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New necessary and sufficient conditions for convergence theorems with respect to non-additive measure

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Contents

1 Introduction

Since Sugeno [16] introduced the concept of non-additive measure, which he called a fuzzy measure, non-additive measure theory has been constructed along the lines of the classical measure theory [1, 14, 24]. Generally, theorems in the classical measure theory no longer hold in non-additive measure theory, so that to find necessary and/or sufficient conditions for such theorems to hold is very important for the construction of non-additive measure theory.

Sugeno defined a fuzzy measure μ as a set function on a measurable space on $(X, \mathscr{S})$ satisfying the following five conditions:

(FM1) lower boundary condition: $\mu(\emptyset) = 0$;

(FM2) upper boundary condition: $\mu(X) = 1$;

(FM3) monotonicity: $A, B \in \mathscr{S},\ A \subset B \Rightarrow \mu(A) \leq \mu(B)$;

(FM4) continuity from below: $\{A_n\} \subset \mathscr{S},\ A_n \nearrow A \Rightarrow \mu(A_n) \nearrow \mu(A)$;

(FM5) continuity from above: $\{A_n\} \subset \mathscr{S},\ A_n \searrow A \Rightarrow \mu(A_n) \searrow \mu(A)$.

However, (FM2) is not essential from the viewpoint of mathematics. Therefore, some authors define a fuzzy measure as a set function satisfying the conditions (FM1), (FM3), (FM4) and the following condition (FM5') in place of the condition (FM5) [5, 9, 10, 17, 18, 22, 24].

(FM5') conditional continuity from above:

$$\{A_n\} \subset \mathscr{S},\ A_n \searrow A,\ \mu(A_k) < \infty \text{ for some } k \Rightarrow \mu(A_n) \searrow \mu(A)$$

On the other hand, others define a fuzzy measure as a set function satisfying (FM1) and (FM3) [6, 21]. Without (FM4) and (FM5'), it is possible to define the Sugeno integral and the Choquet integral which are integrals with respect to fuzzy measures. In addition, several useful set functions in fuzzy theory do not satisfy the continuity conditions. For example, a necessity measure does not satisfy (FM4) and a possibility measure does not satisfy (FM5'). For this reason, some outhors have studied set functions satisfying (FM1) and (FM3) and called such a set function a fuzzy measure.

In this dissertation, we deal with set functions satisfying (FM1) and (FM3), and, in order to avoid confusion, we call such a set function a non-additive measure instead of a fuzzy measure.

In the classical measure theory, there are several different convergences of a sequence of measurable functions such as almost everywhere convergence, almost uniform convergence, and convergence in measure, and theorems that describe implication relationship between such convergence concepts (e.g, the Egoroff, Lebesgue, and Riesz theorems) are fundamental and important. In non-additive measure theory, these theorems do not hold without additional conditions.

This dissertation discusses necessary and sufficient conditions for the implications between almost everywhere convergence, pseudo-almost everywhere convergence [23], almost uniform convergence, pseudo-almost uniform convergence [23], convergence in measure, and convergence pseudo-in measure [23] in non-additive measure theory. Pseudo-almost everywhere convergence [23], pseudo-almost uniform convergence [23], and pseudo-in measure [23] are newly defined in non-additive measure theory.

This dissertation is organized as follows.

In Chapter 2, we show the definitions of conditions with respect to non-additive measure and convergences in non-additive measure theory. Moreover, we explain Ordinal Duality Principle [11].

So far, most of the implications between the convergence concepts have been established [3, 4, 5, 6, 7, 8, 10, 13, 15, 18, 22, 23, 24]. Chapter 3 of this dissertation clarifies the remaining ones, that is, shows necessary and sufficient conditions for almost uniform convergence to imply pseudo-almost everywhere convergence, pseudo-almost uniform convergence, and convergence pseudo-in measure. Moreover, we summarize necessary and sufficient conditions for implications between the six convergences into two tables and show two figures which illustrate some implications among conditions in the tables.

In Chapter 4, we show the result related with the Egoroff theorem. The Egoroff theorem, which asserts that almost everywhere convergence implies almost uniform convergence, is one of the most important convergence theorems in the classical measure theory. In non-additive measure theory, this theorem does not hold without additional conditions. So far, it has been shown that each of the Egoroff condition [13] and condition (E) [4] is a necessary and sufficient condition for the Egoroff theorem to hold in non-additive measure theory. Both of the conditions are described by a doubly-indexed sequence of measurable sets. This dissertation gives condition (M) which is described by a singly-indexed sequence of measurable sets and shows that condition (M) is equivalent to the Egoroff condition.

In Chapter 5, we state the conclusion of this dissertation and give subjects of future research.

2 Definitions of convergences and conditions with respect to non-additive measure

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers and $(X, \mathscr{S})$ is assumed to be a measurable space. All functions on X are assumed to be measurable. In addition, every measurable function f is assumed to be finite real valued, i.e., $-\infty < f(x) < \infty$ for all $x \in X$.

2.1 Non-additive measure

Definition 1 A *non-additive measure* on $(X, \mathscr{S})$ is a set function $\mu : \mathscr{S} \to [0, \infty]$ satisfying the following two conditions:

(i) $\mu(\emptyset) = 0$,

(ii) $A, B \in \mathscr{S},\ A \subset B \ \Rightarrow \ \mu(A) \leq \mu(B)$.

Unless stated otherwise, all subsets are supposed to belong to $\mathscr{S}$ and μ is assumed to be a non-additive measure on $\mathscr{S}$.

A non-additive measure μ is said to be *finite* if $\mu(X) < \infty$. If μ is a non-additive measure on $(X, \mathscr{S})$, the triplet $(X, \mathscr{S}, \mu)$ is called a *non-additive measure space*. For each $A \in \mathscr{S}$, the restriction $\mu \upharpoonright (\mathscr{S} \cap A)$ is a non-additive measure on $(A, \mathscr{S} \cap A)$, where $\mathscr{S} \cap A = \{E \cap A \mid E \in \mathscr{S}\}$, and $(A, \mathscr{S} \cap A, \mu \upharpoonright (\mathscr{S} \cap A))$ is called a *subspace* of $(X, \mathscr{S}, \mu)$.

2.2 Conditions with respect to non-additive measure

In the following definitions, each label in bold face stands for the corresponding term; for example, "$\downarrow \emptyset$" means "order continuity" (Definition 2 (i). These labels will be used later in Tables 1–3 and Figures 1 and 2.

Definition 2 (i) **$\downarrow \emptyset$:** [2] μ is said to be *order continuous* if $N_n \downarrow \emptyset$ implies $\mu(N_n) \to 0$.

(ii) **$\downarrow$0:** [3] μ is said to be *strongly order continuous* if $N_n \downarrow N$ and $\mu(N) = 0$ together imply $\mu(N_n) \to 0$.

(iii) **$\uparrow$*A*:** μ is said to be *continuous from below at A* if $A_n \uparrow A$ implies $\mu(A_n) \to \mu(A)$.

$\uparrow$: μ is said to be *continuous from below* if μ is continuous from below at every measurable set.

(iv) **$\uparrow \boldsymbol{\mu(A)}$:** [8] μ is said to be *strongly continuous from below at A* if $B_n \uparrow B \subset A$ and $\mu(B) = \mu(A)$ together imply $\mu(B_n) \to \mu(B)$.

The value of $\mu(A)$ is not substituted for $\mu(A)$ in "$\uparrow \mu(A)$"; if $\mu(A) = 0.5$ for example, we write not "$\uparrow 0.5$" but "$\uparrow \mu(A)$".

Definition 3 (i) **0-sub. *A*:** μ is said to be *null-subtractive at A* if $\mu(N) = 0$ implies $\mu(A \setminus N) = \mu(A)$.

0-add.: [22] μ is said to be *null-additive* if $\mu(N) = 0$ implies $\mu(A \cup N) = \mu(A)$ for every $A \in \mathscr{S}$.

(ii) **c.0-add. *A*:** μ is said to be *converse-null-additive at A* if $\mu(A) = \mu(A \setminus N)$ implies $\mu(A \cap N) = 0$.

c.0-add.: μ is said to be *converse-null-additive* if μ is converse-null-additive at every measurable set.

In [25] null-subtractivity is defined as null-subtractivity at every measurable set. Null-subtractivity at every measurable set is equivalent to null-additivity [24]. Converse-null-additivity defined above is stronger than the original in [22]: $\mu(A) = \mu(A \setminus N) < \infty$ implies $\mu(A \cap N) = 0$.

Definition 4 (i) **auto.$\uparrow$ *A*:** μ is said to be *autocontinuous from below at A* if $\mu(N_n) \to 0$ implies $\mu(A \setminus N_n) \to \mu(A)$.

auto.$\uparrow$: [22] μ is said to be *autocontinuous from below* if μ is autocontinuous from below at every measurable set.

(ii) **c.auto.$\uparrow$ *A*:** μ is said to be *converse-autocontinuous from below at A* if $\mu(A \setminus N_n) \to \mu(A)$ implies $\mu(A \cap N_n) \to 0$.

c.auto.$\uparrow$: μ is said to be *converse-autocontinuous from below* if μ is converse-autocontinuous from below at every measurable set.

(iii) **m.auto.$\uparrow$ *A*:** μ is said to be *monotone autocontinuous from below at A* if $N_n \downarrow$ and $\mu(N_n) \to 0$ together imply $\mu(A \setminus N_n) \to \mu(A)$.

m.auto.$\uparrow$: μ is said to be *monotone autocontinuous from below* if μ is monotone autocontinuous from below at every measurable set.

(iv) **c.m.auto.$\uparrow$ *A*:** μ is said to be *converse-monotone autocontinuous from below at A* if $N_n \downarrow$ and $\mu(A \setminus N_n) \to \mu(A)$ together imply $\mu(A \cap N_n) \to 0$.

c.m.auto.$\uparrow$: μ is said to be *converse-monotone autocontinuous from below* if μ is converse-monotone autocontinuous from below at every measurable set.

(v) **s.m.auto.$\uparrow$ *A*:** μ is said to be *strongly monotone autocontinuous from below at A* if $N_n \downarrow N$ and $\mu(N) = 0$ together imply $\mu(A \setminus N_n) \to \mu(A)$.

s.m.auto.$\uparrow$: μ is said to be *strongly monotone autocontinuous from below* if μ is strongly monotone autocontinuous from below at every measurable set.

(vi) **s.c.m.auto.$\uparrow$ *A*:** μ is said to be *strongly converse-monotone autocontinuous from below at A* if $N_n \downarrow N$ and $\mu(A \setminus N) = \mu(A)$ together imply $\mu(A \cap N_n) \to 0$.

s.c.m.auto.$\uparrow$: μ is said to be *strongly converse-monotone autocontinuous from below* if μ is strongly converse-monotone autocontinuous from below at every measurable set.

Converse-autocontinuity from below defined above is stronger than the original in [23]: $\mu(A \setminus N_n) \to \mu(A) < \infty$ implies $\mu(A \cap N_n) \to 0$. In [15] strong converse-monotone autocontinuity from below is called pseudo-order continuity.

Definition 5 (i) **(S):** [18] μ said to have *property (S)* if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i}\right) = 0$.

(ii) **(PS) *A*:** μ said to have *property (PS)* at A if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i}\right) = \mu(A)$.

(PS): [17] μ said to have *property (PS)* if μ has property (PS) at every measurable set.

(iii) **(TS) *A*:** μ said to have *property (TS)* at A if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i}\right) = \mu(A)$.

(TS): μ said to have *property (TS)* if μ has property (TS) at every measurable set.

(iv) **(TPS) *A*:** μ said to have *property (TPS)* at A if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i} \cap A\right) = 0$.

(TPS): μ said to have *property (TPS)* if μ has property (TPS) at every measurable set.

Definition 6 (i) **(S1):** [10] μ said to have *property (S1)* if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} N_{n_i}\right) \to 0$ $(k \to \infty)$.

(ii) **(S2) *A*:** μ said to have *property (S2)* at A if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} N_{n_i}\right) \to \mu(A)$ $(k \to \infty)$.

(S2): [10] μ said to have *property (S2)* if μ has property (S2) at every measurable set.

(iii) **(PS1) *A*:** μ said to have *property (PS1)* at A if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} N_{n_i} \cap A\right) \to 0$ $(k \to \infty)$.

(PS1): [10] μ said to have *property (PS1)* if μ has property (PS1) at every measurable set.

(iv) **(PS2) *A*:** μ said to have *property (PS2)* at A if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} N_{n_i}\right) \to \mu(A)$ $(k \to \infty)$.

(PS2): [10] μ said to have *property (PS2)* if μ has property (PS2) at every measurable set.

Definition 7 (i) **(E):** [4] μ is said to satisfy *condition (E)* if $E_n^m \downarrow E^m$ $(n \to \infty)$ for every m and $\mu\left(\bigcup_{m=1}^{\infty} E^m\right) = 0$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{m_i}\right) \to 0$ $(k \to \infty)$.

(ii) **(TE) *A*:** μ is said to satisfy *condition (TE)* at A if $E_n^m \downarrow E^m$ $(n \to \infty)$ for every m and $\mu\left(\bigcup_{m=1}^{\infty} E^m\right) = 0$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} E_{n_i}^{m_i}\right) \to \mu(A)$ $(k \to \infty)$.

(TE): μ is said to satisfy *condition (TE)* if μ satisfies condition (TE) at every measurable set.

(iii) **(PE) *A*:** μ is said to satisfy *condition (PE)* at A if $E_n^m \downarrow E^m$ $(n \to \infty)$ for every m and $\mu\left(A \setminus \bigcup_{m=1}^{\infty} E^m\right) = \mu(A)$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} E_{n_i}^{m_i}\right) \to \mu(A)$ $(k \to \infty)$.

(PE): μ is said to satisfy *condition (PE)* if μ satisfies condition (PE) at every measurable set.

(iv) **(TPE) *A*:** μ is said to satisfy *condition (TPE)* at A if $E_n^m \downarrow E^m$ $(n \to \infty)$ for every m and $\mu\left(A \setminus \bigcup_{m=1}^{\infty} E^m\right) = \mu(A)$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{m_i}\right) \to 0$ $(k \to \infty)$.

(TPE): μ is said to satisfy *condition (TPE)* if μ satisfies condition (TPE) at every measurable set.

In [8] condition (TE) at X is called pseudo-condition (E).

Definition 8 (EC) : [13] μ is said to satisfy *the Egoroff condition* if, for every doubly-indexed sequence $E_{m,n}$ such that $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$ and $\mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}\right) = 0$, and for every positive number ε, there exists a sequence $\{n_m\}$ of positive integers such that $\mu\left(\bigcup_{m=1}^{\infty} E_{m,n_m}\right) < \varepsilon$.

Condition (E) is equivalent to the Egoroff condition. Both Condition (E) and the Egoroff condition are necessary and sufficient conditions for the Egoroff theorem, i.e., each of them is satisfied iff almost everywhere convergence implies almost uniform convergence.

Definition 9 (i) **(M):** μ is said to satisfy *condition (M)* if $\mu(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}E_i) = 0$ implies that for every positive number ε there exists a sequence $\{m_n\}$ of positive integers such that $\mu(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{m_n}E_i) < \varepsilon$.

(ii) **(PM)** ***A*:** μ is said to satisfy *condition (PM)* at A if $\mu(A \setminus \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}E_i) = \mu(A)$ implies that for every positive number $\xi < \mu(A)$ there exists a sequence $\{m_n\}$ of positive integers such that $\mu(A \setminus \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{m_n}E_i) > \xi$.

(PM): μ is said to satisfy *condition (PM)* if μ satisfies condition (PM) at every measurable set.

We define condition of Definition 9 newly and show that condition (M) is equivalent to the Egoroff condition in Chapter 4.

Let P be a condition concerning a non-additive measure space, and $(X,\mathscr{S},\mu)$ a non-additive measure space. We write the condition P concerning $(X,\mathscr{S},\mu)$ as $P(X,\mathscr{S},\mu)$. For example, if P is the null-subtractivity at the whole set, and if A is a measurable set in a non-additive measure space $(X,\mathscr{S},\mu)$, then $P(A,\mathscr{S}\cap A,\mu \restriction (\mathscr{S}\cap A))$ means the null-subtractivity at A.

Let P be a condition concerning a non-additive measure space, and $(X,\mathscr{S},\mu)$ a non-additive measure space. Then $\forall A; P(A,\mathscr{S}\cap A,\mu \restriction (\mathscr{S}\cap A))$ is also a condition concerning $(X,\mathscr{S},\mu)$. We call the condition $\forall A; P(A,\mathscr{S}\cap A,\mu \restriction (\mathscr{S}\cap A))$ the *universal* of $P(X,\mathscr{S},\mu)$. For example, the universal of the null-subtractivity at the whole set is equivalent to the null-additivity.

2.3 Definitions of convergences

In this section, we show the definitions of the convergences treated in this dissertation.

Definition 10

a.e.: $\{f_n\}$ is said to converge to f *almost everywhere*, written $f_n \xrightarrow{\text{a.e.}} f$, if there exists N such that $\mu(N) = 0$ and $\{f_n(x)\}$ converges to $f(x)$ for all $x \in X \setminus N$.

p.a.e.: [23]$\{f_n\}$ is said to converge to f *pseudo-almost everywhere*, written $f_n \xrightarrow{\text{p.a.e.}} f$, if there exists N such that $\mu(X \setminus N) = \mu(X)$ and $\{f_n(x)\}$ converges to $f(x)$ for all $x \in X \setminus N$.

a.u.: $\{f_n\}$ is said to converge to f *almost uniformly*, written $f_n \xrightarrow{\text{a.u.}} f$, if for every $\varepsilon > 0$ there exists N_ε such that $\mu(N_\varepsilon) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on $X \setminus N_\varepsilon$.

p.a.u.: [23] $\{f_n\}$ is said to converge to f *pseudo-almost uniformly*, written $f_n \xrightarrow{\text{p.a.u.}} f$, if for every $\xi < \mu(X)$ there exists N_ξ such that $\xi < \mu(X \setminus N_\xi)$ and $\{f_n\}$ converges to f uniformly on $X \setminus N_\xi$.

in meas.: $\{f_n\}$ is said to converge to f *in measure*, written $f_n \xrightarrow{\mu} f$, if $\mu(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ for every $\varepsilon > 0$.

p. in meas.: [23] $\{f_n\}$ is said to converge to f *pseudo-in measure*, written $f_n \xrightarrow{\text{p.}\mu} f$, if $\mu(\{x \mid |f_n(x) - f(x)| < \varepsilon\}) \to \mu(X)$ for every $\varepsilon > 0$.

Let $A \in \mathscr{S}$. For each convergence defined above, if $\{f_n \restriction A\}$ convergences to $f \restriction A$ on the subspace $(A, \mathscr{S} \cap A, \mu|(\mathscr{S} \cap A))$, we say $\{f_n\}$ convergences to f on A and write $f_n \xrightarrow[A]{*} f$, where $f \restriction A$ denotes the restriction of f to A and $*$ stands for a.e., p.a.e., a.u., p.a.u., μ, or p.μ.

In measure theory, a null set is defined to be a measurable set N such that $\mu(N) = 0$, so if a set N is a null set, then $\mu(X \setminus N) = \mu(X)$. Moreover, on a finite measure space, a set N is a null set iff $\mu(X \setminus N) = \mu(X)$. Therefore, in measure theory $f_n \xrightarrow{\text{a.e.}} f$ implies $f_n \xrightarrow{\text{p.a.e.}} f$, $f_n \xrightarrow{\text{a.u.}} f$ implies $f_n \xrightarrow{\text{p.a.u.}} f$, and $f_n \xrightarrow{\mu} f$ implies $f_n \xrightarrow{\text{p.}\mu} f$. Moreover, on a finite measure space $f_n \xrightarrow{\text{a.e.}} f$ is equivalent to $f_n \xrightarrow{\text{p.a.e.}} f$, $f_n \xrightarrow{\text{a.u.}} f$ is equivalent to $f_n \xrightarrow{\text{p.a.u.}} f$, and $f_n \xrightarrow{\mu} f$ is equivalent to $f_n \xrightarrow{\text{p.}\mu} f$.

However, in non-additive measure theory, it does not necessarily hold that for every set N, $\mu(N) = 0$ implies $\mu(X \setminus N) = \mu(X)$, and neither does the converse proposition hold. Therefore, in non-additive measure theory $f_n \xrightarrow{\text{a.e.}} f$ does not imply $f_n \xrightarrow{\text{p.a.e.}} f$, $f_n \xrightarrow{\text{a.u.}} f$ does not imply $f_n \xrightarrow{\text{p.a.u.}} f$, $f_n \xrightarrow{\mu} f$ does not imply $f_n \xrightarrow{\text{p.}\mu} f$, $f_n \xrightarrow{\text{p.a.e.}} f$ does not imply $f_n \xrightarrow{\text{a.e.}} f$, $f_n \xrightarrow{\text{p.a.u.}} f$ does not imply $f_n \xrightarrow{\text{a.u.}} f$, and $f_n \xrightarrow{\text{p.}\mu} f$ does not imply $f_n \xrightarrow{\mu} f$. For this reason, pseudo-almost everywhere convergence, pseudo-almost uniform convergence, and pseudo-in measure are newly defined in non-additive measure theory.

Example 1 shows that $f_n \xrightarrow{\text{a.e.}} f$ does not imply $f_n \xrightarrow{\text{p.a.e.}} f$.

Example 1 Let $X = \{0, 1\}$, μ be the non-additive measure on the power set 2^X of X defined as

$$\begin{cases} \mu(X) = 2 \\ \mu(\{1\}) = 1 \\ \mu(\{0\}) = 0 \end{cases}$$

and f_n and f be measurable functions defined by

$$\text{for every } n,\ \ f_n(x) = \begin{cases} 0 & \text{if } x = 1, \\ 2 & \text{if } x = 0, \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\{f_n\}$ converges to f almost everywhere, but $\{f_n\}$ does not converge to f pseudo-almost everywhere.

2.4 Convergence theorems

In measure theory, a convergence theorem for a sequence of measurable functions is represented as follows.

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{*} f \Rightarrow f_n \xrightarrow{**} f\ \right]$$

$*$, $**$ stands for a.e., a.u., or μ, and $*$ is different from $**$. We call the convergence theorem written above of type X.

In non-additive measure theory, the following theorem is also considered.

$$\left[\text{for any } A,\ \text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow[A]{*} f \Rightarrow f_n \xrightarrow[A]{**} f\right]$$

$*$, $**$ stands for a.e., p.a.e., a.u., p.a.u., μ, or p.μ., and $*$ is different from $**$. We call the convergence theorem written above of type $\mathscr{S}$.

In non-additive measure theory, if both $*$ and $**$ are a.e., a.u., or μ which are defined in measure theory, then type X is equivalent to type $\mathscr{S}$. For example,

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{\text{a.u.}} f\ \right]$$

is equivalent to

$$\left[\text{for any } A,\ \text{for any } \{f_n\},\ \text{for any} f,\ f_n \xrightarrow[A]{\text{a.e.}} f \Rightarrow f_n \xrightarrow[A]{\text{a.u.}} f\right].$$

However, If either $*$ or $**$ is p.a.e., p.a.u., or p.μ, then it does not necessarily hold that type X implies type $\mathscr{S}$. For example,

$$\left[\text{for any } \{f_n\}, \text{for any} f,\ f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{\text{p.a.e.}} f\ \right]$$

does not imply

$$\left[\text{for any } A,\ \text{for any } \{f_n\},\ \text{for any} f,\ f_n \xrightarrow[A]{\text{a.e.}} f \Rightarrow f_n \xrightarrow[A]{\text{p.a.e.}} f\right].$$

Therefore, in non-additive measure theory, if both $*$ and $**$ are a.e., a.u., or μ, then consider type X, and if either $*$ or $**$ is p.a.e., p.a.u., or p.μ, then consider type $\mathscr{S}$.

Example 2 shows that there exists $\{f_n\}$ and f such that

$$\left[f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{\text{p.a.e.}} f \right]$$

is true, but

$$\left[\text{for any } A,\ f_n \xrightarrow[A]{\text{a.e.}} f \Rightarrow f_n \xrightarrow[A]{\text{p.a.e.}} f \right].$$

is false.

Example 2 Let $X = \{0,1,2\}$, μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 2 & \text{if } A = \{0,1\}, \{0,2\}, \text{or } X, \\ 0 & \text{otherwise} \end{cases}$$

and,

$$\text{for every } n,\ \ f_n(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 1, \\ 2 & \text{if } x = 2, \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 1, \\ 1 & \text{if } x = 2. \end{cases}$$

Then, $\left[f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{\text{p.a.e.}} f \right]$ is true. However $\{f_n \restriction \{0,2\}\}$ converges to $f \restriction \{0,2\}$ almost everywhere. While, $\{f_n \restriction \{0,2\}\}$ does not converge to $f \restriction \{0,2\}$ pseudo-almost everywhere.

2.5 Duality and ordinality

Definition 11 The *conjugate* $\overline{\mu}$ of a finite non-additive measure μ on $(X, \mathscr{S})$ is defined by

$$\overline{\mu}(A) \ = \ \mu(X) - \mu(X \setminus A) \qquad (A \in \mathscr{S}).$$

For every finite non-additive μ on $(X, \mathscr{S})$, its conjugate $\overline{\mu}$ is a finite non-additive measure on $(X, \mathscr{S})$ and $\overline{\overline{\mu}} = \mu$.

We denote the class of all non-additive measure spaces by **NAMS**, and the class of all finite non-additive measure spaces by **fNAMS**. Note that **NAMS** and **fNAMS** are proper classes, i.e., they are not sets.

Definition 12 [11] Let P and Q be conditions concerning a non-additive measure space. P is said to be *dual* to Q when, for every $(X, \mathscr{S}, \mu) \in$ **fNAMS**, $(X, \mathscr{S}, \mu)$ satisfies P iff $(X, \mathscr{S}, \overline{\mu})$ satisfies Q.

For each $(X,\mathscr{S},\mu) \in$ **NAMS**, we denote by $\Phi_{(X,\mathscr{S},\mu)}$ the family of continuous, strictly increasing functions $\varphi : [0,\mu(X)] \to [0,\infty]$ satisfying $\varphi(0) = 0$. If $(X,\mathscr{S},\mu) \in$ **NAMS** and $\varphi \in \Phi_{(X,\mathscr{S},\mu)}$, then the composite function $\varphi \circ \mu$ is a non-additive measure on $(X,\mathscr{S})$.

Definition 13 [11] A condition P concerning a non-additive measure space is said to be *ordinal* if $(X,\mathscr{S},\varphi\circ\mu)$ satisfies P for every $\varphi \in \Phi_{(X,\mathscr{S},\mu)}$ whenever $(X,\mathscr{S},\mu)$ satisfies P.

Ordinal Duality Principle [11]: *An ordinal proposition concerning a (not necessarily finite) non-additive measure space holds, then its dual also holds.*

Now we examine the duality and ordinality of concepts defined in the previous section.

Table 1: Dual pairs

(a)	0-sub.X	$\leftrightarrow$	c.0-add.X
(b)	auto.$\uparrow X$	$\leftrightarrow$	c.auto.$\uparrow X$
(c)	m.auto.$\uparrow X$	$\leftrightarrow$	c.m.auto.$\uparrow X$
(d)	s.m.auto.$\uparrow X$	$\leftrightarrow$	s.c.m.auto.$\uparrow X$
(e)	(S)	$\leftrightarrow$	(PS) X
(f)	(TS) X	$\leftrightarrow$	(TPS) X
(g)	(S1)	$\leftrightarrow$	(PS2) X
(h)	(S2) X	$\leftrightarrow$	(PS1) X
(i)	(E)	$\leftrightarrow$	(PE) X
(j)	(TE) X	$\leftrightarrow$	(TPE) X
(k)	$\downarrow \emptyset$	$\leftrightarrow$	$\uparrow X$
(l)	$\downarrow 0$	$\leftrightarrow$	$\uparrow \mu(X)$
(m)	a.e.	$\leftrightarrow$	p.a.e.
(n)	a.u.	$\leftrightarrow$	p.a.u.
(o)	in meas.	$\leftrightarrow$	p. in meas.

Proposition 1 *Each pair in Table 1 is dual.*

For example, (a) in Table 1 means that null-subtractivity at the whole set is dual to converse-null-additivity at the whole set. (k) and (l) are pointed out in [8], and (m)–(o) are in [9].

Proposition 2 *Every concept in Table 1 is ordinal.*

By the above two propositions, Ordinal Duality Principle can apply to the concepts in Table 1.

2.6 An application of Ordinal Duality Principle to convergence theorems

Assume $P(X,\mathscr{S},\mu)$ is dual to $Q(X,\mathscr{S},\mu)$.

If $P(X,\mathscr{S},\mu)$ is equivalent to

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{*} f \Rightarrow f_n \xrightarrow{**} f\ \right],$$

then $Q(X,\mathscr{S},\mu)$ is equivalent to

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{*'} f \Rightarrow f_n \xrightarrow{**'} f\ \right]$$

($*'$ is dual to $*$ and $**'$ is dual to $**$). For example, since the null-subtractivity at the whole set is satisfied iff

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{\text{p.a.e.}} f\right],$$

the converse-null-additivity at the whole set is satisfied iff

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{\text{p.a.e.}} f \Rightarrow f_n \xrightarrow{\text{a.e.}} f\right].$$

Moreover, since $P(X,\mathscr{S},\mu)$ is equivalent to

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{*'} f \Rightarrow f_n \xrightarrow{**'} f\ \right],$$

it holds that $\forall A \in \mathscr{S}; Q(A,\mathscr{S}\cap A,\mu \restriction (\mathscr{S}\cap A))$ which is the universal of $Q(X,\mathscr{S},\mu)$ is equivalent to

$$\left[\text{for any } A,\ \text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow[A]{*'} f \Rightarrow f_n \xrightarrow[A]{**'} f\right].$$

For example, since null-subtractivity at the whole set is satisfied iff

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{\text{p.a.e.}} f\right],$$

it turns out that null-additivity is satisfied iff

$$\left[\text{for any } A,\ \text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow[A]{\text{a.e.}} f \Rightarrow f_n \xrightarrow[A]{\text{p.a.e.}} f\right].$$

A point to notice is that $\forall A \in \mathscr{S}; P(A, \mathscr{S} \cap A, \mu \restriction (\mathscr{S} \cap A))$ is not dual to $\forall A \in \mathscr{S}; Q(A, \mathscr{S} \cap A, \mu \restriction (\mathscr{S} \cap A))$. For example, the proposition that the null-subtractivity at the whole set is satisfied iff

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{\text{p.a.e.}} f\right]$$

is dual to the proposition that convrse-null-additivity at the whole set is satisfied iff

$$\left[\text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow{\text{p.a.e.}} f \Rightarrow f_n \xrightarrow{\text{a.e.}} f\right].$$

However, the proposition that the null-additivity is satisfied iff

$$\left[\text{for any } A,\ \text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow[A]{\text{a.e.}} f \Rightarrow f_n \xrightarrow[A]{\text{p.a.e.}} f\right]$$

is not dual to the proposition that the convrse-null-additivity is satisfied iff

$$\left[\text{for any } A,\ \text{for any } \{f_n\},\ \text{for any } f,\ f_n \xrightarrow[A]{\text{p.a.e.}} f \Rightarrow f_n \xrightarrow[A]{\text{a.e.}} f\right].$$

For this reason, from the viewpoint of mathematics, type X is fundamental.

Example 3 shows that the null-additivity which is the universe of the null-subtractivity at the whole set is not dual to the converse-null-additivity which is the universe of the converse-null-additivity at the whole set.

Example 3 Let $X = \{0,1,2\}$ and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 3 & \text{if } A = \{1,2\}, \text{or } X, \\ 2 & \text{if } A = \{0,1\}, \text{or } \{0,2\}, \\ 1 & \text{if } A = \{1\}, \text{or } \{2\} \\ 0 & \text{if } A = \{0\}. \end{cases}$$

Then, the converse-null-additivity is satisfied obviously. From the definition of μ, $\overline{\mu}$ is given as follows.

$$\overline{\mu}(A) = \begin{cases} 3 & \text{if } A = \{1,2\} \text{or } X, \\ 2 & \text{if } A = \{0,1\} \text{or } \{0,2\}, \\ 1 & \text{if } A = \{1\} \text{or } \{2\} \\ 0 & \text{if } A = \{0\}. \end{cases}$$

Since $\overline{\mu}(\{1\} \cup \{0\}) = 2$, $\overline{\mu}(\{1\}) = 1$, and $\mu(\{0\}) = 0$, the null-additivity is not satisfied.

3 Relations from (pseudo-)almost uniform convergence

The following propositions and corollaries give implication relations of (pseudo-) almost uniform convergence to other convergences. Proposition 3 is obviously derived from [24]. (i) and (ii) in Propositions 3 and 4 are dual to each other; one is derived from the other by Ordinal Duality Principle. On the other hand, (i) and (ii) in Cororallies 1 and 2 are not dual in the sense of Definition 12.

Proposition 3 *(i) Null-subtractivity at the whole set is a necessary and sufficient condition for almost uniform convergence to imply pseudo-almost everywhere convergence; that is,* μ *is null-subtractive at* X *iff* $f_n \xrightarrow{\text{a.u.}} f$ *implies* $f_n \xrightarrow{\text{p.a.e.}} f$.

(ii) Converse-null-additivity at the whole set is a necessary and sufficient condition for pseudo-almost uniform convergence to imply almost everywhere convergence.

Corollary 1 *(i) Null-additivity is a necessary and sufficient condition that, for every measurable set A, almost uniform convergence on A implies pseudo-almost everywhere convergence on A.*

(ii) Converse-null-additivity is a necessary and sufficient condition that, for every measurable set A, pseudo-almost uniform convergence on A implies almost everywhere convergence on A.

Proposition 4 *(i) The following statements are equivalent.*

(a) *The non-additive measure is monotone autocontinuous from below at the whole set.*

(b) *Almost uniform convergence implies pseudo-almost uniform convergence.*

(c) *Almost uniform convergence implies convergence pseudo-in measure.*

(ii) The following statements are equivalent.

(a) *The non-additive measure is converse-monotone autocontinuous from below at the whole set.*

(b) *Pseudo-almost uniform convergence implies almost uniform convergence.*

(c) *Pseudo-almost uniform convergence implies convergence in measure.*

Proof. By Ordinal Duality Principle, it suffices to prove (i).
(a) ⇒ (b). If $f_n \xrightarrow{\text{a.u.}} f$, then for every m there exists a strictly increasing sequence $\{n_k^m\}_{k=1}^{\infty}$ such that

$$\mu\left(\bigcup_{k=1}^{\infty}\bigcup_{i=n_k^m}^{\infty}\left\{x \,\middle|\, |f_i(x)-f(x)| \geq \frac{1}{k}\right\}\right) < \frac{1}{m}. \tag{1}$$

Define a doubly-indexed sequence $\left\{a_k^m\right\}$ by $a_k^m = \max\ \left\{n_k^1, n_k^2, \ldots, n_k^m\right\}$ for each k. We put $N_m = \bigcup_{k=1}^{\infty} \bigcup_{i=a_k^m}^{\infty} \{x \mid |f_i(x) - f(x)| \geq 1/k\}$. If $l \geq m$, then $a_k^l \geq a_k^m$ for all k. Hence $\{N_m\}$ is a decreasing sequence, and from (1) it follows that $\mu(N_m) \to 0$ as $m \to \infty$. By monotone autocontinuity from below at X, we obtain $\mu(X \setminus N_m) \to \mu(X)$ as $m \to \infty$, and obviously f_n converges to f uniformly on $X \setminus N_m$. Therefore we have $f_n \xrightarrow{\text{p.a.u.}} f$.
(b) ⇒ (a). Let $N_n \downarrow N$ and $\mu(N_n) \to 0$ as $n \to \infty$, and define a sequence $\{f_n\}$ of measurable functions by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in X \setminus N_n, \\ 1 & \text{if } x \in N_n, \end{cases} \qquad (n \geq 1)$$

and a measurable function f by

$$f(x) = \begin{cases} 0 & \text{if } x \in X \setminus N, \\ 2 & \text{if } x \in N. \end{cases}$$

Then f_n converges to f uniformly on $X \setminus N_n$, and since $\mu(N_n) \to 0$, we have $f_n \xrightarrow{\text{a.u.}} f$. By hypothesis, it follows that $f_n \xrightarrow{\text{p.a.u.}} f$. Thus, for every $\xi < \mu(X)$, there exists N'_ξ such that $\xi < \mu(X \setminus N'_\xi)$ and f_n converges to f uniformly on $X \setminus N'_\xi$. By the definitions of f_n and f, there exists n such that $X \setminus N'_\xi \subset X \setminus N_n$ and hence $\mu(X \setminus N'_\xi) \leq \mu(X \setminus N_n)$. Therefore $\mu(X \setminus N_n) \to \mu(X)$ as $n \to \infty$.
(a) ⇒ (c). From the proof of (a) ⇒ (b), if $f_n \xrightarrow{\text{a.u.}} f$, then $f_n \xrightarrow{\text{p.a.u.}} f$. From [24], it holds that $f_n \xrightarrow{\text{p.a.u.}} f$ implies $f_n \xrightarrow{\text{p.}\mu} f$ without additional conditions. Therefore $f_n \xrightarrow{\text{a.u.}} f$ implies $f_n \xrightarrow{\text{p.}\mu} f$.
(c) ⇒ (a) It is similar to the proof of (b) ⇒ (a). □

By Proposition 4, we immediately obtain the following corollary.

Corollary 2 *(i) The following statements are equivalent.*

(a) *The non-additive measure is monotone autocontinuous from below at every measurable.*

(b) *For every measurble set A, almost uniform convergence on A implies pseudo-almost uniform convergence on A.*

(c) *For every measurble set A, almost uniform convergence on A implies convergence pseudo-in measure on A.*

(ii) The following statements are equivalent.

(a) *The non-additive measure is converse-monotone autocontinuous from below at every measurable set.*

(b) *For every measurble set A, pseudo-almost uniform convergence on A implies almost uniform convergence on A.*

(c) *For every measurble set A, pseudo-almost uniform convergence on A implies convergence in measure on A.*

The results in the previous propositions and corollaries are summarized together with existing ones [4, 6, 8, 10, 15, 18, 22, 23, 24] into Tables 2 and 3.

Table 2 shows necessary and sufficient conditions for implications between the six convergences on the whole set X. The cell at row `r` and column `c` indicates a necessary and sufficient condition for `r`-type convergence to imply `c`-type convergence; for example, condition (E) is a necessary and sufficient condition for almost everywhere convergence to imply almost uniform convergence. The symbol $\varnothing$ indicates the implication holds unconditionally. A cell ▭ shows a condition for the Riesz-type theorem; for example, property (S) is a necessary and sufficient condition that $f_n \xrightarrow{\mu} f$ implies that there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $f_{n_i} \xrightarrow{\text{a.e.}} f$ as $i \to \infty$.

Table 3 shows necessary and sufficient conditions for implications between the six convergences on every measurable set; for example, condition (E) is a necessary and sufficient condition that, for every $A \in \mathscr{S}$, $f_n \xrightarrow[A]{\text{a.e.}} f$ implies $f_n \xrightarrow[A]{\text{a.u.}} f$. Table 3 is derived from Table 2.

The results indicating (TS) X and (TPS) X in Table 2 and (TS) and (TPS) in Table 3 are derived from [10, Theorem 5 and its proof] by removing the assumption of continuity of non-additive measures. Each of the other results without reference number follows from the result (and its proof) in the corresponding cell of the other table. For example, "(a.e. $\Rightarrow$ p. in meas.) $\Leftrightarrow$ s.m.auto.$\uparrow$ X" in Table 2 is derived from "$\forall A \in \mathscr{S}$ (a.e. on $A \Rightarrow$ p. in meas. on A) $\Leftrightarrow$ (0-add. & $\uparrow$)" in Table 3 and its proof in [15]; in this case, the equivalence "(0-add. & $\uparrow$) $\Leftrightarrow$ s.m.auto. $\uparrow$" holds (Proposition 5 below).

Table 2: Necessary and sufficient conditions for implications between convergences on the whole set X

$\mathtt{r} \Rightarrow \mathtt{c}$	a.e.	p.a.e.	a.u.	p.a.u.	in meas.	p. in meas.
a.e.	$\varnothing$	0-sub. X	(E) [4]	(TE) X [8]	$\downarrow 0$	s.m.auto.$\uparrow X$
p.a.e.	c.0-add. X	$\varnothing$	(TPE) X [8]	(PE) X [8]	s.c.m.auto.$\uparrow X$	$\uparrow \mu(X)$
a.u.	$\varnothing$ [22]	0-sub. X [Prop.3(i)]	$\varnothing$	m.auto.$\uparrow$ X [Prop.4(i)]	$\varnothing$ [22]	m.auto.$\uparrow$ X [Prop.4(i)]
p.a.u.	c.0-add. X [Prop.3(i)]	$\varnothing$	c.m.auto.$\uparrow$ X [Prop.4(ii)]	$\varnothing$	c.m.auto.$\uparrow$ X [Prop.4(ii)]	$\varnothing$
in meas.	(S)	(TS) X	(S1)	(S2) X	$\varnothing$	auto.$\uparrow$ X
p. in meas.	(TPS) X	(PS) X	(PS1) X	(PS2) X	c.auto.$\uparrow$ X	$\varnothing$

Table 3: Necessary and sufficient conditions for implications between convergences on all measurable sets

$\forall A(\mathtt{r} \Rightarrow \mathtt{c})$	a.e.	p.a.e.	a.u.	p.a.u.	in meas.	p. in meas.
a.e.	$\varnothing$	0-add. [23, 24]	(E)	(TE)	$\downarrow 0$ [15]	0-add. & $\uparrow$ [15]
p.a.e.	c.0-add. [23, 24]	$\varnothing$	(TPE)	(PE)	s.c.m.auto.$\uparrow$ [15]	$\uparrow$ [15]
a.u.	$\varnothing$	0-add. [Cor.1(i)]	$\varnothing$	m.auto.$\uparrow$ [Cor.2(i)]	$\varnothing$	m.auto.$\uparrow$ [Cor.2(i)]
p.a.u.	c.0-add. [Cor.1(ii)]	$\varnothing$ [24]	c.m.auto.$\uparrow$ [Cor.2(ii)]	$\varnothing$	c.m.auto.$\uparrow$ [Cor.2(ii)]	$\varnothing$ [24]
in meas.	(S) [18]	(TS)	(S1) [10]	(S2) [10]	$\varnothing$	auto.$\uparrow$ [23]
p. in meas.	(TPS)	(PS) [6]	(PS1) [10]	(PS2) [10]	c.auto.$\uparrow$ [23]	$\varnothing$

Figure 1 shows some implications among conditions in Table 2. Most of implications in the diagram are obvious. For the sake of simplicity, we omit from the diagram some implications obtained similarly to others. For example, "(PS) & c.0-add. $X \Rightarrow$ (TPS) X" is omitted since it is the dual of "(S) & 0-sub. $X \Rightarrow$ (TS) X" in the diagram; "(TS) X & c.0-add. $X \Rightarrow$ (S)" also is omitted since it is obtained by an application of duality principle to the proof of "(S) & 0-sub. $X \Rightarrow$ (TS) X"; similarly, "(TS) X & c.auto.$\uparrow X$ $\Rightarrow$ (PS) X" and "(TPS) X & auto.$\uparrow X \Rightarrow$ (S)" also are omitted. The implication "(S) & $\downarrow 0 \Rightarrow$ (E)" in the diagram is shown in [7, 13], "(E) $\Rightarrow \downarrow 0$" is in [13], and "(S1) $\Rightarrow$ (S)" is in [10]. Figure 2 gives some implications among conditions in Table 3 and is derived from Figure 1; "$\downarrow \emptyset$ & 0-add. $\Rightarrow \downarrow 0$" is pointed out in [3]. These two diagrams may be imcomplete; there may be other implications.

Proposition 5 *Null-additivity and continuity from below are satisfied iff strongly monotone autocontinuity from below is satisfied.*

Proof. By Figure 1 we have that

$$[\text{0-sub.}X \,\&\, \uparrow \mu(X)] \;\Rightarrow\; \text{s.m.auto.}\uparrow X \;\Rightarrow\; [\text{0-sub.}X \,\&\, \uparrow X]. \tag{2}$$

Considering every subspace $(A, \mathscr{S} \cap A, \mu \upharpoonright (\mathscr{S} \cap A))$, we obtain

$$\begin{aligned} \forall A \in \mathscr{S}[\text{0-sub.}A \,\&\, \uparrow \mu(A)] \;&\Rightarrow\; \forall A \in \mathscr{S}[\text{s.m.auto.}\uparrow A] \\ &\Rightarrow\; \forall A \in \mathscr{S}[\text{0-sub.}A \,\&\, \uparrow A]. \end{aligned}$$

Then "$(\text{0-add. } \& \uparrow) \Leftrightarrow$ s.m.auto.$\uparrow$" follows from the facts

$$\begin{aligned} &\forall A \in \mathscr{S}[\text{0-sub.}A] \;\Leftrightarrow\; \text{0-add.}, \\ &\forall A \in \mathscr{S}[\uparrow \mu(A)] \;\Leftrightarrow\; \forall A \in \mathscr{S}[\uparrow A] \;\Leftrightarrow \uparrow, \\ &\forall A \in \mathscr{S}[\text{s.m.auto.}\uparrow A] \;\Leftrightarrow\; \text{s.m.auto.}\uparrow. \qquad \square \end{aligned}$$

Note that in (2) neither converse holds. In addition, the dual-like implications

$$(\text{c.0-add. } \& \downarrow 0) \;\Rightarrow\; \text{s.c.m.auto.}\uparrow \;\Rightarrow\; (\text{c.0-add. } \& \downarrow \emptyset)$$

hold, and neither converse holds.

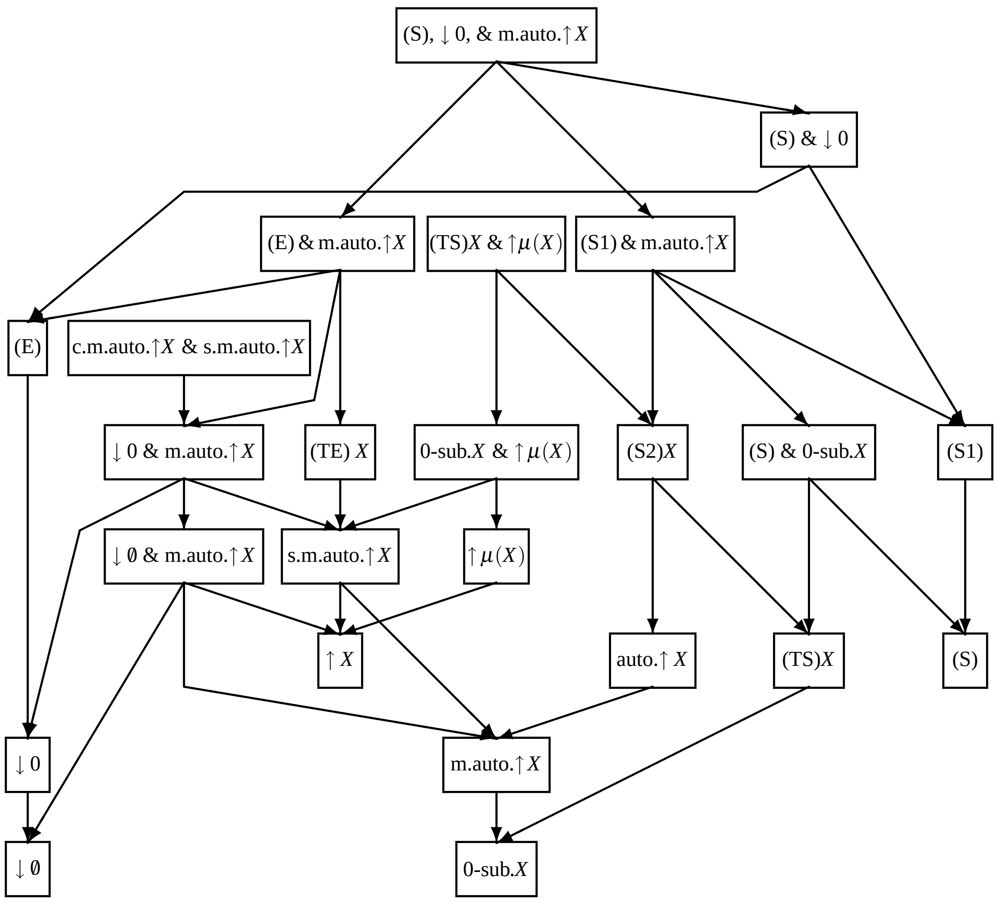

Figure 1: Implication relationship among conditions in Table 2

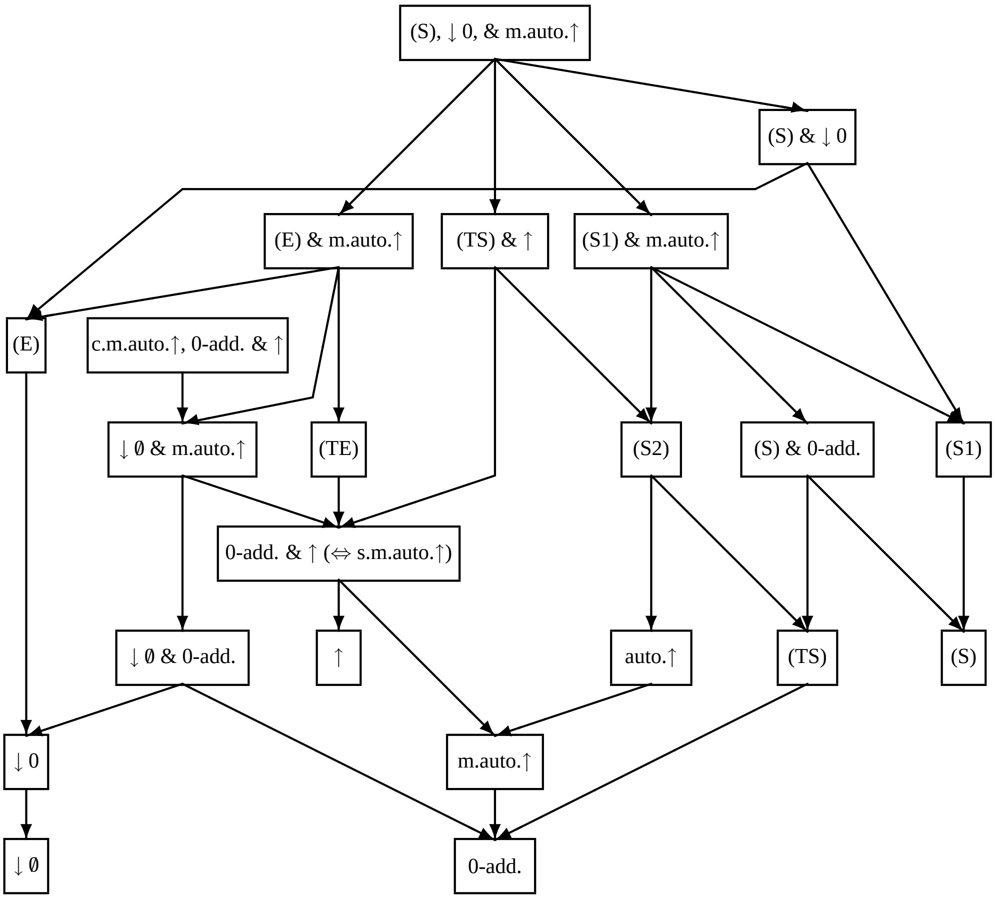

Figure 2: Implication relationship among conditions in Table 3

4 A new neccesary and sufficient condition for the Egoroff theorem

In this chapter, we show a new necessary and suffucient condition for the Egoroff theorem.

Lemma 1 *Let* $\{E_{m,n}\}$ *be a doubly-indexed sequence,* $E_{m,n} \supset E_{m',n'}$ *for* $m \geq m'$ *and* $n \leq n'$*, and* $\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n} = \emptyset$*. Then there exists a sequence* $\{A_n\}$ *of measurable sets such that* $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \emptyset$*, and for every strictly increasing sequence* $\{k_n\}$ *of positive integers, there exists a non-decreasing sequence* $\{n_i\}$ *of positive integers such that* $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$*.*

Proof. Let $\{E_{m,n}\}$ be a doubly-indexed sequence of sets, $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, and $\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n} = \emptyset$.

When there exists a non-decreasing sequence $\{n_i\}$ such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} = \emptyset$, for each positive integer i, let $A_i = E_{i,n_i}$, then it holds that $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \emptyset$ and for every strictly increasing sequence $\{k_n\}$ of positive integers, $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$.

We consider the case that for every non-decreasing sequence $\{n_i\}$ of positive integers, it holds that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \neq \emptyset$. Let $x \in \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,i}$ be fixed arbitrarily. Since $\bigcap_{n=1}^{\infty} E_{m,n} = \emptyset$ for each positive integer m, for every positive integer l, there exists a positive integer k such that $x \notin A_{l,k}$. We denote $j_x^l = \min_k \{k | x \notin A_{l,k}\}$ for each positive integer l. Since for every integer n, $A_{m,n} \uparrow \bigcup_{m=1}^{\infty} A_{m,n}$ $(m \to \infty)$, for every $x \in \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,i}$ and positive integer l, $x \notin A_{l+1,j_x^{l+1}}$ implies $x \notin A_{l,j_x^{l+1}}$. Therefore, for every $x \in \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,i}$, $\{j_x^l\}$ is a non-decreasing sequence of positive integers. In addition, for every $x \in \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,i}$, it holds that $\lim_{l \to \infty} j_x^l = \infty$. Because if it is false, there exists a positive integer k such that for every positive integer l, $k \geq j_x^l$. Therefore for every positive integer j, it holds that $x \notin \bigcap_{i=j}^{\infty} E_{i,i}$. However, this contradicts $x \in \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,i}$, so it holds that $\lim_{l \to \infty} j_x^l = \infty$. For each positive integer n, define A_n by

$$A_n = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} E_{i,i} \setminus \left(\bigcup_{m=1}^{\infty} (E_{m,n-1} \setminus E_{m,n}) \right).$$

For any fixed positive integers l and n, and for any fixed point $x \in \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} E_{i,i}$, $j_x^l = n$ is equivalent to the conjunction of $x \notin E_{l,n}$ and $x \in E_{l,n-1}$. By contraposition, $j_x^l \neq n$ is equivalent to either $x \in E_{l,n}$ or $x \notin E_{l,n-1}$. From this fact, we have the next result.

$$\begin{aligned} x \in A_n \quad &\Leftrightarrow \quad x \in \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty} E_{i,i}, \text{ and for every positive integer } l,\ x \notin E_{l,n-1} \setminus E_{l,n}. \\ &\Leftrightarrow \quad x \in \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty} E_{i,i}, \text{ and for every positive integer } l, \text{ either } x \in E_{l,n} \text{ or } x \notin E_{l,n-1}. \\ &\Leftrightarrow \quad x \in \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty} E_{i,i}, \text{ and for every positive integer } l,\ j_x^l \neq n. \end{aligned}$$

Therefore for each positive integer n,

$$A_n = \{x | x \in \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty} E_{i,i}, \text{ and for every positive integer } l,\ j_x^l \neq n\}.$$

Since $\lim\limits_{l\to\infty} j_x^l = \infty$, for every positive integer n, and for every $x \in A_n$, there exists positive integer k such that $j_x^k \geq n$. Since $x \notin A_{j_x^k}$, it holds that $\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} A_i = \emptyset$. For any fixed strictly increasing sequence $\{k_n\}$ of positive integers, we define sequences $\{a_n\}$ and $\{b_n\}$ by

$$a_1 = 1,\ a_n = b_{n-1} + 1\ (n = 2,\ 3, \ldots),\ b_n = k_{a_n}\ (n = 1,\ 2, \ldots).$$

We will show that

$$\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} E_{i,b_i} \subset \bigcup_{n=1}^{\infty}\bigcap_{i=a_n}^{b_n} A_i.$$

Since for every positive integer i, $i < k_{a_i}$, it holds that $\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} E_{i,b_i} \subset \bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} E_{i,i}$. Let

$$x \in \bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} E_{i,b_i}.$$

At first we consider the case that $j_x^1 > b_1$. Let $i \leq b_1$, then $i < j_x^1$. $\{j_x^l\}$ is a non-decreasing sequence of positive integers. Therefore, for every positive integer l, it holds that $j_x^l \neq i$. Therefore for every $i \leq b_1$, $x \in A_i$, namely, $x \in \bigcap_{i=a_1}^{b_1} A_i$. Secondly we consider the case that $j_x^1 \leq b_1$. Assume that for every positive integer k, there exists a positive integer $i \geq k$ such that $j_x^i \leq b_i$. Since for every positive integer m, $E_{m,n} \downarrow \bigcap_{n=1}^{\infty} E_{m,n}$ (as $n \to \infty$), for every positive integer k, there exists a positive integer $i \geq k$ such that $E_{i,b_i} \subset E_{i,j_x^i}$ by assumption. For every positive integer i, $x \notin E_{i,j_x^i}$, and so for every positive integer k, $x \notin \bigcap_{i=k}^{\infty} E_{i,b_i}$. This contradicts $x \in \bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} E_{i,b_i}$. Therefore, there exists a positive integer k such that for every positive integer $i \geq k$, $j_x^i > b_i$. Let $k' = \min\limits_{k}\{k | \text{for every } i \geq k,\ j_x^i > b_i\}$. Then, it holds that $j_x^{k'-1} \leq b_{k'-1}$. If $k' = 1$, then $j_x^1 > b_1$. This is inconsistent with assumption. Therefore $k' \neq 1$. As a result, $j_x^{k'} > b_{k'} > b_{k'-1} \geq$

$j_x^{k'-1}$. Let j be $b_{k'} \geq j \geq b_{k'-1}+1$. Then, $j_x^{k'} > j > j_x^{k'-1}$. Since $\{j_x^l\}$ is a non-decreasing sequence of positive integers, therefore, $x \in A_j$. Since $a_{k'} = b_{k'-1}+1$, $x \in \bigcap_{i=a'_k}^{b'_k} A_i$. This means that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,b_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=a_n}^{b_n} A_i$. Obviously, $\bigcup_{n=1}^{\infty} \bigcap_{i=a_n}^{b_n} A_i \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$. Therefore, $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,b_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$. $\square$

Next, we show Lemma 2, which is a generalization of Lemma 1.

Lemma 2 *For every doubly-indexed sequence $E_{m,n}$ such that $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, there exists a sequence $\{A_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$, and for every strictly increasing sequence $\{k_n\}$, there exists a non-decreasing sequence $\{n_i\}$ such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$.*

It does not necessarily hold that for every doubly-indexed sequence $E_{m,n}$ such that $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, there exists a non-decreasing sequence $\{n_i\}$ of positive integers such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$. The following example shows that fact.

Example 4 Let $X = [0,1]$. For each positive integer n, define a functin $f_n : \mathbb{N} \to \mathbb{N}$ by $f_n(m)$ is one's digit of $m/10^{n-1}$ in the decimal system. For example, $f_3(9876) = 8$. In addition, for each positive integer n, define $g(m,n) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m,n) = (n+m-1)(n+m-2)/2+n.$$

Then, $g(m,n)$ is a bijection. For each $(n_1,n_2,\dots) \in \mathbb{N}^{\mathbb{N}}$ and positive integer n, define $a_n^{(n_1,n_2,\dots)}$ by

$$a_n^{(n_1,n_2,\dots)} = f_j(n_i),$$

where $(i,j) = g^{-1}(n)$. By using $a_n^{(n_1,n_2,\dots)}$, define $x_{(n_1,n_2,\dots)}$ by

$$x_{(n_1,n_2,\dots)} = \sum_{n=1}^{\infty} a_n^{(n_1,n_2,\dots)}/10^n.$$

Obviously, $x_{(n_1,n_2,\dots)} \in X$. The representation of $x_{(n_1,n_2,\dots)}$ is unique. In other words, for every integer i, there exists a positive integer $j \geq i$ such that $a_j^{(n_1,n_2,\dots)} = 0$. Assume that there exists a positive integer i such that for every positive integer $j \geq i$, $a_j^{(n_1,n_2,\dots)} \neq 0$. Since $g(1,n)$ is strictly increasing for each n, there exists a positive integer h such that for every positive integer $k \geq h$, $g(1,k) \geq i$. Therefore, for every positive integer $k \geq h$, $f_k(n_1) = a_{g(1,k)}^{(n_1,n_2,\dots)} \neq 0$. This contradicts that n_1 is a positive integer. Hence, if $(n_1,n_2,\dots) \neq (m_1,m_2,\dots)$, there exists a positive integer k such that $n_k \neq m_k$. Thus, since there exists a positive integer l such that $f_l(n_k) \neq f_l(m_k)$, it holds that $x_{(n_1,n_2,\dots)} \neq x_{(m_1,m_2,\dots)}$.

For every positive integers m and n, let

$$A_{m,n} = \{x_{(n_1,n_2,\dots)} | n_m = n, \text{ and } \{n_i\} \text{ is a non-decreasing sequence of positive integers}\}.$$

By using $\{A_{m,n}\}$, define $E_{m,n}$ by

$$E_{m,n} = \bigcup_{i=1}^{m} \bigcup_{j=n}^{\infty} A_{i,j}.$$

Obviously, $E_{m,n}$ is increasing for m and decreasing for n. At first, we show that

$$\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n} = \emptyset.$$

Let a positive integer m be fixed arbitrarily. Let $x_{(n_1,n_2,\dots)} \in E_{m,1}$, then $x_{(n_1,n_2,\dots)} \in A_{m,n_m}$. However, for every positive integer i, $x_{(n_1,n_2,\dots)} \notin A_{m,n_m+i}$. Since $\{n_i\}$ is non-decreasing, for every integers $k \leq m$ and i, $x_{(n_1,n_2,\dots)} \notin A_{k,n_m+i}$. Therefore, $x_{(n_1,n_2,\dots)} \notin E_{m,n_m+i}$. This means that for every integer m, $\bigcap_{n=1}^{\infty} E_{m,n} = \emptyset$. Secondly, we show that for each non-decreasing sequence $\{n_i\}$ of positive integers,

$$\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \neq \emptyset.$$

Let $\{n_i\}$ be an arbitrary non-decreasing sequence of positive integers. For each positive integer k, it holds that $x_{(n_1,n_2,\dots)} \in A_{k,n_k} \subset E_{k,n_k}$, and so that

$$x_{(n_1,n_2,\cdots)} \in \bigcap_{i=1}^{\infty} E_{i,n_i}.$$

Obviously $\bigcap_{i=1}^{\infty} E_{i,n_i} \subset \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i}$. Therefore, $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \neq \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$.

By using Lemma 2, we show the next theorem.

Theorem 1 *Condition (M) is equivalent to the Egoroff condition.*

Proof. Assume that condition (M) is satisfied. Let $\{E_{m,n}\}$ be a doubly-indexed sequence of sets, $E_{m,n} \supset E_{m',n'}$ for $m \geq m'$ and $n \leq n'$, and $\mu(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}) = 0$. Assume that there exists a positive number ε such that for every non-decreasing sequence $\{h_i\}$ of positive integers, $\mu(\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,h_i}) \geq \varepsilon$. By Lemma 2, there exists a sequence $\{A_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$, and for every strictly increasing sequence $\{k_n\}$, there exists a non-decreasing sequence $\{n_i\}$ such that $\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} E_{i,n_i} \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{k_n} A_i$. Since $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} E_{m,n}$, we have

$\mu(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}A_i) = 0$. An application of condition (M) to $\{A_n\}$ yields that there exists a strictly increasing sequence $\{l_n\}$ of positive integers such that

$$\mu(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{l_n}A_i) < \varepsilon.$$

There exists a non-decreasing sequence $\{n_i\}$ of positive integers such that $\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty}E_{i,n_i} \subset \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{l_n}A_i$. This contradicts $\mu(\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty}E_{i,n_i}) \geq \varepsilon$. Therefore, for every positive number ε, there exists a non-decreasing sequence $\{h_i\}$ of positive integers such that $\mu(\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty}E_{i,h_i}) < \varepsilon$.

We show that there exists a sequence $\{k_i\}$ of positive integers such that

$$\mu(\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty}E_{i,k_i}) = 0.$$

For each positive integer l, there exists a non-decreasing sequence $\{k_i^l\}$ of positive integers such that

$$\mu(\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty}E_{i,k_i^l}) < \frac{1}{l}.$$

For each positive integer m, define a sequence $\{k_m\}$ by

$$k_m = \sum_{j=1}^{m} k_m^j.$$

Let a positive integer l and a nonnegative integer h be fixed arbitrarily. Since

$$k_{l+h} = \sum_{j=1}^{l+h} k_{l+h}^j > k_{l+h}^l,$$

it follows that

$$E_{l+h,k_{l+h}} \subset E_{l+h,k_{l+h}^l}.$$

Therefore,

$$\bigcap_{i=l+h}^{\infty} E_{i,k_i} \subset \bigcap_{i=l+h}^{\infty} E_{i,k_i^l}.$$

This means that

$$\bigcup_{j=l}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i} \subset \bigcup_{j=l}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i^l}.$$

Since

$$\bigcup_{j=l}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i} = \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i} \text{ and } \bigcup_{j=l}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i^l} = \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i^l},$$

we have

$$\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i} \subset \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i^l}.$$

As a result, for every positive integers l,

$$\mu(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i}) < \frac{1}{l},$$

thus,

$$\mu(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i}) = 0.$$

We show that the Egoroff condition is satisfied. Let a positive number ε be fixed arbitrarily. We can choose a sequence $\{k_n\}$ of positive integers such that $\mu(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{i,k_i}) = 0$. For each positive integer i, let $A_i = E_{i,k_i}$, then $\mu(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}A_i) = 0$. An application of condition (M) to $\{A_n\}$ yields that there exists a sequence $\{h_j\}$ of positive integers such that

$$\mu(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{h_j}A_i) < \varepsilon.$$

Since $E_{m,n}$ is increasing with respect to m and decreasing with respect to n,

$$\bigcap_{i=j}^{h_j}A_i = \bigcap_{i=j}^{h_j}E_{i,k_i} \supset E_{j,k_i} \supset E_{j,k_{h_j}}.$$

For this reason,

$$\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{h_j}A_i \supset \bigcup_{j=1}^{\infty}E_{j,k_{h_j}},$$

and so

$$\mu(\bigcup_{j=1}^{\infty}E_{j,k_{h_j}}) < \varepsilon.$$

Thus, μ satisfies the Egoroff condition.

Conversely, assume that μ satisfies condition (M). Let a sequence $\{A_n\}$ of measurable set satisfy $\mu(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}A_i) = 0$. Define a sequence $\{E_{m,n}\}$ as

$$E_{m,n} = \begin{cases} \bigcup_{i=n}^{m}A_i & \text{if } m > n, \\ \bigcap_{i=m}^{n}A_i & \text{if } m \le n. \end{cases}$$

Then the sequence $\{E_{m,n}\}$ is increasing with respect to m and decreasing with respect to n. Since $\mu(\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}E_{m,n}) = \mu(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}A_i) = 0$, it follows that for every positive number ε, there exists a sequence $\{n_m\}$ such that $\mu(\bigcup_{m=1}^{\infty} E_{m,n_m}) < \varepsilon$. We can let $\{n_m\}$ satisfy $n_m \geq m$. Then $\bigcup_{m=1}^{\infty} E_{m,n_m} = \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{n_m} A_i$, therefore $\mu(\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{n_m} A_i) < \varepsilon$. $\square$

The following proposition and collorary are obvious by Theorem 1 and the duality principle.

Proposition 6 *(i) Null-subtractivity at the whole set and condition (PM) at the whole set are satisfied iff almost everywhere convergence implies pseudo-almost uniformly convergence.*

(ii) Condition (PM) at the whole set is satisfied iff pseudo-almost everywhere convergence implies pseudo-almost uniformly convergence.

(iii) Conversel-null-additivity at the whole set and condition (M) are satisfied iff pseudo-almost everywhere convergence implies almost uniformly convergence.

Corollary 3 *(i) Null-additivity and condition (PM) are satisfied iff for every measurable set A, almost everywhere convergence on A implies pseudo-almost uniformly convergence on A.*

(ii) Condition (PM) is satisfied iff for every measurable set A, pseudo-almost everywhere convergence on A implies pseudo-almost uniformly convergence on A.

(iii) Conversel-null-additivity and condition (M) are satisfied iff for every measurable set A, pseudo-almost everywhere convergence on A implies almost uniformly convergence on A.

5 Conclusion

In this dissertation, we give necessary and sufficient conditions for convergence theorems whose necessary and sufficient conditions have not yet been found. Moreover, we have newly defined condition (M) which is described by a singly-indexed sequence of measurable sets and shown that condition (M) is equivalent to the Egoroff condition.

However, the usability of condition (M) has not yet clear, so it is important to clarify the usability of condition (M).

It is also an important subject to find necessary and sufficient conditions for other versions of convergence theorems. In non-additive measure theory, the concept of "almost everywhere" has the following interpretations:
(ae1) For every $x \in X$ exept x's in a measurable set N such that $\mu(N) = 0$ [5, 14, 24].
(ae2) For every $x \in X$ exept x's in a measurable set N such that $\mu(X \setminus N) = \mu(X)$[5, 24].
(ae3) For every $x \in X$ exept x's in a measurable set N such that $\mu(A \cup N) = \mu(A)$ for every measurable set A[12].

We have adopted (ae1) and (ae2) in this dissertation. Almost everywhere, almost uniformly,and in measure are interpretation of (ae1). Pseudo-almost everywhere, pseudo-almost uniformly,and pseudo-in measure are interpretations of (ae2).

In measure theory, (ae1) is equivalent to (ae3), (ae3) implies (ae2), and on finite measure space, (ae1), (ae2),and (ae3) are equivalent. However, they are not equivalent in non-additive measure theory. Therefore, with respect to almost everywhere, almost uniformly, and in measure, there are 81 different convergence theorems. For example, almost everywhere convergence of (ae1) (i.e. $f_n \xrightarrow{\text{a.e.}} f$) imples almost everywhere convergence of (ae3).

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