtering applications.

4. Chebyshev Polynomials and Triangles

Chebyshev polynomials can be evaluated recursively using the recursion

$$\begin{aligned} \mathcal{T}_N(x) &= 2x\,\mathcal{T}_{N-1}(x) - \mathcal{T}_{N-2}(x), \\ \mathcal{T}_0(x) &= 1, \quad \mathcal{T}_1(x) = x. \end{aligned} \qquad (13)$$

Here we show that these polynomials can also be evaluated using a triangular scheme. To this end, let us express them as

$$\begin{aligned} \mathcal{T}_N(x) &= \mathrm{Re}(x + j\sqrt{1-x^2})^N \\ &= \frac{(x-\sqrt{x^2-1})^N + (x+\sqrt{x^2-1})^N}{2} \\ &= \sum_{i=0}^{N} \frac{1+(-1)^i}{2} \binom{N}{i} x^{N-i} \sqrt{x^2-1}^{\,i}. \end{aligned} \qquad (14)$$

By appropriate selection of L, R and $\{c_i\}$ for the triangle of Fig. 1, we can generalize it to a triangular scheme for evaluating (14). Thus, we set $L = x$, $R = \sqrt{x^2-1}$ and $c_i = \frac{1+(-1)^i}{2}$ as shown in Table 1. All of the Chebyshev polynomials of degrees up to N are obtained from one ∇_N type triangle.

Since $x = \cos(\omega)$ here, the blocks of the structure of Fig. 3 are as

$$y(z) = \frac{1+z^{-2}}{2} u_1(z) + \frac{1-z^{-2}}{2} u_2(z)$$

for a causal Chebyshev structure. It is obvious that the triangular scheme is more expensive comparing (13). However, generalizing the triangle to these important polynomials is worthwhile theoretically. Although the triangular realization of the Chebyshev polynomials can also be used to realize FIRDFs by tapping the resulting structure at appropriate output nodes, our main purpose from deriving the Chebyshev triangles is to show the versatility of the triangles to realize important functions in digital signal processing.

5. General Properties

The triangular digital structure of Fig. 3 corresponds to triangular polynomial evaluation algorithms in different basis functions as discussed in the previous sections. The internal structure of the blocks depends on the type of the basis function. However, in all cases the following general properties can be observed.

1. The blocks are multiplier-less if realized using an appropriate number system, e.g. fixed point one's complement, two's complement or sign-magnitude.

This means that each block consists of bit shift, addition, subtraction and delay operations only. The bit shift operation can be implemented by wiring if the above mentioned binary number systems are used.

2. The blocks are connected in a regular and symmetric manner and only to the neighboring blocks.

3. All of the blocks are identical and thus the structure is highly modular in its triangular part. The multiplication part at the top may deteriorate the overall modularity; multiplier coefficients (c_i) are generally of different values.

4. The computations inside the blocks can be performed concurrently in a multiprocessor environment (one processor associated to one block) by, for example, pipelining the blocks using appropriate delays at the output nodes to achieve parallel processing. In this way, the maximum processing speed can be raised to be proportional to the speed of addition operation.

Some points should be noted here. First, a few words on the overall modularity of Fig. 3. Some special transfer functions, such as MAXFLAT filters in the Bernstein basis triangles, can be realized in a completely multiplier-less manner. On the other hand, Bernstein basis functions can also be used to approximate an arbitrary ideal magnitude response as discussed in Sect. 2.4. A DF's ideal magnitude response generally takes on values of 0 or 1. This enables us to obtain various multiplier-less triangular DF structures (1- or 2-D) since $g(\frac{i}{N}), g(\frac{i}{N_x}, \frac{j}{N_y}) \in \{0,1\}$. In all these cases, the overall structure consists of the triangular processing part(s) only and no multiplication part is necessary. This results in a completely modular structure. In fact, there are also some delay elements at the boundaries of the structure due to the simplification of the blocks as shown in (8). Hence, we have two types of modules: the blocks and the delay elements (cf. [3]). The possibility of discovering other transfer functions that yield triangular structures with favorable properties, under certain basis functions, also exists.

Finally, the structures based on the transposed form (Δ_N-based structures) can also be realized using identical building blocks and possess all properties mentioned above. One difference is that in this type of structures some of the blocks are connected with the neighboring blocks, to their left or right, by adders. These adders can not be merged into the blocks. To pipeline these structures, one, again, needs delays at appropriate locations. For example, the main part of the structure of Fig. 7 can be viewed as an interconnection of blocks with one input, $u(z)$, and two outputs, $y_1(z)$ and $y_2(z)$, satisfying the relations

$$y_1(z) = u(z), \quad y_2(z) = -(\frac{1-z^{-1}}{2})^2\, u(z).$$

The rest of the structure consists of delay elements or identical cascaded blocks of the form $\frac{1+z^{-1}}{2}$.

6. Conclusion

Application of recursive triangles to the realization of FIRDFs was studied. Structures based on the Bernstein form and the power form representations of FIRDFs were presented. It was shown that these structures are of special importance for multiplier-less and modular realization of MAXFLAT filters. Approximation and realization of 2-D filters based on bivariate Bernstein approximation was also considered and a structure composed of two 1-D triangles was proposed. A new Chebyshev structure based on the triangular recursion was also proposed.

Among the most attractive features of these structures are their inherent modularity, parallelism, symmetry, local interconnection and pipelinability that make them suitable for high-speed filtering on special-purpose hardware. Although MAXFLAT filters seem to be the most suitable candidates for these structures, further study should be done to find other types of transfer functions that can be realized efficiently by the triangles.

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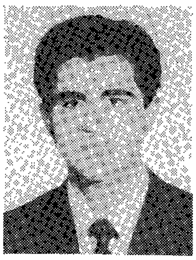

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