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Newton-Okounkov polytopes of Schubert varieties and crystal bases

(シューベルト多様体の Newton-Okounkov 多面体と結晶基底)

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Abstract

The theory of Newton-Okounkov polytopes gives a systematic method of constructing toric degenerations of projective varieties. In this thesis, we study Newton-Okounkov polytopes of Schubert varieties via crystal bases. Such researches were initiated by Kaveh, who realized Berenstein-Littelmann-Zelevinsky's string polytopes as Newton-Okounkov polytopes.

The main results of this thesis are three-fold. First, we prove that Nakashima-Zelevinsky's polyhedral realization of a highest weight crystal basis is identical to the Newton-Okounkov polytope of a Schubert variety associated with a specific valuation. Second, we relate string polytopes and polyhedral realizations with geometrically natural valuations, which are given by counting the orders of zeros along sequences of specific subvarieties. Finally, we apply the folding procedure to Newton-Okounkov polytopes, which relates Newton-Okounkov polytopes of Schubert varieties of different Dynkin types.

Contents

Introduction

Background and main results

A Newton-Okounkov body $\Delta(X, \mathcal{L}, v, \tau)$ is a convex body constructed from a polarized variety $(X, \mathcal{L})$ with a valuation v on its function field $\mathbb{C}(X)$ and with a nonzero section $\tau \in H^0(X, \mathcal{L})$; this generalizes the notion of Newton polytopes for toric varieties. The theory of Newton-Okounkov bodies was introduced by Okounkov [**54, 55, 56**], and afterward developed independently by Kaveh-Khovanskii [**33**] and by Lazarsfeld-Mustata [**41**]. If a Newton-Okounkov body is a convex polytope, then we call it a Newton-Okounkov polytope. A Newton-Okounkov body (polytope) $\Delta(X, \mathcal{L}, v, \tau)$ inherits information about algebraic, geometric, and combinatorial properties of the original projective variety X and the line bundle $\mathcal{L}$. Indeed, it encodes numerical equivalence information of the line bundle $\mathcal{L}$ (see [**22, 41**]). In addition, the theory of Newton-Okounkov polytopes gives a systematic method of constructing toric degenerations [**1**, Theorem 1] and integrable systems [**17**, Theorem B] (see Theorems 1.1.8, 1.1.10).

In this thesis, we study Newton-Okounkov polytopes of Schubert varieties via crystal bases in representation theory. Such researches were initiated by Kaveh [**32**], who proved that Berenstein-Littelmann-Zelevinsky's string polytope constructed from the string parametrization for a Demazure crystal is identical to the Newton-Okounkov polytope of a Schubert variety associated with a specific valuation. The main results of this thesis are three-fold (1)–(3).

(1) Newton-Okounkov polytopes and
polyhedral realizations of crystal bases:

The Kashiwara embedding gives a parametrization of a highest weight crystal basis, which yields an explicit description of Kashiwara operators. Under some technical assumptions, Nakashima described the image of the Kashiwara embedding as the set of lattice points in some explicit rational convex polytope; this description is called Nakashima-Zelevinsky's polyhedral realization of a crystal basis.

In this thesis, we relate the Kashiwara embedding with a specific valuation on the function field of a Schubert variety. From this, we deduce that Nakashima-Zelevinsky's polyhedral realization of a highest weight crystal basis is identical to the Newton-Okounkov polytope of a Schubert variety associated with a specific valuation. This result gives a new class of specific examples of Newton-Okounkov polytopes of Schubert varieties, which we can compute explicitly. As an application of this approach, we show without any assumptions that the image of the Kashiwara embedding is identical to the set of lattice points in some rational convex polytope. In addition, by combining our result with Kaveh's result, we see that Kashiwara's involution corresponds to a change of specific valuations. This is based on joint work with Satoshi Naito.

To be more precise, let G be a connected, simply-connected semisimple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, W the Weyl group, and I an index set for the vertices of the Dynkin diagram. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote by $\mathfrak{b}$ (resp., $\mathfrak{t}$) the Lie algebra of B (resp., T), and by $X(w) \subset G/B$ the Schubert variety corresponding to $w \in W$. It is well-known that $X(w)$ is an irreducible normal projective variety. Let $U^- \subset G$ be the unipotent radical of the opposite Borel subgroup B^-, $\{\alpha_i \mid i \in I\} \subset \mathfrak{t}^*$ the set of simple roots, $\{h_i \mid i \in I\} \subset \mathfrak{t}$ the set of simple coroots, and $P_+ \subset \mathfrak{t}^*$ the set of dominant integral weights for $\mathfrak{g}$. A dominant integral weight $\lambda \in P_+$ gives a line bundle $\mathcal{L}_\lambda$ on G/B generated by global sections; by restricting this bundle, we obtain a line bundle on $X(w)$, which we denote by the same symbol $\mathcal{L}_\lambda$. From the Borel-Weil type theorem, we know that the space $H^0(X(w), \mathcal{L}_\lambda)$ of global sections is a B-module isomorphic to the dual module $V_w(\lambda)^*$ of the Demazure module $V_w(\lambda)$ corresponding to w and λ. Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$, which induces the following birational morphism:

$$\mathbb{C}^r \to X(w),\ (t_1, \ldots, t_r) \mapsto \exp(t_1 f_{i_1}) \exp(t_2 f_{i_2}) \cdots \exp(t_r f_{i_r}) \bmod B,$$

where $e_i, f_i, h_i \in \mathfrak{g}$, $i \in I$, denote the Chevalley generators such that $\{e_i, h_i \mid i \in I\} \subset \mathfrak{b}$ and $\{f_i \mid i \in I\} \subset \mathfrak{u}^- := \mathrm{Lie}(U^-)$. By using this birational morphism, we identify the function field $\mathbb{C}(X(w))$ with the rational function field $\mathbb{C}(t_1, \ldots, t_r)$. Define a valuation $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$ on $\mathbb{C}(X(w))$ with values in $\mathbb{Z}^r$ to be the highest term valuation on $\mathbb{C}(t_1, \ldots, t_r)$ with respect to the lexicographic order $t_r \succ \cdots \succ t_1$ (see Definition 1.1.3). For $\lambda \in P_+$, let $\tau_\lambda \in H^0(G/B, \mathcal{L}_\lambda)$ be a lowest weight vector; by restricting this section, we obtain a section in $H^0(X(w), \mathcal{L}_\lambda)$, which we denote by the same symbol τ_λ. In this setting, we study the Newton-Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$.

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra, and $\mathcal{B}(\infty)$ the crystal basis of its negative half $U_q(\mathfrak{u}^-)$. Denote by $\mathcal{B}(\lambda)$ the crystal basis of the irreducible highest weight $U_q(\mathfrak{g})$-module $V_q(\lambda)$ with highest weight λ, and by $\mathcal{B}_w(\lambda) \subset \mathcal{B}(\lambda)$ the Demazure crystal corresponding to $w \in W$. In the theory of crystal bases, it is important to give their concrete realizations. Until now, many useful realizations have been discovered; the theory of Nakashima-Zelevinsky's polyhedral realizations is one of them. Take an infinite sequence $\mathbf{j} = (\ldots, j_k, \ldots, j_2, j_1)$ in I such that $j_k \neq j_{k+1}$ for all $k \in \mathbb{Z}_{>0}$, and such that the cardinality of $\{k \in \mathbb{Z}_{>0} \mid j_k = i\}$ is ∞ for each $i \in I$. Then, we can associate to $\mathbf{j}$ a crystal structure on

$$\mathbb{Z}^\infty := \{(\ldots, a_k, \ldots, a_2, a_1) \mid a_k \in \mathbb{Z} \text{ for } k \in \mathbb{Z}_{>0},\ \text{and } a_k = 0 \text{ for } k \gg 0\},$$

and obtain a strict embedding of crystals $\Psi_{\mathbf{j}} \colon \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^\infty$, called the Kashiwara embedding with respect to $\mathbf{j}$ (see Sect. 2.1). Nakashima-Zelevinsky

[**53**] described explicitly the image of $\mathcal{B}(\infty)$ under some technical assumptions on $\mathbf{j}$. Afterward, Nakashima [**50, 51**] gave a similar description of the Demazure crystal $\mathcal{B}_w(\lambda)$ under the assumption that $(\mathbf{j}, \lambda)$ is ample (see Definition 2.4.1). These descriptions of crystal bases are called polyhedral realizations. Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$, and extend it to an infinite sequence $\mathbf{j} = (\ldots, j_k, \ldots, j_2, j_1)$ as above, that is, $(j_r, \ldots, j_1) = (i_1, \ldots, i_r)$. For the Kashiwara embedding $\Psi_{\mathbf{j}}$ with respect to $\mathbf{j}$, the following holds (see Sect. 2.1):

$$\Psi_{\mathbf{j}}(\mathcal{B}_w(\lambda)) \subset \{(\ldots, a_k, \ldots, a_2, a_1) \in \mathbb{Z}^\infty \mid a_k = 0 \text{ for all } k > r\}.$$

We define $\Psi_{\mathbf{i}} \colon \mathcal{B}_w(\lambda) \hookrightarrow \mathbb{Z}^r$ by: $\Psi_{\mathbf{i}}(b) = (a_1, \ldots, a_r)$ if and only if $\Psi_{\mathbf{j}}(b) = (\ldots, 0, 0, a_1, \ldots, a_r)$. From the injective map $\Psi_{\mathbf{i}}$, we obtain a subset $\widetilde{\Delta}_{\mathbf{i}}(\lambda) \subset \mathbb{R}^r$ (see Definition 2.1.8). We call $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ the Nakashima-Zelevinsky polytope. As we will see, this is indeed a rational convex polytope (Corollary 2 below); here, we need not assume that $(\mathbf{j}, \lambda)$ is ample. In addition, by the theory of generalized string polytopes (see [**12**] for the definition), we deduce that $\widetilde{\Delta}_{\mathbf{i}}(\lambda) \cap \mathbb{Z}^r = \Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$ (see Corollary 2.1.13).

In order to relate the Newton-Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$ with the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$, we use the theory of perfect bases. Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$ be a perfect basis of $\mathbb{C}[U^-]$ (see Definition 2.2.3); this induces a $\mathbb{C}$-basis $\{\Xi^{\mathrm{up}}_{\lambda,w}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ of the space $H^0(X(w), \mathcal{L}_\lambda)$ of global sections (see Corollary 2.2.22). Write $\mathbf{a}^{\mathrm{op}} := (a_r, \ldots, a_1)$ for an element $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$, and $H^{\mathrm{op}} := \{\mathbf{a}^{\mathrm{op}} \mid \mathbf{a} \in H\}$ for a subset $H \subset \mathbb{R}^r$. The following is the first main result of this thesis.

Theorem 1 (Theorem 2.3.2). *Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$, and $\lambda \in P_+$.*

(1) *The Kashiwara embedding $\Psi_{\mathbf{i}}(b)$ is equal to $-\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi^{\mathrm{up}}_{\lambda,w}(b)/\tau_\lambda)^{\mathrm{op}}$ for all $b \in \mathcal{B}_w(\lambda)$.*
(2) *The Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is identical to the Newton-Okounkov body $-\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}$.*

Corollary 2 (Corollary 2.3.4). *The Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ and the Newton-Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$ are both rational convex polytopes.*

As an application of Theorem 1, we give an explicit form of the Newton-Okounkov polytope $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$. To be more precise, under the assumption that $(\mathbf{j}, \lambda)$ is ample, Nakashima's description of $\Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$ also gives a system of explicit affine inequalities defining the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda) = -\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}$.

Define a valuation $v_{\mathbf{i}}^{\mathrm{high}}$ on $\mathbb{C}(X(w))$ to be the highest term valuation on $\mathbb{C}(t_1, \ldots, t_r)$ with respect to the lexicographic order $t_1 > \cdots > t_r$ (see Definition 1.1.3). Kaveh [**32**] proved that the value $-v_{\mathbf{i}}^{\mathrm{high}}(\Xi^{\mathrm{up}}_{\lambda,w}(b)/\tau_\lambda)$ for $b \in \mathcal{B}_w(\lambda)$ is equal to the string parametrization $\Phi_{\mathbf{i}}(b)$ of b with respect to $\mathbf{i}$, and that the Newton-Okounkov body $-\Delta(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$

is identical to Berenstein-Littelmann-Zelevinsky's string polytope $\Delta_{\mathbf{i}}(\lambda)$. Let us consider the case that w is the longest element $w_0 \in W$. In this case, the Schubert variety $X(w_0)$ is just the full flag variety G/B, and the Demazure crystal $\mathcal{B}_{w_0}(\lambda)$ is just the crystal basis $\mathcal{B}(\lambda)$. We denote the section $\Xi^{\mathrm{up}}_{\lambda,w_0}(b) \in H^0(G/B, \mathcal{L}_\lambda)$ for $b \in \mathcal{B}(\lambda)$ simply by $\Xi^{\mathrm{up}}_{\lambda}(b)$. Let $\mathbf{i} = (i_1, \ldots, i_N) \in I^N$ be a reduced word for the longest element $w_0 \in W$, and define $\Psi_{\mathbf{i}} \colon \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^N$ as $\Psi_{\mathbf{i}} \colon \mathcal{B}(\lambda) = \mathcal{B}_{w_0}(\lambda) \hookrightarrow \mathbb{Z}^N$. As we will see, the image $\Psi_{\mathbf{i}}(\mathcal{B}(\infty))$ is identical to the set of lattice points in a certain rational convex polyhedral cone $\widetilde{\mathcal{C}}_{\mathbf{i}}$. Our result combined with the result of Kaveh above implies that Kashiwara's involution $* \colon \mathcal{B}(\infty) \to \mathcal{B}(\infty)$ corresponds to the change of valuations from $\tilde{v}^{\mathrm{high}}_{\mathbf{i}}$ to $v^{\mathrm{high}}_{\mathbf{i}^{\mathrm{op}}}$, which gives a geometric interpretation of $*$; here, we write $\mathbf{i}^{\mathrm{op}} := (i_N, \ldots, i_1)$. More precisely, we obtain the following.

Corollary 3 (Corollary 2.5.1). *Let $\mathbf{i} \in I^N$ be a reduced word for the longest element $w_0 \in W$. Then, there uniquely exists a piecewise-linear map $\eta_{\mathbf{i}} \colon \widetilde{\mathcal{C}}_{\mathbf{i}} \to \widetilde{\mathcal{C}}_{\mathbf{i}}$ satisfying the following conditions:*

(i) *the map $\eta_{\mathbf{i}}$ corresponds to Kashiwara's involution $*$ through the Kashiwara embedding $\Psi_{\mathbf{i}}$:*
$$b^* = \Psi_{\mathbf{i}}^{-1} \circ \eta_{\mathbf{i}} \circ \Psi_{\mathbf{i}}(b)$$
for all $b \in \mathcal{B}(\infty)$,

(ii) *the map $\eta_{\mathbf{i}}$ corresponds to the change of valuations from $\tilde{v}^{\mathrm{high}}_{\mathbf{i}}$ to $v^{\mathrm{high}}_{\mathbf{i}^{\mathrm{op}}}$:*
$$\eta_{\mathbf{i}}(-\tilde{v}^{\mathrm{high}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_{\lambda}(b)/\tau_\lambda)^{\mathrm{op}}) = -v^{\mathrm{high}}_{\mathbf{i}^{\mathrm{op}}}(\Xi^{\mathrm{up}}_{\lambda}(b)/\tau_\lambda)^{\mathrm{op}}$$
for all $\lambda \in P_+$ and $b \in \mathcal{B}(\lambda)$,

(iii) *the equality $\eta_{\mathbf{i}}^2 = \mathrm{id}_{\widetilde{\mathcal{C}}_{\mathbf{i}}}$ holds,*

(iv) *the map $\eta_{\mathbf{i}}$ induces a bijective piecewise-linear map from the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda) = -\Delta(G/B, \mathcal{L}_\lambda, \tilde{v}^{\mathrm{high}}_{\mathbf{i}}, \tau_\lambda)^{\mathrm{op}}$ onto the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\lambda)^{\mathrm{op}} = -\Delta(G/B, \mathcal{L}_\lambda, v^{\mathrm{high}}_{\mathbf{i}^{\mathrm{op}}}, \tau_\lambda)^{\mathrm{op}}$ for all $\lambda \in P_+$.*

(2) Geometrically natural valuations and perfect bases with positivity properties:

The specific valuations used by Kaveh and in (1) are defined algebraically to be highest term valuations. Another kind of valuation, which is geometrically natural, is given by counting the orders of zeros along a sequence of subvarieties. One is often focused on Newton-Okounkov bodies associated with such valuations (see, for instance, [**39**] and [**41**]). In this thesis, we relate the highest term valuations used by Kaveh and in (1) with such geometrically natural valuations. More precisely, we show that, on a perfect basis with some positivity properties, the highest term valuations are identical to the valuations coming from sequences of specific subvarieties of a Schubert variety; the existence of such a perfect basis follows from Khovanov-Lauda-Rouquier's categorification of the negative half $U_q(\mathfrak{u}^-)$ of the quantized enveloping algebra $U_q(\mathfrak{g})$. From these, we deduce that the associated

Newton-Okounkov polytopes coincide. This result gives new geometric interpretations of string polytopes and polyhedral realizations of crystal bases. This is based on joint work with Hironori Oya.

To be more precise, let X be an irreducible normal projective variety over $\mathbb{C}$ of complex dimension r. We consider a sequence of irreducible normal closed subvarieties

$$X_\bullet \colon X_r \subset X_{r-1} \subset \cdots \subset X_0 = X$$

such that $\dim_{\mathbb{C}}(X_k) = r - k$ for $0 \leq k \leq r$. By the normality assumption, there exists a collection $u_1, \ldots, u_r$ of rational functions on X such that the restriction $u_k|_{X_{k-1}}$ is a not identically zero rational function on X_{k-1} that has a zero of first order on the hypersurface X_k for every k (see Sect. 3.1). Out of such a collection $u_1, \ldots, u_r$ of rational functions, we construct a valuation $v_{X_\bullet} \colon \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^r$, $f \mapsto (a_1, \ldots, a_r)$, as follows. The first coordinate a_1 is the order of zeros of f on X_1. Then, we have $(u_1^{-a_1} f)|_{X_1} \in \mathbb{C}(X_1) \setminus \{0\}$, and the second coordinate a_2 is the order of zeros of $(u_1^{-a_1} f)|_{X_1}$ on X_2. Continuing in this way, we define all a_k. This is the definition of $v_{X_\bullet}$. It is natural to ask whether the valuation used by Kaveh (resp., in (1)) can be realized as a valuation of the form $v_{X_\bullet}$. This question was suggested by Kaveh in [**32**, Introduction (after Theorem 1)]. Our second main result in this thesis gives an answer to this question.

Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$, and set $w_{\geq k} \coloneqq s_{i_k} s_{i_{k+1}} \cdots s_{i_r}$, $w_{\leq k} \coloneqq s_{i_1} s_{i_2} \cdots s_{i_k}$ for $1 \leq k \leq r$, where $s_i \in W$, $i \in I$, denote the simple reflections. Then, we obtain two sequences of subvarieties of $X(w)$:

$$X(w_{\geq \bullet}) \colon X(e) \subset X(w_{\geq r}) \subset \cdots \subset X(w_{\geq 2}) \subset X(w_{\geq 1}) = X(w) \text{ and}$$
$$X(w_{\leq \bullet}) \colon X(e) \subset X(w_{\leq 1}) \subset \cdots \subset X(w_{\leq r-1}) \subset X(w_{\leq r}) = X(w),$$

where $e \in W$ is the identity element. Consider the valuations $v_{X(w_{\geq \bullet})}, v_{X(w_{\leq \bullet})}$ associated with these sequences.

Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$ be a perfect basis of $\mathbb{C}[U^-]$, and assume that this basis satisfies the following positivity conditions:

- $(\mathrm{P})_1$ the element $(-f_i) \cdot \Xi^{\mathrm{up}}(b)$ belongs to $\sum_{b' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}(b')$ for all $i \in I$ and $b \in \mathcal{B}(\infty)$;
- $(\mathrm{P})_2$ the product $\Xi^{\mathrm{up}}(b) \cdot \Xi^{\mathrm{up}}(b')$ belongs to $\sum_{b'' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}(b'')$ for all $b, b' \in \mathcal{B}(\infty)$ such that $\mathrm{wt}(b) \in \{-\alpha_i \mid i \in I\}$.

The existence of a perfect basis with the positivity properties $(\mathrm{P})_1$ and $(\mathrm{P})_2$ follows from a categorification of the negative half $U_q(\mathfrak{u}^-)$ of $U_q(\mathfrak{g})$ (see Proposition 3.2.3). Recall that this basis induces a $\mathbb{C}$-basis $\{\Xi^{\mathrm{up}}_{\lambda,w}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ of the space $H^0(X(w), \mathcal{L}_\lambda)$ of global sections, and that $\tau_\lambda \in H^0(X(w), \mathcal{L}_\lambda)$ is the restriction of a lowest weight vector in $H^0(G/B, \mathcal{L}_\lambda)$. The following is the second main result of this thesis.

Theorem 4 (see Theorem 2.3.2, Proposition 3.1.3, and Corollaries 2.2.21, 3.3.2). *Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$, $\lambda \in P_+$, and $b \in \mathcal{B}_w(\lambda)$.*

(1) *The value $v_{X(w_{\geq \bullet})}(\Xi^{\mathrm{up}}_{\lambda,w}(b)/\tau_\lambda)$ is equal to the Kashiwara embedding $\Psi_{\mathbf{i}}(b)$.*

(2) *The value* $v_{X(w_{\leq \bullet})}(\Xi^{\mathrm{up}}_{\lambda,w}(b)/\tau_\lambda)^{\mathrm{op}}$ *is equal to the string parametrization* $\Phi_{\mathbf{i}}(b)$.

COROLLARY 5 (see Theorems 1.4.6, 2.3.2, Proposition 3.1.3, and Corollary 3.3.3). *Let* $\mathbf{i} \in I^r$ *be a reduced word for* $w \in W$*, and* $\lambda \in P_+$.

(1) *The Newton-Okounkov body* $\Delta(X(w), \mathcal{L}_\lambda, v_{X(w_{\geq \bullet})}, \tau_\lambda)$ *is identical to the Nakashima-Zelevinsky polytope* $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$.
(2) *The Newton-Okounkov body* $\Delta(X(w), \mathcal{L}_\lambda, v_{X(w_{\leq \bullet})}, \tau_\lambda)^{\mathrm{op}}$ *is identical to Berenstein-Littelmann-Zelevinsky's string polytope* $\Delta_{\mathbf{i}}(\lambda)$.

(3) Folding procedure for
Newton-Okounkov polytopes of Schubert varieties:

Finally, we apply the folding procedure to Newton-Okounkov polytopes, which relates Newton-Okounkov polytopes of Schubert varieties of different Dynkin types. Since string polytopes and polyhedral realizations are realized as Newton-Okounkov polytopes of Schubert varieties, we can apply to these polytopes the folding procedure for Schubert varieties and also that for crystal bases. The folding procedure for Schubert varieties (resp., for crystal bases) relates these polytopes for a simply-laced semisimple Lie algebra with those for its fixed point Lie subalgebra (resp., for its orbit Lie algebra); the orbit Lie algebra is the Langlands dual of the fixed point Lie subalgebra. Since the simple Lie algebra of type B (resp., type C) is a fixed point Lie subalgebra of that of type D (resp., type A), and also is an orbit Lie algebra of that of type A (resp., type D), we obtain relations among Newton-Okounkov polytopes of Schubert varieties of types A, B, C, D. This leads to a new interpretation of Kashiwara's similarity between crystal bases in type B and those in type C.

To be more precise, assume that $\mathfrak{g}$ is of simply-laced type, and let $\omega\colon I \to I$ be a Dynkin diagram automorphism. In this thesis, for technical reasons, we always assume the following condition on ω:

(O) any two vertices of the Dynkin diagram in the same ω-orbit are not joined.

Such an ω induces a Lie algebra automorphism $\omega\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$, which preserves the Cartan subalgebra $\mathfrak{t}$. We know that the fixed point Lie subalgebra $\mathfrak{g}^\omega := \{x \in \mathfrak{g} \mid \omega(x) = x\}$ is also a semisimple Lie algebra. Fix a complete set $\breve{I}$ of representatives for the ω-orbits in I; the set $\breve{I}$ is identified with an index set for the vertices of the Dynkin diagram of $\mathfrak{g}^\omega$. There exists a natural injective group homomorphism $\Theta\colon \breve{W} \hookrightarrow W$ from the Weyl group of $\mathfrak{g}^\omega$ to that of $\mathfrak{g}$. If $\mathbf{i} = (i_1, \ldots, i_r) \in \breve{I}^r$ is a reduced word for $w \in \breve{W}$, then

$$\Theta(\mathbf{i}) := (i_{1,1}, \ldots, i_{1,m_{i_1}}, \ldots, i_{r,1}, \ldots, i_{r,m_{i_r}}) \in I^{m_{i_1} + \cdots + m_{i_r}}$$

is a reduced word for $\Theta(w)$, where we set

$$m_i := \min\{k \in \mathbb{Z}_{>0} \mid \omega^k(i) = i\}$$

for $i \in \check{I}$ and $i_{k,l} := \omega^{l-1}(i_k)$ for $1 \leq k \leq r$, $1 \leq l \leq m_{i_k}$. Let $\omega^* \colon \mathfrak{t}^* \xrightarrow{\sim} \mathfrak{t}^*$ be the dual of the $\mathbb{C}$-linear automorphism $\omega \colon \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}$, and set

$$(\mathfrak{t}^*)^0 := \{\lambda \in \mathfrak{t}^* \mid \omega^*(\lambda) = \lambda\}.$$

Note that an element $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$ naturally induces a dominant integral weight $\hat{\lambda}$ for $\mathfrak{g}^\omega$. Recall that Kaveh's result [**32**] and the result in (1) imply that

$$\Delta_{\mathbf{i}}(\hat{\lambda}) = -\Delta(X(w), \mathcal{L}_{\hat{\lambda}}, v_{\mathbf{i}}^{\mathrm{high}}, \tau_{\hat{\lambda}}), \qquad \Delta_{\Theta(\mathbf{i})}(\lambda) = -\Delta(X(\Theta(w)), \mathcal{L}_\lambda, v_{\Theta(\mathbf{i})}^{\mathrm{high}}, \tau_\lambda),$$
$$\widetilde{\Delta}_{\mathbf{i}}(\hat{\lambda}) = -\Delta(X(w), \mathcal{L}_{\hat{\lambda}}, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_{\hat{\lambda}})^{\mathrm{op}}, \quad \widetilde{\Delta}_{\Theta(\mathbf{i})}(\lambda) = -\Delta(X(\Theta(w)), \mathcal{L}_\lambda, \tilde{v}_{\Theta(\mathbf{i})}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}$$

for $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$, where $\mathbf{i} = (i_1, \ldots, i_r) \in \check{I}^r$ is a reduced word for $w \in \check{W}$. Define an $\mathbb{R}$-linear surjective map $\Omega_{\mathbf{i}} = \Omega_{\mathbf{i}}^{(\omega)} \colon \mathbb{R}^{m_{i_1}+\cdots+m_{i_r}} \twoheadrightarrow \mathbb{R}^r$ by

$$\begin{aligned}&\Omega_{\mathbf{i}}(a_{1,1}, \ldots, a_{1,m_{i_1}}, \ldots, a_{r,1}, \ldots, a_{r,m_{i_r}})\\ &:= (a_{1,1} + \cdots + a_{1,m_{i_1}}, \ldots, a_{r,1} + \cdots + a_{r,m_{i_r}}).\end{aligned}$$

The following is the third main result of this thesis.

Theorem 6 (Theorem 4.2.7). *Let $\mathfrak{g}$ be a simply-laced semisimple Lie algebra, $\omega \colon I \to I$ a Dynkin diagram automorphism satisfying condition* (O) *above, $\mathbf{i} = (i_1, \ldots, i_r) \in \check{I}^r$ a reduced word for $w \in \check{W}$, and $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$. Then, the following equalities hold*:

$$\Omega_{\mathbf{i}}(\Delta(X(\Theta(w)), \mathcal{L}_\lambda, v_{\Theta(\mathbf{i})}^{\mathrm{high}}, \tau_\lambda)) = \Delta(X(w), \mathcal{L}_{\hat{\lambda}}, v_{\mathbf{i}}^{\mathrm{high}}, \tau_{\hat{\lambda}}), \text{ and}$$
$$\Omega_{\mathbf{i}}(\Delta(X(\Theta(w)), \mathcal{L}_\lambda, \tilde{v}_{\Theta(\mathbf{i})}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}) = \Delta(X(w), \mathcal{L}_{\hat{\lambda}}, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_{\hat{\lambda}})^{\mathrm{op}}.$$

In our proof of this theorem, we use another simply-laced semisimple Lie algebra $\mathfrak{g}'$ having a Dynkin diagram automorphism $\omega' \colon I' \to I'$ satisfying the following conditions:

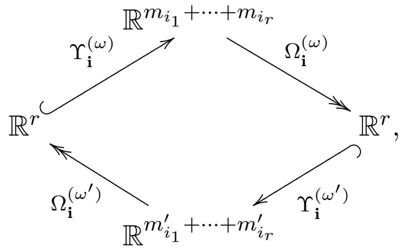
- $(\mathrm{C})_1$ the fixed point Lie subalgebra $(\mathfrak{g}')^{\omega'}$ is isomorphic to the orbit Lie algebra $\breve{\mathfrak{g}}$ associated with ω; this condition implies that the index set $\check{I}$ for $\breve{\mathfrak{g}}$ is identified with an index set $\check{I}'$ $(= (\check{I}'))$ for $(\mathfrak{g}')^{\omega'}$;
- $(\mathrm{C})_2$ if we set $m'_i := \min\{k \in \mathbb{Z}_{>0} \mid (\omega')^k(i) = i\}$, $i \in \check{I}'$, then the product $L := m_i \cdot m'_i$ is independent of the choice of $i \in \check{I} \simeq \check{I}'$.

Let $\mathbf{i} = (i_1, \ldots, i_r) \in \check{I}^r \simeq (\check{I}')^r$ be a reduced word. It is well-known that $P_+ \cap (\mathfrak{t}^*)^0$ is identified with the set of dominant integral weights for the orbit Lie algebra $\breve{\mathfrak{g}}$ associated with ω; let $\check{\lambda}$ denote the dominant integral weight for $\breve{\mathfrak{g}}$ corresponding to $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$. Now we define an $\mathbb{R}$-linear injective map $\Upsilon_{\mathbf{i}} = \Upsilon_{\mathbf{i}}^{(\omega)} \colon \mathbb{R}^r \hookrightarrow \mathbb{R}^{m_{i_1}+\cdots+m_{i_r}}$ by

$$\Upsilon_{\mathbf{i}}(a_1, \ldots, a_r) := (\underbrace{a_1, \ldots, a_1}_{m_{i_1}}, \ldots, \underbrace{a_r, \ldots, a_r}_{m_{i_r}}).$$

By using the theory of crystal bases, we see that Berenstein-Littelmann-Zelevinsky's string polytope (resp., the Nakashima-Zelevinsky polytope) for $\breve{\mathfrak{g}}$ with respect to $\mathbf{i}$ and $\check{\lambda}$ is identified with a slice of $\Delta_{\Theta(\mathbf{i})}(\lambda)$ (resp., $\widetilde{\Delta}_{\Theta(\mathbf{i})}(\lambda)$)

through $\Upsilon_{\mathbf{i}}$ (see Corollary 4.1.11 for more details). Now we obtain the following diagram:



in which the composite maps $\Omega_{\mathbf{i}}^{(\omega)} \circ \Upsilon_{\mathbf{i}}^{(\omega)} \circ \Omega_{\mathbf{i}}^{(\omega')} \circ \Upsilon_{\mathbf{i}}^{(\omega')}$ and $\Omega_{\mathbf{i}}^{(\omega')} \circ \Upsilon_{\mathbf{i}}^{(\omega')} \circ \Omega_{\mathbf{i}}^{(\omega)} \circ \Upsilon_{\mathbf{i}}^{(\omega)}$ are both identical to $L \cdot \mathrm{id}_{\mathbb{R}^r}$, where L is the positive integer in $(\mathrm{C})_2$. This diagram plays an important role in our proof of Theorem 6 above. If $\mathfrak{g}$ is of type A_{2n-1} and ω is its Dynkin diagram automorphism of order two, then $\mathfrak{g}^\omega$ is of type C_n, and $(\mathfrak{g}', \omega')$ is given uniquely by the pair of the simple Lie algebra of type D_{n+1} and its Dynkin diagram automorphism of order two; the fixed point Lie subalgebra $(\mathfrak{g}')^{\omega'}$ is of type B_n. Thus the diagram above relates Newton-Okounkov polytopes of Schubert varieties of types A, B, C, D. A remarkable fact is that the composite map $\Omega_{\mathbf{i}} \circ \Upsilon_{\mathbf{i}}$ is identical to the map coming from a similarity of crystal bases. This gives a new interpretation of the similarity of crystal bases in terms of the folding procedure.

For simplicity, we deal with only finite type case in this thesis, but our results (Theorems 1, 4, 6 and Corollaries 2, 5 above) can be extended to symmetrizable Kac-Moody case without much difficulty. Note that in the case $\mathfrak{g}$ is infinite dimensional, there is no $w \in W$ such that $X(w) = G/B$. Indeed, the full flag variety G/B is infinite dimensional while the Schubert variety $X(w)$ is finite dimensional. Hence in this case, we cannot replace $X(w)$ in Theorems and Corollaries above with G/B. See [**38**] for more precise treatment.

Finally, we mention some previous works. There are other researches which ensure that the theory of Newton-Okounkov bodies is deeply connected with representation theory. For instance, Feigin-Fourier-Littelmann [**9**] described Feigin-Fourier-Littelmann-Vinberg polytopes as Newton-Okounkov polytopes, which is defined by using Dyck paths. Note that this Newton-Okounkov body is not unimodularly equivalent to the ones associated with the valuations $v_{X(w_{\geq \bullet})}, v_{X(w_{\leq \bullet})}$ in general. In the paper [**37**], Kiritchenko considered the valuation associated with the sequence of translated Schubert varieties:

$$wX(e) \subset w_{\leq r-1}X(w_{\geq r}) \subset \cdots \subset w_{\leq 1}X(w_{\geq 2}) \subset eX(w_{\geq 1}) = X(w).$$

In the case that G is of type A and $\mathbf{i}$ is a specific reduced word for the longest element $w_0 \in W$, she proved that the corresponding Newton-Okounkov body is identical to the Feigin-Fourier-Littelmann-Vinberg polytope. In addition, the Lusztig parametrization of the canonical basis also appears in the theory of Newton-Okounkov polytopes (see [**8**]). Furthermore, the author [**12**] extended Kaveh's result [**32**] on string polytopes to Bott-Samelson varieties.

The computation of the Newton-Okounkov body associated with the valuation $v_{X(w_{\leq \bullet})}$ was partially done by Okounkov [**55**]. In the case that G is of type C and $\mathbf{i}$ is a specific reduced word for the longest element $w_0 \in W$, he proved that the Newton-Okounkov body associated with $v_{X(w_{\leq \bullet})}$ is identical (after an explicit unimodular transformation) to the type C Gelfand-Zetlin polytope, which coincides (after an explicit unimodular transformation) with the corresponding string polytope by [**42**, Corollary 7]. Since the collection $u_1, \ldots, u_r$ of rational functions used in [**55**] is different from ours, the Newton-Okounkov body computed in [**55**] is not identical to ours, but they are unimodular equivalent. Note that our approach in this thesis is quite different from his.

Organization of this thesis

This thesis is divided into four chapters. In Ch. 1, we review some basic facts about Newton-Okounkov polytopes, Schubert varieties, and crystal bases. We also recall the definition of string polytopes and the main result of [**32**].

In Ch. 2, we relate Nakashima-Zelevinsky's polyhedral realizations of crystal bases with Newton-Okounkov polytopes. In Sect. 2.1, we recall some basic facts about polyhedral realizations of crystal bases. Sect. 2.2 is devoted to the study of perfect bases. In Sect. 2.3, we prove Theorem 1 and Corollary 2 above. Sect. 2.4 is devoted to the study of explicit forms of Newton-Okounkov polytopes. In Sect. 2.5, we prove Corollary 3 above by combining our result with the main result of [**32**].

In Ch. 3, we discuss geometrically natural valuations, which are given by counting the orders of zeros along sequences of specific subvarieties. In Sect. 3.1, we recall the definition of such valuations. Sect. 3.2 is devoted to explaining properties of perfect bases satisfying positivity conditions $(\mathrm{P})_1$ and $(\mathrm{P})_2$ above. In Sect. 3.3, we prove Theorem 4 and Corollary 5 above.

In Ch. 4, we apply the folding procedure to Newton-Okounkov polytopes of Schubert varieties. Sect. 4.1 is devoted to the study of the folding procedure for crystal bases. In Sect. 4.2, we prove Theorem 6 above. In Sect. 4.3, we study the relation with a similarity of crystal bases. Finally, in Sect. 4.4, we give the list of nontrivial pairs of automorphisms of simply-laced affine Dynkin diagrams satisfying conditions $(\mathrm{C})_1$ and $(\mathrm{C})_2$ above.

CHAPTER 1

Newton-Okounkov polytopes and crystal bases

In this chapter, we review some basic facts about Newton-Okounkov polytopes, Schubert varieties, and crystal bases. We also recall the definition of string polytopes and the main result of [**32**].

1.1. Newton-Okounkov polytopes

First of all, we recall the definition of Newton-Okounkov polytopes, following [**17, 32, 33, 34**]. Let R be a $\mathbb{C}$-algebra without nonzero zero-divisors, and fix a total order $<$ on $\mathbb{Z}^r$, $r \in \mathbb{Z}_{>0}$, respecting the addition.

Definition 1.1.1. A map $v\colon R \setminus \{0\} \to \mathbb{Z}^r$ is called a *valuation* on R if the following hold: for every $\sigma, \tau \in R \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$,

(i) $v(\sigma \cdot \tau) = v(\sigma) + v(\tau)$,
(ii) $v(c \cdot \sigma) = v(\sigma)$,
(iii) $v(\sigma + \tau) \geq \min\{v(\sigma), v(\tau)\}$ unless $\sigma + \tau = 0$.

Note that we need to fix a total order on $\mathbb{Z}^r$ whenever we consider a valuation. The following is a fundamental property of valuations.

Proposition 1.1.2 (see, for instance, [**32**, Proposition 1.8]). *Let v be a valuation on R. For $\sigma_1, \ldots, \sigma_s \in R \setminus \{0\}$, assume that $v(\sigma_1), \ldots, v(\sigma_s)$ are all distinct.*

(1) *The elements $\sigma_1, \ldots, \sigma_s$ are linearly independent over $\mathbb{C}$.*
(2) *For $c_1, \ldots, c_s \in \mathbb{C}$ such that $\sigma \coloneqq c_1\sigma_1 + \cdots + c_s\sigma_s \neq 0$, the following equality holds*:

$$v(\sigma) = \min\{v(\sigma_t) \mid 1 \leq t \leq s,\ c_t \neq 0\}.$$

For $\mathbf{a} \in \mathbb{Z}^r$ and a valuation v on R with values in $\mathbb{Z}^r$, we set

$$R_{\mathbf{a}} \coloneqq \{\sigma \in R \setminus \{0\} \mid v(\sigma) \geq \mathbf{a}\} \cup \{0\};$$

this is a $\mathbb{C}$-subspace of R. The *leaf* above $\mathbf{a} \in \mathbb{Z}^r$ is defined to be the quotient space $\widehat{R}_{\mathbf{a}} \coloneqq R_{\mathbf{a}} / \bigcup_{\mathbf{a} < \mathbf{b}} R_{\mathbf{b}}$. A valuation v is said to have *one-dimensional leaves* if $\dim_{\mathbb{C}}(\widehat{R}_{\mathbf{a}}) = 0$ or 1 for all $\mathbf{a} \in \mathbb{Z}^r$.

Definition 1.1.3. We define two lexicographic orders $<$ and $\prec$ on $\mathbb{Z}^r$, $r \in \mathbb{Z}_{>0}$, by $(a_1, \ldots, a_r) < (a'_1, \ldots, a'_r)$ (resp., $(a_1, \ldots, a_r) \prec (a'_1, \ldots, a'_r)$) if and only if there exists $1 \leq k \leq r$ such that $a_1 = a'_1, \ldots, a_{k-1} = a'_{k-1}$, $a_k < a'_k$ (resp., $a_r = a'_r, \ldots, a_{k+1} = a'_{k+1}$, $a_k < a'_k$). Let $\mathbb{C}(t_1, \ldots, t_r)$ denote the rational function field in r variables. The lexicographic order $<$ on $\mathbb{Z}^r$ induces a total order (denoted by the same symbol $<$) on the set of monomials in the polynomial ring $\mathbb{C}[t_1, \ldots, t_r]$ as follows: $t_1^{a_1} \cdots t_r^{a_r} < t_1^{a'_1} \cdots t_r^{a'_r}$ if and only if $(a_1, \ldots, a_r) < (a'_1, \ldots, a'_r)$. Let us define two maps $v^{\mathrm{high}}, v^{\mathrm{low}} \colon \mathbb{C}(t_1, \ldots, t_r) \setminus \{0\} \to \mathbb{Z}^r$ by $v^{\mathrm{high}}(f/g) \coloneqq v^{\mathrm{high}}(f) - v^{\mathrm{high}}(g)$, $v^{\mathrm{low}}(f/g) \coloneqq v^{\mathrm{low}}(f) - v^{\mathrm{low}}(g)$ for $f, g \in \mathbb{C}[t_1, \ldots, t_r] \setminus \{0\}$, and by

$$v^{\mathrm{high}}(f) \coloneqq -(a_1, \ldots, a_r), \quad v^{\mathrm{low}}(f) \coloneqq (a'_1, \ldots, a'_r)$$

for

$$\begin{aligned} f &= c t_1^{a_1} \cdots t_r^{a_r} + (\text{lower terms}) \\ &= c' t_1^{a'_1} \cdots t_r^{a'_r} + (\text{higher terms}) \\ &\in \mathbb{C}[t_1, \ldots, t_r] \setminus \{0\}, \end{aligned}$$

respectively, where $c, c' \in \mathbb{C} \setminus \{0\}$, and by "lower terms" (resp., "higher terms"), we mean a linear combination of monomials smaller than $t_1^{a_1} \cdots t_r^{a_r}$ (resp., bigger than $t_1^{a'_1} \cdots t_r^{a'_r}$) with respect to the total order $<$. It is obvious that these maps $v^{\mathrm{high}}, v^{\mathrm{low}}$ are valuations with one-dimensional leaves with respect to the total order $<$. Since the total order $<$ on the set of monomials satisfies $t_1 > \cdots > t_r$, we call the valuation v^{high} (resp., v^{low}) on $\mathbb{C}(t_1, \ldots, t_r)$ the *highest term valuation* (resp., the *lowest term valuation*) with respect to the lexicographic order $t_1 > \cdots > t_r$. Similarly, the lexicographic order $\prec$ on $\mathbb{Z}^r$ induces a total order $\prec$ on the set of monomials satisfying $t_r \succ \cdots \succ t_1$. By using the total order $\prec$, we define the *highest term valuation* $\tilde{v}^{\mathrm{high}}$ and the *lowest term valuation* $\tilde{v}^{\mathrm{low}}$ with respect to the lexicographic order $t_r \succ \cdots \succ t_1$ by

$$\tilde{v}^{\mathrm{high}}(f) \coloneqq -(a_r, \ldots, a_1), \quad \tilde{v}^{\mathrm{low}}(f) \coloneqq (a'_r, \ldots, a'_1)$$

for

$$\begin{aligned} f &= c t_1^{a_1} \cdots t_r^{a_r} + (\text{lower terms}) \\ &= c' t_1^{a'_1} \cdots t_r^{a'_r} + (\text{higher terms}) \\ &\in \mathbb{C}[t_1, \ldots, t_r] \setminus \{0\}, \end{aligned}$$

respectively, where $c, c' \in \mathbb{C} \setminus \{0\}$; note that these maps $\tilde{v}^{\mathrm{high}}, \tilde{v}^{\mathrm{low}}$ are valuations with one-dimensional leaves with respect to the total order $<$ (not $\prec$).

lexicographic order	highest term valuation	lowest term valuation
$t_1 > \cdots > t_r$	v^{high}	v^{low}
$t_r \succ \cdots \succ t_1$	$\tilde{v}^{\mathrm{high}}$	$\tilde{v}^{\mathrm{low}}$

EXAMPLE 1.1.4. If $r = 3$ and $f = t_1 t_2 + t_3^2 \in \mathbb{C}[t_1, t_2, t_3]$, then it follows that $v^{\mathrm{high}}(f) = -(1, 1, 0)$, $v^{\mathrm{low}}(f) = (0, 0, 2)$, $\tilde{v}^{\mathrm{high}}(f) = -(2, 0, 0)$, and $\tilde{v}^{\mathrm{low}}(f) = (0, 1, 1)$.

DEFINITION 1.1.5 (see [**32**, Sect. 1.2] and [**34**, Definition 1.10]). Let X be an irreducible normal projective variety over $\mathbb{C}$ of complex dimension r, and $\mathcal{L}$ a line bundle on X generated by global sections. Take a valuation

$v\colon \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^r$ with one-dimensional leaves, and fix a nonzero section $\tau \in H^0(X, \mathcal{L})$. We define a subset $S(X, \mathcal{L}, v, \tau) \subset \mathbb{Z}_{>0} \times \mathbb{Z}^r$ by

$$S(X, \mathcal{L}, v, \tau) \coloneqq \bigcup_{k \in \mathbb{Z}_{>0}} \{(k, v(\sigma/\tau^k)) \mid \sigma \in H^0(X, \mathcal{L}^{\otimes k}) \setminus \{0\}\},$$

and denote by $C(X, \mathcal{L}, v, \tau) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^r$ the smallest real closed cone containing $S(X, \mathcal{L}, v, \tau)$, that is,

$$C(X, \mathcal{L}, v, \tau) \coloneqq \overline{\{c \cdot (k, \mathbf{a}) \mid c \in \mathbb{R}_{>0} \text{ and } (k, \mathbf{a}) \in S(X, \mathcal{L}, v, \tau)\}},$$

where $\overline{H}$ means the closure of $H \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^r$ with respect to the Euclidean topology. Let us define a subset $\Delta(X, \mathcal{L}, v, \tau) \subset \mathbb{R}^r$ by

$$\Delta(X, \mathcal{L}, v, \tau) \coloneqq \{\mathbf{a} \in \mathbb{R}^r \mid (1, \mathbf{a}) \in C(X, \mathcal{L}, v, \tau)\};$$

this is called the *Newton-Okounkov body* of X associated with $\mathcal{L}$, v, and τ. If the set $\Delta(X, \mathcal{L}, v, \tau)$ is a polytope, that is, it is the convex hull of a finite number of points, then we call it a *Newton-Okounkov polytope*.

We see by the definition of valuations that $S(X, \mathcal{L}, v, \tau)$ is a semigroup. Hence it follows that $C(X, \mathcal{L}, v, \tau)$ is a closed convex cone, and that $\Delta(X, \mathcal{L}, v, \tau)$ is a convex set. Moreover, we deduce by [**34**, Theorem 2.30] that $\Delta(X, \mathcal{L}, v, \tau)$ is a convex body, i.e., a compact convex set. If $\mathcal{L}$ is very ample, then it follows from [**34**, Corollary 3.2] that the real dimension of $\Delta(X, \mathcal{L}, v, \tau)$ is equal to r; this is not necessarily the case if $\mathcal{L}$ is not very ample. If the semigroup $S(X, \mathcal{L}, v, \tau)$ is finitely generated, then $\Delta(X, \mathcal{L}, v, \tau)$ is a rational convex polytope, i.e., the convex hull of a finite number of rational points; note that $\Delta(X, \mathcal{L}, v, \tau)$ is not a polytope in general.

Remark 1.1.6. If $\mathcal{L}$ is a very ample line bundle on X, then we obtain a closed embedding $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$ such that $\mathcal{L}$ is the pullback of the twisting sheaf $\mathcal{O}(1)$ of Serre. Denote by $R = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} R_k$ the corresponding homogeneous coordinate ring. Newton-Okounkov bodies are sometimes defined by using R_k instead of $H^0(X, \mathcal{L}^{\otimes k})$ (see [**17**]). However, since X is normal, we deduce by [**18**, Ch. II Ex. 5.14] that $R_k = H^0(X, \mathcal{L}^{\otimes k})$ for all $k \gg 0$; we need not assume the projective normality. In addition, since $S(X, \mathcal{L}, v, \tau)$ is a semigroup, the real closed cone $C(X, \mathcal{L}, v, \tau)$ is identical to the smallest real closed cone containing

$$\bigcup_{k > k'} \{(k, v(\sigma/\tau^k)) \mid \sigma \in H^0(X, \mathcal{L}^{\otimes k}) \setminus \{0\}\}$$

for each $k' \in \mathbb{Z}_{\geq 0}$. Therefore, R_k and $H^0(X, \mathcal{L}^{\otimes k})$ are interchangeable in the definition of Newton-Okounkov bodies.

Remark 1.1.7. If we take another section $\tau' \in H^0(X, \mathcal{L}) \setminus \{0\}$, then $S(X, \mathcal{L}, v, \tau')$ is the shift of $S(X, \mathcal{L}, v, \tau)$ by $kv(\tau/\tau')$ in $\{k\} \times \mathbb{Z}^r$ for $k \in \mathbb{Z}_{>0}$, that is,

$$S(X, \mathcal{L}, v, \tau') \cap (\{k\} \times \mathbb{Z}^r) = S(X, \mathcal{L}, v, \tau) \cap (\{k\} \times \mathbb{Z}^r) + (0, kv(\tau/\tau')).$$

Hence it follows that $\Delta(X, \mathcal{L}, v, \tau') = \Delta(X, \mathcal{L}, v, \tau) + v(\tau/\tau')$. Thus, the Newton-Okounkov body $\Delta(X, \mathcal{L}, v, \tau)$ does not essentially depend on the choice of $\tau \in H^0(X, \mathcal{L}) \setminus \{0\}$; hence it is also denoted simply by $\Delta(X, \mathcal{L}, v)$.

In the rest of this section, we review remarkable applications of Newton-Okounkov polytopes to toric degenerations and integrable systems, following [**1, 17**]. We say that X admits a *flat degeneration* to a variety X_0 if there exists a flat morphism

$$\pi\colon \mathfrak{X} \to \mathrm{Spec}(\mathbb{C}[t])$$

of schemes such that the scheme-theoretic fiber $\pi^{-1}(t)$ (resp., $\pi^{-1}(0)$) over a closed point $t \in \mathbb{C} \setminus \{0\}$ (resp., the origin $0 \in \mathbb{C}$) is isomorphic to X (resp., X_0). If X_0 is a toric variety, then this degeneration is called a *toric degeneration*.

THEOREM 1.1.8 (see [**1**, Theorem 1] and [**17**, Corollary 3.14]). *Assume that $\mathcal{L}$ is very ample, and that the semigroup $S(X, \mathcal{L}, v, \tau)$ is finitely generated; hence the Newton-Okounkov body $\Delta(X, \mathcal{L}, v, \tau)$ is a rational convex polytope. Then, there exists a flat degeneration of X to a (not necessarily normal) toric variety*

$$X_0 \coloneqq \mathrm{Proj}(\mathbb{C}[S(X, \mathcal{L}, v, \tau)]),$$

where the $\mathbb{Z}_{>0}$-grading of $S(X, \mathcal{L}, v, \tau)$ induces a $\mathbb{Z}_{\geq 0}$-grading of $\mathbb{C}[S(X, \mathcal{L}, v, \tau)]$; note that the normalization of X_0 is the normal toric variety corresponding to the Newton-Okounkov polytope $\Delta(X, \mathcal{L}, v, \tau)$.

Assume that X is nonsingular, and regard X as a complex manifold. If $\mathcal{L}$ is very ample, then we obtain a closed embedding $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$ such that $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$. Fix a Hermitian product on $H^0(X, \mathcal{L})^*$, and consider the corresponding Fubini-Study Kähler form ω_{FS} on $\mathbb{P}(H^0(X, \mathcal{L})^*)$. By restricting ω_{FS}, we obtain a Kähler form on X, which induces a Poisson bracket $\{\cdot, \cdot\}$ on the set $C^\infty(U)$ of C^∞-functions on an open subset U of X. Recall that r is the complex dimension of X.

DEFINITION 1.1.9 ([**17**, Definition 2.1]). A collection $\{F_1, \ldots, F_r\}$ of real-valued continuous functions on X is called a (*completely*) *integrable system* on X if there exists an open dense subset U of X such that the following conditions hold:

(i) $F_1, \ldots, F_r \in C^\infty(U)$,
(ii) the differentials $dF_1, \ldots, dF_r$ are linearly independent on U over $\mathbb{R}$,
(iii) $\{F_i, F_j\} = 0$ in $C^\infty(U)$ for all $1 \leq i, j \leq r$.

We call $\mu \coloneqq (F_1, \ldots, F_r)\colon X \to \mathbb{R}^r$ the *moment map* of the integrable system.

THEOREM 1.1.10 ([**17**, Theorem B]). *Assume that X is nonsingular. If $\mathcal{L}$ is very ample and $S(X, \mathcal{L}, v, \tau)$ is finitely generated, then there exists a completely integrable system $\{F_1, \ldots, F_r\}$ on X such that the image of the moment map $\mu \coloneqq (F_1, \ldots, F_r)\colon X \to \mathbb{R}^r$ is identical to the Newton-Okounkov polytope $\Delta(X, \mathcal{L}, v, \tau)$.*

1.2. Schubert varieties

Here, we recall some basic facts about Schubert varieties, following [**20, 38**]. Let G be a connected, simply-connected semisimple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, and I an index set for the vertices of the Dynkin diagram. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Then, the full flag variety is defined to be a quotient space G/B, which is a nonsingular projective variety. Denote by $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of T, by $\mathfrak{t}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ the dual space of $\mathfrak{t}$, and by $\langle \cdot, \cdot \rangle \colon \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{C}$ the canonical pairing. Let $P \subset \mathfrak{t}^*$ be the weight lattice for $\mathfrak{g}$, $P_+ \subset P$ the set of dominant integral weights, $\{\alpha_i \mid i \in I\} \subset P$ the set of simple roots, and $\{h_i \mid i \in I\} \subset \mathfrak{t}$ the set of simple coroots. Denote by $B^- \subset G$ the opposite Borel subgroup, by $N_G(T)$ the normalizer of T in G, and by $W := N_G(T)/T$ the Weyl group of $\mathfrak{g}$.

Definition 1.2.1 (see, for instance, [**20**, Sect. I.5.8]). Given $\lambda \in P$, we define a line bundle $\mathcal{L}_\lambda$ on G/B by

$$\mathcal{L}_\lambda := (G \times \mathbb{C})/B,$$

where B acts on $G \times \mathbb{C}$ on the right as follows:

$$(g, c) \cdot b = (gb, \lambda(b)c)$$

for $g \in G$, $c \in \mathbb{C}$, and $b \in B$.

Proposition 1.2.2 (see, for instance, [**20**, Sects. II.2.6, II.4.4]). *For $\lambda \in P$, the following hold.*

(1) *The line bundle $\mathcal{L}_\lambda$ on G/B is generated by global sections if and only if $\lambda \in P_+$.*
(2) *The line bundle $\mathcal{L}_\lambda$ on G/B is very ample if and only if λ is a regular dominant integral weight, that is, $\langle \lambda, h_i \rangle \in \mathbb{Z}_{>0}$ for all $i \in I$.*

For $\lambda \in P_+$, let $V(\lambda)$ be the irreducible highest weight G-module over $\mathbb{C}$ with highest weight λ and with highest weight vector v_λ. If we define a morphism $\rho_\lambda \colon G/B \to \mathbb{P}(V(\lambda))$ by:

$$g \bmod B \mapsto \mathbb{C} g v_\lambda,$$

then we obtain $\rho_\lambda^*(\mathcal{O}(1)) = \mathcal{L}_\lambda$. Hence the morphism ρ_λ induces a $\mathbb{C}$-linear map

$$\rho_\lambda^* \colon H^0(\mathbb{P}(V(\lambda)), \mathcal{O}(1)) \to H^0(G/B, \mathcal{L}_\lambda).$$

Note that for an arbitrary finite-dimensional G-module V over $\mathbb{C}$, the space $H^0(\mathbb{P}(V), \mathcal{O}(1))$ of global sections is identified with the dual G-module $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. From this and the Borel-Weil theorem (see, for instance, [**38**, Sect. 8.1.21 and Corollary 8.1.26]), we know that the $\mathbb{C}$-linear map ρ_λ^* gives an isomorphism of G-modules from $V(\lambda)^*$ to $H^0(G/B, \mathcal{L}_\lambda)$.

Definition 1.2.3 (see, for instance, [**20**, Sect. II.13.3] and [**38**, Definition 7.1.13]). Denote by $X(w)$ for $w \in W$ the Zariski closure of $B\tilde{w}B/B$ in G/B, where $\tilde{w} \in N_G(T)$ denotes a lift for w; note that the closed subvariety $X(w)$ is independent of the choice of a lift $\tilde{w}$. The $X(w)$ is called the *Schubert variety* corresponding to $w \in W$.

It is well-known that the Schubert variety $X(w)$ is an irreducible normal projective variety (see, for instance, [**20**, Sect. II.14.15]). By restricting the line bundle $\mathcal{L}_\lambda$ on G/B, we obtain a line bundle on $X(w)$, which we denote by the same symbol $\mathcal{L}_\lambda$.

Definition 1.2.4 (see, for instance, [**38**, Definition 8.1.22]). For $w \in W$ and $\lambda \in P_+$, let $v_{w\lambda} \in V(\lambda)$ be a weight vector of weight $w\lambda$, which is called an *extremal weight vector*. Define a B-submodule $V_w(\lambda) \subset V(\lambda)$ by

$$V_w(\lambda) \coloneqq \sum_{b \in B} \mathbb{C} b v_{w\lambda};$$

this is called the *Demazure module* corresponding to $w \in W$.

From the Borel-Weil type theorem (see, for instance, [**38**, Corollary 8.1.26]), we know that the isomorphism $\rho_\lambda^* \colon V(\lambda)^* \xrightarrow{\sim} H^0(G/B, \mathcal{L}_\lambda)$ induces an isomorphism $V_w(\lambda)^* \simeq H^0(X(w), \mathcal{L}_\lambda)$ of B-modules, where $V_w(\lambda)^* \coloneqq \mathrm{Hom}_{\mathbb{C}}(V_w(\lambda), \mathbb{C})$ is the dual B-module. Denote by U^- the unipotent radical of B^- with Lie algebra $\mathfrak{u}^-$. Let $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebra of B, and $e_i, f_i, h_i \in \mathfrak{g}$, $i \in I$, the Chevalley generators such that $\{e_i, h_i \mid i \in I\} \subset \mathfrak{b}$ and $\{f_i \mid i \in I\} \subset \mathfrak{u}^-$. We regard U^- as an affine open subvariety of G/B by the following open embedding:

$$U^- \hookrightarrow G/B, \ u \mapsto u \bmod B.$$

Consider the set-theoretic intersection $U^- \cap X(w)$ in G/B; this is nonempty since it contains $e \bmod B$, where $e \in G$ denotes the identity element. Since the intersection is an open subset of $X(w)$, it acquires an open subvariety structure from $X(w)$. Note that this is identical to the closed subvariety structure on $U^- \cap X(w)$ induced from U^-, since a reduced subscheme structure on the locally closed subset $U^- \cap X(w) \subset G/B$ is unique. The Weyl group W is generated by the set $\{s_i \mid i \in I\}$ of simple reflections. We call $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ a *reduced word* for $w \in W$ if $w = s_{i_1} \cdots s_{i_r}$ and if r is minimum in such expressions of w; in this case, the expression $w = s_{i_1} \cdots s_{i_r}$ is said to be *reduced*, and its length r is called the *length* of w. It is well-known that the complex dimension of $X(w)$ equals the length of w. Let $P_i \subset G$ (resp., $U_i^- \subset U^-$) denote the minimal parabolic subgroup (resp., the opposite root subgroup) corresponding to an index $i \in I$, and set $\mathfrak{u}_i^- \coloneqq \mathrm{Lie}(U_i^-) = \mathbb{C} f_i$. Take a reduced word $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ for $w \in W$. We define the corresponding *Bott-Samelson variety* $Z_{\mathbf{i}}$ by

$$Z_{\mathbf{i}} \coloneqq (P_{i_1} \times \cdots \times P_{i_r})/B^r,$$

where B^r acts on $P_{i_1} \times \cdots \times P_{i_r}$ on the right by

$$(p_1, \ldots, p_r) \cdot (b_1, \ldots, b_r) \coloneqq (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{r-1}^{-1} p_r b_r)$$

for $p_1 \in P_{i_1}, \ldots, p_r \in P_{i_r}$, and $b_1, \ldots, b_r \in B$. Then, the product map

$$Z_{\mathbf{i}} \to G/B, \ (p_1, \ldots, p_r) \bmod B^r \mapsto p_1 \cdots p_r \bmod B, \tag{1.2.1}$$

induces a birational morphism onto the Schubert variety $X(w) \subset G/B$ (see, for instance, [**20**, Ch. II.13]); therefore, the function field $\mathbb{C}(X(w))$ is identified with $\mathbb{C}(Z_{\mathbf{i}})$. We regard $U_{i_1}^- \times \cdots \times U_{i_r}^-$ as an affine open subvariety of

$Z_{\mathbf{i}}$ by the following open embedding:

$$U_{i_1}^- \times \cdots \times U_{i_r}^- \hookrightarrow Z_{\mathbf{i}},\ (u_1, \ldots, u_r) \mapsto (u_1, \ldots, u_r) \bmod B^r. \tag{1.2.2}$$

By using the isomorphism $\mathbb{C}^r \xrightarrow{\sim} U_{i_1}^- \times \cdots \times U_{i_r}^-$ of varieties given by

$$(t_1, \ldots, t_r) \mapsto (\exp(t_1 f_{i_1}), \ldots, \exp(t_r f_{i_r})),$$

we identify the function field $\mathbb{C}(X(w)) = \mathbb{C}(Z_{\mathbf{i}}) = \mathbb{C}(U_{i_1}^- \times \cdots \times U_{i_r}^-)$ with the rational function field $\mathbb{C}(t_1, \ldots, t_r)$. Now we define valuations $v_{\mathbf{i}}^{\mathrm{high}}, v_{\mathbf{i}}^{\mathrm{low}}, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}$ on $\mathbb{C}(X(w))$ to be $v^{\mathrm{high}}, v^{\mathrm{low}}, \tilde{v}^{\mathrm{high}}, \tilde{v}^{\mathrm{low}}$ on $\mathbb{C}(t_1, \ldots, t_r)$, respectively (see Definition 1.1.3). The highest term valuation $v_{\mathbf{i}}^{\mathrm{high}}$ can be described in terms of the Chevalley generators. We review this description, following [**32**]. Consider the left action of $U_{i_k}^-$ on $U_{i_k}^- \times \cdots \times U_{i_r}^-$ given by

$$u \cdot (u_k, \ldots, u_r) \coloneqq (uu_k, u_{k+1}, \ldots, u_r)$$

for $u, u_k \in U_{i_k}^-$, $u_{k+1} \in U_{i_{k+1}}^-, \ldots, u_r \in U_{i_r}^-$; this induces left actions of $U_{i_k}^-$ and $\mathfrak{u}_{i_k}^-$ on $\mathbb{C}[t_k, \ldots, t_r] = \mathbb{C}[U_{i_k}^- \times \cdots \times U_{i_r}^-]$, which are given by:

$$\exp(sf_{i_k}) \cdot f(t_k, \ldots, t_r) = f(t_k - s, t_{k+1}, \ldots, t_r), \text{ and hence}$$

$$f_{i_k} \cdot f(t_k, \ldots, t_r) = -\frac{\partial}{\partial t_k} f(t_k, \ldots, t_r) \tag{1.2.3}$$

for $s \in \mathbb{C}$ and $f(t_k, \ldots, t_r) \in \mathbb{C}[t_k, \ldots, t_r]$ (see [**32**, Proposition 2.2]).

Proposition 1.2.5 (see the proof of [**32**, Theorem 4.1]). *For a nonzero polynomial* $f(t_1, \ldots, t_r) \in \mathbb{C}[t_1, \ldots, t_r]$, *write* $v_{\mathbf{i}}^{\mathrm{high}}(f(t_1, \ldots, t_r)) = -(a_1, \ldots, a_r)$. *Then, the following equalities hold*:

$$a_1 = \max\{a \in \mathbb{Z}_{\geq 0} \mid f_{i_1}^a \cdot f(t_1, \ldots, t_r) \neq 0\},$$
$$a_2 = \max\{a \in \mathbb{Z}_{\geq 0} \mid f_{i_2}^a \cdot (f_{i_1}^{a_1} \cdot f(t_1, \ldots, t_r))|_{t_1=0} \neq 0\},$$
$$\vdots$$
$$a_r = \max\{a \in \mathbb{Z}_{\geq 0} \mid f_{i_r}^a \cdot (\cdots (f_{i_2}^{a_2} \cdot (f_{i_1}^{a_1} \cdot f(t_1, \ldots, t_r))|_{t_1=0}) \cdots)|_{t_{r-1}=0} \neq 0\}.$$

Example 1.2.6. Let $G = SL_3(\mathbb{C})$ (of type A_2), $I = \{1, 2\}$, $\mathbf{i} = (1, 2, 1) \in I^3$, a reduced word for the longest element $w_0 \in W$, and $\lambda = \alpha_1 + \alpha_2 \in P_+$. Then, the Schubert variety $X(w_0)$ is identical to the full flag variety G/B. The coordinate ring $\mathbb{C}[U^-] = \mathbb{C}[U^- \cap X(w_0)]$ is regarded as a $\mathbb{C}$-subalgebra of the polynomial ring $\mathbb{C}[t_1, t_2, t_3]$ by the following birational morphism:

$$\mathbb{C}^3 \to U^-,\ (t_1, t_2, t_3) \mapsto \exp(t_1 f_1) \exp(t_2 f_2) \exp(t_3 f_1),$$

where we set

$$f_1 \coloneqq \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ f_2 \coloneqq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since we have

$$\exp(t_1 f_1) \exp(t_2 f_2) \exp(t_3 f_1) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix},$$

the coordinate ring $\mathbb{C}[U^-]$ is identical to the $\mathbb{C}$-subalgebra $\mathbb{C}[t_1+t_3, t_2, t_2t_3]$ of $\mathbb{C}[t_1, t_2, t_3]$. In addition, by standard monomial theory (see, for instance, [**60**, Sect. 2]), we deduce that for a specific section $\tau_\lambda \in H^0(G/B, \mathcal{L}_\lambda)$, the $\mathbb{C}$-subspace $\{\sigma/\tau_\lambda \mid \sigma \in H^0(G/B, \mathcal{L}_\lambda)\}$ of $\mathbb{C}(U^-)$ is spanned by

$$\{1, t_1+t_3, t_2, t_1t_2, t_2t_3, t_1t_2(t_1+t_3), t_2^2t_3, t_1t_2^2t_3\}.$$

Now we obtain the following list.

valuation	1	t_1+t_3	t_2	t_1t_2
$v_{\mathbf{i}}^{\mathrm{high}}$	$(0,0,0)$	$-(1,0,0)$	$-(0,1,0)$	$-(1,1,0)$
$v_{\mathbf{i}}^{\mathrm{low}}$	$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(1,1,0)$
$\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$	$(0,0,0)$	$-(1,0,0)$	$-(0,1,0)$	$-(0,1,1)$
$\tilde{v}_{\mathbf{i}}^{\mathrm{low}}$	$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(0,1,1)$

valuation	t_2t_3	$t_1t_2(t_1+t_3)$	$t_2^2t_3$	$t_1t_2^2t_3$
$v_{\mathbf{i}}^{\mathrm{high}}$	$-(0,1,1)$	$-(2,1,0)$	$-(0,2,1)$	$-(1,2,1)$
$v_{\mathbf{i}}^{\mathrm{low}}$	$(0,1,1)$	$(1,1,1)$	$(0,2,1)$	$(1,2,1)$
$\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$	$-(1,1,0)$	$-(1,1,1)$	$-(1,2,0)$	$-(1,2,1)$
$\tilde{v}_{\mathbf{i}}^{\mathrm{low}}$	$(1,1,0)$	$(0,1,2)$	$(1,2,0)$	$(1,2,1)$

For $v_{\mathbf{i}} \in \{v_{\mathbf{i}}^{\mathrm{high}}, v_{\mathbf{i}}^{\mathrm{low}}, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}\}$, the Newton-Okounkov body $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{i}}, \tau_\lambda)$ is identical to the convex hull of the corresponding eight points in the list above; see Figures 1–4. Hence we deduce that

$$\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda) = -\Delta(G/B, \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}, \text{ and}$$
$$\Delta(G/B, \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda) = -\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}},$$

where we write $H^{\mathrm{op}} := \{(a_3, a_2, a_1) \mid (a_1, a_2, a_3) \in H\}$ for a subset $H \subset \mathbb{R}^3$. Our second main result (Corollary 3.3.3) states that these coincidences of Newton-Okounkov polytopes hold also for arbitrary G, $\mathbf{i}$, and λ; only restriction is that we need to take a specific section τ_λ.

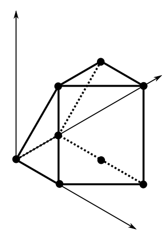


FIGURE 1. $-\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$

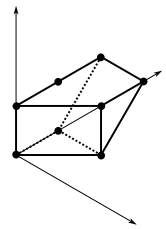


FIGURE 2. $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$

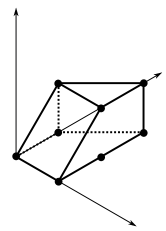


FIGURE 3. $-\Delta(G/B, \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$

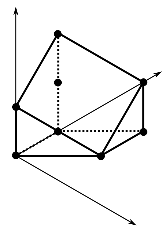


FIGURE 4. $\Delta(G/B, \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$

1.3. Crystal bases

Lusztig [**43, 44, 46**] and Kashiwara [**26, 27**] constructed a specific $\mathbb{C}$-basis of $V(\lambda)$ via the quantized enveloping algebra associated with $\mathfrak{g}$. This is called (the specialization at $q = 1$ of) the *lower global basis* (= the *canonical basis*), and parametrized by *Kashiwara's crystal basis.* In this section, we review some basic facts about crystal bases and lower global bases, following [**26, 27, 28, 29**]; see [**30**] for a survey on this topic. We start with recalling the definition of abstract crystals introduced in [**29**].

DEFINITION 1.3.1 ([**29**, Definition 1.2.1]). A *crystal* $\mathcal{B}$ is a set equipped with maps

$\mathrm{wt}\colon \mathcal{B} \to P$,
$\varepsilon_i, \varphi_i\colon \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ for $i \in I$, and
$\tilde{e}_i, \tilde{f}_i\colon \mathcal{B} \to \mathcal{B} \cup \{0\}$ for $i \in I$,

satisfying the following conditions:

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), h_i \rangle$ for $i \in I$,

(ii) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ for $i \in I$ and $b \in \mathcal{B}$ such that $\tilde{e}_i b \in \mathcal{B}$,
(iii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ for $i \in I$ and $b \in \mathcal{B}$ such that $\tilde{f}_i b \in \mathcal{B}$,
(iv) $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$ for $i \in I$ and $b, b' \in \mathcal{B}$,
(v) $\tilde{e}_i b = \tilde{f}_i b = 0$ for $i \in I$ and $b \in \mathcal{B}$ such that $\varphi_i(b) = -\infty$,

where $-\infty$ and 0 are additional elements that are not contained in $\mathbb{Z}$ and $\mathcal{B}$, respectively.

Definition 1.3.2 ([**29**, Sect. 1.2]). Let $\mathcal{B}_1, \mathcal{B}_2$ be two crystals. A map

$$\psi \colon \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$$

is called a *strict morphism* of crystals from $\mathcal{B}_1$ to $\mathcal{B}_2$ if it satisfies the following conditions:

(i) $\psi(0) = 0$,
(ii) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ for $i \in I$ and $b \in \mathcal{B}_1$ such that $\psi(b) \in \mathcal{B}_2$,
(iii) $\tilde{e}_i \psi(b) = \psi(\tilde{e}_i b)$ and $\tilde{f}_i \psi(b) = \psi(\tilde{f}_i b)$ for $i \in I$ and $b \in \mathcal{B}_1$;

here, if $\psi(b) = 0$, then we set $\tilde{e}_i \psi(b) = \tilde{f}_i \psi(b) = 0$. An injective strict morphism is called a *strict embedding* of crystals.

Consider the total order $<$ on $\mathbb{Z} \cup \{-\infty\}$ given by the usual order on $\mathbb{Z}$, and by $-\infty < s$ for all $s \in \mathbb{Z}$. For two crystals $\mathcal{B}_1, \mathcal{B}_2$, we can define another crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$, called the *tensor product* of $\mathcal{B}_1$ and $\mathcal{B}_2$, as follows (see [**29**, Sect. 1.3]):

$$\begin{aligned}
\mathcal{B}_1 \otimes \mathcal{B}_2 &= \{b_1 \otimes b_2 \mid b_1 \in \mathcal{B}_1,\ b_2 \in \mathcal{B}_2\},\\
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2),\\
\varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i \rangle\},\\
\varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i \rangle\},\\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2),\\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}\\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),\\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}
\end{aligned}$$

where $b_1 \otimes b_2$ stands for an ordered pair (b_1, b_2), and we set $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

Example 1.3.3. For $\lambda \in P$, let $R_\lambda = \{r_\lambda\}$ be a crystal consisting of only one element, given by: $\mathrm{wt}(r_\lambda) = \lambda$, $\varepsilon_i(r_\lambda) = -\langle \lambda, h_i \rangle$, $\varphi_i(r_\lambda) = 0$, and $\tilde{e}_i r_\lambda = \tilde{f}_i r_\lambda = 0$.

Define a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{t}^*$ by $2(\alpha_j, \alpha_i) = (\alpha_i, \alpha_i) \cdot \langle \alpha_j, h_i \rangle$ for all $i, j \in I$, and by $(\alpha_i, \alpha_i) = 2$ for all short simple roots α_i. We set $(c_{i,j})_{i,j\in I} \coloneqq (\langle \alpha_j, h_i\rangle)_{i,j\in I}$, the Cartan matrix of $\mathfrak{g}$, and also set

$$q_i \coloneqq q^{(\alpha_i,\alpha_i)/2} \text{ for } i \in I,$$

$$[s]_i \coloneqq \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}} \text{ for } i \in I,\ s \in \mathbb{Z},$$

$$[s]_i! \coloneqq [s]_i[s-1]_i \cdots [1]_i \text{ for } i \in I,\ s \in \mathbb{Z}_{\geq 0},$$

$$\begin{bmatrix} s \\ k \end{bmatrix}_i \coloneqq \frac{[s]_i[s-1]_i \cdots [s-k+1]_i}{[k]_i[k-1]_i \cdots [1]_i} \text{ for } i \in I,\ s,k \in \mathbb{Z}_{\geq 0} \text{ such that } k \leq s,$$

where $[0]_i! \coloneqq 1$ and $\begin{bmatrix} s \\ 0 \end{bmatrix}_i \coloneqq 1$.

Definition 1.3.4. For a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, the *quantized enveloping algebra* $U_q(\mathfrak{g})$ is the unital associative $\mathbb{Q}(q)$-algebra with generators $\{E_i, F_i, K_i, K_i^{-1} \mid i \in I\}$, and relations:

(i) $K_iK_i^{-1} = K_i^{-1}K_i = 1$ and $K_iK_j = K_jK_i$ for $i, j \in I$,
(ii) $K_iE_jK_i^{-1} = q_i^{c_{i,j}}E_j$ and $K_iF_jK_i^{-1} = q_i^{-c_{i,j}}F_j$ for $i, j \in I$,
(iii) $E_iF_i - F_iE_i = (K_i - K_i^{-1})/(q_i - q_i^{-1})$ for $i \in I$,
(iv) $E_iF_j = F_jE_i$ for $i, j \in I$ such that $i \neq j$,
(v) $\sum_{s=0}^{1-c_{i,j}} (-1)^s E_i^{(s)} E_j E_i^{(1-c_{i,j}-s)} = \sum_{s=0}^{1-c_{i,j}} (-1)^s F_i^{(s)} F_j F_i^{(1-c_{i,j}-s)} = 0$ for $i, j \in I$ such that $i \neq j$,

where $E_i^{(s)} \coloneqq E_i^s/[s]_i!$, $F_i^{(s)} \coloneqq F_i^s/[s]_i!$ for $i \in I$ and $s \in \mathbb{Z}_{\geq 0}$.

Let us denote by $U_q(\mathfrak{u})$ (resp., $U_q(\mathfrak{u}^-)$) the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i \mid i \in I\}$ (resp., $\{F_i \mid i \in I\}$). Define a $\mathbb{Q}$-algebra involution $^-$ on $U_q(\mathfrak{g})$ by:

$$\overline{E_i} = E_i,\ \overline{F_i} = F_i,\ \overline{K_i} = K_i^{-1},\ \overline{q} = q^{-1};$$

the involution $^-$ is called the *bar involution.* Note that this preserves $U_q(\mathfrak{u})$ and $U_q(\mathfrak{u}^-)$. For $i \in I$ and $u \in U_q(\mathfrak{u}^-)$, we see by [**27**, Lemma 3.4.1] that there exist unique elements $e_i'(u), e_i''(u) \in U_q(\mathfrak{u}^-)$ such that

$$E_iu - uE_i = \frac{K_ie_i''(u) - K_i^{-1}e_i'(u)}{q_i - q_i^{-1}}.$$

Then, it follows by [**27**, Proposition 3.2.1] that

$$U_q(\mathfrak{u}^-) = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \{F_i^{(k)} \cdot u \mid u \in U_q(\mathfrak{u}^-) \text{ with } e_i'(u) = 0\}$$

for each $i \in I$. Following [**27**, Sect. 3], we define operators $\tilde{e}_i, \tilde{f}_i$, $i \in I$, on $U_q(\mathfrak{u}^-)$ by

$$\tilde{e}_i(F_i^{(k)} \cdot u) \coloneqq F_i^{(k-1)} \cdot u \text{ and } \tilde{f}_i(F_i^{(k)} \cdot u) \coloneqq F_i^{(k+1)} \cdot u$$

for $u \in U_q(\mathfrak{u}^-)$ with $e_i'(u) = 0$, and $k \in \mathbb{Z}_{\geq 0}$, where $F_i^{(-1)} \cdot u \coloneqq 0$. These operators $\tilde{e}_i, \tilde{f}_i$, $i \in I$, are called the *Kashiwara operators.* Let $A \subset \mathbb{Q}(q)$ denote the $\mathbb{Q}$-subalgebra of $\mathbb{Q}(q)$ consisting of rational functions regular at

$q = 0$. Then, we define an A-submodule $L(\infty) \subset U_q(\mathfrak{u}^-)$ and a subset $\mathcal{B}(\infty) \subset L(\infty)/qL(\infty)$ by

$$L(\infty) \coloneqq \sum_{\substack{l \in \mathbb{Z}_{\geq 0}, \\ i_1, \ldots, i_l \in I}} A\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} 1,$$

$$\mathcal{B}(\infty) \coloneqq \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} 1 \bmod qL(\infty) \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \ldots, i_l \in I\}.$$

Proposition 1.3.5 (see [**27**, Theorem 4]). *The following hold.*

1. *The set $\mathcal{B}(\infty)$ forms a $\mathbb{Q}$-basis of $L(\infty)/qL(\infty)$.*
2. *$\tilde{e}_i L(\infty) \subset L(\infty)$ and $\tilde{f}_i L(\infty) \subset L(\infty)$ for all $i \in I$; hence $\tilde{e}_i, \tilde{f}_i$, $i \in I$, act on $L(\infty)/qL(\infty)$.*
3. *$\tilde{e}_i \mathcal{B}(\infty) \subset \mathcal{B}(\infty) \cup \{0\}$ and $\tilde{f}_i \mathcal{B}(\infty) \subset \mathcal{B}(\infty)$ for all $i \in I$.*
4. *Define maps $\varepsilon_i, \varphi_i \colon \mathcal{B}(\infty) \to \mathbb{Z}$ for $i \in I$ by*

$$\varepsilon_i(b) \coloneqq \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) \coloneqq \varepsilon_i(b) + \langle \mathrm{wt}(b), h_i \rangle.$$

Then, the sextuple $(\mathcal{B}(\infty);\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$ provides a crystal structure on $\mathcal{B}(\infty)$.

The pair $(L(\infty), \mathcal{B}(\infty))$ is called the *lower crystal basis* of $U_q(\mathfrak{u}^-)$. Define a $\mathbb{Q}(q)$-algebra anti-involution $*$ on $U_q(\mathfrak{g})$ by:

$$E_i^* = E_i,\ F_i^* = F_i,\ K_i^* = K_i^{-1}$$

for $i \in I$; note that $* \circ {}^- = {}^- \circ *$. We see by [**27**, Proposition 5.2.4] that $L(\infty)^* = L(\infty)$, and by [**29**, Theorem 2.1.1] that $\mathcal{B}(\infty)^* = \mathcal{B}(\infty)$. The involution $* \colon \mathcal{B}(\infty) \to \mathcal{B}(\infty)$ is called *Kashiwara's involution.* Set

$$\varepsilon_i^* \coloneqq \varepsilon_i \circ *,\ \varphi_i^* \coloneqq \varphi_i \circ *,\ \tilde{e}_i^* \coloneqq * \circ \tilde{e}_i \circ *, \text{ and } \tilde{f}_i^* \coloneqq * \circ \tilde{f}_i \circ *$$

for $i \in I$. Then, the sextuple $(\mathcal{B}(\infty);\ \mathrm{wt}, \{\varepsilon_i^*\}_i, \{\varphi_i^*\}_i, \{\tilde{e}_i^*\}_i, \{\tilde{f}_i^*\}_i)$ provides another crystal structure on $\mathcal{B}(\infty)$. For $\lambda \in P_+$, let $V_q(\lambda)$ denote the irreducible highest weight $U_q(\mathfrak{g})$-module over $\mathbb{Q}(q)$ with highest weight λ and with highest weight vector $v_{q,\lambda}$. By the standard representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ (see, for instance, [**19**, Ch. 2]), we have

$$V_q(\lambda) = \bigoplus_{\substack{k \in \mathbb{Z}_{\geq 0}, \\ 0 \leq l \leq k}} \{F_i^{(l)} \cdot v \mid v \in V_q(\lambda),\ E_i \cdot v = 0, \text{ and } K_i \cdot v = q_i^k v\}$$

for each $i \in I$. Following [**27**, Sect. 2.2], we define operators $\tilde{e}_i, \tilde{f}_i$, $i \in I$, on $V_q(\lambda)$ by

$$\tilde{e}_i(F_i^{(l)} \cdot v) \coloneqq F_i^{(l-1)} \cdot v \text{ and } \tilde{f}_i(F_i^{(l)} \cdot v) \coloneqq F_i^{(l+1)} \cdot v$$

for $v \in V_q(\lambda)$ and $l \in \mathbb{Z}_{\geq 0}$ such that $E_i \cdot v = 0$, $K_i \cdot v = q_i^k v$ for some $k \in \mathbb{Z}_{\geq 0}$, and $l \leq k$, where $F_i^{(-1)} \cdot v \coloneqq 0$. These operators $\tilde{e}_i, \tilde{f}_i$, $i \in I$, are also called the *Kashiwara operators*. Then, we define an A-submodule $L(\lambda) \subset V_q(\lambda)$

and a subset $\mathcal{B}(\lambda) \subset L(\lambda)/qL(\lambda)$ by

$$L(\lambda) := \sum_{\substack{l\in\mathbb{Z}_{\geq 0},\\ i_1,\ldots,i_l\in I}} A\tilde{f}_{i_1}\cdots\tilde{f}_{i_l}v_{q,\lambda},$$

$$\mathcal{B}(\lambda) := \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_l}v_{q,\lambda} \bmod qL(\lambda) \mid l\in\mathbb{Z}_{\geq 0},\ i_1,\ldots,i_l\in I\}\setminus\{0\}.$$

PROPOSITION 1.3.6 (see [**27**, Theorem 2]). *For $\lambda \in P_+$, the following hold.*

(1) *The set $\mathcal{B}(\lambda)$ forms a $\mathbb{Q}$-basis of $L(\lambda)/qL(\lambda)$.*
(2) *$\tilde{e}_iL(\lambda) \subset L(\lambda)$ and $\tilde{f}_iL(\lambda) \subset L(\lambda)$ for all $i \in I$; hence $\tilde{e}_i, \tilde{f}_i$, $i \in I$, act on $L(\lambda)/qL(\lambda)$.*
(3) *$\tilde{e}_i\mathcal{B}(\lambda) \subset \mathcal{B}(\lambda)\cup\{0\}$ and $\tilde{f}_i\mathcal{B}(\lambda) \subset \mathcal{B}(\lambda)\cup\{0\}$ for all $i \in I$.*
(4) *Define maps $\varepsilon_i, \varphi_i\colon \mathcal{B}(\lambda) \to \mathbb{Z}$ for $i \in I$ by*

$$\varepsilon_i(b) := \max\{k\in\mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) := \max\{k\in\mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k b \neq 0\}.$$

Then, the sextuple $(\mathcal{B}(\lambda);\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$ provides a crystal structure on $\mathcal{B}(\lambda)$.

The pair $(L(\lambda), \mathcal{B}(\lambda))$ is called the *lower crystal basis* of $V_q(\lambda)$. The crystals $\mathcal{B}(\infty)$ and $\mathcal{B}(\lambda)$ are related as follows.

PROPOSITION 1.3.7 (see [**27**, Theorem 5]). *For $\lambda \in P_+$, let $\pi_\lambda\colon U_q(\mathfrak{u}^-) \twoheadrightarrow V_q(\lambda)$ denote the surjective $U_q(\mathfrak{u}^-)$-module homomorphism given by $u \mapsto uv_{q,\lambda}$.*

(1) *The equality $\pi_\lambda(L(\infty)) = L(\lambda)$ holds; hence π_λ induces a surjective $\mathbb{Q}$-linear map $L(\infty)/qL(\infty) \twoheadrightarrow L(\lambda)/qL(\lambda)$, denoted also by π_λ.*
(2) *The $\mathbb{Q}$-linear map π_λ induces a surjective map $\pi_\lambda\colon \mathcal{B}(\infty) \twoheadrightarrow \mathcal{B}(\lambda)\cup\{0\}$. In addition, for*

$$\widetilde{\mathcal{B}}(\lambda) := \{b\in\mathcal{B}(\infty) \mid \pi_\lambda(b)\neq 0\},$$

the restriction map $\pi_\lambda\colon \widetilde{\mathcal{B}}(\lambda) \to \mathcal{B}(\lambda)$ is bijective.
(3) *$\tilde{f}_i\pi_\lambda(b) = \pi_\lambda(\tilde{f}_ib)$ for all $i \in I$ and $b \in \mathcal{B}(\infty)$.*
(4) *$\tilde{e}_i\pi_\lambda(b) = \pi_\lambda(\tilde{e}_ib)$ for all $i \in I$ and $b \in \widetilde{\mathcal{B}}(\lambda)$.*
(5) *$\varepsilon_i(\pi_\lambda(b)) = \varepsilon_i(b)$ and $\varphi_i(\pi_\lambda(b)) = \varphi_i(b) + \langle\lambda, h_i\rangle$ for all $i \in I$ and $b \in \widetilde{\mathcal{B}}(\lambda)$.*

Let $U_{q,\mathbb{Z}}(\mathfrak{u}^-)$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_q(\mathfrak{u}^-)$ generated by $\{F_i^{(k)} \mid i\in I,\ k\in\mathbb{Z}_{\geq 0}\}$, and set $V_{q,\mathbb{Z}}(\lambda) := \pi_\lambda(U_{q,\mathbb{Z}}(\mathfrak{u}^-))$. We also set $U_{q,\mathbb{Q}}(\mathfrak{u}^-) := U_{q,\mathbb{Z}}(\mathfrak{u}^-)\otimes_{\mathbb{Z}}\mathbb{Q}$ and $V_{q,\mathbb{Q}}(\lambda) := V_{q,\mathbb{Z}}(\lambda)\otimes_{\mathbb{Z}}\mathbb{Q}$. Define a $\mathbb{Q}$-involution $^-$ on $V_q(\lambda)$ by: $\overline{u\cdot v_{q,\lambda}} = \overline{u}\cdot v_{q,\lambda}$ for $u \in U_q(\mathfrak{g})$. Then, the natural maps

$$L(\infty)\cap\overline{L(\infty)}\cap U_{q,\mathbb{Q}}(\mathfrak{u}^-) \to L(\infty)/qL(\infty) \text{ and}$$
$$L(\lambda)\cap\overline{L(\lambda)}\cap V_{q,\mathbb{Q}}(\lambda) \to L(\lambda)/qL(\lambda)$$

are $\mathbb{Q}$-linear isomorphisms [**27**, Theorem 6]. If we denote the inverses of these isomorphisms by

$$G_q^{\rm low}\colon L(\infty)/qL(\infty) \to L(\infty)\cap\overline{L(\infty)}\cap U_{q,\mathbb{Q}}(\mathfrak{u}^-) \text{ and}$$
$$G_{q,\lambda}^{\rm low}\colon L(\lambda)/qL(\lambda) \to L(\lambda)\cap\overline{L(\lambda)}\cap V_{q,\mathbb{Q}}(\lambda),$$

respectively, then the sets $\{G_q^{\rm low}(b) \mid b\in\mathcal{B}(\infty)\}$ and $\{G_{q,\lambda}^{\rm low}(b) \mid b\in\mathcal{B}(\lambda)\}$ form $\mathbb{Z}[q,q^{-1}]$-bases of $U_{q,\mathbb{Z}}(\mathfrak{u}^-)$ and $V_{q,\mathbb{Z}}(\lambda)$, respectively (see [**27**, Theorem 7]); these are called the *lower global bases*. The following is a fundamental property of these bases.

PROPOSITION 1.3.8 (see [**27**, Lemma 7.3.2], [**28**, Sect. 5.3], and [**29**, equation (3.1.2)]). *For $\lambda\in P_+$, the following hold.*

(1) $\pi_\lambda(G_q^{\rm low}(b)) = G_{q,\lambda}^{\rm low}(\pi_\lambda(b))$ *for all* $b\in\mathcal{B}(\infty)$.

(2) *For all $i\in I$, $b\in\mathcal{B}(\lambda)$, and $k\in\mathbb{Z}_{\geq 0}$,*

$$E_i^{(k)}\cdot G_{q,\lambda}^{\rm low}(b)\in \begin{bmatrix}\varphi_i(b)+k\\ k\end{bmatrix}_i G_{q,\lambda}^{\rm low}(\tilde{e}_i^k b) + \sum_{\substack{b'\in\mathcal{B}(\lambda);\\ {\rm wt}(b')={\rm wt}(b)+k\alpha_i,\\ \varphi_i(b')>\varphi_i(b)+k}} \mathbb{Z}[q,q^{-1}]G_{q,\lambda}^{\rm low}(b'),$$

$$F_i^{(k)}\cdot G_{q,\lambda}^{\rm low}(b)\in \begin{bmatrix}\varepsilon_i(b)+k\\ k\end{bmatrix}_i G_{q,\lambda}^{\rm low}(\tilde{f}_i^k b) + \sum_{\substack{b'\in\mathcal{B}(\lambda);\\ {\rm wt}(b')={\rm wt}(b)-k\alpha_i,\\ \varepsilon_i(b')>\varepsilon_i(b)+k}} \mathbb{Z}[q,q^{-1}]G_{q,\lambda}^{\rm low}(b').$$

(3) *For all $i\in I$, $b\in\mathcal{B}(\infty)$, and $k\in\mathbb{Z}_{\geq 0}$,*

$$F_i^{(k)}\cdot G_q^{\rm low}(b)\in \begin{bmatrix}\varepsilon_i(b)+k\\ k\end{bmatrix}_i G_q^{\rm low}(\tilde{f}_i^k b) + \sum_{\substack{b'\in\mathcal{B}(\infty);\\ {\rm wt}(b')={\rm wt}(b)-k\alpha_i,\\ \varepsilon_i(b')>\varepsilon_i(b)+k}} \mathbb{Z}[q,q^{-1}]G_q^{\rm low}(b').$$

PROOF. Parts (1), (2) are immediate consequences of [**27**, Lemma 7.3.2] and [**29**, equation (3.1.2)], respectively (see also [**28**, Sect. 5.3]). Then, part (3) follows from parts (1), (2) by taking $\lambda\in P_+$ such that $\langle\lambda,h_i\rangle$, $i\in I$, are sufficiently large for fixed $b\in\mathcal{B}(\infty)$. □

Recall that the involution $*\colon U_q(\mathfrak{g})\to U_q(\mathfrak{g})$ induces Kashiwara's involution $*\colon\mathcal{B}(\infty)\to\mathcal{B}(\infty)$; the lower global basis $\{G_q^{\rm low}(b)\mid b\in\mathcal{B}(\infty)\}$ is stable under the involution $*$ as follows.

PROPOSITION 1.3.9 (see [**27, 29**]). *The equality $G_q^{\rm low}(b)^* = G_q^{\rm low}(b^*)$ holds for all $b\in\mathcal{B}(\infty)$.*

PROOF. It follows from the equality $*\circ{}^- = {}^-\circ *$ that $\overline{L(\infty)}$ and hence $L(\infty)\cap\overline{L(\infty)}\cap U_{q,\mathbb{Q}}(\mathfrak{u}^-)$ are invariant under the involution $*$, which implies the assertion of the proposition. □

COROLLARY 1.3.10. *For all $i\in I$, $b\in\mathcal{B}(\infty)$, and $k\in\mathbb{Z}_{\geq 0}$,*

$$G_q^{\rm low}(b)\cdot F_i^{(k)}\in \begin{bmatrix}\varepsilon_i^*(b)+k\\ k\end{bmatrix}_i G_q^{\rm low}((\tilde{f}_i^*)^k b) + \sum_{\substack{b'\in\mathcal{B}(\infty);\\ {\rm wt}(b')={\rm wt}(b)-k\alpha_i,\\ \varepsilon_i^*(b')>\varepsilon_i^*(b)+k}} \mathbb{Z}[q,q^{-1}]G_q^{\rm low}(b').$$

PROOF. The assertion of the corollary follows from Proposition 1.3.8 (3) and from the equalities

$$\begin{aligned}(G_q^{\mathrm{low}}(b)\cdot F_i^{(k)})^* &= (F_i^{(k)})^*\cdot G_q^{\mathrm{low}}(b)^* \\ &\quad \text{(since $*$ is a $\mathbb{Q}(q)$-algebra anti-involution)} \\ &= F_i^{(k)}\cdot G_q^{\mathrm{low}}(b^*) \quad \text{(by Proposition 1.3.9).}\end{aligned}$$

□

DEFINITION 1.3.11. For $w \in W$ and $\lambda \in P_+$, fix a weight vector $v_{q,w\lambda} \in V_q(\lambda)$ of weight $w\lambda$, called an *extremal weight vector*. Then, the $U_q(\mathfrak{u})$-submodule $V_{q,w}(\lambda) \coloneqq U_q(\mathfrak{u}) \cdot v_{q,w\lambda} \subset V_q(\lambda)$ is called the *Demazure module* corresponding to w.

By [**29**, Proposition 3.2.3 (i)], there uniquely exists a subset $\mathcal{B}_w(\lambda) \subset \mathcal{B}(\lambda)$ such that the set $\{G_{q,\lambda}^{\mathrm{low}}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ forms a $\mathbb{Q}(q)$-basis of $V_{q,w}(\lambda)$; this subset $\mathcal{B}_w(\lambda)$ is called a *Demazure crystal*. We set

$$b_\lambda \coloneqq v_{q,\lambda} \bmod qL(\lambda) \in \mathcal{B}(\lambda).$$

The following is a fundamental property of Demazure crystals.

PROPOSITION 1.3.12 (see [**29**, Proposition 3.2.3]). *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ *be a reduced word for* $w \in W$*, and* $\lambda \in P_+$*.*

(1) *The following equality holds*:
$$\mathcal{B}_w(\lambda) = \{\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\lambda \mid a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}.$$
(2) $\tilde{e}_i \mathcal{B}_w(\lambda) \subset \mathcal{B}_w(\lambda) \cup \{0\}$ *for all* $i \in I$*.*

Denote by $b_\infty \in \mathcal{B}(\infty)$ the element corresponding to $1 \in U_q(\mathfrak{u}^-)$, that is, $b_\infty \coloneqq 1 \bmod qL(\infty)$.

PROPOSITION 1.3.13 (see [**29**, Proposition 3.2.5]). *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ *be a reduced word for* $w \in W$*, and* $\lambda \in P_+$*.*

(1) *The subset*
$$\mathcal{B}_w(\infty) \coloneqq \{\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\infty \mid a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}\} \subset \mathcal{B}(\infty)$$
is independent of the choice of a reduced word $\mathbf{i}$*.*
(2) $\tilde{e}_i \mathcal{B}_w(\infty) \subset \mathcal{B}_w(\infty) \cup \{0\}$ *for all* $i \in I$*.*
(3) *The equality* $\pi_\lambda(\mathcal{B}_w(\infty)) = \mathcal{B}_w(\lambda) \cup \{0\}$ *holds*; *hence* π_λ *induces a bijective map* $\pi_\lambda \colon \widetilde{\mathcal{B}}_w(\lambda) \to \mathcal{B}_w(\lambda)$*, where* $\widetilde{\mathcal{B}}_w(\lambda) \coloneqq \mathcal{B}_w(\infty) \cap \widetilde{\mathcal{B}}(\lambda)$*.*

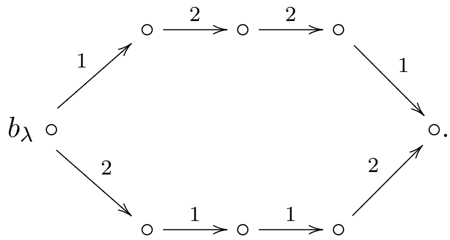
The subset $\mathcal{B}_w(\infty)$ is also called a *Demazure crystal*. Since

$$\mathcal{B}(\infty) = \bigcup_{\lambda \in P_+} \widetilde{\mathcal{B}}(\lambda) \tag{1.3.1}$$

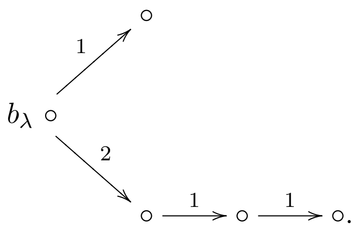
by [**27**, Corollary 4.4.5], we deduce that

$$\begin{aligned}\mathcal{B}_w(\infty) &= \bigcup_{\lambda \in P_+} (\mathcal{B}_w(\infty) \cap \widetilde{\mathcal{B}}(\lambda)) \\ &= \bigcup_{\lambda \in P_+} \widetilde{\mathcal{B}}_w(\lambda).\end{aligned}$$

EXAMPLE 1.3.14. Let $G = SL_3(\mathbb{C})$, and $\lambda = \alpha_1 + \alpha_2 \in P_+$. Then, the crystal basis $\mathcal{B}(\lambda)$ is given as follows, where $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$:



In addition, for $w = s_1 s_2 \in W$, the following directed graph gives the Demazure crystal $\mathcal{B}_w(\lambda)$:



If we define a $\mathbb{Z}[q, q^{-1}]$-module structure on $\mathbb{C}$ by $q \mapsto 1$, then the $\mathbb{C}$-algebra $U_{q,\mathbb{Z}}(\mathfrak{u}^-) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ is isomorphic to the universal enveloping algebra $U(\mathfrak{u}^-)$ of $\mathfrak{u}^-$ by $F_i^{(k)} \otimes 1 \mapsto f_i^k / k!$; this process is called the *specialization at* $q = 1$. For $b \in \mathcal{B}(\infty)$, denote by $G^{\mathrm{low}}(b) \in U(\mathfrak{u}^-)$ the specialization of $G_q^{\mathrm{low}}(b)$ at $q = 1$, that is, $G^{\mathrm{low}}(b) := G_q^{\mathrm{low}}(b) \otimes 1 \in U_{q,\mathbb{Z}}(\mathfrak{u}^-) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \simeq U(\mathfrak{u}^-)$. Note that the $U_{q,\mathbb{Z}}(\mathfrak{u}^-)$-submodule $V_{q,\mathbb{Z}}(\lambda)$ of $V_q(\lambda)$ is invariant under the action of E_i, F_i, and $(K_i - K_i^{-1})/(q_i - q_i^{-1})$ for all $i \in I$. The $\mathbb{C}$-vector space $V_{q,\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ has a $\mathfrak{g}$-module structure given by

$$e_i(v \otimes c) := (E_i v) \otimes c,\ f_i(v \otimes c) := (F_i v) \otimes c,\ h_i(v \otimes c) := \left(\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} v \right) \otimes c$$

for $i \in I$, $v \in V_{q,\mathbb{Z}}(\lambda)$, and $c \in \mathbb{C}$; this $\mathfrak{g}$-module is isomorphic to $V(\lambda)$ (see, for instance, [**19**, Lemma 5.14]). We denote by $G_\lambda^{\mathrm{low}}(b) \in V(\lambda)$ the specialization of $G_{q,\lambda}^{\mathrm{low}}(b)$ at $q = 1$, that is, $G_\lambda^{\mathrm{low}}(b) := G_{q,\lambda}^{\mathrm{low}}(b) \otimes 1 \in V_{q,\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \simeq V(\lambda)$. For $s, k \in \mathbb{Z}_{\geq 0}$ such that $k \leq s$, let $\binom{s}{k}$ denote the usual binomial coefficient. The following is easily seen by Proposition 1.3.8, Corollary 1.3.10, and [**29**, Remark 3.2.6].

COROLLARY 1.3.15 (see [**27, 28, 29**]). *For* $\lambda \in P_+$, *let* $\pi_\lambda \colon U(\mathfrak{u}^-) \twoheadrightarrow V(\lambda)$ *denote the surjective* $U(\mathfrak{u}^-)$*-module homomorphism given by* $u \mapsto u v_\lambda$.

(1) $\pi_\lambda(G^{\mathrm{low}}(b)) = G_\lambda^{\mathrm{low}}(\pi_\lambda(b))$ *for all* $b \in \widetilde{\mathcal{B}}(\lambda)$, *and* $\pi_\lambda(G^{\mathrm{low}}(b)) = 0$ *for all* $b \in \mathcal{B}(\infty) \setminus \widetilde{\mathcal{B}}(\lambda)$.

(2) *For all* $i \in I$, $b \in \mathcal{B}(\lambda)$, *and* $k \in \mathbb{Z}_{\geq 0}$,

$$e_i^{(k)} \cdot G_\lambda^{\mathrm{low}}(b) \in \binom{\varphi_i(b)+k}{k} G_\lambda^{\mathrm{low}}(\tilde{e}_i^k b) + \sum_{\substack{b' \in \mathcal{B}(\lambda);\\ \mathrm{wt}(b')=\mathrm{wt}(b)+k\alpha_i,\\ \varphi_i(b')>\varphi_i(b)+k}} \mathbb{Z} G_\lambda^{\mathrm{low}}(b'),$$

$$f_i^{(k)} \cdot G_\lambda^{\mathrm{low}}(b) \in \binom{\varepsilon_i(b)+k}{k} G_\lambda^{\mathrm{low}}(\tilde{f}_i^k b) + \sum_{\substack{b' \in \mathcal{B}(\lambda);\\ \mathrm{wt}(b')=\mathrm{wt}(b)-k\alpha_i,\\ \varepsilon_i(b')>\varepsilon_i(b)+k}} \mathbb{Z} G_\lambda^{\mathrm{low}}(b'),$$

where $e_i^{(k)} := e_i^k/k!$, $f_i^{(k)} := f_i^k/k!$ *for* $i \in I$, $k \in \mathbb{Z}_{\geq 0}$, *and* $G_\lambda^{\mathrm{low}}(0) := 0$.

(3) *For all* $i \in I$, $b \in \mathcal{B}(\infty)$, *and* $k \in \mathbb{Z}_{\geq 0}$,

$$f_i^{(k)} \cdot G^{\mathrm{low}}(b) \in \binom{\varepsilon_i(b)+k}{k} G^{\mathrm{low}}(\tilde{f}_i^k b) + \sum_{\substack{b' \in \mathcal{B}(\infty);\\ \mathrm{wt}(b')=\mathrm{wt}(b)-k\alpha_i,\\ \varepsilon_i(b')>\varepsilon_i(b)+k}} \mathbb{Z} G^{\mathrm{low}}(b'),$$

$$G^{\mathrm{low}}(b) \cdot f_i^{(k)} \in \binom{\varepsilon_i^*(b)+k}{k} G^{\mathrm{low}}((\tilde{f}_i^*)^k b) + \sum_{\substack{b' \in \mathcal{B}(\infty);\\ \mathrm{wt}(b')=\mathrm{wt}(b)-k\alpha_i,\\ \varepsilon_i^*(b')>\varepsilon_i^*(b)+k}} \mathbb{Z} G^{\mathrm{low}}(b').$$

(4) *For all* $i \in I$ *and* $k \in \mathbb{Z}_{\geq 0}$,

$$f_i^k U(\mathfrak{u}^-) = \bigoplus_{b \in \mathcal{B}(\infty);\ \varepsilon_i(b) \geq k} \mathbb{C} G^{\mathrm{low}}(b), \text{ and}$$

$$U(\mathfrak{u}^-) f_i^k = \bigoplus_{b \in \mathcal{B}(\infty);\ \varepsilon_i^*(b) \geq k} \mathbb{C} G^{\mathrm{low}}(b).$$

(5) *For all* $w \in W$, *the set* $\{G_\lambda^{\mathrm{low}}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ *forms a* $\mathbb{C}$*-basis of the Demazure module* $V_w(\lambda)$.

It is well-known that the kernel of the map $\pi_\lambda \colon U(\mathfrak{u}^-) \twoheadrightarrow V(\lambda)$ is equal to $\sum_{i \in I} U(\mathfrak{u}^-) f_i^{\langle \lambda, h_i \rangle + 1}$. Hence, by Corollary 1.3.15 (4), the set $\widetilde{\mathcal{B}}(\lambda)$ is described in terms of ε_i^* as follows.

Corollary 1.3.16. *For* $\lambda \in P_+$, *the following equality holds*:

$$\widetilde{\mathcal{B}}(\lambda) = \{b \in \mathcal{B}(\infty) \mid \varepsilon_i^*(b) \leq \langle \lambda, h_i \rangle \text{ for all } i \in I\}.$$

1.4. String polytopes

Here, we recall the definition of Berenstein-Littelmann-Zelevinsky's string polytopes, and also review the main result of [**32**]. In the theory of crystal bases, it is important to give their concrete parametrizations. In this thesis, we use two parametrizations: Berenstein-Littelmann-Zelevinsky's string parametrization [**4, 5, 6, 42**] and the Kashiwara embedding [**29, 53**].

Definition 1.4.1 (see [**42**, Sect. 1]). Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$, and $b \in \mathcal{B}_w(\infty)$. Define $\Phi_{\mathbf{i}}(b) = (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 0}^r$ by

$$\begin{aligned} a_1 &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_1}^{a} b \neq 0\}, \\ a_2 &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_2}^{a} \tilde{e}_{i_1}^{a_1} b \neq 0\}, \\ &\vdots \\ a_r &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_r}^{a} \tilde{e}_{i_{r-1}}^{a_{r-1}} \cdots \tilde{e}_{i_1}^{a_1} b \neq 0\}. \end{aligned}$$

The $\Phi_{\mathbf{i}}(b)$ is called *Berenstein-Littelmann-Zelevinsky's string parametrization* of b with respect to $\mathbf{i}$.

The map $\Phi_{\mathbf{i}} \colon \mathcal{B}_w(\infty) \to \mathbb{Z}_{\geq 0}^r$ is indeed injective. By the bijective map $\pi_\lambda \colon \widetilde{\mathcal{B}}_w(\lambda) \xrightarrow{\sim} \mathcal{B}_w(\lambda)$ in Proposition 1.3.13 (3), the map $\Phi_{\mathbf{i}}$ induces a map $\Phi_{\mathbf{i}} \colon \mathcal{B}_w(\lambda) \to \mathbb{Z}_{\geq 0}^r$, called the *string parametrization* of $\mathcal{B}_w(\lambda)$ with respect to $\mathbf{i}$. Let $\mathcal{C}_{\mathbf{i}} \subset \mathbb{R}^r$ denote the smallest real closed cone containing $\Phi_{\mathbf{i}}(\mathcal{B}_w(\infty))$; the $\mathcal{C}_{\mathbf{i}}$ is called the *string cone* for $\mathcal{B}_w(\infty)$ with respect to $\mathbf{i}$. A subset $\mathcal{C} \subset \mathbb{R}^r$ is said to be a *rational convex polyhedral cone* if there exists a finite number of rational points $\mathbf{a}_1, \ldots, \mathbf{a}_l \in \mathbb{Q}^r$ such that $\mathcal{C} = \mathbb{R}_{\geq 0}\mathbf{a}_1 + \cdots + \mathbb{R}_{\geq 0}\mathbf{a}_l$. The following is a fundamental property of $\mathcal{C}_{\mathbf{i}}$.

Proposition 1.4.2 (see [**6**, Sect. 3.2 and Theorem 3.10] and [**42**, Sect. 1]). *Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$. Then, the string cone $\mathcal{C}_{\mathbf{i}}$ is a rational convex polyhedral cone, and the equality $\Phi_{\mathbf{i}}(\mathcal{B}_w(\infty)) = \mathcal{C}_{\mathbf{i}} \cap \mathbb{Z}^r$ holds.*

Definition 1.4.3 (see [**32**, Definition 3.5] and [**42**, Sect. 1]). Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$, and $\lambda \in P_+$. Define a subset $\mathcal{S}_{\mathbf{i}}(\lambda) \subset \mathbb{Z}_{>0} \times \mathbb{Z}^r$ by

$$\mathcal{S}_{\mathbf{i}}(\lambda) \coloneqq \bigcup_{k \in \mathbb{Z}_{>0}} \{(k, \Phi_{\mathbf{i}}(b)) \mid b \in \mathcal{B}_w(k\lambda)\},$$

and denote by $\mathcal{C}_{\mathbf{i}}(\lambda) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^r$ the smallest real closed cone containing $\mathcal{S}_{\mathbf{i}}(\lambda)$. Then, we define a subset $\Delta_{\mathbf{i}}(\lambda) \subset \mathbb{R}^r$ by

$$\Delta_{\mathbf{i}}(\lambda) \coloneqq \{\mathbf{a} \in \mathbb{R}^r \mid (1, \mathbf{a}) \in \mathcal{C}_{\mathbf{i}}(\lambda)\}.$$

This subset $\Delta_{\mathbf{i}}(\lambda)$ is called *Berenstein-Littelmann-Zelevinsky's string polytope* for $\mathcal{B}_w(\lambda)$ with respect to $\mathbf{i}$.

A subset $\Delta \subset \mathbb{R}^r$ is said to be a *rational convex polytope* if it is the convex hull of a finite number of rational points. For $\lambda \in P_+$, we see by [**42**, Sect. 1] that the image $\Phi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$ is identical to the set of $(a_1, \ldots, a_r) \in \Phi_{\mathbf{i}}(\mathcal{B}_w(\infty))$ satisfying the following inequalities:

$$\begin{aligned} &0 \leq a_r \leq \langle \lambda, h_{i_r} \rangle, \\ &0 \leq a_{r-1} \leq \langle \lambda - a_r \alpha_{i_r}, h_{i_{r-1}} \rangle, \\ &\vdots \\ &0 \leq a_1 \leq \langle \lambda - a_2 \alpha_{i_2} - \cdots - a_r \alpha_{i_r}, h_{i_1} \rangle. \end{aligned}$$

Hence we obtain the following by Proposition 1.4.2.

PROPOSITION 1.4.4 (see [**6**, Sect. 3.2 and Theorem 3.10] and [**42**, Sect. 1]). *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ *be a reduced word for* $w \in W$, *and* $\lambda \in P_+$.

(1) *The real closed cone* $\mathcal{C}_{\mathbf{i}}(\lambda)$ *is a rational convex polyhedral cone*; *in addition, the following equality holds*:
$$\mathcal{S}_{\mathbf{i}}(\lambda) = \mathcal{C}_{\mathbf{i}}(\lambda) \cap (\mathbb{Z}_{>0} \times \mathbb{Z}^r).$$

(2) *The set* $\Delta_{\mathbf{i}}(\lambda)$ *is identical to the set of* $(a_1, \ldots, a_r) \in \mathcal{C}_{\mathbf{i}}$ *satisfying the following inequalities*:
$$\begin{aligned}
&0 \le a_r \le \langle \lambda, h_{i_r} \rangle, \\
&0 \le a_{r-1} \le \langle \lambda - a_r \alpha_{i_r}, h_{i_{r-1}} \rangle, \\
&\vdots \\
&0 \le a_1 \le \langle \lambda - a_2 \alpha_{i_2} - \cdots - a_r \alpha_{i_r}, h_{i_1} \rangle.
\end{aligned}$$

(3) *The set* $\Delta_{\mathbf{i}}(\lambda)$ *is a rational convex polytope*; *in addition, the following equality holds*:
$$\Phi_{\mathbf{i}}(\mathcal{B}_w(\lambda)) = \Delta_{\mathbf{i}}(\lambda) \cap \mathbb{Z}^r.$$

REMARK 1.4.5. A system of explicit linear inequalities defining $\mathcal{C}_{\mathbf{i}}$ is given in [**6**, Theorem 3.10]; hence we obtain an explicit description of the string polytope $\Delta_{\mathbf{i}}(\lambda)$ by Proposition 1.4.4 (2).

Let $\tau_\lambda \in H^0(X(w), \mathcal{L}_\lambda) = V_w(\lambda)^*$ denote the nonzero section given by
$$\tau_\lambda(G_\lambda^{\mathrm{low}}(b)) \coloneqq \begin{cases} 1 & \text{if } b = b_\lambda, \\ 0 & \text{if } b \ne b_\lambda \end{cases}$$
for $b \in \mathcal{B}_w(\lambda)$ (see Corollary 1.3.15 (5)). We define an $\mathbb{R}$-linear automorphism $\eta \colon \mathbb{R} \times \mathbb{R}^r \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}^r$ by $\eta(k, \mathbf{a}) \coloneqq (k, -\mathbf{a})$. The following is the main result of [**32**].

THEOREM 1.4.6 (see [**32**, Sect. 4]). *Let* $\mathbf{i} \in I^r$ *be a reduced word for* $w \in W$, *and* $\lambda \in P_+$. *Then, the following equalities hold*:
$$\mathcal{S}_{\mathbf{i}}(\lambda) = \eta(S(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)),\ \mathcal{C}_{\mathbf{i}}(\lambda) = \eta(C(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)),\ \textit{and}$$
$$\Delta_{\mathbf{i}}(\lambda) = -\Delta(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda).$$

CHAPTER 2

Newton-Okounkov polytopes and polyhedral realizations of crystal bases

In this chapter, we prove that Nakashima-Zelevinsky's polyhedral realization of a highest weight crystal basis is identical to the Newton-Okounkov polytope of a Schubert variety associated with the highest term valuation $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$ defined in Sect. 1.2. This chapter except Sect. 2.2 is based on joint work with Satoshi Naito [**14**]; Sect. 2.2 is based on the paper [**15**].

2.1. Polyhedral realizations of crystal bases

In this section, we recall some fundamental properties of Nakashima-Zelevinsky's polyhedral realizations of crystal bases, following [**50, 51, 53**]. Let G be a connected, simply-connected semisimple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, W the Weyl group, and I an index set for the vertices of the Dynkin diagram. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote by $U^- \subset G$ the unipotent radical of the opposite Borel subgroup B^-, by $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of T, by $\mathfrak{t}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ the dual space of $\mathfrak{t}$, and by $\langle \cdot, \cdot \rangle \colon \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{C}$ the canonical pairing. Let $P \subset \mathfrak{t}^*$ be the weight lattice for $\mathfrak{g}$, $P_+ \subset P$ the set of dominant integral weights, $\{\alpha_i \mid i \in I\} \subset P$ the set of simple roots, and $\{h_i \mid i \in I\} \subset \mathfrak{t}$ the set of simple coroots. Consider an infinite sequence $\mathbf{j} = (\ldots, j_k, \ldots, j_2, j_1)$ in I such that $j_k \neq j_{k+1}$ for all $k \in \mathbb{Z}_{>0}$, and such that the cardinality of $\{k \in \mathbb{Z}_{>0} \mid j_k = i\}$ is ∞ for each $i \in I$. Following [**29, 53**], we associate to $\mathbf{j}$ a crystal structure on

$$\mathbb{Z}^\infty := \{(\ldots, a_k, \ldots, a_2, a_1) \mid a_k \in \mathbb{Z} \text{ for } k \in \mathbb{Z}_{>0}, \text{ and } a_k = 0 \text{ for } k \gg 0\}$$

as follows. For $k \in \mathbb{Z}_{>0}$, $i \in I$, and $\mathbf{a} = (\ldots, a_l, \ldots, a_2, a_1) \in \mathbb{Z}^\infty$, we set

$$\begin{aligned}
\sigma_k(\mathbf{a}) &:= a_k + \sum_{l>k} \langle \alpha_{j_l}, h_{j_k} \rangle a_l \in \mathbb{Z}, \\
\sigma^{(i)}(\mathbf{a}) &:= \max\{\sigma_k(\mathbf{a}) \mid k \in \mathbb{Z}_{>0},\ j_k = i\} \in \mathbb{Z}, \text{ and} \\
M^{(i)}(\mathbf{a}) &:= \{k \in \mathbb{Z}_{>0} \mid j_k = i,\ \sigma_k(\mathbf{a}) = \sigma^{(i)}(\mathbf{a})\}.
\end{aligned}$$

Since $a_l = 0$ for $l \gg 0$, the integers $\sigma_k(\mathbf{a}), \sigma^{(i)}(\mathbf{a})$ are well-defined; also, we have $\sigma^{(i)}(\mathbf{a}) \geq 0$. Moreover, $M^{(i)}(\mathbf{a})$ is a finite set if and only if $\sigma^{(i)}(\mathbf{a}) > 0$.

Define a crystal structure on $\mathbb{Z}^\infty$ by

$$\mathrm{wt}(\mathbf{a}) \coloneqq -\sum_{k\in\mathbb{Z}_{>0}} a_k\alpha_{j_k},\ \varepsilon_i(\mathbf{a}) \coloneqq \sigma^{(i)}(\mathbf{a}),\ \varphi_i(\mathbf{a}) \coloneqq \varepsilon_i(\mathbf{a}) + \langle \mathrm{wt}(\mathbf{a}), h_i\rangle,\ \text{and}$$

$$\tilde{e}_i\mathbf{a} \coloneqq \begin{cases}(a_k - \delta_{k,\max M^{(i)}(\mathbf{a})})_{k\in\mathbb{Z}_{>0}} & \text{if } \sigma^{(i)}(\mathbf{a}) > 0,\\ 0 & \text{otherwise,}\end{cases}$$

$$\tilde{f}_i\mathbf{a} \coloneqq (a_k + \delta_{k,\min M^{(i)}(\mathbf{a})})_{k\in\mathbb{Z}_{>0}}$$

for $i \in I$ and $\mathbf{a} = (\ldots, a_k, \ldots, a_2, a_1) \in \mathbb{Z}^\infty$, where $\delta_{k,l}$ is the Kronecker delta for $k, l \in \mathbb{Z}_{>0}$; we denote this crystal by $\mathbb{Z}^\infty_{\mathbf{j}}$.

Proposition 2.1.1 (see [**53**, Sect. 2.4]). *The following hold.*

(1) *There exists a unique strict embedding of crystals* $\Psi_{\mathbf{j}}\colon \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^\infty_{\mathbf{j}}$ *such that* $\Psi_{\mathbf{j}}(b_\infty) = (\ldots, 0, \ldots, 0, 0)$.
(2) *Write* $\Psi_{\mathbf{j}}(b) = (\ldots, a_k, \ldots, a_2, a_1)$ *for* $b \in \mathcal{B}(\infty)$. *Then, the following equalities hold*:

$$b^* = \tilde{f}_{i_1}^{a_1}\tilde{f}_{i_2}^{a_2}\cdots b_\infty,\ \textit{and}\ \tilde{e}_{i_{k-1}}\tilde{f}_{i_k}^{a_k}\tilde{f}_{i_{k+1}}^{a_{k+1}}\cdots b_\infty = 0\ \textit{for all}\ k \in \mathbb{Z}_{>1}.$$

The embedding $\Psi_{\mathbf{j}}$ is called the *Kashiwara embedding* with respect to $\mathbf{j}$. Recall the crystal R_λ in Example 1.3.3. By [**50**, Theorem 3.1], there exists a unique strict embedding of crystals

$$\Omega_\lambda\colon \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\infty) \otimes R_\lambda$$

such that $\Omega_\lambda(b_\lambda) = b_\infty \otimes r_\lambda$. Note that $\Omega_\lambda(\mathcal{B}(\lambda)) = \{b \otimes r_\lambda \mid b \in \widetilde{\mathcal{B}}(\lambda)\}$, and that $\Omega_\lambda(\pi_\lambda(b)) = b \otimes r_\lambda$ for all $b \in \widetilde{\mathcal{B}}(\lambda)$, where $\pi_\lambda\colon \widetilde{\mathcal{B}}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda)$ is the bijective map given in Proposition 1.3.7 (2).

Theorem 2.1.2 ([**50**, Theorem 3.2]). *For* $\lambda \in P_+$*, there exists a unique strict embedding of crystals*

$$\Psi_{\mathbf{j}}\colon \mathcal{B}(\lambda) \xhookrightarrow{\Omega_\lambda} \mathcal{B}(\infty) \otimes R_\lambda \xhookrightarrow{\Psi_{\mathbf{j}}\otimes \mathrm{id}} \mathbb{Z}^\infty_{\mathbf{j}} \otimes R_\lambda$$

such that $\Psi_{\mathbf{j}}(b_\lambda) = (\ldots, 0, \ldots, 0, 0) \otimes r_\lambda$.

The embedding $\Psi_{\mathbf{j}}\colon \mathcal{B}(\lambda) \hookrightarrow \mathbb{Z}^\infty_{\mathbf{j}} \otimes R_\lambda$ is also called the *Kashiwara embedding* with respect to $\mathbf{j}$.

In the following, we give a parametrization of Demazure crystals. Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$, and extend it to an infinite sequence $\mathbf{j} = (\ldots, j_k, \ldots, j_2, j_1)$ in I as above, that is, $(j_r, \ldots, j_1) = (i_1, \ldots, i_r)$.

Proposition 2.1.3 ([**51**, Propositions 3.1, 3.3 (i)]). *For* $\lambda \in P_+$*, the following equalities hold*:

$$\Psi_{\mathbf{j}}(\mathcal{B}_w(\infty)) = \{(a_k)_{k\in\mathbb{Z}_{>0}} \in \Psi_{\mathbf{j}}(\mathcal{B}(\infty)) \mid a_k = 0\ \textit{for all}\ k > r\},$$
$$\Psi_{\mathbf{j}}(\mathcal{B}_w(\lambda)) = \{(a_k)_{k\in\mathbb{Z}_{>0}} \otimes r_\lambda \in \Psi_{\mathbf{j}}(\mathcal{B}(\lambda)) \mid a_k = 0\ \textit{for all}\ k > r\}.$$

DEFINITION 2.1.4. Define $\Psi_{\mathbf{i}}\colon \mathcal{B}_w(\infty) \hookrightarrow \mathbb{Z}^r$ by $\Psi_{\mathbf{i}}(b) := (a_1, \ldots, a_r)$ when $\Psi_{\mathbf{j}}(b) = (\ldots, 0, 0, a_1, \ldots, a_{r-1}, a_r)$; this is also called the *Kashiwara embedding* with respect to $\mathbf{i}$. The *Kashiwara embedding* $\Psi_{\mathbf{i}}\colon \mathcal{B}_w(\lambda) \hookrightarrow \mathbb{Z}^r$ with respect to $\mathbf{i}$ is similarly defined.

The maps $\Psi_{\mathbf{i}}\colon \mathcal{B}_w(\infty) \hookrightarrow \mathbb{Z}^r$ and $\Psi_{\mathbf{i}}\colon \mathcal{B}_w(\lambda) \hookrightarrow \mathbb{Z}^r$ are independent of the choice of an extension $\mathbf{j}$ by Proposition 2.1.1 (2). Note that the bijective map $\pi_\lambda\colon \widetilde{\mathcal{B}}_w(\lambda) \xrightarrow{\sim} \mathcal{B}_w(\lambda)$ in Proposition 1.3.13 (3) preserves the values of $\Psi_{\mathbf{i}}$, that is, $\Psi_{\mathbf{i}}(\pi_\lambda(b)) = \Psi_{\mathbf{i}}(b)$ for all $b \in \widetilde{\mathcal{B}}_w(\lambda)$.

REMARK 2.1.5. Under some conditions on $\mathbf{i}$, a system of explicit linear inequalities defining $\Psi_{\mathbf{i}}(\mathcal{B}_w(\infty))$ is given in [**53**, Theorem 3.1].

REMARK 2.1.6. Let $w_0 \in W$ be the longest element. By [**29**, Proposition 3.2.5 (ii)], it follows that $\tilde{f}_i\mathcal{B}_{w_0}(\infty) \subset \mathcal{B}_{w_0}(\infty)$ for all $i \in I$, and hence that $\mathcal{B}_{w_0}(\infty) = \mathcal{B}(\infty)$. Similarly, we have $\mathcal{B}_{w_0}(\lambda) = \mathcal{B}(\lambda)$ by [**29**, Proposition 3.2.3 (iii)]. From these, if $\mathbf{i} = (i_1, \ldots, i_N) \in I^N$ is a reduced word for w_0, then we obtain the Kashiwara embeddings $\Psi_{\mathbf{i}}\colon \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^N$ and $\Psi_{\mathbf{i}}\colon \mathcal{B}(\lambda) \hookrightarrow \mathbb{Z}^N$.

Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$, and recall the string parametrization $\Phi_{\mathbf{i}}$ in Sect. 1.4. By [**29**, Proposition 3.3.1], we have $\mathcal{B}_w(\infty)^* = \mathcal{B}_{w^{-1}}(\infty)$; hence the map $\Phi_{\mathbf{i}^{\mathrm{op}}} \circ *\colon \mathcal{B}_w(\infty) \to \mathbb{Z}^r$ is well-defined, where $\mathbf{i}^{\mathrm{op}} := (i_r, \ldots, i_1)$ is a reduced word for w^{-1}. The following is an immediate consequence of Proposition 2.1.1 (2).

COROLLARY 2.1.7. *Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$. Then, the equality*

$$\Psi_{\mathbf{i}}(b) = \Phi_{\mathbf{i}^{\mathrm{op}}}(b^*)^{\mathrm{op}}$$

holds for all $b \in \mathcal{B}_w(\infty)$, where $\mathbf{a}^{\mathrm{op}} := (a_r, \ldots, a_1)$ for $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$.

DEFINITION 2.1.8 (see [**50**, Sects. 3, 4], [**51**, Sect. 3.1], and [**53**, Sect. 3]). Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$, and $\lambda \in P_+$. Define a subset $\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda) \subset \mathbb{Z}_{>0} \times \mathbb{Z}^r$ by

$$\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda) := \bigcup_{k \in \mathbb{Z}_{>0}} \{(k, \Psi_{\mathbf{i}}(b)) \mid b \in \mathcal{B}_w(k\lambda)\},$$

and denote by $\widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^r$ the smallest real closed cone containing $\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda)$. Then, we define a subset $\widetilde{\Delta}_{\mathbf{i}}(\lambda) \subset \mathbb{R}^r$ by

$$\widetilde{\Delta}_{\mathbf{i}}(\lambda) := \{\mathbf{a} \in \mathbb{R}^r \mid (1, \mathbf{a}) \in \widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda)\}.$$

We call this subset $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ the *Nakashima-Zelevinsky polytope* for $\mathcal{B}_w(\lambda)$ associated with $\mathbf{i}$.

REMARK 2.1.9. In [**14, 15**], the set $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is called Nakashima-Zelevinsky's polyhedral realization. However, the word "polyhedral realization" is originally used in [**50, 51, 53**] to mean the realization of a crystal basis as the set of lattice points in some explicit rational convex polyhedron. Hence the terminology in [**14, 15**] is slightly inaccurate.

We will prove that the set $\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda)$ is identical to the set of lattice points in $\widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda) \cap (\mathbb{R}_{>0} \times \mathbb{R}^r)$, and that the image $\Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$ is identical to the set of lattice points in $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$. Recall the string cone $\mathcal{C}_{\mathbf{i}} \subset \mathbb{R}^r$ in Sect. 1.4. By Corollary 2.1.7, we have

$$\Psi_{\mathbf{i}}(\mathcal{B}_w(\infty)) = \Phi_{\mathbf{i}^{\mathrm{op}}}(\mathcal{B}_{w^{-1}}(\infty))^{\mathrm{op}} = \mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}} \cap \mathbb{Z}^r,$$

where $\mathbf{a}^{\mathrm{op}} := (a_r, \ldots, a_1)$ for $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$, and $H^{\mathrm{op}} := \{\mathbf{a}^{\mathrm{op}} \mid \mathbf{a} \in H\}$ for $H \subset \mathbb{R}^r$.

PROPOSITION 2.1.10. *There exists a piecewise-linear function $\psi_{\mathbf{i},i}(\mathbf{a})$ on $\mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}}$ for each $i \in I$ such that*

$$\varepsilon_i(\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\infty) = \psi_{\mathbf{i},i}(\mathbf{a})$$

for all $\mathbf{a} = (a_1, \ldots, a_r) \in \Psi_{\mathbf{i}}(\mathcal{B}_w(\infty))$.

PROOF. For $(a_1, \ldots, a_r) \in \Psi_{\mathbf{i}}(\mathcal{B}_w(\infty))$, Proposition 1.3.7 (3) implies that $\pi_\lambda(\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\infty) = \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\lambda$. Therefore, if we take $\lambda \in P_+$ such that $\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\lambda \neq 0$, then we deduce by Proposition 1.3.7 (5) that

$$\varepsilon_i(\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\infty) = \varepsilon_i(\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\lambda)$$

for all $i \in I$. From this and [**12**, Remark 5.4 and Corollary 5.20], the assertion of the proposition follows immediately. □

COROLLARY 2.1.11. *Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$, and $\lambda \in P_+$.*

(1) *The set $\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda)$ is identical to the set of $(k, \mathbf{a}) \in \mathbb{Z}_{>0} \times \mathbb{Z}^r$ such that $\mathbf{a} \in \mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}}$, and such that $\psi_{\mathbf{i},i}(\mathbf{a}) \leq \langle k\lambda, h_i \rangle$ for all $i \in I$. In particular, the real closed cone $\widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda)$ is identical to the set of $(k, \mathbf{a}) \in \mathbb{R}_{\geq 0} \times \mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}}$ such that $\psi_{\mathbf{i},i}(\mathbf{a}) \leq \langle k\lambda, h_i \rangle$ for all $i \in I$.*

(2) *The real closed cone $\widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda)$ is a finite union of rational convex polyhedral cones, and the equality $\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda) = \widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda) \cap (\mathbb{Z}_{>0} \times \mathbb{Z}^r)$ holds.*

PROOF. Part (2) is an immediate consequence of part (1); hence it is sufficient to prove part (1). By Corollary 1.3.16 and Proposition 2.1.1 (2), we deduce that

$$\begin{aligned}
&\Psi_{\mathbf{i}}(\mathcal{B}_w(k\lambda)) \\
&= \{(a_1, \ldots, a_r) \in \Psi_{\mathbf{i}}(\mathcal{B}_w(\infty)) \mid \varepsilon_i(\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} b_\infty) \leq \langle k\lambda, h_i \rangle \text{ for all } i \in I\} \\
&= \{\mathbf{a} \in \mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}} \cap \mathbb{Z}^r \mid \psi_{\mathbf{i},i}(\mathbf{a}) \leq \langle k\lambda, h_i \rangle \text{ for all } i \in I\} \\
&\text{(by Proposition 2.1.10 since } \Psi_{\mathbf{i}}(\mathcal{B}_w(\infty)) = \mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}} \cap \mathbb{Z}^r)
\end{aligned}$$

for all $k \in \mathbb{Z}_{>0}$. This implies the first assertion of part (1). Then, since $\psi_{\mathbf{i},i}$ is piecewise-linear, the second assertion of part (1) follows immediately. This proves the corollary. □

By Corollary 2.1.11 (1) and the definition of $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$, we obtain the following.

COROLLARY 2.1.12. *The set $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is identical to the set of $\mathbf{a} \in \mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}}$ such that $\psi_{\mathbf{i},i}(\mathbf{a}) \leq \langle \lambda, h_i \rangle$ for all $i \in I$.*

COROLLARY 2.1.13. *The set $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is a finite union of rational convex polytopes, and the equality $\widetilde{\Delta}_{\mathbf{i}}(\lambda) \cap \mathbb{Z}^r = \Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$ holds.*

PROOF. The second assertion is an immediate consequence of Corollary 2.1.11 (2); hence it suffices to prove the first assertion. By Corollary 1.3.16 and the crystal structure on $\mathbb{Z}^{\infty}_{\mathbf{j}}$, we deduce that

$$0 \leq a_k \leq \langle \lambda, h_{i_k} \rangle + \sum_{k<l\leq r} |\langle \alpha_{i_l}, h_{i_k} \rangle| a_l$$

for all $(a_1, \ldots, a_r) \in \widetilde{\Delta}_{\mathbf{i}}(\lambda)$ and $1 \leq k \leq r$; therefore, the set $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is bounded, and hence compact. Also, by Corollary 2.1.12, the set $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is given by a finite number of piecewise-linear inequalities. These imply the first assertion of the corollary. □

We will prove in Sect. 2.3 that the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is identical to the Newton-Okounkov polytope $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}^{\mathrm{high}}_{\mathbf{i}}, \tau_\lambda)$, and that these are indeed rational convex polytopes; here, we need not assume that $(\mathbf{j}, \lambda)$ is ample (see Definition 2.4.1 for the definition). When it is ample, we obtain a system of explicit affine inequalities defining $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ (see Corollary 2.4.3).

2.2. Perfect bases

Here, we review some fundamental properties of perfect bases of the space $H^0(G/B, \mathcal{L}_\lambda)$ of global sections and of the coordinate ring $\mathbb{C}[U^-]$. They are convenient tools for calculating Newton-Okounkov polytopes of Schubert varieties. Recall that $V(\lambda) = H^0(G/B, \mathcal{L}_\lambda)^*$ for $\lambda \in P_+$ is the irreducible highest weight G-module with highest weight λ and with highest weight vector v_λ. For $\mu \in P$, we set

$$V(\lambda)_\mu \coloneqq \{v \in V(\lambda) \mid h \cdot v = \langle \mu, h \rangle v \text{ for all } h \in \mathfrak{t}\}.$$

Note that the action of $\mathfrak{g}$ on the dual space $V(\lambda)^*$ is given by $\langle x \cdot f, v \rangle = -\langle f, x \cdot v \rangle$ for $x \in \mathfrak{g}$, $f \in V(\lambda)^*$, and $v \in V(\lambda)$, where $\langle \cdot, \cdot \rangle \colon V(\lambda)^* \times V(\lambda) \to \mathbb{C}$ is the canonical pairing. Since $V(\lambda) = \bigoplus_{\mu \in P} V(\lambda)_\mu$, the dual space $V(\lambda)^*_\mu \coloneqq \mathrm{Hom}_{\mathbb{C}}(V(\lambda)_\mu, \mathbb{C})$ is regarded as a $\mathbb{C}$-subspace of $V(\lambda)^*$. Let $e_i, f_i, h_i \in \mathfrak{g}$, $i \in I$, be the Chevalley generators such that $\{e_i, h_i \mid i \in I\} \subset \mathfrak{b} \coloneqq \mathrm{Lie}(B)$ and $\{f_i \mid i \in I\} \subset \mathfrak{u}^- \coloneqq \mathrm{Lie}(U^-)$. For $i \in I$, we define $\varepsilon_i \colon V(\lambda)^* \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ by

$$\varepsilon_i(f) \coloneqq \begin{cases} \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k \cdot f \neq 0\} & \text{if } f \in V(\lambda)^* \setminus \{0\}, \\ -\infty & \text{if } f = 0 \in V(\lambda)^*. \end{cases}$$

For $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, set

$$(V(\lambda)^*)^{<k,i} \coloneqq \{f \in V(\lambda)^* \mid \varepsilon_i(f) < k\}.$$

DEFINITION 2.2.1 (see [**3**, Definition 5.30] and [**24**, Definition 2.5]). Let $\lambda \in P_+$. A $\mathbb{C}$-basis $\mathbf{B}^{\mathrm{up}}(\lambda) \subset H^0(G/B, \mathcal{L}_\lambda) = V(\lambda)^*$ is said to be *perfect* if the following conditions hold:

(i) $\mathbf{B}^{\mathrm{up}}(\lambda) = \coprod_{\mu \in P} \mathbf{B}^{\mathrm{up}}(\lambda)_\mu$, where $\mathbf{B}^{\mathrm{up}}(\lambda)_\mu \coloneqq \mathbf{B}^{\mathrm{up}}(\lambda) \cap V(\lambda)^*_\mu$,
(ii) $\mathbf{B}^{\mathrm{up}}(\lambda)_\lambda = \{\tau_\lambda\}$, where $\langle \tau_\lambda, v_\lambda \rangle = 1$,

(iii) for $i \in I$ and $\tau \in \mathbf{B}^{\mathrm{up}}(\lambda)$ with $f_i \cdot \tau \neq 0$, there exists a unique element $\tilde{e}_i(\tau) \in \mathbf{B}^{\mathrm{up}}(\lambda)$ such that

$$f_i \cdot \tau \in \mathbb{C}^\times \tilde{e}_i(\tau) + (V(\lambda)^*)^{<\varepsilon_i(\tau)-1,i},$$

where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$,

(iv) for $\tau, \tau' \in \mathbf{B}^{\mathrm{up}}(\lambda)$, if there exists $i \in I$ such that $\tilde{e}_i(\tau) = \tilde{e}_i(\tau')$, then we have $\tau = \tau'$.

Let $U(\mathfrak{u}^-)$ be the universal enveloping algebra of $\mathfrak{u}^-$. The algebra $U(\mathfrak{u}^-)$ has a Hopf algebra structure given by the following coproduct Δ, counit ε, and antipode S:

$$\Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i,\ \varepsilon(f_i) = 0,\ \text{and}\ S(f_i) = -f_i$$

for $i \in I$. In addition, we regard $U(\mathfrak{u}^-)$ as a multigraded $\mathbb{C}$-algebra:

$$U(\mathfrak{u}^-) = \bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geq 0}^I} U(\mathfrak{u}^-)_{\mathbf{d}},$$

where the homogeneous component $U(\mathfrak{u}^-)_{\mathbf{d}}$ for $\mathbf{d} = (d_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$ is defined to be the $\mathbb{C}$-subspace of $U(\mathfrak{u}^-)$ spanned by all those elements $f_{j_1} \cdots f_{j_{|\mathbf{d}|}}$ such that the cardinality of $\{1 \leq k \leq |\mathbf{d}| \mid j_k = i\}$ is equal to d_i for every $i \in I$; here, we set $|\mathbf{d}| := \sum_{i \in I} d_i$. Let

$$U(\mathfrak{u}^-)^*_{\mathrm{gr}} = \bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geq 0}^I} U(\mathfrak{u}^-)^*_{\mathrm{gr},\mathbf{d}} := \bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geq 0}^I} \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{u}^-)_{\mathbf{d}}, \mathbb{C})$$

be the graded dual of $U(\mathfrak{u}^-)$ endowed with the dual Hopf algebra structure. Note that the coordinate ring $\mathbb{C}[U^-]$ also has a Hopf algebra structure given by the following coproduct Δ, counit ε, and antipode S:

$$\Delta(f)(u_1, u_2) = f(u_1 u_2),\ \varepsilon(f) = f(e),\ \text{and}\ S(f)(u) = f(u^{-1})$$

for $f \in \mathbb{C}[U^-]$ and $u, u_1, u_2 \in U^-$, where $e \in U^-$ denotes the identity element. It is well-known that this Hopf algebra $\mathbb{C}[U^-]$ is isomorphic to the dual Hopf algebra $U(\mathfrak{u}^-)^*_{\mathrm{gr}}$ as follows.

Lemma 2.2.2 (see, for instance, [**16**, Proposition 5.1]). *Define a map* $\Upsilon \colon U(\mathfrak{u}^-)^*_{\mathrm{gr}} \to \mathbb{C}[U^-]$ *by*

$$\Upsilon(\rho)(\exp(x)) := \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{\rho(x^l)}{l!}$$

for $\rho \in U(\mathfrak{u}^-)^*_{\mathrm{gr}}$ *and* $x \in \mathfrak{u}^-$; *here,* $\exp(x) \in U^-$ *and* $x^l \in U(\mathfrak{u}^-)$ *for* $l \in \mathbb{Z}_{\geq 0}$. *Then, the map* Υ *is an isomorphism of Hopf algebras.*

Let $\langle \cdot, \cdot \rangle \colon U(\mathfrak{u}^-)^*_{\mathrm{gr}} \times U(\mathfrak{u}^-) \to \mathbb{C}$ denote the canonical pairing. We define a $U(\mathfrak{u}^-)$-bimodule structure on $U(\mathfrak{u}^-)^*_{\mathrm{gr}}$ by

$$\langle x \cdot \rho, y \rangle := -\langle \rho, x \cdot y \rangle,\ \text{and}$$
$$\langle \rho \cdot x, y \rangle := -\langle \rho, y \cdot x \rangle$$

for $x \in \mathfrak{u}^-$, $\rho \in U(\mathfrak{u}^-)^*_{\mathrm{gr}}$, and $y \in U(\mathfrak{u}^-)$. Note that the coordinate ring $\mathbb{C}[U^-]$ has a natural U^--bimodule structure, which is given by

$$(u_1 \cdot f)(u_2) := f(u_1^{-1}u_2), \text{ and}$$
$$(f \cdot u_1)(u_2) := f(u_2 u_1^{-1})$$

for $u_1, u_2 \in U^-$ and $f \in \mathbb{C}[U^-]$. This induces a $U(\mathfrak{u}^-)$-bimodule structure on $\mathbb{C}[U^-]$. It is easily seen that the isomorphism $\Upsilon \colon U(\mathfrak{u}^-)^*_{\mathrm{gr}} \xrightarrow{\sim} \mathbb{C}[U^-]$ of Hopf algebras is compatible with the $U(\mathfrak{u}^-)$-bimodule structures. In this paper, we always identify $U(\mathfrak{u}^-)^*_{\mathrm{gr}}$ with $\mathbb{C}[U^-]$. Define a $\mathbb{C}$-algebra anti-involution $*$ on $U(\mathfrak{u}^-)$ by $f_i^* := f_i$ for all $i \in I$. This map is a $\mathbb{C}$-coalgebra involution; hence it induces a $\mathbb{C}$-algebra involution on $U(\mathfrak{u}^-)^*_{\mathrm{gr}} = \mathbb{C}[U^-]$, denoted also by $*$. For $i \in I$, we define $\varepsilon_i \colon \mathbb{C}[U^-] \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ by

$$\varepsilon_i(f) := \begin{cases} \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k \cdot f \neq 0\} & \text{if } f \in \mathbb{C}[U^-] \setminus \{0\}, \\ -\infty & \text{if } f = 0 \in \mathbb{C}[U^-]. \end{cases}$$

For $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, set

$$\mathbb{C}[U^-]^{<k,i} := \{f \in \mathbb{C}[U^-] \mid \varepsilon_i(f) < k\}.$$

Definition 2.2.3 (see [**3**, Definition 5.30] and [**25**, Definition 4.5]). A $\mathbb{C}$-basis $\mathbf{B}^{\mathrm{up}} \subset \mathbb{C}[U^-] = U(\mathfrak{u}^-)^*_{\mathrm{gr}}$ is said to be *perfect* if the following conditions hold:

(i) $\mathbf{B}^{\mathrm{up}} = \coprod_{\mathbf{d} \in \mathbb{Z}^I_{\geq 0}} \mathbf{B}^{\mathrm{up}}_{\mathbf{d}}$, where $\mathbf{B}^{\mathrm{up}}_{\mathbf{d}} := \mathbf{B}^{\mathrm{up}} \cap U(\mathfrak{u}^-)^*_{\mathrm{gr},\mathbf{d}}$,

(ii) $\mathbf{B}^{\mathrm{up}}_{(0,\ldots,0)} = \{\tau_\infty\}$, where $\langle \tau_\infty, 1\rangle = 1$,

(iii) for $i \in I$ and $\tau \in \mathbf{B}^{\mathrm{up}}$ with $f_i \cdot \tau \neq 0$, there exists a unique element $\tilde{e}_i(\tau) \in \mathbf{B}^{\mathrm{up}}$ such that
$$f_i \cdot \tau \in \mathbb{C}^\times \tilde{e}_i(\tau) + \mathbb{C}[U^-]^{<\varepsilon_i(\tau)-1,i},$$

(iv) for $\tau, \tau' \in \mathbf{B}^{\mathrm{up}}$, if there exists $i \in I$ such that $\tilde{e}_i(\tau) = \tilde{e}_i(\tau')$, then we have $\tau = \tau'$.

In addition, we always impose the following $*$-stability condition on a perfect basis:

(v) $(\mathbf{B}^{\mathrm{up}})^* = \mathbf{B}^{\mathrm{up}}$.

We list some examples of perfect bases.

Example 2.2.4. Recall that $\{G^{\mathrm{low}}_\lambda(b) \mid b \in \mathcal{B}(\lambda)\} \subset V(\lambda)$ and $\{G^{\mathrm{low}}(b) \mid b \in \mathcal{B}(\infty)\} \subset U(\mathfrak{u}^-)$ are (the specializations at $q = 1$ of) the lower global bases. Let $\{G^{\mathrm{up}}_\lambda(b) \mid b \in \mathcal{B}(\lambda)\} \subset H^0(G/B, \mathcal{L}_\lambda)$ and $\{G^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ denote their dual bases, respectively; these are called the *upper global bases* (= the *dual canonical bases*). They are perfect bases by Proposition 1.3.9 and Corollary 1.3.15 (2), (3).

Example 2.2.5. When $\mathfrak{g}$ is of simply-laced type, Lusztig [**45**] constructed a specific $\mathbb{C}$-basis of $U(\mathfrak{u}^-)$, called the *semicanonical basis*. The dual basis of the semicanonical basis, called the *dual semicanonical basis*, is a perfect basis by [**45**, Sect. 2.9 and Theorem 3.8].

EXAMPLE 2.2.6. Khovanov-Lauda [**35, 36**] and Rouquier [**57**] introduced a family $\{R_{\mathbf{d}} \mid \mathbf{d} \in \mathbb{Z}_{\geq 0}^I\}$ of $\mathbb{Z}$-graded algebras, called *Khovanov-Lauda-Rouquier algebras* or *quiver Hecke algebras*, which categorifies the negative half $U_q(\mathfrak{u}^-)$ of the quantized enveloping algebra $U_q(\mathfrak{g})$. To be more precise, let $G_0(R_{\mathbf{d}}\text{-gmod})$ denote the Grothendieck group of finite-dimensional $\mathbb{Z}$-graded $R_{\mathbf{d}}$-modules. Then, the direct sum

$$\bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geq 0}^I} G_0(R_{\mathbf{d}}\text{-gmod})$$

has a natural $\mathbb{Z}[q, q^{-1}]$-algebra structure whose product comes from the induction functor (see [**35**, Proposition 3.1]), where the action of q is induced from the grading shift functor. In addition, the $\mathbb{Z}[q, q^{-1}]$-algebra is isomorphic to a certain $\mathbb{Z}[q, q^{-1}]$-form $\widetilde{U}_{q,\mathbb{Z}}(\mathfrak{u}^-)$ of $U_q(\mathfrak{u}^-)$ (see [**35**, Proposition 3.4 and Theorem 3.17] and the diagram written before [**25**, Lemma 5.3]), which becomes the coordinate ring $\mathbb{C}[U^-]$ if we apply the functor $- \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$. The $\mathbb{Z}[q, q^{-1}]$-algebra $\bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geq 0}^I} G_0(R_{\mathbf{d}}\text{-gmod})$ has a $\mathbb{Z}[q, q^{-1}]$-basis consisting of the classes of self-dual graded simple modules; we call this $\mathbb{Z}[q, q^{-1}]$-basis the *KLR-basis*. The specialization of the KLR-basis at $q = 1$ is known to be a perfect basis of $\mathbb{C}[U^-]$ by [**25**, Lemmas 3.13 and 5.3] (cf. [**40**, Sect. 2.5.1]). The condition (v) holds since the involution $*$ is induced from the twist of $R_{\mathbf{d}}$-modules by the involutive automorphism σ of $R_{\mathbf{d}}$ in [**35**, Sect. 2.1] (see also [**47**, Sect. 12]).

In the following, we give the definition of crystals associated with perfect bases $\mathbf{B}^{\mathrm{up}}(\lambda)$ and $\mathbf{B}^{\mathrm{up}}$. As we will see below (Proposition 2.2.7), they are isomorphic to $\mathcal{B}(\lambda)$ and $\mathcal{B}(\infty)$ as crystals, respectively. Let $\lambda \in P_+$, and $\mathbf{B}^{\mathrm{up}}(\lambda)$ (resp., $\mathbf{B}^{\mathrm{up}}$) a perfect basis of $H^0(G/B, \mathcal{L}_\lambda)$ (resp., $\mathbb{C}[U^-]$). For $i \in I$ and $\tau \in \mathbf{B}^{\mathrm{up}}(\lambda)_\mu$ (resp., $\tau \in \mathbf{B}^{\mathrm{up}}_{(d_i)_{i \in I}}$), set

$$\begin{aligned}
&\mathrm{wt}(\tau) \coloneqq \mu \quad (\text{resp., } \mathrm{wt}(\tau) \coloneqq -\sum_{i \in I} d_i \alpha_i),\\
&\varphi_i(\tau) \coloneqq \varepsilon_i(\tau) + \langle \mathrm{wt}(\tau), h_i \rangle,\\
&\tilde{f}_i(\tau) \coloneqq \begin{cases} \tau' & \text{if } \tilde{e}_i(\tau') = \tau \text{ for some } \tau', \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Then, we see that the sextuple $(\mathbf{B}^{\mathrm{up}}(\lambda);\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$ (resp., $(\mathbf{B}^{\mathrm{up}};\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$) satisfies the axiom of crystals.

PROPOSITION 2.2.7 (see [**3**, Main Theorem 5.37] and [**25**, Theorem 4.19]). *The following hold.*

1. *For $\lambda \in P_+$, the crystal $(\mathbf{B}^{\mathrm{up}}(\lambda);\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$ is canonically isomorphic to the crystal $(\mathcal{B}(\lambda);\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$, that is, there exists a unique bijective map $\mathbf{B}^{\mathrm{up}}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda)$ that commutes with the maps $\{\tilde{e}_i \mid i \in I\}$, $\{\tilde{f}_i \mid i \in I\}$, and preserves the values of* wt, $\{\varepsilon_i \mid i \in I\}$, $\{\varphi_i \mid i \in I\}$.
2. *The crystal $(\mathbf{B}^{\mathrm{up}};\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$ is canonically isomorphic to the crystal $(\mathcal{B}(\infty);\ \mathrm{wt}, \{\varepsilon_i\}_i, \{\varphi_i\}_i, \{\tilde{e}_i\}_i, \{\tilde{f}_i\}_i)$.*

In this paper, by Proposition 2.2.7, we write perfect bases of $H^0(G/B, \mathcal{L}_\lambda)$ and $\mathbb{C}[U^-]$ as $\{\Xi^{\mathrm{up}}_\lambda(b) \mid b \in \mathcal{B}(\lambda)\}$ and $\{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$, respectively.

EXAMPLE 2.2.8. Let $\mathbf{e}_i \in \mathbb{Z}^I_{\geq 0}$ denote the unit vector corresponding to $i \in I$. Since $U(\mathfrak{u}^-)_{k\mathbf{e}_i} = \mathbb{C}f_i^k$ for $k \in \mathbb{Z}_{\geq 0}$, we have $U(\mathfrak{u}^-)^*_{\mathrm{gr},k\mathbf{e}_i} = \mathbb{C}\Xi^{\mathrm{up}}(\tilde{f}_i^k b_\infty)$.

Now, condition (iii) in Definition 2.2.3 gives the following property of $\{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$:

(iii)$'$ for all $i \in I$, $b \in \mathcal{B}(\infty)$, and $k \in \mathbb{Z}_{\geq 0}$,

$$f_i^k \cdot \Xi^{\mathrm{up}}(b) \in \mathbb{C}^\times \Xi^{\mathrm{up}}(\tilde{e}_i^k b) + \sum_{\substack{b' \in \mathcal{B}(\infty);\ \mathrm{wt}(b') = \mathrm{wt}(b) + k\alpha_i, \\ \varepsilon_i(b') < \varepsilon_i(b) - k}} \mathbb{C}\Xi^{\mathrm{up}}(b'),$$

where $\Xi^{\mathrm{up}}(0) := 0$.

In particular, we have

$$f_i^{\varepsilon_i(b)} \cdot \Xi^{\mathrm{up}}(b) \in \mathbb{C}^\times \Xi^{\mathrm{up}}(\tilde{e}_i^{\varepsilon_i(b)} b), \text{ and}$$
$$f_i^k \cdot \Xi^{\mathrm{up}}(b) = 0 \text{ for } k > \varepsilon_i(b).$$

A perfect basis $\mathbf{B}^{\mathrm{up}}(\lambda)$ also has similar properties, but we do not use them in this thesis. Let $\Xi^{\mathrm{up}}_w(b) \in \mathbb{C}[U^- \cap X(w)]$ denote the restriction of $\Xi^{\mathrm{up}}(b) \in \mathbb{C}[U^-]$ for $w \in W$ and $b \in \mathcal{B}(\infty)$. We obtain the following by [**32**, Sect. 4].

THEOREM 2.2.9 (see [**32**, Sect. 4]). *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis, and* $\mathbf{i} \in I^r$ *a reduced word for* $w \in W$. *Then, Berenstein-Littelmann-Zelevinsky's string parametrization* $\Phi_{\mathbf{i}}(b)$ *is equal to* $-v^{\mathrm{high}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b))$ *for all* $b \in \mathcal{B}_w(\infty)$.

Since $\Phi_{\mathbf{i}}(b)$, $b \in \mathcal{B}_w(\infty)$, are all distinct, we obtain the following by Proposition 1.1.2 (1) and Theorem 2.2.9.

COROLLARY 2.2.10. *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis, and* $w \in W$. *Then, the elements* $\Xi^{\mathrm{up}}_w(b)$, $b \in \mathcal{B}_w(\infty)$, *are linearly independent over* $\mathbb{C}$.

In addition, the following is an immediate consequence of Proposition 1.1.2 (2) and Theorem 2.2.9.

LEMMA 2.2.11. *Let* $\mathbf{B}^{\mathrm{up}}_k = \{\Xi^{\mathrm{up}}_k(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be perfect bases for* $k = 1, 2$, *and* $\mathbf{i} \in I^N$ *a reduced word for the longest element* $w_0 \in W$. *Then, the following holds*:

$$\Xi^{\mathrm{up}}_1(b) \in \mathbb{C}^\times \Xi^{\mathrm{up}}_2(b) + \sum_{b' \in \mathcal{B}(\infty);\ \Phi_{\mathbf{i}}(b') < \Phi_{\mathbf{i}}(b)} \mathbb{C}\Xi^{\mathrm{up}}_2(b'),$$

where the order $<$ *is the one defined in Definition* 1.1.3.

REMARK 2.2.12. Theorem 2.2.9, Corollary 2.2.10, and Lemma 2.2.11 hold also for perfect bases which do not necessarily satisfy condition (v). In addition, we need not assume condition (D) below (see the proof of Theorem 2.3.2).

We denote the dual basis of a perfect basis $\mathbf{B}^{\mathrm{up}}(\lambda)$ (resp., $\mathbf{B}^{\mathrm{up}}$) by $\mathbf{B}^{\mathrm{low}}(\lambda) = \{\Xi_\lambda^{\mathrm{low}}(b) \mid b \in \mathcal{B}(\lambda)\} \subset V(\lambda)$ (resp., $\mathbf{B}^{\mathrm{low}} = \{\Xi^{\mathrm{low}}(b) \mid b \in \mathcal{B}(\infty)\} \subset U(\mathfrak{u}^-)$), which is called a *lower perfect basis*.

PROPOSITION 2.2.13. *For all $i \in I$, $b \in \mathcal{B}(\infty)$, and $k \in \mathbb{Z}_{\geq 0}$, the following holds*:

$$f_i^k \cdot \Xi^{\mathrm{low}}(b) \in \mathbb{C}^\times \Xi^{\mathrm{low}}(\tilde{f}_i^k b) + \sum_{\substack{b' \in \mathcal{B}(\infty);\ \mathrm{wt}(b') = \mathrm{wt}(b) - k\alpha_i, \\ \varepsilon_i(b') > \varepsilon_i(b) + k}} \mathbb{C}\Xi^{\mathrm{low}}(b').$$

PROOF. For $b' \in \mathcal{B}(\infty)$, the coefficient of $\Xi^{\mathrm{low}}(b')$ in $f_i^k \cdot \Xi^{\mathrm{low}}(b)$ is equal to

$$\langle \Xi^{\mathrm{up}}(b'), f_i^k \cdot \Xi^{\mathrm{low}}(b) \rangle = (-1)^k \langle f_i^k \cdot \Xi^{\mathrm{up}}(b'), \Xi^{\mathrm{low}}(b) \rangle.$$

If this is not equal to 0, then property (iii)$'$ implies that $b = \tilde{e}_i^k b'$ or $\varepsilon_i(b) < \varepsilon_i(b') - k$, that is, $b' = \tilde{f}_i^k b$ or $\varepsilon_i(b') > \varepsilon_i(b) + k$. In addition, if $\Xi^{\mathrm{low}}(b) \in U(\mathfrak{u}^-)_{\mathbf{d}}$, then we have $f_i^k \cdot \Xi^{\mathrm{low}}(b) \in U(\mathfrak{u}^-)_{\mathbf{d}+k\mathbf{e}_i}$. From these, we deduce that

$$f_i^k \cdot \Xi^{\mathrm{low}}(b) \in \mathbb{C}\Xi^{\mathrm{low}}(\tilde{f}_i^k b) + \sum_{\substack{b' \in \mathcal{B}(\infty);\ \mathrm{wt}(b') = \mathrm{wt}(b) - k\alpha_i, \\ \varepsilon_i(b') > \varepsilon_i(b) + k}} \mathbb{C}\Xi^{\mathrm{low}}(b');$$

the coefficient of $\Xi^{\mathrm{low}}(\tilde{f}_i^k b)$ is not equal to 0 since $\langle f_i^k \cdot \Xi^{\mathrm{up}}(\tilde{f}_i^k b), \Xi^{\mathrm{low}}(b) \rangle \neq 0$ by property (iii)$'$. This proves the proposition. □

REMARK 2.2.14. Baumann introduced the notion of *bases of canonical type* in [**2**]. The axiom of bases of canonical type is slightly stronger than our conditions on lower perfect bases because he imposed an additional condition on the coefficient of $\Xi^{\mathrm{low}}(\tilde{f}_i^k b)$ in Proposition 2.2.13.

Recall the involution $*\colon U(\mathfrak{u}^-) \to U(\mathfrak{u}^-)$ and Kashiwara's involution $*\colon \mathcal{B}(\infty) \to \mathcal{B}(\infty)$. We see by Proposition 1.3.9 that $G^{\mathrm{low}}(b)^* = G^{\mathrm{low}}(b^*)$, $G^{\mathrm{up}}(b)^* = G^{\mathrm{up}}(b^*)$ for all $b \in \mathcal{B}(\infty)$. In addition, all perfect bases have such a property as follows.

PROPOSITION 2.2.15. *Let $\{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$ be a perfect basis of $\mathbb{C}[U^-]$. Then, the equality $\Xi^{\mathrm{up}}(b)^* = \Xi^{\mathrm{up}}(b^*)$ holds for all $b \in \mathcal{B}(\infty)$; hence the equality $\Xi^{\mathrm{low}}(b)^* = \Xi^{\mathrm{low}}(b^*)$ also holds for all $b \in \mathcal{B}(\infty)$.*

PROOF. For $b \in \mathcal{B}(\infty)$, there exists $b^\star \in \mathcal{B}(\infty)$ such that $\Xi^{\mathrm{up}}(b)^* = \Xi^{\mathrm{up}}(b^\star)$ by condition (v). Suppose that there exists $b \in \mathcal{B}(\infty)$ such that $(b^*)^\star \neq b$. Let $\mathbf{i} \in I^N$ be a reduced word for the longest element $w_0 \in W$, and $b_0 \in \mathcal{B}(\infty)$ an element such that $(b_0^*)^\star \neq b_0$ and such that $\Phi_{\mathbf{i}}(b_0) \geq \Phi_{\mathbf{i}}(b)$ for all $b \in \mathcal{B}(\infty)$ with $\mathrm{wt}(b) = \mathrm{wt}(b_0)$ and with $(b^*)^\star \neq b$. Then, we have

$$\langle \Xi^{\mathrm{up}}((b_0^*)^\star), G^{\mathrm{low}}(b_0) \rangle = \langle \Xi^{\mathrm{up}}(b_0^*)^*, G^{\mathrm{low}}(b_0) \rangle = \langle \Xi^{\mathrm{up}}(b_0^*), G^{\mathrm{low}}(b_0^*) \rangle \neq 0$$

by Lemma 2.2.11. Hence, by Lemma 2.2.11 again, it follows that $\Phi_{\mathbf{i}}(b_0) < \Phi_{\mathbf{i}}((b_0^*)^\star)$. By this and the assumption on b_0, we obtain the equality $(b_0^*)^\star = (((b_0^*)^\star)^*)^\star$, which is equivalent to $b_0 = (b_0^*)^\star$. This contradicts the assumption on b_0. Hence the equality $(b^*)^\star = b$ holds for all $b \in \mathcal{B}(\infty)$. This proves the proposition. □

By condition (iii)′ and Propositions 2.2.13, 2.2.15, we obtain the following (see also the proof of Corollary 1.3.10).

Proposition 2.2.16. *For all $i \in I$, $b \in \mathcal{B}(\infty)$, and $k \in \mathbb{Z}_{\geq 0}$, the following hold:*

$$\Xi^{\mathrm{low}}(b) \cdot f_i^k \in \mathbb{C}^\times \Xi^{\mathrm{low}}((\tilde{f}_i^*)^k b) + \sum_{\substack{b' \in \mathcal{B}(\infty);\ \mathrm{wt}(b') = \mathrm{wt}(b) - k\alpha_i, \\ \varepsilon_i^*(b') > \varepsilon_i^*(b) + k}} \mathbb{C}\Xi^{\mathrm{low}}(b'),$$

$$\Xi^{\mathrm{up}}(b) \cdot f_i^k \in \mathbb{C}^\times \Xi^{\mathrm{up}}((\tilde{e}_i^*)^k b) + \sum_{\substack{b' \in \mathcal{B}(\infty);\ \mathrm{wt}(b') = \mathrm{wt}(b) + k\alpha_i, \\ \varepsilon_i^*(b') < \varepsilon_i^*(b) - k}} \mathbb{C}\Xi^{\mathrm{up}}(b').$$

In particular, the following hold for all $i \in I$ and $b \in \mathcal{B}(\infty)$:

$$\Xi^{\mathrm{up}}(b) \cdot f_i^{\varepsilon_i^*(b)} \in \mathbb{C}^\times \Xi^{\mathrm{up}}((\tilde{e}_i^*)^{\varepsilon_i^*(b)} b), \text{ and}$$
$$\Xi^{\mathrm{up}}(b) \cdot f_i^k = 0 \quad \text{for } k > \varepsilon_i^*(b).$$

In the following, we prove that a perfect basis $\mathbf{B}^{\mathrm{up}}$ of $\mathbb{C}[U^-]$ induces a perfect basis $\mathbf{B}^{\mathrm{up}}(\lambda)$ of $H^0(G/B, \mathcal{L}_\lambda)$. For $i \in I$, denote by $\mathfrak{g}_i$ the Lie subalgebra of $\mathfrak{g}$ generated by e_i, f_i, h_i, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ as a Lie algebra. Recall that $\pi_\lambda \colon U(\mathfrak{u}^-) \twoheadrightarrow V(\lambda)$ is the surjective $U(\mathfrak{u}^-)$-module homomorphism given by $u \mapsto uv_\lambda$. We set $\Xi_\lambda^{\mathrm{low}}(\pi_\lambda(b)) \coloneqq \pi_\lambda(\Xi^{\mathrm{low}}(b))$ for $b \in \widetilde{\mathcal{B}}(\lambda)$. For $i \in I$ and $\ell \in \mathbb{Z}_{\geq 0}$, let $I_i^\ell(V(\lambda))$ be the sum of $(\ell+1)$-dimensional irreducible $U(\mathfrak{g}_i)$-submodules of $V(\lambda)$, and write

$$W_i^\ell(V(\lambda)) \coloneqq \bigoplus_{\ell' \geq \ell} I_i^{\ell'}(V(\lambda)) \subset V(\lambda),$$
$$I_i^\ell(\mathcal{B}(\lambda)) \coloneqq \{b \in \mathcal{B}(\lambda) \mid \varepsilon_i(b) + \varphi_i(b) = \ell\},$$
$$W_i^\ell(\mathcal{B}(\lambda)) \coloneqq \{b \in \mathcal{B}(\lambda) \mid \varepsilon_i(b) + \varphi_i(b) \geq \ell\}.$$

A lower perfect basis of $U(\mathfrak{u}^-)$ is compatible with $V(\lambda)$ for all $\lambda \in P_+$ and with their $U(\mathfrak{g}_i)$-submodules $W_i^\ell(V(\lambda))$ as follows.

Proposition 2.2.17. *For $\lambda \in P_+$, the following hold.*

1. *The set $\{\Xi_\lambda^{\mathrm{low}}(b) \mid b \in \mathcal{B}(\lambda)\}$ forms a $\mathbb{C}$-basis of $V(\lambda)$, and the equality $\pi_\lambda(\Xi^{\mathrm{low}}(b)) = 0$ holds for all $b \in \mathcal{B}(\infty) \setminus \widetilde{\mathcal{B}}(\lambda)$.*
2. *For $i \in I$ and $\ell \in \mathbb{Z}_{\geq 0}$,*

$$W_i^\ell(V(\lambda)) = \sum_{b \in W_i^\ell(\mathcal{B}(\lambda))} \mathbb{C}\Xi_\lambda^{\mathrm{low}}(b).$$

In addition, for $b \in I_i^\ell(\mathcal{B}(\lambda))$ *and* $k \in \mathbb{Z}_{\geq 0}$,

$$f_i^k \cdot \Xi_\lambda^{\mathrm{low}}(b) \in \mathbb{C}^\times \Xi_\lambda^{\mathrm{low}}(\tilde{f}_i^k b) + W_i^{\ell+1}(V(\lambda)),$$
$$e_i^k \cdot \Xi_\lambda^{\mathrm{low}}(b) \in \mathbb{C}^\times \Xi_\lambda^{\mathrm{low}}(\tilde{e}_i^k b) + W_i^{\ell+1}(V(\lambda)),$$

where $\Xi_\lambda^{\mathrm{low}}(0) := 0$.

PROOF. We first prove that the set $\{\Xi^{\mathrm{low}}(b) \mid b \in \mathcal{B}(\infty),\ \varepsilon_i^*(b) \geq k\}$ forms a $\mathbb{C}$-basis of $U(\mathfrak{u}^-)f_i^k$, which implies part (1) by Corollary 1.3.16. Set

$$\widetilde{U}_{i,k} := \sum_{b \in \mathcal{B}(\infty);\ \varepsilon_i^*(b) \geq k} \mathbb{C}\Xi^{\mathrm{low}}(b).$$

Then, we have $U(\mathfrak{u}^-)f_i^k \subset \widetilde{U}_{i,k}$ by Proposition 2.2.16. Also, Corollary 1.3.15 (4) implies that $\dim_{\mathbb{C}}(U(\mathfrak{u}^-)f_i^k \cap U(\mathfrak{u}^-)_{\mathbf{d}})$ equals the cardinality of $\{b \in \mathcal{B}(\infty)_{\mathbf{d}} \mid \varepsilon_i^*(b) \geq k\}$ for all $\mathbf{d} = (d_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$, where $\mathcal{B}(\infty)_{\mathbf{d}} := \{b \in \mathcal{B}(\infty) \mid \mathrm{wt}(b) = -\sum_{i \in I} d_i \alpha_i\}$. This completes a proof of the assertion. By Propositions 1.3.7 (3), (5), 2.2.13, we have

$$f_i^k \cdot \Xi_\lambda^{\mathrm{low}}(b) \in \mathbb{C}^\times \Xi_\lambda^{\mathrm{low}}(\tilde{f}_i^k b) + \sum_{\substack{b' \in \mathcal{B}(\lambda);\ \mathrm{wt}(b') = \mathrm{wt}(b) - k\alpha_i, \\ \varepsilon_i(b') > \varepsilon_i(b) + k}} \mathbb{C}\Xi_\lambda^{\mathrm{low}}(b') \tag{2.2.1}$$

for all $i \in I$, $b \in \mathcal{B}(\lambda)$, and $k \in \mathbb{Z}_{\geq 0}$. Fix $i \in I$, and let $\ell_0 \in \mathbb{Z}_{\geq 0}$ be the maximum integer $\ell \in \mathbb{Z}_{\geq 0}$ such that $W_i^\ell(V(\lambda)) \neq 0$, which implies that $W_i^{\ell_0}(V(\lambda)) = I_i^{\ell_0}(V(\lambda))$. Since we have $\varepsilon_i(b) = 0$ for all $b \in \mathcal{B}(\lambda)$ with $\langle \mathrm{wt}(b), h_i \rangle = \ell_0$, it follows by (2.2.1) that

$$f_i^k \cdot \Xi_\lambda^{\mathrm{low}}(b) \in \mathbb{C}^\times \Xi_\lambda^{\mathrm{low}}(\tilde{f}_i^k b)$$

for all $b \in \mathcal{B}(\lambda)$ and $k \in \mathbb{Z}_{\geq 0}$ with $\langle \mathrm{wt}(b), h_i \rangle = \ell_0$. From these, we see that $W_i^{\ell_0}(V(\lambda))$ is spanned by the elements $\{\Xi_\lambda^{\mathrm{low}}(b) \mid b \in W_i^{\ell_0}(\mathcal{B}(\lambda))\}$. By descending induction on ℓ and by replacing $V(\lambda)$ with $V(\lambda)/W_i^{\ell+1}(V(\lambda))$ in the argument above, we prove that $W_i^\ell(V(\lambda))$ is spanned by the elements $\{\Xi_\lambda^{\mathrm{low}}(b) \mid b \in W_i^\ell(\mathcal{B}(\lambda))\}$ for all $\ell \in \mathbb{Z}_{\geq 0}$. This proves the first assertion of part (2). The second assertion of part (2) follows by (2.2.1), the first assertion of part (2), and the standard representation theory of $\mathfrak{sl}_2(\mathbb{C})$. □

The space $H^0(G/B, \mathcal{L}_\lambda)$ of global sections is regarded as a $\mathbb{C}$-subspace of $\mathbb{C}[U^-]$ by:

$$H^0(G/B, \mathcal{L}_\lambda) = V(\lambda)^* \xhookrightarrow{\pi_\lambda^*} U(\mathfrak{u}^-)_{\mathrm{gr}}^* = \mathbb{C}[U^-],$$

where $\pi_\lambda^* \colon V(\lambda)^* \hookrightarrow U(\mathfrak{u}^-)_{\mathrm{gr}}^*$ denotes the dual map of π_λ. Let $\mathbf{B}^{\mathrm{up}}(\lambda) = \{\Xi_\lambda^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\lambda)\} \subset H^0(G/B, \mathcal{L}_\lambda) = V(\lambda)^*$ be the dual basis of $\{\Xi_\lambda^{\mathrm{low}}(b) \mid b \in \mathcal{B}(\lambda)\} \subset V(\lambda)$. Then, we obtain $\pi_\lambda^*(\Xi_\lambda^{\mathrm{up}}(\pi_\lambda(b))) = \Xi^{\mathrm{up}}(b)$ for all $b \in \widetilde{\mathcal{B}}(\lambda)$.

PROPOSITION 2.2.18. *For* $\lambda \in P_+$, *the following hold.*

(1) *The* $\mathbb{C}$-*basis* $\mathbf{B}^{\mathrm{up}}(\lambda) \subset H^0(G/B, \mathcal{L}_\lambda)$ *is a perfect basis.*
(2) *Set* $\tau_\lambda := \Xi_\lambda^{\mathrm{up}}(b_\lambda) \in H^0(G/B, \mathcal{L}_\lambda)$. *Then, the section* τ_λ *does not vanish on* U^- ($\hookrightarrow G/B$); *in particular, the restriction* $(\tau/\tau_\lambda)|_{U^-}$ *belongs to* $\mathbb{C}[U^-]$ *for all* $\tau \in H^0(G/B, \mathcal{L}_\lambda)$.

(3) *A map* $\iota_\lambda\colon H^0(G/B,\mathcal{L}_\lambda)\to\mathbb{C}[U^-]$ *defined by* $\tau\mapsto(\tau/\tau_\lambda)|_{U^-}$ *is injective.*
(4) *The equality* $(\tau/\tau_\lambda)|_{U^-}=\pi_\lambda^*(\tau)$ *holds in* $\mathbb{C}[U^-]$ *for all* $\tau\in H^0(G/B,\mathcal{L}_\lambda)$. *In particular, the element* $\Xi^{\mathrm{up}}(b)\in\mathbb{C}[U^-]$ *is identical to* $\iota_\lambda(\Xi_\lambda^{\mathrm{up}}(\pi_\lambda(b)))$ *for all* $b\in\widetilde{\mathcal{B}}(\lambda)$.
(5) *The following equalities hold*:

$$\{\Xi^{\mathrm{up}}(b)\mid b\in\mathcal{B}(\infty)\}=\bigcup_{\lambda\in P_+}\{\iota_\lambda(\Xi_\lambda^{\mathrm{up}}(b))\mid b\in\mathcal{B}(\lambda)\},\ \textit{and}$$

$$\mathbb{C}[U^-]=\bigcup_{\lambda\in P_+}\iota_\lambda(H^0(G/B,\mathcal{L}_\lambda)).$$

Proof. Part (1) is an immediate consequence of (2.2.1) and the definition of $\mathbf{B}^{\mathrm{up}}(\lambda)$. Because $\exp(x)\cdot v_\lambda\in v_\lambda+x\cdot V(\lambda)$ and $\tau_\lambda(x\cdot V(\lambda))=\{0\}$ for $x\in\mathfrak{u}^-$, we have $\tau_\lambda(\exp(x)\cdot v_\lambda)=1$. Hence as an element of $H^0(G/B,\mathcal{L}_\lambda)$, the section τ_λ does not vanish on U^- ($\hookrightarrow G/B$), which implies part (2). Then, since U^- is regarded as an open subvariety of G/B, we have $(\tau/\tau_\lambda)|_{U^-}\neq 0$ for all nonzero sections $\tau\in H^0(G/B,\mathcal{L}_\lambda)$, which implies part (3). For all $\tau\in H^0(G/B,\mathcal{L}_\lambda)$ and $x\in\mathfrak{u}^-$, we see that

$$\begin{aligned}(\tau/\tau_\lambda)(\exp(x))&=\tau(\exp(x)\cdot v_\lambda)/\tau_\lambda(\exp(x)\cdot v_\lambda)\\&\qquad\text{(by the definition of the isomorphism } \rho_\lambda^* \text{ in Sect. 1.2)}\\&=\tau(\exp(x)\cdot v_\lambda)\quad(\text{since } \tau_\lambda(\exp(x)\cdot v_\lambda)=1).\end{aligned}$$

Also, we have

$$\begin{aligned}((\Upsilon\circ\pi_\lambda^*)(\tau))(\exp(x))&=\sum_{l\in\mathbb{Z}_{\geq 0}}(\pi_\lambda^*(\tau))(x^l)/l!\quad(\text{by the definition of }\Upsilon)\\&=\sum_{l\in\mathbb{Z}_{\geq 0}}\tau(x^l\cdot v_\lambda)/l!\quad(\text{since }\pi_\lambda(x^l)=x^l\cdot v_\lambda)\\&=\tau(\exp(x)\cdot v_\lambda).\end{aligned}$$

From these, part (4) follows immediately. Since $\{\Xi^{\mathrm{up}}(b)\mid b\in\mathcal{B}(\infty)\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[U^-]$, part (5) follows by part (4) and equation (1.3.1). This proves the proposition. □

Recall that $U^-\cap X(w)$ is a closed subvariety of U^-, and that $\Xi_w^{\mathrm{up}}(b)\in\mathbb{C}[U^-\cap X(w)]$ denotes the restriction of $\Xi^{\mathrm{up}}(b)\in\mathbb{C}[U^-]$ for $b\in\mathcal{B}(\infty)$. By abuse of notation, let $\tau_\lambda\in H^0(X(w),\mathcal{L}_\lambda)$ denote the restriction of $\tau_\lambda:=\Xi_\lambda^{\mathrm{up}}(b_\lambda)\in H^0(G/B,\mathcal{L}_\lambda)$. Since $U^-\cap X(w)$ is an open subvariety of $X(w)$, we obtain the following by Proposition 2.2.18 (2).

Lemma 2.2.19. *The section* $\tau_\lambda\in H^0(X(w),\mathcal{L}_\lambda)$ *does not vanish on* $U^-\cap X(w)$; *in particular, an injective map* $\iota_\lambda\colon H^0(X(w),\mathcal{L}_\lambda)\hookrightarrow\mathbb{C}[U^-\cap X(w)]$, $\tau\mapsto(\tau/\tau_\lambda)|_{(U^-\cap X(w))}$, *is well-defined.*

Let $\eta_w\colon\mathbb{C}[U^-]\twoheadrightarrow\mathbb{C}[U^-\cap X(w)]$ (resp., $\eta_w\colon H^0(G/B,\mathcal{L}_\lambda)\twoheadrightarrow H^0(X(w),\mathcal{L}_\lambda)$) denote the restriction map. We set

$$\Xi_{\lambda,w}^{\mathrm{up}}(b):=\eta_w(\Xi_\lambda^{\mathrm{up}}(b))\in H^0(X(w),\mathcal{L}_\lambda)$$

for $b\in\mathcal{B}(\lambda)$.

Corollary 2.2.20. *Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ be a perfect basis, $w \in W$, and $\lambda \in P_+$. Then, the element $\Xi_w^{\mathrm{up}}(b) \in \mathbb{C}[U^- \cap X(w)]$ is identical to $\iota_\lambda(\Xi_{\lambda,w}^{\mathrm{up}}(\pi_\lambda(b)))$ for all $b \in \widetilde{\mathcal{B}}_w(\lambda)$. In addition, the following equality holds*:

$$\mathbb{C}[U^- \cap X(w)] = \bigcup_{\lambda \in P_+} \iota_\lambda(H^0(X(w), \mathcal{L}_\lambda)).$$

Proof. Consider the following diagram of varieties:

$$\begin{array}{ccc} U^- & \hookrightarrow & G/B \\ \uparrow & & \uparrow \\ U^- \cap X(w) & \hookrightarrow & X(w). \end{array}$$

From this, we see that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}[U^-] & \xleftarrow{\ \iota_\lambda\ } & H^0(G/B, \mathcal{L}_\lambda) \\ \downarrow{\scriptstyle \eta_w} & & \downarrow{\scriptstyle \eta_w} \\ \mathbb{C}[U^- \cap X(w)] & \xleftarrow{\ \iota_\lambda\ } & H^0(X(w), \mathcal{L}_\lambda). \end{array}$$

Hence the first assertion of the corollary is an immediate consequence of Proposition 2.2.18 (4) and of the definition of $\Xi_{\lambda,w}^{\mathrm{up}}(b)$. Also, we see that

$$\begin{aligned} \mathbb{C}[U^- \cap X(w)] &= \eta_w(\mathbb{C}[U^-]) \\ &= \bigcup_{\lambda \in P_+} \eta_w(\iota_\lambda(H^0(G/B, \mathcal{L}_\lambda))) \quad \text{(by Proposition 2.2.18 (5))} \\ &= \bigcup_{\lambda \in P_+} \iota_\lambda(\eta_w(H^0(G/B, \mathcal{L}_\lambda))) \\ &= \bigcup_{\lambda \in P_+} \iota_\lambda(H^0(X(w), \mathcal{L}_\lambda)). \end{aligned}$$

This proves the second assertion of the corollary. □

The following is an immediate consequence of Theorem 2.2.9 and the first assertion of Corollary 2.2.20.

Corollary 2.2.21 (see [**32**, Sect. 4]). *Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ be a perfect basis, $\mathbf{i} \in I^r$ a reduced word for $w \in W$, and $\lambda \in P_+$. Then, the string parametrization $\Phi_{\mathbf{i}}(b)$ is equal to $-v_{\mathbf{i}}^{\mathrm{high}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda)$ for all $b \in \mathcal{B}_w(\lambda)$.*

Since $\Phi_{\mathbf{i}}(b)$, $b \in \mathcal{B}_w(\lambda)$, are all distinct, and the dimension of $H^0(X(w), \mathcal{L}_\lambda)$ equals the cardinality of $\mathcal{B}_w(\lambda)$ by Corollary 1.3.15 (5), we obtain the following by Proposition 1.1.2 (1) and Corollary 2.2.21.

Corollary 2.2.22. *Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ be a perfect basis, $w \in W$, and $\lambda \in P_+$. Then, the set $\{\Xi_{\lambda,w}^{\mathrm{up}}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ forms a $\mathbb{C}$-basis of $H^0(X(w), \mathcal{L}_\lambda)$.*

We consider the following condition (D) on a perfect basis $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$ (see also Proposition 2.2.17 (1)):

(D) the set $\{\Xi^{\mathrm{low}}_{\lambda}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ forms a $\mathbb{C}$-basis of the Demazure module $V_w(\lambda)$.

EXAMPLE 2.2.23. The upper global basis and the dual semicanonical basis satisfy condition (D) by Corollary 1.3.15 (5) and [**59**, Theorem 7.1], respectively. We show in Sect. 3.2 (Proposition 3.2.5) that the specialization of the KLR-basis at $q = 1$ also satisfies condition (D).

If $\mathbf{B}^{\mathrm{up}}$ satisfies condition (D), then the $\mathbb{C}$-basis $\{\Xi^{\mathrm{up}}_{\lambda,w}(b) \mid b \in \mathcal{B}_w(\lambda)\} \subset H^0(X(w), \mathcal{L}_\lambda) = V_w(\lambda)^*$ is identical to the dual basis of $\{\Xi^{\mathrm{low}}_{\lambda}(b) \mid b \in \mathcal{B}_w(\lambda)\} \subset V_w(\lambda)$.

COROLLARY 2.2.24. *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis satisfying condition* (D).

(1) *The set* $\{\Xi^{\mathrm{up}}_{w}(b) \mid b \in \mathcal{B}_w(\infty)\}$ *forms a* $\mathbb{C}$*-basis of* $\mathbb{C}[U^- \cap X(w)]$.
(2) *The equality* $\Xi^{\mathrm{up}}_{w}(b) = 0$ *holds unless* $b \in \mathcal{B}_w(\infty)$.

PROOF. Since $\{\Xi^{\mathrm{up}}_{\lambda,w}(\pi_\lambda(b)) \mid b \in \widetilde{\mathcal{B}}_w(\lambda)\}$ forms a $\mathbb{C}$-basis of $H^0(X(w), \mathcal{L}_\lambda)$, we deduce by Corollary 2.2.20 that $\{\Xi^{\mathrm{up}}_{w}(b) \mid b \in \mathcal{B}_w(\infty)\}$ spans $\mathbb{C}[U^- \cap X(w)]$. For an arbitrary finite subset $\{b_1, \ldots, b_k\} \subset \mathcal{B}_w(\infty)$, take $\lambda \in P_+$ such that $b_1, \ldots, b_k \in \widetilde{\mathcal{B}}(\lambda)$. Since the elements $\Xi^{\mathrm{up}}_{\lambda,w}(\pi_\lambda(b_1)), \ldots, \Xi^{\mathrm{up}}_{\lambda,w}(\pi_\lambda(b_k))$ are linearly independent, it follows by the first assertion of Corollary 2.2.20 that $\Xi^{\mathrm{up}}_{w}(b_1), \ldots, \Xi^{\mathrm{up}}_{w}(b_k)$ are also linearly independent. From these, we obtain part (1). For $b \in \mathcal{B}(\infty) \setminus \mathcal{B}_w(\infty)$, we take $\lambda \in P_+$ such that $b \in \widetilde{\mathcal{B}}(\lambda)$. Since $\pi_\lambda \colon \widetilde{\mathcal{B}}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda)$ is bijective and $\pi_\lambda(\widetilde{\mathcal{B}}_w(\lambda)) = \mathcal{B}_w(\lambda)$ by Proposition 1.3.13 (3), we have $\pi_\lambda(b) \notin \mathcal{B}_w(\lambda)$, which implies that $\eta_w(\Xi^{\mathrm{up}}_{\lambda}(\pi_\lambda(b))) = 0$ by condition (D). Hence it follows that

$$\begin{aligned}
\Xi^{\mathrm{up}}_{w}(b) &= \eta_w(\Xi^{\mathrm{up}}(b)) \\
&= \eta_w(\iota_\lambda(\Xi^{\mathrm{up}}_{\lambda}(\pi_\lambda(b)))) \quad \text{(by Proposition 2.2.18 (4))} \\
&= \iota_\lambda(\eta_w(\Xi^{\mathrm{up}}_{\lambda}(\pi_\lambda(b)))) \\
&= \iota_\lambda(0) = 0,
\end{aligned}$$

which implies part (2). This proves the corollary. □

REMARK 2.2.25. Some formulas with respect to the character of $\mathbb{C}[U^- \cap X(w)]$ are given in [**38**, Sect. 12.1]. By Corollary 2.2.24 (1), these formulas can be regarded as those with respect to the formal character of $\mathcal{B}_w(\infty)$ (see [**21**, Sect. 4.7]).

The following is an immediate consequence of Corollaries 2.2.10, 2.2.24 (1).

COROLLARY 2.2.26. *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis* (*not necessarily satisfying condition* (D)). *Then, the set* $\{\Xi^{\mathrm{up}}_{w}(b) \mid b \in \mathcal{B}_w(\infty)\}$ *forms a* $\mathbb{C}$*-basis of* $\mathbb{C}[U^- \cap X(w)]$ *for* $w \in W$.

Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$. Since $\Phi_{\mathbf{i}}(b)$, $b \in \mathcal{B}_w(\infty)$, are all distinct, we obtain the following corollary by Definition 1.1.1, Proposition 1.1.2 (2), Theorem 2.2.9, and Corollary 2.2.26.

Corollary 2.2.27 (see, for instance, [**32**, Sect. 6]). *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis* (*not necessarily satisfying condition* (D))*, and* $\mathbf{i} \in I^r$ *a reduced word for* $w \in W$*. Then, the following holds*:

$$\Xi_w^{\mathrm{up}}(b_1) \cdot \Xi_w^{\mathrm{up}}(b_2) \in \mathbb{C}^\times \Xi_w^{\mathrm{up}}(b) + \sum_{b' \in \mathcal{B}_w(\infty);\ \Phi_{\mathbf{i}}(b') < \Phi_{\mathbf{i}}(b)} \mathbb{C}\Xi_w^{\mathrm{up}}(b')$$

for all $b_1, b_2 \in \mathcal{B}_w(\infty)$*, where* $b \in \mathcal{B}_w(\infty)$ *is the unique element such that* $\Phi_{\mathbf{i}}(b) = \Phi_{\mathbf{i}}(b_1) + \Phi_{\mathbf{i}}(b_2)$.

2.3. First main result

In this section, we relate Nakashima-Zelevinsky's polyhedral realizations of highest weight crystal bases with Newton-Okounkov polytopes. We start with describing the highest term valuation $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$ in terms of the Chevalley generators, which is a counterpart of Proposition 1.2.5. Take a reduced word $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ for $w \in W$, and recall that we identify the function field $\mathbb{C}(X(w))$ with the rational function field $\mathbb{C}(t_1, \ldots, t_r)$ by using the morphisms in (1.2.1) and (1.2.2). If we set $w_{\leq k} \coloneqq s_{i_1} s_{i_2} \cdots s_{i_k}$ for $1 \leq k \leq r$, then we obtain a sequence of subvarieties:

$$X(w_{\leq 1}) \subset X(w_{\leq 2}) \subset \cdots \subset X(w_{\leq r}) = X(w).$$

We write $\mathbf{i}_{\leq k} \coloneqq (i_1, \ldots, i_k) \in I^k$ for $1 \leq k \leq r$, and denote by $Z_{\mathbf{i}_{\leq k}}$ the corresponding Bott-Samelson variety. Then, the morphism $Z_{\mathbf{i}} \twoheadrightarrow X(w)$ given in (1.2.1) induces a surjective birational morphism $Z_{\mathbf{i}_{\leq k}} \twoheadrightarrow X(w_{\leq k})$, and the morphism $U_{i_1}^- \times \cdots \times U_{i_r}^- \hookrightarrow Z_{\mathbf{i}}$ given in (1.2.2) gives an open embedding $U_{i_1}^- \times \cdots \times U_{i_k}^- \hookrightarrow Z_{\mathbf{i}_{\leq k}}$; hence the function field $\mathbb{C}(X(w_{\leq k}))$ is identified with the rational function field $\mathbb{C}(t_1, \ldots, t_k)$. Consider the right action of $U_{i_k}^-$ on $U_{i_1}^- \times \cdots \times U_{i_k}^-$ given by:

$$(u_1, \ldots, u_k) \cdot u = (u_1, \ldots, u_{k-1}, u_k u)$$

for $u_1 \in U_{i_1}^-, \ldots, u_{k-1} \in U_{i_{k-1}}^-$, and $u, u_k \in U_{i_k}^-$; this induces right actions of $U_{i_k}^-$ and $\mathfrak{u}_{i_k}^-$ on $\mathbb{C}[t_1, \ldots, t_k] = \mathbb{C}[U_{i_1}^- \times \cdots \times U_{i_k}^-]$, which are given by:

$$f(t_1, \ldots, t_k) \cdot \exp(s f_{i_k}) = f(t_1, \ldots, t_{k-1}, t_k - s), \text{ and hence}$$

$$f(t_1, \ldots, t_k) \cdot f_{i_k} = -\frac{\partial}{\partial t_k} f(t_1, \ldots, t_k) \tag{2.3.1}$$

for $s \in \mathbb{C}$ and $f(t_1, \ldots, t_k) \in \mathbb{C}[t_1, \ldots, t_k]$.

Proposition 2.3.1. *For a nonzero polynomial* $f(t_1, \ldots, t_r) \in \mathbb{C}[t_1, \ldots, t_r]$*, write* $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(f(t_1, \ldots, t_r)) = -(a_r, \ldots, a_1)$*. Then, the following equalities* *hold*:

$$
\begin{aligned}
a_r &= \max\{a \in \mathbb{Z}_{\geq 0} \mid f(t_1, \ldots, t_r) \cdot f_{i_r}^a \neq 0\},\\
a_{r-1} &= \max\{a \in \mathbb{Z}_{\geq 0} \mid (f(t_1, \ldots, t_r) \cdot f_{i_r}^{a_r})|_{t_r=0} \cdot f_{i_{r-1}}^a \neq 0\},\\
&\vdots\\
a_1 &= \max\{a \in \mathbb{Z}_{\geq 0} \mid (\cdots((f(t_1, \ldots, t_r) \cdot f_{i_r}^{a_r})|_{t_r=0} \cdot f_{i_{r-1}}^{a_{r-1}}) \cdots)|_{t_2=0} \cdot f_{i_1}^a \neq 0\}.
\end{aligned}
$$

Proof. It follows from the definition of $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$ that a_r is equal to the degree of $f(t_1, \ldots, t_r)$ with respect to the variable t_r. Hence we deduce that

$$
\begin{aligned}
a_r &= \max\{a \in \mathbb{Z}_{\geq 0} \mid \frac{\partial^a}{\partial t_r^a} f(t_1, \ldots, t_r) \neq 0\}\\
&= \max\{a \in \mathbb{Z}_{\geq 0} \mid f(t_1, \ldots, t_r) \cdot f_{i_r}^a \neq 0\} \quad \text{(by equation (2.3.1))}.
\end{aligned}
$$

Since the polynomial $f(t_1, \ldots, t_r) \cdot f_{i_r}^{a_r}$ does not contain the variable t_r, the specialization $(f(t_1, \ldots, t_r) \cdot f_{i_r}^{a_r})|_{t_r=0}$ is identical to $f(t_1, \ldots, t_r) \cdot f_{i_r}^{a_r} \in \mathbb{C}[t_1, \ldots, t_{r-1}]$ as a polynomial in the variables $t_1, \ldots, t_{r-1}$. Hence we see by the definition of $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$ that

$$\tilde{v}_{\mathbf{i}_{\leq r-1}}^{\mathrm{high}}((f(t_1, \ldots, t_r) \cdot f_{i_r}^{a_r})|_{t_r=0}) = -(a_{r-1}, \ldots, a_1),$$

where $\tilde{v}_{\mathbf{i}_{\leq r-1}}^{\mathrm{high}}$ denotes the valuation on $\mathbb{C}(X(w_{\leq r-1}))$ defined to be the highest term valuation on $\mathbb{C}(t_1, \ldots, t_{r-1})$ $(= \mathbb{C}(U_{i_1}^- \times \cdots \times U_{i_{r-1}}^-))$ with respect to the lexicographic order $t_{r-1} \succ \cdots \succ t_1$. By induction on r, the assertion of the proposition follows. □

Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ be a perfect basis, and $w \in W$. Recall that $\Xi_w^{\mathrm{up}}(b) \in \mathbb{C}[U^- \cap X(w)]$ (resp., $\Xi_{\lambda,w}^{\mathrm{up}}(b) \in H^0(X(w), \mathcal{L}_\lambda)$) denotes the restriction of $\Xi^{\mathrm{up}}(b) \in \mathbb{C}[U^-]$ for $b \in \mathcal{B}(\infty)$ (resp., $\Xi_\lambda^{\mathrm{up}}(b) \in H^0(G/B, \mathcal{L}_\lambda)$ for $b \in \mathcal{B}(\lambda)$). We set $\tau_\lambda := \Xi_{\lambda,w}^{\mathrm{up}}(b_\lambda)$, and define an $\mathbb{R}$-linear automorphism $\tilde{\eta}\colon \mathbb{R} \times \mathbb{R}^r \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}^r$ by $\tilde{\eta}(k, \mathbf{a}) := (k, -\mathbf{a}^{\mathrm{op}})$, where $\mathbf{a}^{\mathrm{op}} := (a_r, \ldots, a_1)$ for $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$. The following is the first main result of this thesis.

Theorem 2.3.2. *Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ be a perfect basis, $\mathbf{i} \in I^r$ a reduced word for $w \in W$, and $\lambda \in P_+$.*

1. *The Kashiwara embedding $\Psi_{\mathbf{i}}(b)$ is equal to $-\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b))^{\mathrm{op}}$ for all $b \in \mathcal{B}_w(\infty)$.*
2. *The Kashiwara embedding $\Psi_{\mathbf{i}}(b)$ is equal to $-\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda)^{\mathrm{op}}$ for all $b \in \mathcal{B}_w(\lambda)$.*
3. *The following equalities hold:*

$$\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda) = \tilde{\eta}(S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)),\ \widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda) = \tilde{\eta}(C(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)),\ \textit{and}$$
$$\widetilde{\Delta}_{\mathbf{i}}(\lambda) = -\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}.$$

Remark 2.3.3. In Theorem 2.3.2, we need not assume condition (D) for a perfect basis.

Before proving this theorem, we give some corollaries.

Corollary 2.3.4. *Let $\mathbf{i} \in I^r$ be a reduced word for $w \in W$, and $\lambda \in P_+$.*

(1) *The sets $\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda)$ and $S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$ are both finitely generated semigroups.*

(2) *The real closed cones $\widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda)$ and $C(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$ are both rational convex polyhedral cones, and the following equality holds:*

$$S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda) = C(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda) \cap (\mathbb{Z}_{>0} \times \mathbb{Z}^r).$$

(3) *The sets $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ and $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$ are both rational convex polytopes, and the following equality holds:*

$$\Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda)) = -\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}} \cap \mathbb{Z}^r.$$

Proof. Part (2) follows by Corollary 2.1.11 (2) and Theorem 2.3.2 (3) since $C(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)$ is convex. Then, part (3) is an immediate consequence of part (2), Corollary 2.1.13, and Theorem 2.3.2 (3). Finally, part (1) follows by part (2), Theorem 2.3.2 (3), and Gordan's lemma (see, for instance, [**7**, Proposition 1.2.17]). □

Since $\Psi_{\mathbf{i}}(b)$, $b \in \mathcal{B}_w(\infty)$, are all distinct, we obtain the following by Proposition 1.1.2 (2), Corollary 2.2.26, and Theorem 2.3.2 (1).

Corollary 2.3.5. *The image $\Psi_{\mathbf{i}}(\mathcal{B}_w(\infty))$ is identical to $-\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\mathbb{C}[U^- \cap X(w)] \setminus \{0\})^{\mathrm{op}}$.*

In addition, the following corollary holds by Definition 1.1.1, Proposition 1.1.2 (2), Corollary 2.2.26, and Theorem 2.3.2 (1).

Corollary 2.3.6. *Let $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ be a perfect basis, and $\mathbf{i} \in I^r$ a reduced word for $w \in W$. Then, the following holds:*

$$\Xi_w^{\mathrm{up}}(b_1) \cdot \Xi_w^{\mathrm{up}}(b_2) \in \mathbb{C}^\times \Xi_w^{\mathrm{up}}(b) + \sum_{b' \in \mathcal{B}_w(\infty);\ \Psi_{\mathbf{i}}(b') \prec \Psi_{\mathbf{i}}(b)} \mathbb{C}\Xi_w^{\mathrm{up}}(b')$$

for all $b_1, b_2 \in \mathcal{B}_w(\infty)$, where $b \in \mathcal{B}_w(\infty)$ is the unique element such that $\Psi_{\mathbf{i}}(b) = \Psi_{\mathbf{i}}(b_1) + \Psi_{\mathbf{i}}(b_2)$.

Remark 2.3.7. Corollary 2.3.6 is also obtained from Corollary 2.2.27 by applying the involution $*$.

Proof of Theorem 2.3.2. By the first assertion of Corollary 2.2.20, we have $\Xi_w^{\mathrm{up}}(b) = \Xi_{\lambda,w}^{\mathrm{up}}(\pi_\lambda(b))/\tau_\lambda$ in $\mathbb{C}[U^- \cap X(w)]$ for all $b \in \widetilde{\mathcal{B}}_w(\lambda)$. Hence part (2) follows immediately from part (1).

We set $\Psi_{\mathbf{i}}(b) = (a_1, \ldots, a_r)$, $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b))^{\mathrm{op}} = -(a'_1, \ldots, a'_r)$, and

$$\begin{aligned}
a''_r &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid \Xi^{\mathrm{up}}(b) \cdot f_{i_r}^a \neq 0\},\\
a''_{r-1} &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid (\Xi^{\mathrm{up}}(b) \cdot f_{i_r}^{a''_r}) \cdot f_{i_{r-1}}^a \neq 0\},\\
&\ \vdots\\
a''_1 &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid ((\cdots((\Xi^{\mathrm{up}}(b) \cdot f_{i_r}^{a''_r}) \cdot f_{i_{r-1}}^{a''_{r-1}}) \cdots) \cdot f_{i_2}^{a''_2}) \cdot f_{i_1}^a \neq 0\}
\end{aligned}$$

for $b \in \mathcal{B}_w(\infty)$. We will prove that $(a_1, \dots, a_r) = (a_1'', \dots, a_r'')$, and that $(a_1', \dots, a_r') = (a_1'', \dots, a_r'')$. First, it follows by the definition of a_r'' that

$$\begin{aligned} a_r'' &= \max\{a \in \mathbb{Z}_{\geq 0} \mid (\tilde{e}_{i_r}^*)^a b \neq 0\} \\ &\quad \text{(by the second assertion of Proposition 2.2.16)} \\ &= a_r \quad \text{(by Proposition 2.1.1 (2))}, \end{aligned}$$

and hence that

$$\begin{aligned} &\Xi^{\mathrm{up}}(b) \cdot f_{i_r}^{a_r''} \in \mathbb{C}^\times \Xi^{\mathrm{up}}((\tilde{e}_{i_r}^*)^{a_r} b) \\ &\qquad\qquad \text{(by the assertion of Proposition 2.2.16 for } \Xi^{\mathrm{up}}(b) \cdot f_i^{\varepsilon_i^*(b)}). \end{aligned}$$

By the definition of $a_{r-1}'', \dots, a_1''$, this implies the following equalities:

$$\begin{aligned} a_{r-1}'' &= \max\{a \in \mathbb{Z}_{\geq 0} \mid \Xi^{\mathrm{up}}((\tilde{e}_{i_r}^*)^{a_r} b) \cdot f_{i_{r-1}}^a \neq 0\}, \\ a_{r-2}'' &= \max\{a \in \mathbb{Z}_{\geq 0} \mid (\Xi^{\mathrm{up}}((\tilde{e}_{i_r}^*)^{a_r} b) \cdot f_{i_{r-1}}^{a_{r-1}''}) \cdot f_{i_{r-2}}^a \neq 0\}, \\ &\vdots \\ a_1'' &= \max\{a \in \mathbb{Z}_{\geq 0} \mid ((\cdots(\Xi^{\mathrm{up}}((\tilde{e}_{i_r}^*)^{a_r} b) \cdot f_{i_{r-1}}^{a_{r-1}''}) \cdots) \cdot f_{i_2}^{a_2''}) \cdot f_{i_1}^a \neq 0\}. \end{aligned}$$

Since $(\tilde{e}_{i_r}^*)^{a_r} b \in \mathcal{B}_{w_{\leq r-1}}(\infty)$ and $\Psi_{\mathbf{i}_{\leq r-1}}((\tilde{e}_{i_r}^*)^{a_r} b) = (a_1, \dots, a_{r-1})$, by repeating this argument, with $\mathbf{i}$ and b replaced by $\mathbf{i}_{\leq r-1}$ and $(\tilde{e}_{i_r}^*)^{a_r} b$, respectively, we deduce that $(a_1, \dots, a_r) = (a_1'', \dots, a_r'')$, and that

$$\begin{aligned} &((\cdots((\Xi^{\mathrm{up}}(b) \cdot f_{i_r}^{a_r''}) \cdot f_{i_{r-1}}^{a_{r-1}''}) \cdots) \cdot f_{i_2}^{a_2''}) \cdot f_{i_1}^{a_1''} \\ &\in \mathbb{C}^\times \Xi^{\mathrm{up}}((\tilde{e}_{i_1}^*)^{a_1} \cdots (\tilde{e}_{i_r}^*)^{a_r} b) = \mathbb{C}^\times \Xi^{\mathrm{up}}(b_\infty). \end{aligned}$$

Since $1 \in U^- \cap X(e)$ and $\langle \Xi^{\mathrm{up}}(b_\infty), 1\rangle = 1$ by the definition of perfect bases, we see that

$$(((\cdots((\Xi^{\mathrm{up}}(b) \cdot f_{i_r}^{a_r''}) \cdot f_{i_{r-1}}^{a_{r-1}''}) \cdots) \cdot f_{i_2}^{a_2''}) \cdot f_{i_1}^{a_1''})|_{U^- \cap X(e)} \neq 0. \tag{2.3.2}$$

Recall that the coordinate ring $\mathbb{C}[U^- \cap X(w)]$ is identified with a $\mathbb{C}$-subalgebra of $\mathbb{C}[t_1, \dots, t_r] = \mathbb{C}[U_{i_1}^- \times \cdots \times U_{i_r}^-]$ by the following birational morphism:

$$U_{i_1}^- \times \cdots \times U_{i_r}^- \to U^- \cap X(w),\ (u_1, \dots, u_r) \mapsto u_1 \cdots u_r \bmod B.$$

Since the image $U_{i_1}^- \cdots U_{i_r}^-$ ($\subset U^-$) is stable under the right action of $U_{i_r}^-$ on U^-, its Zariski closure $U^- \cap X(w) = \overline{U_{i_1}^- \cdots U_{i_r}^-}$ in U^- is also stable. Similarly, the intersection $U^- \cap X(w_{\leq k})$ for $1 \leq k \leq r$ is stable under the right action of $U_{i_k}^-$ on U^-, and the restriction map $\mathbb{C}[U^-] \twoheadrightarrow \mathbb{C}[U^- \cap X(w_{\leq k})]$ is compatible with the right actions of $U_{i_k}^-$. Note that the induced right action of f_{i_k} on $\mathbb{C}[U^- \cap X(w_{\leq k})]$ ($\hookrightarrow \mathbb{C}[t_1, \dots, t_k]$) is identical to that of f_{i_k} on $\mathbb{C}[t_1, \dots, t_k]$ discussed in (2.3.1). Now we see that

$$\begin{aligned} &(((\cdots((\Xi^{\mathrm{up}}(b) \cdot f_{i_r}^{a_r''}) \cdot f_{i_{r-1}}^{a_{r-1}''}) \cdots) \cdot f_{i_2}^{a_2''}) \cdot f_{i_1}^{a_1''})|_{U^- \cap X(e)} \\ &= ((\cdots((\Xi_w^{\mathrm{up}}(b) \cdot f_{i_r}^{a_r''})|_{U^- \cap X(w_{\leq r-1})} \cdot f_{i_{r-1}}^{a_{r-1}''}) \cdots)|_{U^- \cap X(w_{\leq 1})} \cdot f_{i_1}^{a_1''})|_{U^- \cap X(e)}, \end{aligned}$$

and hence by (2.3.2) that

$$((\cdots(\Xi_w^{\mathrm{up}}(b) \cdot f_{i_r}^{a_r''})|_{U^- \cap X(w_{\leq r-1})} \cdots)|_{U^- \cap X(w_{\leq k+1})} \cdot f_{i_{k+1}}^{a_{k+1}''})|_{U^- \cap X(w_{\leq k})} \cdot f_{i_k}^{a_k''} \neq 0$$

for all $1 \le k \le r$. Similarly, we deduce that

$$\begin{aligned}
&((\cdots(\Xi_w^{\rm up}(b)\cdot f_{i_r}^{a''_r})|_{U^-\cap X(w_{\le r-1})}\cdots)|_{U^-\cap X(w_{\le k+1})}\cdot f_{i_{k+1}}^{a''_{k+1}})|_{U^-\cap X(w_{\le k})}\cdot f_{i_k}^{a''_k+1}\\
&=(((\cdots((\Xi^{\rm up}(b)\cdot f_{i_r}^{a''_r})\cdot f_{i_{r-1}}^{a''_{r-1}})\cdots)\cdot f_{i_{k+1}}^{a''_{k+1}})\cdot f_{i_k}^{a''_k+1})|_{U^-\cap X(w_{\le k})}\\
&=0 \quad (\text{by the definition of } a''_k).
\end{aligned}$$

From these, it follows that a''_k for $1 \le k \le r$ equals the maximum of $a \in \mathbb{Z}_{\ge 0}$ such that

$$((\cdots(\Xi_w^{\rm up}(b)\cdot f_{i_r}^{a''_r})|_{U^-\cap X(w_{\le r-1})}\cdots)|_{U^-\cap X(w_{\le k+1})}\cdot f_{i_{k+1}}^{a''_{k+1}})|_{U^-\cap X(w_{\le k})}\cdot f_{i_k}^{a}\neq 0.$$

Also, since the restriction map $\mathbb{C}[U^-\cap X(w_{\le k})] \twoheadrightarrow \mathbb{C}[U^-\cap X(w_{\le k-1})]$ is given by $t_k \mapsto 0$, we see by Proposition 2.3.1 that a'_k for $1 \le k \le r$ equals the maximum of $a \in \mathbb{Z}_{\ge 0}$ such that

$$((\cdots(\Xi_w^{\rm up}(b)\cdot f_{i_r}^{a'_r})|_{U^-\cap X(w_{\le r-1})}\cdots)|_{U^-\cap X(w_{\le k+1})}\cdot f_{i_{k+1}}^{a'_{k+1}})|_{U^-\cap X(w_{\le k})}\cdot f_{i_k}^{a}\neq 0.$$

These imply that $(a'_1, \dots, a'_r) = (a''_1, \dots, a''_r)$. This proves part (1).

Finally, we prove part (3). Since $\mathcal{L}_\lambda^{\otimes k} = \mathcal{L}_{k\lambda}$ and $\tau_\lambda^k = \tau_{k\lambda}$ in $H^0(X(w), \mathcal{L}_{k\lambda})$ for all $k \in \mathbb{Z}_{>0}$, it follows that

$$S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\rm high}, \tau_\lambda) = \bigcup_{k\in\mathbb{Z}_{>0}} \{(k, \tilde{v}_{\mathbf{i}}^{\rm high}(\sigma/\tau_{k\lambda})) \mid \sigma \in H^0(X(w), \mathcal{L}_{k\lambda}) \setminus \{0\}\}.$$

Also, since $\Psi_{\mathbf{i}}(b)$, $b \in \mathcal{B}_w(k\lambda)$, are all distinct, we deduce by part (2), Proposition 1.1.2 (2), and Corollary 2.2.22 that

$$\{\Psi_{\mathbf{i}}(b) \mid b \in \mathcal{B}_w(k\lambda)\} = \{-\tilde{v}_{\mathbf{i}}^{\rm high}(\sigma/\tau_{k\lambda})^{\rm op} \mid \sigma \in H^0(X(w), \mathcal{L}_{k\lambda}) \setminus \{0\}\}$$

for all $k \in \mathbb{Z}_{>0}$, which implies that $\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda) = \tilde{\eta}(S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\rm high}, \tau_\lambda))$. From this equality, the other assertions of part (3) follow immediately by the definitions. This completes the proof of Theorem 2.3.2. □

2.4. Explicit forms of Newton-Okounkov polytopes

Under the assumption that $(\mathbf{j}, \lambda)$ is ample (see Definition 2.4.1 below), the image $\Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$ is given by a system of explicit affine inequalities. In order to obtain an explicit form of $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\rm high}, \tau_\lambda)$, we recall the description of $\Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$, following [**50, 51**]. Consider the following infinite-dimensional $\mathbb{R}$-vector space:

$$\mathbb{R}^\infty = \{\mathbf{a} = (\dots, a_k, \dots, a_2, a_1) \mid a_k \in \mathbb{R} \text{ for } k \in \mathbb{Z}_{>0}, \text{ and } a_k = 0 \text{ for } k \gg 0\},$$

and write an affine function ψ on $\mathbb{R}^\infty$ as $\psi(\mathbf{a}) = \psi_0 + \sum_{k\in\mathbb{Z}_{>0}} \psi_k a_k$ with $\psi_k \in \mathbb{R}$ for $k \in \mathbb{Z}_{\ge 0}$. Recall that $\mathbf{i} = (i_1, \dots, i_r) \in I^r$ is a reduced word for $w \in W$, and that $\mathbf{j} = (\dots, j_k, \dots, j_2, j_1)$ is an extension of $\mathbf{i}$ such that $j_k \neq j_{k+1}$ for all $k \in \mathbb{Z}_{>0}$, and such that the cardinality of $\{k \in \mathbb{Z}_{>0} \mid j_k = i\}$ is ∞ for each $i \in I$. For $k \in \mathbb{Z}_{>0}$, we set

$$k^{(+)} := \min\{l > k \mid j_l = j_k\}, \text{ and}$$

$$k^{(-)} := \begin{cases} \max\{l < k \mid j_l = j_k\} & \text{if it exists,} \\ 0 & \text{otherwise.} \end{cases}$$

For $k \in \mathbb{Z}_{>0}$ and $i \in I$, let $\beta_k^{(\pm)}(\mathbf{a}), \lambda^{(i)}(\mathbf{a})$ denote the affine functions on $\mathbb{R}^\infty$ given by

$$\beta_k^{(+)}(\mathbf{a}) := a_k + \sum_{k<l<k^{(+)}} \langle \alpha_{j_l}, h_{j_k} \rangle a_l + a_{k^{(+)}},$$

$$\beta_k^{(-)}(\mathbf{a}) := \begin{cases} a_{k^{(-)}} + \sum_{k^{(-)}<l<k} \langle \alpha_{j_l}, h_{j_k} \rangle a_l + a_k & \text{if } k^{(-)} > 0, \\ -\langle \lambda, h_{j_k} \rangle + \sum_{1 \le l<k} \langle \alpha_{j_l}, h_{j_k} \rangle a_l + a_k & \text{if } k^{(-)} = 0, \end{cases}$$

$$\lambda^{(i)}(\mathbf{a}) := \langle \lambda, h_i \rangle - \sum_{1 \le l < \mathbf{j}^{(i)}} \langle \alpha_{j_l}, h_i \rangle a_l - a_{\mathbf{j}^{(i)}},$$

where we write $\mathbf{j}^{(i)} := \min\{k \in \mathbb{Z}_{>0} \mid j_k = i\}$. Define operators $\widehat{S}_k$, $k \in \mathbb{Z}_{>0}$, on the set of affine functions on $\mathbb{R}^\infty$ by

$$\widehat{S}_k(\psi) := \begin{cases} \psi - \psi_k \beta_k^{(+)} & \text{if } \psi_k > 0, \\ \psi - \psi_k \beta_k^{(-)} & \text{if } \psi_k \le 0, \end{cases}$$

and let $\Xi_{\mathbf{j}}[\lambda]$ denote the set of affine functions generated by $\widehat{S}_k$, $k \in \mathbb{Z}_{>0}$, from the functions a_l, $l \in \mathbb{Z}_{>0}$, and $\lambda^{(i)}(\mathbf{a})$, $i \in I$; namely,

$$\Xi_{\mathbf{j}}[\lambda] := \{\widehat{S}_{l_k} \cdots \widehat{S}_{l_1} a_{l_0} \mid k \in \mathbb{Z}_{\ge 0} \text{ and } l_0, \ldots, l_k \in \mathbb{Z}_{>0}\}$$
$$\cup \{\widehat{S}_{l_k} \cdots \widehat{S}_{l_1} \lambda^{(i)}(\mathbf{a}) \mid k \in \mathbb{Z}_{\ge 0},\ i \in I, \text{ and } l_1, \ldots, l_k \in \mathbb{Z}_{>0}\}.$$

DEFINITION 2.4.1 (see [**50**, Sect. 4.2]). Set

$$\Sigma_{\mathbf{j}}[\lambda] := \{\mathbf{a} \in \mathbb{Z}^\infty \mid \psi(\mathbf{a}) \ge 0 \text{ for all } \psi \in \Xi_{\mathbf{j}}[\lambda]\};$$

a pair $(\mathbf{j}, \lambda)$ is called *ample* if $(\ldots, 0, \ldots, 0, 0) \in \Sigma_{\mathbf{j}}[\lambda]$.

PROPOSITION 2.4.2 (see [**50**, Theorem 4.1] and [**51**, Proposition 3.1]). *Assume that* $(\mathbf{j}, \lambda)$ *is ample. Then, the image* $\Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda))$ *is identical to the following set*:

$$\{(a_1, \ldots, a_r) \in \mathbb{Z}^r \mid (\ldots, 0, 0, a_1, \ldots, a_r) \in \Sigma_{\mathbf{j}}[\lambda]\}.$$

For all $\psi \in \Xi_{\mathbf{j}}[\lambda]$, the constant term $\psi(\ldots, 0, \ldots, 0, 0)$ is regarded as a linear function of λ by the definition of $\Xi_{\mathbf{j}}[\lambda]$; hence, for a fixed dominant integral weight $\lambda \in P_+$, we can regard an element of $\Xi_{\mathbf{j}}[k\lambda]$ as a linear function of k and a_l for $l \in \mathbb{Z}_{>0}$. Thus, we obtain the following by Definition 2.1.8, Theorem 2.3.2 (3), and Proposition 2.4.2.

COROLLARY 2.4.3. *Assume that* $(\mathbf{j}, \lambda)$ *is ample. Then, the following equalities hold*:

$$\widetilde{\mathcal{S}}_{\mathbf{i}}(\lambda) = \tilde{\eta}(S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda))$$
$$= \{(k, a_1, \ldots, a_r) \in \mathbb{Z}_{>0} \times \mathbb{Z}^r \mid \psi(\ldots, 0, 0, a_1, \ldots, a_r) \ge 0 \textit{ for all } \psi \in \Xi_{\mathbf{j}}[k\lambda]\},$$
$$\widetilde{\mathcal{C}}_{\mathbf{i}}(\lambda) = \tilde{\eta}(C(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda))$$
$$= \{(k, a_1, \ldots, a_r) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^r \mid \psi(\ldots, 0, 0, a_1, \ldots, a_r) \ge 0 \textit{ for all } \psi \in \Xi_{\mathbf{j}}[k\lambda]\},$$
$$\widetilde{\Delta}_{\mathbf{i}}(\lambda) = -\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}$$
$$= \{(a_1, \ldots, a_r) \in \mathbb{R}^r \mid \psi(\ldots, 0, 0, a_1, \ldots, a_r) \ge 0 \textit{ for all } \psi \in \Xi_{\mathbf{j}}[\lambda]\}.$$

EXAMPLE 2.4.4. Let $G = SL_{n+1}(\mathbb{C})$, and $\lambda \in P_+$. We consider a specific reduced word $\mathbf{i} = (1, 2, 1, 3, 2, 1, \dots, n, n-1, \dots, 1) \in I^N$ for the longest element $w_0 \in W$, where $N := \frac{n(n+1)}{2}$. Then, by Corollary 2.4.3 (see also [**52**, Corollary 2.7]), the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is identical to the set of $(a_1^{(1)}, a_2^{(2)}, a_2^{(1)}, \dots, a_n^{(n)}, \dots, a_n^{(1)}) \in \mathbb{R}^N$ satisfying the following conditions:

$$\begin{array}{ccccccccc}
\lambda_{\geq 1} & & \lambda_{\geq 2} & & \cdots & & \lambda_{\geq n} & & 0, \\
& a_n^{(1)} + \lambda_{\geq 2} & & a_{n-1}^{(1)} + \lambda_{\geq 3} & & \cdots & & a_1^{(1)} & \\
& & a_n^{(2)} + \lambda_{\geq 3} & & \cdots & & a_2^{(2)} & & \\
& & & \ddots & & & & & \\
& & & & a_n^{(n)} & & & &
\end{array}$$

where $\lambda_{\geq k} := \sum_{k \leq l \leq n} \langle \lambda, h_l \rangle \in \mathbb{Z}_{\geq 0}$ for $1 \leq k \leq n$, and the notation

$$\begin{array}{ccc} a & & c \\ & b & \end{array}$$

means that $a \geq b \geq c$. This implies that the translation

$$\widetilde{\Delta}_{\mathbf{i}}(\lambda) + (0, \underbrace{0, \lambda_{\geq n}}_{2}, \underbrace{0, \lambda_{\geq n}, \lambda_{\geq n-1}}_{3}, \dots, \underbrace{0, \lambda_{\geq n}, \lambda_{\geq n-1}, \dots, \lambda_{\geq 2}}_{n})$$

of the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is identical to the Gelfand-Zetlin polytope associated to the non-increasing sequence $(\lambda_{\geq 1}, \lambda_{\geq 2}, \dots, \lambda_{\geq n}, 0)$.

2.5. Relation with Kashiwara's involution

This section is devoted to describing Kashiwara's involution $*$ in terms of valuations on the function field $\mathbb{C}(G/B)$. Let $\mathbf{i} = (i_1, \dots, i_N) \in I^N$ be a reduced word for the longest element $w_0 \in W$, and $\Phi_{\mathbf{i}} \colon \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^N$ the corresponding string parametrization (see Definition 1.4.1 and Remark 2.1.6). Recall that we identify the function field $\mathbb{C}(G/B) = \mathbb{C}(X(w_0))$ with the rational function field $\mathbb{C}(t_1, \dots, t_N)$, and that the valuation $v_{\mathbf{i}}^{\mathrm{high}} \colon \mathbb{C}(G/B) \setminus \{0\} \to \mathbb{Z}^N$ is defined to be the highest term valuation on $\mathbb{C}(t_1, \dots, t_N)$ with respect to the lexicographic order $t_1 > \cdots > t_N$ (see Definition 1.1.3). Then, Theorem 2.2.9 implies that

$$\Phi_{\mathbf{i}}(b) = -v_{\mathbf{i}}^{\mathrm{high}}(\Xi^{\mathrm{up}}(b))$$

for all $b \in \mathcal{B}(\infty)$. Since $\Phi_{\mathbf{i}}(b)$, $b \in \mathcal{B}(\infty)$, are all distinct, we deduce by Proposition 1.1.2 (2) that

$$\Phi_{\mathbf{i}}(\mathcal{B}(\infty)) = -v_{\mathbf{i}}^{\mathrm{high}}(\mathbb{C}[U^-] \setminus \{0\}).$$

Now Corollary 2.1.7 implies that this set is also identical to the following set:

$$\Psi_{\mathbf{i}^{\mathrm{op}}}(\mathcal{B}(\infty))^{\mathrm{op}} = -\tilde{v}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{high}}(\mathbb{C}[U^-] \setminus \{0\}) \quad \text{(by Corollary 2.3.5)},$$

where $\mathbf{i}^{\mathrm{op}} := (i_N, \dots, i_1) \in I^N$, which is a reduced word for $w_0^{-1} = w_0$. Let $\eta_{\mathbf{i}} \colon \Psi_{\mathbf{i}}(\mathcal{B}(\infty)) \to \Phi_{\mathbf{i}^{\mathrm{op}}}(\mathcal{B}(\infty))^{\mathrm{op}}$ $(= \Psi_{\mathbf{i}}(\mathcal{B}(\infty)))$ be the transition map given by $\eta_{\mathbf{i}}(\Psi_{\mathbf{i}}(b)) = \Phi_{\mathbf{i}^{\mathrm{op}}}(b)^{\mathrm{op}}$ for $b \in \mathcal{B}(\infty)$. Then, we see that Kashiwara's involution $*$ corresponds to the map $\eta_{\mathbf{i}}$ through the Kashiwara embedding $\Psi_{\mathbf{i}}$:

$$b^* = \Psi_{\mathbf{i}}^{-1} \circ \eta_{\mathbf{i}} \circ \Psi_{\mathbf{i}}(b)$$

for all $b \in \mathcal{B}(\infty)$. Take an extension $\mathbf{j} = (\ldots, j_k, \ldots, j_2, j_1)$ of $\mathbf{i}$ as in Sect. 2.1. It follows by the definition of the crystal $\mathbb{Z}_{\mathbf{j}}^{\infty}$ that

$$\tilde{e}_i^{\max}\mathbf{a} = \left(a_k - \delta_{i,i_k} \max\{0, \sigma_k(\mathbf{a}) - \sigma^{(i)}(\mathbf{a}_{\geq k+1})\}\right)_{k \in \mathbb{Z}_{>0}}$$

for $i \in I$ and $\mathbf{a} = (\ldots, a_k, \ldots, a_2, a_1) \in \mathbb{Z}_{\mathbf{j}}^{\infty}$, where we set $\mathbf{a}_{\geq k} \coloneqq (\ldots, a_{k+1}, a_k) \in \mathbb{Z}^{\infty}_{(\ldots, j_{k+1}, j_k)}$ for $k \in \mathbb{Z}_{>0}$, and $\tilde{e}_i^{\max}\mathbf{a} \coloneqq \tilde{e}_i^{\varepsilon_i(\mathbf{a})}\mathbf{a}$. In addition, if we set $\mathbf{a}' \coloneqq (\ldots, 0, 0, a_1, \ldots, a_N) \in \mathbb{Z}_{\mathbf{j}}^{\infty}$ for $\mathbf{a} = (a_1, \ldots, a_N) \in \Psi_{\mathbf{i}}(\mathcal{B}(\infty))$, then we deduce that

$$\begin{aligned}
\eta_{\mathbf{i}}(\mathbf{a}) &= \Phi_{\mathbf{i}^{\mathrm{op}}}(\Psi_{\mathbf{j}}^{-1}(\mathbf{a}'))^{\mathrm{op}} \\
&\quad \text{(by Definition 2.1.4)} \\
&= (\varepsilon_{i_1}(\tilde{e}_{i_2}^{\max} \cdots \tilde{e}_{i_N}^{\max}\Psi_{\mathbf{j}}^{-1}(\mathbf{a}')), \ldots, \varepsilon_{i_{N-1}}(\tilde{e}_{i_N}^{\max}\Psi_{\mathbf{j}}^{-1}(\mathbf{a}')), \varepsilon_{i_N}(\Psi_{\mathbf{j}}^{-1}(\mathbf{a}'))) \\
&\quad \text{(by the definition of } \Phi_{\mathbf{i}^{\mathrm{op}}}) \\
&= (\varepsilon_{i_1}(\tilde{e}_{i_2}^{\max} \cdots \tilde{e}_{i_N}^{\max}\mathbf{a}'), \ldots, \varepsilon_{i_{N-1}}(\tilde{e}_{i_N}^{\max}\mathbf{a}'), \varepsilon_{i_N}(\mathbf{a}')) \\
&\quad \text{(since } \Psi_{\mathbf{j}} \text{ is a strict embedding of crystals)} \\
&= (\sigma^{(i_1)}(\tilde{e}_{i_2}^{\max} \cdots \tilde{e}_{i_N}^{\max}\mathbf{a}'), \ldots, \sigma^{(i_{N-1})}(\tilde{e}_{i_N}^{\max}\mathbf{a}'), \sigma^{(i_N)}(\mathbf{a}')) \\
&\quad \text{(by the definition of the crystal structure on } \mathbb{Z}_{\mathbf{j}}^{\infty}).
\end{aligned}$$

From these, it follows that the map $\eta_{\mathbf{i}} \colon \Psi_{\mathbf{i}}(\mathcal{B}(\infty)) \to \Psi_{\mathbf{i}}(\mathcal{B}(\infty))$ is naturally extended to a piecewise-linear map from the string cone $\widetilde{\mathcal{C}}_{\mathbf{i}} \coloneqq \mathcal{C}_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{op}}$ (see Sect. 1.4 for the definition) to itself, which is also denoted by $\eta_{\mathbf{i}}$; we see by the equality $\widetilde{\mathcal{C}}_{\mathbf{i}} \cap \mathbb{Z}^N = \Psi_{\mathbf{i}}(\mathcal{B}(\infty))$ that such an extension is unique. Since $*^2 = \mathrm{id}_{\mathcal{B}(\infty)}$, we deduce that $(\eta_{\mathbf{i}}|_{\Psi_{\mathbf{i}}(\mathcal{B}(\infty))})^2 = \mathrm{id}_{\Psi_{\mathbf{i}}(\mathcal{B}(\infty))}$, and hence that $\eta_{\mathbf{i}}^2 = \mathrm{id}_{\widetilde{\mathcal{C}}_{\mathbf{i}}}$. Thus, we obtain the following.

Corollary 2.5.1. *Let $\mathbf{i} \in I^N$ be a reduced word for the longest element $w_0 \in W$, and $\eta_{\mathbf{i}} \colon \widetilde{\mathcal{C}}_{\mathbf{i}} \to \widetilde{\mathcal{C}}_{\mathbf{i}}$ a unique piecewise-linear map such that $b^* = \Psi_{\mathbf{i}}^{-1} \circ \eta_{\mathbf{i}} \circ \Psi_{\mathbf{i}}(b)$ for all $b \in \mathcal{B}(\infty)$.*

(1) *The map $\eta_{\mathbf{i}}$ corresponds to the change of valuations from $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$ to $v_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{high}}$:*

$$\eta_{\mathbf{i}}(-\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi^{\mathrm{up}}(b))^{\mathrm{op}}) = -v_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{high}}(\Xi^{\mathrm{up}}(b))^{\mathrm{op}}$$

for all $b \in \mathcal{B}(\infty)$.

(2) *The equality $\eta_{\mathbf{i}}^2 = \mathrm{id}_{\widetilde{\mathcal{C}}_{\mathbf{i}}}$ holds.*

(3) *The map $\eta_{\mathbf{i}}$ induces a bijective piecewise-linear map from the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda) = -\Delta(G/B, \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}$ onto the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\lambda)^{\mathrm{op}} = -\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{i}^{\mathrm{op}}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}}$ for all $\lambda \in P_+$.*

Example 2.5.2. Let $G = SL_3(\mathbb{C})$ (of type A_2), $I = \{1, 2\}$, and $\mathbf{i} = (1, 2, 1) \in I^3$, a reduced word for the longest element $w_0 \in W$. We deduce by [**53**, Theorem 3.1] that the image $\Psi_{\mathbf{i}}(\mathcal{B}(\infty))$ is identical to the following semigroup:

$$\{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0} \mid a_1 \leq a_2\}.$$

Recall from Example 1.2.6 that the coordinate ring $\mathbb{C}[U^-]$ is regarded as a $\mathbb{C}$-subalgebra $\mathbb{C}[t_1 + t_3, t_2, t_2 t_3]$ of $\mathbb{C}[t_1, t_2, t_3]$; hence we see by the definition of $\tilde{v}^{\mathrm{high}}_{\mathbf{i}}$ that

$$-\tilde{v}^{\mathrm{high}}_{\mathbf{i}}(\mathbb{C}[U^-] \setminus \{0\})^{\mathrm{op}} = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0} \mid a_1 \leq a_2\},$$

which is indeed identical to $\Psi_{\mathbf{i}}(\mathcal{B}(\infty))$. Now it follows by the definition of the crystal $\mathbb{Z}^{\infty}_{\mathbf{j}}$ that

$$\varepsilon_1(\mathbf{a}') = \max\{a_1, 2a_1 - a_2 + a_3\},\ \tilde{e}_1^{\max}\mathbf{a}' = (\ldots, 0, 0, a_2, \min\{a_3, a_2 - a_1\}),$$
$$\varepsilon_2(\tilde{e}_1^{\max}\mathbf{a}') = a_2,\ \tilde{e}_2^{\max}\tilde{e}_1^{\max}\mathbf{a}' = (\ldots, 0, 0, \min\{a_3, a_2 - a_1\}),\ \text{and}$$
$$\varepsilon_1(\tilde{e}_2^{\max}\tilde{e}_1^{\max}\mathbf{a}') = \min\{a_3, a_2 - a_1\}$$

for $\mathbf{a}' = (\ldots, 0, 0, a_1, a_2, a_3) \in \Psi_{\mathbf{j}}(\mathcal{B}(\infty))$. Therefore, we obtain

$$\eta_{\mathbf{i}}(\mathbf{a}) = (\min\{a_3, a_2 - a_1\}, a_2, \max\{a_1, 2a_1 - a_2 + a_3\})$$

for $\mathbf{a} = (a_1, a_2, a_3) \in \Psi_{\mathbf{i}}(\mathcal{B}(\infty))$. This piecewise-linear map induces a bijective map from the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ onto the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\lambda)^{\mathrm{op}}$. Indeed, by Corollary 2.4.3, we deduce that

$$\widetilde{\Delta}_{\mathbf{i}}(\lambda) = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid 0 \leq a_3 \leq \lambda_1,\ 0 \leq a_1 \leq \lambda_2,\ a_1 \leq a_2 \leq a_3 + \lambda_2\},$$

where $\lambda_i \coloneqq \langle \lambda, h_i \rangle$ for $i = 1, 2$. In particular, if $\lambda = \alpha_1 + \alpha_2 \in P_+$, then we have

$$\widetilde{\Delta}_{\mathbf{i}}(\alpha_1 + \alpha_2) = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid 0 \leq a_3 \leq 1,\ 0 \leq a_1 \leq 1,\ a_1 \leq a_2 \leq a_3 + 1\};$$

see Figure 2 in Example 1.2.6. Also, we deduce by [**42**, Sect. 1] that for $\lambda \in P_+$, the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\lambda)^{\mathrm{op}}$ is identical to the following polytope:

$$\{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid 0 \leq a_1 \leq \lambda_1,\ a_1 \leq a_2 \leq a_1 + \lambda_2,\ 0 \leq a_3 \leq a_2 - 2a_1 + \lambda_1\}.$$

In particular, if $\lambda = \alpha_1 + \alpha_2 \in P_+$, then the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\alpha_1 + \alpha_2)^{\mathrm{op}}$ is identical to the following polytope:

$$\{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid 0 \leq a_1 \leq 1,\ a_1 \leq a_2 \leq a_1 + 1,\ 0 \leq a_3 \leq a_2 - 2a_1 + 1\};$$

see Figure 4 in Example 1.2.6.

EXAMPLE 2.5.3. Let $G = Sp_4(\mathbb{C})$ (of type C_2), and identify I with $\{1, 2\}$ such that $\langle \alpha_2, h_1 \rangle = -2$ and $\langle \alpha_1, h_2 \rangle = -1$. We consider a reduced word $\mathbf{i} = (2, 1, 2, 1) \in I^4$ for the longest element $w_0 \in W$. Then, we see by [**53**, Theorem 3.1] that the image $\Psi_{\mathbf{i}}(\mathcal{B}(\infty))$ is identical to the following semigroup:

$$\{(a_1, \ldots, a_4) \in \mathbb{Z}^4_{\geq 0} \mid 2a_1 \leq a_2 \leq 2a_3\}.$$

Now, by the definition of the crystal $\mathbb{Z}^{\infty}_{\mathbf{j}}$, we see that $\eta_{\mathbf{i}}(\mathbf{a})$ is given by

$$(a_1, \min\{2a_1 + a_4, 2a_1 - a_2 + 2a_3\}, a_3, \max\{-2a_1 + 2a_2 - 2a_3 + a_4, a_2 - 2a_1\}).$$

This piecewise-linear map induces a bijective map from the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ onto the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\lambda)^{\mathrm{op}}$. Indeed, by

Corollary 2.4.3, we deduce that the polytope $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$ is identical to the set of $(a_1, \ldots, a_4) \in \mathbb{R}^4$ satisfying the following inequalities:

$$0 \leq a_4 \leq \lambda_1,\ 0 \leq a_3 \leq a_4 + \lambda_2,$$
$$0 \leq a_2 \leq \min\{a_3 + \lambda_2, 2a_3\},\ 0 \leq 2a_1 \leq \min\{2\lambda_2, a_2\},$$

where $\lambda_i := \langle \lambda, h_i \rangle$ for $i = 1, 2$. In particular, if we define $\rho \in P_+$ by $\langle \rho, h_1 \rangle = \langle \rho, h_2 \rangle = 1$, then the Nakashima-Zelevinsky polytope $\widetilde{\Delta}_{\mathbf{i}}(\rho)$ is given by the following inequalities:

$$0 \leq a_4 \leq 1,\ 0 \leq a_3 \leq a_4 + 1,$$
$$0 \leq a_2 \leq \min\{a_3 + 1, 2a_3\},\ 0 \leq 2a_1 \leq \min\{2, a_2\}.$$

Also, we deduce by [**42**, Sect. 1] that the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\lambda)^{\mathrm{op}}$ is identical to the set of $(a_1, \ldots, a_4) \in \mathbb{R}^4$ satisfying the following inequalities:

$$0 \leq a_1 \leq \lambda_2,\ 2a_1 \leq a_2 \leq 2a_1 + \lambda_1,$$
$$a_2 \leq 2a_3 \leq 2a_2 - 4a_1 + 2\lambda_2,\ 0 \leq a_4 \leq 2a_1 - 2a_2 + 2a_3 + \lambda_1.$$

In particular, if $\lambda = \rho$, then the string polytope $\Delta_{\mathbf{i}^{\mathrm{op}}}(\rho)^{\mathrm{op}}$ is given by the following inequalities:

$$0 \leq a_1 \leq 1,\ 2a_1 \leq a_2 \leq 2a_1 + 1,$$
$$a_2 \leq 2a_3 \leq 2a_2 - 4a_1 + 2,\ 0 \leq a_4 \leq 2a_1 - 2a_2 + 2a_3 + 1.$$

CHAPTER 3

Geometrically natural valuations and perfect bases with positivity properties

In this chapter, we relate string polytopes and polyhedral realizations with geometrically natural valuations. This chapter is based on joint work with Hironori Oya [**15**].

3.1. Geometrically natural valuations

Here, we recall the definition of valuations coming from some sequences of subvarieties of a projective variety. Let X be an irreducible normal projective variety over $\mathbb{C}$ of complex dimension r, and consider the following sequence of irreducible normal closed subvarieties:

$$X_\bullet \colon X_r \subset X_{r-1} \subset \cdots \subset X_0 = X$$

such that $\dim_{\mathbb{C}}(X_k) = r - k$ for $0 \leq k \leq r$. Denote by η_k the generic point of X_k for $1 \leq k \leq r$. Since X_k is normal for all $0 \leq k \leq r-1$, the stalk $\mathcal{O}_{\eta_{k+1},X_k}$ of the structure sheaf $\mathcal{O}_{X_k}$ at η_{k+1} is a discrete valuation ring with quotient field $\mathbb{C}(X_k)$. Let $\mathrm{ord}_{X_{k+1}} \colon \mathbb{C}(X_k) \setminus \{0\} \to \mathbb{Z}$ denote the corresponding valuation, and take a generator $u_{k+1} \in \mathbb{C}(X_k)$ of the unique maximal ideal of $\mathcal{O}_{\eta_{k+1},X_k}$.

Definition 3.1.1. Out of $X_\bullet$, we define a valuation $v_{X_\bullet} \colon \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^r$, $f \mapsto (a_1, \ldots, a_r)$, as follows. The first coordinate a_1 is given by $a_1 \coloneqq \mathrm{ord}_{X_1}(f)$. Then, we have $(u_1^{-a_1} f)|_{X_1} \in \mathbb{C}(X_1) \setminus \{0\}$, and the second coordinate a_2 is given by $a_2 \coloneqq \mathrm{ord}_{X_2}((u_1^{-a_1} f)|_{X_1})$. Continuing in this way, we define all a_k. This is the definition of $v_{X_\bullet}$.

Remark 3.1.2. The valuation $v_{X_\bullet}$ depends on the choice of $u_1, \ldots, u_r$, but the corresponding Newton-Okounkov body is independent up to unimodular equivalence.

Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$. If we set $w_{\geq k} \coloneqq s_{i_k} s_{i_{k+1}} \cdots s_{i_r}$ and $w_{\leq k} \coloneqq s_{i_1} s_{i_2} \cdots s_{i_k}$ for $1 \leq k \leq r$, then we obtain two sequences of subvarieties of $X(w)$:

$$X(w_{\geq \bullet}) \colon X(e) \subset X(w_{\geq r}) \subset \cdots \subset X(w_{\geq 2}) \subset X(w_{\geq 1}) = X(w) \text{ and}$$
$$X(w_{\leq \bullet}) \colon X(e) \subset X(w_{\leq 1}) \subset \cdots \subset X(w_{\leq r-1}) \subset X(w_{\leq r}) = X(w),$$

where $e \in W$ is the identity element. Since Schubert varieties are irreducible normal projective varieties, we obtain two valuations $v_{X(w_{\geq \bullet})}$ and $v_{X(w_{\leq \bullet})}$ out of these sequences.

Proposition 3.1.3. *The following equalities hold*:

$$v_{\mathbf{i}}^{\mathrm{low}} = v_{X(w_{\geq \bullet})} \text{ and } \tilde{v}_{\mathbf{i}}^{\mathrm{low}} = v_{X(w_{\leq \bullet})}.$$

Proof. We prove only the assertion $v_{\mathbf{i}}^{\mathrm{low}} = v_{X(w_{\geq \bullet})}$; a proof of the other assertion is similar. Recall that we identify the function field $\mathbb{C}(X(w))$ with the rational function field $\mathbb{C}(t_1, \ldots, t_r)$ by the following birational morphism:

$$\mathbb{C}^r \to X(w),\ (t_1, \ldots, t_r) \mapsto \exp(t_1 f_{i_1}) \cdots \exp(t_r f_{i_r}) \bmod B.$$

If we set

$$(\mathbb{C}^r)_{\geq k} \coloneqq \{(0, \ldots, 0, t_k, \ldots, t_r) \mid t_k, \ldots, t_r \in \mathbb{C}\} \subset \mathbb{C}^r$$

for $1 \leq k \leq r$ and $(\mathbb{C}^r)_{\geq r+1} \coloneqq \{(0, \ldots, 0)\} \subset \mathbb{C}^r$, then the birational morphism above induces a birational morphism from $(\mathbb{C}^r)_{\geq k}$ to $X(w_{\geq k})$ for $1 \leq k \leq r+1$, where $w_{\geq r+1} \coloneqq e$. Now the assertion is an immediate consequence of the fact that t_k gives a generator of the maximal ideal of the stalk of the structure sheaf $\mathcal{O}_{(\mathbb{C}^r)_{\geq k}}$ at the generic point of $(\mathbb{C}^r)_{\geq k+1}$ for $1 \leq k \leq r$. □

3.2. Perfect bases with positivity properties

Let us consider a perfect basis $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ that satisfies the following positivity conditions:

$(\mathrm{P})_1$ the element $(-f_i) \cdot \Xi^{\mathrm{up}}(b)$ belongs to $\sum_{b' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}(b')$ for all $i \in I$ and $b \in \mathcal{B}(\infty)$;

$(\mathrm{P})_2$ the product $\Xi^{\mathrm{up}}(\tilde{f}_i b_\infty) \cdot \Xi^{\mathrm{up}}(b)$ belongs to $\sum_{b' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}(b')$ for all $i \in I$ and $b \in \mathcal{B}(\infty)$.

Proposition 3.2.1. *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis. Then, positivity conditions* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$ *are equivalent to the following positivity conditions* $(\mathrm{P})_1'$ *and* $(\mathrm{P})_2'$, *respectively*:

$(\mathrm{P})_1'$ *the elements* $(-1)^k f_i^k \cdot \Xi^{\mathrm{up}}(b)$ *and* $(-1)^k \Xi^{\mathrm{up}}(b) \cdot f_i^k$ *both belong to* $\sum_{b' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}(b')$ *for all* $i \in I$, $b \in \mathcal{B}(\infty)$, *and* $k \in \mathbb{Z}_{\geq 0}$;

$(\mathrm{P})_2'$ *the product* $\Xi^{\mathrm{up}}(\tilde{f}_i^k b_\infty) \cdot \Xi^{\mathrm{up}}(b)$ *belongs to* $\sum_{b' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}(b')$ *for all* $i \in I$, $b \in \mathcal{B}(\infty)$, *and* $k \in \mathbb{Z}_{\geq 0}$.

Proof. It follows immediately that condition $(\mathrm{P})_1$ is equivalent to $(\mathrm{P})_1'$; hence it suffices to prove that condition $(\mathrm{P})_2$ implies $(\mathrm{P})_2'$. Since $U(\mathfrak{u}^-)^*_{\mathrm{gr}, k\mathbf{e}_i} = \mathbb{C}\Xi^{\mathrm{up}}(\tilde{f}_i^k b_\infty)$ for $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$ (see Example 2.2.8), we have $\Xi^{\mathrm{up}}(\tilde{f}_i b_\infty)^k \in \mathbb{C}^\times \Xi^{\mathrm{up}}(\tilde{f}_i^k b_\infty)$. Then, positivity condition $(\mathrm{P})_2$ implies that $\Xi^{\mathrm{up}}(\tilde{f}_i b_\infty)^k \in \mathbb{R}_{>0} \Xi^{\mathrm{up}}(\tilde{f}_i^k b_\infty)$; hence we deduce positivity condition $(\mathrm{P})_2'$ by $(\mathrm{P})_2$ again. □

Example 3.2.2. In the case that $\mathfrak{g}$ is of simply-laced type, Lusztig proved that the upper global basis satisfies positivity conditions $(\mathrm{P})_1$ and $(\mathrm{P})_2$ by using the geometric construction of the lower global basis [**44**, Theorem 11.5].

A desired example for general $\mathfrak{g}$ is given by the specialization of the KLR-basis at $q = 1$ (see Example 2.2.6), that is, the following holds.

PROPOSITION 3.2.3 ([**35, 36**]). The specialization of the KLR-basis at $q = 1$ satisfies positivity conditions $(\mathrm{P})_1$ and $(\mathrm{P})_2$.

PROOF. Although this proposition is an immediate consequence of [**35, 36**], we explain a proof for the convenience of the reader. As mentioned in Example 2.2.6, the KLR-basis $\{[S(b)] \mid b \in \mathcal{B}(\infty)\}$ comes from the set $\{S(b) \mid b \in \mathcal{B}(\infty)\}$ consisting of self-dual graded simple modules. The maps

$$(-f_i)\cdot\colon U(\mathfrak{u}^-)^*_{\mathrm{gr},\mathbf{d}} \to U(\mathfrak{u}^-)^*_{\mathrm{gr},\mathbf{d}-\mathbf{e}_i} \text{ and}$$
$$[S(\tilde{f}_i b_\infty)]_{q=1}\cdot\colon U(\mathfrak{u}^-)^*_{\mathrm{gr},\mathbf{d}} \to U(\mathfrak{u}^-)^*_{\mathrm{gr},\mathbf{d}+\mathbf{e}_i}$$

are the specializations at $q = 1$ of the maps induced from a certain restriction functor $\mathrm{Res}\colon R_{\mathbf{d}}\text{-gmod} \to R_{\mathbf{d}-\mathbf{e}_i}\text{-gmod}$ and a certain induction functor $\mathrm{Ind}\colon R_{\mathbf{d}}\text{-gmod} \to R_{\mathbf{d}+\mathbf{e}_i}\text{-gmod}$, respectively (see [**25**, Sect. 5.1] and [**35**, Sects. 2.6, 3.1]). In the Grothendieck groups $G_0(R_{\mathbf{d}\mp\mathbf{e}_i}\text{-gmod})$, we have

$$[\mathrm{Res}(S(b))] = \sum_{b' \in \mathcal{B}(\infty), m \in \mathbb{Z}} c_{i,b}^{(b',m)}[S(b')[m]], \text{ and}$$
$$[\mathrm{Ind}(S(b))] = \sum_{b' \in \mathcal{B}(\infty), m \in \mathbb{Z}} d_{i,b}^{(b',m)}[S(b')[m]]$$

for $i \in I$ and $b \in \mathcal{B}(\infty)$; here, $S(b')[m]$ denotes the grade shift of $S(b')$ by $m \in \mathbb{Z}$, and $c_{i,b}^{(b',m)}$, $d_{i,b}^{(b',m)}$ are the multiplicities of the corresponding graded simple module in composition series of $\mathrm{Res}(S(b))$ and $\mathrm{Ind}(S(b))$, respectively. In particular, the coefficients $c_{i,b}^{(b',m)}$ and $d_{i,b}^{(b',m)}$ are nonnegative integers. Since the specialization at $q = 1$ corresponds to the neglect of grade shifts, we have

$$(-f_i)\cdot[S(b)]_{q=1} = \sum_{b' \in \mathcal{B}(\infty)} \Big(\sum_{m \in \mathbb{Z}} c_{i,b}^{(b',m)}\Big)[S(b')]_{q=1}, \text{ and}$$
$$[S(\tilde{f}_i b_\infty)]_{q=1}\cdot[S(b)]_{q=1} = \sum_{b' \in \mathcal{B}(\infty)} \Big(\sum_{m \in \mathbb{Z}} d_{i,b}^{(b',m)}\Big)[S(b')]_{q=1}$$

in $U(\mathfrak{u}^-)^*_{\mathrm{gr}} = \mathbb{C}[U^-]$. Hence the coefficients $\sum_{m \in \mathbb{Z}} c_{i,b}^{(b',m)}$ and $\sum_{m \in \mathbb{Z}} d_{i,b}^{(b',m)}$ are nonnegative. □

In the following, we prove condition (D) in Sect. 2.2 for a perfect basis $\mathbf{B}^{\mathrm{up}}$ satisfying positivity condition $(\mathrm{P})_1$. By the definition of the $U(\mathfrak{u}^-)$-bimodule structure on $U(\mathfrak{u}^-)^*_{\mathrm{gr}}$, we see that

$$(-1)^k\langle f_i^k \cdot \Xi^{\mathrm{up}}(b), \Xi^{\mathrm{low}}(b')\rangle = \langle \Xi^{\mathrm{up}}(b), f_i^k \cdot \Xi^{\mathrm{low}}(b')\rangle, \text{ and}$$
$$(-1)^k\langle \Xi^{\mathrm{up}}(b) \cdot f_i^k, \Xi^{\mathrm{low}}(b')\rangle = \langle \Xi^{\mathrm{up}}(b), \Xi^{\mathrm{low}}(b') \cdot f_i^k\rangle$$

for all $i \in I$, $b, b' \in \mathcal{B}(\infty)$, and $k \in \mathbb{Z}_{\geq 0}$; hence the following holds.

LEMMA 3.2.4. *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis satisfying positivity condition* $(\mathrm{P})_1$*, and* $\mathbf{B}^{\mathrm{low}} = \{\Xi^{\mathrm{low}}(b) \mid b \in \mathcal{B}(\infty)\} \subset U(\mathfrak{u}^-)$ *its dual basis. Then, the elements* $f_i^k \cdot \Xi^{\mathrm{low}}(b)$ *and* $\Xi^{\mathrm{low}}(b) \cdot f_i^k$ *both belong to* $\sum_{b' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0}\Xi^{\mathrm{low}}(b')$ *for all* $i \in I$*,* $b \in \mathcal{B}(\infty)$*, and* $k \in \mathbb{Z}_{\geq 0}$*.*

PROPOSITION 3.2.5. *A perfect basis* $\mathbf{B}^{\mathrm{up}}$ *with positivity property* $(\mathrm{P})_1$ *satisfies condition* (D) *in Sect.* 2.2.

PROOF. Our proof is similar to the one in [**29**, Sects. 3.1, 3.2]. Set $\mathfrak{u}_i := \mathbb{C}e_i \subset \mathfrak{g}_i$ for $i \in I$. Since we have $V_w(\lambda) = U(\mathfrak{g}_i)V_{s_iw}(\lambda)$ for $i \in I$ and $w \in W$ such that the length of s_iw is smaller than that of w (see [**29**, Lemma 3.2.1]), it suffices to prove that for a $U(\mathfrak{u}_i)$-submodule N of $V(\lambda)$, which is spanned by $\{\Xi_\lambda^{\mathrm{low}}(b) \mid b \in \mathcal{B}_N\}$ for some subset $\mathcal{B}_N \subset \mathcal{B}(\lambda)$, the following equality holds:

$$U(\mathfrak{g}_i)N = \sum_{b \in \widetilde{\mathcal{B}}_N^{(i)}} \mathbb{C}\Xi_\lambda^{\mathrm{low}}(b),$$

where $\widetilde{\mathcal{B}}_N^{(i)} := \{\tilde{f}_i^k b \mid b \in \mathcal{B}_N,\ k \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}$. Since N is a $U(\mathfrak{u}_i)$-submodule of $V(\lambda)$, it follows by the second assertion of Proposition 2.2.17 (2) that $\tilde{e}_i\mathcal{B}_N \subset \mathcal{B}_N \cup \{0\}$. Hence we deduce that

$$\widetilde{\mathcal{B}}_N^{(i)} = \{\tilde{f}_i^k b \mid b \in \mathcal{B}_N \text{ with } \varepsilon_i(b) = 0, \text{ and } k \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}.$$

For $i \in I$, $\ell \in \mathbb{Z}_{\geq 0}$, and a $\mathbb{C}$-subspace $M \subset V(\lambda)$ (resp., a subset $\mathcal{S} \subset \mathcal{B}(\lambda)$), we set

$$\begin{aligned} W_i^\ell(M) &:= W_i^\ell(V(\lambda)) \cap M, \\ I_i^\ell(M) &:= I_i^\ell(V(\lambda)) \cap M \\ (\text{resp.}, W_i^\ell(\mathcal{S}) &:= W_i^\ell(\mathcal{B}(\lambda)) \cap \mathcal{S}); \end{aligned}$$

the $U(\mathfrak{g}_i)$-submodules $W_i^\ell(V(\lambda)), I_i^\ell(V(\lambda))$ and the subset $W_i^\ell(\mathcal{B}(\lambda))$ are defined above Proposition 2.2.17. By the first assertion of Proposition 2.2.17 (2), we have

$$W_i^\ell(N) = \sum_{b \in W_i^\ell(\mathcal{B}_N)} \mathbb{C}\Xi_\lambda^{\mathrm{low}}(b).$$

Let $\ell_0 \in \mathbb{Z}_{\geq 0}$ be the maximum integer $\ell \in \mathbb{Z}_{\geq 0}$ such that $W_i^\ell(U(\mathfrak{g}_i)N) \neq 0$, which implies that $W_i^{\ell_0}(U(\mathfrak{g}_i)N) = I_i^{\ell_0}(U(\mathfrak{g}_i)N)$. Since the following equalities hold:

$$\begin{aligned} W_i^{\ell_0}(U(\mathfrak{g}_i)N) \cap \operatorname{Ker} e_i &= W_i^{\ell_0}(N) \cap \operatorname{Ker} e_i \\ &= \sum_{b \in W_i^{\ell_0}(\mathcal{B}_N);\ \langle \mathrm{wt}(b), h_i\rangle = \ell_0} \mathbb{C}\Xi_\lambda^{\mathrm{low}}(b), \end{aligned}$$

the $U(\mathfrak{g}_i)$-submodule $W_i^{\ell_0}(U(\mathfrak{g}_i)N)$ of $V(\lambda)$ is spanned by

$$\{f_i^k \cdot \Xi_\lambda^{\mathrm{low}}(b) \mid b \in W_i^{\ell_0}(\mathcal{B}_N) \text{ with } \langle \mathrm{wt}(b), h_i\rangle = \ell_0, \text{ and } k \in \mathbb{Z}_{\geq 0}\};$$

note that for $b \in W_i^{\ell_0}(\mathcal{B}_N)$, we have $\varepsilon_i(b) = 0$ if and only if $\langle \mathrm{wt}(b), h_i\rangle = \ell_0$. For $v = \sum_{b \in \mathcal{B}(\lambda)} c_b \Xi_\lambda^{\mathrm{low}}(b) \in V(\lambda) \setminus \{0\}$ with $c_b \in \mathbb{R}_{\geq 0}$, it follows by (2.2.1) and Lemma 3.2.4 that

$$f_i^{\max\{\varphi_i(b) \mid c_b \neq 0\}} \cdot v \neq 0.$$

Hence we deduce by (2.2.1) and Lemma 3.2.4 again that $f_i^k \cdot \Xi_\lambda^{\mathrm{low}}(b) \in \mathbb{R}_{>0}\Xi_\lambda^{\mathrm{low}}(\tilde{f}_i^k b)$ for all $b \in W_i^{\ell_0}(\mathcal{B}_N)$ with $\langle \mathrm{wt}(b), h_i\rangle = \ell_0$ and $0 \leq k \leq \ell_0$.

From these, it follows that $W_i^{\ell_0}(U(\mathfrak{g}_i)N) = \sum_{b \in W_i^{\ell_0}(\widetilde{\mathcal{B}}_N^{(i)})} \mathbb{C}\Xi_\lambda^{\mathrm{low}}(b)$. By descending induction on ℓ and by replacing $U(\mathfrak{g}_i)N$ with $U(\mathfrak{g}_i)N/W_i^{\ell+1}(U(\mathfrak{g}_i)N)$ in the argument above, we obtain the assertion. □

REMARK 3.2.6. In Proposition 3.2.5, we need not assume positivity condition $(\mathrm{P})_2$.

For $w \in W$, we denote the length of w by $\ell(w) \in \mathbb{Z}_{\geq 0}$. Take $i \in I$ (resp., $i' \in I$) and $w \in W$ such that $\ell(s_i w) < \ell(w)$ (resp., $\ell(w s_{i'}) < \ell(w)$). Then, the left action of $\mathfrak{u}_i^-$ (resp., the right action of $\mathfrak{u}_{i'}^-$) on $\mathbb{C}[U^-]$ induces a left action of $\mathfrak{u}_i^-$ (resp., a right action of $\mathfrak{u}_{i'}^-$) on $\mathbb{C}[U^- \cap X(w)]$ by the restriction map $\eta_w \colon \mathbb{C}[U^-] \twoheadrightarrow \mathbb{C}[U^- \cap X(w)]$. The following is an immediate consequence of Corollary 2.2.24 (2) (see also Proposition 3.2.5) and Proposition 3.2.1.

COROLLARY 3.2.7. *Let* $w \in W$, *and* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *a perfect basis satisfying positivity conditions* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$.

(1) *The elements* $(-1)^k f_i^k \cdot \Xi_w^{\mathrm{up}}(b)$ *and* $(-1)^k \Xi_w^{\mathrm{up}}(b) \cdot f_{i'}^k$ *both belong to* $\sum_{b' \in \mathcal{B}_w(\infty)} \mathbb{R}_{\geq 0} \Xi_w^{\mathrm{up}}(b')$ *for all* $i, i' \in I$, $b \in \mathcal{B}_w(\infty)$, *and* $k \in \mathbb{Z}_{\geq 0}$ *such that* $\ell(s_i w) < \ell(w)$, $\ell(w s_{i'}) < \ell(w)$.
(2) *The product* $\Xi_w^{\mathrm{up}}(\tilde{f}_i^k b_\infty) \cdot \Xi_w^{\mathrm{up}}(b)$ *belongs to* $\sum_{b' \in \mathcal{B}_w(\infty)} \mathbb{R}_{\geq 0} \Xi_w^{\mathrm{up}}(b')$ *for all* $i \in I$, $b \in \mathcal{B}_w(\infty)$, *and* $k \in \mathbb{Z}_{\geq 0}$.

Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a reduced word for $w \in W$, and regard the coordinate ring $\mathbb{C}[U^- \cap X(w)]$ as a $\mathbb{C}$-subalgebra of $\mathbb{C}[U_{i_1}^- \times \cdots \times U_{i_r}^-] = \mathbb{C}[t_1, \ldots, t_r]$.

PROPOSITION 3.2.8. *The coefficient of* $t_1^{a_1} \cdots t_r^{a_r}$ *in* $\Xi_w^{\mathrm{up}}(b)$ *is a nonnegative real number for all* $b \in \mathcal{B}_w(\infty)$ *and* $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}$.

PROOF. For $b \in \mathcal{B}_w(\infty)$ and $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}$, denote by $A_b^{(a_1,\ldots,a_r)} \in \mathbb{C}$ the coefficient of $t_1^{a_1} \cdots t_r^{a_r}$ in $\Xi_w^{\mathrm{up}}(b)$. Then, we know from equation (1.2.3) that $A_b^{(a_1,\ldots,a_r)}$ is equal to the value

$$\frac{(-1)^{a_1+\cdots+a_r}}{a_1! \cdots a_r!} (f_{i_r}^{a_r} \cdot (\cdots (f_{i_2}^{a_2} \cdot (f_{i_1}^{a_1} \cdot \Xi_w^{\mathrm{up}}(b))|_{t_1=0})|_{t_2=0} \cdots)|_{t_{r-1}=0})|_{t_r=0}.$$

If we write $w_{\geq k} \coloneqq s_{i_k} s_{i_{k+1}} \cdots s_{i_r}$ for $1 \leq k \leq r$ and $w_{\geq r+1} \coloneqq e$, the identity element of W, then the restriction map $\eta_{k,k+1} \colon \mathbb{C}[U^- \cap X(w_{\geq k})] \twoheadrightarrow \mathbb{C}[U^- \cap X(w_{\geq k+1})]$ is given by $t_k \mapsto 0$; hence we see that $A_b^{(a_1,\ldots,a_r)}$ equals the value

$$\frac{(-1)^{a_1+\cdots+a_r}}{a_1! \cdots a_r!} \eta_{r,r+1}(f_{i_r}^{a_r} \cdot (\eta_{r-1,r}(\cdots (\eta_{2,3}(f_{i_2}^{a_2} \cdot (\eta_{1,2}(f_{i_1}^{a_1} \cdot \Xi_w^{\mathrm{up}}(b))))) \cdots))),$$

where the coordinate ring $\mathbb{C}[U^- \cap X(w_{\geq r+1})] = \mathbb{C}[U^- \cap X(e)]$ is identified with $\mathbb{C}$ by $\Xi_e^{\mathrm{up}}(b_\infty) \mapsto 1$ (recall condition (ii) in Definition 2.2.3). Now by using Corollaries 2.2.24 (2) and 3.2.7 (1) repeatedly, we conclude that $A_b^{(a_1,\ldots,a_r)}$ is a nonnegative real number. This proves the proposition. □

3.3. Second main result

We write $\mathbf{a}^{\mathrm{op}} := (a_r, \ldots, a_1)$ for an element $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$, and $H^{\mathrm{op}} := \{\mathbf{a}^{\mathrm{op}} \mid \mathbf{a} \in H\}$ for a subset $H \subset \mathbb{R}^r$. The following is the second main result of this thesis.

THEOREM 3.3.1. *Let* $\mathbf{i} \in I^r$ *be a reduced word for* $w \in W$, *and* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *a perfect basis satisfying positivity conditions* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$. *Then, the following equalities hold*:

$$v_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b)) = -\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b))^{\mathrm{op}}, \ \textit{and}$$
$$\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b)) = -v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b))^{\mathrm{op}}$$

for all $b \in \mathcal{B}_w(\infty)$.

Before proving Theorem 3.3.1, we give some corollaries. The following corollary is an immediate consequence of the first assertion of Corollary 2.2.20 and Theorem 3.3.1.

COROLLARY 3.3.2. *Let* $\mathbf{i} \in I^r$ *be a reduced word for* $w \in W$, $\lambda \in P_+$, *and* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *a perfect basis satisfying positivity conditions* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$. *Then, the following equalities hold*:

$$v_{\mathbf{i}}^{\mathrm{low}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda) = -\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda)^{\mathrm{op}}, \ \textit{and}$$
$$\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda) = -v_{\mathbf{i}}^{\mathrm{high}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda)^{\mathrm{op}}$$

for all $b \in \mathcal{B}_w(\lambda)$.

Define an $\mathbb{R}$-linear automorphism $\tilde{\eta}\colon \mathbb{R} \times \mathbb{R}^r \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}^r$ by $\tilde{\eta}(k, \mathbf{a}) := (k, -\mathbf{a}^{\mathrm{op}})$.

COROLLARY 3.3.3. *Let* $\mathbf{i} \in I^r$ *be a reduced word for* $w \in W$, *and* $\lambda \in P_+$. *Then, the following equalities hold*:

$$S(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda) = \tilde{\eta}(S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)),$$
$$C(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda) = \tilde{\eta}(C(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)),$$
$$\Delta(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda) = -\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{high}}, \tau_\lambda)^{\mathrm{op}};$$

in addition, similar equalities hold for the other pair $(\tilde{v}_{\mathbf{i}}^{\mathrm{low}}, v_{\mathbf{i}}^{\mathrm{high}})$ *of valuations.*

PROOF. We prove only the assertion for the pair $(v_{\mathbf{i}}^{\mathrm{low}}, \tilde{v}_{\mathbf{i}}^{\mathrm{high}})$ of valuations; a proof of the assertion for the other pair $(\tilde{v}_{\mathbf{i}}^{\mathrm{low}}, v_{\mathbf{i}}^{\mathrm{high}})$ is similar. Let $\{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$ be a perfect basis satisfying positivity conditions $(\mathrm{P})_1$ and $(\mathrm{P})_2$; the existence of such a perfect basis is guaranteed by Proposition 3.2.3. We see by Theorem 2.3.2 (2) and Corollary 3.3.2 that

$$v_{\mathbf{i}}^{\mathrm{low}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda) = -\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\Xi_{\lambda,w}^{\mathrm{up}}(b)/\tau_\lambda)^{\mathrm{op}} = \Psi_{\mathbf{i}}(b)$$

for all $b \in \mathcal{B}_w(\lambda)$. Note that $\{\Xi_{\lambda,w}^{\mathrm{up}}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ is a $\mathbb{C}$-basis of $H^0(X(w), \mathcal{L}_\lambda)$, and that $\Psi_{\mathbf{i}}(b)$, $b \in \mathcal{B}_w(\lambda)$, are all distinct. Hence we deduce by Proposition

1.1.2 (2) that

$$\begin{aligned}&\{v_{\mathbf{i}}^{\mathrm{low}}(\sigma/\tau_\lambda) \mid \sigma \in H^0(X(w), \mathcal{L}_\lambda) \setminus \{0\}\}\\ &= \{-\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\sigma/\tau_\lambda)^{\mathrm{op}} \mid \sigma \in H^0(X(w), \mathcal{L}_\lambda) \setminus \{0\}\}.\end{aligned}$$

This implies the assertion by Definition 1.1.5 (see also the proof of Theorem 2.3.2 (3)). □

The following corollaries (Corollaries 3.3.4 and 3.3.5) are immediate consequences of Proposition 1.4.4 (1), (3), Theorems 1.4.6, 2.2.9, 2.3.2 (1), 3.3.1, Corollaries 2.3.4, 3.3.3, and Gordan's lemma (see, for instance, [**7**, Proposition 1.2.17]).

Corollary 3.3.4. *Let* $\mathbf{i} \in I^r$ *be a reduced word for* $w \in W$, *and* $\lambda \in P_+$.

1. *The semigroups* $S(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$ *and* $S(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$ *are both finitely generated.*
2. *The real closed cones* $C(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$ *and* $C(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$ *are both rational convex polyhedral cones.*
3. *The sets* $\Delta(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$ *and* $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda)$ *are both rational convex polytopes.*

Corollary 3.3.5. *Let* $\mathbf{i} \in I^r$ *be a reduced word for* $w \in W$, $\lambda \in P_+$, *and* $\{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *a perfect basis satisfying positivity conditions* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$. *Then, the following equalities hold*:

$$\Delta(X(w), \mathcal{L}_\lambda, v_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda) \cap \mathbb{Z}^r = \Psi_{\mathbf{i}}(\mathcal{B}_w(\lambda)) = \{v_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b)) \mid b \in \widetilde{\mathcal{B}}_w(\lambda)\}, \text{ and}$$
$$\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_{\mathbf{i}}^{\mathrm{low}}, \tau_\lambda) \cap \mathbb{Z}^r = \Phi_{\mathbf{i}}(\mathcal{B}_w(\lambda))^{\mathrm{op}} = \{\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b)) \mid b \in \widetilde{\mathcal{B}}_w(\lambda)\}.$$

Proof of Theorem 3.3.1. We prove only the assertion for the valuations $\tilde{v}_{\mathbf{i}}^{\mathrm{low}}$ and $v_{\mathbf{i}}^{\mathrm{high}}$; a proof of the other assertion is similar. Define a total order $\lhd$ on $\mathcal{B}_w(\infty)$ by: $b_1 \lhd b_2$ if and only if

$$\begin{cases}|b_1| < |b_2|, \text{ or}\\ |b_1| = |b_2| \text{ and } \Phi_{\mathbf{i}}(b_1) < \Phi_{\mathbf{i}}(b_2) \text{ with respect to the lexicographic order } <;\end{cases}$$

here, $|b| \coloneqq \sum_{i \in I} d_i$ for $b \in \mathcal{B}_w(\infty)$ with $\mathrm{wt}(b) = -\sum_{i \in I} d_i \alpha_i$. We proceed by induction on $r = \ell(w)$ and $b \in \mathcal{B}_w(\infty)$ with respect to the total order $\lhd$. Write $\mathbf{i} = (i_1, \ldots, i_r)$ and $\Phi_{\mathbf{i}}(b) = (a_1, \ldots, a_r)$ for $b \in \mathcal{B}_w(\infty)$.

We first consider the case $b \in \mathcal{B}_{s_{i_1}}(\infty)$, which includes the case $r = 1$. In this case, there exists $a \in \mathbb{Z}_{\geq 0}$ such that $b = \tilde{f}_{i_1}^a b_\infty$. Then, we deduce from the definition of $\Phi_{\mathbf{i}}$ that

$$\begin{aligned}-v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b)) &= \Phi_{\mathbf{i}}(b) \quad \text{(by Theorem 2.2.9)}\\ &= (a, 0, \ldots, 0).\end{aligned}$$

Hence it follows from the definition of $v_{\mathbf{i}}^{\mathrm{high}}$ that $\Xi_w^{\mathrm{up}}(b) = ct_1^a + (\text{other terms})$ for some $c \in \mathbb{C}^\times$, where "other terms" means a linear combination of monomials that are not equal to t_1^a. Since $\Xi^{\mathrm{up}}(\tilde{f}_{i_1}^a b_\infty) \in U(\mathfrak{u}^-)^*_{\mathrm{gr}, a\mathbf{e}_{i_1}}$, it follows that all monomials in "other terms" are of degree a, and hence that they contain t_k for some $2 \leq k \leq r$ as variables. By the definition of $\tilde{v}_{\mathbf{i}}^{\mathrm{low}}$, this implies that

$$\begin{aligned}\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b)) &= (0,\ldots,0,a)\\ &= (a,0,\ldots,0)^{\mathrm{op}}\\ &= -v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b))^{\mathrm{op}}.\end{aligned}$$

We next consider the case $r \geq 2$ and $a_1 = 0$. In this case, b is an element of $\mathcal{B}_{w_{\geq 2}}(\infty)$, where $w_{\geq 2} := s_{i_2} \cdots s_{i_r}$; furthermore, by the definition of $v_{\mathbf{i}}^{\mathrm{high}}$, the equality $a_1 = 0$ implies that t_1 does not appear in $\Xi_w^{\mathrm{up}}(b) \in \mathbb{C}[t_1,\ldots,t_r]$. Hence we deduce that

$$\begin{aligned}\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b)) &= (\tilde{v}_{\mathbf{i}_{\geq 2}}^{\mathrm{low}}(\Xi_{w_{\geq 2}}^{\mathrm{up}}(b)),0)\\ &= -(v_{\mathbf{i}_{\geq 2}}^{\mathrm{high}}(\Xi_{w_{\geq 2}}^{\mathrm{up}}(b))^{\mathrm{op}},0)\\ &\quad \text{(by induction hypothesis concerning } r)\\ &= -(0, v_{\mathbf{i}_{\geq 2}}^{\mathrm{high}}(\Xi_{w_{\geq 2}}^{\mathrm{up}}(b)))^{\mathrm{op}}\\ &= -v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b))^{\mathrm{op}},\end{aligned}$$

where $\mathbf{i}_{\geq 2} := (i_2,\ldots,i_r)$, a reduced word for $w_{\geq 2}$.

Finally, we consider the case $b \notin \mathcal{B}_{s_{i_1}}(\infty)$ and $a_1 > 0$. Set $b_1 := \tilde{f}_{i_1}^{a_1} b_\infty$ and $b_2 := \tilde{f}_{i_2}^{a_2} \cdots \tilde{f}_{i_r}^{a_r} b_\infty$. Then, we have $\Phi_{\mathbf{i}}(b_1) = (a_1,0,\ldots,0)$ and $\Phi_{\mathbf{i}}(b_2) = (0,a_2,\ldots,a_r)$. Hence it follows that

$$\begin{aligned}v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b)) &= -(a_1,\ldots,a_r)\\ &= -\Phi_{\mathbf{i}}(b_1) - \Phi_{\mathbf{i}}(b_2)\\ &= v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b_1)) + v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b_2))\\ &\quad \text{(by Theorem 2.2.9).}\end{aligned} \tag{3.3.1}$$

Now we deduce from the results for the two special cases above that

$$\begin{aligned}&v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b_1)) + v_{\mathbf{i}}^{\mathrm{high}}(\Xi_w^{\mathrm{up}}(b_2))\\ &= -(\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b_1))^{\mathrm{op}} + \tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b_2))^{\mathrm{op}})\\ &= -\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b_1)\cdot \Xi_w^{\mathrm{up}}(b_2))^{\mathrm{op}}\\ &\text{(by Definition 1.1.1).}\end{aligned} \tag{3.3.2}$$

From these, it suffices to prove the equality $\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b_1)\cdot\Xi_w^{\mathrm{up}}(b_2)) = \tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b))$. We know from Corollary 3.2.7 (2) that

$$\Xi_w^{\mathrm{up}}(b_1)\cdot \Xi_w^{\mathrm{up}}(b_2) = \sum_{b_3 \in \mathcal{B}_w(\infty)} C_{b_1,b_2}^{(b_3)} \cdot \Xi_w^{\mathrm{up}}(b_3) \tag{3.3.3}$$

for some coefficients $C_{b_1,b_2}^{(b_3)} \in \mathbb{R}_{\geq 0}$, $b_3 \in \mathcal{B}_w(\infty)$. Since $C_{b_1,b_2}^{(b_3)}$ is nonnegative for all $b_3 \in \mathcal{B}_w(\infty)$, Proposition 3.2.8 implies that any cancellation of monomials does not occur in the right hand side of (3.3.3). From this, we deduce by the definition of $\tilde{v}_{\mathbf{i}}^{\mathrm{low}}$ that

$$\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b_1)\cdot \Xi_w^{\mathrm{up}}(b_2)) = \min\{\tilde{v}_{\mathbf{i}}^{\mathrm{low}}(\Xi_w^{\mathrm{up}}(b_3)) \mid b_3 \in \mathcal{B}_w(\infty),\ C_{b_1,b_2}^{(b_3)} \neq 0\},$$

where "min" means the minimum with respect to the lexicographic order $<$. Also, it follows by Corollary 2.2.27 that $C^{(b)}_{b_1,b_2} \neq 0$, and that if $C^{(b_3)}_{b_1,b_2} \neq 0$ and $b_3 \neq b$, then $\mathrm{wt}(b_3) = \mathrm{wt}(b)$ and $-v^{\mathrm{high}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b_3)) < -v^{\mathrm{high}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b))$; in particular, it holds that

$$\begin{aligned}\tilde{v}^{\mathrm{low}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b_1)\cdot\Xi^{\mathrm{up}}_w(b_2))^{\mathrm{op}} &= -v^{\mathrm{high}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b)) \quad \text{(by equations (3.3.1) and (3.3.2))}\\ &> -v^{\mathrm{high}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b_3))\\ &= \tilde{v}^{\mathrm{low}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b_3))^{\mathrm{op}}\\ &\quad \text{(by induction hypothesis concerning } b).\end{aligned}$$

From these, we obtain that $\tilde{v}^{\mathrm{low}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b_1)\cdot\Xi^{\mathrm{up}}_w(b_2)) = \tilde{v}^{\mathrm{low}}_{\mathbf{i}}(\Xi^{\mathrm{up}}_w(b))$. This proves the theorem. □

Remark 3.3.6. Since Corollary 3.3.3 follows from Corollary 3.3.2, it is natural to ask why we consider not only $\{\Xi^{\mathrm{up}}_{\lambda,w}(b) \mid b \in \mathcal{B}_w(\lambda)\}$ but also $\{\Xi^{\mathrm{up}}_w(b) \mid b \in \mathcal{B}_w(\infty)\}$. The reason is that in order to prove the assertion of Corollary 3.3.2 for $\{\Xi^{\mathrm{up}}_{\lambda,w}(b) \mid b \in \mathcal{B}_w(\lambda)\} \subset H^0(X(w), \mathcal{L}_\lambda)$, we have to consider an element of $\mathbb{C}[U^- \cap X(w)]$ that does not belong to $\iota_\lambda(H^0(X(w), \mathcal{L}_\lambda))$. In our proof of Theorem 3.3.1, we use the elements $b_1, b_2 \in \mathcal{B}_w(\infty)$ determined from $b \in \mathcal{B}_w(\infty)$ with $b \notin \mathcal{B}_{s_{i_1}}(\infty)$ and $a_1 > 0$. An important point is that, even if $b \in \widetilde{\mathcal{B}}_w(\lambda)$ for some $\lambda \in P_+$, the element b_1 is not necessarily an element of $\widetilde{\mathcal{B}}_w(\lambda)$. Let us see this with a specific example. Take $G, \mathbf{i}, \lambda$ as in Example 1.2.6. Then, the set $\Phi_{\mathbf{i}}(\mathcal{B}(\lambda)) = \Phi_{\mathbf{i}}(\widetilde{\mathcal{B}}(\lambda))$ of string parametrizations is identical to

$$\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,1,1),(2,1,0),(0,2,1),(1,2,1)\}.$$

For $b \in \widetilde{\mathcal{B}}(\lambda)$ such that $\Phi_{\mathbf{i}}(b) = (2,1,0)$, the element $b_1 \in \mathcal{B}(\infty)$ satisfies $\Phi_{\mathbf{i}}(b_1) = (2,0,0)$, which implies that $b_1 \notin \widetilde{\mathcal{B}}(\lambda)$.

CHAPTER 4

Folding procedure for Newton-Okounkov polytopes of Schubert varieties

In this chapter, we apply the folding procedure to Newton-Okounkov polytopes of Schubert varieties, which relates Newton-Okounkov polytopes of Schubert varieties of different Dynkin types. This chapter is based on the paper [**13**].

4.1. Orbit Lie algebras

In this section, we apply the folding procedure to crystal bases. First we recall from [**10, 11**] the definition of orbit Lie algebras. Let G be a connected, simply-connected semisimple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, W the Weyl group, and I an index set for the vertices of the Dynkin diagram. We further assume that G is of simply-laced type. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote by $U^- \subset G$ the unipotent radical of the opposite Borel subgroup B^-, by $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of T, by $\mathfrak{t}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ the dual space of $\mathfrak{t}$, and by $\langle \cdot, \cdot \rangle \colon \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{C}$ the canonical pairing. Let $P \subset \mathfrak{t}^*$ be the weight lattice for $\mathfrak{g}$, $P_+ \subset P$ the set of dominant integral weights, $\{\alpha_i \mid i \in I\} \subset P$ the set of simple roots, $\{h_i \mid i \in I\} \subset \mathfrak{t}$ the set of simple coroots, and $C = (c_{i,j})_{i,j \in I} := (\langle \alpha_j, h_i \rangle)_{i,j \in I}$, the Cartan matrix of $\mathfrak{g}$. We define $U_q(\mathfrak{g}), U_q(\mathfrak{u}^-), \mathcal{B}(\infty), \mathcal{B}(\lambda), b_\infty, b_\lambda$, and $\{\varepsilon_i, \tilde{e}_i, \tilde{f}_i \mid i \in I\}$ as in Sect. 1.3. Let $\omega \colon I \to I$ be a bijective map of order L satisfying $c_{\omega(i),\omega(j)} = c_{i,j}$ for all $i, j \in I$; such a bijective map ω is called a *Dynkin diagram automorphism*. It induces a Lie algebra automorphism $\omega \colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ of order L defined by:

$$\omega(e_i) = e_{\omega(i)},\ \omega(f_i) = f_{\omega(i)},\ \omega(h_i) = h_{\omega(i)}$$

for $i \in I$, where $e_i, f_i, h_i \in \mathfrak{g}$, $i \in I$, denote the Chevalley generators such that $\{e_i, h_i \mid i \in I\} \subset \mathfrak{b} := \mathrm{Lie}(B)$ and $\{f_i \mid i \in I\} \subset \mathfrak{u}^- := \mathrm{Lie}(U^-)$; note that the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is stable under ω. In this thesis, we always impose the following orthogonality condition on ω:

(O) $c_{i,j} = 0$ for all $i \neq j$ in the same ω-orbit.

Let us fix a complete set $\check{I} \subset I$ of representatives for the ω-orbits in I. We set $m_i := \min\{k \in \mathbb{Z}_{>0} \mid \omega^k(i) = i\}$ for $i \in I$, and then set

$$\check{c}_{i,j} := \sum_{0 \leq k < m_j} c_{i,\omega^k(j)}$$

for $i, j \in \check{I}$. Then, the matrix $\check{C} := (\check{c}_{i,j})_{i,j \in \check{I}}$ is a Cartan matrix of finite type (see Table 1).

Definition 4.1.1 ([**10, 11**]). The finite-dimensional semisimple Lie algebra $\breve{\mathfrak{g}}$ with Cartan matrix $\breve{C}$ is called the *orbit Lie algebra* associated with ω.

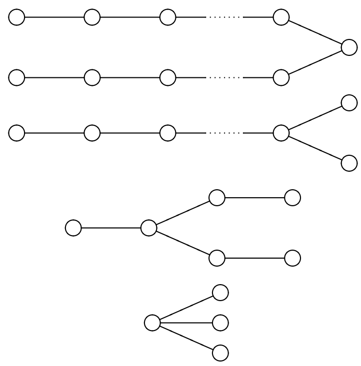
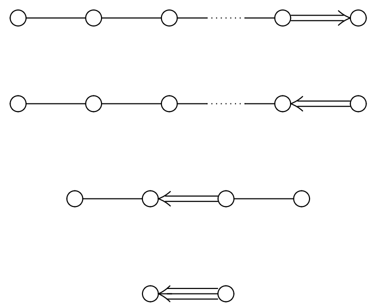
Dynkin diagram of $\mathfrak{g}$	Dynkin diagram of $\breve{\mathfrak{g}}$
	

Table 1. The list of nontrivial automorphisms of connected Dynkin diagrams satisfying condition (O).

Let $U_q(\breve{\mathfrak{g}})$ be the quantized enveloping algebra of $\breve{\mathfrak{g}}$ with generators $\breve{E}_i, \breve{F}_i, \breve{K}_i, \breve{K}_i^{-1}$, $i \in \breve{I}$, and $U_q(\breve{\mathfrak{u}}^-)$ the $\mathbb{Q}(q)$-subalgebra of $U_q(\breve{\mathfrak{g}})$ generated by $\{\breve{F}_i \mid i \in \breve{I}\}$. Denote by $\breve{\mathcal{B}}(\infty)$ the crystal basis of $U_q(\breve{\mathfrak{u}}^-)$, by $\breve{b}_\infty \in \breve{\mathcal{B}}(\infty)$ the element corresponding to $1 \in U_q(\breve{\mathfrak{u}}^-)$, and by $\tilde{e}_i, \tilde{f}_i \colon \breve{\mathcal{B}}(\infty) \cup \{0\} \to \breve{\mathcal{B}}(\infty) \cup \{0\}$, $i \in \breve{I}$, the Kashiwara operators. Then, the crystal $\breve{\mathcal{B}}(\infty)$ is realized as a specific subset of $\mathcal{B}(\infty)$; we recall this realization, following [**48, 49, 58**]. The Dynkin diagram automorphism ω induces a $\mathbb{Q}(q)$-algebra automorphism $\omega\colon U_q(\mathfrak{g}) \xrightarrow{\sim} U_q(\mathfrak{g})$ of order L defined by:

$$\omega(E_i) = E_{\omega(i)},\ \omega(F_i) = F_{\omega(i)},\ \omega(K_i) = K_{\omega(i)}$$

for $i \in I$; note that ω preserves $U_q(\mathfrak{u}^-)$. We see from [**48**, Sect. 3.4] that this automorphism induces a natural bijective map $\omega\colon \mathcal{B}(\infty) \to \mathcal{B}(\infty)$ such that

$$\omega \circ \tilde{e}_i = \tilde{e}_{\omega(i)} \circ \omega \text{ and } \omega \circ \tilde{f}_i = \tilde{f}_{\omega(i)} \circ \omega \tag{4.1.1}$$

for all $i \in I$. Let us define operators $\tilde{e}_i^\omega, \tilde{f}_i^\omega \colon \mathcal{B}(\infty) \cup \{0\} \to \mathcal{B}(\infty) \cup \{0\}$ for $i \in I$ by:

$$\tilde{e}_i^\omega = \prod_{0 \le k < m_i} \tilde{e}_{\omega^k(i)} \text{ and } \tilde{f}_i^\omega = \prod_{0 \le k < m_i} \tilde{f}_{\omega^k(i)}; \tag{4.1.2}$$

note that the operators $\tilde{e}_i, \tilde{e}_{\omega(i)}, \ldots, \tilde{e}_{\omega^{m_i-1}(i)}$ (resp., $\tilde{f}_i, \tilde{f}_{\omega(i)}, \ldots, \tilde{f}_{\omega^{m_i-1}(i)}$) commute with each other by condition (O). These operators $\tilde{e}_i^\omega, \tilde{f}_i^\omega$ are called the ω-*Kashiwara operators.* Let $\breve{\mathfrak{t}} \subset \breve{\mathfrak{g}}$ be a Cartan subalgebra, $\{\breve{\alpha}_i \in \breve{\mathfrak{t}}^* \mid i \in \breve{I}\}$ the set of simple roots, $\{\breve{h}_i \in \breve{\mathfrak{t}} \mid i \in \breve{I}\}$ the set of simple coroots, and then set

$$\mathfrak{t}^0 := \{h \in \mathfrak{t} \mid \omega(h) = h\},$$
$$(\mathfrak{t}^*)^0 := \{\lambda \in \mathfrak{t}^* \mid \omega^*(\lambda) = \lambda\},$$

where we define $\omega^*\colon \mathfrak{t}^* \xrightarrow{\sim} \mathfrak{t}^*$ by: $(\omega^*(\lambda))(h) = \lambda(\omega^{-1}(h))$ for $\lambda \in \mathfrak{t}^*$ and $h \in \mathfrak{t}$. As in [**10**, Sect. 2], we obtain $\mathbb{C}$-linear isomorphisms $P_\omega\colon \mathfrak{t}^0 \xrightarrow{\sim} \breve{\mathfrak{t}}$ and $P_\omega^*\colon \breve{\mathfrak{t}}^* \xrightarrow{\sim} (\mathfrak{t}^0)^* \simeq (\mathfrak{t}^*)^0$ such that

$$P_\omega^{-1}(\breve{h}_i) = \frac{1}{m_i}\sum_{0\le k<m_i} h_{\omega^k(i)},\ P_\omega^*(\breve{\alpha}_i) = \sum_{0\le k<m_i} \alpha_{\omega^k(i)},\ \text{and}$$
$$(P_\omega^*(\breve{\lambda}))(h) = \breve{\lambda}(P_\omega(h))$$

for $i \in \breve{I}$, $\breve{\lambda} \in \breve{\mathfrak{t}}^*$, and $h \in \mathfrak{t}^0$. Denote by $\breve{W}$ the Weyl group of $\breve{\mathfrak{g}}$, and set

$$\widetilde{W} := \{w \in W \mid \omega^* \circ w = w \circ \omega^* \text{ on } \mathfrak{t}^*\}.$$

Then, we see from [**10**, Sect. 3] that there exists a group isomorphism $\Theta\colon \breve{W} \xrightarrow{\sim} \widetilde{W}$ such that $\Theta(\breve{w}) = P_\omega^* \circ \breve{w} \circ (P_\omega^*)^{-1}$ on $(\mathfrak{t}^*)^0$ for all $\breve{w} \in \breve{W}$.

Proposition 4.1.2 ([**48**, Theorem 3.4.1]). *Let*

$$\mathcal{B}^0(\infty) := \{b \in \mathcal{B}(\infty) \mid \omega(b) = b\}$$

denote the fixed point subset by ω.

(1) *The set* $\mathcal{B}^0(\infty) \cup \{0\}$ *is stable under the* ω*-Kashiwara operators* $\tilde{e}_i^\omega, \tilde{f}_i^\omega$ *for all* $i \in I$.
(2) *There exists a unique bijective map* $P_\infty\colon \mathcal{B}^0(\infty)\cup\{0\} \to \breve{\mathcal{B}}(\infty)\cup\{0\}$ *such that*

$$P_\infty(b_\infty) = \breve{b}_\infty,\ P_\infty \circ \tilde{e}_i^\omega = \tilde{e}_i \circ P_\infty,\ \text{and } P_\infty \circ \tilde{f}_i^\omega = \tilde{f}_i \circ P_\infty$$

for all $i \in \breve{I}$.
(3) *The equality*

$$P_\infty(\mathcal{B}^0_{\Theta(w)}(\infty)) = \breve{\mathcal{B}}_w(\infty)$$

holds for every $w \in \breve{W}$, *where* $\mathcal{B}^0_{\Theta(w)}(\infty) := \mathcal{B}^0(\infty) \cap \mathcal{B}_{\Theta(w)}(\infty)$, *and* $\breve{\mathcal{B}}_w(\infty) \subset \breve{\mathcal{B}}(\infty)$ *is the corresponding Demazure crystal.*

For $i \in \breve{I}$ and $b \in \mathcal{B}^0(\infty)$, we set

$$\varepsilon_i^\omega(b) := \max\{a \in \mathbb{Z}_{\ge 0} \mid (\tilde{e}_i^\omega)^a b \ne 0\}.$$

The properties of P_∞ in Proposition 4.1.2 (2) imply the equality

$$\varepsilon_i^\omega(b) = \varepsilon_i(P_\infty(b))$$

for every $i \in \breve{I}$ and $b \in \mathcal{B}^0(\infty)$.

Proposition 4.1.3. *The equality*

$$\varepsilon_i^\omega(b) = \varepsilon_{\omega^k(i)}(b)$$

holds for every $i \in \breve{I}$, $b \in \mathcal{B}^0(\infty)$, *and* $k \in \mathbb{Z}_{\ge 0}$.

Proof. Although this is proved in [**49**, Lemma 2.3.2], we give a proof for the convenience of the reader. By replacing $\breve{I}$ if necessary, we may assume that $k = 0$. Since $(\tilde{e}_i^\omega)^a = \tilde{e}^a_{\omega^{m_i-1}(i)} \cdots \tilde{e}^a_{\omega(i)}\tilde{e}^a_i$ for $a \in \mathbb{Z}_{\ge 0}$ by condition

(O), the condition $(\tilde{e}_i^\omega)^{\varepsilon_i^\omega(b)} b \neq 0$ implies that $\tilde{e}_i^{\varepsilon_i^\omega(b)} b \neq 0$. Suppose, for a contradiction, that $\tilde{e}_i^{\varepsilon_i^\omega(b)+1} b \neq 0$. Then, we have

$$\begin{aligned} \tilde{e}_{\omega^k(i)}^{\varepsilon_i^\omega(b)+1} b &= \tilde{e}_{\omega^k(i)}^{\varepsilon_i^\omega(b)+1} \omega^k(b) \quad (\text{since } b \in \mathcal{B}^0(\infty)) \\ &= \omega^k(\tilde{e}_i^{\varepsilon_i^\omega(b)+1} b) \quad (\text{by equation (4.1.1)}) \\ &\neq 0, \end{aligned}$$

from which we deduce by condition (O) that

$$\tilde{e}_{\omega^k(i)}^{\varepsilon_i^\omega(b)+1} \cdots \tilde{e}_{\omega(i)}^{\varepsilon_i^\omega(b)+1} \tilde{e}_i^{\varepsilon_i^\omega(b)+1} b \neq 0$$

for any $0 \leq k \leq m_i - 1$; this contradicts the equality $(\tilde{e}_i^\omega)^{\varepsilon_i^\omega(b)+1} b = 0$. Therefore, the equality $\tilde{e}_i^{\varepsilon_i^\omega(b)+1} b = 0$ holds, which implies that $\varepsilon_i(b) = \varepsilon_i^\omega(b)$. This proves the proposition. □

Note that $\breve{P} := (P_\omega^*)^{-1}(P \cap (\mathfrak{t}^*)^0) \subset \breve{\mathfrak{t}}^*$ is identical to the weight lattice for $\breve{\mathfrak{g}}$. For $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$, we have a natural bijective map $\omega \colon \mathcal{B}(\lambda) \to \mathcal{B}(\lambda)$, induced by the $\mathbb{Q}(q)$-algebra automorphism $\omega \colon U_q(\mathfrak{g}) \xrightarrow{\sim} U_q(\mathfrak{g})$, such that

$$\omega \circ \tilde{e}_i = \tilde{e}_{\omega(i)} \circ \omega \text{ and } \omega \circ \tilde{f}_i = \tilde{f}_{\omega(i)} \circ \omega \tag{4.1.3}$$

for all $i \in I$ (see [**48**, Sect. 3.2] and [**58**, Sect. 3]). Here we recall that $\pi_\lambda \colon \mathcal{B}(\infty) \twoheadrightarrow \mathcal{B}(\lambda) \cup \{0\}$ is the canonical map induced from the natural surjective map $U_q(\mathfrak{u}^-) \twoheadrightarrow V_q(\lambda)$. If we set

$$\mathcal{B}^0(\lambda) := \{b \in \mathcal{B}(\lambda) \mid \omega(b) = b\},$$

then it is easily checked that $\pi_\lambda(\mathcal{B}^0(\infty)) = \mathcal{B}^0(\lambda) \cup \{0\}$. For $\breve{\lambda} \in (P_\omega^*)^{-1}(P_+ \cap (\mathfrak{t}^*)^0)$, let $\breve{V}_q(\breve{\lambda})$ denote the irreducible highest weight $U_q(\breve{\mathfrak{g}})$-module with highest weight $\breve{\lambda}$, $\breve{\mathcal{B}}(\breve{\lambda})$ the crystal basis of $\breve{V}_q(\breve{\lambda})$, $b_{\breve{\lambda}} \in \breve{\mathcal{B}}(\breve{\lambda})$ the element corresponding to a highest weight vector in $\breve{V}_q(\breve{\lambda})$, and $\tilde{e}_i, \tilde{f}_i \colon \breve{\mathcal{B}}(\breve{\lambda}) \cup \{0\} \to \breve{\mathcal{B}}(\breve{\lambda}) \cup \{0\}$, $i \in \breve{I}$, the Kashiwara operators.

Proposition 4.1.4 ([**48**, Proposition 3.2.1]). *Let $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$, and $\breve{\lambda} := (P_\omega^*)^{-1}(\lambda)$.*

1. *The set $\mathcal{B}^0(\lambda) \cup \{0\}$ is stable under the ω-Kashiwara operators $\tilde{e}_i^\omega, \tilde{f}_i^\omega \colon \mathcal{B}(\lambda) \cup \{0\} \to \mathcal{B}(\lambda) \cup \{0\}$ for all $i \in I$, defined in the same way as ω-Kashiwara operators for $\mathcal{B}(\infty)$.*
2. *There exists a unique bijective map $P_\lambda \colon \mathcal{B}^0(\lambda) \cup \{0\} \to \breve{\mathcal{B}}(\breve{\lambda}) \cup \{0\}$ such that*
$$P_\lambda(b_\lambda) = b_{\breve{\lambda}},\ P_\lambda \circ \tilde{e}_i^\omega = \tilde{e}_i \circ P_\lambda, \text{ and } P_\lambda \circ \tilde{f}_i^\omega = \tilde{f}_i \circ P_\lambda$$
for all $i \in \breve{I}$.
3. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{B}^0(\infty) & \xrightarrow{\pi_\lambda} & \mathcal{B}^0(\lambda) \cup \{0\} \\ {\scriptstyle P_\infty}\downarrow & & \downarrow{\scriptstyle P_\lambda} \\ \breve{\mathcal{B}}(\infty) & \xrightarrow{\pi_{\breve{\lambda}}} & \breve{\mathcal{B}}(\breve{\lambda}) \cup \{0\}, \end{array}$$

where $\pi_{\check{\lambda}}$ *is the map induced from the natural surjective map* $U_q(\check{\mathfrak{u}}^-) \twoheadrightarrow \check{V}_q(\check{\lambda})$.

(4) *The equality*

$$P_\lambda(\mathcal{B}^0_{\Theta(w)}(\lambda)) = \check{\mathcal{B}}_w(\check{\lambda})$$

holds for all $w \in \check{W}$, *where* $\mathcal{B}^0_{\Theta(w)}(\lambda) := \mathcal{B}^0(\lambda) \cap \mathcal{B}_{\Theta(w)}(\lambda)$ *and* $\check{\mathcal{B}}_w(\check{\lambda}) \subset \check{\mathcal{B}}(\check{\lambda})$ *is the corresponding Demazure crystal.*

Remark 4.1.5. The composite maps $\check{\mathcal{B}}(\infty) \xrightarrow{P_\infty^{-1}} \mathcal{B}^0(\infty) \hookrightarrow \mathcal{B}(\infty)$ and $\check{\mathcal{B}}(\check{\lambda}) \xrightarrow{P_\lambda^{-1}} \mathcal{B}^0(\lambda) \hookrightarrow \mathcal{B}(\lambda)$ are identical to the maps arising from a similarity of crystal bases (see [**31**, Sect. 5]). This similarity is a variant of what we consider in Sect. 4.3.

It is easily seen that $\omega \circ * = * \circ \omega$ on $U_q(\mathfrak{g})$, which implies the same equality on $\mathcal{B}(\infty)$. Hence it follows that $\mathcal{B}^0(\infty)^* = \mathcal{B}^0(\infty)$. We denote by $*\colon \check{\mathcal{B}}(\infty) \to \check{\mathcal{B}}(\infty)$ Kashiwara's involution on $\check{\mathcal{B}}(\infty)$.

Proposition 4.1.6 ([**49**, Theorem 1]). *The following diagram is commutative*:

$$\begin{array}{ccc} \mathcal{B}^0(\infty) & \xrightarrow{\ *\ } & \mathcal{B}^0(\infty) \\ {\scriptstyle P_\infty}\downarrow & & {\scriptstyle P_\infty}\downarrow \\ \check{\mathcal{B}}(\infty) & \xrightarrow{\ *\ } & \check{\mathcal{B}}(\infty). \end{array}$$

The following is an immediate consequence of Propositions 4.1.3 and 4.1.6.

Corollary 4.1.7. *The equality*

$$\varepsilon_i(P_\infty(b)^*) = \varepsilon_{\omega^k(i)}(b^*)$$

holds for all $i \in \check{I}$, $b \in \mathcal{B}^0(\infty)$, *and* $k \in \mathbb{Z}_{\geq 0}$.

Let $\{s_i \mid i \in I\} \subset W$ (resp., $\{s_i \mid i \in \check{I}\} \subset \check{W}$) be the set of simple reflections. If we take a reduced word $\mathbf{i} = (i_1, \ldots, i_r) \in \check{I}^r$ for $w \in \check{W}$, then we have

$$\Theta(w) = \Theta(s_{i_1}) \cdots \Theta(s_{i_r}) = s_{i_{1,1}} \cdots s_{i_{1,m_{i_1}}} \cdots s_{i_{r,1}} \cdots s_{i_{r,m_{i_r}}},$$

where we set $i_{k,l} := \omega^{l-1}(i_k)$ for $1 \leq k \leq r$ and $1 \leq l \leq m_{i_k}$. It is easily verified that this is a reduced expression of $\Theta(w)$; we denote by $\Theta(\mathbf{i})$ the corresponding reduced word $(i_{1,1}, \ldots, i_{1,m_{i_1}}, \ldots, i_{r,1}, \ldots, i_{r,m_{i_r}})$.

Corollary 4.1.8. *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in \check{I}^r$ *be a reduced word for* $w \in \check{W}$. *Define an* $\mathbb{R}$*-linear injective map* $\Upsilon_{\mathbf{i}}\colon \mathbb{R}^r \hookrightarrow \mathbb{R}^{m_{i_1}+\cdots+m_{i_r}}$ *by:*

$$\Upsilon_{\mathbf{i}}(a_1, \ldots, a_r) = (\underbrace{a_1, \ldots, a_1}_{m_{i_1}}, \ldots, \underbrace{a_r, \ldots, a_r}_{m_{i_r}}).$$

Then, the equalities

$$\Upsilon_{\mathbf{i}}(\Phi_{\mathbf{i}}(b)) = \Phi_{\Theta(\mathbf{i})}(P_\infty^{-1}(b)) \text{ and } \Upsilon_{\mathbf{i}}(\Psi_{\mathbf{i}}(b)) = \Psi_{\Theta(\mathbf{i})}(P_\infty^{-1}(b))$$

hold for all $b \in \breve{\mathcal{B}}_w(\infty)$. *In particular, the following equalities hold*:

$$\Upsilon_{\mathbf{i}}(\Phi_{\mathbf{i}}(\breve{\mathcal{B}}_w(\infty))) = \Phi_{\Theta(\mathbf{i})}(\mathcal{B}^0_{\Theta(w)}(\infty)), \text{ and } \Upsilon_{\mathbf{i}}(\Psi_{\mathbf{i}}(\breve{\mathcal{B}}_w(\infty))) = \Psi_{\Theta(\mathbf{i})}(\mathcal{B}^0_{\Theta(w)}(\infty)).$$

Proof. We take $b \in \breve{\mathcal{B}}_w(\infty)$, and write $\Phi_{\mathbf{i}}(b)$ as $(a_1, \ldots, a_r)$. We will show that

$$\Phi_{\Theta(\mathbf{i})}(P_\infty^{-1}(b)) = (\underbrace{a_1, \ldots, a_1}_{m_{i_1}}, \ldots, \underbrace{a_r, \ldots, a_r}_{m_{i_r}}).$$

It follows by condition (O) and Proposition 4.1.3 that

$$\varepsilon_{i_{1,k}}(\tilde{e}^{a_1}_{i_{1,k-1}} \cdots \tilde{e}^{a_1}_{i_{1,1}} P_\infty^{-1}(b)) = \varepsilon_{i_{1,k}}(P_\infty^{-1}(b)) = a_1$$

for all $1 \le k \le m_{i_1}$ (see also the proof of Proposition 4.1.3). Therefore, the following equality holds:

$$\Phi_{\Theta(\mathbf{i})}(P_\infty^{-1}(b)) = (\underbrace{a_1, \ldots, a_1}_{m_{i_1}}, \Phi_{\Theta(\mathbf{i}_{\ge 2})}(P_\infty^{-1}(b'))),$$

where $\mathbf{i}_{\ge 2} \coloneqq (i_2, \ldots, i_r)$ and $b' \coloneqq \tilde{e}^{a_1}_{i_1} b$. Moreover, by induction on r, we deduce that

$$\Phi_{\Theta(\mathbf{i}_{\ge 2})}(P_\infty^{-1}(b')) = (\underbrace{a_2, \ldots, a_2}_{m_{i_2}}, \ldots, \underbrace{a_r, \ldots, a_r}_{m_{i_r}}).$$

From these, we obtain the assertion for $\Phi_{\mathbf{i}}$. The assertion for $\Psi_{\mathbf{i}}$ is shown similarly by using Corollary 4.1.7 instead of Proposition 4.1.3. □

If $b \in \mathcal{B}_{\Theta(w)}(\infty)$ satisfies $\Phi_{\Theta(\mathbf{i})}(b) = \Upsilon_{\mathbf{i}}(a_1, \ldots, a_r)$ for some $(a_1, \ldots, a_r) \in \mathbb{Z}^r_{\ge 0}$, then it is easily seen that $b \in \mathcal{B}^0_{\Theta(w)}(\infty)$. Hence we obtain the following.

Corollary 4.1.9. *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in \breve{I}^r$ *be a reduced word for* $w \in \breve{W}$. *Then, the following equalities hold*:

$$\begin{aligned}
&\Upsilon_{\mathbf{i}}(\Phi_{\mathbf{i}}(\breve{\mathcal{B}}_w(\infty))) \\
&= \{(a_{k,l})_{1 \le k \le r, 1 \le l \le m_{i_k}} \in \Phi_{\Theta(\mathbf{i})}(\mathcal{B}_{\Theta(w)}(\infty)) \mid a_{k,1} = \cdots = a_{k,m_{i_k}},\ 1 \le k \le r\}, \\
&\Upsilon_{\mathbf{i}}(\Psi_{\mathbf{i}}(\breve{\mathcal{B}}_w(\infty))) \\
&= \{(a_{k,l})_{1 \le k \le r, 1 \le l \le m_{i_k}} \in \Psi_{\Theta(\mathbf{i})}(\mathcal{B}_{\Theta(w)}(\infty)) \mid a_{k,1} = \cdots = a_{k,m_{i_k}},\ 1 \le k \le r\}.
\end{aligned}$$

Similarly, we obtain the following (see Proposition 4.1.4 (3), (4)).

Corollary 4.1.10. *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in \breve{I}^r$ *be a reduced word for* $w \in \breve{W}$, $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$, *and* $\breve{\lambda} \coloneqq (P^*_\omega)^{-1}(\lambda)$. *Then, the following equalities hold*:

$$\begin{aligned}
&\Upsilon_{\mathbf{i}}(\Phi_{\mathbf{i}}(\breve{\mathcal{B}}_w(\breve{\lambda}))) \\
&= \{(a_{k,l})_{1 \le k \le r, 1 \le l \le m_{i_k}} \in \Phi_{\Theta(\mathbf{i})}(\mathcal{B}_{\Theta(w)}(\lambda)) \mid a_{k,1} = \cdots = a_{k,m_{i_k}},\ 1 \le k \le r\}, \\
&\Upsilon_{\mathbf{i}}(\Psi_{\mathbf{i}}(\breve{\mathcal{B}}_w(\breve{\lambda}))) \\
&= \{(a_{k,l})_{1 \le k \le r, 1 \le l \le m_{i_k}} \in \Psi_{\Theta(\mathbf{i})}(\mathcal{B}_{\Theta(w)}(\lambda)) \mid a_{k,1} = \cdots = a_{k,m_{i_k}},\ 1 \le k \le r\}.
\end{aligned}$$

By the definitions of Berenstein-Littelmann-Zelevinsky's string polytopes and Nakashima-Zelevinsky polytopes, we obtain the following as an immediate consequence of Corollary 4.1.10.

COROLLARY 4.1.11. *Let* $\mathbf{i} = (i_1, \dots, i_r) \in \breve{I}^r$ *be a reduced word for* $w \in \breve{W}$, $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$, *and* $\breve{\lambda} := (P_\omega^*)^{-1}(\lambda)$. *Then, the following equalities hold*:

$$\begin{aligned}&\Upsilon_{\mathbf{i}}(\Delta_{\mathbf{i}}(\breve{\lambda}))\\ &= \{(a_{k,l})_{1\le k\le r, 1\le l\le m_{i_k}} \in \Delta_{\Theta(\mathbf{i})}(\lambda) \mid a_{k,1} = \cdots = a_{k,m_{i_k}},\ 1 \le k \le r\},\\ &\Upsilon_{\mathbf{i}}(\widetilde{\Delta}_{\mathbf{i}}(\breve{\lambda}))\\ &= \{(a_{k,l})_{1\le k\le r, 1\le l\le m_{i_k}} \in \widetilde{\Delta}_{\Theta(\mathbf{i})}(\lambda) \mid a_{k,1} = \cdots = a_{k,m_{i_k}},\ 1 \le k \le r\}.\end{aligned}$$

REMARK 4.1.12. Corollary 4.1.11 is naturally extended to string polytopes for generalized Demazure modules, defined in [**12**].

4.2. Third main result

In this section, we prove our third main result. Let us consider the fixed point Lie subalgebra by ω:

$$\mathfrak{g}^\omega = \{x \in \mathfrak{g} \mid \omega(x) = x\}.$$

Define $e_i', f_i', h_i' \in \mathfrak{g}^\omega$ and $\alpha_i' \in (\mathfrak{t}^*)^0$ for $i \in \breve{I}$ by

$$\begin{aligned}&e_i' := \sum_{0\le k<m_i} e_{\omega^k(i)},\ f_i' := \sum_{0\le k<m_i} f_{\omega^k(i)},\ h_i' := \sum_{0\le k<m_i} h_{\omega^k(i)},\ \text{and}\\ &\alpha_i' := \frac{1}{m_i}\sum_{0\le k<m_i} \alpha_{\omega^k(i)}.\end{aligned}$$

We set $c_{i,j}' := \langle \alpha_j', h_i' \rangle$ for $i, j \in \breve{I}$. Then, it is easily checked that $\breve{c}_{i,j} = c_{j,i}'$ for all $i, j \in \breve{I}$; namely, the matrix $C' := (c_{i,j}')_{i,j\in\breve{I}}$ is the transpose of $\breve{C}$. In particular, the matrix C' is a Cartan matrix of finite type.

PROPOSITION 4.2.1 (see [**23**, Proposition 8.3]). *The fixed point Lie subalgebra* $\mathfrak{g}^\omega$ *is the semisimple Lie algebra with Cartan matrix* C' *and with Chevalley generators* e_i', f_i', h_i', $i \in \breve{I}$; *in particular, the orbit Lie algebra* $\breve{\mathfrak{g}}$ *associated with* ω *is the* (*Langlands*) *dual Lie algebra of* $\mathfrak{g}^\omega$.

Recall that G is the connected, simply-connected semisimple algebraic group with $\mathrm{Lie}(G) = \mathfrak{g}$. The Lie algebra automorphism $\omega\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ induces an algebraic group automorphism $\omega\colon G \xrightarrow{\sim} G$ such that $\omega(\exp(x)) = \exp(\omega(x))$ for all $x \in \mathfrak{g}$. It is easily seen that the fixed point subgroup

$$G^\omega := \{g \in G \mid \omega(g) = g\}$$

is a Zariski closed subgroup of G with $\mathrm{Lie}(G^\omega) = \mathfrak{g}^\omega$; in addition, we see by Table 1 in Sect. 4.1 and a case-by-case argument that G^ω is a connected, simply-connected semisimple algebraic group. Since the fixed point subgroup $(U^-)^\omega := U^- \cap G^\omega$ is a Zariski closed subgroup of U^-, the coordinate ring $\mathbb{C}[(U^-)^\omega]$ is a quotient of $\mathbb{C}[U^-]$; denote by

$$\pi^\omega \colon \mathbb{C}[U^-] \twoheadrightarrow \mathbb{C}[(U^-)^\omega]$$

the quotient map. We set $B^\omega := B \cap G^\omega$, and consider the full flag variety G^ω/B^ω. Let $\iota^\omega \colon G^\omega/B^\omega \hookrightarrow G/B$ denote the natural injective map. Since $\omega(B) = B$, the automorphism $\omega \colon G \xrightarrow{\sim} G$ induces a variety automorphism $\omega \colon G/B \xrightarrow{\sim} G/B$, and the image of ι^ω is identical to the fixed point subvariety $(G/B)^\omega$. In addition, the map ι^ω induces a $\mathbb{C}$-linear isomorphism from the tangent space of G^ω/B^ω at $e \bmod B^\omega$ to that of $(G/B)^\omega$ at $e \bmod B$, where $e \in G^\omega$ $(\subset G)$ is the identity element; note that both of these tangent spaces are identified with the Lie subalgebra of $\mathfrak{g}^\omega$ generated by $\{f'_i \mid i \in \breve{I}\}$. Therefore, the map $\iota^\omega \colon G^\omega/B^\omega \to (G/B)^\omega$ is an isomorphism of varieties (see, for instance, [**61**, Theorem 5.3.2 (iii)]). Here we note that since $\mathfrak{g}^\omega$ is the (Langlands) dual Lie algebra of $\breve{\mathfrak{g}}$, the Weyl group $\breve{W}$ of $\breve{\mathfrak{g}}$ is identified with that of $\mathfrak{g}^\omega$. We consider the Schubert variety $X(w) \subset G^\omega/B^\omega \simeq (G/B)^\omega$ corresponding to $w \in \breve{W}$; this is identified with a Zariski closed subvariety of $X(\Theta(w))$. Let us regard $(U^-)^\omega$ as an affine open subvariety of G^ω/B^ω, and take the intersection $(U^-)^\omega \cap X(w)$ in G^ω/B^ω for $w \in \breve{W}$; this intersection is identified with a Zariski closed subvariety of $U^- \cap X(\Theta(w))$. Let

$$\pi_w^\omega \colon \mathbb{C}[U^- \cap X(\Theta(w))] \twoheadrightarrow \mathbb{C}[(U^-)^\omega \cap X(w)]$$

be the restriction map for $w \in \breve{W}$. We take a reduced word $\mathbf{i} = (i_1, \ldots, i_r) \in \breve{I}^r$ for $w \in \breve{W}$, and regard the coordinate ring $\mathbb{C}[(U^-)^\omega \cap X(w)]$ as a $\mathbb{C}$-subalgebra of the polynomial ring $\mathbb{C}[t_1, \ldots, t_r]$ by the following birational morphism:

$$\mathbb{C}^r \to (U^-)^\omega \cap X(w),\ (t_1, \ldots, t_r) \mapsto \exp(t_1 f'_{i_1}) \cdots \exp(t_r f'_{i_r}).$$

Since $\Theta(\mathbf{i}) = (i_{1,1}, \ldots, i_{1,m_{i_1}}, \ldots, i_{r,1}, \ldots, i_{r,m_{i_r}})$ is a reduced word for $\Theta(w) \in W$, the coordinate ring $\mathbb{C}[U^- \cap X(\Theta(w))]$ is regarded as a $\mathbb{C}$-subalgebra of the polynomial ring $\mathbb{C}[t_{k,l} \mid 1 \le k \le r,\ 1 \le l \le m_{i_k}]$ by the following birational morphism:

$$\begin{aligned} \mathbb{C}^{m_{i_1}+\cdots+m_{i_r}} &\to U^- \cap X(\Theta(w)),\\ (t_{1,1}, \ldots, t_{r,m_{i_r}}) &\mapsto \exp(t_{1,1} f_{i_{1,1}}) \cdots \exp(t_{r,m_{i_r}} f_{i_{r,m_{i_r}}}). \end{aligned}$$

Also, under the inclusion map $(U^-)^\omega \cap X(w) \hookrightarrow U^- \cap X(\Theta(w))$, we have

$$\exp(t f'_{i_k}) \mapsto \exp(t f_{i_{k,1}}) \cdots \exp(t f_{i_{k,m_{i_k}}})$$

for $t \in \mathbb{C}$ and $1 \le k \le r$. Hence we obtain the following.

Lemma 4.2.2. *Define a surjective $\mathbb{C}$-algebra homomorphism $\pi_{\mathbf{i}}^\omega \colon \mathbb{C}[t_{k,l} \mid 1 \le k \le r,\ 1 \le l \le m_{i_k}] \twoheadrightarrow \mathbb{C}[t_1, \ldots, t_r]$ by $\pi_{\mathbf{i}}^\omega(t_{k,l}) := t_k$ for $1 \le k \le r$ and*

$1 \leq l \leq m_{i_k}$. *Then, the following diagram is commutative*:

$$\begin{array}{ccc} \mathbb{C}[U^- \cap X(\Theta(w))] & \hookrightarrow & \mathbb{C}[t_{k,l} \mid 1 \leq k \leq r,\ 1 \leq l \leq m_{i_k}] \\ \downarrow \pi_w^\omega & & \downarrow \pi_{\mathbf{i}}^\omega \\ \mathbb{C}[(U^-)^\omega \cap X(w)] & \hookrightarrow & \mathbb{C}[t_1, \ldots, t_r]. \end{array}$$

Let us consider a perfect basis $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ that satisfies positivity conditions $(\mathrm{P})_1$ and $(\mathrm{P})_2$ in Sect. 3.2. Recall that $\mathfrak{g}$ is of simply-laced type; hence the upper global basis $\{G^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$ satisfies positivity conditions $(\mathrm{P})_1$ and $(\mathrm{P})_2$.

Lemma 4.2.3. *Let* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *be a perfect basis satisfying positivity conditions* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$. *Then, the product*

$$\Xi^{\mathrm{up}}_{\Theta(w)}(\tilde{f}_i^{a_0} \tilde{f}_{\omega(i)}^{a_1} \cdots \tilde{f}_{\omega^{m_i-1}(i)}^{a_{m_i-1}} b_\infty) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b)$$

belongs to $\sum_{b' \in \mathcal{B}_{\Theta(w)}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}_{\Theta(w)}(b')$ *for all* $w \in \breve{W}$, $i \in I$, $b \in \mathcal{B}_{\Theta(w)}(\infty)$, *and* $a_0, a_1, \ldots, a_{m_i-1} \in \mathbb{Z}_{\geq 0}$.

Proof. Our proof is similar to that of Proposition 3.2.1. By Corollary 2.2.24 (2) and Proposition 3.2.5, it suffices to prove that the product $\Xi^{\mathrm{up}}(\tilde{f}_i^{a_0} \tilde{f}_{\omega(i)}^{a_1} \cdots \tilde{f}_{\omega^{m_i-1}(i)}^{a_{m_i-1}} b_\infty) \cdot \Xi^{\mathrm{up}}(b)$ belongs to $\sum_{b' \in \mathcal{B}(\infty)} \mathbb{R}_{\geq 0} \Xi^{\mathrm{up}}(b')$ for all $i \in I$, $b \in \mathcal{B}(\infty)$, and $a_0, a_1, \ldots, a_{m_i-1} \in \mathbb{Z}_{\geq 0}$. Set $\mathbf{d} := \sum_{0 \leq k < m_i} a_k \mathbf{e}_{\omega^k(i)}$. Since

$$U(\mathfrak{u}^-)^*_{\mathrm{gr},\mathbf{d}} = \mathbb{C}\Xi^{\mathrm{up}}(\tilde{f}_i^{a_0} \tilde{f}_{\omega(i)}^{a_1} \cdots \tilde{f}_{\omega^{m_i-1}(i)}^{a_{m_i-1}} b_\infty)$$

by condition (O) in Sect. 4.1, we see that

$$\begin{aligned} &\Xi^{\mathrm{up}}(\tilde{f}_i b_\infty)^{a_0} \cdot \Xi^{\mathrm{up}}(\tilde{f}_{\omega(i)} b_\infty)^{a_1} \cdots \Xi^{\mathrm{up}}(\tilde{f}_{\omega^{m_i-1}(i)} b_\infty)^{a_{m_i-1}} \\ &= C \cdot \Xi^{\mathrm{up}}(\tilde{f}_i^{a_0} \tilde{f}_{\omega(i)}^{a_1} \cdots \tilde{f}_{\omega^{m_i-1}(i)}^{a_{m_i-1}} b_\infty) \end{aligned}$$

for some coefficient $C \in \mathbb{C}^\times$. Then, positivity condition $(\mathrm{P})_2$ implies that $C \in \mathbb{R}_{>0}$; hence we deduce the assertion by $(\mathrm{P})_2$ again. □

Define an $\mathbb{R}$-linear surjective map $\Omega_{\mathbf{i}} \colon \mathbb{R}^{m_{i_1} + \cdots + m_{i_r}} \twoheadrightarrow \mathbb{R}^r$ by:

$$\begin{aligned} &\Omega_{\mathbf{i}}(a_{1,1}, \ldots, a_{1,m_{i_1}}, \ldots, a_{r,1}, \ldots, a_{r,m_{i_r}}) \\ &= (a_{1,1} + \cdots + a_{1,m_{i_1}}, \ldots, a_{r,1} + \cdots + a_{r,m_{i_r}}). \end{aligned}$$

Theorem 4.2.4. *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in \breve{I}^r$ *be a reduced word for* $w \in \breve{W}$, *and* $\mathbf{B}^{\mathrm{up}} = \{\Xi^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\} \subset \mathbb{C}[U^-]$ *a perfect basis satisfying* $(\mathrm{P})_1$ *and* $(\mathrm{P})_2$. *Then, the following equalities hold for all* $b \in \mathcal{B}_{\Theta(w)}(\infty)$:

$$\begin{aligned} &v_{\mathbf{i}}^{\mathrm{high}}(\pi_w^\omega(\Xi^{\mathrm{up}}_{\Theta(w)}(b))) = \Omega_{\mathbf{i}}(v_{\Theta(\mathbf{i})}^{\mathrm{high}}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))), \text{ and} \\ &\tilde{v}_{\mathbf{i}}^{\mathrm{high}}(\pi_w^\omega(\Xi^{\mathrm{up}}_{\Theta(w)}(b)))^{\mathrm{op}} = \Omega_{\mathbf{i}}(\tilde{v}_{\Theta(\mathbf{i})}^{\mathrm{high}}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))^{\mathrm{op}}). \end{aligned}$$

Proof. We prove the assertion only for $v_{\mathbf{i}}^{\mathrm{high}}$ and $v_{\Theta(\mathbf{i})}^{\mathrm{high}}$; a proof of the assertion for $\tilde{v}_{\mathbf{i}}^{\mathrm{high}}$ and $\tilde{v}_{\Theta(\mathbf{i})}^{\mathrm{high}}$ is similar. We imitate the proof of Theorem 3.3.1.

We write $\Phi_{\Theta(\mathbf{i})}(b) = (a_{1,1}, \ldots, a_{1,m_{i_1}}, \ldots, a_{r,1}, \ldots, a_{r,m_{i_r}})$ for $b \in \mathcal{B}_{\Theta(w)}(\infty)$, and proceed by induction on $r = \ell(w)$ and $a_{1,1} + \cdots + a_{r,m_{i_r}}$.

We first consider the case $b \in \mathcal{B}_{s_{i_{1,1}} \cdots s_{i_{1,m_{i_1}}}}(\infty)$, which includes the case $r = 1$. In this case, there exist $a_1, \ldots, a_{m_{i_1}} \in \mathbb{Z}_{\geq 0}$ such that $b = \tilde{f}_{i_{1,1}}^{a_1} \cdots \tilde{f}_{i_{1,m_{i_1}}}^{a_{m_{i_1}}} b_\infty$. Then, it follows by the definition of $\Phi_{\Theta(\mathbf{i})}$ and condition (O) in Sect. 4.1 that

$$\begin{aligned} -v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b)) &= \Phi_{\Theta(\mathbf{i})}(b) \quad \text{(by Theorem 2.2.9)} \\ &= (a_1, \ldots, a_{m_{i_1}}, 0, \ldots, 0). \end{aligned}$$

Hence we deduce by the definition of $v^{\mathrm{high}}_{\Theta(\mathbf{i})}$ that $\Xi^{\mathrm{up}}_{\Theta(w)}(b) = c t_{1,1}^{a_1} \cdots t_{1,m_{i_1}}^{a_{m_{i_1}}} +$ (other terms) for some $c \in \mathbb{C}^\times$, where "other terms" means a linear combination of monomials of degree $a_1 + \cdots + a_{m_{i_1}}$ that are not equal to $t_{1,1}^{a_1} \cdots t_{1,m_{i_1}}^{a_{m_{i_1}}}$. Here, Proposition 3.2.8 implies that $c \in \mathbb{R}_{>0}$, and that the coefficients of the "other terms" are also positive real numbers. Therefore, we see from Lemma 4.2.2 that $\pi^\omega_w(\Xi^{\mathrm{up}}_{\Theta(w)}(b)) = c' t_1^{a_1 + \cdots + a_{m_{i_1}}} +$ (other terms) for some $c' \in \mathbb{R}_{>0}$, where "other terms" means a linear combination of monomials in $\mathbb{C}[t_1, \ldots, t_r]$ of degree $a_1 + \cdots + a_{m_{i_1}}$ that are not equal to $t_1^{a_1 + \cdots + a_{m_{i_1}}}$. This implies by the definition of $v^{\mathrm{high}}_{\mathbf{i}}$ that

$$\begin{aligned} v^{\mathrm{high}}_{\mathbf{i}}(\pi^\omega_w(\Xi^{\mathrm{up}}_{\Theta(w)}(b))) &= -(a_1 + \cdots + a_{m_{i_1}}, 0, \ldots, 0) \\ &= \Omega_{\mathbf{i}}(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))). \end{aligned}$$

We next consider the case $r \geq 2$ and $a_{1,1} = \cdots = a_{1,m_{i_1}} = 0$. In this case, b is an element of $\mathcal{B}_{\Theta(w_{\geq 2})}(\infty)$, where $w_{\geq 2} \coloneqq s_{i_2} \cdots s_{i_r}$. By the definition of $v^{\mathrm{high}}_{\Theta(\mathbf{i})}$, the equalities $a_{1,1} = \cdots = a_{1,m_{i_1}} = 0$ imply that $t_{1,1}, \ldots, t_{1,m_{i_1}}$ do not appear in $\Xi^{\mathrm{up}}_{\Theta(w)}(b)$, and hence that t_1 does not appear in $\pi^\omega_w(\Xi^{\mathrm{up}}_{\Theta(w)}(b)) \in \mathbb{C}[t_1, \ldots, t_r]$. From this, we deduce that

$$\begin{aligned} v^{\mathrm{high}}_{\mathbf{i}}(\pi^\omega_w(\Xi^{\mathrm{up}}_{\Theta(w)}(b))) &= (0, v^{\mathrm{high}}_{\mathbf{i}_{\geq 2}}(\pi^\omega_{w_{\geq 2}}(\Xi^{\mathrm{up}}_{\Theta(w_{\geq 2})}(b)))) \\ &= (0, \Omega_{\mathbf{i}_{\geq 2}}(v^{\mathrm{high}}_{\Theta(\mathbf{i}_{\geq 2})}(\Xi^{\mathrm{up}}_{\Theta(w_{\geq 2})}(b)))) \\ &\quad \text{(by induction hypothesis concerning } r) \\ &= \Omega_{\mathbf{i}}(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))), \end{aligned}$$

where $\mathbf{i}_{\geq 2} \coloneqq (i_2, \ldots, i_r)$, a reduced word for $w_{\geq 2}$.

Finally, we consider the case $b \notin \mathcal{B}_{s_{i_{1,1}} \cdots s_{i_{1,m_{i_1}}}}(\infty)$ and $(a_{1,1}, \ldots, a_{1,m_{i_1}}) \neq (0, \ldots, 0)$. Set $b_1 \coloneqq \tilde{f}_{i_{1,1}}^{a_{1,1}} \cdots \tilde{f}_{i_{1,m_{i_1}}}^{a_{1,m_{i_1}}} b_\infty$ and $b_2 \coloneqq \tilde{f}_{i_{2,1}}^{a_{2,1}} \cdots \tilde{f}_{i_{r,m_{i_r}}}^{a_{r,m_{i_r}}} b_\infty$. Then, it follows by the definition of $\Phi_{\Theta(\mathbf{i})}$ that

$$\begin{aligned} &\Phi_{\Theta(\mathbf{i})}(b_1) = (a_{1,1}, \ldots, a_{1,m_{i_1}}, 0, \ldots, 0), \text{ and} \\ &\Phi_{\Theta(\mathbf{i})}(b_2) = (0, \ldots, 0, a_{2,1}, \ldots, a_{r,m_{i_r}}); \end{aligned}$$

here we have used condition (O) in Sect. 4.1. Hence Theorem 2.2.9 implies that

$$\begin{aligned} v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b)) &= -(a_{1,1}, \dots, a_{r,m_{i_r}}) \\ &= -(a_{1,1}, \dots, a_{1,m_{i_1}}, 0, \dots, 0) - (0, \dots, 0, a_{2,1}, \dots, a_{r,m_{i_r}}) \\ &= v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1)) + v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_2)). \end{aligned}$$

Also, we deduce from induction hypothesis concerning $a_{1,1} + \cdots + a_{r,m_{i_r}}$ that

$$\begin{aligned} &\Omega_{\mathbf{i}}\left(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1)) + v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_2))\right) \\ &= \Omega_{\mathbf{i}}\left(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1))\right) + \Omega_{\mathbf{i}}\left(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_2))\right) \\ &= v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1))) + v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_2))) \\ &= v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b_2))) \end{aligned}$$

(since $v^{\mathrm{high}}_{\mathbf{i}}$ is a valuation and π^{ω}_{w} is a $\mathbb{C}$-algebra homomorphism).

From these, it follows that

$$v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b_2))) = \Omega_{\mathbf{i}}(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))). \tag{4.2.1}$$

Here, by Corollary 2.2.27 and Lemma 4.2.3, we have

$$\Xi^{\mathrm{up}}_{\Theta(w)}(b_1) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b_2) = \sum_{b_3 \in \mathcal{B}_{\Theta(w)}(\infty)} C^{(b_3)}_{b_1,b_2} \Xi^{\mathrm{up}}_{\Theta(w)}(b_3) \tag{4.2.2}$$

for some $C^{(b_3)}_{b_1,b_2} \in \mathbb{R}_{\geq 0}$, $b_3 \in \mathcal{B}_{\Theta(w)}(\infty)$, with $C^{(b)}_{b_1,b_2} \neq 0$. By applying π^{ω}_{w} to (4.2.2), we obtain

$$\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b_2)) = \sum_{b_3 \in \mathcal{B}_{\Theta(w)}(\infty)} C^{(b_3)}_{b_1,b_2} \pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_3)). \tag{4.2.3}$$

Since $C^{(b_3)}_{b_1,b_2} \in \mathbb{R}_{\geq 0}$ for all $b_3 \in \mathcal{B}_{\Theta(w)}(\infty)$, Proposition 3.2.8 and Lemma 4.2.2 imply that no cancellations of monomials occur in the sum on the right hand side of (4.2.3). Therefore, we deduce by the definition of $v^{\mathrm{high}}_{\mathbf{i}}$ that

$$\begin{aligned} &- v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b_2))) \\ &= \max\{-v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_3))) \mid b_3 \in \mathcal{B}_{\Theta(w)}(\infty),\ C^{(b_3)}_{b_1,b_2} \neq 0\}, \end{aligned}$$

where "max" means the maximum with respect to the lexicographic order $<$ in Definition 1.1.3. Since $C^{(b)}_{b_1,b_2} \neq 0$, we obtain

$$-v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))) \leq -v^{\mathrm{high}}_{\mathbf{i}}(\pi^{\omega}_{w}(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b_2))). \tag{4.2.4}$$

Now, by the definition of $v^{\mathrm{high}}_{\Theta(\mathbf{i})}$ together with the equality $-v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b)) = (a_{1,1}, \dots, a_{r,m_{i_r}})$, the monomial $t_{1,1}^{a_{1,1}} \cdots t_{r,m_{i_r}}^{a_{r,m_{i_r}}}$ appears in the polynomial $\Xi^{\mathrm{up}}_{\Theta(w)}(b) \in \mathbb{C}[t_{1,1}, \dots, t_{r,m_{i_r}}]$. We see by Proposition 3.2.8 and Lemma 4.2.2 that the monomial

$$t_1^{a_{1,1}+\cdots+a_{1,m_{i_1}}} \cdots t_r^{a_{r,1}+\cdots+a_{r,m_{i_r}}}$$

appears in the polynomial $\pi_w^\omega(\Xi^{\mathrm{up}}_{\Theta(w)}(b)) \in \mathbb{C}[t_1, \ldots, t_r]$, which implies that

$$\begin{aligned} -\Omega_{\mathbf{i}}(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))) &= (a_{1,1} + \cdots + a_{1,m_{i_1}}, \ldots, a_{r,1} + \cdots + a_{r,m_{i_r}}) \\ &\leq -v^{\mathrm{high}}_{\mathbf{i}}(\pi_w^\omega(\Xi^{\mathrm{up}}_{\Theta(w)}(b))). \end{aligned} \tag{4.2.5}$$

By combining (4.2.1), (4.2.4), and (4.2.5), we conclude that

$$\begin{aligned} \Omega_{\mathbf{i}}(v^{\mathrm{high}}_{\Theta(\mathbf{i})}(\Xi^{\mathrm{up}}_{\Theta(w)}(b))) &= v^{\mathrm{high}}_{\mathbf{i}}(\pi_w^\omega(\Xi^{\mathrm{up}}_{\Theta(w)}(b))) \\ &= v^{\mathrm{high}}_{\mathbf{i}}(\pi_w^\omega(\Xi^{\mathrm{up}}_{\Theta(w)}(b_1) \cdot \Xi^{\mathrm{up}}_{\Theta(w)}(b_2))). \end{aligned}$$

This proves the theorem. □

Denote by $P' \subset (\mathfrak{t}^*)^0$ the subgroup generated by $\varpi'_i := \frac{1}{m_i} \sum_{0 \leq k < m_i} \varpi_{\omega^k(i)}$, $i \in \breve{I}$. Since the set $\{h'_i \mid i \in \breve{I}\}$ is regarded as the set of simple coroots of $\mathfrak{g}^\omega$, the subgroup P' is identified with the weight lattice for $\mathfrak{g}^\omega$; in particular, an element $\lambda \in P \cap (\mathfrak{t}^*)^0$ gives an integral weight $\hat{\lambda}$ for $\mathfrak{g}^\omega$. Recall that for $w \in \breve{W}$, the Schubert variety $X(w) \subset G^\omega/B^\omega \simeq (G/B)^\omega$ is identified with a Zariski closed subvariety of $X(\Theta(w))$. The inclusion map $X(w) \hookrightarrow X(\Theta(w))$ induces a B^ω-module homomorphism $H^0(X(\Theta(w)), \mathcal{L}_\lambda) \to H^0(X(w), \mathcal{L}_{\hat{\lambda}})$ (denoted also by π_w^ω) for $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$. Now we define $\mathbb{C}$-linear injective maps $\iota_\lambda \colon H^0(X(\Theta(w)), \mathcal{L}_\lambda) \hookrightarrow \mathbb{C}[U^- \cap X(\Theta(w))]$ and $\iota_{\hat{\lambda}} \colon H^0(X(w), \mathcal{L}_{\hat{\lambda}}) \hookrightarrow \mathbb{C}[(U^-)^\omega \cap X(w)]$ as in Lemma 2.2.19. The following is an immediate consequence of the definitions.

Proposition 4.2.5. *For $w \in \breve{W}$ and $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$, the following diagram is commutative*:

$$\begin{array}{ccc} \mathbb{C}[U^- \cap X(\Theta(w))] & \xrightarrow{\pi_w^\omega} & \mathbb{C}[(U^-)^\omega \cap X(w)] \\ {\scriptstyle \iota_\lambda}\big\uparrow & & \big\uparrow{\scriptstyle \iota_{\hat{\lambda}}} \\ H^0(X(\Theta(w)), \mathcal{L}_\lambda) & \xrightarrow{\pi_w^\omega} & H^0(X(w), \mathcal{L}_{\hat{\lambda}}). \end{array}$$

From this proposition, we obtain the following by the first assertion of Corollary 2.2.20 and Theorem 4.2.4.

Corollary 4.2.6. *The following hold*:

$$\Omega_{\mathbf{i}}(\Delta(X(\Theta(w)), \mathcal{L}_\lambda, v^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_\lambda)) \subset \Delta(X(w), \mathcal{L}_{\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{\hat{\lambda}}), \text{ and}$$
$$\Omega_{\mathbf{i}}(\Delta(X(\Theta(w)), \mathcal{L}_\lambda, \tilde{v}^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_\lambda)^{\mathrm{op}}) \subset \Delta(X(w), \mathcal{L}_{\hat{\lambda}}, \tilde{v}^{\mathrm{high}}_{\mathbf{i}}, \tau_{\hat{\lambda}})^{\mathrm{op}}.$$

The following is the third main result of this thesis.

Theorem 4.2.7. *Let $\mathbf{i} = (i_1, \ldots, i_r) \in \breve{I}^r$ be a reduced word for $w \in \breve{W}$, and $\lambda \in P_+ \cap (\mathfrak{t}^*)^0$. Then, the following equalities hold*:

$$\Omega_{\mathbf{i}}(\Delta(X(\Theta(w)), \mathcal{L}_\lambda, v^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_\lambda)) = \Delta(X(w), \mathcal{L}_{\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{\hat{\lambda}}), \text{ and}$$
$$\Omega_{\mathbf{i}}(\Delta(X(\Theta(w)), \mathcal{L}_\lambda, \tilde{v}^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_\lambda)^{\mathrm{op}}) = \Delta(X(w), \mathcal{L}_{\hat{\lambda}}, \tilde{v}^{\mathrm{high}}_{\mathbf{i}}, \tau_{\hat{\lambda}})^{\mathrm{op}}.$$

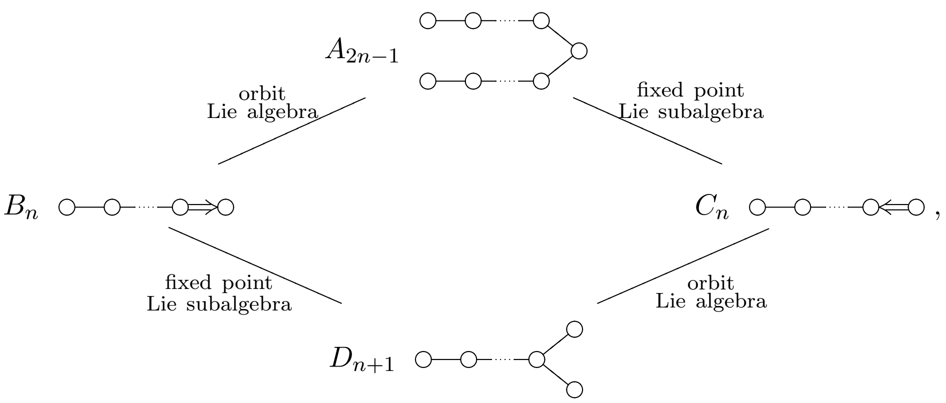
In order to prove this theorem, we consider another simply-laced semisimple Lie algebra $\mathfrak{g}'$ and its Dynkin diagram automorphism $\omega' \colon I' \to I'$. We assume that the pair $((\mathfrak{g}, \omega), (\mathfrak{g}', \omega'))$ satisfies the following conditions:

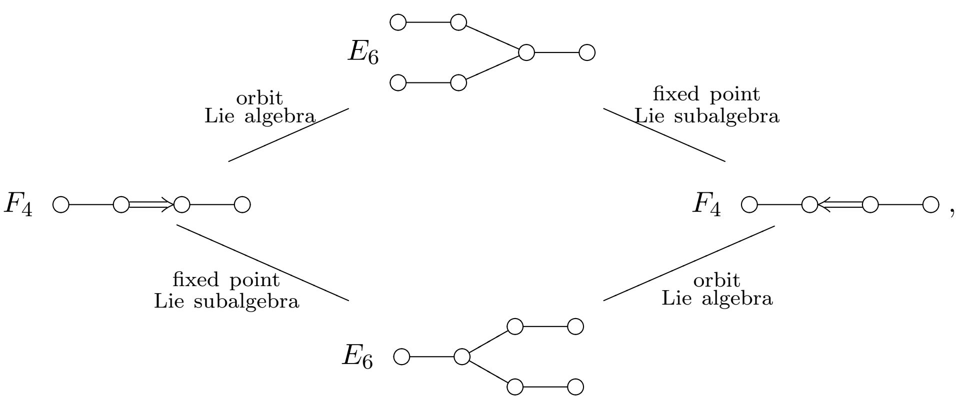
$(\mathrm{C})_1$ the fixed point Lie subalgebra $(\mathfrak{g}')^{\omega'}$ is isomorphic to the orbit Lie algebra $\breve{\mathfrak{g}}$ associated with ω; this condition implies that the index set $\breve{I}$ for $\breve{\mathfrak{g}}$ is identified with the index set $\breve{I}'$ $(= (\breve{I'}))$ for $(\mathfrak{g}')^{\omega'}$;

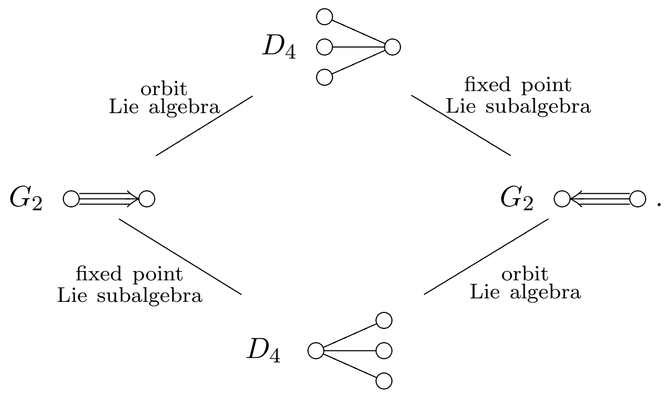
$(\mathrm{C})_2$ if we set $m_i := \min\{k \in \mathbb{Z}_{>0} \mid \omega^k(i) = i\}$, $i \in \breve{I}$, and $m_i' := \min\{k \in \mathbb{Z}_{>0} \mid (\omega')^k(i) = i\}$, $i \in \breve{I}'$, then the product $m_i \cdot m_i'$ is independent of the choice of $i \in \breve{I} \simeq \breve{I}'$.

Remark 4.2.8. Since the orbit Lie algebra $\breve{\mathfrak{g}}$ associated with ω is the (Langlands) dual Lie algebra of the fixed point Lie subalgebra $\mathfrak{g}^{\omega}$, a pair $((\mathfrak{g}, \omega), (\mathfrak{g}', \omega'))$ satisfies conditions $(\mathrm{C})_1$ and $(\mathrm{C})_2$ if and only if the reversed pair $((\mathfrak{g}', \omega'), (\mathfrak{g}, \omega))$ satisfies these.

The following three figures give the list of nontrivial pairs of automorphisms of connected Dynkin diagrams satisfying conditions $(\mathrm{C})_1$ and $(\mathrm{C})_2$:



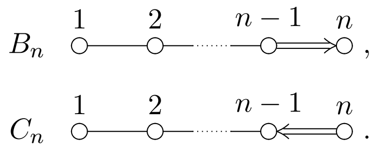




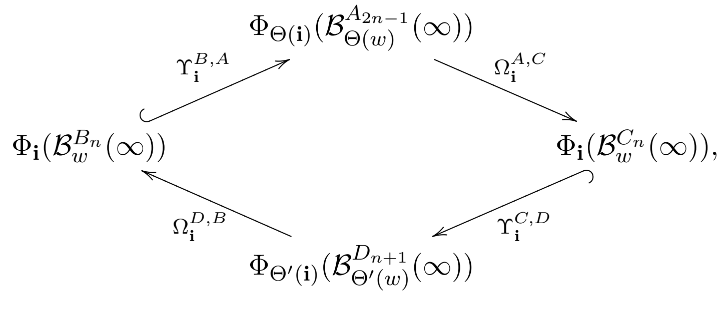
By this list and Table 1 in Sect. 4.1, we obtain the following.

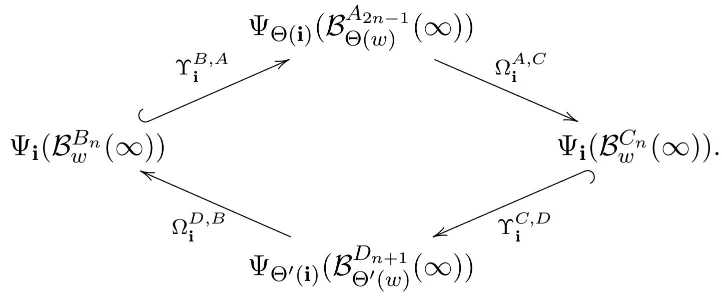
Proposition 4.2.9. *For a simply-laced semisimple Lie algebra* $\mathfrak{g}$ *with a Dynkin diagram automorphism* ω*, there exists a simply-laced semisimple Lie algebra* $\mathfrak{g}'$ *with a Dynkin diagram automorphism* ω' *such that* $((\mathfrak{g},\omega),(\mathfrak{g}',\omega'))$ *satisfies conditions* $(\mathrm{C})_1$ *and* $(\mathrm{C})_2$.

For simplicity, we consider only the pair (A_{2n-1}, D_{n+1}); we note that all the arguments below carry over to the other pairs. Denote the Weyl group of type A_{2n-1} by $W^{A_{2n-1}}$, the Schubert variety of type A_{2n-1} by $X^{A_{2n-1}}(w)$, and so on. We identify $\breve{I} \coloneqq \{1,\ldots,n\}$ with the set of vertices of the Dynkin diagram of type B_n, and also with that of type C_n as follows:



Note that the Weyl group W^{B_n} is isomorphic to the Weyl group W^{C_n}. As we have seen in Sect. 4.1, the Weyl group W^{B_n} ($\simeq W^{C_n}$) is regarded as a specific subgroup of $W^{A_{2n-1}}$ (resp., of $W^{D_{n+1}}$); let $\Theta\colon W^{B_n} \hookrightarrow W^{A_{2n-1}}$ (resp., $\Theta'\colon W^{B_n} \hookrightarrow W^{D_{n+1}}$) be the inclusion map. Take a reduced word $\mathbf{i} = (i_1,\ldots,i_r) \in \breve{I}^r$ for $w \in W^{B_n} \simeq W^{C_n}$. The reduced word $\mathbf{i}$ induces a reduced word $\Theta(\mathbf{i})$ (resp., $\Theta'(\mathbf{i})$) for $\Theta(w)$ (resp., for $\Theta'(w)$); see Sect. 4.1. By Corollary 4.1.8 and Theorem 4.2.4, we obtain the following diagrams; we denote the map $\Omega_{\mathbf{i}}\colon \Phi_{\Theta(\mathbf{i})}(\mathcal{B}^{A_{2n-1}}_{\Theta(w)}(\infty)) \to \Phi_{\mathbf{i}}(\mathcal{B}^{C_n}_w(\infty))$ by $\Omega^{A,C}_{\mathbf{i}}$, the map $\Upsilon_{\mathbf{i}}\colon \Phi_{\mathbf{i}}(\mathcal{B}^{B_n}_w(\infty)) \to \Phi_{\Theta(\mathbf{i})}(\mathcal{B}^{A_{2n-1}}_{\Theta(w)}(\infty))$ by $\Upsilon^{B,A}_{\mathbf{i}}$, and so on.





Proof of Theorem 4.2.7. We give a proof of the assertion only for the map

$$\Omega^{A,C}_{\mathbf{i}} : \Delta(X^{A_{2n-1}}(\Theta(w)), \mathcal{L}_\lambda, v^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_\lambda) \to \Delta(X^{C_n}(w), \mathcal{L}_{\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{\hat{\lambda}});$$

proofs for the other cases are similar. Because

$$\Delta(X^{A_{2n-1}}(\Theta(w)), \mathcal{L}_{2\lambda}, v^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_{2\lambda}) = 2\Delta(X^{A_{2n-1}}(\Theta(w)), \mathcal{L}_\lambda, v^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_\lambda) \text{ and}$$
$$\Delta(X^{C_n}(w), \mathcal{L}_{2\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{2\hat{\lambda}}) = 2\Delta(X^{C_n}(w), \mathcal{L}_{\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{\hat{\lambda}}),$$

it suffices to prove that the map

(4.2.6)
$$\Omega^{A,C}_{\mathbf{i}} : \Delta(X^{A_{2n-1}}(\Theta(w)), \mathcal{L}_{2\lambda}, v^{\mathrm{high}}_{\Theta(\mathbf{i})}, \tau_{2\lambda}) \to \Delta(X^{C_n}(w), \mathcal{L}_{2\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{2\hat{\lambda}})$$

is surjective. By the definitions of $\Omega_{\mathbf{i}}$ and $\Upsilon_{\mathbf{i}}$, we see that $\Omega^{A,C}_{\mathbf{i}} \circ \Upsilon^{B,A}_{\mathbf{i}}(a_1, \ldots, a_r) = (a'_1, \ldots, a'_r)$ and $\Omega^{D,B}_{\mathbf{i}} \circ \Upsilon^{C,D}_{\mathbf{i}}(a_1, \ldots, a_r) = (a''_1, \ldots, a''_r)$ for $(a_1, \ldots, a_r) \in \mathbb{R}^r$, where

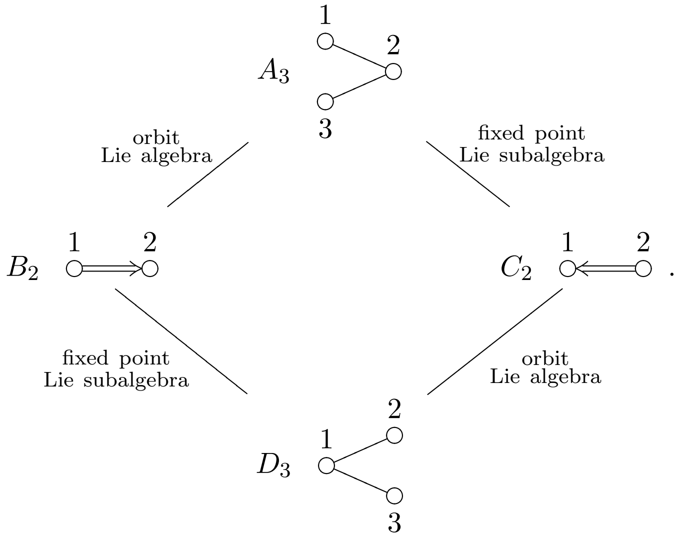
(4.2.7)
$$a'_k := \begin{cases} 2a_k & \text{if } i_k = 1, \ldots, n-1, \\ a_k & \text{if } i_k = n, \end{cases}$$
$$a''_k := \begin{cases} a_k & \text{if } i_k = 1, \ldots, n-1, \\ 2a_k & \text{if } i_k = n \end{cases}$$

for $k = 1, \ldots, r$. From these, it follows that the composite map $\Omega^{A,C}_{\mathbf{i}} \circ \Upsilon^{B,A}_{\mathbf{i}} \circ \Omega^{D,B}_{\mathbf{i}} \circ \Upsilon^{C,D}_{\mathbf{i}}$ is identical to $2 \cdot \mathrm{id}_{\mathbb{R}^r}$. This implies that the map

$$\Omega^{A,C}_{\mathbf{i}} \circ \Upsilon^{B,A}_{\mathbf{i}} \circ \Omega^{D,B}_{\mathbf{i}} \circ \Upsilon^{C,D}_{\mathbf{i}} : \Delta(X^{C_n}(w), \mathcal{L}_{\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{\hat{\lambda}}) \to \Delta(X^{C_n}(w), \mathcal{L}_{2\hat{\lambda}}, v^{\mathrm{high}}_{\mathbf{i}}, \tau_{2\hat{\lambda}})$$

doubles each of the coordinates, and hence is surjective. Therefore, the map (4.2.6) is also surjective. This proves the theorem. □

EXAMPLE 4.2.10. Consider the case $n = 2$:



Set $\mathbf{i} := (1, 2, 1) \in \breve{I}^3$; this is a reduced word for $w := s_1 s_2 s_1 \in W^{B_2} \simeq W^{C_2}$. By the definitions of Θ and Θ', we have $\Theta(\mathbf{i}) = (1, 3, 2, 1, 3)$ and $\Theta'(\mathbf{i}) = (1, 2, 3, 1)$. Then, it follows from [**42**, Sect. 1] that

$$\Phi_{\Theta(\mathbf{i})}(\mathcal{B}^{A_3}_{\Theta(w)}(\infty)) = \{(a_1, \dots, a_5) \in \mathbb{Z}^5_{\geq 0} \mid a_4 \leq a_3,\ a_5 \leq a_3\},$$
$$\Phi_{\mathbf{i}}(\mathcal{B}^{B_2}_w(\infty)) = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0} \mid a_3 \leq a_2\},$$
$$\Phi_{\mathbf{i}}(\mathcal{B}^{C_2}_w(\infty)) = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0} \mid a_3 \leq 2a_2\},$$
$$\Phi_{\Theta'(\mathbf{i})}(\mathcal{B}^{D_3}_{\Theta'(w)}(\infty)) = \{(a_1, \dots, a_4) \in \mathbb{Z}^4_{\geq 0} \mid a_4 \leq a_2 + a_3\}.$$

In addition, the maps $\Omega^{A,C}_{\mathbf{i}} \colon \mathbb{R}^5 \twoheadrightarrow \mathbb{R}^3$, $\Upsilon^{B,A}_{\mathbf{i}} \colon \mathbb{R}^3 \hookrightarrow \mathbb{R}^5$, $\Omega^{D,B}_{\mathbf{i}} \colon \mathbb{R}^4 \twoheadrightarrow \mathbb{R}^3$, and $\Upsilon^{C,D}_{\mathbf{i}} \colon \mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ are given by

$$\Omega^{A,C}_{\mathbf{i}}(a_1, \dots, a_5) := (a_1 + a_2, a_3, a_4 + a_5),$$
$$\Upsilon^{B,A}_{\mathbf{i}}(a_1, a_2, a_3) := (a_1, a_1, a_2, a_3, a_3),$$
$$\Omega^{D,B}_{\mathbf{i}}(a_1, \dots, a_4) := (a_1, a_2 + a_3, a_4),$$
$$\Upsilon^{C,D}_{\mathbf{i}}(a_1, a_2, a_3) := (a_1, a_2, a_2, a_3).$$

Through the map $\Omega^{A,C}_{\mathbf{i}}$, the conditions $a_4 \leq a_3,\ a_5 \leq a_3$ for $\Phi_{\Theta(\mathbf{i})}(\mathcal{B}^{A_3}_{\Theta(w)}(\infty))$ correspond to the condition $a_3 \leq 2a_2$ for $\Phi_{\mathbf{i}}(\mathcal{B}^{C_2}_w(\infty))$; hence we see that $\Omega^{A,C}_{\mathbf{i}}(\Phi_{\Theta(\mathbf{i})}(\mathcal{B}^{A_3}_{\Theta(w)}(\infty))) = \Phi_{\mathbf{i}}(\mathcal{B}^{C_2}_w(\infty))$. Similarly, we observe that the following equalities hold:

$$\Omega^{D,B}_{\mathbf{i}}(\Phi_{\Theta'(\mathbf{i})}(\mathcal{B}^{D_3}_{\Theta'(w)}(\infty))) = \Phi_{\mathbf{i}}(\mathcal{B}^{B_2}_w(\infty)),$$
$$\Upsilon^{B,A}_{\mathbf{i}}(\Phi_{\mathbf{i}}(\mathcal{B}^{B_2}_w(\infty))) = \{(a_1, \dots, a_5) \in \Phi_{\Theta(\mathbf{i})}(\mathcal{B}^{A_3}_{\Theta(w)}(\infty)) \mid a_1 = a_2,\ a_4 = a_5\},$$
$$\Upsilon^{C,D}_{\mathbf{i}}(\Phi_{\mathbf{i}}(\mathcal{B}^{C_2}_w(\infty))) = \{(a_1, \dots, a_4) \in \Phi_{\Theta'(\mathbf{i})}(\mathcal{B}^{D_3}_{\Theta'(w)}(\infty)) \mid a_2 = a_3\}.$$

Take $\lambda \in P^{A_3}_+ \cap (\mathfrak{t}^*)^0$ and set $\lambda_i := \langle \lambda, h^{A_3}_i \rangle$ for $i = 1, 2, 3$. The condition $\lambda \in (\mathfrak{t}^*)^0$ implies that $\lambda_1 = \lambda_3$. By the definition of $\hat{\lambda}$, it follows that $\langle \hat{\lambda}, h_1^{C_2} \rangle = 2\lambda_1 = 2\lambda_3$ and $\langle \hat{\lambda}, h_2^{C_2} \rangle = \lambda_2$. Therefore, we see from Theorem 1.4.6 and [**42**, Sect. 1] that $-\Delta(X^{A_3}(\Theta(w)), \mathcal{L}_\lambda, v_{\Theta(\mathbf{i})}^{\mathrm{high}}, \tau_\lambda)$ (resp., $-\Delta(X^{C_2}(w), \mathcal{L}_{\hat{\lambda}}, v_{\mathbf{i}}^{\mathrm{high}}, \tau_{\hat{\lambda}}))$ is given by the following conditions:

$$\begin{aligned} &(a_1, \dots, a_5) \in \mathbb{R}^5_{\geq 0},\ a_4 \leq a_3,\ a_5 \leq a_3,\ a_5 \leq \lambda_1,\ a_4 \leq \lambda_1, \\ &a_3 \leq \lambda_2 + a_4 + a_5,\ a_2 \leq \lambda_1 + a_3 - 2a_5,\ a_1 \leq \lambda_1 + a_3 - 2a_4 \end{aligned}$$

$$\begin{aligned} (\text{resp., } &(a_1, a_2, a_3) \in \mathbb{R}^3_{\geq 0},\ a_3 \leq 2a_2,\ a_3 \leq 2\lambda_1, \\ &a_2 \leq \lambda_2 + a_3,\ a_1 \leq 2\lambda_1 + 2a_2 - 2a_3). \end{aligned}$$

Hence it follows that

$$\Omega_{\mathbf{i}}^{A,C}(\Delta(X^{A_3}(\Theta(w)), \mathcal{L}_\lambda, v_{\Theta(\mathbf{i})}^{\mathrm{high}}, \tau_\lambda)) = \Delta(X^{C_2}(w), \mathcal{L}_{\hat{\lambda}}, v_{\mathbf{i}}^{\mathrm{high}}, \tau_{\hat{\lambda}}).$$

4.3. Relation with similarity of crystal bases

In this section, we study a relation of the folding procedure discussed in Sects. 4.1, 4.2 with a similarity of crystal bases.

First we review (a variant of) a similarity property of crystal bases, following [**31**, Sect. 5]. Let $\mathfrak{g}, I, P, \{\alpha_i, h_i \mid i \in I\}$ be as in Sect. 4.1, and take $m_i \in \mathbb{Z}_{>0}$ for every $i \in I$. We set $\tilde{\alpha}_i := m_i\alpha_i$, $\tilde{h}_i := \frac{1}{m_i}h_i$ for $i \in I$, and denote by $\widetilde{P} \subset P$ the set of those $\lambda \in P$ such that $\langle \lambda, \tilde{h}_i \rangle \in \mathbb{Z}$ for all $i \in I$. We impose the following condition on $\{m_i \mid i \in I\}$:

$$\tilde{\alpha}_i \in \widetilde{P} \text{ for all } i \in I.$$

Then, it is easily seen that the matrix $(\langle \tilde{\alpha}_j, \tilde{h}_i \rangle)_{i,j \in I}$ is a Cartan matrix of finite type. Let $\mathfrak{g}'$ be the corresponding semisimple Lie algebra. Note that the set $\widetilde{P}$ is identified with the weight lattice for $\mathfrak{g}'$. Let us write $\mathcal{B}(\infty)$ for $\mathfrak{g}$ as $\mathcal{B}^{\mathfrak{g}}(\infty)$, $\mathcal{B}(\lambda)$ for $\mathfrak{g}$ as $\mathcal{B}^{\mathfrak{g}}(\lambda)$, and so on.

Proposition 4.3.1 (see the proof of [**31**, Theorem 5.1]). *There exists a unique map $S_\infty \colon \mathcal{B}^{\mathfrak{g}'}(\infty) \to \mathcal{B}^{\mathfrak{g}}(\infty)$ satisfying the following conditions*:

(i) $S_\infty(b_\infty^{\mathfrak{g}'}) = b_\infty^{\mathfrak{g}}$,
(ii) $S_\infty(\tilde{e}_i b) = \tilde{e}_i^{m_i} S_\infty(b)$ *and* $S_\infty(\tilde{f}_i b) = \tilde{f}_i^{m_i} S_\infty(b)$ *for all* $i \in I$ *and* $b \in \mathcal{B}^{\mathfrak{g}'}(\infty)$, *where* $S_\infty(0) := 0$.

If $\mathfrak{g}$ is of type B_n and $(m_1, \dots, m_{n-1}, m_n) = (1, \dots, 1, 2)$, then $\mathfrak{g}'$ is the simple Lie algebra of type C_n. Conversely, if $\mathfrak{g}$ is of type C_n and $(m_1, \dots, m_{n-1}, m_n) = (2, \dots, 2, 1)$, then $\mathfrak{g}'$ is the simple Lie algebra of type B_n. Hence we obtain the following.

Corollary 4.3.2. *The following hold.*

(1) *There exists a unique map* $S_\infty^{B,C} \colon \mathcal{B}^{B_n}(\infty) \to \mathcal{B}^{C_n}(\infty)$ *satisfying the following conditions*:
 (i) $S_\infty^{B,C}(b_\infty^{B_n}) = b_\infty^{C_n}$,
 (ii) *for all* $1 \leq i \leq n-1$ *and* $b \in \mathcal{B}^{B_n}(\infty)$,

$$\begin{aligned} S_\infty^{B,C}(\tilde{e}_i b) &= \tilde{e}_i^2 S_\infty^{B,C}(b), & S_\infty^{B,C}(\tilde{f}_i b) &= \tilde{f}_i^2 S_\infty^{B,C}(b), \\ S_\infty^{B,C}(\tilde{e}_n b) &= \tilde{e}_n S_\infty^{B,C}(b), & S_\infty^{B,C}(\tilde{f}_n b) &= \tilde{f}_n S_\infty^{B,C}(b), \end{aligned}$$

where $S_\infty^{B,C}(0) \coloneqq 0$.

(2) *There exists a unique map* $S_\infty^{C,B} \colon \mathcal{B}^{C_n}(\infty) \to \mathcal{B}^{B_n}(\infty)$ *satisfying the following conditions*:

(i) $S_\infty^{C,B}(b_\infty^{C_n}) = b_\infty^{B_n}$,

(ii) *for all* $1 \le i \le n-1$ *and* $b \in \mathcal{B}^{C_n}(\infty)$,

$$S_\infty^{C,B}(\tilde{e}_i b) = \tilde{e}_i S_\infty^{C,B}(b), \qquad S_\infty^{C,B}(\tilde{f}_i b) = \tilde{f}_i S_\infty^{C,B}(b),$$
$$S_\infty^{C,B}(\tilde{e}_n b) = \tilde{e}_n^2 S_\infty^{C,B}(b), \qquad S_\infty^{C,B}(\tilde{f}_n b) = \tilde{f}_n^2 S_\infty^{C,B}(b),$$

where $S_\infty^{C,B}(0) \coloneqq 0$.

It is easily seen that the composite map $S_\infty^{C,B} \circ S_\infty^{B,C}$ is identical to the map $S_2^B \colon \mathcal{B}^{B_n}(\infty) \to \mathcal{B}^{B_n}(\infty)$ given by the following conditions:

(i) $S_2^B(b_\infty^{B_n}) = b_\infty^{B_n}$,

(ii) $S_2^B(\tilde{e}_i b) = \tilde{e}_i^2 S_2^B(b)$ and $S_2^B(\tilde{f}_i b) = \tilde{f}_i^2 S_2^B(b)$ for all $1 \le i \le n$ and $b \in \mathcal{B}^{B_n}(\infty)$, where $S_2^B(0) \coloneqq 0$,

(iii) $\varepsilon_i(S_2^B(b)) = 2\varepsilon_i(b)$ and $\varphi_i(S_2^B(b)) = 2\varphi_i(b)$ for all $1 \le i \le n$ and $b \in \mathcal{B}^{B_n}(\infty)$;

see also [**31**, Theorem 3.1]. A similar result holds for the composite map $S_\infty^{B,C} \circ S_\infty^{C,B} \colon \mathcal{B}^{C_n}(\infty) \to \mathcal{B}^{C_n}(\infty)$. Recall that the Weyl group of type B_n is isomorphic to that of type C_n. By conditions (i) and (ii) in Corollary 4.3.2 (1) (resp., (2)), it follows that

$$S_\infty^{B,C}(\mathcal{B}_w^{B_n}(\infty)) \subset \mathcal{B}_w^{C_n}(\infty) \text{ (resp., } S_\infty^{C,B}(\mathcal{B}_w^{C_n}(\infty)) \subset \mathcal{B}_w^{B_n}(\infty))$$

for all $w \in W^{B_n} \simeq W^{C_n}$.

PROPOSITION 4.3.3. *Let* $\mathbf{i} = (i_1, \ldots, i_r) \in \{1, \ldots, n\}^r$ *be a reduced word for* $w \in W^{B_n} \simeq W^{C_n}$. *Then, the following equalities hold for all* $b \in \mathcal{B}_w^{B_n}(\infty)$ *and* $b' \in \mathcal{B}_w^{C_n}(\infty)$:

$$\Phi_{\mathbf{i}}(S_\infty^{B,C}(b)) = \Omega_{\mathbf{i}}^{A,C} \circ \Upsilon_{\mathbf{i}}^{B,A}(\Phi_{\mathbf{i}}(b)), \quad \Phi_{\mathbf{i}}(S_\infty^{C,B}(b')) = \Omega_{\mathbf{i}}^{D,B} \circ \Upsilon_{\mathbf{i}}^{C,D}(\Phi_{\mathbf{i}}(b')),$$
$$\Psi_{\mathbf{i}}(S_\infty^{B,C}(b)) = \Omega_{\mathbf{i}}^{A,C} \circ \Upsilon_{\mathbf{i}}^{B,A}(\Psi_{\mathbf{i}}(b)), \quad \Psi_{\mathbf{i}}(S_\infty^{C,B}(b')) = \Omega_{\mathbf{i}}^{D,B} \circ \Upsilon_{\mathbf{i}}^{C,D}(\Psi_{\mathbf{i}}(b')).$$

PROOF. We prove the assertion only for $S_\infty^{B,C}$; a proof of the assertion for $S_\infty^{C,B}$ is similar. By equation (4.2.7) in the proof of Theorem 4.2.7, it suffices to prove that

$$\varepsilon_i(S_\infty^{B,C}(b)) = \begin{cases} 2\varepsilon_i(b) & \text{if } i = 1, \ldots, n-1, \\ \varepsilon_i(b) & \text{if } i = n, \end{cases}$$
$$\varepsilon_i(S_\infty^{B,C}(b)^*) = \begin{cases} 2\varepsilon_i(b^*) & \text{if } i = 1, \ldots, n-1, \\ \varepsilon_i(b^*) & \text{if } i = n \end{cases}$$

for all $b \in \mathcal{B}^{B_n}(\infty)$. The assertion for $\varepsilon_i(S_\infty^{B,C}(b)^*)$ follows immediately from the proof of [**31**, Theorem 5.1]. We will prove the assertion for $\varepsilon_i(S_\infty^{B,C}(b))$. If $i = n$, then this is obvious by condition (ii) in Corollary 4.3.2 (1). For $i = 1, \ldots, n-1$, we see by condition (ii) in Corollary 4.3.2 (1) that

$$\tilde{e}_i^{2\varepsilon_i(b)} S_\infty^{B,C}(b) = S_\infty^{B,C}(\tilde{e}_i^{\varepsilon_i(b)} b) \neq 0.$$

Suppose, for a contradiction, that $\tilde{e}_i^{2\varepsilon_i(b)+1} S_\infty^{B,C}(b) \neq 0$. Then, we have

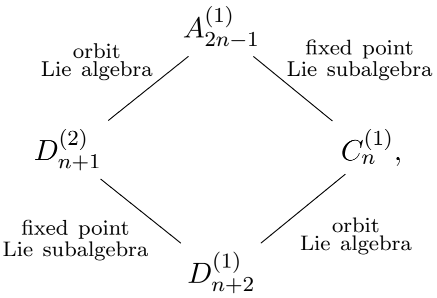
$$
\begin{aligned}
\tilde{e}_i^{2\varepsilon_i(b)+1} S_2^B(b) &= \tilde{e}_i^{2\varepsilon_i(b)+1} S_\infty^{C,B} \circ S_\infty^{B,C}(b) \\
&= S_\infty^{C,B}(\tilde{e}_i^{2\varepsilon_i(b)+1} S_\infty^{B,C}(b)) \\
&\text{(by condition (ii) in Corollary 4.3.2 (2))} \\
&\neq 0,
\end{aligned}
$$

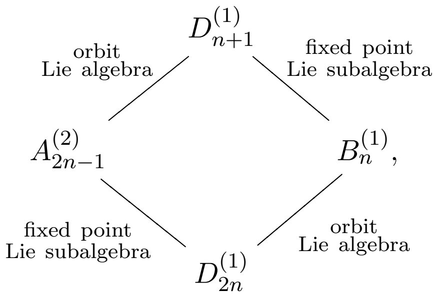
which contradicts condition (iii) for S_2^B above. Therefore, the equality $\tilde{e}_i^{2\varepsilon_i(b)+1} S_\infty^{B,C}(b) = 0$ holds. From these, we deduce that $\varepsilon_i(S_\infty^{B,C}(b)) = 2\varepsilon_i(b)$. This proves the proposition. □

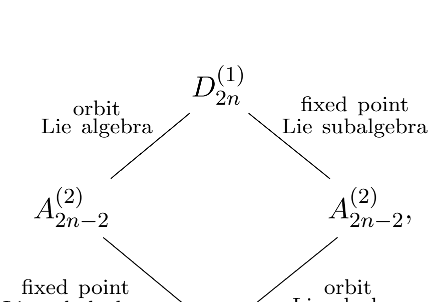
Remark 4.3.4. Proposition 4.3.3 is naturally extended to an arbitrary pair $((\mathfrak{g}, \omega), (\mathfrak{g}', \omega'))$ satisfying conditions $(\mathrm{C})_1$ and $(\mathrm{C})_2$ in Sect. 4.2.

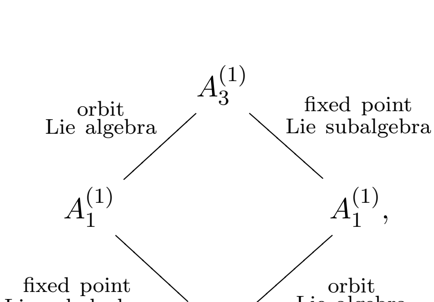
4.4. Case of affine Lie algebras

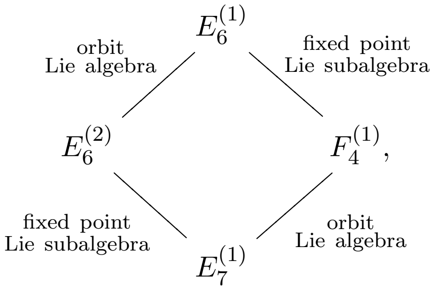
Our results (Corollary 4.1.11 and Theorem 4.2.7) in this chapter are naturally extended to symmetrizable Kac-Moody algebras. The following figures give the list of nontrivial pairs of automorphisms of simply-laced affine Dynkin diagrams satisfying conditions $(\mathrm{C})_1$ and $(\mathrm{C})_2$ in Sect. 4.2; we have used Kac's notation, and some automorphisms of $A_n^{(1)}$ have been omitted.

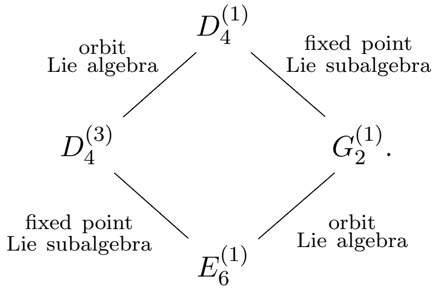









$E_6^{(1)}$
orbit
Lie algebra
fixed point
Lie subalgebra
$E_6^{(2)}$
$F_4^{(1)}$,
fixed point
Lie subalgebra
orbit
Lie algebra
$E_7^{(1)}$


$D_4^{(1)}$
orbit
Lie algebra
fixed point
Lie subalgebra
$D_4^{(3)}$
$G_2^{(1)}$.
fixed point
Lie subalgebra
orbit
Lie algebra
$E_6^{(1)}$

Bibliography

[1] D. Anderson, Okounkov bodies and toric degenerations, Math. Ann. **356** (2013), 1183–1202.

[2] P. Baumann, The canonical basis and the quantum Frobenius morphism, preprint 2012, arXiv:1201.0303v2.

[3] A. Berenstein and D. Kazhdan, Geometric and unipotent crystals II: From unipotent bicrystals to crystal bases, in Quantum Groups, Contemp. Math. Vol. 433, Amer. Math. Soc., Providence, RI, 2007, 13–88.

[4] A. Berenstein and A. Zelevinsky, String bases for quantum groups of type A_r, in I. M. Gelfand Seminar, Adv. Soviet Math. Vol. 16, Part 1, Amer. Math. Soc., Providence, RI, 1993, 51–89.

[5] A. Berenstein and A. Zelevinsky, Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics, Duke Math. J. **82** (1996), 473–502.

[6] A. Berenstein and A. Zelevinsky, Tensor product multiplicities, canonical bases and totally positive varieties, Invent. Math. **143** (2001), 77–128.

[7] D. Cox, J. Little, and H. Schenck, Toric Varieties, Graduate Studies in Mathematics Vol. 124, Amer. Math. Soc., Providence, RI, 2011.

[8] X. Fang, G. Fourier, and P. Littelmann, Essential bases and toric degenerations arising from birational sequences, Adv. Math. **312** (2017), 107–149.

[9] E. Feigin, G. Fourier, and P. Littelmann, Favourable modules: filtrations, polytopes, Newton-Okounkov bodies and flat degenerations, Transform. Groups **22** (2017), 321–352.

[10] J. Fuchs, U. Ray, and C. Schweigert, Some automorphisms of generalized Kac-Moody algebras, J. Algebra **191** (1997), 518–540.

[11] J. Fuchs, B. Schellekens, and C. Schweigert, From Dynkin diagram symmetries to fixed point structures, Comm. Math. Phys. **180** (1996), 39–97.

[12] N. Fujita, Newton-Okounkov bodies for Bott-Samelson varieties and string polytopes for generalized Demazure modules, preprint 2015, arXiv:1503.08916v2.

[13] N. Fujita, Folding procedure for Newton-Okounkov polytopes of Schubert varieties, preprint 2017, arXiv:1703.03144v1. To appear in Comm. Algebra.

[14] N. Fujita and S. Naito, Newton-Okounkov convex bodies of Schubert varieties and polyhedral realizations of crystal bases, Math. Z. **285** (2017), 325–352.

[15] N. Fujita and H. Oya, A comparison of Newton-Okounkov polytopes of Schubert varieties, J. Lond. Math. Soc. (2) **96** (2017), 201–227.

[16] C. Geiss, B. Leclerc, and J. Schröer, Preprojective algebras and cluster algebras, in Trends in Representation Theory of Algebras and Related Topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, 253–283.

[17] M. Harada and K. Kaveh, Integrable systems, toric degenerations, and Okounkov bodies, Invent. Math. **202** (2015), 927–985.

[18] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics Vol. 52, Springer, New York, 1977.

[19] J. C. Jantzen, Lectures on Quantum Groups, Graduate Studies in Mathematics Vol. 6, Amer. Math. Soc., Providence, RI, 1996.

[20] J. C. Jantzen, Representations of Algebraic Groups, 2nd ed., Math. Surveys Monographs Vol. 107, Amer. Math. Soc., Providence, RI, 2003.

[21] A. Joseph, A decomposition theorem for Demazure crystals, J. Algebra **265** (2003), 562–578.

[22] S.-Y. Jow, Okounkov bodies and restricted volumes along very general curves, Adv. Math. **223** (2010), 1356–1371.
[23] V. G. Kac, Infinite-Dimensional Lie Algebras, 3rd ed., Cambridge University Press, Cambridge, 1990.
[24] S. J. Kang, S. J. Oh, and E. Park, Perfect bases for integrable modules over generalized Kac-Moody algebras, Algebr. Represent. Theory **14** (2011), 571–587.
[25] S. J. Kang, S. J. Oh, and E. Park, Categorification of quantum generalized Kac-Moody algebras and crystal bases, Internat. J. Math. **23** (2012), 1250116.
[26] M. Kashiwara, Crystallizing the q-analogue of universal enveloping algebras, Comm. Math. Phys. **133** (1990), 249–260.
[27] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. **63** (1991), 465–516.
[28] M. Kashiwara, Global crystal bases of quantum groups, Duke Math. J. **69** (1993), 455–485.
[29] M. Kashiwara, The crystal bases and Littelmann's refined Demazure character formula, Duke Math. J. **71** (1993), 839–858.
[30] M. Kashiwara, On crystal bases, in Representations of Groups (Banff, AB, 1994), CMS Conf. Proc. Vol. 16, Amer. Math. Soc., Providence, RI, 1995, 155–197.
[31] M. Kashiwara, Similarity of crystal bases, Contemp. Math. **194** (1996), 177–186.
[32] K. Kaveh, Crystal bases and Newton-Okounkov bodies, Duke Math. J. **164** (2015), 2461–2506.
[33] K. Kaveh and A. G. Khovanskii, Convex bodies and algebraic equations on affine varieties, preprint 2008, arXiv:0804.4095v1; a short version with title *Algebraic equations and convex bodies* appeared in Perspectives in Analysis, Geometry, and Topology, Progr. Math. Vol. 296, Birkhäuser/Springer, New York, 2012, 263–282.
[34] K. Kaveh and A. G. Khovanskii, Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory, Ann. of Math. **176** (2012), 925–978.
[35] M. Khovanov and A. D. Lauda, A diagrammatic approach to categorification of quantum groups I, Represent. Theory **13** (2009), 309–347.
[36] M. Khovanov and A. D. Lauda, A diagrammatic approach to categorification of quantum groups II, Trans. Amer. Math. Soc. **363** (2011), 2685–2700.
[37] V. Kiritchenko, Newton-Okounkov polytopes of flag varieties, Transform. Groups **22** (2017), 387–402.
[38] S. Kumar, Kac-Moody Groups, their Flag Varieties and Representation Theory, Progr. Math. Vol. 204, Birkhäuser Boston, Boston, 2002.
[39] A. Küronya, V. Lozovanu, and C. Maclean, Convex bodies appearing as Okounkov bodies of divisors, Adv. Math. **229** (2012), 2622–2639.
[40] A. D. Lauda and M. Vazirani, Crystals from categorified quantum groups, Adv. Math. **228** (2011), 803–861.
[41] R. Lazarsfeld and M. Mustata, Convex bodies associated to linear series, Ann. Sci. de I'ENS **42** (2009), 783–835.
[42] P. Littelmann, Cones, crystals, and patterns, Transform. Groups **3** (1998), 145–179.
[43] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. **3** (1990), 447–498.
[44] G. Lusztig, Quivers, perverse sheaves, and quantized enveloping algebras, J. Amer. Math. Soc. **4** (1991), 365–421.
[45] G. Lusztig, Semicanonical bases arising from enveloping algebras, Adv. Math. **151** (2000), 129–139.
[46] G. Lusztig, Introduction to Quantum Groups, reprint of the 1994 edition, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010.
[47] P. J. McNamara, Folding KLR algebras, preprint 2016, arXiv:1603.05768v1.
[48] S. Naito and D. Sagaki, Crystal bases and diagram automorphisms, in Representation Theory of Algebraic Groups and Quantum Groups, Adv. Stud. Pure Math. Vol. 40, Math Soc. Japan, Tokyo, 2004, 321–341.
[49] S. Naito and D. Sagaki, Crystal base elements of an extremal weight module fixed by a diagram automorphism, Algebr. Represent. Theory **8** (2005), 689–707.

[50] T. Nakashima, Polyhedral realizations of crystal bases for integrable highest weight modules, J. Algebra **219** (1999), 571–597.

[51] T. Nakashima, Polytopes for crystallized Demazure modules and extremal vectors, Comm. Algebra **30** (2002), 1349–1367.

[52] T. Nakashima, Decorated geometric crystals, polyhedral and monomial realizations of crystal bases, in Recent Developments in Algebraic and Combinatorial Aspects of Representation Theory, Contemp. Math. Vol. 602, Amer. Math. Soc., Providence, RI, 2013, 143–163.

[53] T. Nakashima and A. Zelevinsky, Polyhedral realizations of crystal bases for quantized Kac-Moody algebras, Adv. Math. **131** (1997), 253–278.

[54] A. Okounkov, Brunn-Minkowski inequality for multiplicities, Invent. Math. **125** (1996), 405–411.

[55] A. Okounkov, Multiplicities and Newton polytopes, in Kirillov's Seminar on Representation Theory, Amer. Math. Soc. Transl. Ser. 2 Vol. 181, Adv. Math. Sci. Vol. 35, Amer. Math. Soc., Providence, RI, 1998, 231–244.

[56] A. Okounkov, Why would multiplicities be log-concave?, in The Orbit Method in Geometry and Physics, Progr. Math. Vol. 213, Birkhäuser Boston, Boston, 2003, 329–347.

[57] R. Rouquier, 2-Kac-Moody algebras, preprint 2008, arXiv:0812.5023v1.

[58] D. Sagaki, Crystal bases, path models, and a twining character formula for Demazure modules, Publ. Res. Inst. Math. Sci. **38** (2002), 245–264.

[59] A. Savage, Quiver varieties and Demazure modules, Math. Ann. **335** (2006), 31–46.

[60] C. S. Seshadri, Introduction to the Theory of Standard Monomials, 2nd ed., Texts and Readings in Mathematics Vol. 46, Hindustan Book Agency, New Delhi, 2014.

[61] T. A. Springer, Linear Algebraic Groups, reprint of the 1998 second edition, Modern Birkhäuser Classics, Birkhäuser Boston, Boston, 2009.