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Specialization of nonsymmetric Macdonald polynomials at $t = \infty$ and level-zero representations of quantum affine algebras

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Abstract

In this paper, we establish an explicit description of the specialization $E_{w\lambda}(q,\infty)$ of the nonsymmetric Macdonald polynomials $E_{w\lambda}(q,t)$ at $t=\infty$ in terms of the quantum Bruhat graph, where λ is a dominant weight and w is an element of a finite Weyl group W. As an application of this explicit formula, we give a representation-theoretic interpretation of the specialization $E_{w_\circ\lambda}(q,\infty)$ in terms of the Demazure submodule $V_{w_\circ}^{-}(\lambda)$ of the level-zero extremal weight module $V(\lambda)$ over a quantum affine algebra of untwisted type; here, $w_\circ$ denotes the longest element of the finite Weyl group W. Also, we give a representation-theoretic proof of Cherednik-Orr's recursion formula of Demazure type for the specialization at $t=\infty$ of nonsymmetric Macdonald polynomials.

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Chapter 1

Introduction

Symmetric Macdonald polynomials with two parameters q and t were introduced by Macdonald [M2] as a family of orthogonal symmetric polynomials, which include as special or limiting cases almost all the classical families of orthogonal symmetric polynomials. This family of polynomials are characterized in terms of the double affine Hecke algebra (DAHA) introduced by Cherednik ([C1], [C2]). In fact, there exists another family of orthogonal polynomials, called nonsymmetric Macdonald polynomials, which are simultaneous eigenfunctions of Y-operators acting on the polynomial representation of the DAHA; by "symmetrizing" nonsymmetric Macdonald polynomials, we obtain symmetric Macdonald polynomials (see [M1]).

Based on the characterization above of nonsymmetric Macdonald polynomials, Ram-Yip [RY] obtained a combinatorial formula expressing symmetric or nonsymmetric Macdonald polynomials associated to an arbitrary untwisted affine root system; this formula is described in terms of alcove walks, which are certain strictly combinatorial objects. In addition, Orr-Shimozono [OS] refined the Ram-Yip formula above, and generalized it to an arbitrary affine root system (including the twisted case); also, they specialized their formula at $t = 0$, $t = \infty$, $q = 0$, and $q = \infty$.

As for representation-theoretic interpretations of the specialization of symmetric or nonsymmetric Macdonald polynomials at $t = 0$, we know the following. Ion [I] proved that for a dominant integral weight λ and an element x of a finite Weyl group W, the specialization $E_{x\lambda}(q, 0)$ of the nonsymmetric Macdonald polynomial $E_{x\lambda}(q, t)$ at $t = 0$ is equal to the graded character of a certain Demazure submodule of an irreducible highest weight module over an affine Lie algebra of untwisted simply-laced type or twisted non-simply-laced type. As for the relation with level-zero representations of quantum affine algebras, Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS2] proved that for a dominant integral weight λ, the set $\mathrm{QLS}(\lambda)$ of all quantum Lakshmibai-Seshadri (QLS) paths of shape λ provides a realization of the crystal basis of a special quantum Weyl module over a quantum affine algebra $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ (without degree operator) of an arbitrary untwisted type, and also proved that its graded character equals the specialization $E_{w_\circ\lambda}(q, 0)$ at $t = 0$, where $w_\circ$ denotes the longest element of W. Here a QLS path is obtained from an affine level-zero Lakshmibai-Seshadri path through the projection $\mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{aff}} \to \mathbb{R} \otimes_{\mathbb{Z}} P$, which factors the null root δ of an affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$, and is described in terms of (the parabolic version of) the quantum Bruhat graph, introduced by Brenti-Fomin-Postnikov [BFP]; the set of QLS paths is endowed with an affine crystal structure in a way similar to the one for the set of ordinary LS paths introduced by Littelmann [L1]. Moreover, Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS3] obtained a formula for the specialization $E_{x\lambda}(q,0)$, $x \in W$, at $t=0$ in an arbitrary untwisted affine type, which is described in terms of QLS paths of shape λ, and also proved that the specialization $E_{x\lambda}(q,0)$ is just the graded character of a certain Demazure-type submodule of the special quantum Weyl module. The crucial ingredient in the proof of this result is a graded character formula obtained in [NS4] for the Demazure submodule $V_e^-(\lambda)$ of the level-zero extremal weight module $V(\lambda)$ of extremal weight λ over a quantum affine algebra $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$, where e is the identity element of W. More precisely, in [NS4], Naito and Sagaki proved that the graded character $\mathrm{gch}\, V_e^-(\lambda)$ of $V_e^-(\lambda) \subset V(\lambda)$ is identical to $\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} E_{w_\circ\lambda}(q^{-1},0)$, where λ is a dominant integral weight of the from $\sum_{i\in I} m_i\varpi_i$, with ϖ_i, $i \in I$, the fundamental weights. The graded character $\mathrm{gch}\, V_e^-(\lambda)$ is obtained from the ordinary character of $V_e^-(\lambda)$ by replacing e^δ by q, with δ the null root of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$.

The purpose of this thesis is to establish the relation between the specialization $E_{x\lambda}(q,\infty)$ for $x \in W$ of the nonsymmetric Macdonald polynomial $E_{x\lambda}(q,t)$ at $t=\infty$ and the level-zero extremal weight module $V(\lambda)$ over $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$. First, we prove an explicit formula for the specialization $E_{x\lambda}(q,\infty)$, which is described in terms of (a specific subset $\mathrm{QLS}^{x\lambda,\infty}(\lambda)$ of) $\mathrm{QLS}(\lambda)$. By using this formula, we give a representation-theoretic interpretation of the specialization $E_{w_\circ\lambda}(q,\infty)$ in terms of the Demazure submodule $V_{w_\circ}^-(\lambda)$ of $V(\lambda)$. More precisely, we prove that the graded character $\mathrm{gch}\, V_{w_\circ}^-(\lambda)$ of $V_{w_\circ}^-(\lambda)$ is identical to $\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} E_{w_\circ\lambda}(q,\infty)$, where λ is a dominant integral weight of the form $\sum_{i\in I} m_i\varpi_i$. Next, we define a certain (finite-dimensional) quotient module $V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)$, and prove that the graded character $\mathrm{gch}\, V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)$ of $V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)$ is identical to $E_{w_\circ\lambda}(q,\infty)$. Also, as an application of the explicit formula above, we give a representation-theoretic (or rather, crystal-theoretic) proof of Cherednik-Orr's recursion formula of Demazure type for the specialization $E_{x\lambda}(q,\infty)$, $x \in W$; in the course of the proof of this result, we obtain a recursive relation for the subsets $\mathrm{QLS}^{x\lambda,\infty}(\lambda)$, $x \in W$, of $\mathrm{QLS}(\lambda)$, which determines these subsets inductively in terms of the tilted Bruhat order by starting with the equality $\mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda) = \mathrm{QLS}(\lambda)$.

This thesis is organized as follows. In Chapter 2, we fix our notation, and review the definitions and some of the properties of nonsymmetric Macdonald polynomials and level-zero extremal weight modules over $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$. In Chapter 3, we first prove an explicit formula for the specialization $E_{x\lambda}(q,\infty)$, $x \in W$, described in terms of QLS paths. Next, using this result, we give a representation-theoretic interpretation of the specialization $E_{w_\circ\lambda}(q,\infty)$ in terms of the Demazure submodule $V_{w_\circ}^-(\lambda)$ of $V(\lambda)$. In Chapter 4, we give a crystal-theoretic proof of Cherednik-Orr's recursion formula of Demazure type for the specialization $E_{x\lambda}(q,\infty)$, $x \in W$.

Chapter 2

Preliminaries

2.1 Root systems of finite types

Throughout this thesis, we use the following notation.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, I the vertex set for the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i\}_{i\in I}$ (resp., $\{\alpha_i^\vee\}_{i\in I}$) the set of all simple roots (resp., coroots) of $\mathfrak{g}$, $\mathfrak{h} = \bigoplus_{i\in I}\mathbb{C}\alpha_i^\vee$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^* = \bigoplus_{i\in I}\mathbb{C}\alpha_i$ the dual space of $\mathfrak{h}$, and $\mathfrak{h}^*_{\mathbb{R}} = \bigoplus_{i\in I}\mathbb{R}\alpha_i$ the real form of $\mathfrak{h}^*$; the canonical pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ is denoted by $\langle\cdot,\cdot\rangle : \mathfrak{h}^*\times\mathfrak{h}\to\mathbb{C}$. Let $Q = \sum_{i\in I}\mathbb{Z}\alpha_i \subset \mathfrak{h}^*_{\mathbb{R}}$ denote the root lattice of $\mathfrak{g}$, $Q^\vee = \sum_{i\in I}\mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_{\mathbb{R}}$ the coroot lattice of $\mathfrak{g}$, and $P = \sum_{i\in I}\mathbb{Z}\varpi_i \subset \mathfrak{h}^*_{\mathbb{R}}$ the weight lattice of $\mathfrak{g}$, where the ϖ_i, $i\in I$, are the fundamental weights for $\mathfrak{g}$, i.e., $\langle\varpi_i,\alpha_j^\vee\rangle = \delta_{ij}$ for $i,j\in I$; we set $P^+ := \sum_{i\in I}\mathbb{Z}_{\geq 0}\varpi_i$, and call an elements λ of P^+ a dominant weight. Let us denote by Δ the set of all roots and by Δ^+ (resp., Δ^-) the set of all positive (resp., negative) roots. Also, let $W := \langle s_i \mid i\in I\rangle$ be the Weyl group of $\mathfrak{g}$, where s_i, $i\in I$, are the simple reflections acting on $\mathfrak{h}^*$ and on $\mathfrak{h}$:

$$\begin{aligned} s_i\nu &= \nu - \langle\nu,\alpha_i^\vee\rangle\alpha_i, \quad \nu\in\mathfrak{h}^*, \\ s_ih &= h - \langle\alpha_i,h\rangle\alpha_i^\vee, \quad h\in\mathfrak{h}; \end{aligned}$$

we denote the identity element and the longest element of W by e and $w_\circ$, respectively. If $\alpha\in\Delta$ is written as $\alpha = w\alpha_i$ for $w\in W$ and $i\in I$, then we define $\alpha^\vee$ to be $w\alpha_i^\vee$; note that $s_\alpha = s_{\alpha^\vee} = ws_iw^{-1}$. For $u\in W$, the length of u is denoted by $\ell(u)$, which equals $\#(\Delta^+\cap u^{-1}\Delta^-)$.

2.2 Nonsymmetric Macdonald polynomials

In this section, we recall the definition of nonsymmetric Macdonald polynomials in untwisted affine types. Although nonsymmetric Macdonald polynomials have at most six parameters (q and five t's) in general, we consider nonsymmetric Macdonald polynomials with two parameters q and t since we focus on the specialization at $t=\infty$ (see [M1] for the general case).

For $\mu\in P$, we denote by $\underline{v}(\mu)$ the shortest element in W such that $\underline{v}(\mu)\mu$ is an antidominant weight. Then we define a partial order $<$ on P as follows. For $\mu,\nu\in W$, $\mu\geq\nu$ if either of the conditions (1), (2) below holds:

(1) $0 \neq \underline{v}(\mu)\mu - \underline{v}(\nu)\nu \in \sum_{i\in I} \mathbb{Z}_{\leq 0}\alpha_i$.

(2) $\underline{v}(\mu)\mu = \underline{v}(\nu)\nu$, and $\underline{v}(\nu) \geq \underline{v}(\mu)$ with respect to the Bruhat order on W.

Let $K = \mathbb{Q}(q,t)$ be the rational function field in indeterminates q and t over $\mathbb{Q}$. We denote by A the group algebra of P over K, and by $\widehat{A}$ the formal completion of A. We define an involution $\overline{\cdot}$ on K by $\overline{q} = q^{-1}$ and $\overline{t} = t^{-1}$, and set $\overline{f} := \sum_{\mu\in P} \overline{f_\mu} e^{-\mu}$ for $f = \sum_{\mu\in P} f_\mu e^{\mu}$, with $f_\mu \in K$. Also, for $\nu \in P$ and $f = \sum_{\mu\in P} f_\mu e^{\mu} \in \widehat{A}$, with $f_\mu \in K$, we set

$$[f : e^{\nu}] := f_\nu \in K, \quad \mathrm{ct}(f) := f_0 \in K.$$

Now we set

$$\nabla := \prod_{\alpha\in\Delta^+} \prod_{j=0}^{\infty} \frac{(1-e^{\alpha}q^{j})(1-e^{-\alpha}q^{j+1})}{(1-e^{\alpha}tq^{j})(1-e^{-\alpha}tq^{j+1})} \in \widehat{A},$$

and define a scalar product $(\cdot,\cdot) : A \times A \to K$ by $(f,g) := \mathrm{ct}(f\overline{g}\nabla)/\mathrm{ct}(\nabla)$, $f,g \in A$. Indeed, this scalar product is a nondegenerate, Hermitian sesquilinear form; namely, $(kf,g) = k(f,g) = (f,\overline{k}g)$ and $(f,g) = \overline{(g,f)}$ for $f,g \in A$ and $k \in K$.

It is known that there exists a (unique) basis $\{E_\mu(q,t)\}_{\mu\in P}$ of A over K satisfying the conditions:

(1) $[E_\mu(q,t) : e^{\mu}] = 1$, and if $[E_\mu(q,t) : e^{\nu}] \neq 0$, then $\mu \geq \nu$;

(2) for $\nu \in P$ such that $\nu < \mu$, $(E_\mu, e^{\nu}) = 0$.

The basis elements $E_\mu(q,t)$, $\mu \in P$, are called the nonsymmetric Macdonald polynomials. We denote by $E_\mu(q,\infty)$ the specialization

$$\lim_{t\to\infty} E_\mu(q,t) := \sum_{\nu\in P} \lim_{t\to\infty} [E_\mu(q,t) : e^{\nu}] e^{\nu}; \tag{2.2.1}$$

this specialization is studied in [CO] in simply-laced types and twisted non-simply-laced types.

2.3 Extremal weight modules over the quantum affine algebra $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$

In this section, we recall the definition of extremal weight vectors and extremal weight modules over the quantum affine algebra $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$, and some of the basic properties of extremal weight modules.

First, we fix the notation for untwisted affine root data; see §3.4.1 for more details. Let $\mathfrak{g}_{\mathrm{aff}}$ be the untwisted affine Lie algebra over $\mathbb{C}$ associated to the finite-dimensional simple Lie algebra $\mathfrak{g}$, and $\mathfrak{h}_{\mathrm{aff}} = \left(\bigoplus_{j\in I_{\mathrm{aff}}} \mathbb{C}\alpha_j^{\vee}\right) \oplus \mathbb{C}D$ its Cartan subalgebra, where $\left\{\alpha_j^{\vee}\right\}_{j\in I_{\mathrm{aff}}} \subset \mathfrak{h}_{\mathrm{aff}}$ is the set of simple coroots, with $I_{\mathrm{aff}} = I \sqcup \{0\}$, and $D \in \mathfrak{h}_{\mathrm{aff}}$ is the degree operator. We denote by $\left\{\alpha_j\right\}_{j\in I_{\mathrm{aff}}} \subset \mathfrak{h}_{\mathrm{aff}}^{*}$ the set of simple roots, and by $\Lambda_j \in \mathfrak{h}_{\mathrm{aff}}^{*}$, $j \in I_{\mathrm{aff}}$, the fundamental weights. Note that $\langle \alpha_j, D\rangle = \delta_{j,0}$ and $\langle \Lambda_j, D\rangle = 0$ for $j \in I_{\mathrm{aff}}$, where $\langle\cdot\,,\cdot\rangle : \mathfrak{h}_{\mathrm{aff}}^{*} \times \mathfrak{h}_{\mathrm{aff}} \to \mathbb{C}$ denotes the canonical pairing between $\mathfrak{h}_{\mathrm{aff}}$ and $\mathfrak{h}_{\mathrm{aff}}^{*} := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{aff}}, \mathbb{C})$. Also, let

$\delta = \sum_{j \in I_{\mathrm{aff}}} a_j \alpha_j \in \mathfrak{h}_{\mathrm{aff}}^*$ and $c = \sum_{j \in I_{\mathrm{aff}}} a_j^\vee \alpha_j^\vee \in \mathfrak{h}_{\mathrm{aff}}$ denote the null root and the canonical central element of $\mathfrak{g}_{\mathrm{aff}}$, respectively. We take a weight lattice P_{aff} for $\mathfrak{g}_{\mathrm{aff}}$ as follows: $P_{\mathrm{aff}} = \left(\bigoplus_{j \in I_{\mathrm{aff}}} \mathbb{Z}\Lambda_j\right) \oplus \mathbb{Z}\delta \subset \mathfrak{h}_{\mathrm{aff}}^*$.

In what follows, we mainly follow the notation of [NS4, §3]. Let M be an integrable $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module. A vector $u \in M$ of weight $\lambda \in P_{\mathrm{aff}}$ is said to be extremal (see [Kas2, §3.1]) if there exists a family $\{v_x\}_{x \in W_{\mathrm{aff}}}$ of weight vectors satisfying the following:

(1) $v_e = v$;

(2) for every $j \in I_{\mathrm{aff}}$ and $x \in W_{\mathrm{aff}}$ such that $n := \langle x\lambda, \alpha_j^\vee \rangle \geq 0$, the equalities $E_j v_x = 0$ and $F_j^{(n)} v_x = v_{s_j x}$ hold;

(3) for every $j \in I_{\mathrm{aff}}$ and $x \in W_{\mathrm{aff}}$ such that $n := \langle x\lambda, \alpha_j^\vee \rangle \leq 0$, the equalities $F_j v_x = 0$ and $E_j^{(-n)} v_x = v_{s_j x}$ hold;

here E_j, F_j, $j \in I_{\mathrm{aff}}$, are the Chevalley generators, and $E_j^{(k)}$ and $F_j^{(k)}$ for $k \in \mathbb{Z}_{\geq 0}$ are divided powers of E_i and F_j, respectively. We denote v_x by $S_x v$ for $x \in W_{\mathrm{aff}}$.

For $\lambda \in P_{\mathrm{aff}}$, the extremal weight module $V(\lambda)$ is the integrable $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module generated by the weight vector v_λ of weight λ with the defining relations that v_λ is an extremal weight vector of weight λ. We know that if $\lambda \in P_{\mathrm{aff}}$ is a dominant (resp., antidominant) weight, then $V(\lambda)$ is isomorphic to the irreducible highest (resp., lowest) weight module of weight λ. Moreover, for $w \in W_{\mathrm{aff}}$, there exists an isomorphism $V(\lambda) \to V(w\lambda)$ of $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-modules given by $v_\lambda \mapsto S_{w^{-1}} v_{w\lambda}$. Therefore,

(1) if $\lambda \in P_{\mathrm{aff}}$ has a positive level, i.e., $\langle \lambda, c \rangle > 0$, then there exists $x \in W_{\mathrm{aff}}$ such that $x\lambda$ is a dominant weight, and hence $V(\lambda)$ is isomorphic to the irreducible highest weight module of weight $x\lambda$;

(2) if $\lambda \in P_{\mathrm{aff}}$ has a negative level, i.e., $\langle \lambda, c \rangle < 0$, then there exists $x \in W_{\mathrm{aff}}$ such that $x\lambda$ is an antidominant weight, and hence $V(\lambda)$ is isomorphic to the irreducible lowest weight module of weight $x\lambda$.

Thus, studies on extremal weight modules are mainly focused on the case when λ is a level-zero weight, i.e., $\langle \lambda, c \rangle = 0$; for more details about the structure of $V(\lambda)$ for a weight λ of level-zero, see §3.4.5.

Chapter 3

Specialization of nonsymmetric Macdonald polynomials at $t = \infty$ and Demazure submodules of level-zero extremal weight modules

3.1 Introduction

Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS2] proved that for a dominant integral weight λ, the set $\mathrm{QLS}(\lambda)$ of all quantum Lakshmibai-Seshadri (QLS) paths of shape λ provides a realization of the crystal basis of a special quantum Weyl module over a quantum affine algebra $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ (without degree operator) of an arbitrary untwisted type, and also proved that its graded character equals the specialization $E_{w_\circ\lambda}(q, 0)$ of the nonsymmetric Macdonald polynomials $E_{w_\circ\lambda}(q, t)$ at $t = 0$, where $w_\circ$ denotes the longest element of W. Here a QLS path is obtained from an affine level-zero Lakshmibai-Seshadri path through the projection $\mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{aff}} \to \mathbb{R} \otimes_{\mathbb{Z}} P$, which factors the null root δ of an affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$, and is described in terms of (the parabolic version of) the quantum Bruhat graph, introduced by Brenti-Fomin-Postnikov [BFP]; the set of QLS paths is endowed with an affine crystal structure in a way similar to the one for the set of ordinary LS paths introduced by Littelmann [L1]. Moreover, Lenart-Naito-Sagaki-Schilling-Shimozono [LNSSS3] obtained a formula for the specialization $E_{x\lambda}(q, 0)$, $x \in W$, of the nonsymmetric Macdonald polynomials $E_{x\lambda}(q, t)$ at $t = 0$ in an arbitrary untwisted affine type, which is described in terms of QLS paths of shape λ, and also proved that the specialization $E_{x\lambda}(q, 0)$ is just the graded character of a certain Demazure-type submodule of the special quantum Weyl module. The crucial ingredient in the proof of this result is a graded character formula obtained in [NS4] for the Demazure submodule $V_e^-(\lambda)$ of the level-zero extremal weight module $V(\lambda)$ of extremal weight λ over a quantum affine algebra $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$, where e is the identity element of W. More precisely, in [NS4], Naito and Sagaki proved that the graded character $\mathrm{gch}\, V_e^-(\lambda)$ of $V_e^-(\lambda) \subset V(\lambda)$ is identical to

$\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} E_{w_\circ\lambda}(q^{-1},0)$, where λ is a dominant integral weight of the form $\sum_{i\in I} m_i\varpi_i$, with ϖ_i, $i\in I$, the fundamental weights. The graded character $\operatorname{gch} V_e^-(\lambda)$ is obtained from the ordinary character of $V_e^-(\lambda)$ by replacing e^δ by q, with δ the null root of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$.

The aim of this chapter is to give a representation-theoretic interpretation of the specialization $E_{w_\circ\lambda}(q,\infty)$ of the nonsymmetric Macdonald polynomial $E_{w_\circ\lambda}(q,t)$ at $t=\infty$ in terms of the Demazure submodule $V_{w_\circ}^-(\lambda)$ of $V(\lambda)$; here we remark that $V_{w_\circ}^-(\lambda)\subset V_e^-(\lambda)$. More precisely, we prove the following.

Theorem A (= Theorem 3.5.2). *Let $\lambda=\sum_{i\in I} m_i\varpi_i$ be a dominant integral weight. Then, the graded character $\operatorname{gch} V_{w_\circ}^-(\lambda)$ of the Demazure submodule $V_{w_\circ}^-(\lambda)$ of $V(\lambda)$ is identical to*

$$\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} E_{w_\circ\lambda}(q,\infty).$$

In order to prove Theorem A, we first rewrite the Orr-Shimozono formula for the specialization $E_{x\lambda}(q,\infty)$ for $x\in W$ (originally described in terms of quantum alcove walks) in terms of QLS paths by use of an explicit bijection sending quantum alcove walks to QLS paths that preserves weights and degrees; in some ways, this bijection generalizes a similar one in [LNSSS2]. In particular, for $x=w_\circ$, the Orr-Shimozono formula rewritten in terms of QLS paths states that

$$E_{w_\circ\lambda}(q,\infty)=\sum_{\psi\in\mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{w_\circ\lambda}(\psi)}, \tag{$*$}$$

where $\mathrm{QLS}(\lambda)$ is the set of all QLS paths of shape λ, and for $\psi\in\mathrm{QLS}(\lambda)$, $\deg_{w_\circ\lambda}(\psi)$ is a certain nonpositive integer, which is explicitly described in terms of the quantum Bruhat graph; see §3.3.2 for details.

Next, using the explicit realization, obtained in [INS], of the crystal basis $\mathcal{B}(\lambda)$ of $V(\lambda)$ by semi-infinite LS paths of shape λ, we compute the graded character $\operatorname{gch} V_x^-(\lambda)$ of the Demazure submodule $V_x^-(\lambda)$ for $x\in W$, and prove the following.

Theorem B (= Theorem 3.5.1). *Let $\lambda=\sum_{i\in I} m_i\varpi_i$ be a dominant integral weight, and x an element of the finite Weyl group W. Then, the graded character $\operatorname{gch} V_x^-(\lambda)$ of $V_x^-(\lambda)$ is identical to*

$$\left(\prod_{i\in I}\prod_{r=1}^{m_i}(1-q^{-r})\right)^{-1} \sum_{\psi\in\mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}.$$

In the proof of Theorem B, we make use of the surjective strict morphism of crystals from the set of all semi-infinite LS paths of shape λ onto $\mathrm{QLS}(\lambda)$, which is obtained by factoring the null root δ of $\mathfrak{g}_{\mathrm{aff}}$. By combining the special case $x=w_\circ$ of Theorem B with equation $(*)$ above, we obtain Theorem A.

Finally, for $x \in W$, we define a certain (finite-dimensional) quotient module $V_x^-(\lambda)/X_x^-(\lambda)$ of $V_x^-(\lambda)$, and then prove that its graded character $\mathrm{gch}\,(V_x^-(\lambda)/X_x^-(\lambda))$ is identical to $\sum_{\psi \in \mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}$. Hence it follows that under the specialization $e^{\delta} = q = 1$, all the modules $V_x^-(\lambda)/X_x^-(\lambda)$, $x \in W$, have the same character; in particular, they have the same dimension. Also, in the case $x = w_\circ$, we have $\mathrm{gch}\left(V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)\right) = E_{w_\circ\lambda}(q, \infty)$; note that in the case $x = e$, the quotient module $V_e^-(\lambda)/X_e^-(\lambda)$ is just the one in [NS4, §7.2], and hence we have $\mathrm{gch}\,(V_e^-(\lambda)/X_e^-(\lambda)) = E_{w_\circ\lambda}(q^{-1}, 0)$ (see [LNSSS3, §3] and [NS4, §6.4]). Based on these results together with [Kat, Theorem 5.1] for the classical limit, we can think of the quotient modules $V_x^-(\lambda)/X_x^-(\lambda)$, $x \in W$, as a quantum analog of "generalized Weyl modules" introduced in [FM]; see [No] for details.

This chapter is organized as follows. In Section 3.2, we fix our notation, and recall some basic facts about the (parabolic) quantum Bruhat graph. Also, we briefly review the Orr-Shimozono formula for the specialization $E_{x\lambda}(q, \infty)$ at $t = \infty$ for $x \in W$. In Section 3.3, we prove equation $(*)$ above, or more generally Theorem 3.3.19. This theorem gives the description of the specialization $E_{x\lambda}(q, \infty)$ at $t = \infty$ for $x \in W$ in terms of QLS paths of shape λ. In Section 3.4, we compute the graded character $\mathrm{gch}\,V_x^-(\lambda)$ for an arbitrary $x \in W$, and prove Theorem B. By combining the special case $x = w_\circ$ of Theorem B with equation $(*)$, we obtain Theorem A. Finally, for $x \in W$, we define a certain (finite-dimensional) quotient module $V_x^-(\lambda)/X_x^-(\lambda)$ of $V_x^-(\lambda)$, and compute its graded character. In the special case $x = w_\circ$, we obtain the equality $\mathrm{gch}\left(V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)\right) = E_{w_\circ\lambda}(q, \infty)$.

This chapter is based on the joint work [NNS1] with Satoshi Naito and Daisuke Sagaki.

3.2 (Parabolic) quantum Bruhat graph and the Orr-Shimozono formula

3.2.1 (Parabolic) quantum Bruhat graph

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. In this chapter, we follow the notation of §2.1.

Definition 3.2.1 ([BFP, Definition 6.1])**.** The quantum Bruhat graph, denoted by $\mathrm{QBG}(W)$, is the directed graph with vertex set W whose directed edges are labeled by positive roots as follows. For $u, v \in W$, and $\beta \in \Delta^+$, an arrow $u \xrightarrow{\beta} v$ is an edge of $\mathrm{QBG}(W)$ if the following hold:

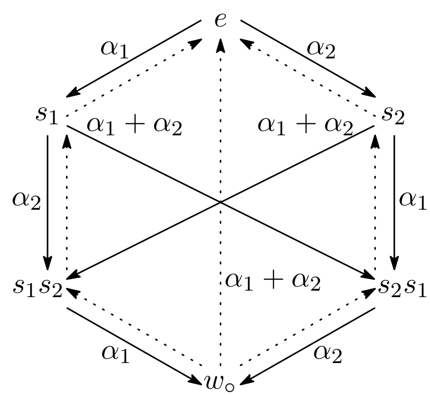
(1) $v = us_\beta$, and

(2) either (2a): $\ell(v) = \ell(u) + 1$ or (2b): $\ell(v) = \ell(u) - 2\langle \rho, \beta^\vee \rangle + 1$,

where $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$. An edge satisfying (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

Remark 3.2.2. The quantum Bruhat graph defined above is a "right-handed" version, while the one defined in [BFP] is a "left-handed" version. We remark that the results of [BFP] used in this thesis (such as Proposition 3.2.5) are unaffected by this difference (cf. [Po]).

Example 3.2.3. Let $\mathfrak{g}$ be of type A_2. Then, W is $\mathfrak{S}_3$, and the quantum Bruhat graph $\mathrm{QBG}(W)$ is as follows:



Here, plain (resp., dotted) directed edges indicate Bruhat (resp., quantum) edges.

For an edge $u \xrightarrow{\beta} v$ of $\mathrm{QBG}(W)$, we set

$$\mathrm{wt}(u \to v) := \begin{cases} 0 & \text{if } u \xrightarrow{\beta} v \text{ is a Bruhat edge,} \\ \beta^\vee & \text{if } u \xrightarrow{\beta} v \text{ is a quantum edge.} \end{cases}$$

Also, for $u, v \in W$, we take a shortest directed path $u = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = v$ in $\mathrm{QBG}(W)$, and set

$$\mathrm{wt}(u \Rightarrow v) := \mathrm{wt}(x_0 \to x_1) + \cdots + \mathrm{wt}(x_{r-1} \to x_r) \in Q^\vee;$$

we know from [Po, Lemma 1 (2), (3)] that this definition does not depend on the choice of a shortest directed path from u to v in $\mathrm{QBG}(W)$. For a dominant weight $\lambda \in P^+$, we set $\mathrm{wt}_\lambda(u \Rightarrow v) := \langle \lambda, \mathrm{wt}(u \Rightarrow v) \rangle$, and call it the λ-weight of a directed path from u to v in $\mathrm{QBG}(W)$.

Lemma 3.2.4. *If $x \xrightarrow{\beta} y$ is a Bruhat (resp., quantum) edge of $\mathrm{QBG}(W)$, then $yw_\circ \xrightarrow{-w_\circ\beta} xw_\circ$ is also a Bruhat (resp., quantum) edge of $\mathrm{QBG}(W)$.*

Proof. This follows easily from equalities $\ell(y) - \ell(x) = \ell(xw_\circ) - \ell(yw_\circ)$ and $\langle \rho, -w_\circ\beta^\vee \rangle = \langle \rho, \beta^\vee \rangle$. □

Let $w \in W$. We take (and fix) reduced expressions $w = s_{i_1} \cdots s_{i_p}$ and $w_\circ w^{-1} = s_{i_{-q}} \cdots s_{i_0}$. Note that

$$w_\circ = s_{i_{-q}} \cdots s_{i_0} s_{i_1} \cdots s_{i_p}$$

is also a reduced expression for the longest element $w_\circ$. Now we set

$$\beta_k := s_{i_p} \cdots s_{i_{k+1}} \alpha_{i_k}, \quad -q \leq k \leq p; \tag{3.2.1}$$

we have $\{\beta_{-q}, \dots, \beta_0, \dots, \beta_p\} = \Delta^+$. Then we define a total order $\prec$ on Δ^+ by

$$\beta_{-q} \prec \beta_{-q+1} \prec \cdots \prec \beta_p. \tag{3.2.2}$$

Note that this total order is a weak reflection order in the sense of Definition 3.3.2 below.

Proposition 3.2.5 ([BFP, Theorem 6.4])**.** *Let u and v be elements in W.*

(1) *There exists a unique directed path from u to v in* QBG(W) *for which the edge labels are strictly increasing* (*resp., strictly decreasing*) *in the total order $\prec$ above.*

(2) *The unique label-increasing* (*resp., label-decreasing*) *path*

$$u = u_0 \xrightarrow{\gamma_1} u_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} u_r = v$$

from u to v in QBG(W) *is a shortest directed path from u to v. Moreover, it is lexicographically minimal* (*resp., lexicographically maximal*) *among all shortest directed paths from u to v*; *namely, for an arbitrary shortest directed path*

$$u = u'_0 \xrightarrow{\gamma'_1} u'_1 \xrightarrow{\gamma'_2} \cdots \xrightarrow{\gamma'_r} u'_r = v$$

from u to v in QBG(W)*, there exists $1 \le j \le r$ such that $\gamma_j \prec \gamma'_j$ (resp., $\gamma_j \succ \gamma'_j$), and $\gamma_k = \gamma'_k$ for $1 \le k \le j-1$.*

For a subset $S \subset I$, we set $W_S := \langle s_i \mid i \in S \rangle$; notice that S may be the empty set $\emptyset$. We denote the longest element of W_S by $w_\circ(S)$. Also, we set $\Delta_S := Q_S \cap \Delta$, where $Q_S := \sum_{i \in S} \mathbb{Z}\alpha_i$, and then $\Delta_S^+ := \Delta_S \cap \Delta^+$, $\Delta_S^- := \Delta_S \cap \Delta^-$. Let W^S denote the set of all minimal-length coset representatives for the cosets in W/W_S. For $w \in W$, we denote the minimal-length coset representative of the coset wW_S by $\lfloor w \rfloor$, and for a subset $U \subset W$, we set $\lfloor U \rfloor := \{\lfloor w \rfloor \mid w \in U\} \subset W^S$.

Definition 3.2.6 ([LNSSS1, §4.3])**.** The parabolic quantum Bruhat graph, denoted by QBG(W^S), is the directed graph with vertex set W^S whose directed edges are labeled by positive roots in $\Delta^+ \setminus \Delta_S^+$ as follows. For $u, v \in W^S$, and $\beta \in \Delta^+ \setminus \Delta_S^+$, an arrow $u \xrightarrow{\beta} v$ is an edge of QBG(W^S) if the following hold:

(1) $v = \lfloor u s_\beta \rfloor$, and

(2) either (2a): $\ell(v) = \ell(u) + 1$ or (2b): $\ell(v) = \ell(u) - 2\langle \rho - \rho_S, \beta^\vee \rangle + 1$,

where $\rho_S := \frac{1}{2} \sum_{\alpha \in \Delta_S^+} \alpha$. An edge satisfying (2a) (resp., (2b)) is called a Bruhat (resp., quantum) edge.

For an edge $u \xrightarrow{\beta} v$ in QBG(W^S), we set

$$\mathrm{wt}^S(u \to v) := \begin{cases} 0 & \text{if } u \xrightarrow{\beta} v \text{ is a Bruhat edge,} \\ \beta^\vee & \text{if } u \xrightarrow{\beta} v \text{ is a quantum edge.} \end{cases}$$

Also, for $u, v \in W^S$, we take a shortest directed path $\mathbf{p} : u = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = v$ in $\mathrm{QBG}(W^S)$ (such a path always exists by [LNSSS1, Lemma 6.12]), and set

$$\mathrm{wt}^S(\mathbf{p}) := \mathrm{wt}^S(x_0 \to x_1) + \cdots + \mathrm{wt}^S(x_{r-1} \to x_r) \in Q^\vee.$$

We know from [LNSSS1, Proposition 8.1] that if $\mathbf{q}$ is another shortest directed path from u to v in $\mathrm{QBG}(W^S)$, then $\mathrm{wt}^S(\mathbf{p}) - \mathrm{wt}^S(\mathbf{q}) \in Q_S^\vee := \sum_{i \in S} \mathbb{Z}\alpha_i^\vee$.

Now, we take and fix an arbitrary dominant weight $\lambda \in P^+$, and set

$$S = S_\lambda := \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}.$$

By the remark just above, for $u, v \in W^S$, the value $\langle \lambda, \mathrm{wt}^S(\mathbf{p}) \rangle$ does not depend on the choice of a shortest directed path $\mathbf{p}$ from u to v in $\mathrm{QBG}(W^S)$; this value is called the λ-weight of a directed path from u to v in $\mathrm{QBG}(W^S)$. Moreover, we know from [LNSSS2, Lemma 7.2] that the value $\langle \lambda, \mathrm{wt}^S(\mathbf{p}) \rangle$ is equal to the value $\mathrm{wt}_\lambda(x \Rightarrow y) = \langle \lambda, \mathrm{wt}(x \Rightarrow y) \rangle$ for all $x \in uW_S$ and $y \in vW_S$. In view of this fact, for $u, v \in W^S$, we also write $\mathrm{wt}_\lambda(u \Rightarrow v)$ for the value $\langle \lambda, \mathrm{wt}^S(\mathbf{p}) \rangle$ by abuse of notation; hence, in this notation, we have

$$\mathrm{wt}_\lambda(x \Rightarrow y) = \mathrm{wt}_\lambda(\lfloor x \rfloor \Rightarrow \lfloor y \rfloor) \tag{3.2.3}$$

for all $x, y \in W$.

Definition 3.2.7 ([LNSSS2, §3.2])**.** Let $\lambda \in P^+$ be a dominant weight and $\sigma \in \mathbb{Q} \cap [0, 1]$, and set $S = S_\lambda$. We denote by $\mathrm{QBG}_{\sigma\lambda}(W)$ (resp., $\mathrm{QBG}_{\sigma\lambda}(W^S)$) the subgraph of $\mathrm{QBG}(W)$ (resp., $\mathrm{QBG}(W^S)$) with the same vertex set but having only the edges: $u \xrightarrow{\beta} v$ with $\sigma\langle \lambda, \beta^\vee \rangle \in \mathbb{Z}$.

Lemma 3.2.8 ([LNSSS2, Lemma 6.2])**.** *Let $\sigma \in \mathbb{Q} \cap [0, 1]$; notice that σ may be 1. If $u \xrightarrow{\beta} v$ is an edge of $\mathrm{QBG}_{\sigma\lambda}(W)$, then there exists a directed path from $\lfloor u \rfloor$ to $\lfloor v \rfloor$ in $\mathrm{QBG}_{\sigma\lambda}(W^S)$.*

Also, for $u, v \in W$, let $\ell(u \Rightarrow v)$ denote the length of a shortest directed path in $\mathrm{QBG}(W)$ from u to v. For $w \in W$, as in [BFP], we define the w-tilted Bruhat order $\leq_w$ on W as follows: for $u, v \in W$,

$$u \leq_w v \overset{\mathrm{def}}{\Leftrightarrow} \ell(w \Rightarrow v) = \ell(w \Rightarrow u) + \ell(u \Rightarrow v).$$

We remark that the w-tilted Bruhat order on W is a partial order with the unique minimal element w.

Lemma 3.2.9 ([LNSSS1, Theorem 7.1], [LNSSS2, Lemma 6.6])**.** *Let $u, v \in W^S$, and $w \in W_S$.*

(1) *There exists a unique minimal element in the coset vW_S in the uw-tilted Bruhat order $\leq_{uw}$. We denote it by $\min(vW_S, \leq_{uw})$.*

(2) *There exists a unique directed path from uw to some $x \in vW_S$ in $\mathrm{QBG}(W)$ whose edge labels are increasing in the total order $\prec$ on Δ^+, defined in (3.2.2), and lie in $\Delta^+ \setminus \Delta_S^+$. This path ends with $\min(vW_S, \leq_{uw})$.*

(3) *Let $\sigma \in \mathbb{Q} \cap [0, 1]$, and $\lambda \in P$ a dominant weight. If there exists a directed path from u to v in $\mathrm{QBG}_{\sigma\lambda}(W^S)$, then the directed path in part (2) is in $\mathrm{QBG}_{\sigma\lambda}(W)$.*

3.2.2 Orr-Shimozono formula

In this subsection, we review a formula [OS, Proposition 5.4] for the specialization of nonsymmetric Macdonald polynomials at $t = \infty$.

Let $\widetilde{\mathfrak{g}}$ denote the finite-dimensional simple Lie algebra whose root datum is dual to that of $\mathfrak{g}$; the set of simple roots is $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}$, and the set of simple coroots is $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$. We denote the set of all roots of $\widetilde{\mathfrak{g}}$ by $\widetilde{\Delta} = \{\alpha^\vee \mid \alpha \in \Delta\}$, and the set of all positive (resp., negative) roots of $\widetilde{\mathfrak{g}}$ by $\widetilde{\Delta}^+$ (resp., $\widetilde{\Delta}^-$). Also, for a subset $S \subset I$, we set $\widetilde{Q}_S := \sum_{i\in S} \mathbb{Z}\alpha_i^\vee$, $\widetilde{\Delta}_S := \widetilde{\Delta} \cap \widetilde{Q}_S$, $\widetilde{\Delta}_S^+ = \widetilde{\Delta}_S \cap \widetilde{\Delta}^+$, and $\widetilde{\Delta}_S^- = \widetilde{\Delta}_S \cap \widetilde{\Delta}^-$.

We consider the untwisted affinization of the root datum of $\widetilde{\mathfrak{g}}$. Let us denote by $\widetilde{\Delta}_{\mathrm{aff}}$ the set of all real roots, and by $\widetilde{\Delta}_{\mathrm{aff}}^+$ (resp., $\widetilde{\Delta}_{\mathrm{aff}}^-$) the set of all positive (resp., negative) real roots. Then we have $\widetilde{\Delta}_{\mathrm{aff}} = \{\alpha^\vee + a\widetilde{\delta} \mid \alpha \in \Delta, a \in \mathbb{Z}\}$, with $\widetilde{\delta}$ the null root. We set $\alpha_0^\vee := \widetilde{\delta} - \varphi^\vee$, where $\varphi \in \Delta$ denotes the highest short root, and set $I_{\mathrm{aff}} := I \sqcup \{0\}$. Then, $\{\alpha_i^\vee\}_{i\in I_{\mathrm{aff}}}$ is the set of all simple roots. Also, for $\beta \in \mathfrak{h} \oplus \mathbb{C}\widetilde{\delta}$, we define $\deg(\beta) \in \mathbb{C}$ and $\overline{\beta} \in \mathfrak{h}$ by

$$\beta = \overline{\beta} + \deg(\beta)\widetilde{\delta}. \tag{3.2.4}$$

We denote the Weyl group of $\widetilde{\mathfrak{g}}$ by $\widetilde{W}$; we identify $\widetilde{W}$ and W through the identification of the simple reflections of the same index for each $i \in I$. For $\nu \in \mathfrak{h}^*$, let $t(\nu)$ denote the translation in $\mathfrak{h}^*$: $t(\nu)\gamma = \gamma + \nu$ for $\gamma \in \mathfrak{h}^*$. The corresponding affine Weyl group and the extended affine Weyl group are defined by $\widetilde{W}_{\mathrm{aff}} := t(Q) \rtimes W$ and $\widetilde{W}_{\mathrm{ext}} := t(P) \rtimes W$, respectively. Also, we define $s_0 : \mathfrak{h}^* \to \mathfrak{h}^*$ by $\nu \mapsto \nu - (\langle \nu, \varphi^\vee\rangle - 1)\varphi$. Then, $\widetilde{W}_{\mathrm{aff}} = \langle s_i \mid i \in I_{\mathrm{aff}}\rangle$; note that $s_0 = t(\varphi)s_\varphi$. The extended affine Weyl group $\widetilde{W}_{\mathrm{ext}}$ acts on $\mathfrak{h} \oplus \mathbb{C}\widetilde{\delta}$ as linear transformations, and on $\mathfrak{h}^*$ as affine transformations: for $v \in W$, $t(\nu) \in t(P)$,

$$\begin{aligned} vt(\nu)(\overline{\beta} + r\widetilde{\delta}) &= v\overline{\beta} + (r - \langle \nu, \overline{\beta}\rangle)\widetilde{\delta}, \quad \overline{\beta} \in \mathfrak{h}, r \in \mathbb{C}, \\ vt(\nu)\gamma &= v\nu + v\gamma, \qquad \gamma \in \mathfrak{h}^*. \end{aligned}$$

An element $u \in \widetilde{W}_{\mathrm{ext}}$ can be written as

$$u = t(\mathrm{wt}(u))\mathrm{dir}(u), \tag{3.2.5}$$

where $\mathrm{wt}(u) \in P$ and $\mathrm{dir}(u) \in W$, according to the decomposition $\widetilde{W}_{\mathrm{ext}} = t(P) \rtimes W$. For $w \in \widetilde{W}_{\mathrm{ext}}$, we denote the length of w by $\ell(w)$, which equals $\#\left(\widetilde{\Delta}_{\mathrm{aff}}^+ \cap w^{-1}\widetilde{\Delta}_{\mathrm{aff}}^-\right)$. Also, we set $\Omega := \{w \in \widetilde{W}_{\mathrm{ext}} \mid \ell(w) = 0\}$.

For $\mu \in P$, we denote the shortest element in the coset $t(\mu)W$ by $m_\mu \in \widetilde{W}_{\mathrm{ext}}$. In the following, we fix $\mu \in P$, and take a reduced expression $m_\mu = us_{\ell_1} \cdots s_{\ell_L} \in \widetilde{W}_{\mathrm{ext}} = \Omega \ltimes \widetilde{W}_{\mathrm{aff}}$, where $u \in \Omega$ and $\ell_1, \ldots, \ell_L \in I_{\mathrm{aff}}$.

For each $J = \{j_1 < j_2 < j_3 < \cdots < j_r\} \subset \{1, \ldots, L\}$, we define an alcove path $p_J^{\mathrm{OS}} = \left(m_\mu = z_0^{\mathrm{OS}}, z_1^{\mathrm{OS}}, \ldots, z_r^{\mathrm{OS}}; \beta_{j_1}^{\mathrm{OS}}, \ldots, \beta_{j_r}^{\mathrm{OS}}\right)$ as follows: we set $\beta_k^{\mathrm{OS}} :=$

$s_{\ell_L} \cdots s_{\ell_{k+1}} \alpha_{\ell_k}^{\vee} \in \widetilde{\Delta}_{\mathrm{aff}}^{+}$ for $1 \le k \le L$, and set

$$\begin{aligned} z_0^{\mathrm{OS}} &:= m_\mu, \\ z_1^{\mathrm{OS}} &:= m_\mu s_{\beta_{j_1}^{\mathrm{OS}}}, \\ z_2^{\mathrm{OS}} &:= m_\mu s_{\beta_{j_1}^{\mathrm{OS}}} s_{\beta_{j_2}^{\mathrm{OS}}}, \\ &\ \vdots \\ z_r^{\mathrm{OS}} &:= m_\mu s_{\beta_{j_1}^{\mathrm{OS}}} \cdots s_{\beta_{j_r}^{\mathrm{OS}}}. \end{aligned}$$

Also, following [OS, §3.3], we set $\mathrm{B}(e; m_\mu) := \left\{ p_J^{\mathrm{OS}} \mid J \subset \{1, \ldots, L\} \right\}$ and $\mathrm{end}(p_J^{\mathrm{OS}}) := z_r^{\mathrm{OS}} \in \widetilde{W}_{\mathrm{ext}}$. Then we define $\overleftarrow{\mathrm{QB}}(e; m_\mu)$ to be the following subset of $\mathrm{B}(e; m_\mu)$:

$$\left\{ p_J^{\mathrm{OS}} \in \mathrm{B}(e; m_\mu) \;\middle|\; \begin{array}{l} \mathrm{dir}(z_i^{\mathrm{OS}}) \xleftarrow{-\overline{\beta_{j_{i+1}}^{\mathrm{OS}}}^{\vee}} \mathrm{dir}(z_{i+1}^{\mathrm{OS}}) \\ \qquad \text{is a directed edge of } \mathrm{QBG}(W),\ 0 \le i \le r-1 \end{array} \right\}.$$

Remark 3.2.10 ([M1, (2.4.7)]). If $j \in \{1, \ldots, L\}$, then $-\overline{\beta_j^{\mathrm{OS}}}^{\vee} \in \Delta^+$.

For $p_J^{\mathrm{OS}} \in \overleftarrow{\mathrm{QB}}(e; m_\mu)$, we define $\mathrm{qwt}^*(p_J^{\mathrm{OS}})$ as follows. Let $J^+ \subset J$ denote the set of all indices $j_i \in J$ for which $\mathrm{dir}(z_{i-1}^{\mathrm{OS}}) \xleftarrow{-\overline{\beta_{j_i}^{\mathrm{OS}}}^{\vee}} \mathrm{dir}(z_i^{\mathrm{OS}})$ is a quantum edge. Then we set

$$\mathrm{qwt}^*(p_J^{\mathrm{OS}}) := \sum_{j \in J^+} \beta_j^{\mathrm{OS}}.$$

For $\mu \in P$, we denote by $E_\mu(q, t)$ the nonsymmetric Macdonald polynomial, and by $E_\mu(q, \infty)$ the specialization $\lim_{t \to \infty} E_\mu(q, t)$ at $t = \infty$; this specialization is studied in [CO] in simply-laced types and twisted non-simply-laced types.

We know the following formula for the specialization $E_\mu(q, \infty)$ at $t = \infty$.

Proposition 3.2.11 ([OS, Proposition 5.4]). *Let $\mu \in P$. Then,*

$$E_\mu(q, \infty) = \sum_{p_J^{\mathrm{OS}} \in \overleftarrow{\mathrm{QB}}(e; m_\mu)} q^{-\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}}))} e^{\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}}))}.$$

3.3 Orr-Shimozono formula in terms of QLS paths

3.3.1 Weak reflection orders

Let $\lambda \in P^+$ be a dominant weight, $\mu \in W\lambda$, and set $S := S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^{\vee} \rangle = 0\}$. We denote by $v(\mu) \in W^S$ the minimal-length coset representative for the coset $\{w \in W \mid w\lambda = \mu\}$ in W/W_S. We have $\ell(v(\mu)w) = \ell(v(\mu)) + \ell(w)$ for all $w \in W_S$. In particular, we have $\ell(v(\mu)w_\circ(S)) = \ell(v(\mu)) + \ell(w_\circ(S))$. When $\mu = \lambda_- := w_\circ \lambda$, it is clear that $w_\circ \in \{w \in W \mid w\lambda = \lambda_-\}$. Since $w_\circ$ is the longest element of W, we have

$$w_\circ = v(\lambda_-) w_\circ(S), \tag{3.3.1}$$

and $\ell(v(\lambda_-)w_\circ(S)) = \ell(v(\lambda_-)) + \ell(w_\circ(S))$; note that $v(\lambda_-) = w_\circ w_\circ(S) = \lfloor w_\circ \rfloor$. The following lemma follows from [M1, Chap. 2].

Lemma 3.3.1.

(1) $\mathrm{dir}(m_\mu) = v(\mu)v(\lambda_-)^{-1}$ *and* $\ell(\mathrm{dir}(m_\mu)) + \ell(v(\mu)) = \ell(v(\lambda_-))$; *hence*

$$m_\mu = t(\mu)v(\mu)v(\lambda_-)^{-1}. \tag{3.3.2}$$

(2) $v(\mu)v(\lambda_-)^{-1}w_\circ = v(\mu)w_\circ(S)$.

(3) $\bigl(v(\lambda_-)v(\mu)^{-1}\bigr)\, m_\mu = m_{\lambda_-}$, *and* $\ell(v(\lambda_-)v(\mu)^{-1}) + \ell(m_\mu) = \ell(m_{\lambda_-})$.

(4) $\ell(v(\lambda_-)v(\mu)^{-1}) + \ell(v(\mu)) = \ell(v(\lambda_-))$.

In this subsection, we give a particular reduced expression for m_{λ_-} $(= t(\lambda_-)$ by (3.3.2)), and then study some of its properties.

First of all, we recall the notion of a weak reflection order on Δ^+.

Definition 3.3.2. A total order $\prec$ on Δ^+ is called a weak reflection order on Δ^+ if it satisfies the following condition: if $\alpha, \beta, \gamma \in \Delta^+$ with $\gamma^\vee = \alpha^\vee + \beta^\vee$, then $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$.

The following result is well-known (see [Pa, Theorem on p. 662] for example).

Proposition 3.3.3. *For a total order $\prec$ on Δ^+, the following are equivalent*:

(1) *the order $\prec$ is a weak reflection order*;

(2) *there exists a (unique) reduced expression $w_\circ = s_{i_1} \cdots s_{i_N}$ for $w_\circ$ such that $s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k} \prec s_{i_N} \cdots s_{i_{j+1}} \alpha_{i_j}$ for $1 \le k < j \le N$.*

Next, we recall from [Pa, pp. 661–662] the notion and some properties of a weak reflection order on a finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$; we remark that arguments in [Pa] also work in the general setting of Kac-Moody algebras.

Definition 3.3.4. Let T be a finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$, and $\prec'$ a total order on T. We say that the order $\prec'$ is a weak reflection order on T if it satisfies the following conditions:

(1) if $\theta_1, \theta_2 \in T$ satisfy $\theta_1 \prec' \theta_2$ and $\theta_1 + \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$, then $\theta_1 + \theta_2 \in T$ and $\theta_1 \prec' \theta_1 + \theta_2 \prec' \theta_2$;

(2) if $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$ satisfy $\theta_1 + \theta_2 \in T$, then $\theta_1 \in T$ and $\theta_1 + \theta_2 \prec' \theta_1$, or $\theta_2 \in T$ and $\theta_1 + \theta_2 \prec' \theta_2$.

We remark that there does not necessarily exist a weak reflection order on an arbitrary finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$.

Proposition 3.3.5. *Let T be a finite subset of $\widetilde{\Delta}^+_{\mathrm{aff}}$ and $\prec'$ a weak reflection order on T. We write T as $\{\gamma_1 \prec' \gamma_2 \prec' \cdots \prec' \gamma_p\}$. Then there exists $w \in \widetilde{W}_{\mathrm{aff}}$ such that $\widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = T$. Moreover, there exists a (unique) reduced expression $w = s_{\ell_1} \cdots s_{\ell_p}$ for w such that $s_{\ell_p} \cdots s_{\ell_{j+1}} \alpha^\vee_{\ell_j} = \gamma_j$ for $1 \le j \le p$.*

The converse of Proposition 3.3.5 also holds.

Proposition 3.3.6. *Let $w \in \widetilde{W}_{\mathrm{aff}}$, and let $w = s_{\ell_1} \cdots s_{\ell_p}$ be a reduced expression. We set a $\gamma_j := s_{\ell_p} \cdots s_{\ell_{j+1}} \alpha_{\ell_j}^{\vee}$ for $1 \le j \le p$, and define a total order $\prec'$ on $\widetilde{\Delta}_{\mathrm{aff}}^{+} \cap w^{-1}\widetilde{\Delta}_{\mathrm{aff}}^{-}$ as follows: for $1 \le j,k \le p$, $\gamma_j \prec' \gamma_k \overset{\text{def}}{\Leftrightarrow} j < k$. Then, the total order $\prec'$ is a weak reflection order on $\widetilde{\Delta}_{\mathrm{aff}}^{+} \cap w^{-1}\widetilde{\Delta}_{\mathrm{aff}}^{-}$.*

Remark 3.3.7. Let

$$\begin{aligned} v(\lambda_-) &= s_{i_1} \cdots s_{i_M}, \\ w_\circ(S) &= s_{i_{M+1}} \cdots s_{i_N}, \\ w_\circ &= s_{i_1} \cdots s_{i_M} s_{i_{M+1}} \cdots s_{i_N} \end{aligned}$$

be reduced expressions for $v(\lambda_-)$, $w_\circ(S)$, and $w_\circ = v(\lambda_-)w_\circ(S)$, respectively, where $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^{\vee} \rangle = 0\}$; recall that $w_\circ(S)$ is the longest element of W_S. We set $\beta_j := s_{i_N} \cdots s_{i_{j+1}} \alpha_{i_j}$, $1 \le j \le N$. By Proposition 3.3.3, we have $\Delta^+ \setminus \Delta_S^+ = \{\beta_1 \prec \beta_2 \prec \cdots \prec \beta_M\}$ and $\Delta_S^+ = \{\beta_{M+1} \prec \beta_{M+2} \prec \cdots \prec \beta_N\}$, where $\prec$ is the weak reflection order on Δ^+ determined by the reduced expression above for $w_\circ$. In particular, we have

$$\theta_1 \prec \theta_2 \text{ for } \theta_1 \in \Delta^+ \setminus \Delta_S^+ \text{ and } \theta_2 \in \Delta_S^+. \tag{3.3.3}$$

Conversely, if a weak reflection order on Δ^+ satisfies (3.3.3), then the reduced expression $w_\circ = s_{\ell_1} \cdots s_{\ell_N}$ for $w_\circ$ corresponding to this weak reflection order is given by concatenating a reduced expression for $v(\lambda_-)$ with a reduced expression for $w_\circ(S)$. Moreover, if we alter a reduced expression for $w_\circ(S)$ with a reduced expression for $v(\lambda_-)$ unchanged, then the restriction to $\Delta^+ \setminus \Delta_S^+$ of the weak reflection order on Δ^+ does not change. Thus, the restriction to $\Delta^+ \setminus \Delta_S^+$ of the weak reflection order on Δ^+ satisfying (3.3.3) depends only on a reduced expression for $v(\lambda_-)$.

First let us take a reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ and a weak reflection order $\prec$ on Δ^+ such that the restriction to $\Delta^+ \setminus \Delta_S^+$ of this weak reflection order $\prec$ is determined by the reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ as in Remark 3.3.7. Also, we define an injective map Φ by:

$$\begin{aligned} \Phi : \widetilde{\Delta}_{\mathrm{aff}}^{+} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}_{\mathrm{aff}}^{-} &\to \mathbb{Q}_{\ge 0} \times (\Delta^+ \setminus \Delta_S^+), \\ \beta = \overline{\beta} + \deg(\beta)\widetilde{\delta} &\mapsto \left(\frac{\langle \lambda_-, \overline{\beta} \rangle - \deg(\beta)}{\langle \lambda_-, \overline{\beta} \rangle}, w_\circ \overline{\beta}^{\vee} \right); \end{aligned}$$

note that $\langle \lambda_-, \overline{\beta} \rangle > 0$, $\langle \lambda_-, \overline{\beta} \rangle - \deg(\beta) \ge 0$, and $w_\circ \overline{\beta}^{\vee} \in \Delta^+ \setminus \Delta_S^+$ since we know from [M1, (2.4.7) (i)] that

$$\widetilde{\Delta}_{\mathrm{aff}}^{+} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}_{\mathrm{aff}}^{-} = \{\alpha^{\vee} + a\widetilde{\delta} \mid \alpha \in \Delta^-, 0 < a \le \langle \lambda_-, \alpha^{\vee} \rangle\}. \tag{3.3.4}$$

We now consider the lexicographic order $<$ on $\mathbb{Q}_{\ge 0} \times (\Delta^+ \setminus \Delta_S^+)$ induced by the usual total order on $\mathbb{Q}_{\ge 0}$ and the restriction to $\Delta^+ \setminus \Delta_S^+$ of the weak reflection order $\prec$ on Δ^+; that is, for $(a, \alpha), (b, \beta) \in \mathbb{Q}_{\ge 0} \times (\Delta^+ \setminus \Delta_S^+)$,

$$(a, \alpha) < (b, \beta) \text{ if and only if } a < b, \text{ or } a = b \text{ and } \alpha \prec \beta.$$

Then we denote by $\prec'$ the total order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ induced by the lexicographic order on $\mathbb{Q}_{\geq 0} \times (\Delta^+ \setminus \Delta^+_S)$ through the map Φ, and write $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ as $\{\gamma_1 \prec' \cdots \prec' \gamma_L\}$.

Proposition 3.3.8. *Keep the notation and setting above. Then, there exists a unique reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-}, $u \in \Omega$, $\{\ell_1, \dots, \ell_L\} \subset I_{\mathrm{aff}}$, such that $\beta_j^{\mathrm{OS}} \left(= s_{\ell_L} \cdots s_{\ell_{j+1}} \alpha_{\ell_j}^\vee\right) = \gamma_j$ for $1 \leq j \leq L$.*

Proof. We will show that the total order $\prec'$ is a weak reflection order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$.

We check condition (1) in Definition 3.3.4. Assume that $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ satisfy $\theta_1 \prec' \theta_2$ and $\theta_1 + \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$. Then it is clear that $\theta_1 + \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$.

Consider the case that the first component of $\Phi(\theta_1)$ is less than that of $\Phi(\theta_2)$ (i.e., $\frac{\langle \lambda_-, \overline{\theta_1}\rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1}\rangle} < \frac{\langle \lambda_-, \overline{\theta_2}\rangle - \deg(\theta_2)}{\langle \lambda_-, \overline{\theta_2}\rangle}$). In this case, the first component of $\Phi(\theta_1+\theta_2)$ is equal to $\frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle}$, which lies between the first components of $\Phi(\theta_1)$ and $\Phi(\theta_2)$. Hence we have $\Phi(\theta_1) < \Phi(\theta_1+\theta_2) < \Phi(\theta_2)$.

Consider the case that the first component of $\Phi(\theta_1)$ is equal to that of $\Phi(\theta_2)$. In this case, we have $w_\circ \overline{\theta_1}^\vee \prec w_\circ \overline{\theta_2}^\vee$, where $\prec$ is the restriction to $\Delta^+ \setminus \Delta^+_S$ of the weak reflection order on Δ^+. Note that the first component of $\Phi(\theta_1+\theta_2)$ is equal to $\frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle}$, which is equal to both of the first components of $\Phi(\theta_1)$ and $\Phi(\theta_2)$. Moreover, since $\theta_1+\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, we have $w_\circ \left(\overline{\theta_1+\theta_2}\right)^\vee \in \Delta^+ \setminus \Delta^+_S$. It follows from the definition of the weak reflection order $\prec$ on Δ^+ that $w_\circ \overline{\theta_1}^\vee \prec w_\circ \left(\overline{\theta_1+\theta_2}\right)^\vee \prec w_\circ \overline{\theta_2}^\vee$. Hence we have $\Phi(\theta_1) < \Phi(\theta_1+\theta_2) < \Phi(\theta_2)$. Thus, the total order $\prec'$ satisfies condition (1).

We check condition (2) in Definition 3.3.4. If $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ and $\theta_1+\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}}$, then it is clear that $\theta_1+\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$. Hence we may assume that $\theta_1 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ and $\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$; indeed, if $\theta_1, \theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, then the assertion is obvious by condition (1). Since $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = \{\alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta^-, 0 < a \leq \langle \lambda_-, \alpha^\vee\rangle\}$, we have $0 < \deg(\theta_1) \leq \langle \lambda_-, \overline{\theta_1}\rangle$ and $0 < \deg(\theta_1+\theta_2) \leq \langle \lambda_-, \overline{\theta_1+\theta_2}\rangle$. Also, since $\theta_2 \in \widetilde{\Delta}^+_{\mathrm{aff}} \setminus m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, we find that $\langle \lambda_-, \overline{\theta_2}\rangle < 0 \leq \deg(\theta_2)$, $\deg(\theta_2) > \langle \lambda_-, \overline{\theta_2}\rangle \geq 0$, or $\langle \lambda_-, \overline{\theta_2}\rangle = \deg(\theta_2) = 0$; if $0 > \deg(\theta_2)$, then we have $\theta_2 \in \widetilde{\Delta}^-_{\mathrm{aff}}$, a contradiction.

In the case that $\langle \lambda_-, \overline{\theta_2}\rangle < 0 \leq \deg(\theta_2)$, the first component of $\Phi(\theta_1+\theta_2)$, which is $\frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle}$, satisfies the inequalities

$$\begin{aligned}\frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} &\leq \frac{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} \\ &= 1 - \frac{\deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1+\theta_2}\rangle} < 1 - \frac{\deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1}\rangle} = \frac{\langle \lambda_-, \overline{\theta_1}\rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1}\rangle}.\end{aligned}$$

Therefore, we deduce that the first component of $\Phi(\theta_1+\theta_2)$ is less than that of $\Phi(\theta_1)$, and hence $\Phi(\theta_1+\theta_2) < \Phi(\theta_1)$.

In the case that $\deg(\theta_2) > \langle \lambda_-, \overline{\theta_2} \rangle \geq 0$, the first component of $\Phi(\theta_1+\theta_2)$ satisfies the inequalities

$$\begin{aligned} \frac{\langle \lambda_-, \overline{\theta_1+\theta_2} \rangle - \deg(\theta_1+\theta_2)}{\langle \lambda_-, \overline{\theta_1+\theta_2} \rangle} &= \frac{\bigl(\langle \lambda_-, \overline{\theta_1} \rangle - \deg(\theta_1)\bigr) + \bigl(\langle \lambda_-, \overline{\theta_2} \rangle - \deg(\theta_2)\bigr)}{\langle \lambda_-, \overline{\theta_1+\theta_2} \rangle} \\ &< \frac{\bigl(\langle \lambda_-, \overline{\theta_1} \rangle - \deg(\theta_1)\bigr)}{\langle \lambda_-, \overline{\theta_1+\theta_2} \rangle} \leq \frac{\langle \lambda_-, \overline{\theta_1} \rangle - \deg(\theta_1)}{\langle \lambda_-, \overline{\theta_1} \rangle}. \end{aligned}$$

Therefore, we deduce that the first component of $\Phi(\theta_1 + \theta_2)$ is less than that of $\Phi(\theta_1)$, and hence that $\Phi(\theta_1 + \theta_2) < \Phi(\theta_1)$.

In the case that $\langle \lambda_-, \overline{\theta_2} \rangle = \deg(\theta_2) = 0$, the first component of $\Phi(\theta_1 + \theta_2)$ is equal to that of $\Phi(\theta_1)$. Moreover, since $\langle \lambda_-, \overline{\theta_2} \rangle = \langle \lambda, w_\circ \overline{\theta_2} \rangle = 0$, we have $w_\circ \overline{\theta_2}^\vee \in \Delta_S^+$. Therefore, by (3.3.3), we see that $w_\circ(\overline{\theta_1+\theta_2})^\vee \prec w_\circ \overline{\theta_2}^\vee$. It follows from the definition of the weak reflection order on Δ^+ that $w_\circ \overline{\theta_1}^\vee \prec w_\circ(\overline{\theta_1+\theta_2})^\vee \prec w_\circ \overline{\theta_2}^\vee$, and hence that $\Phi(\theta_1 + \theta_2) < \Phi(\theta_1)$.

Thus, we conclude that $\prec'$ satisfies condition (2), and the total order $\prec'$ is a weak reflection order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$.

Now, by Proposition 3.3.5, there exists $w \in \widetilde{W}_{\mathrm{aff}}$ such that $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}} = \widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$, and there exists a reduced expression $w = s_{\ell_1} \cdots s_{\ell_L}$, $\{\ell_1, \ldots, \ell_L\} \subset I_{\mathrm{aff}}$ for w such that $\gamma_j = s_{\ell_L} \cdots s_{\ell_{j+1}} \alpha_{\ell_j}^\vee$ for $1 \leq j \leq L$. Since $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}} = \widetilde{\Delta}^+_{\mathrm{aff}} \cap w^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$, it follows from [M1, (2.2.6)] that there exists $u \in \Omega$ such that $uw = m_{\lambda_-}$. Thus, we obtain a reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-}, with $\gamma_j = s_{\ell_L} \cdots s_{\ell_{j+1}} \alpha_{\ell_j}^\vee = \beta_j^{\mathrm{OS}}$ for $1 \leq j \leq L$. This completes the proof of the proposition. □

By Remark 3.3.7, the restriction to $\Delta^+ \setminus \Delta_S^+$ of a weak reflection order on Δ^+ satisfying (3.3.3) corresponds bijectively to a reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ for $v(\lambda_-)$. Hence, by Proposition 3.3.8, we can take a reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-} corresponding to each reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ for $v(\lambda_-)$. Conversely, as seen in Lemma 3.3.10, from the reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-}, we obtain a reduced expression for $v(\lambda_-)$, which is identical to the original reduced expression $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ (see Lemma 3.3.10 below).

In the remainder of this subsection, we fix reduced expressions $v(\lambda_-) = s_{i_1} \cdots s_{i_M}$ and $w_\circ(S) = s_{i_{M+1}} \cdots s_{i_N}$, and use the weak reflection order $\prec$ on Δ^+ (which satisfies (3.3.3)) determined by these reduced expressions for $v(\lambda_-)$ and $w_\circ(S)$. Also, we use the total order $\prec'$ on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}}$ defined just before Proposition 3.3.8, and take a reduced expression $m_{\lambda_-} = us_{\ell_1} \cdots s_{\ell_L}$ for m_{λ_-} given by Proposition 3.3.8.

Recall that $\beta_k^{\mathrm{OS}} = s_{\ell_L} \cdots s_{\ell_{k+1}} \alpha_{\ell_k}^\vee$ for $1 \leq k \leq L$. We set $a_k := \deg(\beta_k^{\mathrm{OS}}) \in \mathbb{Z}_{>0}$; since $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1} \widetilde{\Delta}^-_{\mathrm{aff}} = \{\beta_1^{\mathrm{OS}}, \ldots, \beta_L^{\mathrm{OS}}\}$, we see by (3.3.4) that $0 < a_k \leq \langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle$. Also, for $1 \leq j \leq L$, we set $\beta_k^{\mathrm{L}} := us_{\ell_1} \cdots s_{\ell_{k-1}} \alpha_{\ell_k}^\vee$ and $b_k := \deg(\beta_k^{\mathrm{L}}) \in \mathbb{Z}_{\geq 0}$. Then we have $\{\beta_k^{\mathrm{L}} \mid 1 \leq k \leq L\} = \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-} \widetilde{\Delta}^-_{\mathrm{aff}} = \{\alpha^\vee + a\tilde{\delta} \mid \alpha \in \Delta^+, 0 \leq a < -\langle \lambda_-, \alpha^\vee \rangle\}$ (see [M1, (2.4.7) (ii)]).

Remark 3.3.9. For $1 \le k \le L$, we have

$$\begin{aligned} -t(\lambda_-)\beta_k^{\mathrm{OS}} &= -(us_{\ell_1}\cdots s_{\ell_L})(s_{\ell_L}\cdots s_{\ell_{k+1}}\alpha_{\ell_k}^\vee) = -us_{\ell_1}\cdots s_{\ell_{k-1}}s_{\ell_k}\alpha_{\ell_k}^\vee \\ &= -us_{\ell_1}\cdots s_{\ell_{k-1}}(-\alpha_{\ell_k}^\vee) = us_{\ell_1}\cdots s_{\ell_{k-1}}\alpha_{\ell_k}^\vee = \beta_k^{\mathrm{L}} = \overline{\beta_k^{\mathrm{L}}} + b_k\widetilde{\delta}. \end{aligned}$$

From this, together with $-t(\lambda_-)\beta_k^{\mathrm{OS}} = -\overline{\beta_k^{\mathrm{OS}}} - (a_k - \langle\lambda_-, \overline{\beta_k^{\mathrm{OS}}}\rangle)\widetilde{\delta}$, we obtain $\overline{\beta_k^{\mathrm{L}}} = -\overline{\beta_k^{\mathrm{OS}}}$ and $\langle\lambda_-, \overline{\beta_k^{\mathrm{OS}}}\rangle - a_k = b_k$.

Lemma 3.3.10. *Keep the notation and setting above. Since $us_{\ell_k} = s_{i'_k}u$ for some $i'_k \in I_{\mathrm{aff}}$, $1 \le k \le M$, we can rewrite the reduced expression $us_{\ell_1}\cdots s_{\ell_L}$ for m_{λ_-} as $s_{i'_1}\cdots s_{i'_M}us_{\ell_{M+1}}\cdots s_{\ell_L}$. Then, $s_{i'_1}\cdots s_{i'_M}$ is a reduced expression for $v(\lambda_-)$, and $us_{\ell_{M+1}}\cdots s_{\ell_L}$ is a reduced expression for m_λ. Moreover, $i_k = i'_k$ for $1 \le k \le M$.*

Proof. First we show that $\{\beta_k^{\mathrm{L}} \mid 1 \le k \le M\} = -w_\circ\left(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+\right)$. Since $\{\beta_j^{\mathrm{OS}} \mid 1 \le j \le L\} = \{\alpha^\vee + a\widetilde{\delta} \mid \alpha \in \Delta^-, 0 < a \le \langle\lambda_-, \alpha^\vee\rangle\}$, we see that the minimum value of the first components of $\Phi(\beta_k^{\mathrm{OS}})$, i.e., $\frac{\langle\lambda_-, \overline{\beta_k^{\mathrm{OS}}}\rangle - a_k}{\langle\lambda_-, \overline{\beta_k^{\mathrm{OS}}}\rangle}$ for $1 \le k \le L$, is equal to 0. Since $\Phi(\beta_1^{\mathrm{OS}}) < \Phi(\beta_2^{\mathrm{OS}}) < \cdots < \Phi(\beta_L^{\mathrm{OS}})$, where $<$ denotes the lexicographic order on $\mathbb{Q}_{\ge 0} \times (\Delta^+ \setminus \Delta_S^+)$, there exists a positive integer M' such that the first component of $\Phi(\beta_k^{\mathrm{OS}})$ is equal to 0 for $1 \le k \le M'$, and greater than 0 for $M'+1 \le k \le L$. Since $\beta_k^{\mathrm{L}} = \overline{\beta_k^{\mathrm{L}}} + b_k\widetilde{\delta}$ and $\langle\lambda_-, \overline{\beta_k^{\mathrm{OS}}}\rangle - a_k = b_k$ by Remark 3.3.9, we deduce that the first component of $\Phi(\beta_k^{\mathrm{OS}})$ is equal to 0 if and only if $\beta_k^{\mathrm{L}} = \overline{\beta_k^{\mathrm{L}}} \in \widetilde{\Delta}^+$. In this case, we have $\langle\lambda, -w_\circ\beta_k^{\mathrm{L}}\rangle = \langle\lambda_-, -\beta_k^{\mathrm{L}}\rangle \overset{\text{Remark 3.3.9}}{=} \langle\lambda_-, \overline{\beta_k^{\mathrm{OS}}}\rangle > 0$, and hence $\beta_k^{\mathrm{L}} \in -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Therefore, we obtain $\{\beta_k^{\mathrm{L}} \mid 1 \le k \le L\} \cap -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+) = \{\beta_k^{\mathrm{L}} \mid 1 \le k \le M'\} \subset -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Also, because $\{\beta_k^{\mathrm{L}} \mid 1 \le k \le L\} = \widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_{\lambda_-}\widetilde{\Delta}_{\mathrm{aff}}^- = \{\alpha^\vee + a\widetilde{\delta} \mid \alpha \in \Delta^+, 0 \le a < -\langle\lambda_-, \alpha^\vee\rangle\} \supset -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$, we deduce that $\{\beta_k^{\mathrm{L}} \mid 1 \le k \le M'\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Since $\#(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+) = M$, it follows that $M = M'$, and hence $\{\beta_k^{\mathrm{L}} \mid 1 \le k \le M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$.

We show that $i'_k \in I$ for $1 \le k \le M$. We set $\zeta_k^\vee := s_{i'_1}\cdots s_{i'_{k-1}}\alpha_{i'_k}^\vee$ for $1 \le k \le M$. Since $u\alpha_{\ell_k}^\vee = \alpha_{i'_k}^\vee$, we have

$$\beta_k^{\mathrm{L}} = us_{\ell_1}\cdots s_{\ell_{k-1}}\alpha_{\ell_k}^\vee = s_{i'_1}\cdots s_{i'_{k-1}}u\alpha_{\ell_k}^\vee = s_{i'_1}\cdots s_{i'_{k-1}}\alpha_{i'_k}^\vee = \zeta_k^\vee.$$

Hence it follows that $\{\zeta_k^\vee \mid 1 \le k \le M\} = \{\beta_k^{\mathrm{L}} \mid 1 \le k \le M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. If there exists $k \in \{1, \dots, M\}$ such that $i'_k = 0$, then, by choosing the minimum of such k's, we obtain $\zeta_k^\vee = s_{i'_1}\cdots s_{i'_{k-1}}\alpha_{i'_k}^\vee \notin \widetilde{\Delta}^+$, contrary to the equality $\{\zeta_k^\vee \mid 1 \le k \le M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$. Therefore, we have $i'_k \in I$ for $1 \le k \le M$.

Next, we show that $s_{i'_1}\cdots s_{i'_M}$ is a reduced expression for $v(\lambda_-)$ and $us_{\ell_{M+1}}\cdots s_{\ell_L}$ is a reduced expression for m_λ. Since $s_{\ell_1}\cdots s_{\ell_M}$ is a reduced expression, so is $s_{i'_1}\cdots s_{i'_M}$. Therefore, there exist $i'_{M+1}, \dots, i'_N \in I$ such that $w_\circ = s_{i'_1}\cdots s_{i'_M}s_{i'_{M+1}}\cdots s_{i'_N}$ is a reduced expression for $w_\circ$. Because $s_{i'_N}\cdots s_{i'_{M+1}}s_{i'_M}\cdots s_{i'_{k+1}}\alpha_{i'_k}^\vee = -w_\circ\beta_k^{\mathrm{L}}$, $1 \le k \le M$, by using the reduced expression above for $w_\circ$, we obtain

$$\widetilde{\Delta}^+ = \{-w_\circ\beta_1^{\mathrm{L}}, \dots, -w_\circ\beta_M^{\mathrm{L}}, s_{i'_N}\cdots s_{i'_{M+2}}\alpha_{i'_{M+1}}^\vee, \dots, \alpha_{i'_N}^\vee\}.$$

Here, $\{\beta_k^{\mathrm{L}} \mid 1 \le k \le M\} = -w_\circ(\widetilde{\Delta}^+ \setminus \widetilde{\Delta}_S^+)$ implies $\{s_{i'_N} \cdots s_{i'_{M+2}} \alpha^\vee_{i'_{M+1}}, \dots, \alpha^\vee_{i'_N}\} = \widetilde{\Delta}_S^+$. From this by descending induction on $M+1 \le k \le N$, we deduce that $i'_{M+1}, \dots, i'_N \in S$, and $s_{i'_{M+1}} \cdots s_{i'_N}$ is an element of W_S; note that the length of this element is equal to $N-M$, which is the cardinality of $\widetilde{\Delta}_S^+$. Therefore, $s_{i'_{M+1}} \cdots s_{i'_N}$ is the longest element $w_\circ(S)$ of W_S, and hence $s_{i'_1} \cdots s_{i'_M} = w_\circ w_\circ(S) = v(\lambda_-)$, which is a reduced expression for $v(\lambda_-)$. Moreover, because $m_{\lambda_-} = v(\lambda_-) m_\lambda$ with $\ell(m_{\lambda_-}) = \ell(v(\lambda_-)) + \ell(m_\lambda)$ by Lemma 3.3.1 (3) for the case $\mu = \lambda$, $m_\lambda = v(\lambda_-)^{-1} m_{\lambda_-} = u s_{\ell_{M+1}} \cdots s_{\ell_L}$ is a reduced expression for m_λ.

Finally, we show that $i_k = i'_k$ for $1 \le k \le M$. Since $M = M'$ as shown above,

$$\Phi(\beta_k^{\mathrm{OS}}) = \left(\frac{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle - a_k}{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle}, w_\circ \overline{\beta_k^{\mathrm{OS}}}^\vee \right) = \left(0, w_\circ \overline{\beta_k^{\mathrm{OS}}}^\vee \right)$$

for $1 \le k \le M$ by the definition of Φ, and

$$\begin{aligned} w_\circ \overline{\beta_k^{\mathrm{OS}}}^\vee = -w_\circ \overline{\beta_k^{\mathrm{L}}}^\vee &= -w_\circ \zeta_k = -s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_1} s_{i'_1} \cdots s_{i'_{k-1}} \alpha_{i'_k} \\ &= s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k} \end{aligned}$$

by Remark 3.3.9. Thus, for $1 \le k < j \le M$, we have $s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k} \prec s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{j+1}} \alpha_{i'_j}$, where the order $\prec$ is the fixed weak reflection order on Δ^+ defined just before Proposition 3.3.8. Here we recall from Remark 3.3.7 that $\beta_k = s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k}$, $1 \le k \le N$. Because

$$\{\beta_k \mid 1 \le k \le M\} = \{s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k} \mid 1 \le k \le M\} = \Delta^+ \setminus \Delta_S^+,$$

it follows from the definition of the weak reflection order $\prec$ on Δ^+ together with (3.3.3) that

$$\{\beta_1 \prec \cdots \prec \beta_M\} = \left\{ s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_2} \alpha_{i'_1} \prec \cdots \prec s_{i'_N} \cdots s_{i'_{M+1}} \alpha_{i'_M} \right\} = \Delta^+ \setminus \Delta_S^+.$$

Therefore, noting that $\beta_k = s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k}$ for $1 \le k \le N$, we obtain

$$s_{i_N} \cdots s_{i_{k+1}} \alpha_{i_k} = s_{i'_N} \cdots s_{i'_{M+1}} s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k}, \quad \text{for } 1 \le k \le M. \tag{3.3.5}$$

By substituting the equalities $s_{i_{M+1}} \cdots s_{i_N} = w_\circ(S) = s_{i'_{M+1}} \cdots s_{i'_N}$ into (3.3.5), we have $s_{i_M} \cdots s_{i_{k+1}} \alpha_{i_k} = s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k}$ for $1 \le k \le M$. In particular, when $k = M$, we have $\alpha_{i_M} = \alpha_{i'_M}$, which implies that $i_M = i'_M$. If $i_j = i'_j$ for $k+1 \le j \le M$, then it follows from $s_{i_M} \cdots s_{i_{k+1}} \alpha_{i_k} = s_{i'_M} \cdots s_{i'_{k+1}} \alpha_{i'_k}$ that $\alpha_{i_k} = \alpha_{i'_k}$, and hence $i_k = i'_k$. Thus, by descending induction on k, we deduce that $i_k = i'_k$ for $1 \le k \le M$. □

Remark 3.3.11 ([LNSSS2, §6.1]). For $1 \le k \le L$, we set

$$d_k := \frac{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle - a_k}{\langle \lambda_-, \overline{\beta_k^{\mathrm{OS}}} \rangle} = \frac{b_k}{\langle -\lambda_-, \overline{\beta_k^{\mathrm{L}}} \rangle};$$

the second equality follows from Remark 3.3.9; here d_k is just the first component of $\Phi(\beta_k^{\mathrm{OS}}) \in \mathbb{Q}_{\ge 0} \times (\Delta^+ \setminus \Delta_S^+)$. For $1 \le k, j \le L$, $\Phi(\beta_k^{\mathrm{OS}}) < \Phi(\beta_j^{\mathrm{OS}})$ if and only if $k < j$, and hence we have

$$0 \le d_1 \le \cdots \le d_L \lneqq 1. \tag{3.3.6}$$

Lemma 3.3.12. *If $1 \le k < j \le L$ and $d_k = d_j$, then $w_\circ \overline{\beta_k^{\mathrm{OS}}}^\vee \prec w_\circ \overline{\beta_j^{\mathrm{OS}}}^\vee$.*

Proof. By the definitions, we obtain $\Phi(\beta_k^{\mathrm{OS}}) = \left(d_k, w_\circ \overline{\beta_k^{\mathrm{OS}}}^\vee\right)$ and $\Phi(\beta_j^{\mathrm{OS}}) = \left(d_j, w_\circ \overline{\beta_j^{\mathrm{OS}}}^\vee\right)$. Since $d_k = d_j$ and $\Phi(\beta_k^{\mathrm{OS}}) < \Phi(\beta_j^{\mathrm{OS}})$, we have $w_\circ \overline{\beta_k^{\mathrm{OS}}}^\vee \prec w_\circ \overline{\beta_j^{\mathrm{OS}}}^\vee$. □

3.3.2 Orr-Shimozono formula in terms of QLS paths

Let $\lambda \in P^+$ be a dominant weight, and set $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$.

Definition 3.3.13 ([LNSSS2, Definition 3.1])**.** A pair $\psi = (w_1, w_2, \ldots, w_s; \sigma_0, \sigma_1, \ldots, \sigma_s)$ of a sequence $w_1, \ldots, w_s$ of elements in W^S such that $w_k \neq w_{k+1}$ for $1 \le k \le s-1$ and an increasing sequence $0 = \sigma_0 < \cdots < \sigma_s = 1$ of rational numbers is called a quantum Lakshmibai-Seshadri (QLS) path of shape λ if

(C) for every $1 \le i \le s-1$, there exists a directed path from w_{i+1} to w_i in $\mathrm{QBG}_{\sigma_i \lambda}(W)$.

Let $\mathrm{QLS}(\lambda)$ denote the set of all QLS paths of shape λ.

Remark 3.3.14. We know from [LNSSS4, Definition 3.2.2 and Theorem 4.1.1] that condition (C) can be replaced by

(C)' for every $1 \le i \le s-1$, there exists a directed path from w_{i+1} to w_i in $\mathrm{QBG}_{\sigma_i \lambda}(W^S)$ that is also a shortest directed path from w_{i+1} to w_i in $\mathrm{QBG}(W^S)$.

For $\psi = (w_1, w_2, \ldots, w_s; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda)$, we set

$$\mathrm{wt}(\psi) := \sum_{i=0}^{s-1} (\sigma_{i+1} - \sigma_i) w_{i+1} \lambda,$$

and we define a map $\kappa : \mathrm{QLS}(\lambda) \to W^S$ by $\kappa(\psi) := w_s$. Also, for $\mu \in W\lambda$, we define the degree of ψ at μ by

$$\deg_\mu(\psi) := -\sum_{i=1}^{s} \sigma_i \mathrm{wt}_\lambda(w_{i+1} \Rightarrow w_i);$$

here we set $w_{s+1} := v(\mu)$. Note that by Remark 3.3.14, $\sigma_i \mathrm{wt}_\lambda(w_{i+1} \Rightarrow w_i) \in \mathbb{Z}_{\ge 0}$ for $1 \le i \le s-1$. Also, $\sigma_s = 1$ for $i = s$ by the definition of a QLS path. Hence it follows that $\deg_\mu(\psi) \in \mathbb{Z}_{\le 0}$.

Now, we define a subset $\mathrm{EQB}(w)$ of W for each $w \in W$. Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression for w. For each $J = \{j_1 < j_2 < j_3 < \cdots < j_r\} \subset \{1, \ldots, p\}$, we define

$$p_J := (w = z_0, \ldots, z_r; \beta_{j_1}, \ldots, \beta_{j_r})$$

as follows: we set $\beta_k := s_{i_p} \cdots s_{i_{k+1}}(\alpha_{i_k}) \in \Delta^+$ for $1 \le k \le p$, and set

$$\begin{aligned}
z_0 &= w = s_{i_1} \cdots s_{i_p}, \\
z_1 &= w s_{\beta_{j_1}} = s_{i_1} \cdots s_{i_{j_1 - 1}} s_{i_{j_1+1}} \cdots s_{i_p} = s_{i_1} \cdots \breve{s_{i_{j_1}}} \cdots s_{i_p}, \\
z_2 &= w s_{\beta_{j_1}} s_{\beta_{j_2}} = s_{i_1} \cdots s_{i_{j_1-1}} s_{i_{j_1+1}} \cdots s_{i_{j_2-1}} s_{i_{j_2+1}} \cdots s_{i_p} = s_{i_1} \cdots \breve{s_{i_{j_1}}} \cdots \breve{s_{i_{j_2}}} \cdots s_{i_p}, \\
&\ \vdots \\
z_r &= w s_{\beta_{j_1}} \cdots s_{\beta_{j_r}} = s_{i_1} \cdots \breve{s_{i_{j_1}}} \cdots \breve{s_{i_{j_r}}} \cdots s_{i_p},
\end{aligned}$$

where the symbol $\breve{\cdot}$ indicates a term to be omitted; also, we set $\mathrm{end}(p_J) := z_r$. Then we define $\mathrm{B}(w) := \{p_J \mid J \subset \{1, \ldots, p\}\}$, and

$$\mathrm{QB}(w) := \left\{ p_J \in \mathrm{B}(w) \;\middle|\; \begin{array}{r} z_i \xrightarrow{\beta_{j_{i+1}}} z_{i+1} \ \text{ is a directed edge of } \mathrm{QBG}(W) \\ \text{for all } 0 \le i \le r-1 \end{array} \right\}.$$

We remark that J may be the empty set $\emptyset$; in this case, $\mathrm{end}(p_\emptyset) = w$.

Remark 3.3.15. We identify elements in $\mathrm{QB}(w)$ with directed paths in $\mathrm{QBG}(W)$. More precisely, for $p_J = (w = z_0, \ldots, z_r; \beta_{j_1}, \ldots \beta_{j_r}) \in \mathrm{QB}(w)$, we write

$$p_J = (w = z_0, \ldots, z_r; \beta_{j_1}, \ldots \beta_{j_r}) = \left(w = z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_r}} z_r \right).$$

Remark 3.3.16. Let $w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$ be a directed path in $\mathrm{QBG}(W)$. Then we see that

$$1 \le j_1 < j_2 < \cdots < j_r \le p \Leftrightarrow \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w).$$

Also, it follows from Proposition 3.2.5 (1) that the map $\mathrm{end} : \mathrm{QB}(w) \to W$ is injective.

By using the map $\mathrm{end} : \mathrm{B}(w) \to W$ defined above, we set $\mathrm{EQB}(w) := \mathrm{end}(\mathrm{QB}(w))$.

Proposition 3.3.17. *The set* $\mathrm{EQB}(w)$ *is independent of the choice of a reduced expression for* w*.*

Proof. Let us take two reduced expressions for w:

$$\mathbf{I} : w = s_{i_1} \cdots s_{i_p} \text{ and } \mathbf{K} : w = s_{k_1} \cdots s_{k_p}.$$

In this proof, let $\mathrm{EQB}(w)_{\mathbf{I}}$ (resp., $\mathrm{EQB}(w)_{\mathbf{K}}$) denote the set $\mathrm{EQB}(w)$ associated to $\mathbf{I}$ (resp., $\mathbf{K}$).

It suffices to show that $\mathrm{EQB}(w)_{\mathbf{I}} \subset \mathrm{EQB}(w)_{\mathbf{K}}$. From the two reduced expressions above for w, we obtain the following two reduced expressions for $w_\circ$:

$$w_\circ = s_{i_{-q}} \cdots s_{i_0} s_{i_1} \cdots s_{i_p}, \tag{3.3.7}$$

$$w_\circ = s_{i_{-q}} \cdots s_{i_0} s_{k_1} \cdots s_{k_p}. \tag{3.3.8}$$

Using the reduced expression (3.3.7) (resp., (3.3.8)), we define β_m (resp., γ_m), $-q \le m \le p$, as in (3.2.1). Then we have

$$\{\beta_{-q}, \ldots, \beta_p\} = \{\gamma_{-q}, \ldots, \gamma_p\} = \Delta^+, \tag{3.3.9}$$

$$\{\beta_1, \ldots, \beta_p\} = \{\gamma_1, \ldots, \gamma_p\} = \Delta^+ \cap w^{-1}\Delta^-. \tag{3.3.10}$$

Let $z \in \mathrm{EQB}(w)_{\mathbf{I}}$, and

$$p_J = \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w)_{\mathbf{I}}; \tag{3.3.11}$$

recall from Remark 3.3.16 that $1 \le j_1 \le \cdots \le j_r \le p$. It follows from Proposition 3.2.5 (1) that there exists a unique shortest directed path in $\mathrm{QBG}(W)$

$$w = y_0 \xrightarrow{\gamma_{n_1}} y_1 \xrightarrow{\gamma_{n_2}} \cdots \xrightarrow{\gamma_{n_r}} y_r = z, \tag{3.3.12}$$

with $-q \le n_1 < n_2 < \cdots < n_r \le p$; this is a label-increasing directed path with respect to the weak reflection order defined by $\gamma_{-q} \prec \cdots \prec \gamma_p$. To prove that $z \in \mathrm{EQB}(w)_{\mathbf{K}}$, it suffices to show that $1 \le n_1$. It follows from (3.3.9) that for $1 \le u \le r$, there exists $-q \le t_u \le p$ such that $\beta_{t_u} = \gamma_{n_u}$. Therefore, by (3.3.12),

$$w = y_0 \xrightarrow{\beta_{t_1}} y_1 \xrightarrow{\beta_{t_2}} \cdots \xrightarrow{\beta_{t_r}} y_r = z$$

is a directed path in $\mathrm{QBG}(W)$. We see from Proposition 3.2.5 (2) that this path is greater than or equal to the path (3.3.11) in the lexicographic order with respect to the edge labels. In particular, we have $t_1 \ge j_1 \ge 1$. Since $\gamma_{n_1} = \beta_{t_1} \in \Delta^+ \cap w^{-1}\Delta^-$, we deduce that $n_1 \ge 1$ by (3.3.10). This implies that $\mathrm{EQB}(w)_{\mathbf{I}} \subset \mathrm{EQB}(w)_{\mathbf{K}}$. □

Let $\mu \in W\lambda$. Recall that $v(\mu) \in W^S$ is the minimal-length coset representative for the coset $\{w \in W \mid w\lambda = \mu\}$. We set

$$\mathrm{QLS}^{\mu,\infty}(\lambda) := \{\psi \in \mathrm{QLS}(\lambda) \mid \kappa(\psi) \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor\}.$$

Remark 3.3.18. If $w = w_\circ$, then we have $\mathrm{EQB}(w_\circ) = W$ by Proposition 3.2.5 (1), since in this case, we can use all the positive roots as an edge label. If $\mu = \lambda_- = w_\circ\lambda$, then $v(\mu)w_\circ(S) = w_\circ$ by (3.3.1), and hence $\lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor = W^S$. Therefore, we have $\mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda) = \mathrm{QLS}(\lambda)$.

With the notation above, we set

$$\mathrm{gch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) := \sum_{\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_\mu(\psi)}.$$

The following is the main result of this section.

Theorem 3.3.19. *Let $\lambda \in P^+$ be a dominant weight, and $\mu \in W\lambda$. Then,*

$$E_\mu(q, \infty) = \mathrm{gch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda).$$

3.3.3 Proof of Theorem 3.3.19

Let $\lambda \in P^+$ be a dominant weight, $\mu \in W\lambda$, and set $S := S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. In this subsection, in order to prove Theorem 3.3.19, we give a bijection

$$\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$$

that preserves weights and degrees.

We fix reduced expressions

$$\begin{aligned} v(\lambda_-)v(\mu)^{-1} &= s_{i_1} \cdots s_{i_K}, \\ v(\mu) &= s_{i_{K+1}} \cdots s_{i_M}, && (3.3.13) \\ w_\circ(S) &= s_{i_{M+1}} \cdots s_{i_N} && (3.3.14) \end{aligned}$$

for $v(\lambda_-)v(\mu)^{-1}$, $v(\mu)$, and $w_\circ(S)$, respectively; recall that $\lambda_- = w_\circ\lambda$. Then, by Lemma 3.3.1 (4), $v(\lambda_-) = s_{i_1}\cdots s_{i_M}$ is a reduced expression for $v(\lambda_-)$. As in §3.3.1, we use the weak reflection order $\prec$ on Δ^+ introduced in Remark 3.3.7 (which satisfies (3.3.3)) determined by the reduced expressions above for $v(\lambda_-)$ and $w_\circ(S)$. Also, we use the total order $\prec'$ on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ defined just before Proposition 3.3.8 and take the reduced expression $m_{\lambda_-} = us_{\ell_1}\cdots s_{\ell_L}$ for m_{λ_-} given by Proposition 3.3.8; recall that $us_{\ell_k} = s_{i_k}u$ for $1 \le k \le M$. It follows from Lemma 3.3.1 (3) that $\bigl(v(\mu)v(\lambda_-)^{-1}\bigr)\, m_{\lambda_-} = m_\mu$ and $-\ell(v(\mu)v(\lambda_-)^{-1}) + \ell(m_{\lambda_-}) = \ell(m_\mu)$. Moreover, we see that

$$
\begin{aligned}
\bigl(v(\mu)v(\lambda_-)^{-1}\bigr)\, m_{\lambda_-} &= (s_{i_K}\cdots s_{i_1})\, us_{\ell_1}\cdots s_{\ell_L} \\
&\overset{\text{Lemma 3.3.10}}{=} us_{\ell_K}\cdots s_{\ell_1}s_{\ell_1}\cdots s_{\ell_L} = us_{\ell_{K+1}}\cdots s_{\ell_L},
\end{aligned}
$$

and hence $m_\mu = us_{\ell_{K+1}}\cdots s_{\ell_L}$ is a reduced expression for m_μ. In particular, when $\mu = \lambda$ (note that $v(\lambda) = e$), $m_\lambda = us_{\ell_{M+1}}\cdots s_{\ell_L}$ is a reduced expression for m_λ.

Also, recall from Remark 3.3.7 and the beginning of §3.3.1 that $\beta_k = s_{i_N}\cdots s_{i_{k+1}}\alpha_{i_k}$, $1 \le k \le N$, and $\beta^{\mathrm{OS}}_k = s_{\ell_L}\cdots s_{\ell_{k+1}}\alpha^\vee_{\ell_k}$, $1 \le k \le L$.

Remark 3.3.20. Keep the notation above. We have

$$
\begin{aligned}
\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} &= \{\beta^{\mathrm{OS}}_1, \dots, \beta^{\mathrm{OS}}_L\}, \\
\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} &= \{\beta^{\mathrm{OS}}_{K+1}, \dots, \beta^{\mathrm{OS}}_L\}, \\
\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} &= \{\beta^{\mathrm{OS}}_{M+1}, \dots, \beta^{\mathrm{OS}}_L\}.
\end{aligned}
$$

In particular, we have $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} \subset \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} \subset \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\lambda_-}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$.

Lemma 3.3.21 ([M1, (2.4.7) (i)])**.** *If we denote by ς the characteristic function of Δ^-, i.e.,*

$$
\varsigma(\gamma) := \begin{cases} 0 & \text{if } \gamma \in \Delta^+, \\ 1 & \text{if } \gamma \in \Delta^-, \end{cases}
$$

then

$$
\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}} = \{\alpha^\vee + a\widetilde{\delta} \mid \alpha \in \Delta^-, 0 < a < \varsigma(v(\mu)v(\lambda_-)^{-1}\alpha) + \langle \lambda, w_\circ\alpha^\vee\rangle\}.
$$

Remark 3.3.22. Let $\gamma_1, \gamma_2, \dots, \gamma_r \in \widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$, and define a sequence $(y_0, y_1, \dots, y_r; \gamma_1, \gamma_2, \dots, \gamma_r)$ by $y_0 = m_\mu$, and $y_i = y_{i-1}s_{\gamma_i}$ for $1 \le i \le r$. Then, the sequence $(y_0, y_1, \dots, y_r; \gamma_1, \gamma_2, \dots, \gamma_r)$ is an element of $\overleftarrow{\mathrm{QB}}(e; m_\mu)$ if and only if the following conditions hold:

(1) $\gamma_1 \prec' \gamma_2 \prec' \cdots \prec' \gamma_r$, where the order $\prec'$ is the weak reflection order on $\widetilde{\Delta}^+_{\mathrm{aff}} \cap m_{\mu}^{-1}\widetilde{\Delta}^-_{\mathrm{aff}}$ introduced at the beginning of §3.3.3;

(2) $\mathrm{dir}(y_{i-1}) \xleftarrow{-(\overline{\gamma_i})^\vee} \mathrm{dir}(y_i)$ is an edge of $\mathrm{QBG}(W)$ for $1 \le i \le r$.

In the following, we define a map $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$. Let p_J^{OS} be an arbitrary element of $\overleftarrow{\mathrm{QB}}(e; m_\mu)$ of the form

$$p_J^{\mathrm{OS}} = \left(m_\mu = z_0^{\mathrm{OS}}, z_1^{\mathrm{OS}}, \dots, z_r^{\mathrm{OS}}; \beta_{j_1}^{\mathrm{OS}}, \beta_{j_2}^{\mathrm{OS}}, \dots, \beta_{j_r}^{\mathrm{OS}}\right) \in \overleftarrow{\mathrm{QB}}(e; m_\mu),$$

with $J = \{j_1 < \cdots < j_r\} \subset \{K+1, \dots, L\}$. We set $x_k := \mathrm{dir}(z_k^{\mathrm{OS}})$, $0 \le k \le r$. Then, by the definition of $\overleftarrow{\mathrm{QB}}(e; m_\mu)$,

$$v(\mu)v(\lambda_-)^{-1} \overset{\text{Lemma 3.3.1}}{=} x_0 \xleftarrow{-\overline{\beta_{j_1}^{\mathrm{OS}}}^{\vee}} x_1 \xleftarrow{-\overline{\beta_{j_2}^{\mathrm{OS}}}^{\vee}} \cdots \xleftarrow{-\overline{\beta_{j_r}^{\mathrm{OS}}}^{\vee}} x_r \tag{3.3.15}$$

is a directed path in $\mathrm{QBG}(W)$. We take $0 = u_0 \le u_1 < \cdots < u_{s-1} < u_s = r$ and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{s-1} < 1 = \sigma_s$ in such a way that (see (3.3.6))

$$\underbrace{0 = d_{j_1} = \cdots = d_{j_{u_1}}}_{=\sigma_0} < \underbrace{d_{j_{u_1+1}} = \cdots = d_{j_{u_2}}}_{=\sigma_1} < \cdots < \underbrace{d_{j_{u_{s-1}+1}} = \cdots = d_{j_r}}_{=\sigma_{s-1}} < 1 = \sigma_s; \tag{3.3.16}$$

note that $d_{j_1} > 0$ if and only if $u_1 = 0$. We set $w'_p := x_{u_p}$ for $0 \le p \le s-1$, and $w'_s := x_r$. Then, by taking a subsequence of (3.3.15), we obtain the following directed path in $\mathrm{QBG}(W)$ for each $0 \le p \le s-1$:

$$w'_p = x_{u_p} \xleftarrow{-\overline{\beta_{j_{u_p+1}}^{\mathrm{OS}}}^{\vee}} x_{u_p+1} \xleftarrow{-\overline{\beta_{j_{u_p+2}}^{\mathrm{OS}}}^{\vee}} \cdots \xleftarrow{-\overline{\beta_{j_{u_{p+1}}}^{\mathrm{OS}}}^{\vee}} x_{u_{p+1}} = w'_{p+1}.$$

Multiplying this directed path on the right by $w_\circ$, we obtain the following directed path in $\mathrm{QBG}(W)$ for each $0 \le p \le s-1$ (see Lemma 3.2.4):

$$w_p := w'_p w_\circ = x_{u_p} w_\circ \xrightarrow{w_\circ \overline{\beta_{j_{u_p+1}}^{\mathrm{OS}}}^{\vee}} \cdots \xrightarrow{w_\circ \overline{\beta_{j_{u_{p+1}}}^{\mathrm{OS}}}^{\vee}} x_{u_{p+1}} w_\circ = w'_{p+1} w_\circ =: w_{p+1}. \tag{3.3.17}$$

Note that the edge labels of this directed path are increasing in the weak reflection order $\prec$ on Δ^+ introduced at the beginning of §3.3.3 (see Lemma 3.3.12), and lie in $\Delta^+ \setminus \Delta_S^+$; this property will be used to give the inverse to Ξ. Because

$$(1-\sigma_p)\langle \lambda, w_\circ \overline{\beta_{j_u}^{\mathrm{OS}}} \rangle = (1-d_{j_u})\langle \lambda, w_\circ \overline{\beta_{j_u}^{\mathrm{OS}}} \rangle = -\frac{a_{j_u}}{\langle \lambda_-, -\overline{\beta_{j_u}^{\mathrm{OS}}} \rangle} \langle \lambda_-, \overline{\beta_{j_u}^{\mathrm{OS}}} \rangle = a_{j_u} \in \mathbb{Z}$$

for $u_p + 1 \le u \le u_{p+1}$, $0 \le p \le s-1$, we find that (3.3.17) is a directed path in $\mathrm{QBG}_{(1-\sigma_p)\lambda}(W)$ for $0 \le p \le s-1$. Therefore, by Lemma 3.2.8, there exists a directed path in $\mathrm{QBG}_{(1-\sigma_p)\lambda}(W^S)$ from $\lfloor w_p \rfloor$ to $\lfloor w_{p+1} \rfloor$, where $S = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. Also, we claim that $\lfloor w_p \rfloor \ne \lfloor w_{p+1} \rfloor$ for $1 \le p \le s-1$. Suppose, for a contradiction, that $\lfloor w_p \rfloor = \lfloor w_{p+1} \rfloor$ for some p. Then, $w_p W_S = w_{p+1} W_S$, and hence $\min(w_{p+1} W_S, \le_{w_p}) = \min(w_p W_S, \le_{w_p}) = w_p$. Recall that the directed path (3.3.17) is a path in QBG from w_p to w_{p+1} whose labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$. By Lemma 3.2.9 (1), (2), the directed path (3.3.17) is a shortest path in QBG from w_p to $\min(w_{p+1} W_S, \le_{w_p}) = \min(w_p W_S, \le_{w_p}) = w_p$, which implies that the length of the directed path (3.3.17) is equal to 0. Therefore, $\{j_{u_p+1}, \dots, j_{u_{p+1}}\} = \emptyset$, and hence $u_p = u_{p+1}$, which contradicts the fact that $u_p < u_{p+1}$.

Thus we obtain

$$\psi := (\lfloor w_s \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, \dots, 1-\sigma_0) \in \mathrm{QLS}(\lambda). \tag{3.3.18}$$

We now define $\Xi(p_J^{\mathrm{OS}}) := \psi$.

Lemma 3.3.23. *Keep the notation and setting above, and let $s_{i_{K+1}} \cdots s_{i_M} s_{i_{M+1}} \cdots s_{i_N}$ be a reduced expression for $v(\mu)w_\circ(S)$ obtained by concatenating (3.3.13) and (3.3.14). Then, $\lfloor w_1 \rfloor \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor$. Hence we obtain a map $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$.*

Proof. Since it is clear that $v(\mu) \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor$, we may assume that $\lfloor w_1 \rfloor \neq v(\mu)$.

Since $z_0^{\mathrm{OS}} = m_\mu$, we have $w_0' = x_0 = \mathrm{dir}(z_0^{\mathrm{OS}}) = v(\mu)v(\lambda_-)^{-1}$. It follows that $w_0 = w_0' w_\circ = \left(v(\mu)v(\lambda_-)^{-1}\right) w_\circ \overset{\text{Lemma 3.3.1 (2)}}{=} v(\mu)w_\circ(S)$. If $u_1 = 0$, then we obtain $w_1 = w_0 = v(\mu)w_\circ(S)$, contrary to the assumption that $\lfloor w_1 \rfloor \neq v(\mu)$. Hence it follows that $u_1 \geq 1$. This implies that $j_{u_1} \leq M$ by the definition of u_1 in (3.3.16) and the proof of Lemma 3.3.10. Thus, we obtain $K+1 \leq j_1 < j_2 < \cdots < j_{u_1} \leq M$.

Now, consider the directed path (3.3.17) in the case $p = 0$. This is a (non-trivial) directed path in $\mathrm{QBG}(W)$ from $w_0 = v(\mu)w_\circ(S)$ to w_1 whose edge labels are increasing in the weak reflection order $\prec$ on Δ^+ introduced at the beginning of §3.3.3. Because these edge labels are $w_\circ \left(\overline{\beta_{j_k}^{\mathrm{OS}}}\right)^\vee = \beta_{j_k} = s_{i_N} \cdots s_{i_{j_k+1}} \alpha_{i_{j_k}}$ for $1 \leq k \leq u_1$ (the first equality follows from the proof of Lemma 3.3.10), it follows from the fact that $K+1 \leq j_1 < j_2 < \cdots < j_{u_1} \leq M$ and Remark 3.3.16 (recall that we take a reduced expression for $w_\circ$ given by concatenating the reduced expressions for $v(\lambda_-)v(\mu)^{-1}$ and $v(\mu)w_\circ(S)$) that $w_1 \in \mathrm{EQB}(v(\mu)w_\circ(S))$. Hence $\lfloor w_1 \rfloor \in \lfloor \mathrm{EQB}(v(\mu)w_\circ(S)) \rfloor$. □

Proposition 3.3.24. *The map $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$ is bijective.*

Proof. Let us give the inverse to Ξ. Take an arbitrary $\psi = (y_1, \dots, y_s; \tau_0, \dots, \tau_s) \in \mathrm{QLS}^{\mu,\infty}(\lambda)$. By convention, we set $y_{s+1} = v(\mu) \in W^S$. We define the elements v_p, $1 \leq p \leq s+1$, by: $v_{s+1} = v(\mu)w_\circ(S)$, and $v_p = \min(y_p W_S, \leq_{v_{p+1}})$ for $1 \leq p \leq s$.

Because there exists a directed path in $\mathrm{QBG}_{\tau_p\lambda}(W^S)$ from y_{p+1} to y_p for $1 \leq p \leq s-1$, we see from Lemma 3.2.9 (2), (3) that there exists a unique directed path

$$v_p \xleftarrow{-w_\circ\gamma_{p,1}} \cdots \xleftarrow{-w_\circ\gamma_{p,t_p}} v_{p+1} \tag{3.3.19}$$

in $\mathrm{QBG}_{\tau_p\lambda}(W)$ from v_{p+1} to v_p whose edge labels $-w_\circ\gamma_{p,t_p}, \dots, -w_\circ\gamma_{p,1}$ are increasing in the weak reflection order $\prec$ and lie in $\Delta^+ \setminus \Delta_S^+$ for $1 \leq p \leq s-1$; we remark that this is also true for $p = s$, since $\tau_s = 1$. Multiplying the vertices in this directed path on the right by $w_\circ$, we obtain by Lemma 3.2.4 the following directed paths:

$$v_{p,0} := v_p w_\circ \xrightarrow{\gamma_{p,1}} v_{p,1} \xrightarrow{\gamma_{p,2}} \cdots \xrightarrow{\gamma_{p,t_p}} v_{p+1}w_\circ =: v_{p,t_p}, \quad 1 \leq p \leq s.$$

Concatenating these paths for $1 \leq p \leq s$, we obtain the following directed path:

$$\begin{aligned} &v_{1,0} \xrightarrow{\gamma_{1,1}} \cdots \xrightarrow{\gamma_{1,t_1}} v_{1,t_1} = v_{2,0} \xrightarrow{\gamma_{2,1}} \cdots \xrightarrow{\gamma_{s-2,t_{s-2}}} v_{s-2,t_{s-2}} = v_{s-1,0} \xrightarrow{\gamma_{s-1,1}} \cdots \\ &\cdots \xrightarrow{\gamma_{s-1,t_{s-1}}} v_{s-1,t_{s-1}} = v_{s,0} \xrightarrow{\gamma_{s,1}} \cdots \xrightarrow{\gamma_{s,t_s}} v_{s,t_s} = v_{s+1,0} = v(\mu)v(\lambda_-)^{-1} \end{aligned} \tag{3.3.20}$$

in $\mathrm{QBG}(W)$. Now, for $1 \le p \le s$ and $1 \le m \le t_p$, we set $d_{p,m} := 1 - \tau_p \in \mathbb{Q} \cap [0, 1)$, $a_{p,m} := (d_{p,m} - 1)\langle \lambda_-, \gamma_{p,m}^\vee \rangle$, and $\widetilde{\gamma}_{p,m} := a_{p,m}\widetilde{\delta} - \gamma_{p,m}^\vee$.

Claim 1. $\widetilde{\gamma}_{p,m} \in \widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_\mu^{-1}\widetilde{\Delta}_{\mathrm{aff}}^-$.

Proof of Claim 1. Since $\tau_p > 0$, and since the path (3.3.19) is a directed path in $\mathrm{QBG}_{\tau_p\lambda}(W)$ whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$, we obtain $a_{p,m} = -\tau_p\langle \lambda_-, \gamma_{p,m}^\vee \rangle = \tau_p\langle \lambda, -w_\circ\gamma_{p,m}^\vee \rangle \in \mathbb{Z}_{>0}$.

We will show that $a_{p,m} < \varsigma(v(\mu)v(\lambda_-)^{-1}(-\gamma_{p,m})) + \langle \lambda, w_\circ\left(-\gamma_{p,m}^\vee\right)\rangle$. Here we note that the inequality $\langle \lambda, w_\circ\left(-\gamma_{p,m}^\vee\right)\rangle = -\langle \lambda_-, \gamma_{p,m}^\vee \rangle \ge -\tau_p\langle \lambda_-, \gamma_{p,m}^\vee \rangle = a_{p,m}$ holds, with equality if and only if $p = s$. Hence it suffices to consider the case $p = s$. In the case $p = s$, the path (3.3.19) is the unique directed path in $\mathrm{QBG}(W)$ from $v(\mu)w_\circ(S) = v_{s+1}$ to v_s whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$. Also, since $\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$ and $\kappa(\psi) = y_s = \lfloor v_s \rfloor$, we find that there exists $v_s' \in \mathrm{EQB}(v(\mu)w_\circ(S))$ such that $\lfloor v_s' \rfloor = y_s$. By the definition of $\mathrm{EQB}(v(\mu)w_\circ(S))$, there exists a unique directed path in $\mathrm{QBG}(W)$ from $v(\mu)w_\circ(S)$ to v_s' whose edge labels are increasing; we see from (3.3.3) that this directed path is obtained as the concatenation of the following two directed paths: the one whose edge labels lie in $\Delta^+ \setminus \Delta_S^+$, and the one whose edge labels lie in Δ_S^+. Therefore, by removing all the edges whose labels lie in Δ_S^+ from the path above, we obtain a directed path in $\mathrm{QBG}(W)$ from $v(\mu)w_\circ(S)$ to some $v_s'' \in y_sW_S \cap \mathrm{EQB}(v(\mu)w_\circ(S))$ whose edge labels are increasing and lie in $\Delta^+ \setminus \Delta_S^+$. Here, since $\lfloor v_s \rfloor = \lfloor v_s'' \rfloor$ and $v_s = \min(y_sW_S, \le_{v(\mu)w_\circ(S)})$, Lemma 3.2.9 (2) shows that $v_s = v_s''$. Hence we have $v_s \in \mathrm{EQB}(v(\mu)w_\circ(S))$. Moreover, by the definition of $\mathrm{EQB}(v(\mu)w_\circ(S))$, the edge labels $-w_\circ\gamma_{s,1}, \dots, -w_\circ\gamma_{s,t_s}$ in the given directed path in $\mathrm{QBG}(W)$ from $v(\mu)w_\circ(S) = v_{s+1}$ to v_s are elements of $\Delta^+ \cap (v(\mu)w_\circ(S))^{-1}\Delta^-$, and hence $v(\mu)w_\circ(S)\,(-w_\circ\gamma_{s,m}) \overset{\text{Lemma 3.3.1 (2)}}{=} v(\mu)v(\lambda_-)^{-1}(-\gamma_{s,m}) \in \Delta^-$. Therefore, in the case $p = s$, we have $\varsigma(v(\mu)v(\lambda_-)^{-1}(-\gamma_{s,m})) = 1$. Thus we have shown that $a_{s,m} = \langle \lambda, w_\circ\left(-\gamma_{s,m}^\vee\right)\rangle < \varsigma(v(\mu)v(\lambda_-)^{-1}(-\gamma_{s,m})) + \langle \lambda, w_\circ\left(-\gamma_{s,m}^\vee\right)\rangle$. Hence we conclude that $\widetilde{\gamma}_{p,m} \in \widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_\mu^{-1}\widetilde{\Delta}_{\mathrm{aff}}^-$ by Lemma 3.3.21.
■

Claim 2.

(1) *We have*

$$\widetilde{\gamma}_{s,t_s} \prec' \cdots \prec' \widetilde{\gamma}_{s,1} \prec' \widetilde{\gamma}_{s-1,t_{s-1}} \prec' \cdots \prec' \widetilde{\gamma}_{1,1},$$

where $\prec'$ denotes the weak reflection order on $\widetilde{\Delta}_{\mathrm{aff}}^+ \cap m_{\lambda_-}^{-1}\widetilde{\Delta}_{\mathrm{aff}}^-$ introduced at the beginning of §3.3.3; hence we choose $J' = \{j_1', \dots, j_{r'}'\} \subset \{K+1, \dots, L\}$ in such way that

$$\left(\beta_{j_1'}^{\mathrm{OS}}, \dots, \beta_{j_{r'}'}^{\mathrm{OS}}\right) = \left(\widetilde{\gamma}_{s,t_s}, \dots, \widetilde{\gamma}_{s,1}, \widetilde{\gamma}_{s-1,t_{s-1}}, \dots, \widetilde{\gamma}_{1,1}\right).$$

(2) *Let $1 \le k \le r'$, and take $1 \le p \le s$, $0 < m \le t_p$ such that $\left(\beta_{j_1'}^{\mathrm{OS}} \prec' \cdots \prec' \beta_{j_k'}^{\mathrm{OS}}\right) = (\widetilde{\gamma}_{s,t_s} \prec' \cdots \prec' \widetilde{\gamma}_{p,m})$. Then, $\mathrm{dir}(z_k^{\mathrm{OS}}) = v_{p,m-1}$. Moreover, $\mathrm{dir}(z_{k-1}^{\mathrm{OS}}) \xleftarrow{-\overline{\beta_{j_k'}^{\mathrm{OS}}}^\vee} \mathrm{dir}(z_k^{\mathrm{OS}})$ is an edge of $\mathrm{QBG}(W)$.*

Proof of Claim 2. (1) It suffices to show the following:

(i) for $1 \le p \le s$ and $1 < m \le t_p$, we have $\widetilde{\gamma}_{p,m} \prec' \widetilde{\gamma}_{p,m-1}$;

(ii) for $2 \le p \le s$, we have $\widetilde{\gamma}_{p,1} \prec' \widetilde{\gamma}_{p-1,t_{p-1}}$.

(i) Because $\frac{\langle \lambda_-, -\gamma_{p,m}^\vee \rangle - a_{p,m}}{\langle \lambda_-, -\gamma_{p,m}^\vee \rangle} = d_{p,m}$ and $\frac{\langle \lambda_-, -\gamma_{p,m-1}^\vee \rangle - a_{p,m-1}}{\langle \lambda_-, -\gamma_{p,m-1}^\vee \rangle} = d_{p,m-1}$, we have

$$\Phi(\widetilde{\gamma}_{p,m}) = (d_{p,m}, -w_\circ \gamma_{p,m}),$$
$$\Phi(\widetilde{\gamma}_{p,m-1}) = (d_{p,m-1}, -w_\circ \gamma_{p,m-1}).$$

Therefore, the first component of $\Phi(\widetilde{\gamma}_{p,m})$ is equal to that of $\Phi(\widetilde{\gamma}_{p,m-1})$ since $d_{p,m} = 1 - \tau_p = d_{p,m-1}$. Moreover, since $-w_\circ \gamma_{p,m} \prec -w_\circ \gamma_{p,m-1}$, we have $\Phi(\widetilde{\gamma}_{p,m}) < \Phi(\widetilde{\gamma}_{p,m-1})$. This implies that $\widetilde{\gamma}_{p,m} \prec' \widetilde{\gamma}_{p,m-1}$ by Proposition 3.3.8.

(ii) The proof of (ii) is similar to that of (i). The first components of $\Phi(\widetilde{\gamma}_{p,1})$ and $\Phi(\widetilde{\gamma}_{p-1,t_{p-1}})$ are $d_{p,1}$ and $d_{p-1,t_{p-1}}$, respectively. Since $d_{p,1} = 1 - \tau_p < 1 - \tau_{p-1} = d_{p-1,t_{p-1}}$, we have $\Phi(\widetilde{\gamma}_{p,1}) < \Phi(\widetilde{\gamma}_{p-1,t_{p-1}})$. This implies that $\widetilde{\gamma}_{p,1} \prec' \widetilde{\gamma}_{p-1,t_{p-1}}$.

(2) We proceed by induction on k. Since $\mathrm{dir}(z_0^{\mathrm{OS}}) = \mathrm{dir}(m_\mu) = v(\mu)v(\lambda_-)^{-1}$ and $\beta_{j'_1}^{\mathrm{OS}} = \widetilde{\gamma}_{s,t_s}$, we have $\mathrm{dir}(z_1^{\mathrm{OS}}) = \mathrm{dir}(z_0^{\mathrm{OS}}) s_{-\overline{\beta_{j'_1}^{\mathrm{OS}}}} = v(\mu)v(\lambda_-)^{-1} s_{\gamma_{s,t_s}} = v_{s,t_s-1}$. Hence the assertion holds in the case $k = 1$.

Assume that $\mathrm{dir}(z_{k-1}^{\mathrm{OS}}) = v_{p,m}$ for $0 \lneqq m \le t_p$; here we remark that $v_{p,m-1}$ is the predecessor of $v_{p,m}$ in the directed path (3.3.20) since $0 \le m-1 \le t_{p-1}$. Hence we have $\mathrm{dir}(z_k^{\mathrm{OS}}) = \mathrm{dir}(z_{k-1}^{\mathrm{OS}}) s_{-\overline{\beta_{j'_k}^{\mathrm{OS}}}} = v_{p,m} s_{\gamma_{p,m}} \overset{(3.3.20)}{=} v_{p,m-1}$. Also, since (3.3.20) is a directed path in $\mathrm{QBG}(W)$, $v_{p,m} = \mathrm{dir}(z_{k-1}^{\mathrm{OS}}) \xleftarrow{-\overline{\beta_{j'_k}^{\mathrm{OS}}}^\vee} \mathrm{dir}(z_k^{\mathrm{OS}}) = v_{p,m-1}$ is an edge of $\mathrm{QBG}(W)$. ■

Since $J' = \{j_1, \dots, j'_{r'}\} \subset \{K+1, \dots, L\}$, we can define an element $p_{J'}^{\mathrm{OS}}$ to be $\left(m_\mu = z_0^{\mathrm{OS}}, z_1^{\mathrm{OS}}, \dots, z_{r'}^{\mathrm{OS}}; \beta_{j'_1}^{\mathrm{OS}}, \beta_{j'_2}^{\mathrm{OS}}, \dots, \beta_{j'_{r'}}^{\mathrm{OS}}\right)$, where $z_0^{\mathrm{OS}} = m_\mu$, $z_k^{\mathrm{OS}} = z_{k-1}^{\mathrm{OS}} s_{\beta_{j'_k}^{\mathrm{OS}}}$ for $1 \le k \le r'$; it follows from Remark 3.3.22 and Claim 2 that $p_{J'}^{\mathrm{OS}} \in \overleftarrow{\mathrm{QB}}(e; m_\mu)$. Hence we can define a map $\Theta : \mathrm{QLS}^{\mu,\infty}(\lambda) \to \overleftarrow{\mathrm{QB}}(e; m_\mu)$ by $\Theta(\psi) := p_{J'}^{\mathrm{OS}}$.

It remains to show that the map Θ is the inverse to the map Ξ, i.e., the following two claims.

Claim 3. *For $\psi = (y_1, \dots, y_s; \tau_0, \dots, \tau_s) \in \mathrm{QLS}(\lambda)$, we have $\Xi \circ \Theta(\psi) = \psi$.*

Claim 4. *For $p_J^{\mathrm{OS}} = \left(m_\mu = z_0^{\mathrm{OS}}, z_1^{\mathrm{OS}}, \dots, z_r^{\mathrm{OS}}; \beta_{j_1}^{\mathrm{OS}}, \beta_{j_2}^{\mathrm{OS}}, \dots, \beta_{j_r}^{\mathrm{OS}}\right) \in \overleftarrow{\mathrm{QB}}(e; m_\mu)$, we have $\Theta \circ \Xi(p_J^{\mathrm{OS}}) = p_J^{\mathrm{OS}}$.*

Proof of Claim 3. We set $\Theta(\psi) = p_{J'}^{\mathrm{OS}}$, with $J' = \{j'_1, \dots, j'_r\}$. In the following description of $\Theta(\psi) = p_{J'}^{\mathrm{OS}}$, we employ the notation u_p, σ_p, w'_p, and w_p used in the definition of $\Xi(p_J^{\mathrm{OS}})$.

For $1 \le k \le r'$, if we set $\beta_{j'_k}^{\mathrm{OS}} = \widetilde{\gamma}_{p,m}$ with $m > 0$, then we have $d_{j'_k} = 1 + \frac{\deg(\beta_{j'_k}^{\mathrm{OS}})}{\langle \lambda_-, -\overline{\beta_{j'_k}^{\mathrm{OS}}} \rangle} = 1 + \frac{\deg(\widetilde{\gamma}_{p,m})}{\langle \lambda_-, -\overline{\widetilde{\gamma}_{p,m}} \rangle} = 1 + \frac{a_{p,m}}{\langle \lambda_-, \gamma_{p,m}^\vee \rangle} = d_{p,m}$. Therefore, the sequence (3.3.16) determined by $\Theta(\psi)=p^{\mathrm{OS}}_{J'}$ is

$$\underbrace{0=d_{s,t_s}=\cdots=d_{s,1}}_{=1-\tau_s}<\underbrace{d_{s-1,t_{s-1}}=\cdots=d_{s-1,1}}_{=1-\tau_{s-1}}<\cdots<\underbrace{d_{1,t_1}=\cdots=d_{1,1}}_{=1-\tau_1}<1=1-\tau_0. \tag{3.3.21}$$

Because the sequence (3.3.21) of rational numbers is just the sequence (3.3.16) for $\Theta(\psi)=p^{\mathrm{OS}}_{J'}$, we deduce that $\beta^{\mathrm{OS}}_{j'_{u_p}}=\widetilde{\gamma}_{s-p+1,1}$ for $1\le p\le s$, and $\sigma_p=1-\tau_{s-p}$ for $0\le p\le s$. Therefore, we have $w'_p=\mathrm{dir}(z^{\mathrm{OS}}_{u_p})=v_{s-p+1,0}$ and $w_p=v_{s-p+1,0}w_\circ=v_{s-p+1}$. Since $\lfloor w_p\rfloor=\lfloor v_{s-p+1}\rfloor=y_{s-p+1}$, we conclude that $\Xi\circ\Theta(\psi)=(\lfloor w_s\rfloor,\dots,\lfloor w_1\rfloor;1-\sigma_s,\dots,1-\sigma_0)=(y_1,\dots,y_s;\tau_0,\dots,\tau_s)=\psi$. ■

Proof of Claim 4. We set $\psi=\Xi(p^{\mathrm{OS}}_J)$, and write it as $\psi=(y_1,\dots,y_s;\tau_0,\dots,\tau_s)$, where $y_p=\lfloor w_{s+1-p}\rfloor$ for $1\le p\le s$ and $\tau_p=1-\sigma_{s-p}$ for $0\le p\le s$ in the notation of (3.3.18) (and the comment preceding it). Also, in the following description of $\Xi(p^{\mathrm{OS}}_J)=\psi$, we employ the notation $v_{p,m}$, $d_{p,m}$, $a_{p,m}$, $\gamma_{p,m}$, $\widetilde{\gamma}_{p,m}$, and J' used in the definition of $\Theta(\psi)$.

Recall that $w_0=v(\mu)w_\circ(S)=v_{s+1}$. For $0\le p\le s-1$,

$$v_{s-p+1}\xrightarrow{-w_\circ\gamma_{s-p,t_{s-p}}}\cdots\xrightarrow{-w_\circ\gamma_{s-p,1}}v_{s-p}$$

is a directed path in $\mathrm{QBG}(W)$ whose edge labels are increasing and lie in $\Delta^+\setminus\Delta^+_S$ (see (3.3.19)). Now we can show by induction on p that $w_p=v_{s-p+1}$ for $1\le p\le s$. Indeed, if $w_p=v_{s-p+1}$, then both of the path above and the path (3.3.17) start from w_p and end with some element in $w_{p+1}W_S=v_{s-p}W_S$ (this equality follows from the definition of v_{s-p}), and have increasing edge labels lying in $\Delta^+\setminus\Delta^+_S$. Therefore, by Lemma 3.2.9 (2), we deduce that the ends of these two paths are identical, and hence that $w_{p+1}=v_{s-p}$. Moreover, since these two paths are identical, so are the edge labels of them:

$$\left(w_\circ\overline{\beta^{\mathrm{OS}}_{j_{u_p+1}}}^{\vee}\prec\cdots\prec w_\circ\overline{\beta^{\mathrm{OS}}_{j_{u_{p+1}}}}^{\vee}\right)=\left(-w_\circ\gamma_{s-p,t_{s-p}}\prec\cdots\prec -w_\circ\gamma_{s-p,1}\right)$$

for $0\le p\le s-1$. From the above, we have $u_{p+1}-u_p=t_{s-p}$ and $-\overline{\beta^{\mathrm{OS}}_{j_{u_p+k}}}^{\vee}=\gamma_{s-p,t_{s-p}-k+1}$ for $0\le p\le s-1$, $1\le k\le t_{s-p}$. Because $\sigma_p=d_{j_{u_p+1}}=\cdots=d_{j_{u_{p+1}}}$ for $0\le p\le s-1$, $1-\sigma_p=\tau_{s-p}$ for $0\le p\le s$, and $1-\tau_{s-p}=d_{s-p,1}=\cdots=d_{s-p,t_{s-p}}$ for $0\le p\le s-1$, we see that for $1\le k\le t_{s-p}$,

$$\begin{aligned}
\beta^{\mathrm{OS}}_{j_{u_p+k}}&=\overline{\beta^{\mathrm{OS}}_{j_{u_p+k}}}+a_{j_{u_p+k}}\widetilde{\delta}\\
&=\overline{\beta^{\mathrm{OS}}_{j_{u_p+k}}}-(d_{j_{u_p+k}}-1)\langle\lambda_-,\overline{\beta^{\mathrm{OS}}_{j_{u_p+k}}}\rangle\widetilde{\delta}\\
&=-\gamma^\vee_{s-p,t_{s-p}-k+1}+(d_{s-p,t_{s-p}-k+1}-1)\langle\lambda_-,\gamma^\vee_{s-p,t_{s-p}-k+1}\rangle\widetilde{\delta}\\
&=-\gamma^\vee_{s-p,t_{s-p}-k+1}+a_{s-p,t_{s-p}-k+1}\widetilde{\delta}\\
&=\widetilde{\gamma}_{s-p,t_{s-p}-k+1}.
\end{aligned}$$

Therefore, we have

$$\left(\beta^{\mathrm{OS}}_{j_{u_p+1}}\prec'\cdots\prec'\beta^{\mathrm{OS}}_{j_{u_{p+1}}}\right)=\left(\widetilde{\gamma}_{s-p,t_{s-p}}\prec'\cdots\prec'\widetilde{\gamma}_{s-p,1}\right),\quad 0\le p\le s-1.$$

Concatenating the sequences above for $0 \le p \le s-1$, we obtain

$$\begin{aligned}\left(\beta^{\mathrm{OS}}_{j_1} \prec' \cdots \prec' \beta^{\mathrm{OS}}_{j_r}\right) &= \left(\widetilde{\gamma}_{s,t_s} \prec' \cdots \prec' \widetilde{\gamma}_{s,1} \prec' \widetilde{\gamma}_{s-1,t_{s-1}} \prec' \cdots \prec' \widetilde{\gamma}_{1,1}\right) \\ &= \left(\beta^{\mathrm{OS}}_{j'_1} \prec' \cdots \prec' \beta^{\mathrm{OS}}_{j'_{r'}}\right).\end{aligned}$$

Hence the set J' determined by $\Xi(p^{\mathrm{OS}}_J) = \psi$ is identical to J. Thus we conclude that $\Theta \circ \Xi(p^{\mathrm{OS}}_J) = p^{\mathrm{OS}}_{J'} = p^{\mathrm{OS}}_J$. ■

This completes the proof of Proposition 3.3.24. □

We recall from (3.2.4) and (3.2.5) that $\deg(\beta)$ is defined by $\beta = \overline{\beta} + \deg(\beta)\widetilde{\delta}$ for $\beta \in \mathfrak{h} \oplus \mathbb{C}\widetilde{\delta}$, and $\mathrm{wt}(u) \in P$ and $\mathrm{dir}(u)$ are defined by: $u = t(\mathrm{wt}(u))\mathrm{dir}(u)$ for $u \in \widetilde{W}_{\mathrm{ext}} = t(P) \rtimes W$.

Proposition 3.3.25. *The bijection $\Xi : \overleftarrow{\mathrm{QB}}(e; m_\mu) \to \mathrm{QLS}^{\mu,\infty}(\lambda)$ satisfies the following*:

(1) $\mathrm{wt}(\mathrm{end}(p^{\mathrm{OS}}_J)) = \mathrm{wt}(\Xi(p^{\mathrm{OS}}_J))$;

(2) $\deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_J)) = -\deg_\mu(\Xi(p^{\mathrm{OS}}_J))$.

Proof. We proceed by induction on $\#J$.

If $J = \emptyset$, then it is obvious that $\deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_J)) = \deg_\mu(\Xi(p^{\mathrm{OS}}_J)) = 0$ and $\mathrm{wt}(\mathrm{end}(p^{\mathrm{OS}}_J)) = \mathrm{wt}(\Xi(p^{\mathrm{OS}}_J)) = \mu$, since $\Xi(p^{\mathrm{OS}}_J) = (v(\mu)w_\circ(S); 0, 1)$.

Let $J = \{j_1 < j_2 < \cdots < j_r\}$, and set $K := J \setminus \{j_r\}$; assume that $\Xi(p^{\mathrm{OS}}_K)$ is of the form: $\Xi(p^{\mathrm{OS}}_K) = (\lfloor w_s \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, \dots, 1-\sigma_0)$. In the following, we employ the notation w_p, $0 \le p \le s$, used in the definition of the map Ξ. Note that $\mathrm{dir}(p^{\mathrm{OS}}_K) = w_s w_\circ$ and $w_0 = v(\mu)w_\circ(S)$ by the definition of Ξ. Also, observe that if $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, then $\{d_{j_1} \le \cdots \le d_{j_{r-1}} \le d_{j_r}\} = \{d_{j_1} \le \cdots \le d_{j_{r-1}}\}$, and if $d_{j_r} > d_{j_{r-1}} = \sigma_{s-1}$, then $\{d_{j_1} \le \cdots \le d_{j_{r-1}} \le d_{j_r}\} = \{d_{j_1} \le \cdots \le d_{j_{r-1}} < d_{j_r}\}$. From these, we deduce that

$$\Xi(p^{\mathrm{OS}}_J) = \begin{cases} (\lfloor w_s s_{w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}} \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, 1-\sigma_{s-1}, \dots, 1-\sigma_0) \\ \qquad\qquad \text{if } d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}, \\ (\lfloor w_s s_{w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}} \rfloor, \lfloor w_s \rfloor, \lfloor w_{s-1} \rfloor, \dots, \lfloor w_1 \rfloor; 1-\sigma_s, 1-d_{j_r}, 1-\sigma_{s-1} \dots, 1-\sigma_0) \\ \qquad\qquad \text{if } d_{j_r} > d_{j_{r-1}} = \sigma_{s-1}. \end{cases}$$

For the induction step, it suffices to show the following claims.

Claim 1.

(1) *We have*
$$\mathrm{wt}(\Xi(p^{\mathrm{OS}}_J)) = \mathrm{wt}(\Xi(p^{\mathrm{OS}}_K)) - a_{j_r} w_s w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}^\vee.$$

(2) *We have*
$$\deg_\mu(\Xi(p^{\mathrm{OS}}_J)) = \deg_\mu(\Xi(p^{\mathrm{OS}}_K)) - \chi_r a_{j_r},$$
where $\chi_r := 0$ (resp., $\chi_r := 1$) if $w_s s_{w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}} \leftarrow w_s$ is a Bruhat (resp., quantum) edge.

Claim 2.

(1) *We have*

$$\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}})) = \mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})) - a_{j_r} w_s w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}^{\vee}.$$

(2) *We have*

$$\deg(\mathrm{qwt}^*(p_J^{\mathrm{OS}})) = \deg(\mathrm{qwt}^*(p_K^{\mathrm{OS}})) + \chi_r a_{j_r}.$$

Proof of Claim 1. (1) If $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, then we compute:

$$\begin{aligned}
\mathrm{wt}(\Xi(p_J^{\mathrm{OS}})) =& (\sigma_s - \sigma_{s-1}) \lfloor w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \rfloor \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) \lfloor w_p \rfloor \lambda \\
=& (\sigma_s - \sigma_{s-1}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) w_p \lambda \\
=& \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (\sigma_s - \sigma_{s-1}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (\sigma_s - \sigma_{s-1}) w_s \lambda \\
\overset{d_{j_r} = \sigma_{s-1},\ \sigma_s = 1}{=}& \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (1 - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (1 - d_{j_r}) w_s \lambda.
\end{aligned}$$

If $d_{j_r} > d_{j_{r-1}} = \sigma_{s-1}$, then we compute:

$$\begin{aligned}
\mathrm{wt}(\Xi(p_J^{\mathrm{OS}})) =& (\sigma_s - d_{j_r}) \lfloor w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \rfloor \lambda + (d_{j_r} - \sigma_{s-1}) \lfloor w_s \rfloor \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) \lfloor w_p \rfloor \lambda \\
=& (\sigma_s - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda + (d_{j_r} - \sigma_{s-1}) w_s \lambda + \sum_{p=1}^{s-1} (\sigma_p - \sigma_{p-1}) w_p \lambda \\
=& \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda - (\sigma_s - \sigma_{s-1}) w_s \lambda \\
& \quad + (\sigma_s - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda + (d_{j_r} - \sigma_{s-1}) w_s \lambda \\
=& \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (\sigma_s - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (\sigma_s - d_{j_r}) w_s \lambda \\
\overset{\sigma_s = 1}{=}& \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda + (1 - d_{j_r}) w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \lambda - (1 - d_{j_r}) w_s \lambda.
\end{aligned}$$

In both cases above, since

$$\mathrm{wt}(\Xi(p_K^{\mathrm{OS}})) = \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) \lfloor w_p \rfloor \lambda = \sum_{p=1}^{s} (\sigma_p - \sigma_{p-1}) w_p \lambda,$$

and since

$$\begin{aligned}
&(1-d_{j_r})w_s s_{w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}}\lambda - (1-d_{j_r})w_s\lambda \\
&\qquad = -(1-d_{j_r})w_s\langle \lambda, w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}\rangle w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee} \\
&\qquad = -\frac{a_{j_r}}{\langle \lambda_-, \overline{\beta^{\mathrm{OS}}_{j_r}}\rangle}\langle \lambda_-, \overline{\beta^{\mathrm{OS}}_{j_r}}\rangle w_s w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee} \qquad \text{by Remark 3.3.11} \\
&\qquad = -a_{j_r} w_s w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee},
\end{aligned}$$

it follows that

$$\begin{aligned}
\mathrm{wt}(\Xi(p^{\mathrm{OS}}_J)) &= \sum_{p=1}^{s}(\sigma_p-\sigma_{p-1})w_p\lambda + (1-d_{j_r})w_s s_{w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}}\lambda - (1-d_{j_r})w_s\lambda \\
&= \mathrm{wt}(\Xi(p^{\mathrm{OS}}_K)) - a_{j_r} w_s w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee}.
\end{aligned}$$

(2) From the relation between p^{OS}_J and p^{OS}_K, and from the definition of $\overleftarrow{\mathrm{QB}}(e; m_\mu)$, we find that $w_s w_\circ s_{-\overline{\beta^{\mathrm{OS}}_{j_r}}} \xrightarrow{-\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee}} w_s w_\circ$ is an edge of $\mathrm{QBG}(W)$. Hence, by Lemma 3.2.4, $w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}} \xleftarrow{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee}} w_s$ is an edge of $\mathrm{QBG}(W)$.

If $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, then by the definition of $\deg_\mu$ (along with [LNSSS2, Lemma 7.2]), we see that

$$\begin{aligned}
\deg_\mu(\Xi(p^{\mathrm{OS}}_J)) &= -\sum_{p=0}^{s-2}(1-\sigma_p)\mathrm{wt}_\lambda(\lfloor w_{p+1}\rfloor \Leftarrow \lfloor w_p\rfloor) - (1-d_{j_r})\mathrm{wt}_\lambda(\lfloor w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}}\rfloor \Leftarrow \lfloor w_{s-1}\rfloor) \qquad (3.3.22)\\
&= -\sum_{p=0}^{s-2}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1} \Leftarrow w_p) - (1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}} \Leftarrow w_{s-1}).
\end{aligned}$$

Here, $w_0 = v(\mu)w_\circ(S)$ as mentioned in the proof of Lemma 3.3.23, so that $\lfloor w_0\rfloor = v(\mu)$. Since $d_{j_r} = d_{j_{r-1}} = \sigma_{s-1}$, we have $w_\circ\overline{\beta^{\mathrm{OS}}_{j_{r-1}}}^{\vee} \prec w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee}$ by Lemma 3.3.12. Because the (unique) label-increasing directed path in $\mathrm{QBG}(W)$ from w_{s-1} to w_s has the final edge label $w_\circ\overline{\beta^{\mathrm{OS}}_{j_{r-1}}}^{\vee}$, by concatenating this directed path from w_{s-1} to w_s with $w_s \xrightarrow{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}^{\vee}} w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}}$, we obtain a label-increasing (hence shortest) directed path from w_{s-1} to $w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}}$ passing through w_s. Therefore, we deduce that

$$\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}} \Leftarrow w_{s-1}) = \mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}} \leftarrow w_s) + \mathrm{wt}_\lambda(w_s \Leftarrow w_{s-1}). \qquad (3.3.23)$$

It follows from (3.3.22) and (3.3.23) that

$$\deg_\mu(\Xi(p^{\mathrm{OS}}_J)) = -\sum_{p=0}^{s-1}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1} \Leftarrow w_p) - (1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ\overline{\beta^{\mathrm{OS}}_{j_r}}} \leftarrow w_s).$$

If $d_{j_r} > d_{j_{r-1}} = \sigma_{s-1}$, then by the definition of $\deg_\mu$ (along with [LNSSS2, Lemma 7.2]), we see that

$$\deg_\mu(\Xi(p_J^{\mathrm{OS}})) = -\sum_{p=0}^{s-1}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1} \Leftarrow w_p) - (1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s),$$

where $w_0 = v(\mu)w_\circ(S)$. Also, by the definition of $\deg_\mu$ (along with [LNSSS2, Lemma 7.2]), we have

$$\deg_\mu(\Xi(p_K^{\mathrm{OS}})) = -\sum_{p=0}^{s-1}(1-\sigma_p)\mathrm{wt}_\lambda(w_{p+1} \Leftarrow w_p),$$

where $w_0 = v(\mu)w_\circ(S)$.

In both cases above, we deduce that

$$\deg_\mu(\Xi(p_J^{\mathrm{OS}})) = \deg_\mu(\Xi(p_K^{\mathrm{OS}})) - (1-d_{j_r})\mathrm{wt}_\lambda(w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s).$$

If $w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s$ is a Bruhat edge, then we have $\mathrm{wt}_\lambda(w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s) = 0$. If $w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s$ is a quantum edge, then we have $\mathrm{wt}_\lambda(w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \leftarrow w_s) = \langle \lambda, w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}} \rangle$. Note that

$$(1-d_{j_r})\langle \lambda, w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}} \rangle \overset{\text{Remark 3.3.11}}{=} \frac{a_{j_r}}{\langle \lambda_-, \overline{\beta_{j_r}^{\mathrm{OS}}} \rangle} \langle \lambda_-, \overline{\beta_{j_r}^{\mathrm{OS}}} \rangle = a_{j_r}.$$

Therefore, in both cases, we have $\deg_\mu(\Xi(p_J^{\mathrm{OS}})) = \deg_\mu(\Xi(p_K^{\mathrm{OS}})) - \chi_r a_{j_r}$, and Claim 1 (2) is proved. ■

Proof of Claim 2. Let us prove part (1). Note that $\mathrm{end}(p_J^{\mathrm{OS}}) = \mathrm{end}(p_K^{\mathrm{OS}}) s_{\beta_{j_r}^{\mathrm{OS}}}$, and that

$$\mathrm{end}(p_K^{\mathrm{OS}}) = t(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})))\mathrm{dir}(\mathrm{end}(p_K^{\mathrm{OS}})) = t(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})))w_s w_\circ;$$

the second equality follows from the comment at the beginning of the proof of Proposition 3.3.25. Also, we have $s_{\beta_{j_r}^{\mathrm{OS}}} = s_{a_{j_r}\tilde{\delta} + \overline{\beta_{j_r}^{\mathrm{OS}}}} = t\left(-a_{j_r} \overline{\beta_{j_r}^{\mathrm{OS}}}^\vee\right) s_{\overline{\beta_{j_r}^{\mathrm{OS}}}}$. Combining these, we obtain

$$\begin{aligned} \mathrm{end}(p_J^{\mathrm{OS}}) &= \left(t(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})))w_s w_\circ\right)\left(t\left(-a_{j_r}\overline{\beta_{j_r}^{\mathrm{OS}}}^\vee\right) s_{\overline{\beta_{j_r}^{\mathrm{OS}}}}\right) \\ &= t\left(\mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})) - a_{j_r} w_s w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}^\vee\right) w_s w_\circ s_{\overline{\beta_{j_r}^{\mathrm{OS}}}}, \end{aligned}$$

and hence

$$\mathrm{wt}(\mathrm{end}(p_J^{\mathrm{OS}})) = \mathrm{wt}(\mathrm{end}(p_K^{\mathrm{OS}})) - a_{j_r} w_s w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}^\vee.$$

Let us prove part (2). Since $\mathrm{dir}(\mathrm{end}(p_K^{\mathrm{OS}})) = w_s w_\circ$, we have $\mathrm{dir}(\mathrm{end}(p_J^{\mathrm{OS}})) = w_s w_\circ s_{\overline{\beta_{j_r}^{\mathrm{OS}}}}$. If $w_s s_{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}} \xleftarrow{w_\circ \overline{\beta_{j_r}^{\mathrm{OS}}}^\vee} w_s$ is a Bruhat edge, then it follows from Lemma

3.2.4 that $w_s w_\circ s_{-\overline{\beta^{\mathrm{OS}}_{j_r}}} \xrightarrow{-\overline{\beta^{\mathrm{OS}}_{j_r}}^\vee} w_s w_\circ$ is also a Bruhat edge. Hence we obtain $J^+ = K^+$. This implies that $\deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_J)) = \deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_K))$. If $w_s s_{w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}} \xleftarrow{w_\circ \overline{\beta^{\mathrm{OS}}_{j_r}}^\vee} w_s$ is a quantum edge, then it follows from Lemma 3.2.4 that $w_s w_\circ s_{-\overline{\beta^{\mathrm{OS}}_{j_r}}} \xrightarrow{-\overline{\beta^{\mathrm{OS}}_{j_r}}^\vee} w_s w_\circ$ is also a quantum edge. Hence we obtain $J^+ = K^+ \sqcup \{j_r\}$. This implies that $\deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_J)) = \deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_K)) + \deg(\beta^{\mathrm{OS}}_{j_r}) = \deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_K)) + a_{j_r}$. Therefore, in both cases, we have $\deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_J)) = \deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_K)) + \chi_r a_{j_r}$, and Claim 2 (2) is proved. ■

This completes the proof of Proposition 3.3.25. □

Proof of Theorem 3.3.19. We know from Proposition 3.2.11 that

$$E_\mu(q,\infty) = \sum_{p^{\mathrm{OS}}_J \in \overleftarrow{\mathrm{QB}}(e;m_\mu)} e^{\mathrm{wt}(\mathrm{end}(p^{\mathrm{OS}}_J))} q^{-\deg(\mathrm{qwt}^*(p^{\mathrm{OS}}_J))}.$$

Therefore, it follows from Propositions 3.3.24 and 3.3.25 that

$$E_\mu(q,\infty) = \sum_{\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_\mu(\psi)}.$$

Hence we conclude that $E_\mu(q,\infty) = \mathrm{gch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda)$, as desired. □

3.4 Demazure submodules of level-zero extremal weight modules

3.4.1 Untwisted affine root data

As in §2.3, we use the following notation.

Let $\mathfrak{g}_{\mathrm{aff}}$ be the untwisted affine Lie algebra over $\mathbb{C}$ associated to the finite-dimensional simple Lie algebra $\mathfrak{g}$, and $\mathfrak{h}_{\mathrm{aff}} = \left(\bigoplus_{j\in I_{\mathrm{aff}}} \mathbb{C}\alpha_j^\vee\right) \oplus \mathbb{C}D$ its Cartan subalgebra, where $\left\{\alpha_j^\vee\right\}_{j\in I_{\mathrm{aff}}} \subset \mathfrak{h}_{\mathrm{aff}}$ is the set of simple coroots, with $I_{\mathrm{aff}} = I \sqcup \{0\}$, and $D \in \mathfrak{h}_{\mathrm{aff}}$ is the degree operator. We denote by $\left\{\alpha_j\right\}_{j\in I_{\mathrm{aff}}} \subset \mathfrak{h}^*_{\mathrm{aff}}$ the set of simple roots, and by $\Lambda_j \in \mathfrak{h}^*_{\mathrm{aff}}$, $j \in I_{\mathrm{aff}}$, the fundamental weights. Note that $\langle \alpha_j, D\rangle = \delta_{j,0}$ and $\langle \Lambda_j, D\rangle = 0$ for $j \in I_{\mathrm{aff}}$, where $\langle\cdot\,,\cdot\rangle : \mathfrak{h}^*_{\mathrm{aff}} \times \mathfrak{h}_{\mathrm{aff}} \to \mathbb{C}$ denotes the canonical pairing between $\mathfrak{h}_{\mathrm{aff}}$ and $\mathfrak{h}^*_{\mathrm{aff}} := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{aff}}, \mathbb{C})$. Also, let $\delta = \sum_{j\in I_{\mathrm{aff}}} a_j\alpha_j \in \mathfrak{h}^*_{\mathrm{aff}}$ and $c = \sum_{j\in I_{\mathrm{aff}}} a_j^\vee \alpha_j^\vee \in \mathfrak{h}_{\mathrm{aff}}$ denote the null root and the canonical central element of $\mathfrak{g}_{\mathrm{aff}}$, respectively. Here we note that $\mathfrak{h}_{\mathrm{aff}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}D$; if we regard an element $\lambda \in \mathfrak{h}^*$ as an element of $\mathfrak{h}^*_{\mathrm{aff}}$ by: $\langle\lambda, c\rangle = \langle\lambda, D\rangle = 0$, then we have $\varpi_i = \Lambda_i - a_i^\vee\Lambda_0$ for $i \in I$. We take a weight lattice P_{aff} for $\mathfrak{g}_{\mathrm{aff}}$ as follows: $P_{\mathrm{aff}} = \left(\bigoplus_{j\in I_{\mathrm{aff}}} \mathbb{Z}\Lambda_j\right) \oplus \mathbb{Z}\delta \subset \mathfrak{h}^*_{\mathrm{aff}}$, and set $Q_{\mathrm{aff}} := \bigoplus_{j\in I_{\mathrm{aff}}} \mathbb{Z}\alpha_j$.

Remark 3.4.1. We should warn the reader that the root datum of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ is not necessarily dual to that of the untwisted affine Lie algebra associated to $\widetilde{\mathfrak{g}}$ in §3.2.2, though the root datum of $\widetilde{\mathfrak{g}}$ is dual to that of $\mathfrak{g}$. In particular, for the index $0 \in I_{\mathrm{aff}}$, the simple coroot $\alpha_0^\vee = c - \theta^\vee$, with $\theta \in \Delta^+$ the highest root of $\mathfrak{g}$, does not agree with the simple root $\widetilde{\delta} - \varphi^\vee$ in §3.2.2, which is denoted by $\alpha_0^\vee$ there.

The Weyl group W_{aff} of $\mathfrak{g}_{\mathrm{aff}}$ is defined to be the subgroup $\langle s_j \mid j \in I_{\mathrm{aff}} \rangle \subset \mathrm{GL}(\mathfrak{h}_{\mathrm{aff}}^*)$ generated by the simple reflections s_j associated to α_j for $j \in I_{\mathrm{aff}}$, with length function $\ell : W_{\mathrm{aff}} \to \mathbb{Z}_{\geq 0}$ and identity element $e \in W_{\mathrm{aff}}$. For $\xi \in Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$, let $t(\xi) \in W_{\mathrm{aff}}$ denote the translation in $\mathfrak{h}_{\mathrm{aff}}^*$ by ξ (see [Kac, §6.5]). Then we know from [Kac, Proposition 6.5] that $\{t(\xi) \mid \xi \in Q^\vee\}$ forms an abelian normal subgroup of W_{aff} such that $t(\xi)t(\zeta) = t(\xi+\zeta)$, $\xi, \zeta \in Q^\vee$, and $W_{\mathrm{aff}} = W \ltimes \{t(\xi) \mid \xi \in Q^\vee\}$. We denote by Δ_{aff} the set of real roots, i.e., $\Delta_{\mathrm{aff}} := \{x\alpha_j \mid x \in W_{\mathrm{aff}},\, j \in I_{\mathrm{aff}}\}$, and by $\Delta_{\mathrm{aff}}^+ \subset \Delta_{\mathrm{aff}}$ the set of positive real roots; we know from [Kac, Proposition 6.3] that

$$\begin{aligned} \Delta_{\mathrm{aff}} &= \{\alpha + n\delta \mid \alpha \in \Delta,\, n \in \mathbb{Z}\}, \\ \Delta_{\mathrm{aff}}^+ &= \Delta^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta,\, n \in \mathbb{Z}_{>0}\}. \end{aligned}$$

For $\beta \in \Delta_{\mathrm{aff}}$, we denote by $\beta^\vee \in \mathfrak{h}_{\mathrm{aff}}$ the dual root of β, and by $s_\beta \in W_{\mathrm{aff}}$ the reflection with respect to β. Note that if $\beta \in \Delta_{\mathrm{aff}}$ is of the form $\beta = \alpha + n\delta$ with $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then $s_\beta = s_\alpha t(n\alpha^\vee)$.

3.4.2 Peterson's coset representatives

Let S be a subset of I. Following [Pe] (see also [LS, §10]), we set:

$$\begin{aligned} Q_S^\vee &:= \sum_{i \in S} \mathbb{Z}\alpha_i^\vee, && (3.4.1) \\ (\Delta_S)_{\mathrm{aff}} &:= \{\alpha + n\delta \mid \alpha \in \Delta_S, n \in \mathbb{Z}\} \subset \Delta_{\mathrm{aff}}, && (3.4.2) \\ (\Delta_S)_{\mathrm{aff}}^+ &:= (\Delta_S)_{\mathrm{aff}} \cap \Delta_{\mathrm{aff}}^+ = \Delta_S^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta_S,\, n \in \mathbb{Z}_{>0}\}, && (3.4.3) \\ (W_S)_{\mathrm{aff}} &:= W_S \ltimes \{t(\xi) \mid \xi \in Q_S^\vee\} = \langle s_\beta \mid \beta \in (\Delta_S)_{\mathrm{aff}}^+ \rangle, && (3.4.4) \\ (W^S)_{\mathrm{aff}} &:= \{x \in W_{\mathrm{aff}} \mid x\beta \in \Delta_{\mathrm{aff}}^+ \text{ for all } \beta \in (\Delta_S)_{\mathrm{aff}}^+\}. && (3.4.5) \end{aligned}$$

Then we know the following from [Pe] (see also [LS, Lemma 10.6]).

Proposition 3.4.2. *For each $x \in W_{\mathrm{aff}}$, there exist a unique $x_1 \in (W^S)_{\mathrm{aff}}$ and a unique $x_2 \in (W_S)_{\mathrm{aff}}$ such that $x = x_1 x_2$.*

We define a (surjective) map $\Pi^S : W_{\mathrm{aff}} \to (W^S)_{\mathrm{aff}}$ by $\Pi^S(x) := x_1$ if $x = x_1 x_2$ with $x_1 \in (W^S)_{\mathrm{aff}}$ and $x_2 \in (W_S)_{\mathrm{aff}}$.

Lemma 3.4.3 ([Pe]; see also [LS, Proposition 10.10])**.**

(1) $\Pi^S(w) = \lfloor w \rfloor$ *for every* $w \in W$.

(2) $\Pi^S(xt(\xi)) = \Pi^S(x)\Pi^S(t(\xi))$ *for every* $x \in W_{\mathrm{aff}}$ *and* $\xi \in Q^\vee$.

An element $\xi \in Q^\vee$ is said to be S-adjusted if $\langle \gamma, \xi \rangle \in \{-1, 0\}$ for all $\gamma \in \Delta_S^+$ (see [LNSSS1, Lemma 3.8]). Let $Q^{\vee, S\text{-ad}}$ denote the set of S-adjusted elements.

Lemma 3.4.4 ([INS, Lemma 2.3.5])**.**

(1) *For each* $\xi \in Q^\vee$, *there exists a unique* $\phi_S(\xi) \in Q_S^\vee$ *such that* $\xi + \phi_S(\xi) \in Q^{\vee,\, S\text{-ad}}$. *In particular,* $\xi \in Q^{\vee,\, S\text{-ad}}$ *if and only if* $\phi_S(\xi) = 0$.

(2) *For each* $\xi \in Q^\vee$, *the element* $\Pi^S(t(\xi)) \in (W^S)_{\mathrm{aff}}$ *is of the form* $\Pi^S(t(\xi)) = z_\xi t(\xi + \phi_S(\xi))$ *for a specific element* $z_\xi \in W_S$. *Also,* $\Pi^S(wt(\xi)) = \lfloor w \rfloor z_\xi t(\xi + \phi_S(\xi))$ *for every* $w \in W$ *and* $\xi \in Q^\vee$.

(3) *We have*

$$(W^S)_{\mathrm{aff}} = \{ wz_\xi t(\xi) \mid w \in W^S,\, \xi \in Q^{\vee,\, S\text{-ad}} \}. \tag{3.4.6}$$

Remark 3.4.5. (1) Let $\xi, \zeta \in Q^\vee$. If $\xi \equiv \zeta \mod Q_S^\vee$, i.e., $\xi - \zeta \in Q_S^\vee$, then $\Pi^S(t(\xi)) = \Pi^S(t(\zeta))$ since $t(\xi - \zeta) \in (W_S)_{\mathrm{aff}}$. Hence we see by Lemma 3.4.4 (2) that $\xi + \phi_S(\xi) = \zeta + \phi_S(\zeta)$ and $z_\xi = z_\zeta$. In particular, $z_{\xi + \phi_S(\xi)} = z_\xi$ for every $\xi \in Q^\vee$.

(2) Let $x = wz_\xi t(\xi) \in (W^S)_{\mathrm{aff}}$, with $w \in W^S$ and $\xi \in Q^{\vee,\, S\text{-ad}}$; note that $\Pi^S(x) = x$. Then it follows from Lemma 3.4.3 (2) that for every $\zeta \in Q^\vee$,

$$x\Pi^S(t(\zeta)) = \Pi^S(x)\Pi^S(t(\zeta)) = \Pi^S(xt(\zeta)) \in (W^S)_{\mathrm{aff}}. \tag{3.4.7}$$

3.4.3 Parabolic semi-infinite Bruhat graph

In this subsection, we prove some technical lemmas, which we use later.

Definition 3.4.6 ([Pe]). Let $x \in W_{\mathrm{aff}}$, and write it as $x = wt(\xi)$ for $w \in W$ and $\xi \in Q^\vee$. Then we define the semi-infinite length $\ell^{\frac{\infty}{2}}(x)$ of x by $\ell^{\frac{\infty}{2}}(x) := \ell(w) + 2\langle \rho, \xi \rangle$, where $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$.

Let us fix a subset S of I.

Definition 3.4.7. (1) We define the (parabolic) semi-infinite Bruhat graph SiBG^S to be the Δ_{aff}^+-labeled, directed graph with vertex set $(W^S)_{\mathrm{aff}}$ and Δ_{aff}^+-labeled, directed edges of the following form: $x \xrightarrow{\beta} s_\beta x$ for $x \in (W^S)_{\mathrm{aff}}$ and $\beta \in \Delta_{\mathrm{aff}}^+$, where $s_\beta x \in (W^S)_{\mathrm{aff}}$ and $\ell^{\frac{\infty}{2}}(s_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$.

(2) The semi-infinite Bruhat order is a partial order $\preceq$ on $(W^S)_{\mathrm{aff}}$ defined as follows: for $x, y \in (W^S)_{\mathrm{aff}}$, we write $x \preceq y$ if there exists a directed path from x to y in SiBG^S; also, we write $x \prec y$ if $x \preceq y$ and $x \neq y$.

Let $[\,\cdot\,] = [\,\cdot\,]_{I \setminus S} : Q^\vee \twoheadrightarrow Q_{I\setminus S}^\vee$ denote the projection from $Q^\vee$ onto $Q_{I\setminus S}^\vee$ with kernel $Q_S^\vee$. Also, for $\xi, \zeta \in Q^\vee$, we write

$$\xi \geq \zeta \quad \text{if} \quad \xi - \zeta \in Q^{\vee,+} := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee. \tag{3.4.8}$$

The next lemma follows from [NS4, Remark 2.3.3].

Lemma 3.4.8. *Let* $u, v \in W^S$, $\xi, \zeta \in Q^{\vee,\, S\text{-ad}}$, *and* $\beta \in \Delta_{\mathrm{aff}}^+$. *If* $uz_\zeta t(\zeta) \xrightarrow{\beta} vz_\xi t(\xi)$ *in* SiBG^S, *then* $[\xi] \geq [\zeta]$.

Lemma 3.4.9. *Let* $x \in W^S$, *and* $\xi, \zeta \in Q^{\vee,\, S\text{-ad}}$. *Then,* $xz_\xi t(\xi) \succeq xz_\zeta t(\zeta)$ *if and only if* $[\xi] \geq [\zeta]$.

Proof. The "only if" part is obvious by Lemma 3.4.8. We show the "if" part by induction on $\ell(x)$. If $\ell(x) = 0$, i.e., $x = e$, then the assertion $z_\xi t(\xi) \succeq z_\zeta t(\zeta)$ follows from [INS, Lemma 6.2.1] (with $a = 1$, and J replaced by S). Assume now that $\ell(x) > 0$, and take $i \in I$ such that $\ell(s_i x) = \ell(x) - 1$; note that $s_i x \in W^S$ and $-x^{-1}\alpha_i \in \Delta^+ \setminus \Delta_S^+$. By induction hypothesis, we have $s_i x z_\xi t(\xi) \succeq s_i x z_\zeta t(\zeta)$. If we take a dominant weight $\lambda \in P^+$ such that $S_\lambda = \bigl\{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\bigr\} = S$, then we see that

$$\langle s_i x z_\xi t(\xi)\lambda, \alpha_i^\vee \rangle = \langle s_i x z_\zeta t(\zeta)\lambda, \alpha_i^\vee \rangle = \langle s_i x\lambda, \alpha_i^\vee \rangle > 0.$$

Therefore, we deduce from [NS4, Lemma 2.3.6 (3)] that $x z_\xi t(\xi) \succeq x z_\zeta t(\zeta)$, as desired. □

Lemma 3.4.10. *Let $x, y \in (W^S)_{\mathrm{aff}}$ and $\beta \in \Delta_{\mathrm{aff}}^+$ be such that $x \xrightarrow{\beta} y$ in SiBG^S. Then, $\Pi^S(xt(\xi)) \xrightarrow{\beta} \Pi^S(yt(\xi))$ in SiBG^S for every $\xi \in Q^\vee$. Therefore, if $x, y \in (W^S)_{\mathrm{aff}}$ satisfy $x \preceq y$, then $\Pi^S(xt(\xi)) \preceq \Pi^S(yt(\xi))$.*

Proof. We see (3.4.7) that $\Pi^S(xt(\xi)) = x\Pi^S(t(\xi))$ and $\Pi^S(yt(\xi)) = y\Pi^S(t(\xi))$. Since $y = s_\beta x$ by the assumption, we obtain $\Pi^S(yt(\xi)) = s_\beta \Pi^S(xt(\xi))$. Hence it suffices to show that

$$\ell^{\frac{\infty}{2}}(\Pi^S(yt(\xi))) = \ell^{\frac{\infty}{2}}(\Pi^S(xt(\xi))) + 1. \tag{3.4.9}$$

We write $x \in (W^S)_{\mathrm{aff}}$ as $x = w z_\zeta t(\zeta)$, with $w \in W^S$ and $\zeta \in Q^{\vee, S\text{-ad}}$ (see (3.4.6)). Then we see from [INS, Lemma A.2.1 and (A.2.1)] that

$$\begin{aligned}
\ell^{\frac{\infty}{2}}(\Pi^S(xt(\xi))) &= \ell(w) + 2\langle \rho - \rho_S, \zeta + \xi \rangle \\
&= \ell(w) + 2\langle \rho - \rho_S, \zeta \rangle + 2\langle \rho - \rho_S, \xi \rangle \\
&= \ell^{\frac{\infty}{2}}(\Pi^S(x)) + 2\langle \rho - \rho_S, \xi \rangle \\
&= \ell^{\frac{\infty}{2}}(x) + 2\langle \rho - \rho_S, \xi \rangle.
\end{aligned}$$

Similarly, we see that $\ell^{\frac{\infty}{2}}(\Pi^S(yt(\xi))) = \ell^{\frac{\infty}{2}}(y) + 2\langle \rho - \rho_S, \xi \rangle$. Since $\ell^{\frac{\infty}{2}}(y) = \ell^{\frac{\infty}{2}}(x) + 1$ by the assumption, we obtain (3.4.9), as desired. □

Let $x, y \in W^S$, and take a shortest directed path

$$\mathbf{p} : x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} x_2 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_p} x_p = y$$

from x to y in $\mathrm{QBG}(W^S)$. Recall from §3.2.1 that the weight $\mathrm{wt}^S(\mathbf{p})$ of this directed path is defined to be

$$\mathrm{wt}^S(\mathbf{p}) = \sum_{\substack{1 \le k \le p \\ x_{k-1} \xrightarrow{\gamma_k} x_k \text{ is} \\ \text{a quantum edge}}} \gamma_k^\vee \in Q^{\vee,+}.$$

We set

$$\xi_{x,y} := \mathrm{wt}^S(\mathbf{p}) + \phi_S(\mathrm{wt}^S(\mathbf{p})) \in Q^{\vee, S\text{-ad}} \tag{3.4.10}$$

in the notation of Lemma 3.4.4 (1). We now claim that $\xi_{x,y}$ does not depend on the choice of a shortest directed path $\mathbf{p}$ from x to y in $\mathrm{QBG}(W^S)$. Indeed, let

$\mathbf{p}'$ be another directed path from x to y in $\mathrm{QBG}(W^S)$. We know from [LNSSS1, Proposition 8.1] that $\mathrm{wt}^S(\mathbf{p}) = \mathrm{wt}^S(\mathbf{p}') \mod Q_S^\vee$. Therefore, by Remark 3.4.5 (1), we obtain $\mathrm{wt}^S(\mathbf{p}) + \phi_S(\mathrm{wt}^S(\mathbf{p})) = \mathrm{wt}^S(\mathbf{p}') + \phi_S(\mathrm{wt}^S(\mathbf{p}'))$. This proves the claim.

Lemma 3.4.11. *Let $x, y \in W^S$. Then we have $yz_{\xi_{x,y}}t(\xi_{x,y}) \succeq x$.*

Proof. We proceed by induction on the length p of a shortest directed path from x to y in $\mathrm{QBG}(W^S)$. If $p = 0$, i.e., $x = y$, then $\xi_{x,y} = \xi_{x,x} = 0$, and hence $z_{\xi_{x,y}} = t(\xi_{x,y}) = e$. Thus the assertion of the lemma is obvious. Assume now that $p > 0$, and let

$$\mathbf{p} : x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_p} x_p = y$$

be a shortest directed path from x to y in $\mathrm{QBG}(W^S)$. Then we deduce from [INS, Proposition A.1.2] that $x \xrightarrow{\beta} s_\beta x$ in SiBG^S (in particular, $s_\beta x \succeq x$), where

$$\beta := \begin{cases} x_0\gamma_1 & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a Bruhat edge,} \\ x_0\gamma_1 + \delta & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a quantum edge;} \end{cases}$$

note that

$$s_\beta x = s_\beta x_0 = \begin{cases} x_1 & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a Bruhat edge,} \\ x_1 t(\gamma_1^\vee) & \text{if } x = x_0 \xrightarrow{\gamma_1} x_1 \text{ is a quantum edge.} \end{cases}$$

In the case that $x = x_0 \xrightarrow{\gamma_1} x_1$ is a quantum edge, we have $x_1 t(\gamma_1^\vee) = s_\beta x \in (W^S)_{\mathrm{aff}}$, which implies, by (3.4.6) and the fact that $x_1 \in W^S$, that

$$\gamma_1^\vee \in Q^{\vee, S\text{-ad}} \text{ and } z_{\gamma_1^\vee} = e. \tag{3.4.11}$$

Assume first that $x = x_0 \xrightarrow{\gamma_1} x_1$ is a Bruhat edge. Note that $\mathbf{p}' : x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_p} x_p = y$ is a shortest directed path from x_1 to y in $\mathrm{QBG}(W^S)$. Since $\mathrm{wt}^S(\mathbf{p}) = \mathrm{wt}^S(\mathbf{p}')$ by the definition, we deduce that $\xi_{x,y} = \xi_{x_1,y}$. Also, by the induction hypothesis, we have $yz_{\xi_{x_1,y}}t(\xi_{x_1,y}) \succeq x_1$. Combining these, we obtain $yz_{\xi_{x,y}}t(\xi_{x,y}) = yz_{\xi_{x_1,y}}t(\xi_{x_1,y}) \succeq x_1 = s_\beta x \succeq x$, as desired.

Next, assume that $x = x_0 \xrightarrow{\gamma_1} x_1$ is a quantum edge; we have $\mathrm{wt}^S(\mathbf{p}) = \mathrm{wt}^S(\mathbf{p}') + \gamma_1^\vee$, which implies that $\xi_{x,y} \equiv \xi_{x_1,y} + \gamma_1^\vee \mod Q_S^\vee$. We compute

$$\begin{aligned} yz_{\xi_{x,y}}t(\xi_{x,y}) &= y\Pi^S(t(\xi_{x,y})) \quad \text{by Lemma 3.4.4 (2)} \\ &= y\Pi^S(t(\xi_{x_1,y})t(\xi_{x,y} - \xi_{x_1,y})) \\ &= y\Pi^S(t(\xi_{x_1,y}))\Pi^S(t(\xi_{x,y} - \xi_{x_1,y})) \quad \text{by Lemma 3.4.3 (2)} \\ &= yz_{\xi_{x_1,y}}t(\xi_{x_1,y})\Pi^S(t(\xi_{x,y} - \xi_{x_1,y})). \end{aligned}$$

Since $\xi_{x,y} \equiv \xi_{x_1,y} + \gamma_1^\vee \mod Q_S^\vee$, we see from Remark 3.4.5 (1) and (3.4.11) that $\Pi^S(t(\xi_{x,y} - \xi_{x_1,y})) = t(\gamma_1^\vee)$. Therefore, using the induction hypothesis $yz_{\xi_{x_1,y}}t(\xi_{x_1,y}) \succeq x_1$ and Lemma 3.4.10, we deduce that

$$\begin{aligned} \underbrace{yz_{\xi_{x,y}}t(\xi_{x,y})}_{\in (W^S)_{\mathrm{aff}}} &= \big(yz_{\xi_{x_1,y}}t(\xi_{x_1,y})\big)t(\gamma_1^\vee) = \Pi^S\big(\big(yz_{\xi_{x_1,y}}t(\xi_{x_1,y})\big)t(\gamma_1^\vee)\big) \succeq \Pi^S(x_1 t(\gamma_1^\vee)) \\ &= \Pi^S(s_\beta x) = s_\beta x \succeq x. \end{aligned}$$

This proves the lemma. $\square$

Lemma 3.4.12. *Let $x,\ y \in W^S$, and $\zeta \in Q^{\vee,\,S\text{-ad}}$. If $yz_\zeta t(\zeta) \succeq x$, then $[\zeta] \ge [\xi_{x,y}]$.*

Proof. We set

$$\widetilde{s}_j := \begin{cases} s_j & \text{if } j \neq 0, \\ s_\theta & \text{if } j = 0, \end{cases} \qquad \text{and} \qquad \widetilde{\alpha}_j := \begin{cases} \alpha_j & \text{if } j \neq 0, \\ -\theta & \text{if } j = 0. \end{cases}$$

We know from [LNSSS1, Lemma 6.12] that there exist a sequence $x = x_0,\ x_1,\ \ldots,\ x_n = e$ of elements of W^S and a sequence $i_1,\ \ldots,\ i_n \in I_{\mathrm{aff}} = I \sqcup \{0\}$ such that

$$x = x_0 \xrightarrow{x_0^{-1}\widetilde{\alpha}_{i_1}} x_1 \xrightarrow{x_1^{-1}\widetilde{\alpha}_{i_2}} \cdots \xrightarrow{x_{n-1}^{-1}\widetilde{\alpha}_{i_n}} x_n = e \quad \text{in } \mathrm{QBG}(W^S);$$

note that $x_{k-1}^{-1}\widetilde{\alpha}_{i_k} \in \Delta^+ \setminus \Delta_S^+$ for all $1 \le k \le n$. We prove the assertion of the lemma by induction on n.

Assume first that $n = 0$, i.e., $x = e$. Because $y \in W^S$ is greater than or equal to e in the (ordinary) Bruhat order, there exists a directed path $\mathbf{p}$ from e to y in $\mathrm{QBG}(W^S)$ whose edges are all Bruhat edges (see, e.g., [BB, Theorem 2.5.5]); since $\mathrm{wt}^S(\mathbf{p}) = 0$, we obtain $\xi_{e,y} = \mathrm{wt}^S(\mathbf{p}) + \phi_S(\mathrm{wt}^S(\mathbf{p})) = 0$. Also, if $yz_\zeta t(\zeta) \succeq x = e = ez_0 t(0)$, then it follows from Lemma 3.4.8 that $[\zeta] \ge [0] = [\xi_{e,y}]$, which proves the assertion in the case $n = 0$.

Assume next that $n > 0$; we set $i := i_1$ for simplicity of notation. Then, $x^{-1}\widetilde{\alpha}_i = x_0^{-1}\widetilde{\alpha}_i \in \Delta^+ \setminus \Delta_S^+$, and the assertion of the lemma holds for $x_1 = \widetilde{s}_i x_0 = \widetilde{s}_i x$ by the induction hypothesis.

Case (i). Assume that $y^{-1}\widetilde{\alpha}_i \in (-\Delta^+) \cup \Delta_S^+$. We deduce by [LNSSS1, Lemma 7.7 (3)] that

$$\xi_{\widetilde{s}_i x, y} \equiv \xi_{x,y} - \delta_{i,0} x^{-1}\widetilde{\alpha}_i^\vee \mod Q_S^\vee. \tag{3.4.12}$$

Assume first that $i \neq 0$. Let $\zeta \in Q^{\vee,\,S\text{-ad}}$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\alpha_i \in \Delta^+ \setminus \Delta_S^+$ and $y^{-1}\alpha_i \in (-\Delta^+) \cup \Delta_S^+$, we see from [INS, Lemma 4.1.6 (2)] that $yz_\zeta t(\zeta) \succeq s_i x = \widetilde{s}_i x$. Therefore, by the induction hypothesis, we obtain $[\zeta] \ge [\xi_{\widetilde{s}_i x, y}] \overset{(3.4.12)}{=} [\xi_{x,y}]$.

Assume next that $i = 0$. Let $\zeta \in Q^\vee$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\widetilde{\alpha}_0 = -x^{-1}\theta$ (= the finite part $\overline{x^{-1}\alpha_0}$ of $x^{-1}\alpha_0$) $\in \Delta^+ \setminus \Delta_S^+$, and $y^{-1}\widetilde{\alpha}_0 = -y^{-1}\theta$ (= the finite part $\overline{y^{-1}\alpha_0}$ of $y^{-1}\alpha_0$) $\in (-\Delta^+) \cup \Delta_S^+$, we see from [INS, Lemma 4.1.6 (2)] that

$$yz_\zeta t(\zeta) \succeq s_0 x = s_\theta x t(-x^{-1}\theta^\vee) = \underbrace{\widetilde{s}_0 x}_{=x_1} t(x^{-1}\widetilde{\alpha}_0^\vee)$$

Therefore, by Lemma 3.4.10,

$$\begin{aligned} \Pi^S(yz_\zeta t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) &= \Pi^S\big((yz_\zeta t(\zeta))t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) \\ &\succeq \Pi^S(\widetilde{s}_0 x t(x^{-1}\widetilde{\alpha}_0^\vee) t(-x^{-1}\widetilde{\alpha}_0^\vee)) = \Pi^S(\widetilde{s}_0 x) \\ &= \Pi^S(x_1) = x_1 = \widetilde{s}_0 x. \end{aligned}$$

If we write the left-hand side of this inequality as $\Pi^S(yz_\zeta t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) = yz_{\zeta'} t(\zeta')$ for some $\zeta' \in Q^{\vee,\,S\text{-ad}}$ (see Lemma 3.4.4 (2)), then we have $\zeta' \equiv \zeta - x^{-1}\widetilde{\alpha}_0^\vee$ mod

$Q_S^\vee$. Also, by the induction hypothesis, we have $[\zeta'] \ge [\xi_{\widetilde{s}_0 x, y}]$. Combining these, we obtain

$$[\zeta] = [\zeta' + x^{-1}\widetilde{\alpha}_0^\vee] \ge [\xi_{\widetilde{s}_0 x, y} + x^{-1}\widetilde{\alpha}_0^\vee] \overset{(3.4.12)}{=} [\xi_{x,y}],$$

as desired.

Case (ii). Assume that $y^{-1}\widetilde{\alpha}_i \in \Delta^+ \setminus \Delta_S^+$. By [LNSSS1, Lemma 7.7 (4)], we have

$$\xi_{\widetilde{s}_i x, \lfloor \widetilde{s}_i y \rfloor} \equiv \xi_{x,y} - \delta_{i,0} x^{-1}\widetilde{\alpha}_i^\vee + \delta_{i,0} y^{-1}\widetilde{\alpha}_i^\vee \mod Q_S^\vee. \tag{3.4.13}$$

Assume first that $i \ne 0$; note that $\widetilde{s}_i y = s_i y \in W^S$ (see, e.g., [LNSSS1, Proposition 5.10]). Let $\zeta \in Q^\vee$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\alpha_i \in \Delta^+ \setminus \Delta_S^+$ and $y^{-1}\alpha_i \in \Delta^+ \setminus \Delta_S^+$, we see that

$$\widetilde{s}_i y z_\zeta t(\zeta) = s_i y z_\zeta t(\zeta) \succeq s_i x = \widetilde{s}_i x \quad \text{by [NS4, Lemma 2.3.6 (3)]}.$$

Therefore, by the induction hypothesis, we obtain $[\zeta] \ge [\xi_{\widetilde{s}_i x, \widetilde{s}_i y}] \overset{(3.4.13)}{=} [\xi_{x,y}]$.

Assume next that $i = 0$. Let $\zeta \in Q^\vee$ be such that $yz_\zeta t(\zeta) \succeq x$. Because $x^{-1}\widetilde{\alpha}_0 = -x^{-1}\theta$ (= the finite part $\overline{x^{-1}\alpha_0}$ of $x^{-1}\alpha_0$) $\in \Delta^+ \setminus \Delta_S^+$ and $y^{-1}\widetilde{\alpha}_0 = -y^{-1}\theta$ (= the finite part $\overline{y^{-1}\alpha_0}$ of $y^{-1}\alpha_0$) $\in \Delta^+ \setminus \Delta_S^+$, we see from [NS4, Lemma 2.3.6 (3)] that $s_0 y z_\zeta t(\zeta) \succeq s_0 x$. Therefore, by Lemma 3.4.10, we have

$$\Pi^S\big((s_0 y z_\zeta t(\zeta)) t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) \succeq \Pi^S\big((s_0 x) t(-x^{-1}\widetilde{\alpha}_0^\vee)\big).$$

Here we have

$$\Pi^S\big((s_0 x) t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) = \Pi^S\big((\widetilde{s}_0 x t(x^{-1}\widetilde{\alpha}_0^\vee)) t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) = \widetilde{s}_0 x = x_1.$$

Also, using Lemma 3.4.4 (2), we compute

$$\begin{aligned}
\Pi^S\big((s_0 y z_\zeta t(\zeta)) t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) &= \Pi^S(s_0 y z_\zeta t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) \\
&= \Pi^S(s_0 y z_\zeta) \Pi^S(t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) = \Pi^S(s_0 y) \Pi^S(t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) \\
&= \Pi^S(\widetilde{s}_0 y t(y^{-1}\widetilde{\alpha}_0^\vee)) \Pi^S(t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) = \Pi^S(\widetilde{s}_0 y t(y^{-1}\widetilde{\alpha}_0^\vee) t(\zeta - x^{-1}\widetilde{\alpha}_0^\vee)) \\
&= \Pi^S(\widetilde{s}_0 y t(\zeta + y^{-1}\widetilde{\alpha}_0^\vee - x^{-1}\widetilde{\alpha}_0^\vee)).
\end{aligned}$$

If we write this element as $\Pi^S\big((s_0 y z_\zeta t(\zeta)) t(-x^{-1}\widetilde{\alpha}_0^\vee)\big) = \lfloor s_0 y \rfloor z_{\zeta''} t(\zeta'')$ for some $\zeta'' \in Q^{\vee, S\text{-ad}}$ (see Lemma 3.4.4 (2)), we see that $\zeta'' \equiv \zeta + y^{-1}\widetilde{\alpha}_0^\vee - x^{-1}\widetilde{\alpha}_0^\vee \mod Q_S^\vee$. In addition, by the induction hypothesis, we have $[\zeta''] \ge [\xi_{\widetilde{s}_0 x, \lfloor \widetilde{s}_0 y \rfloor}]$. Combining these, we obtain

$$\begin{aligned}
[\zeta] &= [\zeta'' - y^{-1}\widetilde{\alpha}_0^\vee + x^{-1}\widetilde{\alpha}_0^\vee] \\
&\ge [\xi_{\widetilde{s}_0 x, \lfloor \widetilde{s}_0 y \rfloor} - y^{-1}\widetilde{\alpha}_0^\vee + x^{-1}\widetilde{\alpha}_0^\vee] \overset{(3.4.13)}{=} [\xi_{x,y}],
\end{aligned}$$

as desired. This completes the proof of the lemma. $\square$

3.4.4 Semi-infinite Lakshmibai-Seshadri paths

Let $\lambda \in P^+$ be a dominant weight; we set $S := S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\} \subset I$.

Definition 3.4.13. For a rational number $0 < \sigma \leq 1$, define $\mathrm{SiBG}(\lambda\,;\sigma)$ to be the subgraph of SiBG^S with the same vertex set but having only the edges of the form: $x \xrightarrow{\beta} y$ with $\sigma\langle x\lambda, \beta^\vee\rangle \in \mathbb{Z}$; note that $\mathrm{SiBG}(\lambda\,;1) = \mathrm{SiBG}^S$.

Definition 3.4.14. A semi-infinite Lakshmibai-Seshadri (SiLS for short) path of shape λ is, by definition, a pair $\eta = (x_1 \succ \cdots \succ x_s\,; 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1)$ of a (strictly) decreasing sequence $x_1 \succ \cdots \succ x_s$ of elements in $(W^S)_{\mathrm{aff}}$ and an increasing sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers such that there exists a directed path from x_{u+1} to x_u in $\mathrm{SiBG}(\lambda\,;\sigma_u)$ for all $u = 1, 2, \ldots, s-1$. We denote by $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ the set of all SiLS paths of shape λ.

Following [INS, §3.1] (see also [NS4, §2.4]), we endow the set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ with a crystal structure with weights in P_{aff} by the root operators e_i, f_i, $i \in I_{\mathrm{aff}}$, and the map $\mathrm{wt} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to P_{\mathrm{aff}}$ defined by

$$\begin{gathered}\mathrm{wt}(\eta) := \sum_{u=1}^{s}(\sigma_u - \sigma_{u-1})x_u\lambda \in P_{\mathrm{aff}} \\ \text{for } \eta = (x_1, \ldots, x_s\,; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda).\end{gathered} \tag{3.4.14}$$

Let $\mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ denote the set of all connected components of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$, and let $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ denote the connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ containing $\eta_e := (e\,; 0, 1) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$.

Also, we define a surjective map $\mathrm{cl} : (W^S)_{\mathrm{aff}} \twoheadrightarrow W^S$ by

$$\mathrm{cl}(x) = w \quad \text{if } x = wz_\xi t(\xi), \text{ with } w \in W^S \text{ and } \xi \in Q^{\vee, S\text{-ad}},$$

and for $\eta = (x_1, \ldots, x_s\,; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, we set

$$\mathrm{cl}(\eta) := (\mathrm{cl}(x_1), \ldots, \mathrm{cl}(x_s)\,; \sigma_0, \sigma_1, \ldots, \sigma_s);$$

where, for each $1 \leq p < q \leq s$ such that $\mathrm{cl}(x_p) = \cdots = \mathrm{cl}(x_q)$, we drop $\mathrm{cl}(x_p), \ldots, \mathrm{cl}(x_{q-1})$ and $\sigma_p, \ldots, \sigma_{q-1}$. We know from [NS4, §6.2] that $\mathrm{cl}(\eta) \in \mathrm{QLS}(\lambda)$. Thus we obtain a map $\mathrm{cl} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \mathrm{QLS}(\lambda)$.

Remark 3.4.15. Recall that $\psi_e := (e\,; 0, 1) \in \mathrm{QLS}(\lambda)$. We see from the definition that an element in $\mathrm{cl}^{-1}(\psi_e)$ is of the form:

$$(z_{\xi_1}t(\xi_1), z_{\xi_2}t(\xi_2), \ldots, z_{\xi_{s-1}}t(\xi_{s-1}), z_{\xi_s}t(\xi_s)\,; \sigma_0, \sigma_1, \ldots, \sigma_{s-1}, \sigma_s) \tag{3.4.15}$$

for some $s \geq 1$ and $\xi_1, \xi_2, \ldots, \xi_s \in Q^{\vee, S\text{-ad}}$.

The final direction of $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is defined to be

$$\kappa(\eta) := x_s \in (W^S)_{\mathrm{aff}} \quad \text{if } \eta = (x_1, \ldots, x_s\,; \sigma_0, \sigma_1, \ldots, \sigma_s). \tag{3.4.16}$$

Then, for $x \in (W^S)_{\mathrm{aff}}$, we set

$$\mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) := \{\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \mid \kappa(\eta) \succeq x\}. \tag{3.4.17}$$

The next lemma follows from [INS, Lemma 7.1.4].

Lemma 3.4.16. *Let $\eta \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$, and let X be a monomial in root operators such that $\eta = X\eta_e$. Assume that $\eta_0 \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is of the form* (3.4.15)*. Then, $\kappa(X\eta_0) = \kappa(\eta)\kappa(\eta_0)$.*

Now, we recall from §3.3.2 the degree function $\deg_\lambda : \mathrm{QLS}(\lambda) \to \mathbb{Z}_{\le 0}$ for the case $\mu = \lambda$. We know the following lemma from [NS4, Lemma 6.2.3].

Lemma 3.4.17. *For each $\psi \in \mathrm{QLS}(\lambda)$, there exists a unique $\eta_\psi \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ such that $\mathrm{cl}(\eta_\psi) = \psi$ and $\kappa(\eta_\psi) \in W^S$.*

Let $\psi \in \mathrm{QLS}(\lambda)$. We know from [NS4, (6.2.5)] that $\mathrm{wt}(\eta_\psi)$ is of the form:

$$\mathrm{wt}(\eta_\psi) = \underbrace{\lambda - \gamma}_{=\mathrm{wt}(\psi)} + K\delta \quad \text{for some } \gamma \in Q^+ \text{ and } K \in \mathbb{Z}_{\le 0}. \tag{3.4.18}$$

Also, we know from [LNSSS2, Corollary 4.8] (see also the comment after [NS4, (6.2.5)]) that

$$K = -\sum_{u=1}^{s-1} \sigma_u \mathrm{wt}_\lambda(w_{u+1} \Rightarrow w_u) = \deg_\lambda(\psi) \tag{3.4.19}$$

for $\psi = (w_1, \ldots, w_s\,; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda)$. Here we should note that in the definition of $\deg_\lambda(\psi)$, $w_{s+1} = v(\lambda) = e$, and hence that $\mathrm{wt}_\lambda(w_{s+1} \Rightarrow w_s) = \mathrm{wt}_\lambda(e \Rightarrow w_s) = 0$.

Let us write a dominant weight $\lambda \in P^+$ as $\lambda = \sum_{i\in I} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\ge 0}$ for $i \in I$, and define $\overline{\mathrm{Par}(\lambda)}$ (resp., $\mathrm{Par}(\lambda)$) to be the set of I-tuples $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I}$ of partitions such that $\rho^{(i)}$ is a partition of length less than or equal to m_i (resp., strictly less than m_i) for each $i \in I$. A partition of length less than 0 is understood to be the empty partition $\emptyset$; note that $\mathrm{Par}(\lambda) \subset \overline{\mathrm{Par}(\lambda)}$. Also, for $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I} \in \overline{\mathrm{Par}(\lambda)}$, we set $|\boldsymbol{\rho}| := \sum_{i\in I} |\rho^{(i)}|$, where for a partition $\chi = (\chi_1 \ge \chi_2 \ge \cdots \ge \chi_m)$, we set $|\chi| := \chi_1 + \cdots + \chi_m$. Following [INS, (3.2.2)], we endow the set $\mathrm{Par}(\lambda)$ with a crystal structure with weights in P_{aff}; note that $\mathrm{wt}(\boldsymbol{\rho}) = -|\boldsymbol{\rho}|\delta$.

Proposition 3.4.18. *Keep the notation above.*

(1) *Each connected component $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ contains a unique element of the form:*

$$\eta^C = (z_{\xi_1} t(\xi_1), z_{\xi_2} t(\xi_2), \ldots, z_{\xi_{s-1}} t(\xi_{s-1}), e\,; \sigma_0, \sigma_1, \ldots, \sigma_{s-1}, \sigma_s) \tag{3.4.20}$$

for some $s \ge 1$ and $\xi_1, \xi_2, \ldots, \xi_{s-1} \in Q^{\vee, S\text{-ad}}$ (see [INS, Proposition 7.1.2]*).*

(2) *There exists a bijection $\Theta : \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)) \to \mathrm{Par}(\lambda)$ such that $\mathrm{wt}(\eta^C) = \lambda - |\Theta(C)|\delta$ (see* [INS, Proposition 7.2.1 and its proof]*).*

(3) *Let $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$. Then, there exists an isomorphism $C \xrightarrow{\sim} \{\Theta(C)\} \otimes \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ of crystals that maps η^C to $\Theta(C) \otimes \eta_e$. Consequently, $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is isomorphic as a crystal to $\mathrm{Par}(\lambda) \otimes \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ (see* [INS, Proposition 3.2.4 and its proof]*).*

3.4.5 Extremal weight modules

In this and the next subsection, we mainly follow the notation of [NS4, §4 and §5]; we use the symbol "v" for the quantum parameter in order to distinguish it from $q = e^{\delta}$. Let $\lambda \in P^+$ be a dominant weight. We denote by $V(\lambda)$ the extremal weight module of extremal weight λ over a quantum affine algebra $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$. This is the integrable $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module generated by a single element v_λ with the defining relation that v_λ is an "extremal weight vector" of weight λ (for details, see [Kas1, §8] and [Kas2, §3]). We know from [Kas1, Proposition 8.2.2] that $V(\lambda)$ has a crystal basis $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ with global basis $\{G(b) \mid b \in \mathcal{B}(\lambda)\}$. Denote by u_λ the element of $\mathcal{B}(\lambda)$ such that $G(u_\lambda) = v_\lambda \in V(\lambda)$, and by $\mathcal{B}_0(\lambda)$ the connected component of $\mathcal{B}(\lambda)$ containing u_λ.

Let $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}}) \subset U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ denote the a quantum affine algebra without the degree operator. We know the following from [Kas2] (see also [NS4, §5.2]):

(i) for each $i \in I$, there exists a $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module automorphism $z_i : V(\varpi_i) \to V(\varpi_i)$ that maps v_{ϖ_i} to $v_{\varpi_i}^{[1]} := G(u_{\varpi_i}^{[1]})$, where $u_{\varpi_i}^{[1]} \in \mathcal{B}(\varpi_i)$ is a (unique) element of weight $\varpi_i + \delta$;

(ii) the map $z_i : V(\varpi_i) \to V(\varpi_i)$ induces a bijection $z_i : \mathcal{B}(\varpi_i) \to \mathcal{B}(\varpi_i)$ that maps u_{ϖ_i} to $u_{\varpi_i}^{[1]}$; this map commutes with the Kashiwara operators e_j, f_j, $j \in I_{\mathrm{aff}}$, on $\mathcal{B}(\varpi_i)$.

Let us write a dominant weight $\lambda \in P^+$ as $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I$. We fix an arbitrary total ordering on I, and then set $\widetilde{V}(\lambda) := \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i}$. By [BN, eq. (4.8) and Corollary 4.15], there exists a $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module embedding $\Phi_\lambda : V(\lambda) \hookrightarrow \widetilde{V}(\lambda)$ that maps v_λ to $\widetilde{v}_\lambda := \bigotimes_{i \in I} v_{\varpi_i}^{\otimes m_i}$. Also, for each $i \in I$ and $1 \leq k \leq m_i$, we define $z_{i,k}$ to be the $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module automorphism of $\widetilde{V}(\lambda)$ that acts as z_i only on the k-th factor of $V(\varpi_i)^{\otimes m_i}$ in $\widetilde{V}(\lambda)$, and as the identity map on the other factors of $\widetilde{V}(\lambda)$; these $z_{i,k}$'s, $i \in I$, $1 \leq k \leq m_i$, commute with each other. Now, for $\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}$, we set

$$s_{\boldsymbol{\rho}}(z^{-1}) := \prod_{i \in I} s_{\rho^{(i)}}(z_{i,1}^{-1}, \ldots, z_{i,m_i}^{-1}). \tag{3.4.21}$$

Here, for a partition $\rho = (\rho_1 \geq \cdots \geq \rho_{m-1} \geq 0)$ of length less than $m \in \mathbb{Z}_{\geq 1}$, $s_\rho(x) = s_\rho(x_1, \ldots, x_m)$ denotes the Schur polynomial in the variables $x_1, \ldots, x_m$ corresponding to the partition ρ. We can easily show (see [NS4, §7.3]) that $s_{\boldsymbol{\rho}}(z^{-1})(\operatorname{Img}\Phi_\lambda) \subset \operatorname{Img}\Phi_\lambda$ for each $\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}$. Hence we can define a $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-module homomorphism $z_{\boldsymbol{\rho}} : V(\lambda) \to V(\lambda)$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\Phi_\lambda} & \widetilde{V}(\lambda) \\ {\scriptstyle z_{\boldsymbol{\rho}}}\big\downarrow & & \big\downarrow{\scriptstyle s_{\boldsymbol{\rho}}(z^{-1})} \\ V(\lambda) & \xrightarrow{\Phi_\lambda} & \widetilde{V}(\lambda); \end{array} \tag{3.4.22}$$

note that $z_{\boldsymbol{\rho}} v_\lambda = S_{\boldsymbol{\rho}}^{-} v_\lambda$ in the notation of [BN] (and [NS4]). The map $z_{\boldsymbol{\rho}} : V(\lambda) \to V(\lambda)$ induces a $\mathbb{C}$-linear map $z_{\boldsymbol{\rho}} : \mathcal{L}(\lambda)/\mathsf{v}\mathcal{L}(\lambda) \to \mathcal{L}(\lambda)/\mathsf{v}\mathcal{L}(\lambda)$; this map commutes with Kashiwara operators. It follows from [BN, p. 371] that

$$\mathcal{B}(\lambda) = \{z_{\boldsymbol{\rho}} b \mid \boldsymbol{\rho} \in \mathrm{Par}(\lambda),\, b \in \mathcal{B}_0(\lambda)\}; \tag{3.4.23}$$

for $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$, we set

$$u^{\boldsymbol{\rho}} := z_{\boldsymbol{\rho}} u_\lambda \in \mathcal{B}(\lambda). \tag{3.4.24}$$

Remark 3.4.19. We see from [BN, Theorem 4.16 (ii)] (see also the argument after [NS4, (7.3.8)]) that $z_{\boldsymbol{\rho}} G(b) = G(z_{\boldsymbol{\rho}} b)$ for $b \in \mathcal{B}_0(\lambda)$ and $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)}$.

3.4.6 Demazure submodules

Let $\lambda \in P^+$ be a dominant weight. For each $x \in W_{\mathrm{aff}}$, we set

$$V_x^-(\lambda) := U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{aff}}) S_x^{\mathrm{norm}} v_\lambda \subset V(\lambda), \tag{3.4.25}$$

where $S_x^{\mathrm{norm}} v_\lambda$ denotes the extremal weight vector of weight $x\lambda$ (see, e.g., [NS4, (3.2.1)]), and $U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{aff}})$ is the negative part of $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$. Since $V_x^-(\lambda) = V_{\Pi^S(x)}^-(\lambda)$ for $x \in W_{\mathrm{aff}}$ by [NS4, Lemma 4.1.2], we consider Demazure submodules $V_x^-(\lambda)$ only for $x \in (W^S)_{\mathrm{aff}}$ in what follows. We know from [Kas3, §2.8] and [NS4, §4.1] that $V_x^-(\lambda)$ is "compatible" with the global basis of $V(\lambda)$; namely, there exists a subset $\mathcal{B}_x^-(\lambda) \subset \mathcal{B}(\lambda)$ such that

$$V_x^-(\lambda) = \bigoplus_{b \in \mathcal{B}_x^-(\lambda)} \mathbb{C}(\mathsf{v}) G(b) \subset V(\lambda) = \bigoplus_{b \in \mathcal{B}(\lambda)} \mathbb{C}(\mathsf{v}) G(b). \tag{3.4.26}$$

We know the following theorem from [INS, Theorem 3.2.1] and [NS4, Theorem 4.2.1].

Theorem 3.4.20. *Let $\lambda \in P^+$ be a dominant weight. There exists an isomorphism $\Psi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of crystals such that*

(a) $\Psi_\lambda(u^{\boldsymbol{\rho}}) = \eta^{\Theta^{-1}(\boldsymbol{\rho})}$ *for all* $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$ (*in particular,* $\Psi_\lambda(u_\lambda) = \eta_e$);

(b) $\Psi_\lambda(\mathcal{B}_x^-(\lambda)) = \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$ *for all* $x \in (W^S)_{\mathrm{aff}}$.

3.4.7 Affine Weyl group action

Let $\mathcal{B}$ be a regular crystal for $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ in the sense of [Kas2, §2.2] (or [Kas1, p. 389]); in particular, as a crystal for $U_{\mathsf{v}}(\mathfrak{g}) \subset U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$, it decomposes into a disjoint union of ordinary highest weight crystals. By [Kas1, §7], the Weyl group W_{aff} acts on $\mathcal{B}$ by

$$s_j \cdot b := \begin{cases} f_j^n b & \text{if } n := \langle \mathrm{wt} b,\, \alpha_j^\vee \rangle \geq 0, \\ e_j^{-n} b & \text{if } n := \langle \mathrm{wt} b,\, \alpha_j^\vee \rangle \leq 0 \end{cases} \tag{3.4.27}$$

for $b \in \mathcal{B}$ and $j \in I_{\mathrm{aff}}$. Here we note that $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is a regular crystal for $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$ for a dominant weight $\lambda \in P^+$.

Remark 3.4.21 ([NS4, Remark 3.5.2]). Recall from Remark 3.4.15 that every element $\eta \in \mathrm{cl}^{-1}(\psi_e)$ is of the form (3.4.15). Then, for each $x \in W_{\mathrm{aff}}$,

$$x \cdot \eta = \bigl(\Pi^S(xz_{\xi_1}t(\xi_1)), \ldots, \Pi^S(xz_{\xi_s}t(\xi_s))\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s\bigr), \tag{3.4.28}$$

where $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. In particular, we see by (3.4.28) and the uniqueness of η^C that $\eta = (z_{\xi_s}t(\xi_s)) \cdot \eta^C$, with $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ the connected component containing the η.

Remark 3.4.22. Let $\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \in \overline{\mathrm{Par}(\lambda)}$. Denote by $c_i \in \mathbb{Z}_{\geq 0}$, $i \in I$, the number of columns of length m_i in the Young diagram corresponding to the partition $\rho^{(i)}$, and set $\xi := \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$; note that $c_i = 0$ for all $i \in S$. Also, for $i \in I$, let $\varrho^{(i)}$ denote the partition corresponding to the Young diagram obtained from that of $\rho^{(i)}$ by removing all columns of length m_i (i.e., the first c_i columns), and set $\boldsymbol{\varrho} := (\varrho^{(i)})_{i \in I}$; note that $\boldsymbol{\varrho} \in \mathrm{Par}(\lambda)$. Then we deduce from [BN, Lemma 4.14 and its proof] that

$$z_{\boldsymbol{\rho}} u_\lambda = t(\xi) \cdot (z_{\boldsymbol{\varrho}} u_\lambda) = t(\xi) \cdot u^{\boldsymbol{\varrho}}. \tag{3.4.29}$$

3.5 Graded character formulas for Demazure submodules and their certain quotients

3.5.1 Graded character formula for Demazure submodules

Fix a dominant weight $\lambda \in P^+$; recall that $S = S_\lambda = \bigl\{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\bigr\}$.

Because every weight space of the Demazure submodule $V_x^-(\lambda)$ corresponding to $x \in W^S = W \cap (W^S)_{\mathrm{aff}}$ is finite-dimensional, we can define the (ordinary) character $\mathrm{ch}\, V_x^-(\lambda)$ of $V_x^-(\lambda)$ by

$$\mathrm{ch}\, V_x^-(\lambda) := \sum_{\beta \in Q_{\mathrm{aff}}} \dim V_x^-(\lambda)_{\lambda-\beta}\, e^{\lambda-\beta},$$

where $V_x^-(\lambda)_{\lambda-\beta}$ denotes the $(\lambda-\beta)$-weight space of $V_x^-(\lambda)$. Here we recall that an element $\beta \in Q_{\mathrm{aff}}$ can be written uniquely in the form: $\beta = \gamma + k\delta$ for $\gamma \in Q$ and $k \in \mathbb{Z}$. If we set $q := e^\delta$, then $e^{\lambda-\beta} = e^{\lambda-\gamma}q^{-k}$. Now we define the graded character $\mathrm{gch}\, V_x^-(\lambda)$ of $V_x^-(\lambda)$ to be

$$\mathrm{gch}\, V_x^-(\lambda) := \sum_{\gamma \in Q,\, k \in \mathbb{Z}} \dim V_x^-(\lambda)_{\lambda-\gamma-k\delta}\, e^{\lambda-\gamma} q^{-k},$$

which is obtained from the ordinary character $\mathrm{ch}\, V_x^-(\lambda)$ by replacing e^δ with q.

Theorem 3.5.1. *Keep the notation and setting above. Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$, and $x \in W^S$. The graded character $\mathrm{gch}\, V_x^-(\lambda)$ of $V_x^-(\lambda)$ can be expressed as*

$$\mathrm{gch}\, V_x^-(\lambda) = \left(\prod_{i \in I} \prod_{r=1}^{m_i} (1-q^{-r})^{-1}\right) \sum_{\psi \in \mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}. \tag{3.5.1}$$

By combining the special case $x = \lfloor w_\circ \rfloor \in W^S$ of Theorem 3.5.1 with the special case $\mu = w_\circ \lambda$ of Theorem 3.3.19, we obtain the following theorem; recall from Remark 3.3.18 that $\mathrm{QLS}^{w_\circ \lambda, \infty}(\lambda) = \mathrm{QLS}(\lambda)$.

Theorem 3.5.2. *Let $\lambda \in P^+$ be a dominant weight of the from $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$. Then, the graded character $\mathrm{gch}\, V_{w_\circ}^-(\lambda)$ is equal to*

$$\left(\prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1} \right) E_{w_\circ \lambda}(q, \infty).$$

Remark 3.5.3 ([NS4, Theorem 6.1.1]). We know from [LNSSS2, Theorem 7.9] that

$$P_\lambda(q^{-1}, 0) = \sum_{\psi \in \mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_\lambda(\psi)},$$

where $P_\lambda(q^{-1}, 0)$ is the specialization of the symmetric Macdonald polynomial $P_\lambda(q^{-1}, t)$ at $t = 0$. Also, by [LNSSS2, Lemma 7.7], we have $E_{w_\circ \lambda}(q^{-1}, 0) = P_\lambda(q^{-1}, 0)$. Therefore, it follows from the special case $x = e$ of Theorem 3.5.1 that the graded character $\mathrm{gch}\, V_e^-(\lambda)$ is equal to

$$\left(\prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1} \right) E_{w_\circ \lambda}(q^{-1}, 0).$$

Note that we have $V_{w_\circ}^-(\lambda) \subset V_e^-(\lambda)$ by [NS4, Corollary 5.2.5].

3.5.2 Proof of Theorem 3.5.1

We see from Theorem 3.4.20 that

$$\mathrm{ch}\, V_x^-(\lambda) = \sum_{\eta \in \mathbb{B}_{\succeq x}^{\frac{\infty}{2}}(\lambda)} e^{\mathrm{wt}(\eta)};$$

since

$$\mathbb{B}_{\succeq x}^{\frac{\infty}{2}}(\lambda) = \bigsqcup_{\psi \in \mathrm{QLS}(\lambda)} \left(\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}_{\succeq x}^{\frac{\infty}{2}}(\lambda) \right),$$

we deduce that

$$\mathrm{ch}\, V_x^-(\lambda) = \sum_{\psi \in \mathrm{QLS}(\lambda)} \underbrace{\left(\sum_{\eta \in \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}_{\succeq x}^{\frac{\infty}{2}}(\lambda)} e^{\mathrm{wt}(\eta)} \right)}_{(*)}. \tag{3.5.2}$$

In order to obtain the graded character formula (3.5.1) for $V_x^-(\lambda)$, we will compute the sum $(*)$ of the terms $e^{\mathrm{wt}(\eta)}$ over all $\eta \in \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}_{\succeq x}^{\frac{\infty}{2}}(\lambda)$ for each $\psi \in \mathrm{QLS}(\lambda)$. Let $\psi \in \mathrm{QLS}(\lambda)$, and take $\eta_\psi \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ as in Lemma 3.4.17. Let X be a monomial in root operators such that $\eta_\psi = X\eta_e$, where $\eta_e = (e\,; 0, 1)$. We see by [NS4, Lemma 6.2.2] that

$$\mathrm{cl}^{-1}(\psi) = \bigl\{X(t(\zeta)\cdot\eta^C) \mid C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)),\ \zeta \in Q^\vee\bigr\}; \tag{3.5.3}$$

for the definition of η^C, see (3.4.20). We claim that

$$\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) = \left\{ X(t(\zeta)\cdot\eta^C) \;\middle|\; \begin{array}{l} C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)), \\ \zeta \in Q^\vee,\ [\zeta] \geq [\xi_{x,\kappa(\psi)}] \end{array} \right\}. \tag{3.5.4}$$

We first show the inclusion $\subset$. Let $\eta \in \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, and write it as $\eta = X(t(\zeta)\cdot\eta^C)$ for some $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ and some $\zeta \in Q^\vee$ (see (3.5.3)). Also, we set $y := \kappa(\psi) = \kappa(\eta_\psi) \in W^S$. We see by (3.4.28) that $t(\zeta)\cdot\eta^C$ is of the form (3.4.15), with $\kappa(t(\zeta)\cdot\eta^C) = \Pi^S(t(\zeta)) = z_\zeta t(\zeta + \phi_S(\zeta))$. Therefore, we deduce from Lemma 3.4.16 that $\kappa(X(t(\zeta)\cdot\eta^C)) = \kappa(\eta_\psi)\kappa(t(\zeta)\cdot\eta^C) = yz_\zeta t(\zeta + \phi_S(\zeta))$. Since $\eta = X(t(\zeta)\cdot\eta^C) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$ by the assumption, we have $yz_\zeta t(\zeta + \phi_S(\zeta)) \succeq x$. Hence it follows from Lemma 3.4.12 that $[\zeta] = [\zeta + \phi_S(\zeta)] \geq [\xi_{x,y}] = [\xi_{x,\kappa(\psi)}]$. Thus, η is contained in the set on the right-hand side of (3.5.4).

For the opposite inclusion $\supset$, let $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$, and let $\zeta \in Q^\vee$ be such that $[\zeta] \geq [\xi_{x,\kappa(\psi)}]$. It is obvious by (3.5.3) that $X(t(\zeta)\cdot\eta^C) \in \mathrm{cl}^{-1}(\psi)$. Hence it suffices to show that $X(t(\zeta)\cdot\eta^C) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$. The same argument as above shows that $\kappa(X(t(\zeta)\cdot\eta^C)) = yz_\zeta t(\zeta + \phi_S(\zeta))$, with $y := \kappa(\psi) \in W^S$. Therefore, we see that

$$\begin{aligned} \kappa(X(t(\zeta)\cdot\eta^C)) &= yz_\zeta t(\zeta + \phi_S(\zeta)) \succeq yz_{\xi_{x,y}} t(\xi_{x,y}) \quad \text{by Lemma 3.4.9} \\ &\succeq x \quad \text{by Lemma 3.4.11}, \end{aligned}$$

which implies that $X(t(\zeta)\cdot\eta^C) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$. This proves (3.5.4).

Let $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$, and write $\Theta(C) \in \mathrm{Par}(\lambda)$ as $\Theta(C) = (\rho^{(i)})_{i\in I}$, with $\rho^{(i)} = (\rho^{(i)}_1 \geq \cdots \geq \rho^{(i)}_{m_i-1})$ for each $i \in I$. Also, let $\zeta \in Q^\vee$ be such that $[\zeta] \geq [\xi_{x,\kappa(\psi)}]$, and write the difference $[\zeta] - [\xi_{x,\kappa(\psi)}] \in Q^{\vee,+}$ as

$$[\zeta] - [\xi_{x,\kappa(\psi)}] = \sum_{i\in I} c_i\alpha_i^\vee;$$

note that $c_i = 0$ for all $i \in S$. Now, for each $i \in I$, we set $c_i + \rho^{(i)} := (c_i + \rho^{(i)}_1 \geq \cdots \geq c_i + \rho^{(i)}_{m_i-1} \geq c_i)$, which is a partition of length less than or equal to m_i, and then set

$$(c_i)_{i\in I} + \Theta(C) := (c_i + \rho^{(i)})_{i\in I} \in \overline{\mathrm{Par}(\lambda)}. \tag{3.5.5}$$

Noting that $\langle \lambda,\, Q_S^\vee \rangle = \{0\}$, we compute:

$$\begin{aligned}
\mathrm{wt}(t(\zeta)\cdot \eta^C) &= t(\zeta)(\mathrm{wt}(\eta^C)) \\
&= t(\zeta)\bigl(\lambda - |(\rho^{(i)})_{i\in I}|\delta\bigr) \quad \text{by Proposition 3.4.18 (2)} \\
&= \lambda - \langle \lambda,\, \zeta\rangle\delta - |(\rho^{(i)})_{i\in I}|\delta \\
&= \lambda - \langle \lambda,\, \xi_{x,\kappa(\psi)}\rangle\delta - \left\langle \lambda,\, \sum_{i\in I} c_i \alpha_i^\vee \right\rangle \delta - |(\rho^{(i)})_{i\in I}|\delta \\
&= \lambda - \mathrm{wt}_\lambda\bigl(x \Rightarrow \kappa(\psi)\bigr)\delta - \left(\sum_{i\in I} m_i c_i\right)\delta - |(\rho^{(i)})_{i\in I}|\delta \\
&= \mathrm{wt}(\eta_e) - \mathrm{wt}_\lambda\bigl(x \Rightarrow \kappa(\psi)\bigr)\delta - |(c_i + \rho^{(i)})_{i\in I}|\delta.
\end{aligned}$$

From this computation, together with (3.4.18), we deduce that

$$\begin{aligned}
\mathrm{wt}(X(t(\zeta)\cdot \eta^C)) &= \mathrm{wt}(X\eta_e) - \mathrm{wt}_\lambda\bigl(x \Rightarrow \kappa(\psi)\bigr)\delta - |(c_i + \rho^{(i)})_{i\in I}|\delta \\
&= \mathrm{wt}(\eta_\psi) - \mathrm{wt}_\lambda\bigl(x \Rightarrow \kappa(\psi)\bigr)\delta - |(c_i + \rho^{(i)})_{i\in I}|\delta \\
&= \mathrm{wt}(\psi) + \bigl(\deg_\lambda(\psi) - \mathrm{wt}_\lambda\bigl(x \Rightarrow \kappa(\psi)\bigr)\bigr)\,\delta - |(c_i + \rho^{(i)})_{i\in I}|\delta.
\end{aligned} \tag{3.5.6}$$

Because $\deg_\lambda(\psi) - \mathrm{wt}_\lambda\bigl(x \Rightarrow \kappa(\psi)\bigr) = \deg_{x\lambda}(\psi)$ by the definitions of $\deg_{x\lambda}(\psi)$ and $\deg_\lambda(\psi)$, we obtain

$$\mathrm{wt}(X(t(\zeta)\cdot \eta^C)) = \mathrm{wt}(\psi) + \bigl(\deg_{x\lambda}(\psi) - |(c_i + \rho^{(i)})_{i\in I}|\bigr)\delta.$$

Summarizing, we find that for each $\psi \in \mathrm{QLS}(\lambda)$,

$$\begin{aligned}
\sum_{\eta \in \mathrm{cl}^{-1}(\psi)\cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)} e^{\mathrm{wt}(\eta)} &\overset{(3.5.4)}{=} \sum_{\substack{C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)) \\ \zeta \in Q^\vee,\, [\zeta] \geq [\xi_{x,\kappa(\psi)}]}} e^{\mathrm{wt}(X(t(\zeta)\cdot\eta^C))} \\
&= e^{\mathrm{wt}(\psi)} e^{\deg_{x\lambda}(\psi)\delta} \sum_{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)}} x^{-|\boldsymbol{\rho}|\delta} \overset{e^\delta = q}{=} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)} \sum_{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)}} q^{-|\boldsymbol{\rho}|} \\
&= e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)} \prod_{i\in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1}.
\end{aligned}$$

Substituting this into (3.5.2), we finally obtain (3.5.1). This completes the proof of Theorem 3.5.1.

3.5.3 Graded character formula for certain quotients of Demazure submodules

Let $\lambda \in P^+$ be a dominant weight; recall that $S = S_\lambda = \bigl\{ i \in I \mid \langle \lambda,\, \alpha_i^\vee \rangle = 0 \bigr\}$.

For each $x \in W^S = W \cap (W^S)_{\mathrm{aff}}$, we set

$$X_x^-(\lambda) := \sum_{\substack{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \\ \boldsymbol{\rho} \neq (\emptyset)_{i \in I}}} U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{aff}}) S_x^{\mathrm{norm}} z_{\boldsymbol{\rho}} v_\lambda = \sum_{\substack{\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \\ \boldsymbol{\rho} \neq (\emptyset)_{i \in I}}} z_{\boldsymbol{\rho}} \Big(V_x^-(\lambda) \Big); \tag{3.5.7}$$

for the definition of $z_{\boldsymbol{\rho}} : V(\lambda) \to V(\lambda)$, see (3.4.22).

For $\psi \in \mathrm{QLS}(\lambda)$, we take and fix a monomial X_ψ in root operators such that $X_\psi \eta_e = \eta_\psi$, and set

$$\eta_\psi \cdot t(\xi) := X_\psi(t(\xi) \cdot \eta_e) \quad \text{for } \xi \in Q^\vee.$$

Remark 3.5.4. Note that $t(\xi) \cdot \eta_e = (\Pi^S(t(\xi))\,;\, 0,\, 1)$ (see (3.4.28)). We deduce from [INS, Lemma 7.1.4] that if $\eta_\psi = X_\psi \eta_e$ is of the form $\eta_\psi = (x_1, \ldots, x_s\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s)$, then

$$\eta_\psi \cdot t(\xi) = X_\psi(t(\xi) \cdot \eta_e) = (x_1 \Pi^S(t(\xi)), \ldots, x_s \Pi^S(t(\xi))\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s).$$

In particular, the element $\eta_\psi \cdot t(\xi)$ does not depend on the choice of X_ψ. Also, since $x_u \Pi^S(t(\xi))\lambda = x_u \lambda - \langle \lambda, \xi \rangle \delta$ for all $1 \le u \le s$, we see by (3.4.14) that

$$\begin{aligned} \mathrm{wt}(\eta_\psi \cdot t(\xi)) &= \mathrm{wt}(\eta_\psi) - \langle \lambda, \xi \rangle \delta \\ &\overset{(3.4.18)}{=} \mathrm{wt}(\psi) + \big(\deg_\lambda(\psi) - \langle \lambda, \xi \rangle\big)\delta, \end{aligned} \tag{3.5.8}$$

and that

$$\mathrm{cl}(\eta_\psi \cdot t(\xi)) = \psi. \tag{3.5.9}$$

Theorem 3.5.5. *Keep the notation and setting above. For each $x \in W^S$, there exists a subset $\mathcal{B}(X_x^-(\lambda))$ of $\mathcal{B}(\lambda)$ such that*

$$X_x^-(\lambda) = \bigoplus_{b \in \mathcal{B}(X_x^-(\lambda))} \mathbb{C}(\mathsf{v}) G(b). \tag{3.5.10}$$

Moreover, under the isomorphism $\Psi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of crystals in Theorem 3.4.20, *the subset $\mathcal{B}(X_x^-(\lambda)) \subset \mathcal{B}(\lambda)$ is mapped to the following subset of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$:*

$$\mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \setminus \big\{ \eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \mid \psi \in \mathrm{QLS}(\lambda) \big\}. \tag{3.5.11}$$

From Theorem 3.5.5, we immediately obtain the following corollary; cf. [NS4, Theorem 6.1.1 combined with Proposition 6.2.4] for the case $x = e$.

Corollary 3.5.6. *For each $x \in W^S$, there holds the equality*

$$\mathrm{gch}(V_x^-(\lambda)/X_x^-(\lambda)) = \sum_{\psi \in \mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_{x\lambda}(\psi)}. \tag{3.5.12}$$

By combing the special case $x = \lfloor w_\circ \rfloor \in W^S$ of Corollary 3.5.6 with the special case $\mu = w_\circ \lambda$ of Theorem 3.3.19, we obtain the equality

$$\mathrm{gch}(V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)) = E_{w_\circ \lambda}(q, \infty).$$

Remark 3.5.7. We recall from Remark 3.5.3 that

$$E_{w_\circ\lambda}(q^{-1},0) = \sum_{\psi\in\mathrm{QLS}(\lambda)} e^{\mathrm{wt}(\psi)} q^{\deg_\lambda(\psi)}.$$

Hence it follows from the special case $x = e$ of Corollary 3.5.6 that

$$\mathrm{gch}(V_e^-(\lambda)/X_e^-(\lambda)) = E_{w_\circ\lambda}(q^{-1},0);$$

cf. [LNSSS3, Theorem 35]. Here we have $V_{w_\circ}^-(\lambda) \subset V_e^-(\lambda)$, as mentioned in Remark 3.5.3. However, we can easily show that $X_e^-(\lambda) \cap V_{w_\circ}^-(\lambda) \supsetneq X_{w_\circ}^-(\lambda)$ (except for some trivial cases). Therefore, there is no inclusion relation between the quotient modules $V_{w_\circ}^-(\lambda)/X_{w_\circ}^-(\lambda)$ and $V_e^-(\lambda)/X_e^-(\lambda)$; this can be also observed from the comparison of some explicit computations of $E_{w_\circ\lambda}(q^{-1},0)$ and $E_{w_\circ\lambda}(q,\infty)$.

3.5.4 Proof of Theorem 3.5.5

Lemma 3.5.8 (cf. (3.4.23))**.** *Let $x \in W^S$. Then, we have*

$$\mathcal{B}_x^-(\lambda) = \bigl\{ z_{\boldsymbol{\rho}} b \mid \boldsymbol{\rho} \in \mathrm{Par}(\lambda),\ b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda) \bigr\}. \tag{3.5.13}$$

Moreover, for every $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)}$ and $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$, the element $z_{\boldsymbol{\rho}} b$ is contained in $\mathcal{B}_x^-(\lambda)$.

Proof. We first prove the inclusion $\supset$. Let $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$, and write it as $b = Xu_\lambda$ for a monomial X in Kashiwara operators. For $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$, we have $z_{\boldsymbol{\rho}} b = X z_{\boldsymbol{\rho}} u_\lambda = X u^{\boldsymbol{\rho}}$ since $z_{\boldsymbol{\rho}}$ commutes with Kashiwara operators (see §3.4.5). Now we set $\eta := \Psi_\lambda(b)$ and $\eta' := \Psi_\lambda(z_{\boldsymbol{\rho}} b)$, where $\Psi_\lambda : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is the isomorphism of crystals in Theorem 3.4.20. Then, we have $\eta = X\eta_e$ and $\eta' = X\Psi_\lambda(u^{\boldsymbol{\rho}}) = X\eta^C$, with $C := \Theta^{-1}(\boldsymbol{\rho}) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$. Therefore, noting that $\kappa(\eta^C) = e$, we deduce from Lemma 3.4.16 that $\kappa(\eta') = \kappa(\eta)\kappa(\eta^C) = \kappa(\eta)$. Also, since $b \in \mathcal{B}_x^-(\lambda)$, it follows that $\kappa(\eta) \succeq x$, and hence $\kappa(\eta') = \kappa(\eta) \succeq x$. Hence we obtain $\eta' \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, which implies that $z_{\boldsymbol{\rho}} b \in \mathcal{B}_x^-(\lambda)$.

Next we prove the opposite inclusion $\subset$. Let $b' \in \mathcal{B}_x^-(\lambda)$, and write it as $b' = z_{\boldsymbol{\rho}} b$ for some $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$ and $b \in \mathcal{B}_0(\lambda)$ (see (3.4.23)); we need to show that $b \in \mathcal{B}_x^-(\lambda)$. We set $\eta := \Psi_\lambda(b) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $\eta' := \Psi_\lambda(b') \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$. Then, the same argument as above shows that $\kappa(\eta) = \kappa(\eta') \succeq x$. Hence we obtain $\eta \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, which implies that $b \in \mathcal{B}_x^-(\lambda)$.

For the second assertion, let $\boldsymbol{\rho} = (\rho^{(i)})_{i\in I} \in \overline{\mathrm{Par}(\lambda)}$ and $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$; remark that

$$z_{\boldsymbol{\rho}} b \in \mathcal{B}_x^-(\lambda) \iff \Psi_\lambda(z_{\boldsymbol{\rho}} b) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \iff \kappa(\Psi_\lambda(z_{\boldsymbol{\rho}} b)) \succeq x.$$

We write b as $b = Xu_\lambda$ for a monomial X in Kashiwara operators. Also, define $\boldsymbol{\varrho} := (\varrho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)$ and $\xi := \sum_{i\in I} c_i \alpha_i^\vee \in Q^{\vee,+}$ as in Remark 3.4.22. Then it follows that $z_{\boldsymbol{\rho}} b = z_{\boldsymbol{\rho}} X u_\lambda = X z_{\boldsymbol{\rho}} u_\lambda \overset{(3.4.29)}{=} X(t(\xi) \cdot u^{\boldsymbol{\varrho}})$. If we set $C := \Theta^{-1}(\boldsymbol{\varrho}) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$, then we have

$$\Psi_\lambda(z_{\boldsymbol{\rho}} b) = \Psi_\lambda\bigl(X(t(\xi) \cdot u^{\boldsymbol{\varrho}})\bigr) = X\bigl(t(\xi) \cdot \Psi_\lambda(u^{\boldsymbol{\varrho}})\bigr) = X\bigl(t(\xi) \cdot \eta^C\bigr);$$

note that $t(\xi)\cdot\eta^C$ is of the form (3.4.15) with $\kappa(t(\xi)\cdot\eta^C) = \Pi^S(t(\xi))$ by Remark 3.4.21 and the fact that $\kappa(\eta^C) = e$. Therefore, we see from Lemma 3.4.16 that

$$\kappa(\Psi_\lambda(z_{\boldsymbol{\rho}} b)) = \kappa(X(t(\xi)\cdot\eta^C)) = \kappa(X\eta_e)\Pi^S(t(\xi)). \tag{3.5.14}$$

Here we recall that $\kappa(X\eta_e) \succeq x$ since $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. Also, recall that $\xi \in Q^{\vee,+}$. From these, we deduce that

$$\begin{aligned}\kappa(\Psi_\lambda(z_{\boldsymbol{\rho}} b)) &= \kappa(X\eta_e)\Pi^S(t(\xi)) \succeq \kappa(X\eta_e) \quad \text{by Lemma 3.4.9}\\ &\succeq x.\end{aligned}$$

This proves the lemma. □

Proof of Theorem 3.5.5. We will prove that if we set

$$\mathcal{B} := \left\{ z_{\boldsymbol{\rho}} b \mid \boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I},\, b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda) \right\} \subset \mathcal{B}(\lambda), \tag{3.5.15}$$

then

$$X_x^-(\lambda) = \bigoplus_{b\in\mathcal{B}} \mathbb{C}(\mathsf{v}) G(b). \tag{3.5.16}$$

We first show the inclusion $\supset$ in (3.5.16). Let $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I}$ and $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. We see from Remark 3.4.19 that $G(z_{\boldsymbol{\rho}} b) = z_{\boldsymbol{\rho}} G(b)$. Since $G(b) \in V_x^-(\lambda)$ and

$$X_x^-(\lambda) = \sum_{\substack{\boldsymbol{\rho}\in\overline{\mathrm{Par}(\lambda)}\\ \boldsymbol{\rho}\neq(\emptyset)_{i\in I}}} z_{\boldsymbol{\rho}}\Big(V_x^-(\lambda)\Big)$$

by the definition, we have $G(z_{\boldsymbol{\rho}} b) = z_{\boldsymbol{\rho}} G(b) \in X_x^-(\lambda)$. Thus we have shown the inclusion $\supset$ in (3.5.16). Next we show the opposite inclusion $\subset$ in (3.5.16). Since $\left\{G(b) \mid b \in \mathcal{B}_x^-(\lambda)\right\}$ is a $\mathbb{C}(\mathsf{v})$-basis of $V_x^-(\lambda)$, we deduce from (3.5.7) that

$$X_x^-(\lambda) = \mathrm{Span}_{\mathbb{C}(\mathsf{v})}\left\{ z_{\boldsymbol{\rho}} G(b) \mid \boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I},\, b \in \mathcal{B}_x^-(\lambda)\right\}. \tag{3.5.17}$$

Let $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i\in I}$ and $b \in \mathcal{B}_x^-(\lambda)$. By Lemma 3.5.8, we can write the b as $b = z_{\boldsymbol{\rho}'} b'$ for some $\boldsymbol{\rho}' \in \mathrm{Par}(\lambda)$ and $b' \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. It follows that $z_{\boldsymbol{\rho}} b = z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'} b'$. Because $z_{\boldsymbol{\rho}}$ and $z_{\boldsymbol{\rho}'}$ are defined to be a certain product of Schur polynomials (see (3.4.21)), the element $z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'}$ can be expressed as:

$$z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'} = \sum_{\substack{\boldsymbol{\rho}''\in\overline{\mathrm{Par}(\lambda)}\\ |\boldsymbol{\rho}''|=|\boldsymbol{\rho}|+|\boldsymbol{\rho}'|}} n_{\boldsymbol{\rho}''} z_{\boldsymbol{\rho}''}, \quad \text{with } n_{\boldsymbol{\rho}''} \in \mathbb{Z};$$

here we remark that $|\boldsymbol{\rho}| + |\boldsymbol{\rho}'| \geq 1$ since $\boldsymbol{\rho} \neq (\emptyset)_{i\in I}$. Therefore, we deduce that

$$\begin{aligned} z_{\boldsymbol{\rho}} G(b) &= z_{\boldsymbol{\rho}} G(z_{\boldsymbol{\rho}'} b') = z_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}'} G(b') \\ &= \sum_{\substack{\boldsymbol{\rho}''\in\overline{\mathrm{Par}(\lambda)}\\ |\boldsymbol{\rho}''|=|\boldsymbol{\rho}|+|\boldsymbol{\rho}'|}} n_{\boldsymbol{\rho}''} G(z_{\boldsymbol{\rho}''} b') \in \bigoplus_{b\in\mathcal{B}} \mathbb{C}(\mathsf{v}) G(b). \end{aligned}$$

From this, together with (3.5.17), we obtain the inclusion $X_x^-(\lambda) \subset \bigoplus_{b\in\mathcal{B}} \mathbb{C}(\mathsf{v})G(b)$ in (3.5.16). Thus, we obtain (3.5.16), as desired. In what follows, we write $\mathcal{B}(X_x^-(\lambda))$ for the subset $\mathcal{B} \subset \mathcal{B}(\lambda)$ in (3.5.15).

Furthermore, we will prove that

$$\Psi_\lambda\bigl(\mathcal{B}(X_x^-(\lambda))\bigr) = \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \setminus \bigl\{\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \mid \psi \in \mathrm{QLS}(\lambda)\bigr\}.$$

For this purpose, it suffices to show that for each $\psi \in \mathrm{QLS}(\lambda)$,

$$\mathrm{cl}^{-1}(\psi) \cap \Psi_\lambda\bigl(\mathcal{B}(X_x^-(\lambda))\bigr) = \Bigl(\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)\Bigr) \setminus \bigl\{\eta_\psi \cdot t(\xi_{x,\kappa(\psi)})\bigr\}. \tag{3.5.18}$$

Let $\psi \in \mathrm{QLS}(\lambda)$; recall that X_ψ is a monomial in root operators such that $\eta_\psi = X_\psi \eta_e$. Then we know from (3.5.4) that

$$\begin{aligned}
&\mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \\
&\quad = \bigl\{X_\psi(t(\zeta)\cdot \eta^C) \mid C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)),\, \zeta \in Q^\vee,\, [\zeta] \ge [\xi_{x,\kappa(\psi)}]\bigr\}.
\end{aligned}$$

We first show the inclusion $\supset$ in (3.5.18). Let η be an element in the set on the right-hand side of (3.5.18), and write it as $\eta = X_\psi(t(\zeta)\cdot\eta^C)$ for some $C \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda))$ and $\zeta \in Q^\vee$ such that $[\zeta] \ge [\xi_{x,\kappa(\psi)}]$. We write the difference $[\zeta] - [\xi_{x,\kappa(\psi)}] \in Q^{\vee,+}$ as $[\zeta] - [\xi_{x,\kappa(\psi)}] = \sum_{i\in I} c_i \alpha_i^\vee$ with $c_i \in \mathbb{Z}_{\ge 0}$ for $i \in I$ (note that $c_i = 0$ for all $i \in S$), and define $\boldsymbol{\rho} := (c_i)_{i\in I} + \Theta(C) \in \overline{\mathrm{Par}(\lambda)}$ as in (3.5.5). We claim that $\boldsymbol{\rho} \ne (\emptyset)_{i\in I}$. Suppose, for a contradiction, that $\boldsymbol{\rho} = (\emptyset)_{i\in I}$. Then we have $\Theta(C) = (\emptyset)_{i\in I}$ and $c_i = 0$ for all $i \in I$, and hence

$$\begin{aligned}
\eta = X_\psi(t(\zeta)\cdot\eta^C) &= X_\psi(t(\zeta)\cdot\eta_e) = X_\psi(\Pi^S(t(\zeta))\,;\,0,\,1) \\
&= X_\psi(\Pi^S(t(\xi_{x,\kappa(\psi)}))\,;\,0,\,1) \quad \text{since } [\zeta] = [\xi_{x,\kappa(\psi)}] \\
&= X_\psi(t(\xi_{x,\kappa(\psi)})\cdot\eta_e) = \eta_\psi \cdot t(\xi_{x,\kappa(\psi)}),
\end{aligned}$$

which contradicts the assumption that η is an element in the set on the right-hand side of (3.5.18). Thus we obtain $\boldsymbol{\rho} \ne (\emptyset)_{i\in I}$. Now, we set

$$b := \Psi_\lambda^{-1}(\eta_\psi \cdot t(\xi_{x,\kappa(\psi)})) = \Psi_\lambda^{-1}\bigl(X_\psi(t(\xi_{x,\kappa(\psi)})\cdot\eta_e)\bigr) \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda);$$

note that $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$ by (3.5.4), and that $b = X_\psi(t(\xi_{x,\kappa(\psi)})\cdot u_\lambda)$. Then we see by (3.5.15) that $z_{\boldsymbol{\rho}} b \in \mathcal{B}(X_x^-(\lambda))$. Also, we have

$$\begin{aligned}
z_{\boldsymbol{\rho}} b = z_{\boldsymbol{\rho}}\bigl(X_\psi(t(\xi_{x,\kappa(\psi)})\cdot u_\lambda)\bigr) &= X_\psi\bigl(t(\xi_{x,\kappa(\psi)})\cdot(z_{\boldsymbol{\rho}} u_\lambda)\bigr) \\
&= X_\psi\bigl(t(\xi_{x,\kappa(\psi)})\cdot t([\zeta]-[\xi_{x,\kappa(\psi)}])\cdot u^{\Theta(C)}\bigr) \quad \text{by Remark 3.4.22} \\
&= X_\psi(t(\zeta+\gamma)\cdot u^{\Theta(C)}) \quad \text{for some } \gamma \in Q_S^\vee \\
&= X_\psi(t(\zeta)\cdot u^{\Theta(C)}).
\end{aligned}$$

Therefore, $\Psi_\lambda(z_{\boldsymbol{\rho}} b) = X_\psi(t(\zeta)\cdot\eta^C) = \eta$, which implies that η is contained in $\Psi_\lambda(\mathcal{B}(X_x^-(\lambda)))$. Thus we have shown the inclusion $\supset$ in (3.5.18).

Next we show the opposite inclusion $\subset$ in (3.5.18). Since $\mathcal{B}(X_x^-(\lambda)) \subset \mathcal{B}_x^-(\lambda)$, it follows that

$$\mathrm{cl}^{-1}(\psi) \cap \Psi_\lambda\bigl(\mathcal{B}(X_x^-(\lambda))\bigr) \subset \mathrm{cl}^{-1}(\psi) \cap \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda).$$

Hence it suffices to show that $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) \notin \Psi_\lambda\bigl(\mathcal{B}(X_x^-(\lambda))\bigr)$. Suppose, for a contradiction, that there exists $b' \in \mathcal{B}(X_x^-(\lambda))$ such that $\Psi_\lambda(b') = \eta_\psi \cdot t(\xi_{x,\kappa(\psi)})$. By (3.5.15), we can write it as $b' = z_{\boldsymbol{\rho}} b$ for some $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i \in I}$ and $b \in \mathcal{B}_x^-(\lambda) \cap \mathcal{B}_0(\lambda)$. We set $\eta := \Psi_\lambda^{-1}(b) \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda) \cap \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$ and write $\kappa(\eta) \in (W^S)_{\mathrm{aff}}$ as $\kappa(\eta) = yz_\xi t(\xi)$ for some $y \in W^S$ and $\xi \in Q^{\vee, S\text{-ad}}$. Then, $\kappa(\eta) = yz_\xi t(\xi) \succeq x$ since $\eta \in \mathbb{B}^{\frac{\infty}{2}}_{\succeq x}(\lambda)$, and hence

$$[\xi] \geq [\xi_{x,y}] \qquad \text{by Lemma 3.4.12.} \tag{3.5.19}$$

Let us write b as $b = Yu_\lambda$ for some monomial Y in Kashiwara operators (note that $\eta = Y\eta_e$), and define $\zeta = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$ and $\boldsymbol{\varrho} = (\varrho^{(i)})_{i \in I} \in \mathrm{Par}(\lambda)$ in such a way that $\boldsymbol{\rho} = (c_i)_{i \in I} + \boldsymbol{\varrho}$ (see Remark 3.4.22 and (3.5.5)); note that $c_i = 0$ for all $i \in S$. Then, by (3.4.29), we have

$$b' = z_{\boldsymbol{\rho}} b = z_{\boldsymbol{\rho}} Y u_\lambda = Y z_{\boldsymbol{\rho}} u_\lambda = Y(t(\zeta) \cdot u^{\boldsymbol{\varrho}}).$$

Therefore, we see that

$$\begin{aligned} &\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) = \Psi_\lambda(b') = \Psi_\lambda\bigl(Y(t(\zeta) \cdot u^{\boldsymbol{\varrho}})\bigr) = Y(t(\zeta) \cdot \eta^C), \\ &\qquad \text{with } C := \Theta^{-1}(\boldsymbol{\varrho}) \in \mathrm{Conn}(\mathbb{B}^{\frac{\infty}{2}}(\lambda)). \end{aligned} \tag{3.5.20}$$

Since $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) = X_\psi(t(\xi_{x,\kappa(\psi)}) \cdot \eta_e) \in \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$, it follows that $\eta^C = \eta_e$, and hence $\boldsymbol{\varrho} = (\emptyset)_{i \in I}$. Hence we obtain $\eta_\psi \cdot t(\xi_{x,\kappa(\psi)}) = Y(t(\zeta) \cdot \eta_e)$. Since $t(\zeta) \cdot \eta_e = (\Pi^S(t(\zeta))\,;\, 0, 1)$, we see from Lemma 3.4.16 that $\kappa(Y(t(\zeta) \cdot \eta_e)) = \kappa(\eta)\kappa(t(\zeta) \cdot \eta_e) = yz_\xi t(\xi)\Pi^S(t(\zeta))$. Similarly, we see that $\kappa(\eta_\psi \cdot t(\xi_{x,\kappa(\psi)})) = \kappa(\psi)\Pi^S(t(\xi_{x,\kappa(\psi)}))$. Combining these equalities, we obtain $\kappa(\psi)\Pi^S(t(\xi_{x,\kappa(\psi)})) = yz_\xi t(\xi)\Pi^S(t(\zeta))$, and hence ($y = \kappa(\psi)$ and) $[\zeta + \xi] = [\xi_{x,\kappa(\psi)}]$. Since $[\xi] \geq [\xi_{x,y}]$ by (3.5.19) and $\zeta \in Q^{\vee,+}$, it follows that ($[\xi] = [\xi_{x,y}]$ and) $[\zeta] = 0$, which implies that $c_i = 0$ for all $i \in I \setminus S$; recall that $c_i = 0$ for all $i \in S$ by the definition. Therefore, we conclude that $\boldsymbol{\rho} = (c_i)_{i \in I} + \boldsymbol{\varrho} = (\emptyset)_{i \in I}$; this contradicts our assumption that $\boldsymbol{\rho} \in \overline{\mathrm{Par}(\lambda)} \setminus (\emptyset)_{i \in I}$. Thus we have shown the inclusion $\subset$ in (3.5.18). This completes the proof of Theorem 3.5.5. $\square$

Chapter 4

Representation-theoretic interpretation of Cherednik-Orr's recursion formula for the specialization of nonsymmetric Macdonald polynomials at $t = \infty$

4.1 Introduction

In Chapter 3, we proved that for a dominant weight λ and $\mu \in W\lambda$, the specialization $E_\mu(q, \infty)$ of the nonsymmetric Macdonald polynomial $E_\mu(q, t)$ at $t = \infty$ is identical to a certain graded character of a specific subset $\mathrm{QLS}^{\mu,\infty}(\lambda)$ of the set $\mathrm{QLS}(\lambda)$ of quantum Lakshmibai-Seshadri (QLS for short) paths of shape λ; here, we recall that the subset $\mathrm{QLS}^{\mu,\infty}(\lambda)$ is determined by the subset $\mathrm{EQB}(\overline{v}(\mu))$ of W, where $\overline{v}(\mu)$ denotes the maximal-length coset representative for the coset $\{w \in W \mid w\lambda = \mu\}$. We remark that the set $\mathrm{QLS}(\lambda)$ provides an explicit realization of the crystal basis of a special quantum Weyl module $W_{\mathsf{v}}(\lambda)$ over the quantum affine algebra $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$, where $\mathfrak{g}_{\mathrm{aff}}$ is the untwisted affine Lie algebra associated to $\mathfrak{g}$ (for details, see [NS1], [NS2], [NS3], [LNSSS1], [LNSSS2], and [Na]). However, the description of the subset $\mathrm{EQB}(\overline{v}(\mu)) \subset W$ is not very explicit.

The aim of this chapter is to give a representation-theoretic (or rather, crystal-theoretic) proof of Cherednik-Orr's recursion formula for the specialization $E_\mu(q, \infty)$ at $t = \infty$, which is described in terms of Demazure type operators $T_i^\dagger := \frac{1}{1-e^{-\alpha_i}}(s_i - 1)$, $i \in I$. More precisely, we prove the following.

Theorem C (= Theorem 4.4.2; see also [CO, Proposition 3.5 (iii)])**.** *Let λ be a dominant weight, $\mu \in W\lambda$, and $i \in I$ be such that $\langle \mu, \alpha_i^\vee \rangle < 0$.*

(a) *If* $-\overline{v}(\mu)^{-1}\alpha_i$ *is not a simple root, then*

$$T_i^{\dagger} E_\mu(q,\infty) = E_{s_i\mu}(q,\infty).$$

(b) *If* $-\overline{v}(\mu)^{-1}\alpha_i$ *is a simple root, then*

$$T_i^{\dagger} E_\mu(q,\infty) = \left(1 - q^{\langle \lambda,\, \overline{v}(\mu)^{-1}\alpha_i^\vee \rangle}\right) E_{s_i\mu}(q,\infty).$$

We give a proof of this theorem by using a canonical $U_\mathsf{v}'(\mathfrak{g}_{\mathrm{aff}})$-crystal structure on $\mathrm{QLS}(\lambda)$, that is, by means of the root operators e_i, f_i, for $i \in I$; in contrast to the proof of the recursion formula of Demazure type for the specialization $E_\mu(q,0)$ at $t=0$, given in the appendix of [LNSSS3], our proof is much more difficult because of the appearance of the factor $1 - q^{\langle \lambda,\, \overline{v}(\mu)^{-1}\alpha_i^\vee \rangle}$ in case (b). Moreover, in the course of our proof, we obtain a recursive relation for the subsets $\mathrm{EQB}(w)$, $w \in W$, which determines these subsets inductively in terms of the tilted Bruhat order (see §4.3.2 for details) by starting with the equality $\mathrm{EQB}(w_\circ) = W$.

We should mention that in [Kat], Kato gave an algebro-geometric interpretation of the specialization $E_\mu(q,\infty)$ at $t=\infty$ in terms of Schubert varieties of semi-infinite flag manifolds.

This chapter is organized as follows. In Section 4.2, we fix our notation, and review Theorem 3.3.19 in Chapter 3. In Section 4.3, we prove the recursive relation for the subsets $\mathrm{EQB}(w)$, $w \in W$. In Section 4.4, we recall a canonical $U_\mathsf{v}'(\mathfrak{g}_{\mathrm{aff}})$-crystal structure on $\mathrm{QLS}(\lambda)$, and prove a variation of the string property of the subset $\mathrm{QLS}^{\mu,\infty}(\lambda) \subset \mathrm{QLS}(\lambda)$ for $\mu \in W\lambda$. Also, we study the behavior of the quantity $\deg_\mu(\psi)$ for $\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$ under root operators. Finally, by combining these results with Theorem 3.3.19, we establish Theorem C.

This chapter is based on the joint work [NNS2] with Satoshi Naito and Daisuke Sagaki.

4.2 Specialization of nonsymmetric Macdonald polynomials at $t=\infty$ in terms of QLS paths

In this chapter, we follow the notation of §2.1 and §3.2.1 for the root system of finite types and the (parabolic) quantum Bruhat graphs and use some properties in §3.2.1 such as Proposition 3.2.5.

4.2.1 Subsets $\mathrm{EQB}(w)$ of W

As in §3.3.2, for each $w \in W$, we define a subset $\mathrm{EQB}(w)$ of W. Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression for w. For each $J = \{j_1 < j_2 < j_3 < \cdots < j_r\} \subset \{1, \dots, p\}$, we define

$$p_J := (w = z_0, \dots, z_r; \beta_{j_1}, \dots, \beta_{j_r})$$

as follows: we set $\beta_k := s_{i_p} \cdots s_{i_{k+1}} \alpha_{i_k} \in \Delta^+$ for $1 \le k \le p$, and set

$$\begin{aligned}
&z_0 = w = s_{i_1} \cdots s_{i_p}, \\
&z_1 = w s_{\beta_{j_1}} = s_{i_1} \cdots s_{i_{j_1-1}} s_{i_{j_1+1}} \cdots s_{i_p} = s_{i_1} \cdots \overset{\smile}{s_{i_{j_1}}} \cdots s_{i_p}, \\
&z_2 = w s_{\beta_{j_1}} s_{\beta_{j_2}} = s_{i_1} \cdots s_{i_{j_1-1}} s_{i_{j_1+1}} \cdots s_{i_{j_2-1}} s_{i_{j_2+1}} \cdots s_{i_p} = s_{i_1} \cdots \overset{\smile}{s_{i_{j_1}}} \cdots \overset{\smile}{s_{i_{j_2}}} \cdots s_{i_p}, \\
&\quad \vdots \\
&z_r = w s_{\beta_{j_1}} \cdots s_{\beta_{j_r}} = s_{i_1} \cdots \overset{\smile}{s_{i_{j_1}}} \cdots \overset{\smile}{s_{i_{j_r}}} \cdots s_{i_p},
\end{aligned}$$

where the symbol $\overset{\smile}{\cdot}$ indicates a term to be omitted; also, we set $\mathrm{end}(p_J) := z_r$. Then we define $\mathrm{B}(w) := \{p_J \mid J \subset \{1, \dots, p\}\}$, and

$$\mathrm{QB}(w)$$
$$:= \{p_J \in \mathrm{B}(w) \mid z_i \xrightarrow{\beta_{j_{i+1}}} z_{i+1} \text{ is a directed edge of } \mathrm{QBG}(W) \text{ for all } 0 \le i \le r-1\}.$$

We remark that J may be the empty set; in this case, $\mathrm{end}(p_\emptyset) = w$. Finally, we set $\mathrm{EQB}(w) := \{\mathrm{end}(p_J) \mid p_J \in \mathrm{QB}(w)\}$.

Remark 4.2.1 (= Remark 3.3.15). We identify elements in $\mathrm{QB}(w)$ with directed paths in $\mathrm{QBG}(W)$. More precisely, for $p_J = (w = z_0, \dots, z_r; \beta_{j_1}, \dots \beta_{j_r}) \in \mathrm{QB}(w)$, we write

$$p_J = (w = z_0, \dots, z_r; \beta_{j_1}, \dots \beta_{j_r}) = \left(w = z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_r}} z_r \right).$$

Remark 4.2.2 (= Remark 3.3.16). We take and fix a reduced expression $w_\circ w^{-1} = s_{i_{-q}} \cdots s_{i_0}$ for $w_\circ w^{-1}$, and set $\beta_k := s_{i_p} \cdots s_{i_{k+1}} \alpha_{i_k}$, $-q \le k \le p$. Let $w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$, $-q \le j_k \le p$, $1 \le k \le r$, be a directed path in $\mathrm{QBG}(W)$. Then

$$1 \le j_1 < j_2 < \cdots < j_r \le p \Leftrightarrow \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w).$$

Also, it follows from Proposition 3.2.5 (1) that the map $\mathrm{end} : \mathrm{QB}(w) \to W$, $p_J \mapsto \mathrm{end}(p_J)$, is injective.

Remark 4.2.3. (1) If $w = w_\circ$, then we have $\mathrm{EQB}(w_\circ) = W$ by Proposition 3.2.5 (1), since in this case, we can use all the positive roots as edge labels.

(2) The set $\mathrm{EQB}(w)$ does not depend on the choice of a reduced expression for w (see Proposition 3.3.17).

Example 4.2.4. Let $\mathfrak{g}$ be of type A_2. Then, $\mathrm{EQB}(w_\circ) = W$ by Remark 4.2.3 (1). Also, the elements p_J of $\mathrm{QB}(s_1 s_2)$ are as follows (see Example 3.2.3):

J	p_J	$\mathrm{end}(p_J)$
$\emptyset$	$(s_1 s_2)$	$s_1 s_2$
$\{2\}$	$(s_1 s_2 \xrightarrow{\alpha_2} s_1)$	s_1

From this, we have $\mathrm{EQB}(s_1 s_2) = \{s_1 s_2, s_1\}$. Similarly, we have $\mathrm{EQB}(s_2) = \{s_2, e\}$.

4.2.2 Nonsymmetric Macdonald polynomials at $t = \infty$ in terms of QLS paths

In this subsection, we briefly recall Theorem 3.3.19. We follow the notation of §3.3.2 for QLS paths as follows.

Definition 4.2.5 (= Theorem 3.3.13; [LNSSS2, Definition 3.1])**.** Let $\lambda \in P^+$ be a dominant weight, and set $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. A pair $\psi = (w_1, w_2, \dots, w_s; \tau_0, \tau_1, \dots, \tau_s)$ of a sequence $w_1, \dots, w_s$ of elements in W^S such that $w_k \neq w_{k+1}$ for $1 \le k \le s-1$ and an increasing sequence $0 = \tau_0 < \cdots < \tau_s = 1$ of rational numbers, is called a quantum Lakshmibai-Seshadri (QLS) path of shape λ if

(C) for every $1 \le i \le s-1$, there exists a directed path from w_{i+1} to w_i in $\mathrm{QBG}_{\tau_i \lambda}(W^S)$.

Let $\mathrm{QLS}(\lambda)$ denote the set of all QLS paths of shape λ.

Remark 4.2.6. We know from [LNSSS4, Definition 3.2.2 and Theorem 4.1.1] that condition (C) can be replaced by the condition:

(C)' for every $1 \le i \le s-1$, there exists a shortest directed path in $\mathrm{QBG}(W^S)$ from w_{i+1} to w_i that is also a directed path in $\mathrm{QBG}_{\tau_i \lambda}(W^S)$.

For $\psi = (w_1, w_2, \dots, w_s; \tau_0, \tau_1, \dots, \tau_s) \in \mathrm{QLS}(\lambda)$, we set

$$\mathrm{wt}(\psi) := \sum_{i=0}^{s-1} (\tau_{i+1} - \tau_i) w_{i+1} \lambda \in P,$$

and $\kappa(\psi) := w_s \in W^S$; we call the element $\kappa(\psi)$ the final direction of ψ.

Let $\lambda \in P^+$ be a dominant weight, and $\mu \in W\lambda$. We denote by $\overline{v}(\mu) \in W$ the maximal-length coset representative for the coset $\{w \in W \mid w\lambda = \mu\}$ in W/W_S. We set

$$\mathrm{QLS}^{\mu,\infty}(\lambda) := \{\psi \in \mathrm{QLS}(\lambda) \mid \kappa(\psi) \in \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor\}.$$

Remark 4.2.7 (= Remark 3.3.18). If $w = w_\circ$, then we have $\mathrm{EQB}(w_\circ) = W$ by Remark 4.2.3 (1). If $\mu = w_\circ \lambda$, then $\overline{v}(\mu) = w_\circ$ since $w_\circ$ is the maximal-length coset representative for the coset $\{w \in W \mid w\lambda = w_\circ \lambda\}$. Therefore, we deduce that $\lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor = W^S$, and hence $\mathrm{QLS}^{w_\circ \lambda, \infty}(\lambda) = \mathrm{QLS}(\lambda)$.

For $\psi = (w_1, \dots, w_s; \tau_0, \dots, \tau_s) \in \mathrm{QLS}(\lambda)$, we define the degree of ψ at $\mu \in W\lambda$ to be

$$\deg_\mu(\psi) := -\sum_{i=1}^{s} \tau_i \mathrm{wt}_\lambda(w_{i+1} \Rightarrow w_i);$$

here we set $w_{s+1} := \lfloor \overline{v}(\mu) \rfloor$, which is the minimal-length coset representative for the coset $\{w \in W \mid w\lambda = \mu\}$ in W/W_S. Note that by Remark 4.2.6, it holds that $\tau_i \mathrm{wt}_\lambda(w_{i+1} \Rightarrow w_i) \in \mathbb{Z}_{\ge 0}$ for $1 \le i \le s-1$. Also, $\tau_s = 1$ by the definition of QLS paths. Hence it follows that $\deg_\mu(\psi) \in \mathbb{Z}_{\le 0}$. Now, for a subset Y of $\mathrm{QLS}^{\mu,\infty}(\lambda)$, we define the graded character of Y at $\mu \in W\lambda$ to be

$$\mathrm{gch}_\mu Y := \sum_{\psi \in Y} q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)}. \tag{4.2.1}$$

Now, for $\mu \in P$, let $E_\mu(q,t)$ denote the nonsymmetric Macdonald polynomial, and set $E_\mu(q,\infty) := \lim_{t\to\infty} E_\mu(q,t)$, which is the specialization at $t=\infty$.

We know the following formula for the specialization $E_\mu(q,\infty)$ at $t=\infty$.

Theorem 4.2.8 (= Theorem 3.3.19). *Let $\lambda \in P^+$ be a dominant weight, and $\mu \in W\lambda$. Then, we have the equality*

$$E_\mu(q,\infty) = \mathrm{gch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda).$$

Example 4.2.9. Let $\mathfrak{g}$ be of type A_2, and let $\lambda = \varpi_1 + \varpi_2$. Then, the elements ψ of $\mathrm{QLS}(\lambda)$, together with their weights and degrees, are as follows (see Example 3.2.3): Since $\mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda) = \mathrm{QLS}(\lambda)$ by Remark 4.2.7, we have

ψ	$\mathrm{wt}(\psi)$	$\deg_{w_\circ\lambda}(\psi)$	$\deg_{s_1s_2\lambda}(\psi)$	$\deg_{s_2\lambda}(\psi)$
$(e;0,1)$	λ	-2	-2	-1
$(s_1;0,1)$	$s_1\lambda$	-2	-1	-1
$(s_2;0,1)$	$s_2\lambda$	-2	-2	0
$(s_1s_2;0,1)$	$s_1s_2\lambda$	-1	0	0
$(s_2s_1;0,1)$	$s_2s_1\lambda$	-1	-1	0
$(w_\circ;0,1)$	$w_\circ\lambda$	0	0	0
$(s_2s_1,s_1;0,1/2,1)$	0	-2	-1	-1
$(s_1s_2,s_2;0,1/2,1)$	0	-2	-2	0
$(e,w_\circ;0,1/2,1)$	0	-1	-1	-1

$$E_{w_\circ\lambda}(q,\infty) = e^{w_\circ\lambda}+q^{-1}e^{s_1s_2\lambda}+q^{-1}e^{s_2s_1\lambda}+q^{-2}e^{s_2\lambda}+q^{-2}e^{s_1\lambda}+q^{-2}e^{\lambda}+(q^{-1}+2q^{-2})e^{0}.$$

Also, recall from Example 4.2.4 that $\mathrm{EQB}(s_1s_2) = \{s_1s_2, s_1\}$ and $\mathrm{EQB}(s_2) = \{s_2, e\}$. Therefore, we have

$$\begin{aligned}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) &= \{(s_1s_2;0,1),(s_1;0,1),(s_2s_1,s_1;0,1/2,1)\},\\ \mathrm{QLS}^{s_2\lambda,\infty}(\lambda) &= \{(s_2;0,1),(e;0,1),(s_1s_2,s_2;0,1/2,1)\},\end{aligned}$$

and hence

$$\begin{aligned}E_{s_1s_2\lambda}(q,\infty) &= e^{s_1s_2\lambda}+q^{-1}e^{s_1\lambda}+q^{-1}e^{0},\\ E_{s_2\lambda}(q,\infty) &= e^{s_2\lambda}+q^{-1}e^{\lambda}+e^{0}.\end{aligned}$$

4.3 Properties of subsets $\mathrm{EQB}(w)$

In order to establish Theorem C, we prove a recursive relation for the subsets $\mathrm{EQB}(w)$, $w \in W$. This relation enable us to determine the subset $\mathrm{EQB}(w)$ for an arbitrary $w \in W$ by descending induction on the left weak Bruhat order; recall that $\mathrm{EQB}(w_\circ) = W$ (see Remark 4.2.3 (1)).

4.3.1 Some technical lemmas

For each $w \in W$, we set $I_w := \{j \in I \mid ws_j < w\}$, where we denote by $<$ the Bruhat order on W.

Lemma 4.3.1. *Let $w \in W$ and $i \in I$ be such that $s_i w < w$; note that $-w^{-1}\alpha_i \in \Delta^+$.*

(a) *$s_i w \notin wW_{I_w}$ if and only if $-w^{-1}\alpha_i$ is not a simple root. Moreover, in this case, $I_{s_i w} = I_w$.*

(b) *$s_i w \in wW_{I_w}$ if and only if $-w^{-1}\alpha_i$ is a simple root. Moreover, in this case, $I_{s_i w} = I_w \setminus \{j\}$ for a unique $j \in I_w$ such that $\alpha_j = -w^{-1}\alpha_i$.*

Proof. Suppose that $-w^{-1}\alpha_i$ is a simple root, say α_k. Then, since $w > s_i w = ws_k$, we have $k \in I_w$. Hence $s_i w \in wW_{I_w}$.

Conversely, suppose that $s_i w \in wW_{I_w}$, and that $\beta := -w^{-1}\alpha_i \in \Delta^+$ is not a simple root. Since $s_i w = ws_\beta \in wW_{I_w}$, we have $s_\beta \in W_{I_w}$. Therefore, β can be written in the form $\beta = \sum_{j \in I_w} n_j \alpha_j$ with $n_j \in \mathbb{Z}_{\geq 0}$. Hence we have

$$\alpha_i = w(w^{-1}\alpha_i) = -\sum_{j \in I_w} n_j w\alpha_j; \tag{4.3.1}$$

here, $\#\{j \in I_w \mid n_j \neq 0\} \geq 2$ since β is not a simple root. If $j \in I_w$, then $ws_j < w$, and hence $w\alpha_j \in \Delta^-$. It follows from equation (4.3.1) that α_i can be written as a sum of two or more positive roots, which is impossible. This proves the first assertions of (a) and (b).

Let us prove the second assertions of (a) and (b). Let $j \in I_{s_i w}$. Then we have $s_i w s_j < s_i w$. Therefore, we have $\ell(w) - \ell(ws_j) \geq \ell(w) - \ell(s_i w s_j) - 1 = \ell(w) - \ell(s_i w) = 1 > 0$, and hence $ws_j < w$, which implies that $j \in I_w$. Thus we obtain $I_{s_i w} \subset I_w$. Let $j \in I_w \setminus I_{s_i w}$. Since $s_i w s_j > s_i w$ and $ws_j < w$, we see that

$$s_i w\alpha_j \in \Delta^+ \text{ and } w\alpha_j \in \Delta^-.$$

Therefore, we deduce that $w\alpha_j = -\alpha_i$, and hence $\alpha_j = -w^{-1}\alpha_i$. In case (a), there does not exist such a j, and hence $I_{s_i w} = I_w$. In case (b), there exists a unique j such that $\alpha_j = -w^{-1}\alpha_i$, and hence $I_{s_i w} = I_w \setminus \{j\}$. This proves the lemma. □

Remark 4.3.2. Let $w \in W$ and $i \in I$ be such that $s_i w < w$. Since $s_i w = ws_{-w^{-1}\alpha_i}$ and $\ell(w) - \ell(s_i w) = 1$, we see that $s_i w \xrightarrow{-w^{-1}\alpha_i} w$ is a Bruhat edge. Also, we claim that

$$w \xrightarrow{-w^{-1}\alpha_i} s_i w \text{ is a (quantum) edge if and only if } s_i w \in wW_{I_w}.$$

This is shown as follows.

(a) Assume that $s_i w \notin wW_{I_w}$. Since $-w^{-1}\alpha_i$ is not a simple root, we have $2\langle \rho, -w^{-1}\alpha_i^\vee \rangle - 1 > 1$, so that $-1 = \ell(s_i w) - \ell(w) \neq -2\langle \rho, -w^{-1}\alpha_i^\vee \rangle + 1 < -1$. Hence $w \xrightarrow{-w^{-1}\alpha_i} s_i w$ is not a quantum edge; it is clear that this is not a Bruhat edge from the assumption that $s_i w < w$.

(b) Assume that $s_i w \in wW_{I_w}$. Since $-w^{-1}\alpha_i$ is a simple root, we have $2\langle \rho, -w^{-1}\alpha_i^\vee \rangle - 1 = 1$. Hence $w \xrightarrow{-w^{-1}\alpha_i} s_i w$ is a quantum edge.

Lemma 4.3.3. *Let $w \in W$, $\gamma \in \Delta^+$, and $i \in I$. Assume that $w \xrightarrow{w^{-1}\alpha_i} s_i w$ and $s_i w s_\gamma \xrightarrow{-s_\gamma w^{-1}\alpha_i} w s_\gamma$ are Bruhat edges, and that $w \xrightarrow{\gamma} w s_\gamma$ is a quantum or Bruhat edge. Then, $w \xrightarrow{\gamma} w s_\gamma$ is a Bruhat edge, and $w = s_i w s_\gamma$.*

Proof. Suppose, for a contradiction, that $w \xrightarrow{\gamma} w s_\gamma$ is a quantum edge. Then, since $w \xrightarrow{w^{-1}\alpha_i} s_i w$ is a Bruhat edge, it follows from [LNSSS1, Lemma 5.14 (2); the left diagram] that $w s_\gamma \xrightarrow{s_\gamma w^{-1}\alpha_i} s_i w s_\gamma$ is a Bruhat edge, which contradicts the assumption that $s_i w s_\gamma \xrightarrow{-s_\gamma w^{-1}\alpha_i} w s_\gamma$ is a Bruhat edge. Hence $w \xrightarrow{\gamma} w s_\gamma$ is a Bruhat edge.

Also, suppose, for a contradiction, that $w \xrightarrow{\gamma} w s_\gamma$ is a Bruhat edge and $s_i w s_\gamma \neq w$. Then, since $w \xrightarrow{w^{-1}\alpha_i} s_i w$ and $w \xrightarrow{\gamma} w s_\gamma$ are Bruhat edges, it follows from [LNSSS1, Lemma 5.14 (1); the left diagram] that $w s_\gamma \xrightarrow{s_\gamma w^{-1}\alpha_i} s_i w s_\gamma$ is a Bruhat edge, which contradicts the assumption that $s_i w s_\gamma \xrightarrow{-s_\gamma w^{-1}\alpha_i} w s_\gamma$ is a Bruhat edge. Hence $s_i w s_\gamma = w$. This proves the lemma.

□

Lemma 4.3.4. *Let $w \in W$, $\gamma \in \Delta^+$, and $i \in I$. Assume that $w \xrightarrow{w^{-1}\alpha_i} s_i w$ and $w s_\gamma \xrightarrow{s_\gamma w^{-1}\alpha_i} s_i w s_\gamma$ are Bruhat edges. Then, the following conditions are equivalent:*

(1) $w \xrightarrow{\gamma} w s_\gamma$ *is a Bruhat (resp., quantum) edge;*

(2) $s_i w \xrightarrow{\gamma} s_i w s_\gamma$ *is a Bruhat (resp., quantum) edge.*

Proof. From the assumptions, we easily deduce that $\ell(w) - \ell(w s_\gamma) = \ell(s_i w) - \ell(s_i w s_\gamma)$. The desired equivalence follows from this equality.

□

For $u, v \in W$, let $\ell(u \Rightarrow v)$ denote the length of a shortest directed path in $\mathrm{QBG}(W)$ from u to v.

Lemma 4.3.5. *Let $u = u_0 \xrightarrow{\beta_{j_1}} u_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} u_r = v$ be a directed path in $\mathrm{QBG}(W)$ from u to v. Then, we have $\ell(u \Rightarrow v) \equiv r$ modulo 2.*

Proof. From the decomposition $\mathfrak{h}^* = \mathbb{C}\alpha \oplus \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha^\vee \rangle = 0\}$, we see that $\det(s_\alpha) = -1$ for $\alpha \in \Delta$, since $s_\alpha \alpha = -\alpha$, and $s_\alpha \mu = \mu$ if $\mu \in \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha^\vee \rangle = 0\}$. Therefore, if there exists a directed path $u = u_0 \xrightarrow{\beta_{j_1}} u_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} u_r = v$ in $\mathrm{QBG}(W)$ from u to v, then $\det(u^{-1}v) = \det(s_{\beta_{j_1}} \cdots s_{\beta_{j_r}}) = (-1)^r$. Similarly, we have $\det(u^{-1}v) = (-1)^{\ell(u \Rightarrow v)}$ since $\ell(u \Rightarrow v)$ denotes the length of a shortest directed path in $\mathrm{QBG}(W)$ from u to v. From these, we deduce that $(-1)^r = (-1)^{\ell(u \Rightarrow v)}$, and hence that $\ell(u \Rightarrow v) \equiv r$ modulo 2.

□

4.3.2 Recursive relation for subsets $\mathrm{EQB}(w)$

In this subsection, we assume that $s_i w < w$. Under this assumption, we study a relation between $\mathrm{EQB}(w)$ and $\mathrm{EQB}(s_i w)$. Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression for w. By Remark 4.2.3 (2), we can (and do) assume that $i_1 = i$, and $s_i w = s_{i_2} \cdots s_{i_p}$ is a reduced expression for $s_i w$; in this subsection, we fix such a reduced expression for w. Also, we take and fix a reduced expression $w_\circ w^{-1} = s_{i_{-q}} \cdots s_{i_0}$ for $w_\circ w^{-1}$, and set $\beta_k = s_{i_p} \cdots s_{i_{k+1}} \alpha_{i_k}$ for $-q \le k \le p$.

Remark 4.3.6. By Remark 4.2.2, if $s_i w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$ is a directed path in $\mathrm{QBG}(W)$, then

$$\left(s_i w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(s_i w) \ \Leftrightarrow 2 \le j_1 < j_2 < \cdots < j_r \le p.$$

Lemma 4.3.7. *Let $w \in W$ and $i \in I$ be such that $s_i w < w$. Let $z \in \mathrm{EQB}(w)$, and let*

$$p_J = \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w), \tag{4.3.2}$$

where $J = \{1 \le j_1 < \cdots < j_r \le p\}$.

(1) *Assume that $s_i z < z$:*

(1a) *if $s_i z_a < z_a$ for all $0 \le a \le r$, then $s_i z \in \mathrm{EQB}(s_i w)$;*

(1b) *if there exists $1 \le b \le r-1$ such that*

$$\begin{cases} s_i z_b > z_b, \\ s_i z_a < z_a \quad \textit{for } b+1 \le a \le r, \end{cases}$$

then $s_i z \in \mathrm{EQB}(w)$.

(2) *If $s_i z > z$, then $s_i z \in \mathrm{EQB}(w)$. In particular, $s_i z \in \mathrm{EQB}(w) \cup \mathrm{EQB}(s_i w)$.*

Proof. (1) Assume that $s_i z < z$.

(1a) Suppose that $j_1 = 1$. Then, $z_1 = s_{i_1} w = s_i w$, and hence $s_i z_1 = s_i(s_i w) > s_i w = z_1$, contrary to the assumption of (1a). Hence we obtain $j_1 > 1$, so that $j_k \ge j_1 > 1$ for all $1 \le k \le r$. Note that $\beta_{j_k} \ne \beta_1 = -w^{-1}\alpha_i$ for $1 \le k \le r$. Therefore, we can apply Lemma 4.3.4 to the path p_J in (4.3.2), and hence obtain a directed path in $\mathrm{QBG}(W)$:

$$s_i w = s_i z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_r}} s_i z_r = s_i z;$$

note that the edge labels of this path are identical to those of the path p_J in (4.3.2). Since $1 < j_1 < j_2 < \cdots < j_r \le p$, we deduce that $s_i z \in \mathrm{EQB}(s_i w)$.

(1b) We see easily that $z_b \xrightarrow{z_b^{-1}\alpha_i} s_i z_b$ and $s_i z_b s_{\beta_{j_{b+1}}} = s_i z_{b+1} \xrightarrow{-s_{\beta_{j_{b+1}}} z_b^{-1}\alpha_i} z_b s_{\beta_{j_{b+1}}} = z_{b+1}$ are Bruhat edges, and that $z_b \xrightarrow{\beta_{j_{b+1}}} z_{b+1}$ is a directed edge of

QBG(W). Hence it follows from Lemma 4.3.3 that $z_b = s_i z_{b+1}$. Also, applying Lemma 4.3.4 to the directed path

$$z_{b+1} \xrightarrow{\beta_{j_{b+2}}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$$

in QBG(W), we obtain a directed path in QBG(W):

$$s_i z_{b+1} \xrightarrow{\beta_{j_{b+2}}} \cdots \xrightarrow{\beta_{j_r}} s_i z_r = s_i z.$$

Concatenating p_J with this path, we obtain a label-increasing directed path in QBG(W):

$$w = z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_b}} z_b = s_i z_{b+1} \xrightarrow{\beta_{j_{b+2}}} \cdots \xrightarrow{\beta_{j_r}} s_i z_r = s_i z.$$

From this, we deduce that $s_i z \in \mathrm{EQB}(w)$ by Remark 4.2.2.

(2) Assume that $s_i z > z$. By Proposition 3.2.5 (1), there exists a unique label-increasing directed path of the form

$$w = y_0 \xrightarrow{\beta_{k_1}} y_1 \xrightarrow{\beta_{k_2}} \cdots \xrightarrow{\beta_{k_u}} y_u = s_i z$$

from w to $s_i z$ in QBG(W); here, $-q \le k_1 < \cdots < k_u \le p$. By Remark 4.2.2, in order to prove that $s_i z \in \mathrm{EQB}(w)$, it suffices to show that $1 \le k_1$.

Case (i). Suppose that there exists $1 \le b \le u-1$ such that

$$\begin{cases} s_i y_b > y_b, \\ s_i y_a < y_a & \text{for } b+1 \le a \le u. \end{cases}$$

Then, as in the proof of (1b), by Lemma 4.3.3 and Lemma 4.3.4, we obtain a label-increasing directed path of the form

$$w = y_0 \xrightarrow{\beta_{k_1}} \cdots \xrightarrow{\beta_{k_b}} y_b = s_i y_{b+1} \xrightarrow{\beta_{k_{b+2}}} \cdots \xrightarrow{\beta_{k_u}} s_i y_u = z$$

from w to z in QBG(W). By the uniqueness of a label-increasing directed path from w to z in QBG(W), we deduce that $k_1 = j_1 \ge 1$.

Case (ii). Suppose that $s_i y_a < y_a$ for all $1 \le a \le u$. By Lemma 4.3.4, we obtain a label-increasing directed path of the form

$$s_i w = s_i y_0 \xrightarrow{\beta_{k_1}} \cdots \xrightarrow{\beta_{k_u}} s_i y_u = z$$

from $s_i w$ to z in QBG(W). If $j_1 = 1$, then $z_1 = s_{i_1} w = s_i w$. In this case, by removing the first directed edge from the path p_J in (4.3.2), we obtain a label-increasing directed path of the form

$$s_i w = s_i z_0 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$$

from $s_i w$ to z in QBG(W). By the uniqueness of a label-increasing directed path from $s_i w$ to z in QBG(W), we find that $k_1 = j_2 \ge 1$. If $j_1 > 1$, then by concatenating the directed edge $s_i w \xrightarrow{\beta_1} w$ with the path p_J in (4.3.2), we obtain a label-increasing directed path of the form

$$s_i w \xrightarrow{\beta_1} w = z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$$

from $s_i w$ to z in $\mathrm{QBG}(W)$. As in the case $j_1 = 1$, we find that $k_1 = 1$. This proves the lemma.

□

Now, following [BFP], for each $w \in W$, we define the w-tilted Bruhat order $<_w$ on W by:

$$x <_w y \text{ if } \ell(w \Rightarrow y) = \ell(w \Rightarrow x) + \ell(x \Rightarrow y) \text{ for } x, y \in W;$$

recall that $\ell(x \Rightarrow y)$ denotes the length of a shortest directed path from x to y in $\mathrm{QBG}(W)$.

The following proposition shows how the subset $\mathrm{EQB}(w)$ determines the subset $\mathrm{EQB}(s_i w)$ for $w \in W$ and $i \in I$ such that $s_i w < w$. Therefore, starting with the equality $\mathrm{EQB}(w_\circ) = W$ (see Lemma 4.2.3 (1)), we can determine all the subsets $\mathrm{EQB}(w)$, $w \in W$, inductively.

Proposition 4.3.8. *Let $w \in W$ and $i \in I$ be such that $s_i w < w$.*

(1) *If $s_i w \notin w W_{I_w}$, then*

  (1a) $\mathrm{EQB}(w) \cap \mathrm{EQB}(s_i w) = \emptyset$,

  (1b) $\mathrm{EQB}(w) \cup s_i \mathrm{EQB}(w) = \mathrm{EQB}(w) \sqcup \mathrm{EQB}(s_i w)$.

(2) *If $s_i w \in w W_{I_w}$, then*

  (2a) $\mathrm{EQB}(s_i w) = \{z \in \mathrm{EQB}(w) \mid s_i w \leq_w z\}$,

  (2b) $s_i \mathrm{EQB}(w) = \mathrm{EQB}(w)$.

Proof. Recall that $w = s_{i_1} s_{i_2} \cdots s_{i_p}$ is the fixed reduced expression for w with $i_1 = i$ (fixed at the beginning of Section 4.3.2). Note that $\beta_1 = -w^{-1}\alpha_i$.

(1) Assume that $s_i w \notin w W_{I_w}$.

(1a) Suppose, for a contradiction, that $\mathrm{EQB}(w) \cap \mathrm{EQB}(s_i w) \neq \emptyset$, and take $z \in \mathrm{EQB}(w) \cap \mathrm{EQB}(s_i w)$. Let

$$p_J = \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w),$$
$$p_K = \left(s_i w = y_0 \xrightarrow{\beta_{n_1}} y_1 \xrightarrow{\beta_{n_2}} \cdots \xrightarrow{\beta_{n_u}} y_u = z \right) \in \mathrm{QB}(s_i w),$$

with $1 \leq j_1 < j_2 < \cdots < j_r \leq p$ and $2 \leq n_1 < n_2 < \cdots < n_u \leq p$. Since $s_i w \notin w W_{I_w}$, it follows from Remark 4.3.2 that there does not exist a directed edge of $\mathrm{QBG}(W)$ from w to $s_i w = w s_{\beta_1}$, and hence that $j_1 \neq 1$. Also, since $s_i w < w$, it follows that $s_i w \xrightarrow{\beta_1} w$ is a Bruhat edge by Remark 4.3.2. Concatenating this edge with p_J, we obtain a directed path

$$s_i w \xrightarrow{\beta_1} w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$$

in $\mathrm{QBG}(W)$ from $s_i w$ to z, which is a label-increasing one since $j_1 \neq 1$. Here we note that p_K is a label-increasing directed path in $\mathrm{QBG}(W)$ from $s_i w$ to z. Since $n_1 \neq 1$, we have two different label-increasing directed paths in $\mathrm{QBG}(W)$ from $s_i w$ to z, contrary to Proposition 3.2.5 (1). This proves (1a).

(1b) It is easy to verify that $\mathrm{EQB}(w) \cup s_i\mathrm{EQB}(w) \subset \mathrm{EQB}(w) \sqcup \mathrm{EQB}(s_i w)$ by part (1a) and Lemma 4.3.7. Hence it suffices to prove that $\mathrm{EQB}(w) \cup s_i\mathrm{EQB}(w) \supset \mathrm{EQB}(w) \sqcup \mathrm{EQB}(s_i w)$. Since it is clear that $\mathrm{EQB}(w) \cup s_i\mathrm{EQB}(w) \supset \mathrm{EQB}(w)$, we need only prove that $s_i\mathrm{EQB}(w) \supset \mathrm{EQB}(s_i w)$.

Claim. *Let $z \in \mathrm{EQB}(s_i w)$, and let*

$$p_K = \left(s_i w = y_0 \xrightarrow{\beta_{n_1}} y_1 \xrightarrow{\beta_{n_2}} \cdots \xrightarrow{\beta_{n_u}} y_u = z \right) \in \mathrm{QB}(s_i w),$$

with $2 \leq n_1 < n_2 < \cdots < n_u \leq p$. Then, $s_i y_a > y_a$ for all $0 \leq a \leq u$. In particular, for $a = u$, we have $s_i z > z$.

Proof of Claim. Suppose, for a contradiction, that there exists $1 \leq b \leq u$ such that

$$\begin{cases} s_i y_b < y_b, \\ s_i y_a > y_a & \text{for } 0 \leq a \leq b-1. \end{cases}$$

Then, $y_{b-1} \xrightarrow{y_{b-1}^{-1}\alpha_i} s_i y_{b-1}$ and $s_i y_{b-1} s_{\beta_{n_b}} = s_i y_b \xrightarrow{-y_b^{-1}\alpha_i} y_b = y_{b-1} s_{\beta_{n_b}}$ are both Bruhat edges, and $y_{b-1} \xrightarrow{\beta_{j_b}} y_b$ is a directed edge of $\mathrm{QBG}(W)$. Therefore, by Lemma 4.3.3 and Lemma 4.3.4, we obtain a directed path

$$w = s_i y_0 \xrightarrow{\beta_{n_1}} \cdots \xrightarrow{\beta_{n_{b-1}}} s_i y_{b-1} = y_b \xrightarrow{\beta_{n_{b+1}}} \cdots \xrightarrow{\beta_{n_u}} y_u = z$$

in $\mathrm{QBG}(W)$ whose edge labels are increasing. Since the edge labels of this path are increasing, we have $z \in \mathrm{EQB}(w)$. Also, by the assumption of the claim, we have $z \in \mathrm{EQB}(s_i w)$. Thus, $z \in \mathrm{EQB}(w) \cap \mathrm{EQB}(s_i w)$, contrary to Proposition 4.3.8 (1a). ■

Now we take a directed path p_K in the claim above. By Lemma 4.3.4, we obtain

$$w = s_i y_0 \xrightarrow{\beta_{n_1}} \cdots \xrightarrow{\beta_{n_u}} s_i y_u = s_i z,$$

which is a label-increasing directed path such that $n_1 \geq 2$. It follows that $s_i z \in \mathrm{EQB}(w)$, and hence $z \in s_i\mathrm{EQB}(w)$. Thus we have $s_i\mathrm{EQB}(w) \supset \mathrm{EQB}(s_i w)$, as desired.

(2) Assume that $s_i w \in wW_{I_w}$.

(2a) First we prove that $\mathrm{EQB}(s_i w) \subset \{z \in \mathrm{EQB}(w) \mid s_i w \leq_w z\}$. Let $z \in \mathrm{EQB}(s_i w)$, and let

$$p_K = \left(s_i w = y_0 \xrightarrow{\beta_{n_1}} y_1 \xrightarrow{\beta_{n_2}} \cdots \xrightarrow{\beta_{n_u}} y_u = z \right) \in \mathrm{QB}(s_i w).$$

Note that $n_1 \geq 2$ by Remark 4.3.6. Since $s_i w \in wW_{I_w}$, we see that $w \xrightarrow{\beta_1} s_i w$ is a quantum edge by Remark 4.3.2. Hence we obtain a label-increasing directed path

$$w \xrightarrow{\beta_1} s_i w = y_0 \xrightarrow{\beta_{n_1}} y_1 \xrightarrow{\beta_{n_2}} \cdots \xrightarrow{\beta_{n_u}} y_u = z$$

in $\mathrm{QBG}(W)$. This implies that $z \in \mathrm{EQB}(w)$. Moreover, by Proposition 3.2.5 (2), this path is a shortest directed path in $\mathrm{QBG}(W)$ from w to z. It follows that $s_i w \leq_w z$.

Next we prove that $\mathrm{EQB}(s_i w) \supset \{z \in \mathrm{EQB}(w) \mid s_i w \leq_w z\}$. Let $z \in \mathrm{EQB}(w)$ be such that $s_i w \leq_w z$. By the definition of the tilted Bruhat order, there exists a shortest directed path in $\mathrm{QBG}(W)$ from w to z passing through $s_i w$:

$$w \xrightarrow{\beta_{k_1}} \cdots \xrightarrow{\beta_{k_a}} s_i w \xrightarrow{\beta_{k_{a+1}}} \cdots \xrightarrow{\beta_{k_b}} z.$$

Here we recall that $w \xrightarrow{\beta_1} s_i w$ is a quantum edge. Since $w \xrightarrow{\beta_{k_1}} \cdots \xrightarrow{\beta_{k_a}} s_i w$ is a shortest directed path in $\mathrm{QBG}(W)$ from w to $s_i w$, it follows that $a = 1$ and $k_1 = 1$. Hence this path can be written as:

$$w \xrightarrow{\beta_1} s_i w \xrightarrow{\beta_{k_2}} \cdots \xrightarrow{\beta_{k_b}} z. \tag{4.3.3}$$

Since $z \in \mathrm{EQB}(w)$, there exists a label-increasing directed path

$$p_J = \left(w = z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w)$$

from w to z in $\mathrm{QBG}(W)$; it follows from the definition of $p_J \in \mathrm{QB}(w)$ that $j_1 \geq 1$. Also, since p_J is a label-increasing directed path, it follows from Proposition 3.2.5 (2) that p_J is less than or equal to the directed path (4.3.3) in the lexicographic order (with respect to the edge labels), which implies that $j_1 \leq 1$. Therefore, $j_1 = 1$, and hence

$$p_J = \left(w = z_0 \xrightarrow{\beta_1} z_1 = s_i w \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right).$$

From this, it follows that $\left(z_1 = s_i w \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(s_i w)$, and hence $z \in \mathrm{EQB}(s_i w)$. Thus we have proved that $\mathrm{EQB}(s_i w) \supset \{z \in \mathrm{EQB}(w) \mid s_i w \leq_w z\}$, and hence (2a).

Finally, we prove (2b). Recall that $s_i \mathrm{EQB}(w) \subset \mathrm{EQB}(s_i w) \cup \mathrm{EQB}(w)$ by Lemma 4.3.7. Also, by part (2a), we have $\mathrm{EQB}(s_i w) \subset \mathrm{EQB}(w)$, and hence $s_i \mathrm{EQB}(w) \subset \mathrm{EQB}(w)$. From this, we obtain $\mathrm{EQB}(w) \subset s_i \mathrm{EQB}(w)$, since $s_i^2 = 1$. This completes the proof of the proposition.

□

Example 4.3.9. (1) Let $\mathfrak{g}$ be of type A_2, and let $w = s_1 s_2$ and $i = 1$; in this case, the root $-w^{-1}\alpha_1 = \alpha_1 + \alpha_2$ is not a simple root, and hence $s_1 w = s_2 \notin wW_{I_w}$. Recall from Example 4.2.4 that $\mathrm{EQB}(s_1 s_2) = \{s_1 s_2, s_1\}$ and $\mathrm{EQB}(s_2) = \{s_2, e\}$. Hence we have $\mathrm{EQB}(s_1 s_2) \cup s_1 \mathrm{EQB}(s_1 s_2) = \{s_1 s_2, s_1, s_2, e\} = \mathrm{EQB}(s_1 s_2) \sqcup \mathrm{EQB}(s_2)$.

(2) Let $\mathfrak{g}$ be of type A_2, and let $w = w_\circ$ and $i = 2$; in this case, the root $-w^{-1}\alpha_2 = \alpha_1$ is a simple root, and hence $s_2 w = s_1 s_2 \in wW_{I_w}$. Recall from Example 4.2.4 that $\mathrm{EQB}(w_\circ) = W$ and $\mathrm{EQB}(s_1 s_2) = \{s_1 s_2, s_1\} \subset \mathrm{EQB}(w_\circ)$. Moreover, it is easy to check that for $z \in W$, $\ell(w_\circ \Rightarrow z) = \ell(w_\circ \Rightarrow s_1 s_2) + \ell(s_1 s_2 \Rightarrow z)$ if and only if $z \in \mathrm{EQB}(s_1 s_2)$.

(3) Let $\mathfrak{g}$ be of type A_3. We take $w = s_2 w_\circ \in W$, and fix a reduced expression $w = s_1 s_2 s_3 s_2 s_1$ for w. Then, the elements p_J of $\mathrm{QB}(w)$ are as follows:

J	p_J	$\mathrm{end}(p_J)$
$\emptyset$	(w)	$s_1 s_2 s_3 s_2 s_1$
$\{2\}$	$(w \xrightarrow{\alpha_3} s_1 s_3 s_2 s_1)$	$s_1 s_3 s_2 s_1$
$\{3\}$	$(w \xrightarrow{\alpha_1+\alpha_2+\alpha_3} e)$	e
$\{5\}$	$(w \xrightarrow{\alpha_1} s_1 s_2 s_3 s_2)$	$s_1 s_2 s_3 s_2$
$\{2,4\}$	$(w \xrightarrow{\alpha_3} s_1 s_3 s_2 s_1 \xrightarrow{\alpha_1+\alpha_2} s_3)$	s_3
$\{2,5\}$	$(w \xrightarrow{\alpha_3} s_1 s_3 s_2 s_1 \xrightarrow{\alpha_1} s_1 s_3 s_2)$	$s_1 s_3 s_2$
$\{3,5\}$	$(w \xrightarrow{\alpha_1+\alpha_2+\alpha_3} e \xrightarrow{\alpha_1} s_1)$	s_1
$\{2,4,5\}$	$(w \xrightarrow{\alpha_3} s_1 s_3 s_2 s_1 \xrightarrow{\alpha_1+\alpha_2} s_3 \xrightarrow{\alpha_1} s_3 s_1)$	$s_3 s_1$

From this, we have

$$\begin{aligned} \mathrm{EQB}(w) &= \{s_1 s_2 s_3 s_2 s_1, s_1 s_3 s_2 s_1, s_1 s_2 s_3 s_2, s_1 s_3 s_2, s_1 s_3, s_1, s_3, e\}, \\ s_1 \mathrm{EQB}(w) &= \{s_2 s_3 s_2 s_1, s_3 s_2 s_1, s_2 s_3 s_2, s_3 s_2, s_1 s_3, s_1, s_3, e\}. \end{aligned}$$

Also, let $i = 1$ and fix a reduced expression $s_1 w = s_2 s_3 s_2 s_1$ for $s_1 w$; in this case, the root $-w^{-1}\alpha_1 = \alpha_2 + \alpha_3$ is not a simple root, and hence $s_1 w \notin wW_{I_w}$. Then the elements p_J of $\mathrm{QB}(s_1 w)$ are as follows:

J	p_J	$\mathrm{end}(p_J)$
$\emptyset$	$(s_1 w)$	$s_2 s_3 s_2 s_1$
$\{1\}$	$(s_1 w \xrightarrow{\alpha_3} s_3 s_2 s_1)$	$s_3 s_2 s_1$
$\{4\}$	$(s_1 w \xrightarrow{\alpha_1} s_2 s_3 s_2)$	$s_2 s_3 s_2$
$\{1,4\}$	$(s_1 w \xrightarrow{\alpha_3} s_3 s_2 s_1 \xrightarrow{\alpha_1} s_3 s_2)$	$s_3 s_2$

Hence we have $\mathrm{EQB}(s_1 w) = \{s_2 s_3 s_2 s_1, s_3 s_2 s_1, s_2 s_3 s_2, s_3 s_2\}$. Thus, we see that for $w = s_2 w_\circ$ and $i = 1$,

$$\begin{aligned} &\mathrm{EQB}(w) \cup s_1 \mathrm{EQB}(w) = \\ &\{s_1 s_2 s_3 s_2 s_1, s_1 s_3 s_2 s_1, s_1 s_2 s_3 s_2, s_1 s_3 s_2, s_2 s_3 s_2 s_1, s_2 s_3 s_2, s_3 s_2 s_1, s_3 s_2, s_1 s_3, s_1, s_3, e\} \\ &= \mathrm{EQB}(w) \sqcup \mathrm{EQB}(s_1 w). \end{aligned}$$

Lemma 4.3.10. *Let $w \in W$ and $i \in I$ be such that $s_i w < w$. If $z \in \mathrm{EQB}(s_i w)$, then $s_i z > z$.*

Proof. If $s_i w \notin wW_{I_w}$, then the assertion of the lemma follows from the claim in the proof of Proposition 4.3.8.

Suppose now that $s_i w \in wW_{I_w}$. Let $z \in \mathrm{EQB}(s_i w)$, and let

$$p_K = \left(s_i w = y_0 \xrightarrow{\beta_{n_1}} y_1 \xrightarrow{\beta_{n_2}} \cdots \xrightarrow{\beta_{n_u}} y_u = z \right) \in \mathrm{QB}(s_i w);$$

note that $n_1 > 1$. Concatenating the directed edge $w \xrightarrow{\beta_1} s_i w$ of $\mathrm{QBG}(W)$ with this path, we obtain a label-increasing directed path

$$w \xrightarrow{\beta_1} s_i w = y_0 \xrightarrow{\beta_{n_1}} y_1 \xrightarrow{\beta_{n_2}} \cdots \xrightarrow{\beta_{n_u}} y_u = z \tag{4.3.4}$$

in $\mathrm{QBG}(W)$. By Proposition 3.2.5 (2), this path is a shortest directed path in $\mathrm{QBG}(W)$ from w to z of length $u+1$. If $s_i z < z$, then there exists $1 \le b \le u$ such that

$$\begin{cases} s_i y_b < y_b, \\ s_i y_a > y_a \quad \text{for } 0 \le a \le b-1, \end{cases}$$

since $s_i w < s_i(s_i w) = w$. Now, applying Lemma 4.3.3 and Lemma 4.3.4 to p_K, we obtain a directed path

$$w = s_i y_0 \xrightarrow{\beta_{n_1}} \cdots \xrightarrow{\beta_{n_{b-1}}} s_i y_{b-1} = y_b \xrightarrow{\beta_{n_{b+1}}} \cdots \xrightarrow{\beta_{n_u}} y_u = z$$

in $\mathrm{QBG}(W)$ from w to z of length $u-1$. This contradicts the fact that the directed path (4.3.4) is shortest. This proves the lemma. □

Proposition 4.3.11. *Let $w \in W$, $z \in \mathrm{EQB}(w)$, and $i \in I$ be such that $s_i w < w$ and $s_i z > z$. Let $\lambda \in P^+$ be a dominant weight.*

(1) *We have*
$$\mathrm{wt}_\lambda(w \Rightarrow s_i z) = \mathrm{wt}_\lambda(s_i w \Rightarrow z).$$

(2) *If $z \notin \mathrm{EQB}(s_i w)$, then*
$$\mathrm{wt}_\lambda(w \Rightarrow s_i z) = \mathrm{wt}_\lambda(w \Rightarrow z).$$

(3) *If $z \in \mathrm{EQB}(s_i w)$, then*
$$\mathrm{wt}_\lambda(w \Rightarrow s_i z) + \langle \lambda, -w^{-1}\alpha_i^\vee \rangle = \mathrm{wt}_\lambda(w \Rightarrow z).$$

Proof. (1) This is proved by [LNSSS2, Corollary 4.2]; see also equation (3.2.3).

(2) Assume that $z \notin \mathrm{EQB}(s_i w)$.

Case (a). Suppose that $s_i w \notin wW_{I_w}$. Since $z \in \mathrm{EQB}(w)$, there exists a label-increasing directed path

$$p_J = \left(w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w)$$

in $\mathrm{QBG}(W)$ from w to z. Recall that in this case, there does not exist a directed edge of the form $w \xrightarrow{\beta_1} s_i w$, $\beta_1 = -w^{-1}\alpha_i$, in $\mathrm{QBG}(W)$ by Remark 4.3.2. Hence we see that $j_1 > 1$. Since $s_i w \xrightarrow{\beta_1} w$ is a Bruhat edge, we obtain a label-increasing (hence shortest) directed path

$$s_i w \xrightarrow{\beta_1} w = z_0 \xrightarrow{\beta_{j_1}} z_1 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_r}} z_r = z$$

from $s_i w$ to z in $\mathrm{QBG}(W)$. Since $\mathrm{wt}_\lambda(s_i w \xrightarrow{\beta_1} w) = 0$, we deduce that

$$\mathrm{wt}_\lambda(s_i w \Rightarrow z) = \mathrm{wt}_\lambda(w \Rightarrow z).$$

Combining this and the equality in part (1), we obtain

$$\mathrm{wt}_\lambda(w \Rightarrow s_i z) = \mathrm{wt}_\lambda(s_i w \Rightarrow z) = \mathrm{wt}_\lambda(w \Rightarrow z),$$

as desired.

Case (b). Suppose that $s_i w \in wW_{I_w}$. Let

$$s_i w \xrightarrow{\beta_{k_1}} \cdots \xrightarrow{\beta_{k_a}} z_a = z$$

be a label-increasing directed path in $\mathrm{QBG}(W)$, with $-q \le k_1 < \cdots < k_a \le p$. Since $z \notin \mathrm{EQB}(s_i w)$ by the assumption, it follows from Remark 4.2.2 that $k_1 < 2$. Since $w \xrightarrow{\beta_1} s_i w$, $\beta_1 = -w^{-1}\alpha_i$, is a quantum edge by Remark 4.3.2 (b),

$$w \xrightarrow{\beta_1} s_i w \xrightarrow{\beta_{k_1}} \cdots \xrightarrow{\beta_{k_a}} z_a = z \tag{4.3.5}$$

is a directed path in $\mathrm{QBG}(W)$.

Claim. *The directed path* (4.3.5) *is not a shortest directed path in* $\mathrm{QBG}(W)$ *from* w *to* z.

Proof of Claim. Suppose, for a contradiction, that the directed path (4.3.5) is shortest. Since $z \in \mathrm{EQB}(w)$, there exists a (unique) label-increasing directed path

$$w \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_{a+1}}} z \tag{4.3.6}$$

in $\mathrm{QBG}(W)$ from w to z such that $1 \le j_1 < \cdots < j_{a+1} \le p$. Because the directed path (4.3.6) is lexicographically minimal (with respect to the edge labels) among the shortest directed paths from w to z by Proposition 3.2.5 (2), we deduce that $j_1 = 1$ by comparing the first edges of directed paths (4.3.5) and (4.3.6). Also, by comparing the second edges of directed paths (4.3.5) and (4.3.6), we deduce that $j_2 \le k_1$.

However, since $k_1 < 2$ as stated before this claim, we obtain $1 = j_1 < j_2 \le k_1 < 2$, a contradiction. $\blacksquare$

We note that $\ell(w \Rightarrow z) = a - 1$ or $a + 1$ by Lemma 4.3.5, since $w \xrightarrow{\beta_1} s_i w$ and $s_i w \xrightarrow{\beta_1} w$ are directed edges of $\mathrm{QBG}(W)$. Since $\ell(w \Rightarrow z) \neq a + 1$ by the claim above, it follows from Lemma 4.3.5 that $\ell(w \Rightarrow z) = a - 1$. Now, let

$$w \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_{a-1}}} z \tag{4.3.7}$$

be a label-increasing directed path, with $1 \le j_1 < \cdots < j_{a-1} \le p$. Concatenating the directed edge $s_i w \xrightarrow{\beta_1} w$ of $\mathrm{QBG}(W)$ with the directed path (4.3.7), we obtain a directed path

$$s_i w \xrightarrow{\beta_1} w \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_{a-1}}} z \tag{4.3.8}$$

in $\mathrm{QBG}(W)$; since the length of this path is a, this is a shortest directed path in $\mathrm{QBG}(W)$ from $s_i w$ to z. Since $s_i w \xrightarrow{\beta_1} w$ is a Bruhat edge, $\mathrm{wt}_\lambda(s_i w \xrightarrow{\beta_1} w) = 0$. Therefore, by comparing the λ-weights of directed paths (4.3.7) and (4.3.8), we find that

$$\mathrm{wt}_\lambda(s_i w \Rightarrow z) = \mathrm{wt}_\lambda(w \Rightarrow z).$$

Combining this and the equality in part (1), we obtain

$$\mathrm{wt}_\lambda(w \Rightarrow s_i z) = \mathrm{wt}_\lambda(s_i w \Rightarrow z) = \mathrm{wt}_\lambda(w \Rightarrow z),$$

as desired.

(3) By Proposition 4.3.8 (1a), we deduce that $s_i w \in wW_{I_w}$. Since $s_i w \le_w z$ by Proposition 4.3.8 (2a), we have

$$\mathrm{wt}_\lambda(w \Rightarrow z) = \mathrm{wt}_\lambda(w \Rightarrow s_i w) + \mathrm{wt}_\lambda(s_i w \Rightarrow z).$$

Also, since $w \xrightarrow{-w^{-1}\alpha_i} s_i w$ is a quantum edge by Remark 4.3.2 (b), $\mathrm{wt}_\lambda(w \Rightarrow s_i w) = \langle \lambda, -w^{-1}\alpha_i^\vee \rangle$. Therefore,

$$\mathrm{wt}_\lambda(w \Rightarrow z) = \langle \lambda, -w^{-1}\alpha_i^\vee \rangle + \mathrm{wt}_\lambda(s_i w \Rightarrow z).$$

Combining this and the equality in part (1), we obtain

$$\mathrm{wt}_\lambda(w \Rightarrow z) = \langle \lambda, -w^{-1}\alpha_i^\vee \rangle + \mathrm{wt}_\lambda(s_i w \Rightarrow z) = \langle \lambda, -w^{-1}\alpha_i^\vee \rangle + \mathrm{wt}_\lambda(w \Rightarrow s_i z),$$

as desired. This completes the proof of the proposition. $\square$

4.3.3 Some additional properties of subsets $\mathrm{EQB}(w)$

In this subsection, we show some additional properties of the subsets $\mathrm{EQB}(w)$, $w \in W$. In addition, by using Proposition 4.3.8, we obtain a recursive relation for the subsets $\lfloor \mathrm{EQB}(w) \rfloor$, $w \in W$.

Lemma 4.3.12. *For each $w \in W$, the subset* $\mathrm{EQB}(w)$ *decomposes into a disjoint union of some cosets in* W/W_{I_w}.

Proof. Let $z \in \mathrm{EQB}(w)$. It suffices to show that $zs_j \in \mathrm{EQB}(w)$ for all $j \in I_w$. Let $j \in I_w$. Since $ws_j < w$, we can take a reduced expression for w as:

$$w = s_{i_1} \cdots s_{i_p}, \quad \text{with } i_p = j.$$

Since $z \in \mathrm{EQB}(w)$, there exists $J \subset \{1, \dots, p\}$ such that

$$p_J = \left(w = z_0 \xrightarrow{\beta_{j_1}} \cdots \xrightarrow{\beta_{j_r}} z_r = z \right) \in \mathrm{QB}(w).$$

If $j_r = p$, then we set $K = J \setminus \{p\}$; otherwise, we set $K = J \sqcup \{p\}$. In both cases, we have $\mathrm{end}(p_K) = zs_{i_p} = zs_j$. Also, in the case $K = J \setminus \{p\}$, it is clear that $p_K \in \mathrm{QB}(w)$. In the case $K = J \sqcup \{p\}$, $\beta_p = \alpha_{i_p} = \alpha_j$ is a simple root. Therefore, $z \xrightarrow{\beta_p} zs_j$ is a directed edge of $\mathrm{QBG}(W)$, and hence $p_K \in \mathrm{QB}(w)$. Thus we obtain $\mathrm{end}(p_K) \in \mathrm{EQB}(w)$, and hence $zs_j \in \mathrm{EQB}(w)$. This proves the lemma.

□

The next lemma follows from [M1, Chap. 2].

Lemma 4.3.13. *Let $\lambda \in P^+$ be a dominant weight. Let $\mu \in W\lambda$ and $i \in I$ be such that $\langle \mu, \alpha_i^\vee \rangle \neq 0$. Then $s_i \overline{v}(\mu) = \overline{v}(s_i\mu)$. Moreover, the following conditions are equivalent*:

(1) $\langle \mu, \alpha_i^\vee \rangle < 0$;

(2) $s_i \overline{v}(\mu) < \overline{v}(\mu)$.

In what follows, we take and fix a dominant weight $\lambda \in P^+$, and set $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$. Then, for $\mu \in W\lambda$, we have $S \subset I_{\overline{v}(\mu)}$. Therefore, by Lemma 4.3.12, we deduce that

$$\lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor \subset \mathrm{EQB}(\overline{v}(\mu)), \tag{4.3.9}$$

where $\lfloor \ \rfloor$ denotes the surjection $\lfloor \ \rfloor : W \to W^S$, $w \mapsto \lfloor w \rfloor$.

The following is a generalization of Proposition 4.3.8; we use this proposition in Section 4.4.4.

Proposition 4.3.14. *Let $\mu \in W\lambda$ and $i \in I$ be such that $\langle \mu, \alpha_i^\vee \rangle < 0$.*

(1) *Assume that $s_i \overline{v}(\mu) \notin \overline{v}(\mu) W_{I_{\overline{v}(\mu)}}$. Then,*

$$\lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor \cup \lfloor s_i \mathrm{EQB}(\overline{v}(\mu)) \rfloor = \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor \sqcup \lfloor \mathrm{EQB}(\overline{v}(s_i\mu)) \rfloor.$$

(2) *Assume that $s_i \overline{v}(\mu) \in \overline{v}(\mu) W_{I_{\overline{v}(\mu)}}$. Then,*

(2a)

$$\lfloor \mathrm{EQB}(\overline{v}(s_i\mu)) \rfloor = \{z \in \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor \mid \overline{v}(s_i\mu) \leq_{\overline{v}(\mu)} z\},$$

(2b)

$$\lfloor s_i \mathrm{EQB}(\overline{v}(\mu)) \rfloor = \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor.$$

Proof. First of all, by Lemma 4.3.13, we have $\overline{v}(s_i\mu) = s_i\overline{v}(\mu) < \overline{v}(\mu)$.

Let us prove part (1). By Lemma 4.3.8 (1b), we have

$$\mathrm{EQB}(\overline{v}(\mu)) \cup s_i\mathrm{EQB}(\overline{v}(\mu)) = \mathrm{EQB}(\overline{v}(\mu)) \sqcup \mathrm{EQB}(\overline{v}(s_i\mu)).$$

By Lemma 4.3.12, both sides of this equation can be written as a disjoint union of some cosets in $W/W_{I_{\overline{v}(\mu)}}$. Also, since $S \subset I_{\overline{v}(\mu)}$, we find that both sides of this equation can be written as a disjoint union of some cosets in W/W_S. Therefore, by applying the surjection $\lfloor\ \ \rfloor : W \to W^S$, $w \mapsto \lfloor w \rfloor$, to the equation above, we obtain the assertion of part (1).

Part (2) is an immediate consequence of Proposition 4.3.8 (2); indeed, using Lemma 4.3.12, we can easily verify that

$$\lfloor \{z \in \mathrm{EQB}(\overline{v}(\mu)) \mid \overline{v}(s_i\mu) \leq_{\overline{v}(\mu)} z\} \rfloor = \{z \in \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor \mid \overline{v}(s_i\mu) \leq_{\overline{v}(\mu)} z\}.$$

This proves the proposition.

□

The following is a generalization of Lemma 4.3.10; we use this lemma in Sections 4.4.4 and 4.4.5.

Lemma 4.3.15. *Let $\mu \in W\lambda$ and $i \in I$ be such that $\langle \mu, \alpha_i^\vee \rangle < 0$. If $z \in \lfloor \mathrm{EQB}(\overline{v}(s_i\mu)) \rfloor$, then $\lfloor s_i z \rfloor > z$. Moreover, $s_i z \in \mathrm{EQB}(\overline{v}(\mu)) \setminus \mathrm{EQB}(\overline{v}(s_i\mu))$.*

Proof. By Lemma 4.3.10 together with the inclusion (4.3.9), we have $s_i z > z$, and hence $\lfloor s_i z \rfloor \geq z$; here we note that $s_i\overline{v}(\mu) < \overline{v}(\mu)$ by Lemma 4.3.13. Suppose, for a contradiction, that $\lfloor s_i z \rfloor = z$. Then, we see that $s_i z \in zW_S \subset zW_{I_{\overline{v}(s_i\mu)}} \subset \mathrm{EQB}(\overline{v}(s_i\mu))$ by Lemma 4.3.12 and the inclusion (4.3.9). Therefore, by Lemma 4.3.10, we have $z = s_i(s_i z) > s_i z$, which contradicts the fact that $s_i z > z$. Thus, we deduce that $\lfloor s_i z \rfloor > z$, and $s_i z \notin \mathrm{EQB}(\overline{v}(s_i\mu))$. In addition, by Proposition 4.3.8, we obtain $s_i z \in \mathrm{EQB}(\overline{v}(\mu))$ whether $s_i\overline{v}(\mu) \in \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$ or not. This proves the lemma.

□

Remark 4.3.16. We can show that if $w \in W$ and $z \in \mathrm{EQB}(w)$, then z is less than or equal to w in the right weak Bruhat order on W; we omit its proof since we do not use this fact in this chapter.

4.4 Recursion formula for $E_\mu(q, \infty)$

Cherednik and Orr gave a recursion formula ([CO, Proposition 3.5 (iii)]) for the specialization $E_\mu(q, \infty)$ of the nonsymmetric Macdonald polynomial $E_\mu(q, t)$ at $t = \infty$ in terms of Demazure-type operators, for the affine root systems of dual untwisted type. In this section, we give a crystal-theoretic (hence representation-theoretic) proof of this recursion formula for the affine root systems of untwisted type. For this purpose, in view of Theorem 4.2.8, it suffices to prove that for a dominant weight $\lambda \in P^+$, the graded characters $\mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda)$, $\mu \in W\lambda$, satisfy the same recursion formula as the one above with $E_\mu(q,\infty)$ replaced by $\mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda)$; namely, we prove Theorem 4.4.1 below, by making use of a canonical $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$-crystal structure on $\mathrm{QLS}(\lambda)$.

Throughout this section, we take and fix a dominant weight $\lambda \in P^+$, and set $S = S_\lambda = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$.

4.4.1 Demazure-type operators

For $i \in I$, we define a $\mathbb{C}(q)$-linear operator $T_i^\dagger$ on $\mathbb{C}(q)[P]$ by $T_i^\dagger := \frac{1}{1-e^{-\alpha_i}}(s_i - 1)$; note that for $\mu \in P$,

$$T_i^\dagger e^\mu = \begin{cases} e^{\mu+\alpha_i} + e^{\mu+2\alpha_i} + \cdots + e^{s_i\mu} & \text{if } \langle \mu, \alpha_i^\vee \rangle < 0, \\ 0 & \text{if } \langle \mu, \alpha_i^\vee \rangle = 0, \\ -e^\mu - e^{\mu-\alpha_i} - \cdots - e^{s_i\mu+\alpha_i} & \text{if } \langle \mu, \alpha_i^\vee \rangle > 0. \end{cases}$$

We will prove Theorem 4.4.1 in Sections 4.4.4 and 4.4.5; recall that $-\overline{v}(\mu)^{-1}\alpha_i$ is a simple root if and only if $s_i\overline{v}(\mu) \in \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$ (see Proposition 4.3.1).

Theorem 4.4.1. *Let $\mu \in W\lambda$ and $i \in I$ be such that $\langle \mu, \alpha_i^\vee \rangle < 0$.*

(a) *If $-\overline{v}(\mu)^{-1}\alpha_i$ is not a simple root, or equivalently, if $s_i\overline{v}(\mu) \notin \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$, then*

$$T_i^\dagger\ \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) = \mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda).$$

(b) *If $-\overline{v}(\mu)^{-1}\alpha_i$ is a simple root, or equivalently, if $s_i\overline{v}(\mu) \in \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$, then*

$$T_i^\dagger\ \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) = \left(1 - q^{\langle \lambda, \overline{v}(\mu)^{-1}\alpha_i^\vee \rangle}\right) \mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda).$$

By combining this theorem with Theorem 4.2.8, we obtain Cherednik-Orr's recursion formula for $E_\mu(q,\infty)$, $\mu \in W\lambda$; cf. [CO, Proposition 3.5 (iii)] for the affine root systems of dual untwisted type.

Theorem 4.4.2. *Let $\lambda \in P^+$ be a dominant weight. Let $\mu \in W\lambda$ and $i \in I$ be such that $\langle \mu, \alpha_i^\vee \rangle < 0$.*

(a) *If $-\overline{v}(\mu)^{-1}\alpha_i$ is not a simple root, then*

$$T_i^\dagger E_\mu(q,\infty) = E_{s_i\mu}(q,\infty).$$

(b) *If $-\overline{v}(\mu)^{-1}\alpha_i$ is a simple root, then*

$$T_i^\dagger E_\mu(q,\infty) = \left(1 - q^{\langle \lambda, \overline{v}(\mu)^{-1}\alpha_i^\vee \rangle}\right) E_{s_i\mu}(q,\infty).$$

Now, we set $D_i^\dagger := T_i^\dagger + 1$, which is a $\mathbb{C}(q)$-linear operator on $\mathbb{C}(q)[P]$. The next lemma follows easily from the definition of $D_i^\dagger$.

Lemma 4.4.3. *Let $i \in I$.*

(1) $D_i^\dagger e^\mu = \frac{e^\mu - e^{s_i\mu+\alpha_i}}{1-e^{\alpha_i}}$ *for* $\mu \in P$.

(2) *If* $\mu \in P$ *satisfies* $\langle \mu, \alpha_i^\vee \rangle \leq 0$, *then*

$$D_i^\dagger e^\mu = e^\mu + e^{\mu+\alpha_i} + \cdots + e^{s_i\mu}.$$

(3) $(D_i^\dagger)^2 = D_i^\dagger$.

Proof. Because (1) and (3) are immediate from the definition of the operator $D_i^\dagger$, we omit their proofs.

(2) By (1), we have

$$\begin{aligned} D_i^\dagger e^\mu &= \frac{e^\mu\left(1 - e^{(1-\langle \mu,\, \alpha_i^\vee \rangle)\alpha_i}\right)}{1-e^{\alpha_i}} \\ &= e^\mu + e^{\mu+\alpha_i} + \cdots + e^{\mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i}, \end{aligned}$$

where, for the last equality, we have used the assumption that $\langle \mu, \alpha_i^\vee \rangle \leq 0$. This proves the lemma.

□

4.4.2 Crystal structure on QLS(λ)

In this subsection, following [LNSSS4], we endow the set QLS(λ) with a canonical $U_{\mathsf{v}}'(\mathfrak{g}_{\mathrm{aff}})$-crystal structure, where $U_{\mathsf{v}}'(\mathfrak{g}_{\mathrm{aff}})$ denotes the quantum affine algebra associated to the untwisted affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ associated to $\mathfrak{g}$.

We follow the notation of §3.4.1 (or §2.3). In this subsection, as in §3.4.1, we regard an element $\lambda \in \mathfrak{h}^*$ as an element of $\mathfrak{h}^*_{\mathrm{aff}}$ by: $\langle \lambda,\, c \rangle = \langle \lambda,\, D \rangle = 0$, and then we have $\varpi_i = \Lambda_i - a_i^\vee \Lambda_0$ for $i \in I$. Also, as in the proof of Lemma 3.4.12, we set

$$\widetilde{s}_j := \begin{cases} s_j & \text{if } j \neq 0, \\ s_\theta & \text{if } j = 0, \end{cases} \qquad \text{and} \qquad \widetilde{\alpha}_j := \begin{cases} \alpha_j & \text{if } j \neq 0, \\ -\theta & \text{if } j = 0. \end{cases}$$

Remark 4.4.4. We identify an element $\psi = (v_1, \ldots, v_s; \sigma_0, \sigma_1, \ldots, \sigma_s) \in$ QLS(λ) with the following piecewise-linear, continuous map $\psi : [0,1] \to \mathfrak{h}^*_{\mathbb{R}} = \bigoplus_{i\in I} \mathbb{R}\alpha_i$:

$$\psi(t) = \sum_{k=1}^{p-1} (\sigma_k - \sigma_{k-1}) v_k \lambda + (t - \sigma_{p-1}) v_p \lambda \ \text{ for } \sigma_{p-1} \leq t \leq \sigma_p, 1 \leq p \leq s.$$

Let $i \in I_{\mathrm{aff}}$. We define the root operators e_i, f_i : QLS(λ) $\to$ QLS(λ) $\sqcup \{\mathbf{0}\}$ as follows.

First, we define a function $H(t)$ on $[0,1]$ by

$$H(t) = H_i^\psi(t) := \langle \psi(t), \overline{\alpha}_i^\vee \rangle, \ t \in [0,1],$$

and set

$$m = m_i^\psi := \min\{H_i^\psi(t) \mid t \in [0,1]\}.$$

It follows from [LNSSS4, Proposition 4.1.12] that $m \in \mathbb{Z}_{\leq 0}$. If $m = 0$, then we set $e_i\psi := \mathbf{0}$. If $m \leq -1$, then we set

$$\begin{aligned} t_1 &:= \min\{t \in [0,1] \mid H(t) = m\}, \\ t_0 &:= \max\{t \in [0,t_1] \mid H(t) = m+1\}, \end{aligned}$$

and define $e_i\psi$ by

$$e_i\psi(t) := \begin{cases} \psi(t) & \text{for } t \in [0,t_0], \\ \psi(t_0) + \widetilde{s}_i(\psi(t) - \psi(t_0)) & \text{for } t \in [t_0,t_1], \\ \psi(t) + \widetilde{\alpha}_i & \text{for } t \in [t_1,1]. \end{cases}$$

Similarly, we define f_i as follows. If $H(1) - m = 0$, then we set $f_i\psi := \mathbf{0}$. Otherwise, we set

$$\begin{aligned} t'_0 &:= \max\{t \in [0,1] \mid H(t) = m\}, \\ t'_1 &:= \min\{t \in [t'_0,1] \mid H(t) = m+1\}, \end{aligned}$$

and define $f_i\psi$ by

$$f_i\psi(t) := \begin{cases} \psi(t) & \text{for } t \in [0,t'_0], \\ \psi(t'_0) + \widetilde{s}_i(\psi(t) - \psi(t'_0)) & \text{for } t \in [t'_0,t'_1], \\ \psi(t) - \widetilde{\alpha}_i & \text{for } t \in [t'_1,1]. \end{cases}$$

Then, it follows from [LNSSS4, Proposition 4.2.1] that $e_i\psi \in \mathrm{QLS}(\lambda) \sqcup \{\mathbf{0}\}$ and $f_i\psi \in \mathrm{QLS}(\lambda) \sqcup \{\mathbf{0}\}$ for $\psi \in \mathrm{QLS}(\lambda)$.

Also, for $i \in I_{\mathrm{aff}}$, we define $\varepsilon_i, \varphi_i : \mathrm{QLS}(\lambda) \to \mathbb{Z}$ by $\varepsilon_i(\psi) := \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k\psi \neq \mathbf{0}\}$, $\varphi_i(\psi) := \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k\psi \neq \mathbf{0}\}$.

As for the representation theoretic meaning of the set $\mathrm{QLS}(\lambda)$, we know the following; for details, see [NS1], [NS2], [NS3], [LNSSS1], [LNSSS2], [LNSSS4], and [Na].

Proposition 4.4.5 ([LNSSS4, Theorem 4.1.1], [NS3, Theorem 3.2], [Na, Remark 2.15])**.** *The set* $\mathrm{QLS}(\lambda)$*, equipped with the maps* $\mathrm{wt}, e_i, f_i, \varepsilon_i, \varphi_i$*,* $i \in I_{\mathrm{aff}}$*, is a* $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$*-crystal. Moreover, it provides a realization of the crystal basis of a particular quantum Weyl module* $W_{\mathsf{v}}(\lambda)$ *over a quantum affine algebra* $U'_{\mathsf{v}}(\mathfrak{g}_{\mathrm{aff}})$*.*

Example 4.4.6. Let $\mathfrak{g}$ be of type A_2, and let $\lambda = \varpi_1 + \varpi_2$. Then the crystal graph of $\mathrm{QLS}(\lambda)$ is as follows:

$$
\begin{tikzcd}
& \eta_e \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & \\
\eta_{s_1} \arrow[d, "f_2"'] & & \eta_{s_2} \arrow[d, "f_1"] & \\
\eta_1 \arrow[d, "f_2"'] & & \eta_2 \arrow[d, "f_1"] & \eta_3 \arrow[uull, "f_0"'] \\
\eta_{s_2s_1} \arrow[uurr, "f_0"] \arrow[dr, "f_1"'] & & \eta_{s_1s_2} \arrow[uull, "f_0"'] \arrow[dl, "f_2"] & \\
& \eta_{w_\circ} \arrow[uurr, "f_0"'] & &
\end{tikzcd}
$$

Here, $\psi_v = (v; 0, 1)$ for $v \in W$, and

$$\psi_1 := (s_2s_1, s_1; 0, 1/2, 1),\ \ \psi_2 := (s_1s_2, s_2; 0, 1/2, 1),\ \ \psi_3 := (e, w_\circ; 0, 1/2, 1).$$

The next lemma follows from the definition of root operators.

Lemma 4.4.7. *Let $i \in I$ and $\psi \in \mathrm{QLS}(\lambda)$ be such that $f_i\psi = \mathbf{0}$. Then, $\lfloor s_i\kappa(\psi)\rfloor \le \kappa(\psi)$. Moreover, if $e_i\psi \neq \mathbf{0}$, then the following hold:*

(1) *if $\kappa(e_i\psi) = \kappa(\psi)$, then for every $p > 0$ such that $e_i^p\psi \neq \mathbf{0}$, we have*

$$\kappa(e_i^p\psi) = \kappa(\psi);$$

(2) *if $\kappa(e_i\psi) = \lfloor s_i\kappa(\psi)\rfloor < \kappa(\psi)$, then for every $p > 0$ such that $e_i^p\psi \neq \mathbf{0}$, we have*

$$\kappa(e_i^p\psi) = \lfloor s_i\kappa(\psi)\rfloor.$$

Proof. Since $f_i\psi = \mathbf{0}$ by the assumption, it follows from the definition of the root operator f_i that $\max\{t \in [0,1] \mid H_i^\psi(t) = m_i^\psi\} = 1$, and hence that the function $H_i^\psi(t)$ is weakly decreasing in a sufficiently small neighborhood of the point $t = 1$. Therefore, we must have $\langle \kappa(\psi)\lambda, \alpha_i^\vee\rangle \le 0$, and hence $\lfloor s_i\kappa(\psi)\rfloor \le \kappa(\psi)$.

Now, suppose that there exists $p \in \mathbb{Z}_{>0}$ such that $e_i^p\psi \neq \mathbf{0}$ and $\kappa(e_i^p\psi) \neq \kappa(e_i^{p-1}\psi)$; note that if $\kappa(e_i^p\psi) \neq \kappa(e_i^{p-1}\psi)$, then $\kappa(e_i^p\psi) = \lfloor s_i\kappa(e_i^{p-1}\psi)\rfloor$ by the definition of e_i (or, by the definition of f_i). Therefore, if we set $t_0'' := \max\{t \in [0,1] \mid H_i^{e_i^p\psi}(t) = m_i^{e_i^p\psi}\}$, then from the definition of f_i, we deduce that $t_1'' := \min\{t \in [t_0'', 1] \mid H_i^{e_i^p\psi}(t) = m_i^{e_i^p\psi} + 1\} = 1$. Hence, by noting that $m_i^{e_i^{p-1}\psi} = m_i^{e_i^p\psi} - 1$, we obtain $\max\{t \in [0,1] \mid H_i^{e_i^{p-1}\psi}(t) = m_i^{e_i^{p-1}\psi}\} = 1$. This implies that $f_i(e_i^{p-1}\psi) = \mathbf{0}$, and hence $p = 1$. This proves the lemma. $\square$

Remark 4.4.8. For $\psi \in \mathrm{QLS}(\lambda)$ such that $f_i\psi \neq \mathbf{0}$, we obtain

$$\kappa(e_i^{\max}\psi) = \cdots = \kappa(\psi) = \cdots = \kappa(e_i f_i^{\max}\psi) \leq \kappa(f_i^{\max}\psi)$$

by applying Lemma 4.4.7 to $f_i^{\max}\psi = \psi'$, where $f_i^{\max}\psi := f^{\varphi_i(\psi)}(\psi)$ and $e_i^{\max}\psi := e^{\varepsilon_i(\psi)}(\psi)$. Moreover, if $\kappa(\psi) < \lfloor s_i\kappa(\psi)\rfloor$, then $\kappa(f_i^{\max}\psi) = \lfloor s_i\kappa(\psi)\rfloor$; otherwise, $\kappa(f_i^{\max}\psi) = \kappa(\psi)$.

4.4.3 String property

Lemma 4.4.9. *Let $\mu \in W\lambda$ and $i \in I$ be such that $\langle \mu, \alpha_i^\vee\rangle < 0$, or equivalently, $s_i\overline{v}(\mu) = \overline{v}(s_i\mu) < \overline{v}(\mu)$ (see Lemma 4.3.13). If $\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$, then $f_i^{\max}\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$.*

Proof. If $\kappa(f_i^{\max}\psi) = \kappa(\psi)$, then the assertion is obvious. Hence we assume that $\kappa(f_i^{\max}\psi) \neq \kappa(\psi)$; in this case, $\kappa(\psi) < \lfloor s_i\kappa(\psi)\rfloor (\leq s_i\kappa(\psi))$ and $\kappa(f_i^{\max}\psi) = \lfloor s_i\kappa(\psi)\rfloor$ by Remark 4.4.8. Since $\kappa(\psi) \in \lfloor \mathrm{EQB}(\overline{v}(\mu))\rfloor \subset \mathrm{EQB}(\overline{v}(\mu))$ by the assumption, it follows from Lemma 4.3.7 (2) that $s_i\kappa(\psi) \in \mathrm{EQB}(\overline{v}(\mu))$, and hence $\kappa(f_i^{\max}\psi) = \lfloor s_i\kappa(\psi)\rfloor \in \lfloor \mathrm{EQB}(\overline{v}(\mu))\rfloor$. Hence it follows that $f_i^{\max}\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$. This proves the lemma.

□

Proposition 4.4.10. *Let $\mu \in W\lambda$ and $i \in I$ be such that $\langle \mu, \alpha_i^\vee\rangle < 0$. Let $\psi \in \mathrm{QLS}(\lambda)$, and let $S(\psi)$ denote the i-string containing ψ, i.e.,*

$$S(\psi) := \left(\bigcup_{p\geq 0}\{e_i^p\psi\} \cup \bigcup_{q\geq 0}\{f_i^q\psi\}\right) \setminus \{\mathbf{0}\}.$$

(a) *If $s_i\overline{v}(\mu) \notin \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$, then*

$$\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S(\psi) = \emptyset, S(\psi), \text{ or } \{f_i^{\max}\psi\}.$$

(b) *If $s_i\overline{v}(\mu) \in \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$, then*

$$\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S(\psi) = \emptyset \text{ or } S(\psi).$$

Proof. Assume that $\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S(\psi) \neq \emptyset$, and take $\psi' \in \mathrm{QLS}^{\mu,\infty}(\lambda) \cap S(\psi)$. Since $f_i^{\max}\psi' \in \mathrm{QLS}^{\mu,\infty}(\lambda) \cap S(\psi)$ by Lemma 4.4.9, we may assume that ψ' is the lowest element $f_i^{\max}\psi$ of the i-string $S(\psi)$. If $\kappa(e_i\psi') = \kappa(\psi')$, then $\kappa(e_i^p\psi') = \kappa(\psi') \in \lfloor \mathrm{EQB}(\overline{v}(\mu))\rfloor$ for all $p > 0$ such that $e_i^p\psi' \neq \mathbf{0}$ by Lemma 4.4.7 (1). Hence we obtain $S(\psi) \subset \mathrm{QLS}^{\mu,\infty}(\lambda)$.

Now we consider the case that $\kappa(e_i\psi') = \lfloor s_i\kappa(\psi')\rfloor < \kappa(\psi')$; in this case, by Lemma 4.4.7 (2), $\kappa(e_i^p\psi') = \kappa(e_i\psi) = \lfloor s_i\kappa(\psi')\rfloor$ for all $p > 0$ such that $e_i^p\psi' \neq \mathbf{0}$.

Case (i). If $s_i\kappa(\psi') \in \mathrm{EQB}(\overline{v}(\mu))$, then $\kappa(e_i^p\psi') \in \lfloor \mathrm{EQB}(\overline{v}(\mu))\rfloor$ for all $p > 0$ such that $e_i^p\psi' \neq \mathbf{0}$, and hence $S(\psi) \setminus \{\psi'\} \subset \mathrm{QLS}^{\mu,\infty}(\lambda)$. Thus, we obtain $S(\psi) \subset \mathrm{QLS}^{\mu,\infty}(\lambda)$.

Case (ii). If $s_i\kappa(\psi') \notin \mathrm{EQB}(\overline{v}(\mu))$, then $\kappa(e_i^p\psi') \notin \lfloor \mathrm{EQB}(\overline{v}(\mu))\rfloor$ for any $p > 0$ such that $e_i^p\psi' \neq \mathbf{0}$, and hence $(S(\psi) \setminus \{\psi'\}) \cap \mathrm{QLS}^{\mu,\infty}(\lambda) = \emptyset$. Therefore, we obtain

$S(\psi) \cap \mathrm{QLS}^{\mu,\infty}(\lambda) = \{\psi'\} = \{f_i^{\max}\psi\}$.
Also, if $s_i\kappa(\psi') \notin \mathrm{EQB}(\overline{v}(\mu))$, then we have $s_i\overline{v}(\mu) \notin \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$ by Proposition 4.3.8 (2b). Hence we need the extra case that $\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S(\psi) = \{f_i^{\max}\psi\}$ only if $s_i\overline{v}(\mu) \notin \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$. This proves the proposition.

$\square$

Proposition 4.4.11. *Let $\mu_1, \mu_2 \in W\lambda$. Let $\psi \in \mathrm{QLS}(\lambda)$ and $i \in I$ be such that $e_i\psi \neq \mathbf{0}$. Then,*

$$\deg_{\mu_1}(\psi) - \deg_{\mu_2}(e_i\psi) = -\mathrm{wt}_\lambda(\lfloor\overline{v}(\mu_1)\rfloor \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\lfloor\overline{v}(\mu_2)\rfloor \Rightarrow \kappa(e_i\psi)).$$

In particular, if $\kappa(\psi) = \kappa(e_i\psi)$, then $\deg_\mu(\psi) = \deg_\mu(e_i\psi)$ for all $\mu \in W\lambda$.

Proof. If we set $\mathrm{Deg}(\psi) := \deg_\nu(\psi) + \mathrm{wt}_\lambda(\lfloor\overline{v}(\nu)\rfloor \Rightarrow \kappa(\psi))$ for $\nu \in W\lambda$, then it follows from the definition of $\deg_\nu(\psi)$ that $\mathrm{Deg}(\psi)$ does not depend on the choice of $\nu \in W\lambda$, and that it is identical to the right-hand side of the equation in [LNSSS2, Corollary 4.8]. Therefore, by [LNSSS2, Remark 4.4], we have

$$\mathrm{Deg}(e_i\psi) = \mathrm{Deg}(\psi),$$

and hence

$$\deg_{\mu_1}(\psi) + \mathrm{wt}_\lambda(\lfloor\overline{v}(\mu_1)\rfloor \Rightarrow \kappa(\psi)) = \deg_{\mu_2}(e_i\psi) + \mathrm{wt}_\lambda(\lfloor\overline{v}(\mu_2)\rfloor \Rightarrow \kappa(e_i\psi)).$$

This proves the first assertion of Proposition 4.4.11. The second assertion follows from the first one.

$\square$

4.4.4 Proof of the recursion formula in the case $s_i\overline{v}(\mu) \notin \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$

Throughout this subsection, we take and fix $\mu \in W\lambda$ and $i \in I$ such that $\langle\mu, \alpha_i^\vee\rangle < 0$ and $s_i\overline{v}(\mu) \notin \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$. We set $E^\mu(\lambda) := \bigcup_{p\geq 0} e_i^p\mathrm{QLS}^{\mu,\infty}(\lambda) \setminus \{\mathbf{0}\}$.

Lemma 4.4.12. *We have*

$$E^\mu(\lambda) = \mathrm{QLS}^{\mu,\infty}(\lambda) \sqcup \mathrm{QLS}^{s_i\mu,\infty}(\lambda).$$

Proof. First, we note that $\kappa(e_i^p\psi) = \kappa(\psi)$ or $\lfloor s_i\kappa(\psi)\rfloor$ for $\psi \in \mathrm{QLS}(\lambda)$ and $p \in \mathbb{Z}_{\geq 0}$ such that $e_i^p\psi \neq \mathbf{0}$. It follows from Proposition 4.3.14 (1) that the inclusion $\subset$ holds, and that $\mathrm{QLS}^{\mu,\infty}(\lambda) \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda) = \emptyset$. Hence it remains to prove the opposite inclusion $\supset$. It is obvious that $E^\mu(\lambda) \supset \mathrm{QLS}^{\mu,\infty}(\lambda)$. Let $\psi \in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$. Then, since $\kappa(\psi) \in \lfloor\mathrm{EQB}(\overline{v}(s_i\mu))\rfloor$, it follows from Lemma 4.3.15 that $\lfloor s_i\kappa(\psi)\rfloor > \kappa(\psi)$, and $s_i\kappa(\psi) \in \mathrm{EQB}(\overline{v}(\mu))$. Also, by Remark 4.4.8, we have $\kappa(f_i^{\max}\psi) = \lfloor s_i\kappa(\psi)\rfloor$, and hence $f_i^{\max}\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$. Since there exists $p \in \mathbb{Z}_{\geq 0}$ such that $e_i^p(f_i^{\max}\psi) = \psi$, we deduce that $\psi = e_i^p(f_i^{\max}\psi) \in e_i^p\mathrm{QLS}^{\mu,\infty}(\lambda) \setminus \{\mathbf{0}\} \subset E^\mu(\lambda)$, as desired. This proves the lemma.

$\square$

Lemma 4.4.13. *Let $\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$ be such that $f_i\psi = \mathbf{0}$, and let $k \in \mathbb{Z}_{>0}$ be such that $e_i^k\psi \neq \mathbf{0}$. Then*

(1) *if* $e_i^k\psi \in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$, *then* $\deg_{s_i\mu}(e_i^k\psi) = \deg_\mu(\psi)$;

(2) *if* $e_i^k\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$, *then* $\deg_\mu(e_i^k\psi) = \deg_\mu(\psi)$.

Proof. First, we note that $e_i^k\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$ (resp., $\in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$) if and only if $e_i\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$ (resp., $\in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$), since $\kappa(e_i\psi) = \cdots = \kappa(e_i^k\psi)$ by Lemma 4.4.7.

(1) Since $e_i^k\psi \in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$ by the assumption, we have $e_i\psi \in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$. In this case, since $\lfloor \mathrm{EQB}(\overline{v}(\mu))\rfloor \cap \lfloor \mathrm{EQB}(s_i\overline{v}(\mu))\rfloor = \emptyset$ by Proposition 4.3.14 (1), it follows that $\kappa(e_i\psi) \neq \kappa(\psi)$, and hence $\kappa(e_i\psi) = \lfloor s_i\kappa(\psi)\rfloor < \kappa(\psi)$ by Lemma 4.4.7; notice that $s_i\kappa(\psi) < \kappa(\psi)$. Therefore, we see that

$$
\begin{aligned}
&\deg_\mu(\psi) - \deg_{s_i\mu}(e_i\psi) \\
&= -\mathrm{wt}_\lambda(\lfloor \overline{v}(\mu)\rfloor \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\lfloor s_i\overline{v}(\mu)\rfloor \Rightarrow \lfloor s_i\kappa(\psi)\rfloor) && \text{by Proposition 4.4.11} \\
&= -\mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(s_i\overline{v}(\mu) \Rightarrow s_i\kappa(\psi)) && \text{by equation (3.2.3)} \\
&= 0 && \text{by Proposition 4.3.11 (1);}
\end{aligned}
$$

the last equality follows since $s_i\overline{v}(\mu) < \overline{v}(\mu)$ and $s_i\kappa(\psi) < \kappa(\psi)$ by our assumption. Since $\kappa(e_i\psi) = \kappa(e_i^2\psi) = \cdots = \kappa(e_i^k\psi)$ by Lemma 4.4.7 (2), we deduce that $\deg_{s_i\mu}(e_i\psi) = \cdots = \deg_{s_i\mu}(e_i^k\psi)$ by Proposition 4.4.11. This proves the desired equality $\deg_{s_i\mu}(e_i^k\psi) = \deg_\mu(\psi)$.

(2) Since $e_i^k\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$ by the assumption, we have $e_i\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda)$. If $\kappa(e_i\psi) = \kappa(\psi)$, then $\deg_\mu(e_i\psi) = \deg_\mu(\psi)$ by Proposition 4.4.11. If $\kappa(e_i\psi) = \lfloor s_i\kappa(\psi)\rfloor < \kappa(\psi)$, then $\kappa(e_i\psi) = \lfloor s_i\kappa(\psi)\rfloor \in \lfloor \mathrm{EQB}(\overline{v}(\mu))\rfloor$, and hence $s_i\kappa(\psi) \in \mathrm{EQB}(\overline{v}(\mu))$ by Lemma 4.3.12. In this case, it follows from Proposition 4.3.8 (1a) that $s_i\kappa(\psi) \notin \mathrm{EQB}(s_i\overline{v}(\mu))$. Therefore, we see that

$$
\begin{aligned}
&\deg_\mu(\psi) - \deg_\mu(e_i\psi) \\
&= -\mathrm{wt}_\lambda(\lfloor \overline{v}(\mu)\rfloor \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\lfloor \overline{v}(\mu)\rfloor \Rightarrow \lfloor s_i\kappa(\psi)\rfloor) && \text{by Proposition 4.4.11} \\
&= -\mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow s_i\kappa(\psi)) && \text{by equation (3.2.3)} \\
&= 0 && \text{by Proposition 4.3.11 (2).}
\end{aligned}
$$

Thus, in both cases, we have $\deg_\mu(e_i\psi) = \deg_\mu(\psi)$. Since $\kappa(e_i\psi) = \kappa(e_i^2\psi) = \cdots = \kappa(e_i^k\psi)$ by Lemma 4.4.7 (2), we deduce that $\deg_\mu(e_i\psi) = \cdots = \deg_\mu(e_i^k\psi)$ by Proposition 4.4.11. This proves the desired equality $\deg_\mu(e_i^k\psi) = \deg_\mu(\psi)$. □

Lemma 4.4.14. *We have*

$$\mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) + \mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda) = D_i^\dagger \, \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda).$$

Proof. Let $S_1, \dots, S_t$ be all of the distinct i-strings S_j such that $\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j \neq \emptyset$. Then, $\mathrm{QLS}^{\mu,\infty}(\lambda)$ decomposes into a disjoint union of i-strings as follows:

$$\mathrm{QLS}^{\mu,\infty}(\lambda) = (\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_1) \sqcup \cdots \sqcup (\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_t).$$

From this, we deduce that

$$\mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) = \mathrm{ch}_\mu (\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_1) + \cdots + \mathrm{ch}_\mu (\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_t), \qquad (4.4.1)$$

where we use the notation (4.2.1). Applying $D_i^\dagger$ to equation (4.4.1), we obtain

$$D_i^\dagger \operatorname{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) = D_i^\dagger \operatorname{ch}_\mu (\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_1) + \cdots + D_i^\dagger \operatorname{ch}_\mu (\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_t).$$

Here, because $\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j = S_j$ or $\{f_i^{\max}\psi\}$ for some $\psi \in \mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j$ for each $1 \le j \le t$ by Proposition 4.4.10 (a), we see from the definition of $E^\mu(\lambda)$ that

$$E^\mu(\lambda) \cap S_j = S_j \text{ for all } 1 \le j \le t,$$

and hence

$$E^\mu(\lambda) = S_1 \sqcup \cdots \sqcup S_t.$$

Also, since $E^\mu(\lambda) = \mathrm{QLS}^{\mu,\infty}(\lambda) \sqcup \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$ by Lemma 4.4.12, we deduce that

$$\begin{aligned}
&\operatorname{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) + \operatorname{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda) \\
&\qquad = \sum_{j=1}^{t} \left(\operatorname{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) + \operatorname{ch}_{s_i\mu}(\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j)\right).
\end{aligned}$$

Therefore, in order to prove the lemma, it suffices to show that for each $1 \le j \le t$,

$$D_i^\dagger \operatorname{ch}_\mu (\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) = \operatorname{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) + \operatorname{ch}_{s_i\mu}(\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j), \quad (4.4.2)$$

where we use the notation (4.2.1).

Now, let $1 \le j \le t$, and write $S_j = \{\psi, e_i\psi, \dots, e_i^k\psi\}$ for some $k \ge 0$ (depending on j), where ψ is the lowest element of the i-string S_j. Since $f_i\psi = \mathbf{0}$, we have $k = -\langle \mathrm{wt}(\psi), \alpha_i^\vee \rangle$. Hence $\mathrm{wt}(e_i^k\psi) = \mathrm{wt}(\psi) + k\alpha_i = \mathrm{wt}(\psi) - \langle \mathrm{wt}(\psi), \alpha_i^\vee \rangle \alpha_i = s_i \mathrm{wt}(\psi)$. In view of Proposition 4.4.10 (a), we need to consider the following two cases.

Case (i). Assume that $\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j = S_j$. In this case, we have $\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j = \emptyset$ by Lemma 4.4.12, and $\deg_\mu(\psi) = \cdots = \deg_\mu(e_i^k\psi)$ by Lemma 4.4.13 (2). From these, we see that

$$\begin{aligned}
\operatorname{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) + \operatorname{ch}_{s_i\mu}(\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j) &= \operatorname{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) \\
&= q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)} + \cdots + q^{\deg_\mu(e_i^k\psi)} e^{s_i \mathrm{wt}(\psi)} \\
&= q^{\deg_\mu(\psi)} (e^{\mathrm{wt}(\psi)} + \cdots + e^{s_i \mathrm{wt}(\psi)}) \\
&= D_i^\dagger q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)} \qquad \text{by Lemma 4.4.3 (2)},
\end{aligned}$$

and hence

$$\begin{aligned}
D_i^\dagger \operatorname{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) &= (D_i^\dagger)^2 q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)} \\
&= D_i^\dagger q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)} \qquad \text{by Lemma 4.4.3 (3)} \\
&= \operatorname{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) + \operatorname{ch}_{s_i\mu}(\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j).
\end{aligned}$$

Case (ii). Assume that $\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j = \{\psi\}$. In this case, we have $\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j = \{e_i\psi, \dots, e_i^k\psi\}$ by Lemma 4.4.12, and $\deg_\mu(\psi) = \deg_{s_i\mu}(e_i\psi) = \cdots = \deg_{s_i\mu}(e_i^k\psi)$ by Lemma 4.4.13 (1). From these, we see that $\operatorname{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) = q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)}$, and that

$$\begin{aligned}
\operatorname{ch}_{s_i\mu}(\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j) &= q^{\deg_{s_i\mu}(e_i\psi)} e^{\mathrm{wt}(e_i\psi)} + \cdots + q^{\deg_{s_i\mu}(e_i^k\psi)} e^{\mathrm{wt}(e_i^k\psi)} \\
&= q^{\deg_\mu(\psi)} (e^{\mathrm{wt}(\psi)+\alpha_i} + \cdots + e^{s_i \mathrm{wt}(\psi)}).
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
D_i^\dagger \ \mathrm{ch}_\mu\left(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j\right) &= D_i^\dagger q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)} \\
&= q^{\deg_\mu(\psi)}(e^{\mathrm{wt}(\psi)} + \cdots + e^{s_i \mathrm{wt}(\psi)}) \quad \text{by Lemma 4.4.3 (2)} \\
&= \mathrm{ch}_\mu(\mathrm{QLS}^{\mu,\infty}(\lambda) \cap S_j) + \mathrm{ch}_{s_i\mu}(\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S_j).
\end{aligned}$$

Thus, in both cases, we obtain equation (4.4.2), as desired. This proves the lemma.

□

Proof of Theorem 4.4.1 (a). By Lemma 4.4.14, we have

$$D_i^\dagger \ \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) = \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) + \mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda).$$

Since $D_i^\dagger = T_i^\dagger + 1$, we conclude from this equation that $T_i^\dagger \ \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) = \mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$. This proves Theorem 4.4.1 (a).

□

Example 4.4.15. Let $\mathfrak{g}$ be of type A_2, and let $\lambda = \varpi_1 + \varpi_2$, $w = s_1s_2$, $i = 1$; by Example 4.3.9, we have $s_1w = s_2 \notin wW_{I_w}$. Let ψ_v, $v \in W$, and ψ_k, $k = 1, 2, 3$, be as in Example 4.4.6. Recall from Example 4.2.9 that

$$\begin{aligned}
\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) &= \{\psi_{s_1s_2}, \psi_{s_1}, \psi_1\}, \\
\mathrm{QLS}^{s_2\lambda,\infty}(\lambda) &= \{\psi_{s_2}, \psi_e, \psi_2\}.
\end{aligned}$$

Since $e_1\psi_{s_1s_2} = \psi_2$, $e_1^2\psi_{s_1s_2} = \psi_{s_2}$, $e_1\psi_1 = \mathbf{0}$, and $e_1\psi_{s_1} = \psi_e$ by Example 4.4.6, we see that

$$\begin{aligned}
\bigcup_{p \geq 0} e_1^p \mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) \setminus \{\mathbf{0}\} &= \{\psi_{s_1s_2}, \psi_{s_1}, \psi_1, \psi_{s_2}, \psi_e, \psi_2\} \\
&= \mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) \sqcup \mathrm{QLS}^{s_2\lambda,\infty}(\lambda).
\end{aligned}$$

Also, by Example 4.2.9, we have

$$\begin{aligned}
\deg_{s_1s_2\lambda}(\psi_{s_1s_2}) = \deg_{s_2\lambda}(\psi_2) = \deg_{s_2\lambda}(\psi_{s_2}) = 0, \\
\deg_{s_1s_2\lambda}(\psi_{s_1}) = \deg_{s_2\lambda}(\psi_e) = -1, \\
\deg_{s_1s_2\lambda}(\psi_1) = -1.
\end{aligned} \tag{4.4.3}$$

Therefore, by using the data in Example 4.2.9, we compute:

$$
\begin{aligned}
&D_1^\dagger E_{s_1 s_2\lambda}(q,\infty) \\
&= D_1^\dagger \left(\sum_{\psi \in \mathrm{QLS}^{s_1 s_2\lambda,\infty}(\lambda)} q^{\deg_{s_1 s_2\lambda}(\psi)} e^{\mathrm{wt}(\psi)} \right) \\
&= D_1^\dagger q^{\deg_{s_1 s_2\lambda}(\psi_{s_1 s_2})} e^{\mathrm{wt}(\psi_{s_1 s_2})} + D_1^\dagger q^{\deg_{s_1 s_2\lambda}(\psi_{s_1})} e^{\mathrm{wt}(\psi_{s_1})} + D_1^\dagger q^{\deg_{s_1 s_2\lambda}(\psi_1)} e^{\mathrm{wt}(\psi_1)} \\
&= D_1^\dagger q^{\deg_{s_1 s_2\lambda}(\psi_{s_1 s_2})} e^{s_1 s_2\lambda} + D_1^\dagger q^{\deg_{s_1 s_2\lambda}(\psi_{s_1})} e^{s_1\lambda} + D_1^\dagger q^{\deg_{s_1 s_2\lambda}(\psi_1)} e^{0} \\
&= q^{\deg_{s_1 s_2\lambda}(\psi_{s_1 s_2})} \left(e^{s_1 s_2\lambda} + e^0 + e^{s_2\lambda} \right) + q^{\deg_{s_1 s_2\lambda}(\psi_{s_1})} \left(e^{s_1\lambda} + e^{\lambda} \right) + q^{\deg_{s_1 s_2\lambda}(\psi_1)} e^0 \\
&\hspace{22em} \text{(by Lemma 4.4.3 (2))} \\
&= q^{\deg_{s_1 s_2\lambda}(\psi_{s_1 s_2})} \left(e^{\mathrm{wt}(\psi_{s_1 s_2})} + e^{\mathrm{wt}(\psi_1)} + e^{\mathrm{wt}(\psi_{s_2})} \right) \\
&\qquad\qquad + q^{\deg_{s_1 s_2\lambda}(\psi_{s_1})} \left(e^{\mathrm{wt}(\psi_{s_1})} + e^{\mathrm{wt}(\psi_e)} \right) + q^{\deg_{s_1 s_2\lambda}(\psi_1)} e^{\mathrm{wt}(\psi_1)} \\
&= \sum_{\psi \in \mathrm{QLS}^{s_1 s_2\lambda,\infty}(\lambda)} q^{\deg_{s_1 s_2\lambda}(\psi)} e^{\mathrm{wt}(\psi)} + \sum_{\psi \in \mathrm{QLS}^{s_2\lambda,\infty}(\lambda)} q^{\deg_{s_2\lambda}(\psi)} e^{\mathrm{wt}(\psi)} \qquad \text{by (4.4.3)} \\
&= E_{s_1 s_2\lambda}(q,\infty) + E_{s_2\lambda}(q,\infty).
\end{aligned}
$$

4.4.5 Proof of the recursion formula in the case $s_i\overline{v}(\mu) \in \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$

Throughout this subsection, we take and fix $\mu \in W\lambda$ and $i \in I$ such that $\langle \mu, \alpha_i^\vee \rangle < 0$ and $s_i\overline{v}(\mu) \in \overline{v}(\mu)W_{I_{\overline{v}(\mu)}}$.

Lemma 4.4.16. *There exist i-strings $S_1, \dots, S_t \subset \mathrm{QLS}^{\mu,\infty}(\lambda)$ such that*

$$\mathrm{QLS}^{\mu,\infty}(\lambda) = \mathrm{QLS}^{s_i\mu,\infty}(\lambda) \sqcup f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \sqcup S_1 \sqcup \cdots \sqcup S_t,$$

where $f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda) := \{f_i^{\max}\psi \mid \psi \in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)\}$. In particular,

$$
\begin{aligned}
&\mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) \\
= \;&\mathrm{ch}_\mu \mathrm{QLS}^{s_i\mu,\infty}(\lambda) + \mathrm{ch}_\mu(f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) + \mathrm{ch}_\mu(S_1 \sqcup \cdots \sqcup S_t),
\end{aligned}
\tag{4.4.4}
$$

where we use the notation (4.2.1).

Proof. Let $\psi \in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$. Since $\kappa(\psi) \in \lfloor \mathrm{EQB}(s_i\overline{v}(\mu)) \rfloor$, it follows from Lemma 4.3.15 that $\lfloor s_i\kappa(\psi) \rfloor > \kappa(\psi)$, and $s_i\kappa(\psi) \in \mathrm{EQB}(\overline{v}(\mu)) \setminus \mathrm{EQB}(\overline{v}(s_i\mu))$. If we set $\psi' := f_i^{\max}\psi \in f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)$, then we have $S(\psi) = S(\psi') = \{\psi', e_i\psi', \dots, e_i^k\psi'\}$ for some $k \in \mathbb{Z}_{\geq 0}$. Here, by Lemma 4.3.12 and Remark 4.4.8, we have $\kappa(\psi') = \lfloor s_i\kappa(\psi) \rfloor \in \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor \setminus \lfloor \mathrm{EQB}(s_i\overline{v}(\mu)) \rfloor$ and $\kappa(e_i\psi') = \cdots = \kappa(e_i^k\psi') = \kappa(\psi) \in \lfloor \mathrm{EQB}(\overline{v}(s_i\mu)) \rfloor \subset \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor$; notice that $\psi' \neq \psi$, and hence $k > 0$. Thus, we have $\{e_i\psi', \dots, e_i^k\psi'\} = \mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S(\psi)$. Also, since $\psi' = f_i^{\max}\psi$, we see that $\psi' \in f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \cap S(\psi)$. Therefore, it follows that

$$S(\psi) = \{\psi', e_i\psi', \dots, e_i^k\psi'\} = (\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \sqcup f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) \cap S(\psi).$$

The argument above shows that $\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \sqcup f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)$ decomposes into a disjoint union of i-strings. In addition, by Proposition 4.4.10 (b), so does $\mathrm{QLS}^{\mu,\infty}(\lambda)$.

Hence the same is true for

$$\mathrm{QLS}^{\mu,\infty}(\lambda) \setminus (\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \sqcup f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda))\,;$$

here, we remark that $\mathrm{QLS}^{\mu,\infty}(\lambda) \supset \mathrm{QLS}^{s_i\mu,\infty}(\lambda) \sqcup f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)$ since $\kappa(e_i^p\psi') \in \lfloor \mathrm{EQB}(\overline{v}(\mu)) \rfloor$ for all $1 \le p \le k$ if $\psi \in \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$, as seen in the argument above. This proves the first assertion.

The second assertion follows from the first one.

$\square$

Lemma 4.4.17.

(1) *Let* $\psi \in f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)$. *Then, for every* $k \in \mathbb{Z}_{>0}$ *such that* $e_i^k\psi \neq \mathbf{0}$, *we have*

$$\deg_{s_i\mu}(e_i^k\psi) = \deg_\mu(\psi) = \deg_\mu(e_i^k\psi) + \langle \lambda, -\overline{v}(\mu)^{-1}\alpha_i^\vee \rangle.$$

(2) *Let* S_j, $1 \le j \le t$, *be as in Lemma* 4.4.16, *and let* $\psi \in S_j$. *Then, for every* $k, \ell \in \mathbb{Z}_{>0}$ *such that* $e_i^k\psi \neq \mathbf{0}$, *and* $f_i^\ell\psi \neq \mathbf{0}$, *we have*

$$\deg_\mu(e_i^k\psi) = \deg_\mu(f_i^\ell\psi) = \deg_\mu(\psi).$$

Proof. (1) Let $\psi \in f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)$. It follows from the proof of Lemma 4.4.16 that $\kappa(e_i\psi) = \lfloor s_i\kappa(\psi) \rfloor < \kappa(\psi)$, and $\kappa(e_i\psi) \in \lfloor \mathrm{EQB}(s_i\overline{v}(\mu)) \rfloor$; hence $s_i\kappa(\psi) < \kappa(\psi)$. Therefore, we have

$$\begin{aligned}
&\deg_\mu(\psi) - \deg_{s_i\mu}(e_i\psi) \\
&= -\mathrm{wt}_\lambda(\lfloor \overline{v}(\mu) \rfloor \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\lfloor s_i\overline{v}(\mu) \rfloor \Rightarrow \lfloor s_i\kappa(\psi) \rfloor) && \text{by Proposition 4.4.11} \\
&= -\mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(s_i\overline{v}(\mu) \Rightarrow s_i\kappa(\psi)) && \text{by equation (3.2.3)} \\
&= 0 && \text{by Proposition 4.3.11 (1).}
\end{aligned}$$

Also, since $\kappa(e_i\psi) = \kappa(e_i^2\psi) = \cdots = \kappa(e_i^k\psi)$ by Lemma 4.4.7 (2), we see that $\deg_{s_i\mu}(e_i\psi) = \cdots = \deg_{s_i\mu}(e_i^k\psi)$ by Proposition 4.4.11. Hence we obtain $\deg_{s_i\mu}(e_i^k\psi) = \deg_\mu(\psi)$.

Now, since $\lfloor s_i\kappa(\psi) \rfloor = \kappa(e_i\psi) \in \lfloor \mathrm{EQB}(s_i\overline{v}(\mu)) \rfloor$, we have $s_i\kappa(\psi) \in \mathrm{EQB}(s_i\overline{v}(\mu))$ by Lemma 4.3.12. Hence we see that

$$\begin{aligned}
&\deg_\mu(\psi) - \deg_\mu(e_i\psi) \\
&= -\mathrm{wt}_\lambda(\lfloor \overline{v}(\mu) \rfloor \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\lfloor \overline{v}(\mu) \rfloor \Rightarrow \lfloor s_i\kappa(\psi) \rfloor) && \text{by Proposition 4.4.11} \\
&= -\mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow s_i\kappa(\psi)) && \text{by equation (3.2.3)} \\
&= \langle \lambda, -\overline{v}(\mu)^{-1}\alpha_i^\vee \rangle && \text{by Proposition 4.3.11 (3).}
\end{aligned}$$

Since $\kappa(e_i\psi) = \kappa(e_i^2\psi) = \cdots = \kappa(e_i^k\psi)$ as mentioned above, we have $\deg_\mu(e_i\psi) = \cdots = \deg_\mu(e_i^k\psi)$ by repeated application of Proposition 4.4.11. Combining these, we obtain $\deg_\mu(\psi) = \deg_\mu(e_i^k\psi) + \langle \lambda, -\overline{v}(\mu)^{-1}\alpha_i^\vee \rangle$, as desired.

(2) By Lemma 4.4.9, it suffices to show that $\deg_\mu(e_i^k\psi) = \deg_\mu(\psi)$ for $\psi \in S_j$ such that ψ is the lowest element of the i-string S_j.

If $\kappa(e_i\psi) = \kappa(\psi)$, then $\kappa(e_i^k\psi) = \kappa(\psi)$ by Lemma 4.4.7 (1). In this case, applying Proposition 4.4.11 repeatedly, we obtain $\deg_\mu(e_i^k\psi) = \deg_\mu(\psi)$.

If $\kappa(e_i\psi) = \lfloor s_i\kappa(\psi)\rfloor < \kappa(\psi)$ (notice that in this case, we have $s_i\kappa(\psi) < \kappa(\psi)$), then $s_i\kappa(\psi) \notin \mathrm{EQB}(s_i\overline{v}(\mu))$; indeed, since S_j is an i-string such that $S_j \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda) = \emptyset$, we have $e_i\psi \in S_j \backslash \mathrm{QLS}^{s_i\mu,\infty}(\lambda)$, and hence $\kappa(e_i\psi) \notin \lfloor \mathrm{EQB}(s_i\overline{v}(\mu))\rfloor$. Therefore, we have

$$\begin{aligned}
&\deg_\mu(\psi) - \deg_\mu(e_i\psi)\\
&\qquad = -\mathrm{wt}_\lambda(\lfloor\overline{v}(\mu)\rfloor \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\lfloor\overline{v}(\mu)\rfloor \Rightarrow \lfloor s_i\kappa(\psi)\rfloor) && \text{by Proposition 4.4.11}\\
&\qquad = -\mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow \kappa(\psi)) + \mathrm{wt}_\lambda(\overline{v}(\mu) \Rightarrow s_i\kappa(\psi)) && \text{by equation (3.2.3)}\\
&\qquad = 0 && \text{by Proposition 4.3.11 (2).}
\end{aligned}$$

Also, applying Proposition 4.4.11 repeatedly, we have $\deg_\mu(e_i\psi) = \cdots = \deg_\mu(e_i^k\psi)$. Combining these, we obtain $\deg_\mu(e_i^k\psi) = \deg_\mu(\psi)$, as desired. This proves the lemma.

□

Lemma 4.4.18.

(1) $D_i^\dagger\ \mathrm{ch}_\mu(f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) = \mathrm{ch}_\mu(f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) + \mathrm{ch}_{s_i\mu}\,\mathrm{QLS}^{s_i\mu,\infty}(\lambda)$.

(2) $\mathrm{ch}_\mu\,\mathrm{QLS}^{s_i\mu,\infty}(\lambda) = q^{\langle\lambda,\overline{v}(\mu)^{-1}\alpha_i^\vee\rangle}\,\mathrm{ch}_{s_i\mu}\,\mathrm{QLS}^{s_i\mu,\infty}(\lambda)$.

(3) $D_i^\dagger\ \mathrm{ch}_\mu\mathrm{QLS}^{s_i\mu,\infty}(\lambda) = 0$.

(4) *Let S_j, $1 \le j \le t$, be as in Lemma* 4.4.16*. Then,* $D_i^\dagger\ \mathrm{ch}_\mu S_j = \mathrm{ch}_\mu S_j$.

Proof. By the proof of Lemma 4.4.16, there exists i-strings $S_1', \dots, S_u'$ such that

$$\mathrm{QLS}^{s_i\mu,\infty}(\lambda) \sqcup f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda) = S_1' \sqcup \cdots \sqcup S_u'.$$

To prove parts (1), (2), and (3), it suffices to show the following claim.

Claim. *For each $1 \le j \le u$, the following hold*:

(i) $D_i^\dagger\ \mathrm{ch}_\mu(S_j' \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) = \mathrm{ch}_\mu(S_j' \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) + \mathrm{ch}_{s_i\mu}(S_j' \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda))$.

(ii) $\mathrm{ch}_\mu(S_j' \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)) = q^{\langle\lambda,\overline{v}(\mu)^{-1}\alpha_i^\vee\rangle}\,\mathrm{ch}_{s_i\mu}(S_j' \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda))$.

(iii) $D_i^\dagger\ \mathrm{ch}_\mu(S_j' \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)) = 0$.

Proof of Claim. Let $1 \le j \le u$, and write $S_j' = \{\psi, e_i\psi, \dots, e_i^k\psi\}$ for some $k \in \mathbb{Z}_{\ge 0}$ (depending on j), where ψ is the lowest element of the i-string S_j; note that $k > 0$ by the proof of Lemma 4.4.16. Then it follows that $S_j' \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda) = \{\psi\}$ and $S_j' \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda) = \{e_i\psi, \dots, e_i^k\psi\}$.

(i) We have $\mathrm{ch}_\mu(S'_j \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) = q^{\deg_\mu(\psi)}e^{\mathrm{wt}(\psi)}$. Hence it follows from Lemma 4.4.3 (2) (note that $\langle \mathrm{wt}(\psi), \alpha_i^\vee \rangle \le 0$), together with the equality $\mathrm{wt}(e_i^k\psi) = s_i\mathrm{wt}(\psi)$, that

$$\begin{aligned} D_i^\dagger \ \mathrm{ch}_\mu(S'_j \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) &= q^{\deg_\mu(\psi)}(e^{\mathrm{wt}(\psi)} + \cdots + e^{s_i\mathrm{wt}(\psi)}) \\ &= q^{\deg_\mu(\psi)}(e^{\mathrm{wt}(\psi)} + \cdots + e^{\mathrm{wt}(e_i^k\psi)}). \end{aligned}$$

Also, we see that

$$\begin{aligned} &\mathrm{ch}_\mu(S'_j \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) + \mathrm{ch}_{s_i\mu}(S'_j \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)) = \\ &\qquad q^{\deg_\mu(\psi)}e^{\mathrm{wt}(\psi)} + q^{\deg_{s_i\mu}(e_i\psi)}e^{\mathrm{wt}(e_i\psi)} + \cdots + q^{\deg_{s_i\mu}(e_i^k\psi)}e^{\mathrm{wt}(e_i^k\psi)}. \end{aligned}$$

Because $\deg_\mu(\psi) = \deg_{s_i\mu}(e_i\psi) = \cdots = \deg_{s_i\mu}(e_i^k\psi)$ by Lemma 4.4.17 (1), we conclude that

$$\begin{aligned} &D_i^\dagger \ \mathrm{ch}_\mu(S'_j \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) \\ &\quad = \mathrm{ch}_\mu(S'_j \cap f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) + \mathrm{ch}_{s_i\mu}(S'_j \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)), \end{aligned}$$

as desired.

(ii) We deduce that

$$\begin{aligned} &\mathrm{ch}_\mu(S'_j \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)) \\ &= q^{\deg_\mu(e_i\psi)}e^{\mathrm{wt}(e_i\psi)} + \cdots + q^{\deg_\mu(e_i^k\psi)}e^{\mathrm{wt}(e_i^k\psi)} \\ &= q^{\langle \lambda, \overline{v}(\mu)^{-1}\alpha_i^\vee \rangle + \deg_\mu(\psi)}(e^{\mathrm{wt}(e_i\psi)} + \cdots + e^{\mathrm{wt}(e_i^k\psi)}) \qquad \text{by Lemma 4.4.17 (1)} \\ &= q^{\langle \lambda, \overline{v}(\mu)^{-1}\alpha_i^\vee \rangle} \left(q^{\deg_{s_i\mu}(e_i\psi)}e^{\mathrm{wt}(e_i\psi)} + \cdots + q^{\deg_{s_i\mu}(e_i^k\psi)}e^{\mathrm{wt}(e_i^k\psi)} \right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 4.4.17 (1)} \\ &= q^{\langle \lambda, \overline{v}(\mu)^{-1}\alpha_i^\vee \rangle}\ \mathrm{ch}_{s_i\mu}(S'_j \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)), \end{aligned}$$

as desired.

(iii) As in the proof of (ii), we compute:

$$\begin{aligned} &\mathrm{ch}_\mu(S'_j \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)) \\ &\quad = q^{-\langle \lambda, -\overline{v}(\mu)^{-1}\alpha_i^\vee \rangle + \deg_\mu(\psi)}(e^{\mathrm{wt}(e_i\psi)} + \cdots + e^{\mathrm{wt}(e_i^k\psi)}) \qquad \text{by Lemma 4.4.17 (1)} \\ &\quad = q^{-\langle \lambda, -\overline{v}(\mu)^{-1}\alpha_i^\vee \rangle + \deg_\mu(\psi)} \left(\left(e^{\mathrm{wt}(\psi)} + e^{\mathrm{wt}(e_i\psi)} + \cdots + e^{\mathrm{wt}(e_i^k\psi)} \right) - e^{\mathrm{wt}(\psi)} \right) \\ &\quad = q^{-\langle \lambda, -\overline{v}(\mu)^{-1}\alpha_i^\vee \rangle + \deg_\mu(\psi)} \left(D_i^\dagger - 1 \right) (e^{\mathrm{wt}(\psi)}) \qquad \text{by Lemma 4.4.3 (2).} \end{aligned}$$

From this, we deduce that

$$\begin{aligned} D_i^\dagger \ \mathrm{ch}_\mu(S'_j \cap \mathrm{QLS}^{s_i\mu,\infty}(\lambda)) &= q^{-\langle \lambda, -\overline{v}(\mu)^{-1}\alpha_i^\vee \rangle + \deg_\mu(\psi)} \left((D_i^\dagger)^2 - D_i^\dagger \right) e^{\mathrm{wt}(\psi)} \\ &= 0 \qquad \text{by Lemma 4.4.3 (3),} \end{aligned}$$

as desired. ■

(4) Let $1 \le j \le u$, and write $S_j = \{\psi, e_i\psi, \dots, e_i^k\psi\}$ for some $k \ge 0$ (depending on j), where ψ is the lowest element of the i-string S_j. From Lemma 4.4.17 (2) and the equality $\mathrm{wt}(e_i^k\psi) = s_i\mathrm{wt}(\psi)$, we deduce that

$$\begin{aligned}\mathrm{ch}_\mu S_j = q^{\deg_\mu(\psi)}(e^{\mathrm{wt}(\psi)} + \cdots + e^{\mathrm{wt}(e_i^k\psi)}) &= q^{\deg_\mu(\psi)}(e^{\mathrm{wt}(\psi)} + \cdots + e^{s_i\mathrm{wt}(\psi)}) \\ &= D_i^\dagger q^{\deg_\mu(\psi)} e^{\mathrm{wt}(\psi)}.\end{aligned}$$

From this, we see that

$$D_i^\dagger \ \mathrm{ch}_\mu S_j = (D_i^\dagger)^2 \ \mathrm{ch}_\mu\{\psi\} = D_i^\dagger \ \mathrm{ch}_\mu\{\psi\} = \mathrm{ch}_\mu S_j \qquad \text{by Lemma 4.4.3 (3)},$$

which proves part (4).

□

Proof of Theorem 4.4.1 (b). Applying $D_i^\dagger$ to both sides of equation (4.4.4) in Lemma 4.4.16, we deduce that

$$\begin{aligned}&D_i^\dagger \ \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) \\ &= D_i^\dagger \ \mathrm{ch}_\mu \mathrm{QLS}^{s_i\mu,\infty}(\lambda) + D_i^\dagger \ \mathrm{ch}_\mu(f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) + D_i^\dagger \ \mathrm{ch}_\mu(S_1 \sqcup \cdots \sqcup S_t) \qquad (4.4.5) \\ &= \mathrm{ch}_\mu(f_i^{\max}\mathrm{QLS}^{s_i\mu,\infty}(\lambda)) + \mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda) + \mathrm{ch}_\mu(S_1 \sqcup \cdots \sqcup S_t) \\ &\qquad \text{by Lemma 4.4.18 (1), (3), (4).}\end{aligned}$$

By subtracting equation (4.4.4) in Lemma 4.4.16 from equation (4.4.5), we see that

$$\begin{aligned}T_i^\dagger \ \mathrm{ch}_\mu \mathrm{QLS}^{\mu,\infty}(\lambda) &= \mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda) - \mathrm{ch}_\mu \mathrm{QLS}^{s_i\mu,\infty}(\lambda) \\ &= (1 - q^{\langle\lambda, \overline{v}(\mu)^{-1}\alpha_i^\vee\rangle}) \,\mathrm{ch}_{s_i\mu} \mathrm{QLS}^{s_i\mu,\infty}(\lambda) \quad \text{by Lemma 4.4.18 (2),}\end{aligned}$$

which proves Theorem 4.4.1 (b). □

Example 4.4.19. Let $\mathfrak{g}$ be of type A_2, and let $\lambda = \varpi_1 + \varpi_2$, $w = w_\circ$, and $i = 2$; by Example 4.3.9, we have $s_2 w = s_1 s_2 \in wW_{I_w}$. Let ψ_v, $v \in W$, and ψ_k, $k = 1, 2, 3$, be as in Example 4.4.6. Recall from Example 4.2.9 that

$$\begin{aligned}\mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda) &= \mathrm{QLS}(\lambda) = \{\psi_v \mid v \in W\} \sqcup \{\psi_k \mid k = 1, 2, 3\}, \\ \mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) &= \{\psi_{s_1s_2}, \psi_{s_1}, \psi_1\}.\end{aligned}$$

Since $e_2\psi_{w_\circ} = \psi_{s_1s_2}$, $e_2\psi_{s_2s_1} = \psi_1$, $e_2^2\psi_{s_2s_1} = \psi_{s_1}$, $e_2\psi_{s_2} = \psi_e$, and $e_2\psi_{s_1s_2} = e_2\psi_2 = e_2\psi_{s_1} = e_2\psi_e = e_2\psi_3 = \mathbf{0}$ by Example 4.4.6, we have

$$f_2^{\max}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) = \{\psi_{w_\circ}, \psi_{s_2s_1}\}.$$

Hence we see that

$$\mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda) = \mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) \sqcup f_2^{\max}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) \sqcup \{\psi_{s_2}, \psi_e\} \sqcup \{\psi_2\} \sqcup \{\psi_3\};$$

remark that $\{\psi_{s_2}, \psi_e\}$, $\{\psi_2\}$, and $\{\psi_3\}$ are 2-strings. We set

$$S_1 := \{\psi_{w_\circ}, \psi_{s_1s_2}\},\ S_2 := \{\psi_{s_2s_1}, \psi_1, \psi_{s_1}\},\ S_3 := \{\psi_{s_2}, \psi_e\},\ S_4 := \{\psi_2\},\ S_5 := \{\psi_3\}.$$

Then we have

$$\begin{aligned}\mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda) &= S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4 \sqcup S_5,\\ \mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) \sqcup f_2^{\max}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) &= S_1 \sqcup S_2.\end{aligned}$$

In addition, by Example 4.2.9, we have

$$\begin{aligned}\deg_{w_\circ\lambda}(\psi_{w_\circ}) &= \deg_{s_1s_2\lambda}(\psi_{s_1s_2}) = \deg_{w_\circ\lambda}(\psi_{s_1s_2}) + 1,\\ \deg_{w_\circ\lambda}(\psi_{s_2s_1}) &= \deg_{s_1s_2\lambda}(\psi_1) = \deg_{s_1s_2\lambda}(\psi_{s_1})\\ &= \deg_{w_\circ\lambda}(\psi_1) + 1 = \deg_{w_\circ\lambda}(\psi_{s_1}) + 1,\\ \deg_{w_\circ\lambda}(\psi_{s_2}) &= \deg_{w_\circ\lambda}(\psi_e);\end{aligned} \tag{4.4.6}$$

note that $\langle\lambda, -w_\circ^{-1}\alpha_2^\vee\rangle = 1$. Therefore, we compute:

$$\begin{aligned}D_2^\dagger \operatorname{ch}_{w_\circ\lambda} S_1 &= D_2^\dagger\left(q^{\deg_{w_\circ\lambda}(\psi_{w_\circ})}e^{\mathrm{wt}(\psi_{w_\circ})} + q^{\deg_{w_\circ\lambda}(\psi_{s_1s_2})}e^{\mathrm{wt}(\psi_{s_1s_2})}\right)\\ &= D_2^\dagger\left(q^{\deg_{w_\circ\lambda}(\psi_{w_\circ})}e^{w_\circ\lambda} + q^{\deg_{w_\circ\lambda}(\psi_{s_1s_2})}e^{s_1s_2\lambda}\right)\\ &= q^{\deg_{w_\circ\lambda}(\psi_{w_\circ})}D_2^\dagger e^{w_\circ\lambda} && \text{since } D_2^\dagger e^{s_1s_2\lambda} = 0\\ &= q^{\deg_{w_\circ\lambda}(\psi_{w_\circ})}\left(e^{w_\circ\lambda} + e^{s_1s_2\lambda}\right) && \text{by Lemma 4.4.3 (2)}\\ &= q^{\deg_{w_\circ\lambda}(\psi_{w_\circ})}e^{w_\circ\lambda} + q^{\deg_{s_1s_2\lambda}(\psi_{s_1s_2})}e^{s_1s_2\lambda} && \text{by (4.4.6)}\\ &= q^{\deg_{w_\circ\lambda}(\psi_{w_\circ})}e^{\mathrm{wt}(\psi_{w_\circ})} + q^{\deg_{s_1s_2\lambda}(\psi_{s_1s_2})}e^{\mathrm{wt}(\psi_{s_1s_2})}\\ &= \operatorname{ch}_{w_\circ\lambda}(S_1 \cap f_2^{\max}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda)) + \operatorname{ch}_{s_1s_2\lambda}(S_1 \cap \mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda)),\end{aligned}$$

where, for the second and sixth equalities, we have used equalities $\mathrm{wt}(\psi_{w_\circ}) = w_\circ\lambda$ and $\mathrm{wt}(\psi_{s_1s_2}) = s_1s_2\lambda$ in Example 4.2.9. Similarly, we deduce that

$$D_2^\dagger \operatorname{ch}_{w_\circ\lambda} S_2 = \operatorname{ch}_{w_\circ\lambda}(S_2 \cap f_2^{\max}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda)) + \operatorname{ch}_{s_1s_2\lambda}(S_2 \cap \mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda)).$$

Also, it is easy to check that

$$D_2^\dagger \operatorname{ch}_{w_\circ\lambda} S_k = \operatorname{ch}_{w_\circ\lambda} S_k \text{ for } k = 3, 4, 5;$$

note that we use (4.4.6) for $k = 3$. Thus, we obtain

$$\begin{aligned}&D_2^\dagger \operatorname{ch}_{w_\circ\lambda} \mathrm{QLS}^{w_\circ\lambda,\infty}(\lambda)\\ &= \operatorname{ch}_{w_\circ\lambda}(f_2^{\max}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda)) + \operatorname{ch}_{s_1s_2\lambda}\mathrm{QLS}^{s_1s_2\lambda,\infty}(\lambda) + \operatorname{ch}_{w_\circ\lambda}(S_3 \sqcup S_4 \sqcup S_5).\end{aligned}$$

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