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REPRESENTATION THEORY OF QUANTUM SYMMETRIC PAIRS OF TYPE AIII/AIV

HIDEYA WATANABE

Abstract. A quantum symmetric pair is a pair of a quantum group and its coideal subalgebra; the classical limit of the coideal subalgebra becomes the fixed-point subalgebra under an involution on a semisimple Lie algebra. Quantum symmetric pairs have played important roles in many branches of mathematics such as the representation theory of Lie superalgebras, low-dimensional topology, and integrable systems. However, the classification of the irreducible modules, which is one of the most basic problems in representation theory, has not been settled yet. As a first step toward this problem, in this thesis we study the representation theory of a special class of quantum symmetric pairs. The main tools are highest weight theory, crystal basis theory, and the representation theory of Hecke algebras of type B.

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CONTENTS

Introduction

Quantum groups. The quantum groups (also called quantized enveloping algebras) $U_q(\mathfrak{g})$, which are certain unital associative algebras over $\mathbb{Q}(q)$, were introduced in the mid of 1980's by Drinfel'd [Dr85] and Jimbo [J85] independently, with a complex semisimple Lie algebra $\mathfrak{g}$ and a parameter q as inputs. From a quantum group, one can construct the universal R-matrix, which gives a solution to the quantum Yang-Baxter equation. Hence the quantum groups are important in mathematical physics. The universal R-matrix has also an application to knot theory ([RT90], see also [Jo85]).

In representation theory, quantum groups have played a central role since their birth. The classification of finite-dimensional irreducible modules over a quantum group is done in a way similar to the classical case (i.e., complex semisimple Lie algebra case). Namely, the finite-dimensional irreducible $U_q(\mathfrak{g})$-modules are classified by the dominant integral weights. However, there are interesting concepts which are peculiar to the quantum case. Here, we mention Kashiwara's crystal bases, Lusztig's canonical bases, and Jimbo's q-Schur duality.

Crystal bases. Crystal bases were introduced in [K90], [K91] as "local" bases of modules over a quantum group (including the negative part $U_q(\mathfrak{g})^-$ of a quantum group) at $q = 0$. Crystal bases have much information about modules. For example, each finite-dimensional module over $U_q(\mathfrak{g})$ is uniquely determined by its crystal basis.

Crystal bases extract some combinatorial aspects of module structures. By axiomatizing such properties, the notion of (abstract) crystals was introduced in [K93b]. Roughly speaking, a crystal is a finite set with certain structure maps. There is a formula, called the tensor product rule, which makes a pair of crystals into a new crystal. Thus, the category of crystals form a tensor category. Note that each crystal basis of a module is a crystal, and the tensor product of two modules has a crystal basis whose crystal structure is given by the tensor product rule.

One of the advantages of considering abstract crystals is that we can reduce many problems in representation theory to those in combinatorics. For example, when $\mathfrak{g} = \mathfrak{sl}_n$, the special linear Lie algebra over $\mathbb{C}$, each dominant integral weight is identified with a partition. Hence the finite-dimensional irreducible modules over $U_q(\mathfrak{sl}_n)$ are classified by partitions. Also, for each partition λ, the set $\mathrm{SST}(\lambda)$ of semistandard Young tableaux of shape lm is equipped with a crystal structure. Then, the crystal basis of the finite-dimensional irreducible module corresponding to λ is isomorphic to $\mathrm{SST}(\lambda)$. Moreover, the crystal structure of $\mathrm{SST}(\lambda)$ is characterized by a special element, called the highest weight vector with highest weight λ. Hence the problem of decomposing a given finite-dimensional modules into irreducible modules is reduced to the problem of finding the highest weight vectors with various highest weights in the crystal basis of the given module. This is a purely combinatorial problem. Similar combinatorial realizations of crystal bases are known in the other classical types, i.e., for $\mathfrak{g} = \mathfrak{so}_n$, the special orthogonal Lie algebra, and $\mathfrak{sp}_{2m}$, the symplectic Lie algebra ([KN94]).

Now, recall that the crystal bases are "local" bases at $q = 0$, namely, they are not actual bases. In order to obtain genuine bases, we need to "globalize" the crystal bases. This leads to the notion of global crystal bases ([K91], [K93a]). A global crystal basis is a basis whose limit at $q = 0$ becomes a crystal basis. It is not always true that there is a global crystal basis which specializes to a given crystal basis. Hence it is an important problem to clarify when a given crystal basis can be globalized. For example, it is known that the crystal bases of $U_q(\mathfrak{g})^-$ and each finite-dimensional irreducible module have a unique global crystal basis. Once we obtain a global crystal basis of a module, we get more information about the module structure.

Canonical bases. From a geometric point of view, Lusztig constructed canonical bases for quantum groups of type ADE in [L90a], and of general type in [L91]. In order to formulate canonical bases, we need the bar-involution, which is a $\mathbb{Q}$-linear involution on $U_q(\mathfrak{g})$ sending q to q^{-1}. Then, the canonical basis of $U_q(\mathfrak{g})^-$ is a basis whose elements are all invariant under the bar-involution, and which specializes to a "local" basis of $U_q(\mathfrak{g})^-$ as q goes to 0.

Now, let λ be a dominant integral weight, and consider the corresponding finite-dimensional irreducible $U_q(\mathfrak{g})$-module $L(\lambda)$. Let $v_\lambda \in L(\lambda)$ be the highest weight vector. Then, the canonical basis of $L(\lambda)$ is defined to be the image of the canonical basis of $U_q(\mathfrak{g})^-$ under the surjection $U_q(\mathfrak{g})^- \to L(\lambda)$; $x \mapsto xv_\lambda$. Since $L(\lambda)$ is generated by v_λ over $U_q(\mathfrak{g})^-$, it is equipped with a $\mathbb{Q}$-linear involution that fixes v_λ, and is compatible with the bar-involution of $U_q(\mathfrak{g})$. Then, by its construction, the canonical basis of $L(\lambda)$ is invariant under this involution.

In [L90b] for ADE type, and in [GL93] for general type, it is proved that Lusztig's canonical bases coincide with Kashiwara's global crystal bases for both irreducible modules and $U_q(\mathfrak{g})^-$.

q-Schur duality. The quantum group $U_q(\mathfrak{sl}_n)$ of type A is deeply concerned with the Hecke algebras $\mathcal{H}(\mathfrak{S}_d)$ of type A, where $\mathfrak{S}_d$ denotes the d-th symmetric group. Jimbo [J86] discovered that the universal R-matrix gives rise to the $\mathcal{H}(\mathfrak{S}_d)$-module structure on the tensor product $\mathbf{V}^{\otimes d}$ of the natural representation of $U_q(\mathfrak{sl}_n)$, and furthermore, this action and the usual action of $U_q(\mathfrak{sl}_n)$ on $\mathbf{V}^{\otimes d}$ form a double centralizer. This fact induces an exact functor from the category of finite-dimensional $U_q(\mathfrak{sl}_n)$-modules to the category of finite-dimensional $\mathcal{H}(\mathfrak{S}_d)$-modules, and also one in the opposite direction, both of which send irreducible modules to irreducibles or zero.

The tensor power $\mathbf{V}^{\otimes d}$ has two distinguishing bases; one is the canonical basis (equivalently, global crystal basis) as a $U_q(\mathfrak{sl}_n)$-module, and the other is the Kazhdan-Lusztig basis as an $\mathcal{H}(\mathfrak{S}_d)$-module. The Kazhdan-Lusztig basis is a basis invariant under a $\mathbb{Q}$-linear involution of $\mathbf{V}^{\otimes d}$ which is compatible with the bar-involution of $\mathcal{H}(\mathfrak{S}_d)$. In [FKK98], it is proved that these two bases coincide. This result was then used to formulate the Kazhdan-Lusztig theory without using $\mathcal{H}(\mathfrak{S}_d)$ directly.

Quantum symmetric pairs. Recently, the quantum symmetric pairs have been getting attention. A (classical) symmetric pair is a pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ of a complex semisimple Lie algebra $\mathfrak{g}$ and the fixed-point subalgebra $\mathfrak{g}^\theta := \{x \in \mathfrak{g} \mid \theta(x) = x\}$ for some Lie algebra automorphism θ of $\mathfrak{g}$ such that $\theta^2 = \mathrm{id}_{\mathfrak{g}}$. Quantum symmetric pairs were first studied in [NS95], [N96], [NDS97] as pairs $(\mathbf{U}, \mathbf{U}^\imath)$ of a quantum group $\mathbf{U} = U_q(\mathfrak{g})$ and its coideal subalgebra $\mathbf{U}^\imath$ whose classical limit becomes the universal enveloping algebra $U(\mathfrak{g}^\theta)$. Their aim is to study Macdonald polynomials as quantum zonal spherical functions. A uniform construction of the quantum symmetric pairs was achieved for finite type in [Le99], and for any Kac-Moody type in [Ko14].

In an influential paper [BW13], Bao and Wang establish analogs of parts of the theory of quantum groups in the setting of quantum symmetric pairs of type AIII/AIV. Namely, they constructed the universal K-matrix (the counterpart of the universal R-matrix), the bar-involution, and the $\imath$-canonical bases (the counterpart of the canonical bases) of $\mathbf{U}$-modules. These were used to complete the Kazhdan-Lusztig theory for the ortho-symplectic Lie superalgebras $\mathfrak{osp}(m|2n)$, namely, to solve the long-standing problem of determining the irreducible characters in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of $\mathfrak{osp}(m|2n)$ (see also [B17] and [BWW18]).

Since then, it has been discovered that many properties of the quantum groups can be generalized to the quantum symmetric pairs. For example, a systematic construction of the universal K-matrix ([BaKo15b], [BW18a], [BW18b]), bar-involution ([BaKo15a], [BW18a], [BW18b]), and the $\imath$-canonical bases of $\mathbf{U}$-modules and $\mathbf{U}^\imath$ ([BW18a], [BWW18], [BW18b]) have been achieved. Also, for a particular class of the quantum symmetric pairs, a geometric construction of $\mathbf{U}^\imath$ is given in [BKLW18] (see also [BLM90]).

However, we know little about the representation theory of $\mathbf{U}^\imath$. The most important problems such as

- the classification of irreducible modules,
- the complete reducibility of modules,
- the determination of the structure of irreducible modules,

have not been settled yet (see e.g., [GK91] and [AKR17] for partial results). This thesis is a first step toward these problems.

Results. The main results in this thesis are the multiparameter version of q-Schur duality in type B and the representation theory of quantum symmetric pairs of type AIII/AIV. In both results, we treat one of the particular quantum symmetric pairs $(U_q(\mathfrak{sl}_{2r+1}), \mathbf{U}^\jmath)$ and $(U_q(\mathfrak{sl}_{2r+2}), \mathbf{U}^\imath)$. In order to avoid repetition, we only explain results involving $\mathbf{U}^\jmath$ here.

q-Schur duality in type BCD. As we have mentioned above, Bao and Wang [BW13] discovered the type B/C analog of Jimbo's q-Schur duality. In Part 2, we upgrade this duality to the multiparameter setting. Namely, we introduce a new parameter p, which may or may not be independent of q. When $p = q$, our result recovers the one in [BW13], while the $p = 1$ case coincides with the type D analog of the q-Schur duality, which is first discovered in [B17].

Classification of finite-dimensional irreducible modules. In Part 3, we study a certain module category $\mathcal{O}^\jmath_{\mathrm{int}}$, which contains all of finite-dimensional $\mathbf{U}^\jmath$-modules. The first step is the triangular decomposition of $\mathbf{U}^\jmath$, which enables us to establish the highest weight theory. As is well known, the highest weight theory lies at the heart of the classification of irreducible modules over semisimple Lie algebras and quantum groups. Hence it is pleasing that this theory is also applicable to the setting of quantum symmetric pairs. Next, we develop the crystal basis theory for modules in $\mathcal{O}^\jmath_{\mathrm{int}}$. Namely, we introduce the notion of $\jmath$-crystal bases for $\mathbf{U}^\jmath$-modules in $\mathcal{O}^\jmath_{\mathrm{int}}$. By analyzing a $\jmath$-crystal basis of $\mathbf{V}^{\otimes d}$ for various $d \in \mathbb{Z}_{>0}$, we complete the classification of irreducible $\mathbf{U}^\jmath$-modules in $\mathcal{O}^\jmath_{\mathrm{int}}$. As a result, we see that every irreducible module in $\mathcal{O}^\jmath_{\mathrm{int}}$ is finite-dimensional, and that the irreducible modules in $\mathcal{O}^\jmath_{\mathrm{int}}$ are classified by bipartitions (pairs of partitions). In our crystal basis theory, the assumption that p is algebraically independent of q is essential. I hope that we can develop similar theory which is applicable to any p, and to other quantum symmetric pairs in a future work.

New crystals. Like the ordinary crystal basis theory, our new crystal basis theory has many combinatorical aspects. First of all, we introduce the notion of (abstract) $\jmath$-crystals. Then, for each bipartition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^-; \boldsymbol{\lambda}^+)$, we equip the set $\mathrm{SST}(\boldsymbol{\lambda}) = \mathrm{SST}(\boldsymbol{\lambda}^-) \times \mathrm{SST}(\boldsymbol{\lambda}^+)$ of semistandard bitableaux (pairs of semistandard tableaux) of shape $\boldsymbol{\lambda}$ with a $\jmath$-crystal structure. This is done in a purely combinatorial manner. We prove that the $\jmath$-crystal basis of the irreducible module $L(\boldsymbol{\lambda})$ corresponding to $\boldsymbol{\lambda}$ is isomorphic to $\mathrm{SST}(\boldsymbol{\lambda})$ as an abstract $\jmath$-crystal. As a byproduct, we obtain the equality $\dim L(\boldsymbol{\lambda}) = |\,\mathrm{SST}(\boldsymbol{\lambda})|$.

Also, we discover a combinatorial formula (the tensor product rule for $\jmath$-crystals) which makes given two $\jmath$-crystals into a new $\jmath$-crystal. As in the ordinary crystal case, this tells us the $\jmath$-crystal structure of the $\jmath$-crystal basis of the tensor product $M \otimes N$ of a $\mathbf{U}^\jmath$-module M with a $\jmath$-crystal basis, and a $U_q(\mathfrak{sl}_{2r+1})$-module N with an ordinary crystal basis.

Finally, we mention that our new crystal basis theory is compatible with the ordinary crystal basis theory. Let M be a $U_q(\mathfrak{sl}_{2r+1})$-module with a crystal basis. Then, the crystal basis is also the $\jmath$-crystal basis of M, regarded as a $\mathbf{U}^\jmath$-module by restriction. Moreover, the $\jmath$-crystal structure of the $\jmath$-crystal basis is completely determined by its original crystal structure. This observation enables us to solve the problem of how a given irreducible $U_q(\mathfrak{sl}_{2r+1})$-module decomposes into irreducible $\mathbf{U}^\jmath$-modules (branching rule) in terms of semistandard tableaux and bitableaux.

Organization. This thesis is organized as follows.

Part 1 is a preliminary part. In Section 1, we review basic concepts concerning Kac-Moody algebras, Weyl groups, and crystals. Also, we prepare some combinatorial tools which are frequently used throughout this thesis. In Section 2, we define (classical) symmetric pairs, and give an example of type AIII/AIV. Sections 3 to 4 are devoted to quantizing objects which have been introduced so far. Namely, we formulate quantum groups, braid group actions on them, and quantum symmetric pairs. Also, basic results about R-matrices, crystal bases, canonical bases, K-matrices and $\imath$-canonical bases are given. A construction of Poincaré-Birkhoff-Witt-type basis for quantum symmetric pairs is new. In Section 5, we formulate the notion of Kazhdan-Lusztig bases. These bases are then used to construct the left cell representations. The aim of Section 6 is to study the connection between the representation theory of Hecke algebras and that of generalized q-Schur algebras.

Part 2 is the first half of the main body of this thesis. It contains the q-Schur duality in type BCD for $\mathbf{U}^\jmath$ in Section 7, and the one for $\mathbf{U}^\imath$ in Section 8. The results in this part were obtained in [BWW16].

Parts 3 and 4 are the second half of the main body of this thesis. They treat the representation theory of $(U_q(\mathfrak{sl}_{2r+1}), \mathbf{U}^\jmath)$ and $(U_q(\mathfrak{sl}_{2r+2}), \mathbf{U}^\imath)$. We begin with applying the construction of the PBW-type basis to $\mathbf{U}^\jmath$ in Section 9. This enables us to establish highest weight theory. Section 10 is devoted to the detailed study of the representation theory of $(U_q(\mathfrak{sl}_3), \mathbf{U}^\jmath)$. This is the smallest $\mathbf{U}^\jmath$, and hence the results in this section become fundamental tools in the study for a bigger $\mathbf{U}^\jmath$. In Section 11, we prove that each module in $\mathcal{O}^\jmath_{\mathrm{int}}$ is completely reducible, and then start the classification of irreducible modules in $\mathcal{O}^\jmath_{\mathrm{int}}$. In Section 12, we introduce the notion of quasi-$\jmath$-crystal bases, which is a weak version of $\jmath$-crystal bases. The main result here is the tensor product rule for quasi-$\jmath$-crystals. By studying a particular quasi-$\jmath$-crystal basis of $\mathbf{V}^{\otimes d}$, we complete the classification of irreducible modules in $\mathcal{O}^\jmath_{\mathrm{int}}$ in Section 13. Also, we define the notion of $\jmath$-crystal bases, and prove that each irreducible module has a unique $\jmath$-crystal basis whose $\jmath$-crystal structure is described in terms of semistandard bitableaux. In Section 14, we prove the existence and uniqueness of global $\jmath$-crystal basis of an irreducible module in $\mathcal{O}^\jmath_{\mathrm{int}}$. After studying basic properties of global $\jmath$-crystal bases in Section 15, we prove in Section 16 that each global crystal basis of a module in $\mathcal{O}^\jmath_{\mathrm{int}}$ is compatible with a particular filtration. Sections 17 to 21 are the counterparts of Sections 9 to 14 in the $(U_q(\mathfrak{sl}_{2r+2}), \mathbf{U}^\imath)$ case. As the results in Sections 15 and 16 needs so many computations, we skip the counterparts for $\mathbf{U}^\imath$. Instead, we would like to develop the general theory including $\mathbf{U}^\jmath$ and $\mathbf{U}^\imath$ in a future work.

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Part 1. Preliminaries

In this part, we prepare notations and basic notions needed in later parts. The materials in this part, except in Section 4, are elementary, and can be found in standard textbooks.

NOTATION

- $\underline{n} := n - \frac{1}{2}$ for $n \in \mathbb{Z}$. Note that $\underline{-n} \neq -\underline{n}$.
- $\mathrm{Mat}_n(R)$: the ring of $n \times n$ matrices with coefficients in a ring R.
- $\mathfrak{gl}_n = \mathrm{Mat}_n(\mathbb{C})$: the general linear Lie algebra over $\mathbb{C}$.
- $\mathfrak{sl}_n = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}$: the special linear Lie algebra over $\mathbb{C}$.
- $\mathfrak{S}_d$: the d-th symmetric group.
- For $l, m \in \mathbb{Z}$ such that $l \leq m$, set $[l, m] := \{l, l+1, \dots, m\}$.
- For $\underline{l}, \underline{m} \in \mathbb{Z} + \frac{1}{2}$ such that $\underline{l} \leq \underline{m}$, set $[\underline{l}, \underline{m}] := \{\underline{l}, \underline{l} + 1, \dots, \underline{m}\}$.
- Every subsets of $\mathbb{Z}$ or of $\mathbb{Z} + \frac{1}{2}$ are regarded as a totally ordered set with the usual ordering.
- L^{op}: the opposite totally ordered set of a totally ordered set L.
- A^{op}: the opposite algebra of an algebra A.

1. KAC-MOODY ALGEBRAS

A Kac-Moody algebra is a complex Lie algebra constructed from an integer matrix and some ingredients. In this section, we review basic results and related concepts concerning representation theory of Kac-Moody algebras. We mainly follow notations in [HK02]. In Subsection 1.1, we give a definition of Kac-Moody algebras, and the classification of integrable irreducible modules. In Subsection 1.2, we list basic concepts about Weyl groups such as length function, reduced expressions, and Bruhat order. In Subsection 1.3, we introduce the notion of crystals, and give the tensor product rule. Subsection 1.4 is devoted to formulating notations regarding partitions. After reviewing Schensted's bumping algorithm and Robinson-Schensted-Knuth correspondence in Subsection 1.5, we equip some combinatorial objects with crystal structures in Subsection 1.6. Also, we show that RSK correspondence is in fact a morphism of crystals.

1.1. Kac-Moody algebras.

Given a complex semisimple Lie algebra, one gets an integer matrix (Cartan matrix) by extracting partial information about structure of the Lie algebra. Axiomatizing important properties of such matrices leads to the definition of generalized Cartan matrix.

Definition 1.1.1. Let $I = [1, n]$ for some $n \in \mathbb{Z}_{>0}$.

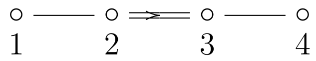
1. A generalized Cartan matrix is a matrix $A = (a_{i,j})_{i,j \in I} \in \mathrm{Mat}_n(\mathbb{Z})$ satisfying the following:
   (a) $a_{i,i} = 2$ for all $i \in I$.
   (b) $a_{i,j} \leq 0$ for all $i \neq j \in I$.
   (c) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.
2. A generalized Cartan matrix A is said to be symmetrizable if there exists a diagonal matrix $D = \mathrm{diag}(d_i)_{i \in I}$ such that $d_i \in \mathbb{Z}_{>0}$ for all $i \in I$, and that DA is a symmetric matrix.
3. A generalized Cartan matix A is said to be of finite type (or, a Cartan matrix) if it is positive definite.
4. A generalized Cartan matrix A is said to be indecomposable if there exist no nontrivial partitions $I = I_1 \sqcup I_2$ such that $a_{i,j} = 0$ for all $i \in I_1$, $j \in I_2$.

Definition 1.1.2. Let A be a generalized Cartan matrix. The Dynkin diagram of A is the (partially) directed graph whose vertices are the elements of I, and $i \neq j \in I$ are joined by $a_{i,j}a_{j,i}$ edges. The edges between $i \neq j$ are directed from i to j if $|a_{i,j}| < |a_{j,i}|$.

Example 1.1.3. Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Then, its Dynkin diagram is



The following is the classification theorem for indecomposable Catan matrices. Although a similar result for generalized Cartan matrices of affine type (i.e., positive semidefinite of corank 1) is known, we do not need here.

Theorem 1.1.4. *Let A be an indecomposable Cartan matrix. Then, its Dynkin diagram is isomorphic to one of the diagrams listed in Figure 1 in page 111.*

Throughout this section, we fix a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. Suppose that we are given the following objects;

- free $\mathbb{Z}$-modules P, $P^\vee$ of finite rank,
- linearly independent elements $\Pi = \{\alpha_i \mid i \in I\} \subset P$, $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$,
- a perfect pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$ such that $\langle h_i, \alpha_j \rangle = a_{i,j}$.

Such a quintuple $(P, P^\vee, \Pi, \Pi^\vee, \langle \cdot, \cdot \rangle)$ is called a Cartan datum associated with A. Also, the elements of Π and $\Pi^\vee$ are called the simple roots and simple coroots, respectively. It is well-known that one can construct a (possibly infinite-dimensional) complex Lie algebra, called a Kac-Moody algebra, from a Cartan datum.

Definition 1.1.5. The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with a Cartan datum $(P, P^\vee, \Pi, \Pi^\vee, \langle \cdot, \cdot \rangle)$ is the complex Lie algebra generated by symbols e_i, f_i, $i \in I$, and $h \in P^\vee$ subject to the following relations; for $i, j \in I$ and $h, h' \in P^\vee$.

$$\begin{aligned} &[h, h'] = 0, \\ &[h, e_i] = \langle h, \alpha_i \rangle e_i, \\ &[h, f_i] = -\langle h, \alpha_i \rangle f_i, \\ &[e_i, f_j] = \delta_{i,j} h_i, \\ &(\operatorname{ad} e_i)^{1-a_{i,j}}(e_j) = 0, \quad i \neq j, \\ &(\operatorname{ad} f_i)^{1-a_{i,j}}(f_j) = 0, \quad i \neq j. \end{aligned}$$

The following is one of the most basic result in Lie theory.

Theorem 1.1.6. *Let A be a symmetrizable generalized Cartan matrix, and $(P, P^\vee, \Pi, \Pi^\vee, \langle \cdot, \cdot \rangle)$ a Cartan datum. Then, $\mathfrak{g}(A)$ is simple (i.e., $\mathfrak{g}(A)$ has no nontrivial ideal) if and only if A is an indecomposable Cartan matrix and $\operatorname{rk} P = \operatorname{rk} P^\vee = n$. In particular, the complex simple Lie algebras are classified by the Dynkin diagrams listed in Figure 1 in page 111.*

Example 1.1.7. Let us give two examples of Cartan datum that are frequently used in the sequel.

(1) Let $A = (a_{i,j})_{1 \leq i,j \leq n-1}$ be such that

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

Then, A is an indecomposable Cartan matrix whose Dynkin diagram is of type A_{n-1} (see Figure 1 in page 111). Let $\epsilon_1, \dots, \epsilon_n \in \mathbb{R}^n$ be the standard basis of the Euclidean space $\mathbb{R}^n$. Set

$$P := \mathbb{Z}^n/\mathbb{Z}(\epsilon_1 + \cdots + \epsilon_n), \quad P^\vee := \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 + \cdots + x_n = 0\},$$

and

$$\alpha_i := \epsilon_i - \epsilon_{i+1} + \mathbb{Z}(\epsilon_1 + \cdots + \epsilon_n) \in P, \quad h_i := \epsilon_i - \epsilon_{i+1} \in P^\vee \quad (i \in [1, n-1]).$$

Let $\langle \cdot, \cdot \rangle$ denote the pairing $P \times P^\vee \to \mathbb{Z}$ induced from the inner product on $\mathbb{R}^n$. Then, $(P, P^\vee, \{\alpha_1, \dots, \alpha_{n-1}\}, \{h_1, \dots, h_{n-1}\}, \langle \cdot, \cdot \rangle)$ is a Cartan datum associated with A. We call this Cartan datum the $\mathfrak{sl}_n$-type. In this case, the associated Kac-Moody algebra is isomorphic to $\mathfrak{sl}_n$.

(2) Let $(P, P^\vee, \{\alpha_1, \dots, \alpha_{m+n-1}\}, \{h_1, \dots, h_{m+n-1}\}, \langle \cdot, \cdot \rangle)$ be the Cartan datum of $\mathfrak{sl}_{m+n}$-type. Then, $(P, P^\vee, \{\alpha_i \mid i \neq m\}, \{h_i \mid i \neq m\}, \langle \cdot, \cdot \rangle)$ is a Cartan datum associated with a Cartan matrix $(a_{i,j})_{i,j \neq m}$. We call this Cartan datum the $\mathfrak{s}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$-type. In this case, the associated Kac-Moody algebra is isomorphic to $\mathfrak{s}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$; the Lie algebra consisting of $X \in \mathrm{Mat}_{m+n}(\mathbb{C})$ such that

$$X = \begin{pmatrix} A & O \\ O & B \end{pmatrix}, \quad A \in \mathrm{Mat}_m(\mathbb{C}), \ B \in \mathrm{Mat}_n \mathbb{C}, \quad \mathrm{tr}(X) = 0.$$

Let $\mathrm{Aut}(\mathfrak{g})$ denote the group of Lie algebra automorphisms on $\mathfrak{g}$. Here, we prepare some important elements of $\mathrm{Aut}(\mathfrak{g})$.

Lemma 1.1.8. *Let $i \in I$.*

(1) *There exists a unique $\omega \in \mathrm{Aut}(\mathfrak{g})$ that sends e_i, f_i, h to $f_i, e_i, -h$ for all $i \in I$, $h \in P^\vee$.*

(2) *The linear endomorphisms $\exp(\mathrm{ad} -e_i)$ and $\exp(\mathrm{ad} f_i)$ are elements of $\mathrm{Aut}(\mathfrak{g})$.*

Set $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} P^\vee \subset \mathfrak{g}$. Then, it is a Cartan subalgebra (i.e., a self-normalizing nilpotent subalgebra) of $\mathfrak{g}$. Since we have a perfect pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$, the dual space $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ is identified with $\mathbb{C} \otimes_{\mathbb{Z}} P$. For $\lambda \in \mathfrak{h}^*$, set

$$\mathfrak{g}_\lambda := \{g \in \mathfrak{g} \mid [h, g] = \lambda(h) g \text{ for all } h \in \mathfrak{h}\}.$$

The elements of the set $\Phi := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ are called the roots of $\mathfrak{g}$. Set $\Phi_+ := \Phi \cap \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, and call it the set of positive roots.

Example 1.1.9. When our Cartan datum is the $\mathfrak{sl}_n$-type, we have

$$\Phi = \{\epsilon_i - \epsilon_j + \mathbb{Z}(\epsilon_1 + \cdots + \epsilon_n) \mid 1 \leq i \neq j \leq n\},$$
$$\Phi_+ = \{\epsilon_i - \epsilon_j + \mathbb{Z}(\epsilon_1 + \cdots + \epsilon_n) \mid 1 \leq i < j \leq n\}.$$

Let $\mathfrak{g}_{\mathbb{Q}}$ denote the Lie subalgebra of $\mathfrak{g}$ over $\mathbb{Q}$ generated by e_i, f_i, and $P^\vee$. Also, set $U_{\mathbb{Q}}(\mathfrak{g})$ to be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{Q}}$, that is the quotient algebra of the tensor algebra of $\mathfrak{g}_{\mathbb{Q}}$ factored by the two-sided ideal generated by $xy - yx - [x, y]$, $x, y \in \mathfrak{g}_{\mathbb{Q}}$.

Now, let us take a look at representation theory. First, set $Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. Let us introduce a partial order on P. For $\lambda, \mu \in P$, we write $\lambda \leq \mu$ if $\mu - \lambda \in Q^+$.

For a $U_{\mathbb{Q}}(\mathfrak{g})$-module M and $\lambda \in \mathfrak{h}^*$, set $M_\lambda := \{m \in M \mid hm = \lambda(h)m \text{ for all } h \in P^\vee\}$, and $\mathrm{Wt}(M) := \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$. Throughout this thesis, a $U_{\mathbb{Q}}(\mathfrak{g})$-module M is always assumed to satisfy the following conditions:

- $M = \bigoplus_{\lambda \in P} M_\lambda$.
- There exist $\lambda_1, \dots, \lambda_l \in P$ such that $\mathrm{Wt}(M) \subset \bigcup_{i=1}^{l} (\lambda_i - Q_+)$.
- f_i acts on M locally nilpotently for all $i \in I$.

A $U_{\mathbb{Q}}(\mathfrak{g})$-module M is said to be a highest weight module with highest weight $\lambda \in P$ if there exists $m \in M$ such that $M = U_{\mathbb{Q}}(\mathfrak{g})m$, $e_i m = 0$ for all $i \in I$, and $hm = \lambda(h)m$ for all $h \in P^\vee$.

Let $P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$. The elements of P_+ are called the dominant integral weights.

Example 1.1.10. When our Cartan datum is the $\mathfrak{sl}_n$-type, we have a bijection (see Subsection 1.4 for the definition of Par_{n-1})

$$P_+ \to \mathrm{Par}_{n-1};\ \lambda \mapsto \left(\sum_{i=1}^{n-1} \langle h_i, \lambda \rangle, \sum_{i=2}^{n-1} \langle h_i, \lambda \rangle, \dots, \langle h_{n-1}, \lambda \rangle \right).$$

The following theorem states that the irreducible $U_{\mathbb{Q}}(\mathfrak{g})$-modules are classified by P_+.

Theorem 1.1.11. *If L is an irreducible $U_{\mathbb{Q}}(\mathfrak{g})$-module, then there exists a unique $\lambda \in P_+$ such that L is a highest weight module with highest weight λ. Conversely, for each $\lambda \in P_+$, there exists a unique (up to isomorphism) irreducible highest weight module $L(\lambda)$ with highest weight λ.*

1.2. **Weyl groups.** Let us introduce the Weyl groups which are important tools in Lie theory. For each $i \in I$, we define a linear transformation s_i on $\mathfrak{h}^*$ by

$$s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The subgroup $W \subset \mathrm{GL}(\mathfrak{h}^*)$ generated by s_i, $i \in I$ is called the Weyl group of $\mathfrak{g}$. W also acts on $\mathfrak{h}$ by

$$s_i(h) = h - \langle h, \alpha_i \rangle h_i, \quad h \in \mathfrak{h}.$$

The action of W on $\mathfrak{h}$ is extended to the whole of $\mathfrak{g}$ by

$$s_i \mapsto \exp(\mathrm{ad}\, f_i) \exp(\mathrm{ad} -e_i) \exp(\mathrm{ad}\, f_i) \in \mathrm{Aut}(\mathfrak{g}), \quad i \in I.$$

For $i \neq j \in I$, let $m_{i,j} \in \mathbb{Z}_{>0} \sqcup \{\infty\}$ denote the order of $s_i s_j$. It is known that

$$m_{i,j} = \begin{cases} 2 & \text{if } a_{i,j}a_{j,i} = 0, \\ 3 & \text{if } a_{i,j}a_{j,i} = 1, \\ 4 & \text{if } a_{i,j}a_{j,i} = 2, \\ 6 & \text{if } a_{i,j}a_{j,i} = 3, \\ \infty & \text{if } a_{i,j}a_{j,i} \geq 4, \end{cases}$$

Then, the defining relations of W are the following:

$$\begin{aligned} & s_i^2 = e, \\ & (s_i s_j)^{m_{i,j}} = e, \quad \text{if } m_{i,j} < \infty. \end{aligned}$$

This implies that W is determined by the Dynkin diagram if the diagram has no edges with multiplicity more than 3.

Since W is generated by $\{s_i\}_{i \in I}$, each $w \in W$ is expressed as $w = s_{i_1} \cdots s_{i_l}$ for some $i_1, \dots, i_l \in I$. When l is minimal among such expressions, it is called the length of w, and is denoted by $\ell(w)$. An expression of the form $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ is called a reduced expression.

Let $T := \{ws_iw^{-1} \mid i \in I,\ w \in W\}$. For $y, w \in W$, we write $y < w$ if there exist $t_1, \dots, t_k \in T$ such that $w = t_k \cdots t_1 y$, and $\ell(t_{i+1} \cdots t_i y) > \ell(t_i \cdots t_1 y)$ for all $i = 0, \dots, k-1$. This defines a partial order on W, called the Bruhat order.

If $|W|$ is finite (equivalently, A is of finite type), then there exists a unique element $w_0 \in W$ whose length is maximum. w_0 is called the longest element of W.

Example 1.2.1. Let $d \in \mathbb{Z}_{>1}$.

(1) The Weyl group of type A_{d-1} (i.e., the Weyl group associated with the Dynkin diagram of type A_{d-1}) is nothing but the d-th symmetric group $\mathfrak{S}_d$. It is realized as the group of permutations of $[1, d]$. Let s_i denote the adjacent transposition $(i, i+1)$. Then, $\mathfrak{S}_d$ is generated by $s_1, \dots, s_{d-1}$, and the defining relations are the following:

$$\begin{aligned} &s_i^2 = 1, \quad 1 \leq i \leq d-1, \\ &s_is_j = s_js_i, \quad |i-j| > 2, \\ &s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad 1 \leq i < d-1. \end{aligned}$$

(2) The Weyl group of type B_d is the group W_d of permutations w of $[-d, d]$ such that $w(-i) = -w(i)$ for all $i = 1, \dots, d$. Clearly, there exists an injective homomorphism $\mathfrak{S}_d \to W_d$ of groups which sends s_i to $(-(i+1), -i)(i, i+1)$ for all $i = 1, \dots, d-1$. We identify $\mathfrak{S}_d$ with its image under this injection. Set $s_0 := (-1, 1)$. Then, W_d is generated by $s_0, s_1, \dots, s_{d-1}$, and the defining relations are the following:

$$\begin{aligned} &s_i^2 = 1, \quad 0 \leq i \leq d-1, \\ &s_is_j = s_js_i, \quad |i-j| > 2, \\ &s_is_js_i = s_js_is_j, \quad 1 \leq i < d-1, \\ &s_0s_1s_0s_1 = s_1s_0s_1s_0. \end{aligned}$$

1.3. **Crystals.** Let A be a symmetrizable generalized Cartan matrix, and $(P, P^\vee, \Pi, \Pi^\vee, \langle\cdot,\cdot\rangle)$ a Cartan datum. Here, we give combinatorial objects, which will be turned out to be closely connected to the representation theory of Kac-Moody algebras or quantum groups.

Definition 1.3.1 (cf. [K93b, 1.2]). (1) A crystal of type $(P, P^\vee, \Pi, \Pi^\vee, \langle\cdot,\cdot\rangle)$ is a set $\mathcal{B}$ with maps $\widetilde{E}_i, \widetilde{F}_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$, $i \in I$, where 0 is a formal symbol and a map $\mathrm{wt} : \mathcal{B} \to P$ satisfying the following:

- (a) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$, where $\varepsilon_i(b) := \max\{k \mid \widetilde{E}_i^k b \neq 0\}$, and $\varphi_i(b) := \max\{k \mid \widetilde{F}_i^k b \neq 0\}$.
- (b) If $b, \widetilde{E}_i b \in \mathcal{B}$, then $\mathrm{wt}(\widetilde{E}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\widetilde{E}_i b) = \varepsilon_i(b) - 1$, and $\varphi_i(\widetilde{E}_i b) = \varphi_i(b) + 1$.
- (c) If $b, \widetilde{F}_i b \in \mathcal{B}$, then $\mathrm{wt}(\widetilde{F}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\widetilde{F}_i b) = \varepsilon_i(b) + 1$, and $\varphi_i(\widetilde{F}_i b) = \varphi_i(b) - 1$.
- (d) For $b, b' \in \mathcal{B}$ and $i \in I$, $b' = \widetilde{E}_i b$ if and only if $b = \widetilde{F}_i b'$.

(2) Let $\mathcal{B}_1, \mathcal{B}_2$ be crystals of the same type. A morphism $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ of crystals is a map $\mathcal{B}_1 \to \mathcal{B}_2 \sqcup \{0\}$ satisfying the following:

- (a) If $b \in \mathcal{B}_1$ and $\psi(b) \in \mathcal{B}_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$.

(b) If $b, \widetilde{E}_i(b) \in \mathcal{B}_1$ and $\psi(b), \psi(\widetilde{E}_i b) \in \mathcal{B}_2$, then $\psi(\widetilde{E}_i b) = \widetilde{E}_i \psi(b)$.
(c) If $b, \widetilde{F}_i(b) \in \mathcal{B}_1$ and $\psi(b), \psi(\widetilde{F}_i b) \in \mathcal{B}_2$, then $\psi(\widetilde{F}_i b) = \widetilde{F}_i \psi(b)$.

A crystal can be regarded as an I-colored directed graph; the vertices are the elements of $\mathcal{B}$, and $b, b' \in \mathcal{B}$ are joined by an i-colored arrow from b to b' if $b' = \widetilde{F}_i b$.

Example 1.3.2. Let $\mathcal{B}$ be a crystal of $\mathfrak{sl}_{m+n}$-type. By forgetting the operators $\widetilde{E}_m, \widetilde{F}_m$, we can regard $\mathcal{B}$ as a crystal of $\mathfrak{s}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$-type. This defines a functor $\mathrm{Res}^{m+n}_{m,n}$ from the category of $\mathfrak{sl}_{m+n}$-type to the category of $\mathfrak{s}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$-type. In terms of the directed graph, $\mathrm{Res}^{m+n}_{m,n}(\mathcal{B})$ is obtained from $\mathcal{B}$ by deleting the arrows colored by m.

Definition 1.3.3. We call $b \in \mathcal{B}$ a highest weight vector if it is a source, i.e., $\widetilde{E}_i b = 0$ for all $i \in I$.

There is a formula (tensor product rule) that makes two crystals into a new crystal.

Definition 1.3.4. Let B_1, B_2 be crystals of type $(P, P^\vee, \Pi, \Pi^\vee, \langle \cdot, \cdot \rangle)$. The tensor product $B_1 \otimes B_2$ of crystals B_1 and B_2 is the crystal whose underlying set is $B_1 \times B_2$ and whose structure maps are defined as follows: For each $i \in I$, $b_1 \in B_1$, $b_2 \in B_2$,

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

$$\widetilde{E}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{E}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{E}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\widetilde{F}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{F}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{F}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

where we write $b_1 \otimes b_2$ instead of $(b_1, b_2) \in B_1 \times B_2$, and we understand $b_1 \otimes b_2 = 0$ if $b_1 = 0$ or $b_2 = 0$.

To memorize this rule, it is convenient to introduce the notion of i-signatures. For $b \in \mathcal{B}$ and $i \in I$, the i-signature $\mathrm{sgn}_i(b)$ of b is the sequence of $\varepsilon_i(b)$ $-$'s followed by $\varphi_i(b)$ $+$'s;

$$\mathrm{sgn}_i(b) = \underbrace{-, \dots, -}_{\varepsilon_i(b)}, \underbrace{+, \dots, +}_{\varphi_i(b)}.$$

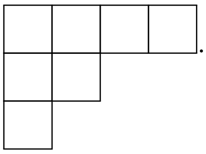
For $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$, by Definition 1.3.4, $\mathrm{sgn}_i(b_1 \otimes b_2)$ is obtained by deleting adjacent pair $(+, -)$ in the concatenation of $\mathrm{sgn}_i(b_1)$ and $\mathrm{sgn}_i(b_2)$ until there are no such pairs. Also, $\widetilde{E}_i$ (resp., $\widetilde{F}_i$) acts on b_k, where k is such that the rightmost $-$ (resp., leftmost $+$) in $\mathrm{sgn}_i(b_1 \otimes b_2)$ is originally in $\mathrm{sgn}_i(b_k)$. This method is useful particularly when we consider the tensor product of many crystals.

1.4. **Partitions.** A partition of $n \in \mathbb{Z}_{\geq 0}$ of length $l \in \mathbb{Z}_{\geq 0}$ is a non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{Z}^l_{\geq 0}$ of nonnegative integers such that $|\lambda| := \sum_{i=1}^l \lambda_i = n$. For a partition λ, we denote its length by $\ell(\lambda)$. Also, $\ell'(\lambda)$ denotes the maximal integer $i \in [1, \ell(\lambda)]$ such that $\lambda_i \neq 0$. For each $n, l \in \mathbb{Z}_{\geq 0}$, let $\mathrm{Par}_l(n)$ denote the set of partitions of n of length l. Also, set $\mathrm{Par}_l := \bigsqcup_{n \geq 0} \mathrm{Par}_l(n)$. It is usual to write $\lambda \vdash n$ in order to represent that λ is a partition of n.

The dominance order $\preceq$ on Par_l is a partial order defined as follows. For $\lambda, \mu \in \mathrm{Par}_l$, we have $\lambda \preceq \mu$ if the following hold:

(1) $|\lambda| = |\mu|$.
(2) $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for all $k = 1, \dots, l$.

A partition $\lambda \in \mathrm{Par}_l$ is often identified with its Young diagram. For example, the Young diagram of a partition $(4,2,1)$ is



The rows (resp., columns) of a Young diagram are indexed by $\{1,2,\dots\}$ from top to bottom (resp., from left to right).

Let $(L,\leq)$ be a totally ordered set. A semistandard Young tableau of shape $\lambda \in \mathrm{Par}_l$ in letters L is a filling of the Young diagram of λ whose entries weakly increase from left to right along the rows and strictly increase from top to bottom along the columns. For a Young tableau T, we denote by $T(i,j) \in L$ the letter in the (i,j)-th box. A semistandard Young tableau is said to be standard if $|L| = d$, and each letter appears exactly once. The following are a semistandard but not standard Young tableau of shape $(4,2,1)$ in letters $[1,7]$, and a standard Young tableau of the same shape in the same letters:

$$\begin{array}{|c|c|c|c|}\hline 2&2&4&7\\ \hline 5&6\\ \cline{1-2} 7\\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\ \hline 2&7\\ \cline{1-2} 5\\ \cline{1-1}\end{array}.$$

We denote by $\mathrm{SST}(\lambda;L)$ (resp., $\mathrm{ST}(\lambda;L)$) the set of semistandard (resp., standard) Young tableaux of shape λ in letters L. When $L = [1,r]$ for some $r \in \mathbb{Z}_{>0}$, we abbreviate $\mathrm{SST}(\lambda;L)$ to $\mathrm{SST}_r(\lambda)$. Also, when $L = [1,|\lambda|]$, we abbreviate $\mathrm{ST}(\lambda;L)$ to $\mathrm{ST}(\lambda)$. Note that $\mathrm{SST}_r(\lambda) = \emptyset$ unless $r \geq \ell'(\lambda)$. For $\lambda \in \mathrm{Par}_l(n)$, define $T_\lambda \in \mathrm{SST}_l(\lambda)$ by $T_\lambda(i,j) := i$ for all i,j.

A bipartition of $n \in \mathbb{Z}_{\geq 0}$ of length $(l^-,l^+) \in \mathbb{Z}^2_{\geq 0}$ is a pair $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^-;\boldsymbol{\lambda}^+)$ of partitions such that $\ell(\boldsymbol{\lambda}^\pm) = l^\pm$, and $|\boldsymbol{\lambda}| := |\boldsymbol{\lambda}^-| + |\boldsymbol{\lambda}^+| = n$. For each $n, l^-, l^+ \in \mathbb{Z}_{\geq 0}$, let $\mathrm{Bip}_{(l^-,l^+)}(n)$ denote the set of bipartitions of n of length (l^-,l^+). Also, set $\mathrm{Bip}_{(l^-,l^+)} := \bigsqcup_{n \geq 0} \mathrm{Bip}_{(l^-,l^+)}(n)$. Note that we have

$$\mathrm{Bip}_{(l^-;l^+)}(n) = \bigsqcup_{n^-+n^+=n} \left(\mathrm{Par}_{l^-}(n^-) \times \mathrm{Par}_{l^+}(n^+)\right).$$

The dominance order $\preceq$ on $\mathrm{Bip}_{(l^-,l^+)}$ is a partial order defined as follows. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Bip}_{(l^-,l^+)}$, we have $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$ if the following hold:

(1) $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$.
(2) $\sum_{i=1}^k \boldsymbol{\lambda}_i^- \leq \sum_{i=1}^k \boldsymbol{\mu}_i^-$ for all $k = 1,\dots,l^-$.
(3) $|\boldsymbol{\lambda}^-| + \sum_{i=1}^k \boldsymbol{\lambda}_i^+ \leq |\boldsymbol{\mu}^-| + \sum_{i=1}^k \boldsymbol{\mu}_i^+$ for all $k = 1,\dots,l^+$.

Also, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Bip}_{(l^-,l^+)}$, we write $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ to indicate that $\boldsymbol{\lambda}^\pm \preceq \boldsymbol{\mu}^\pm$. Then, $\trianglelefteq$ defines a partial order on $\mathrm{Bip}_{(l^-,l^+)}$. Note that $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ implies $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$.

Let $(L^\pm, \leq^\pm)$ be totally ordered sets. A semistandard Young bitableau of shape $\boldsymbol{\lambda} \in \mathrm{Bip}_{(l^-,l^+)}$ in letters $L^\pm$ is a pair $(T^-,T^+) \in \mathrm{SST}(\boldsymbol{\lambda}^-;L^-) \times \mathrm{SST}(\boldsymbol{\lambda}^+;L^+)$. We denote by $\mathrm{SST}(\boldsymbol{\lambda};L^-,L^+)$ the set of semistandard Young bitableaux of shape $\boldsymbol{\lambda}$ in letters $L^\pm$. Namely,

$$\mathrm{SST}(\boldsymbol{\lambda};L^-,L^+) = \mathrm{SST}(\boldsymbol{\lambda}^-;L^-) \times \mathrm{SST}(\boldsymbol{\lambda}^+;L^+).$$

A semistandard Young bitableau is said to be standard if $L^- = L^+$, $|L^-| = d$, and each letter appears exactly once. The following is a standard Young bitableau of shape $(4,2,1;5;2)$ in letters $[1,14]$:

$$\left(\begin{array}{|c|c|c|c|}\hline 4 & 7 & 9 & 12 \\ \hline 5 & 8 \\ \cline{1-2} 11 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 10 & 14 \\ \hline 2 & 13 \\ \cline{1-2} \end{array} \right).$$

For $r \in \mathbb{Z}_{>0}$, we set

$$\mathrm{SST}_{(r,r)}(\boldsymbol{\lambda}) := \mathrm{SST}(\boldsymbol{\lambda}; [-\underline{r}, -\underline{1}]^{\mathrm{op}}, [\underline{1}, \underline{r}]),$$

and

$$\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda}) := \mathrm{SST}(\boldsymbol{\lambda}; [-r, 0]^{\mathrm{op}}, [1, r]).$$

1.5. **Robinson-Schensted-Knuth correspondence and its variants.** Given a semistandard Young tableau T of shape λ in letters L, and $i \in L$, define a new semistandard Young tableau $T \leftarrow \boxed{i}$ as follows;

(1) Let j be the smallest number satisfying $T(j,1) \geq i$.
(2) Replace $\boxed{T(j,1)}$ by $\boxed{i}$; if there is no such j, then put $\boxed{i}$ at the bottom of the first column, and stop the algorithm.
(3) Repeat (1)-(2) for the next column with the role of $\boxed{i}$ replaced by $\boxed{T(j,1)}$.

This algorithm is called Schenstead's bumping algorithm.

Example 1.5.1. Let

$$T = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 4 & 7 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}.$$

Then, $T \leftarrow \boxed{3}$ is computed as follows:

$$\begin{array}{|c|c|c|c|}\hline 2 & 2 & 4 & 7 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 2 & 2 & 4 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 2 & 2 & 4 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 2 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

Suppose that we are given a word $\mathbf{w} = (w_1, \dots, w_d) \in L^d$ and a strictly increasing sequence $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{Z}^d_{>0}$. We define two Young tableaux $P(\mathbf{w})$ and $Q_{\mathbf{r}}(\mathbf{w})$ as follows. First, $P(\mathbf{w})$ is defined to be $(\cdots((\boxed{w_1} \leftarrow \boxed{w_2}) \leftarrow \boxed{w_3}) \cdots) \leftarrow \boxed{w_d}$; hence $P(\mathbf{w})$ is a semistandard Young tableau in letters L of some shape $\lambda \vdash d$. Next, $Q_{\mathbf{r}}(\mathbf{w})$ is defined to be the semistandard Young tableau in letters $\{r_1, \dots, r_d\}$ of the same shape λ whose (i,j)-th entry is r_k if a new box is added in the position (i,j) when we bump the box $\boxed{w_k}$ to the semistandard Young tableau $P(w_1, \dots, w_{k-1})$. We call $P(\mathbf{w})$ the insertion tableau of $\mathbf{w}$, and $Q_{\mathbf{r}}(\mathbf{w})$ the recording tableau of $\mathbf{w}$ in letters $\mathbf{r}$.

Example 1.5.2. Let $L = \mathbb{Z}$, $\mathbf{w} = (1, 3, 2, 2, 3)$, and $\mathbf{r} = (1, 3, 4, 7, 8)$. Then, $P(\mathbf{w})$ and $Q_{\mathbf{r}}(\mathbf{w})$ are computed as follows:

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \mapsto \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} = P(\mathbf{w}),$$

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \mapsto \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \mapsto \begin{array}{|c|c|}\hline 1&4\\\hline 3\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1&4&7\\\hline 3\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1&4&7\\\hline 3\\\cline{1-1} 8\\\cline{1-1}\end{array} = Q_{\mathbf{r}}(\mathbf{w}).$$

Theorem 1.5.3 (Robinson-Schensted-Knuth correspondence, see [S99, Theorem 7.11.5])**.** *For each strictly increasing sequence* $\mathbf{r} \in \mathbb{Z}^d_{>0}$*, the correspondence* $(\mathbf{w}, \mathbf{r}) \mapsto (P(\mathbf{w}), Q_{\mathbf{r}}(\mathbf{w}))$ *gives a bijection from* L^d *to* $\bigsqcup_{\lambda \in \mathrm{Par}_d(d)} (\mathrm{SST}(\lambda; L) \times \mathrm{ST}(\lambda; \{r_1, \ldots, r_d\}))$*. In particular, we have a bijection*

$$L^d \to \bigsqcup_{\lambda \in \mathrm{Par}_d(d)} (\mathrm{SST}(\lambda; L) \times \mathrm{ST}(\lambda)); \;\; \mathbf{w} \mapsto (P(\mathbf{w}), Q_{(1,\ldots,d)}(\mathbf{w})). \tag{1}$$

Remark 1.5.4. When L is finite, the right-hand side of the bijection (1) is also written as

$$\bigsqcup_{\lambda \in \mathrm{Par}_{|L|}(d)} (\mathrm{SST}(\lambda; L) \times \mathrm{ST}(\lambda)) .$$

Let $x \in \mathfrak{S}_d$. Then, the map $x \mapsto (x(1), \ldots, x(d))$ gives an injection $\mathfrak{S}_d \to [1, d]^d$. Set $P(x)$ to be the insertion tableau of $(x(1), \ldots, x(d))$, and $Q(x)$ to be the recording tableau of $(x(1), \ldots, x(d))$ in letters $(1, \ldots, d)$. If we identify $\mathfrak{S}_d$ with its image of the injection above, then the RSK-correspondence restricts to the following bijection:

$$\mathfrak{S}_d \to \bigsqcup_{\lambda \in \mathrm{Par}_d(d)} (\mathrm{ST}(\lambda) \times \mathrm{ST}(\lambda)); \;\; x \mapsto (P(x), Q(x)). \tag{2}$$

Example 1.5.5. When $d = 3$, the set $\mathrm{Par}_3(3)$ consists of

$$\begin{array}{|c|c|c|}\hline &&\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline &\\\hline \\\cline{1-1}\end{array}, \quad \begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array}.$$

The elements of $\mathrm{ST}(\lambda)$ for $\lambda \in \mathrm{Par}_3(3)$ are the following:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}.$$

The bijection (2) are given by the following assignment:

$$e \mapsto \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right), \quad s_1 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right), \quad s_2 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right),$$

$$s_2 s_1 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right), \quad s_1 s_2 \mapsto \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right),$$

$$s_1 s_2 s_1 \mapsto \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right).$$

Lemma 1.5.6. *Let L be a totally ordered set, $\mathbf{w} \in L^d$. Let $x \in \mathfrak{S}_d$ be the maximal element satisfying $w_{x^{-1}(1)} \leq \cdots \leq w_{x^{-1}(d)}$. Then, $Q(\mathbf{w}) = Q_{(1,\dots,d)}(x)$, and $P(\mathbf{w})$ is obtained from $P(x)$ by replacing i with $w_{x^{-1}(i)}$ for each $i = 1, \dots, d$.*

Proof. Clear from the definitions. ☐

Let W_d act on $\mathbb{Z}^d$ by

$$(i_1, \dots, i_d)s_j = \begin{cases} (-i_1, i_2, \dots, i_d) & \text{if } j = 0, \\ (i_1, \dots, i_{j+1}, i_j, \dots, i_d) & \text{if } j \neq 0. \end{cases}$$

Then, for each $w \in W_d$, we have $(i_1, \dots, i_d)w = (i_{w(1)}, \dots, i_{w(d)})$, where we understand $i_{-j} = -i_j$.

Now, let $r \in \mathbb{Z}_{>0}$. Then, $[-r, r]^d$ is decomposed as

$$[-r, r]^d = \bigsqcup_{k=0}^{d} \bigsqcup_{1 \leq i_1 < \cdots < i_k \leq d} \{\mathbf{w} \in [-r, r]^d \mid w_i > 0 \text{ if and only if } i \in \{i_1, \dots, i_k\}\},$$

and each component is in bijection (as sets) with $([-r, 0]^{\mathrm{op}})^{d-k} \times [1, r]^k$. Hence, the map

$$\mathbf{w} \mapsto \left((P(\mathbf{w}^-), P(\mathbf{w}^+)), (Q_{(j_1,\dots,j_{d-k})}(\mathbf{w}^-), Q_{(i_1,\dots,i_k)}(\mathbf{w}^+))\right),$$

where $\{i_1, \dots, i_k\} = \{i \mid w_i > 0\}$, $\{j_1, \dots, j_{d-k}\} = [1, d] \backslash \{i_1, \dots, i_k\}$, $\mathbf{w}^+ = (w_{i_1}, \dots, w_{i_k}) \in [1, r]^k$, and $\mathbf{w}^- = (w_{j_1}, \dots, w_{j_{d-k}}) \in ([-r, 0]^{\mathrm{op}})^{d-k}$, gives a bijection

$$[-r, r]^d \to \bigsqcup_{\boldsymbol{\lambda} \in \mathrm{Bip}_{r+1,r}(d)} \left(\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda}) \times \mathrm{ST}(\boldsymbol{\lambda})\right). \tag{3}$$

We set

$$P^{\pm}(\mathbf{w}) := P(\mathbf{w}^{\pm}), \ Q^-(\mathbf{w}) := Q_{(j_1,\dots,j_{d-k})}(\mathbf{w}^-), \ Q^+(\mathbf{w}) := Q_{(i_1,\dots,i_k)}(\mathbf{w}^+).$$

Note that $P^-(\mathbf{w})$ is obtained from $P(-w_{j_1}, \dots, -w_{j_{d-k}})$ (here, we regard $(-w_{j_1}, \dots, -w_{j_{d-k}}) \in [0, r]^{d-k}$) by multiplying each entries by -1.

Let $y \in W_d$. Then, the map $y \mapsto (y(1), \dots, y(d))$ gives an injection $W_d \to [-d, d]^d$. Set $P^{\pm}(y) := P^{\pm}(y(1), \dots, y(d))$, and $Q^{\pm}(y) := Q^{\pm}(y(1), \dots, y(d))$. Then, the bijection (3) restricts to the bijection

$$W_d \to \bigsqcup_{\boldsymbol{\lambda} \in \mathrm{Bip}_{(d-k,k)}(d)} (\mathrm{ST}(\boldsymbol{\lambda}) \times ST(\boldsymbol{\lambda}))$$

Example 1.5.7. Let $y \in W_5$ be such that $(y(1), y(2), y(3), y(4), y(5)) = (3, -4, -1, 5, 2)$. Then, $(i_1, i_2, i_3) = (1, 4, 5)$, and $(j_1, j_2) = (2, 3)$. The associated tableaux are following:

$$P^-(y) = \begin{array}{|c|c|}\hline -1 & -4 \\ \hline \end{array}, \quad P^+(y) = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 5 \\ \cline{1-1} \end{array}, \quad Q^-(y) = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}, \quad Q^+(y) = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}.$$

Lemma 1.5.8. *Let* $\mathbf{w} \in \mathbb{Z}^d$. *For each* $i \in [1, d]$, *set* $w_{-i} := -w_i$. *Let* $y \in W_d$ *be the maximal element satisfying* $0 \leq w_{y^{-1}(1)} \leq \cdots \leq w_{y^{-1}(d)}$. *Then,* $Q^{\pm}(\mathbf{w}) = Q^{\pm}(y)$, *and* $P^{\pm}(\mathbf{w})$ *is obtained from* $P^{\pm}(y)$ *by replacing* i *with* $w_{y^{-1}(i)}$.

By replacing $[-r, r]$ with $[-\underline{r+1}, \underline{r+1}]$, we obtain similar results.

1.6. **Kashiwara operators on semistandard Young tableaux.** Consider the Cartan datum of $\mathfrak{sl}_n$-type. Let $i \in [1, n-1]$, and $\lambda \in \mathrm{Par}_n(d)$. We define two maps (called the Kashiwara operators) $\widetilde{E}_i, \widetilde{F}_i : \mathrm{SST}_n(\lambda) \to \mathrm{SST}_n(\lambda) \sqcup \{0\}$ as follows, where 0 denotes a formal symbol.

Let $\mathbf{s} = (s_1, \dots, s_d) \in [1, n]^d$. First, delete s_k such that $s_k \neq i, i+1$. Next, delete the adjacent pair $(i, i+1)$. The resulting sequence is of the form $\mathbf{s}_i := (i+1, \dots, i+1, i, \dots, i)$. Then, define $\widetilde{E}_i(\mathbf{s})$ to be the sequence obtained from $\mathbf{s}$ by replacing s_k by i, where s_k is the rightmost $i+1$ in $\mathbf{s}_i$; if there is no $i+1$ in $\mathbf{s}$, then we define $\widetilde{E}_i(\mathbf{s}) = 0$. Also, we define $\widetilde{F}_i(\mathbf{s})$ to be the sequence obtained from $\mathbf{s}$ by replacing s_l by $i+1$, where s_l is the leftmost i in $\mathbf{s}_i$; if there is no i in $\mathbf{s}_i$, then we define $\widetilde{F}_i(\mathbf{s}) = 0$.

Example 1.6.1. Let $\mathbf{s} = (2, 1, 2, 1, 3, 3, 2, 3, 2, 1, 1, 1, 3)$. Then, we have

$$\mathbf{s}_1 = (2, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, 2, 1, 1, 1, \cdot).$$

Hence,

$$\widetilde{e}_1 \mathbf{s}_1 = (2, 1, 2, 1, 3, 3, 2, 3, 1, 1, 1, 1, 3), \text{ and } \widetilde{f}_1 \mathbf{s}_1 = (2, 1, 2, 1, 3, 3, 2, 3, 2, 2, 1, 1, 3).$$

Given $T \in \mathrm{SST}_n(\lambda)$, the sequence

$$\mathrm{ME}(T) := (T(1, \lambda_1), T(1, \lambda_1 - 1), \dots, T(1, 1), T(2, \lambda_2), \dots, T(2, 1), \dots, T(n, \lambda_n), \dots, T(n, 1))$$

is called the Middle-Eastern reading of T. It is known that there exists a unique $T' \in \mathrm{SST}_n(\lambda)$ such that $\mathrm{ME}(T') = \widetilde{E}_i(\mathrm{ME}(T))$ if $\widetilde{E}_i(\mathrm{ME}(T)) \neq 0$. Then, we define

$$\widetilde{E}_i T := \begin{cases} T' & \text{if } \widetilde{E}_i(\mathrm{ME}(T)) \neq 0, \\ 0 & \text{if } \widetilde{E}_i(\mathrm{ME}(T)) = 0. \end{cases}$$

$\widetilde{F}_i T$ is defined similarly.

Also, for $\mathbf{s} \in [1, n]^d$ and $T \in \mathrm{SST}_n(\lambda)$, we set $\mathrm{wt}(\mathbf{s}) := \sum_{i=1}^d \epsilon_{s_i} + \mathbb{Z}(\epsilon_1 + \cdots + \epsilon_n)$, $\mathrm{wt}(T) := \mathrm{wt}(\mathrm{ME}(T))$. The following are fundamental and well-known facts in representation theory of $\mathfrak{sl}_n$.

Proposition 1.6.2 (see e.g., [HK02] or [Kw09])**.** *Let* $\lambda, \mu \in \mathrm{Par}_n(d)$.

1. *The sets* $[1, n]^d$ *and* $\mathrm{SST}_n(\lambda)$ *with maps* $\widetilde{E}_i, \widetilde{F}_i, \mathrm{wt}$ *are crystals of* $\mathfrak{sl}_n$*-type.*
2. *The map* $\mathrm{SST}_n(\lambda) \to [1, n]^d$; $T \mapsto \mathrm{ME}(T)$ *is an injective morphism of crystals.*
3. $\mathrm{SST}_n(\lambda)$ *is connected, and it has a unique highest weight vector* T_λ. *Moreover,* T_λ *is a unique element of weight* $\mathrm{wt}(T_\lambda)$.
4. $\mathrm{SST}_n(\lambda) \simeq \mathrm{SST}_n(\mu)$ *if and only if* $\lambda_i - \mu_i$, $i \in [1, n]$ *are constant.*

Remark 1.6.3. Comparing the tensor product rule (Definition 1.3.4) with the construction above, we see that $[1, n]^d \simeq [1, n]^{\otimes d}$.

Here, recall Robinson-Schensted-Knuth correspondence

$$[1,n]^d \to \bigsqcup_{\lambda \in \mathrm{Par}_n(d)} (\mathrm{SST}_n(\lambda) \times \mathrm{ST}(\lambda)) = \bigsqcup_{\lambda \in \mathrm{Par}_n(d)} \bigsqcup_{Q \in \mathrm{ST}(\lambda)} (\mathrm{SST}_n(\lambda) \times \{Q\}).$$

The right-hand side is equipped with a crystal structure if we identify $\mathrm{SST}_n(\lambda) \times \{Q\}$ with $\mathrm{SST}_n(\lambda)$.

Theorem 1.6.4 (see [Kw09, Theorem 4.6])**.** *Robinson-Schensted-Knuth correspondence*

$$[1,n]^d \to \bigsqcup_{\lambda \in \mathrm{Par}_n(d)} \bigsqcup_{Q \in \mathrm{ST}(\lambda)} (\mathrm{SST}_n(\lambda) \times \{Q\}).$$

is an isomorphism of crystals.

Next, we give to $[-r,r]^d$ and $\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d)$ crystal structures of $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_r)$-type. To do so, it is convenient to shift the indices $[1,2r]$ of the Cartan matrix of $\mathfrak{sl}_{2r+1}$ as $[-\underline{r},\underline{r}]$. Then, $[-r,r]^d$ is equipped with an $\mathfrak{sl}_{2r+1}$-crystal structure. By Example 1.3.2, $[-r,r]^d$ with $\widetilde{E}_i, \widetilde{F}_i, \mathrm{wt}$, $i \in [-\underline{r},\underline{r}] \setminus \{\underline{1}\}$ is an $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_r)$-crystal. Note that for $i \in [1,r]$ and $\mathbf{s} \in [-\underline{r},\underline{r}]^d$, we have

$$\widetilde{E}_{-\underline{i}}(\mathbf{s}) = -\widetilde{F}_{\underline{i}}(-\mathbf{s}), \quad \widetilde{F}_{-\underline{i}}(\mathbf{s}) = -\widetilde{E}_{\underline{i}}(-\mathbf{s}),$$

where $-(i_1,\dots,i_d) := (-i_1,\dots,-i_d)$ for $(i_1,\dots,i_d) \in [-\underline{r},\underline{r}]^d$. Hence, $[1,r]^d$ with $\widetilde{E}_{\underline{i}}, \widetilde{F}_{\underline{i}}, \mathrm{wt}$, $i \in [2,r]$ is an $\mathfrak{sl}_r$-crystal, and $[-r,0]^d$ with $\widetilde{E}_{-\underline{i}}, \widetilde{F}_{-\underline{i}}, \mathrm{wt}$, $i \in [1,r]$ is an $\mathfrak{sl}_{r+1}$-crystal.

For $\mathbf{T} = (\mathbf{T}^-, \mathbf{T}^+) \in \mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$, set $\mathrm{ME}(\mathbf{T}) := (\mathrm{ME}(\mathbf{T}^-), \mathrm{ME}(\mathbf{T}^+)) \in [-r,r]^d$; the concatenation of $\mathrm{ME}(\mathbf{T}^-)$ and $\mathrm{ME}(\mathbf{T}^+)$. Then, for $i \in [2,r]$ and $X \in \{E,F\}$, we have

$$\widetilde{X}_{\underline{i}}(\mathrm{ME}(\mathbf{T})) = (\mathrm{ME}(\mathbf{T}^-), \widetilde{X}_{\underline{i}}(\mathrm{ME}(\mathbf{T}^+))) = \mathrm{ME}(\mathbf{T}^-, \widetilde{X}_{\underline{i}}(\mathbf{T}^+)).$$

Also, for $i \in [1,r]$ and $X \in \{E,F\}$, we have

$$\widetilde{X}_{-\underline{i}}(\mathrm{ME}(\mathbf{T})) = (-\widetilde{Y}_{\underline{i}}(-\mathrm{ME}(\mathbf{T}^-)), \mathrm{ME}(\mathbf{T}^+)) = \mathrm{ME}(-\widetilde{Y}_{\underline{i}}(-\mathbf{T}^-), \mathrm{ME}(\mathbf{T}^+)),$$

where Y is the unique element of $\{E,F\} \setminus \{X\}$, and $-\mathbf{T}^-$ is obtained from $\mathbf{T}^-$ by multiplying the entries by -1. From this observation, we have the following.

Proposition 1.6.5. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Bip}_{(r+1,r)}(d)$.*

1. *The map $\mathrm{ME} : \mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda}) \to [-r,r]^d$ is an injective morphism of crystals.*
2. *$\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$ is connected, and it has $T_{\boldsymbol{\lambda}} := (-T_{\boldsymbol{\lambda}^-}, T_{\boldsymbol{\lambda}^+})$ as a unique element such that*

$$\widetilde{E}_{\underline{i}}(T_{\boldsymbol{\lambda}}) = 0 = \widetilde{F}_{-\underline{j}}(T_{\boldsymbol{\lambda}}) \text{ for all } i \in [2,r], j \in [1,r].$$

   Moreover, $T_{\boldsymbol{\lambda}}$ is a unique element of weight $\mathrm{wt}(T_{\boldsymbol{\lambda}})$.
3. *$\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda}) \simeq \mathrm{SST}_{(r+1,r)}(\boldsymbol{\mu})$ if and only if $\boldsymbol{\lambda}_i^\pm - \boldsymbol{\mu}_i^\pm$ are constant.*
4. *The bijection*

$$[-r,r]^d \to \bigsqcup_{\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d)} \bigsqcup_{\mathbf{Q} \in \mathrm{ST}(\boldsymbol{\lambda})} \left(\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda}) \times \{\mathbf{Q}\}\right).$$

   is an isomorphism of crystals of $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_r)$-type

Also, we have corresponding results for $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_{r+1})$-crystals $[-\underline{r+1}, \underline{r+1}]^d$ and $\mathrm{SST}_{r+1,r+1}(\boldsymbol{\lambda})$.

2. Symmetric pairs

The aim of this section is to define admissible pairs, and give examples of symmetric pairs of type AIII/AIV in Subsection **??**. In Subsection 2.2, we briefly explain how admissible pairs and real semisimple Lie algebras are related to each other.

2.1. Admissible pair.

Recall that we have fixed a Cartan datum $(P, P^\vee, \Pi, \Pi^\vee, \langle \cdot, \cdot \rangle)$ associated with a generalized Cartan matrix $A = (a_{i,,j})_{i,j \in I}$. In this subsection, we assume that A is of finite type. Since $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$ is a perfect pairing, for each $i \in I$, there exists a unique $\varpi_i^\vee \in P^\vee$ such that $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{i,j}$ for all $j \in I$.

Definition 2.1.1. A Dynkin involution of A is a bijection $\tau : I \to I$ such that $\tau^2 = \mathrm{id}_I$, and $a_{\tau(i),\tau(j)} = a_{i,j}$ for all $i, j \in I$.

Example 2.1.2.
(1) id_I is always a Dynkin involution.
(2) For each $i \in I$, there exists a unique $j \in I$ such that $-w_0(\alpha_i) = \alpha_j$. This induces a Dynkin involution, which we denote by $-w_0$.
(3) When our Cartan datum is the $\mathfrak{sl}_n$-type, the map $\tau(i) := n - i$ is a Dynkin involution.

Let $I_\bullet \subset I$. Then, $(P, P^\vee, \{\alpha_i \mid i \in I_\bullet\}, \{h_i \mid i \in I_\bullet\}, \langle \cdot, \cdot \rangle)$ is a Cartan datum associated with $A_\bullet := (a_{i,j})_{i,j \in I_\bullet}$. Note that $A_\bullet$ is also of finite type. We denote $\mathfrak{g}_\bullet$ and $W_\bullet$ the Kac-Moody algebra and the Weyl group associated with this Cartan datum, respectively. Let $w_\bullet \in W_\bullet$ denote the longest element. Also, set $\rho_\bullet^\vee := \frac{1}{2} \sum_{\alpha \in \Phi_{\bullet,+}} \alpha$, where $\Phi_{\bullet,+} := \Phi_+ \cap \sum_{j \in I_\bullet} \mathbb{Z}\alpha_j$.

Definition 2.1.3. Let $I = I_\bullet \sqcup I_\circ$ be a partition of I, and τ a Dynkin involution of A. The pair $(I_\bullet, \tau)$ is said to be admissible if the following conditions are satisfied:
(1) $\tau(I_\bullet) = I_\bullet$.
(2) $-w_\bullet(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in I_\bullet$.
(3) $\langle \rho_\bullet^\vee, \alpha_j \rangle \in \mathbb{Z}$ for all $j \in I_\circ$ such that $\tau(j) = j$.

Example 2.1.4. $(\emptyset, \tau)$, where τ is a Dynkin involution, and $(I, -w_0)$ are admissible pairs.

Remark 2.1.5. The admissible pairs are in bijection with the Satake diagrams, which arises in the classification of real simple Lie algebras ([A62]). See Figure 2 in page 112 for the lists of the Satake diagrams when A is indecomposable. There, the black nodes represents the elements of $I_\bullet$, and the arrows represents the nontrivial orbits of $\tau|_{I_\circ}$.

Let $(I_\bullet, \tau)$ be an admissible pair, and assume that there exists a linear involution $\widetilde{\tau}$ on P and $P^\vee$ such that $\widetilde{\tau}(\alpha_i) = \alpha_{\tau(i)}$, $\widetilde{\tau}(h_i) = h_{\tau(i)}$, and $\langle \widetilde{\tau}(h), \widetilde{\tau}(\lambda) \rangle = \langle h, \lambda \rangle$ for all $i \in I$, $h \in P^\vee$, $\lambda \in P$. Then, $\widetilde{\tau}$ is extended to the whole of $\mathfrak{g}(A)$ by setting $\widetilde{\tau}(e_i) = e_{\tau(i)}$, $\widetilde{\tau}(f_i) = f_{\tau(i)}$ for all $i \in I$. Also, let $s(I_\bullet, \tau) : Q \to \mathbb{C}^\times$ be a group homomorphism such that

- $s(\alpha_i) = 1$ for all $i \in I_\bullet$ or $\tau(i) = i$.
- $s(\alpha_j) \in \{1, -1\}$ for all $i \notin I_\bullet$ and $\tau(i) \neq i$.
- $s(\alpha_j) s(\alpha_{\tau(j)}) = (-1)^{\langle 2\rho_\bullet^\vee, \alpha_j \rangle}$ for all $i \notin I_\bullet$ and $\tau(i) \neq i$.

Then, $s(I_\bullet, \tau)$ induces an automorphism $\mathrm{Ad}(s(I_\bullet, \tau)) \in \mathrm{Aut}(\mathfrak{g})$ defined by

$$\mathrm{Ad}(s(I_\bullet, \tau))(x) = s(I_\bullet, \tau)(\alpha) x, \quad x \in \mathfrak{g}_\alpha.$$

Similarly, let $s : Q \to \mathbb{C}^\times$ be a group homomorphism defined by $s(\alpha_i) = -1$ for all $i \in I$, and define $\mathrm{Ad}(s) \in \mathrm{Aut}(\mathfrak{g})$. Now, set

$$\theta := \mathrm{Ad}(s(I_\bullet, \tau)) \circ w_\bullet \circ \widetilde{\tau} \circ \mathrm{Ad}(s) \circ \omega \in \mathrm{Aut}(\mathfrak{g}(A)).$$

This is a Lie algebra involution on $\mathfrak{g}$. It is known that the fixed-point subalgebra $\mathfrak{g}^\theta := \{x \in \mathfrak{g} \mid \theta(x) = x\}$ is a reductive Lie algebra; in particular, it is the Kac-Moody algebra associated with a Cartan datum for some Cartan matrix.

Definition 2.1.6. A symmetric pair is a pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ for some admissible pair $(I_\bullet, \tau)$.

Since $\mathfrak{g}^\theta$ is reductive, most parts of its algebra structure becomes clear once we recognize its Cartan subalgebra.

Proposition 2.1.7 ([Le17, Theorem 2.5 and 2.7])**.** *Let $(\mathfrak{g}, \mathfrak{g}^\theta)$ be a symmetric pair. Then, there exists a subset $\Gamma_\theta \subset \Phi_+$ such that the subspace*

$$(\mathfrak{g}^\theta \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Gamma_\theta} \mathbb{C}(e_\alpha + f_\alpha)$$

is a Cartan subalgebra of $\mathfrak{g}^\theta$ if we choose $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ appropriately.

Example 2.1.8. Suppose that our Cartan datum is the $\mathfrak{sl}_n$-type. Set $\tau(i) := n - i$. Then, $(\emptyset, \tau)$ is an admissible pair, and the associated involution θ is the restriction of the involution on $\mathfrak{gl}_n$ given by

$$E_{i,j} \mapsto E_{n+1-i,n+1-j} \quad (1 \leq i, j \leq n),$$

where $E_{i,j}$ denotes the matrix having entries 1 at (i, j), and 0 elsewhere. For $1 \leq i < j \leq n$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, set

$$\begin{aligned} B_{i,j} &:= E_{i,j} + E_{n+1-i,n+1-j}, \\ H_k &:= (E_{\lfloor \frac{n}{2} \rfloor + k, \lfloor \frac{n}{2} \rfloor + k} - E_{\lfloor \frac{n}{2} \rfloor + k+1, \lfloor \frac{n}{2} \rfloor + k+1}) - (E_{\lfloor \frac{n}{2} \rfloor - k+1, \lfloor \frac{n}{2} \rfloor - k+1} - E_{\lfloor \frac{n}{2} \rfloor - k+2, \lfloor \frac{n}{2} \rfloor - k+2}). \end{aligned}$$

Then, $\mathfrak{g}^\theta$ is generated by $B_{1,2}, B_{2,3}, \dots, B_{n-1,n}, H_1, \dots, H_{\lfloor \frac{n-1}{2} \rfloor}$, and spanned by $B_{i,j}, H_k$, $1 \leq i < j \leq n$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. For $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$, set

$$\beta_i := \alpha_{\lfloor \frac{n}{2} \rfloor + 1 - i} + \alpha_{\lfloor \frac{n}{2} \rfloor - i} + \cdots + \alpha_{\lfloor \frac{n}{2} \rfloor + 1 - n + i}.$$

Then, $\Gamma_\theta = \{\beta_i \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

(1) Let $n = 2r + 1$ for some $r \in \mathbb{Z}_{>0}$. Then, the subalgebra spanned by

$$H_1, \dots, H_r, B_{r,r+2}, B_{r-1,r+3}, \dots, B_{1,2r+1}$$

is a Cartan subalgebra of $\mathfrak{g}^\theta$.

Set

$$e_i := \begin{cases} B_{r+i,r+i+1} - B_{r-i+2,r+i+1} & \text{if } i = 2, \dots, r, \\ B_{r+1,r+1} & \text{if } i = r+1, \\ B_{i,i+1} + B_{2r-i+2,i+1} & \text{if } i = r+2, \dots, 2r, \end{cases}$$

$$f_i := \begin{cases} B_{r-i+1,r-i+2} - B_{r-i+1,r+i} & \text{if } i = 2, \dots, r, \\ B_{r,r+1} & \text{if } i = r+1, \\ B_{2r-i+1,2r-i+2} + B_{2r-i+1,i} & \text{if } i = r+2, \dots, 2r, \end{cases}$$

$$h_i := \begin{cases} \frac{1}{2}(H_i + B_{r-i+1,r+i+1} - B_{r-i+2,r+i}) & \text{if } i = 2, \dots, r, \\ \frac{1}{2}(H_1 - B_{r,r+2}) & \text{if } i = r+1, \\ \frac{1}{2}(H_{i-r} - B_{2r-i+1,i+1} + B_{2r-i+2,i}) & \text{if } i = r+2, \dots, 2r, \\ \sum_{k=1}^r (kh_{2r-k+1} + (r+1)B_{k,2r-k+2}) & \text{if } i = 1. \end{cases}$$

Then, we have

$$[h_i, e_j] = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ and } (2 \leq i, j \leq r \text{ or } r+1 \leq i, j), \\ 0 & \text{otherwise,} \end{cases}$$

$$[h_i, f_j] = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1 \text{ and } (2 \leq i, j \leq r \text{ or } r+1 \leq i, j), \\ 0 & \text{otherwise.} \end{cases}$$

Since the subalgebra spanned by $h_1, \dots, h_{2r}$ is a Cartan subalgebra of $\mathfrak{g}^\theta$, we conclude that $\mathfrak{g}^\theta \simeq \mathfrak{s}(\mathfrak{gl}_r \oplus \mathfrak{gl}_{r+1})$.

(2) Let $n = 2r+2$ for some $r \in \mathbb{Z}_{>0}$. Then, the subalgebra spanned by

$$H_1, \dots, H_r, B_{r+1,r+2}, B_{r,r+3}, \dots, B_{1,2r+2}$$

is a Cartan subalgebra of $\mathfrak{g}^\theta$. Set

$$e_i := \begin{cases} B_{r+i+1,r+i+2} - B_{r-i+2,r+i+2} & \text{if } i = 1, \dots, r, \\ B_{2r+2-i,i+2} + B_{i+1,i+2} & \text{if } i = r+1, \dots, 2r, \end{cases}$$

$$f_i := \begin{cases} B_{r-i+1,r-i+2} - B_{r-i+1,r+i+1} & \text{if } i = 1, \dots, r, \\ B_{2r-i+1,i+1} + B_{2r-i+1,2r-i+2} & \text{if } i = r+1, \dots, 2r, \end{cases}$$

$$h_i := \begin{cases} \frac{1}{2}(H_i + B_{r-i+1,r+i+2} - B_{r-i+2,r+i+1}) & \text{if } i = 1, \dots, r, \\ \frac{1}{2}(H_{i-r} - B_{2r-i+1,i+2} + B_{2r-i+2,i+1}) & \text{if } i = r+1, \dots, 2r, \\ B_{1,2r+2} + B_{2,2r+1} + \cdots + B_{r+1,r+2} & \text{if } i = 0. \end{cases}$$

Then, we have

$$[h_i, e_j] = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ and } (1 \leq i, j \leq r \text{ or } r+1 \leq i, j), \\ 0 & \text{otherwise,} \end{cases}$$

$$[h_i, f_j] = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1 \text{ and } (1 \leq i, j \leq r \text{ or } r+1 \leq i, j), \\ 0 & \text{otherwise.} \end{cases}$$

Since the subalgebra spanned by $h_0, h_1, \dots, h_{2r}$ is a Cartan subalgebra of $\mathfrak{g}^\theta$, we conclude that $\mathfrak{g}^\theta \simeq \mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_{r+1})$.

Remark 2.1.9. As we have seen in Example 2, the Lie subalgebra $\mathfrak{l}$ of $\mathfrak{sl}_{m+n}$ generated by e_i, f_i, $i \in [1, m+n-1] \setminus \{m\}$ and $h \in P^\vee$ is isomorphic to $\mathfrak{s}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$. However, the subalgebra $\mathfrak{g}^\theta$ in Example 2.1.8, which is also isomorphic to $\mathfrak{s}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$ ($(m,n) = (r+1, r)$ or $(r+1, r+1)$), does not coincide with $\mathfrak{l}$.

Set

$$P_\imath := P/\{\lambda - \theta(\lambda) \mid \lambda \in P\},$$
$$P_\imath^\vee := \{h \in P^\vee \mid \theta(h) = h\}.$$

We have the induced pairing $\langle \cdot, \cdot \rangle : P_\imath^\vee \times P_\imath \to \mathbb{Z}$.

2.2. **Admissible pairs and real semisimple Lie algebras.** Here, we explain the relations between admissible pairs and real semisimple Lie algebras. For detail, see for example [O04]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $(I_\bullet, \tau)$ an admissible pair. Set

$$\theta' := w_\bullet \circ \widetilde{\tau} \circ \omega' \in \mathrm{Aut}(\mathfrak{g}),$$

where $\omega' \in \mathrm{Aut}(\mathfrak{g})$ is defined by $\omega(e_i) = -f_i$ and $\omega(h_i) = -h_i$.

Let ς be the $\mathbb{R}$-algebra automorphism of $\mathfrak{g}$ defined by $\varsigma(e_i) = -f_i$, $\varsigma(h_i) = -h_i$, and $\varsigma(\sqrt{-1}x) = -\sqrt{-1}\varsigma(x)$, $i \in I$, $x \in \mathfrak{g}$. Then, $\theta'\varsigma = \varsigma\theta'$ is an $\mathbb{R}$-algebra automorphism of $\mathfrak{g}$, and hence $\mathfrak{g}^{\theta'\varsigma} := \{x \in \mathfrak{g} \mid \theta'\varsigma(x) = x\}$ is an $\mathbb{R}$-subalgebra of $\mathfrak{g}$. Then, the assignment

$$(I_\bullet, \tau) \mapsto \mathfrak{g}^{\theta'\varsigma}$$

gives a bijection between the set of admissible pairs and the isomorphism classes of real semisimple Lie algebras.

3. Quantum groups

The quantum groups for general simple Lie algebras were first introduced independently by Drinfel'd [Dr85] and Jimbo [J85] in order to solve the quantum Yang-Baxter equation. In this section, we mainly follow the notation of Hong-Kang's book [HK02] and Lusztig's one [L93]. After giving the definition of quantum groups in Subsection 3.1, we state the classification theorem for irreducible modules over quantum groups in a suitable category in Subsection 3.2. Subsection 3.3 is devoted to formulating Lusztig's canonical bases. Subsections 3.4 and 3.5 are about Kashiwara's crystal basis theory. In Subsection 3.6, we explain how to construct canonical bases (equivalently, global crystal bases) of tensor product modules. In Subsection 3.7, we see how the quantum groups are related to the underlying Kac-Moody algebras.

3.1. **Definition.** First, we introduce the quantum integers and related notations.

Definition 3.1.1. Let q be an indeterminate.

(1) For $m \in \mathbb{Z}$, set $[m] := \frac{q^m - q^{-m}}{q - q^{-1}}$.
(2) For $m \in \mathbb{Z}_{\geq 0}$, set $[m]! := \prod_{i=1}^m [i]$; we understand $[0]! = 1$.
(3) For $m \geq n \in \mathbb{Z}_{\geq 0}$, set $\begin{bmatrix} m \\ n \end{bmatrix} := \frac{[m]!}{[n]![m-n]!}$.
(4) For $a \in \mathbb{Z}$, set $[m]_a$ to be the $[m]$ evaluated at $q = q^a$, and define $[m]_a!$ and $\begin{bmatrix} m \\ n \end{bmatrix}_a$ in a similar way.

Let A be a symmetrizable generalized Cartan matrix A with a symmetrizing matrix $D = \mathrm{diag}(d_i)_{i \in I}$ such that d_i's are coprime to each other. For each $i \in I$, set $q_i := q^{d_i}$, and

$$E_i^{(n)} := \frac{1}{[n]_{d_i}!} E_i^n, \quad F_i^{(n)} := \frac{1}{[n]_{d_i}!} F_i^n \quad (n \in \mathbb{Z}_{\geq 0}).$$

Definition 3.1.2. The quantum group $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$ is the unital associative algebra over $\mathbb{Q}(q)$ generated by E_i, F_i, $i \in I$ and K_h, $h \in P^\vee$ subject to the following relations; for $i, j \in I$ and $h, h' \in P^\vee$,

$$\begin{aligned}
&K_0 = 1,\\
&K_h K_{h'} = K_{h+h'},\\
&K_h E_i = q^{\langle h, \alpha_i \rangle} E_i K_h,\\
&K_h F_i = q^{-\langle h, \alpha_i \rangle} F_i K_h,\\
&E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{(we set } K_i := K_{d_i h_i}\text{)},\\
&\sum_{s=0}^{1-a_{i,j}} (-1)^s E_i^{(s)} E_j E_i^{(1-a_{i,j}-s)} = 0, \quad i \neq j,\\
&\sum_{s=0}^{1-a_{i,j}} (-1)^s F_i^{(s)} F_j F_i^{(1-a_{i,j}-s)} = 0, \quad i \neq j.
\end{aligned}$$

We often use the following automorphisms. That they are actually automorphisms is straightforwardly verified by using the presentation above.

Lemma 3.1.3. (1) *There exists a unique $\mathbb{Q}$-algebra automorphism $\overline{\cdot}$ on $U_q(\mathfrak{g})$ that sends E_i, F_i, K_h, q to E_i, F_i, K_{-h}, q^{-1}, respectively for all $i \in I$, $h \in P^\vee$. We call this automorphism the bar-involution on $U_q(\mathfrak{g})$.*

(2) *There exists a unique $\mathbb{Q}(q)$-algebra automorphism ω on $U_q(\mathfrak{g})$ that sends E_i, F_i, K_h to F_i, E_i, K_{-h}, respectively for all $i \in I$, $h \in P^\vee$.*

(3) *There exists a unique $\mathbb{Q}(q)$-algebra anti-automorphism σ on $U_q(\mathfrak{g})$ that sends E_i, F_i, K_h to F_i, E_i, K_h, respectively for all $i \in I$, $h \in P^\vee$.*

(4) *There exists a unique $\mathbb{Q}(q)$-algebra anti-automorphism ϱ on $U_q(\mathfrak{g})$ that sends E_i, F_i, K_h to $q_i F_i K_i^{-1}$, $q_i E_i K_i$, K_h, respectively for all $i \in I$, $h \in P^\vee$.*

Set $Q := \sum_{i \in I} \mathbb{Z}\alpha_i \subset P$, and call it the root lattice. Then, we have the root space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where

$$U_q(\mathfrak{g})_\alpha = \{x \in U_q(\mathfrak{g}) \mid K_h x K_h^{-1} = q^{\langle h, \alpha \rangle} x \text{ for all } h \in P^\vee\}.$$

3.2. Highest weight theory for $U_q(\mathfrak{g})$.

In the representation theory of $U_q(\mathfrak{g})$, highest weight theory is a basic and strong tool. We begin with the triangular decomposition of $U_q(\mathfrak{g})$. Let $U_q(\mathfrak{g})^+, U_q(\mathfrak{g})^0, U_q(\mathfrak{g})^-$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by E_i $(i \in I)$, K_h $(h \in P^\vee)$, F_i $(i \in I)$, respectively. Note that the subalgebra $U_q(\mathfrak{g})^0$ is different from the subspace $U_q(\mathfrak{g})_0$. Then, we have an isomorphism

$$U_q(\mathfrak{g})^- \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^+ \simeq U_q(\mathfrak{g}); \ f \otimes h \otimes e \mapsto fhe$$

of vector spaces. We call this isomorphism the triangular decomposition of $U_q(\mathfrak{g})$.

Definition 3.2.1. Let M be a $U_q(\mathfrak{g})$-module.

(1) For each $\lambda \in P$, the subspace $M_\lambda := \{m \in M \mid K_h m = q^{\langle h, \lambda \rangle} m \text{ for all } h \in P^\vee\}$ is called the λ-weight space of M.

(2) $\lambda \in P$ is said to be a weight of M if $M_\lambda \neq 0$. Let $\mathrm{Wt}(M)$ denote the set of weights of M.

(3) M is said to be a weight module if it is the direct sum of its weight spaces; $M = \bigoplus_{\lambda \in P} M_\lambda$.
(4) M is said to be a highest (resp., lowest) weight module with highest (resp., lowest) weight $\lambda \in P$ if it is generated by a nonzero vector $m_\lambda \in M_\lambda$ such that $E_i m = 0$ (resp., $F_i m = 0$) for all $i \in I$.
(5) M is said to have a bar-involution if there exists a $\mathbb{Q}$-linear involution $\overline{\cdot}$ on M such that $\overline{x \cdot m} = \overline{x} \cdot \overline{m}$ for all $x \in U_q(\mathfrak{g})$ and $m \in M$.

Note that by the triangular decomposition of $U_q(\mathfrak{g})$, each weight of a highest weight module with highest weight $\lambda \in P$ is less than or equal to λ.

Definition 3.2.2. Let $\lambda \in P$.

(1) Let $I(\lambda)$ be the left ideal of $U_q(\mathfrak{g})$ generated by E_i ($i \in I$) and $K_h - q^{\langle h, \lambda \rangle}$ ($h \in P^\vee$). Set
$$V(\lambda) := U_q(\mathfrak{g})/I(\lambda),$$
and call it the Verma module with highest weight λ.
(2) Let $L(\lambda)$ denote the unique irreducible quotient of $V(\lambda)$, and call it the irreducible highest weight module with highest weight λ.

Definition 3.2.3. Let $\mathcal{O}_{\text{int}}$ denote the category of $U_q(\mathfrak{g})$-modules M satisfying the following;

(1) M is a weight module.
(2) There exist $\mu_1, \dots, \mu_m \in P$ such that $\mathrm{Wt}(M) \subset \bigcup_{i=1}^m (\mu_i - Q_+)$.
(3) For each $i \in I$, F_i acts on M locally nilpotently, i.e., for each $m \in M$, there exists $N \in \mathbb{Z}_{>0}$ such that $F_i^N m = 0$.

Note that $V(\lambda)$ and $L(\lambda)$ are not necessarily objects of $\mathcal{O}_{\text{int}}$ (condition (3) may be false). The following are basic results about the category $\mathcal{O}_{\text{int}}$.

Theorem 3.2.4 (see e.g., [HK02]). (1) *The category $\mathcal{O}_{\text{int}}$ is semisimple, i.e., each $M \in \mathcal{O}_{\text{int}}$ is a direct sum of irreducible modules.*
(2) *$L(\lambda) \in \mathcal{O}_{\text{int}}$ if and only if $\lambda \in P_+$.*
(3) *$L(\lambda)$ is finite-dimensional if and only if $\lambda \in P_+$.*
(4) *If $\lambda \in P_+$, then $L(\lambda)$ is a lowest weight module with lowest weight $w_0(\lambda)$.*

3.3. **Canonical bases.** The notion of canonical bases (or, (lower) global crystal bases) for $U_q(\mathfrak{g})$ and its modules were introduced by Lusztig [L90a], [L91] and Kashiwara [K90], [K91] independently. In this subsection, we briefly recall Lusztig's formulation. For simplicity, we assume that our Cartan matrix is of type ADE.

First, we prepare the following.

- $\mathbf{A}_0 := \{f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 0\}$.
- $\mathbf{A} := \mathbb{Q}[q, q^{-1}]$.
- $\mathbf{A}_\infty := \{f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = \infty\}$.

Definition 3.3.1. Let V be a $\mathbb{Q}(q)$-vector space and $x \in \{0, \emptyset, \infty\}$. An $\mathbf{A}_x$-lattice of V is a free $\mathbf{A}_x$-submodule U_x of V such that $\mathbb{Q}(q) \otimes_{\mathbf{A}_x} U_x = V$.

Let $U_q(\mathfrak{g})_{\mathbf{A}}$ be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $E_i^{(n)}$, $F_i^{(n)}$, K_h with $i \in I$, $n \in \mathbb{Z}_{>0}$, $h \in P^\vee$. Also, define $U_q(\mathfrak{g})^{\pm}_{\mathbf{A}}$ to be the $\mathbf{A}$-subalgebra generated by $E_i^{(n)}$ (resp., $F_i^{(n)}$), $i \in I$, $n \in \mathbb{Z}_{>0}$.

Next, we need Lusztig's braid group action on $U_q(\mathfrak{g})$.

Proposition 3.3.2 ([L93, Chapter 37]). *Let $i \in I$ and $e \in \{1, -1\}$. Then, there exist unique algebra automorphisms $T'_{i,e}$ and $T''_{i,e}$ on $U_q(\mathfrak{g})$ satisfying the following: For $i, j \in I$, $h \in P^\vee$,*

$$T'_{i,e}(E_j) = \begin{cases} -K_i^e F_i & \text{if } j = i, \\ \sum_{r+s=-a_{i,j}} (-1)^r q_i^{er} E_i^{(r)} E_j E_i^{(s)} & \text{if } j \neq i \end{cases}$$

$$T'_{i,e}(F_j) = \begin{cases} -E_i K_i^{-e} & \text{if } j = i, \\ \sum_{r+s=-a_{i,j}} (-1)^r q_i^{-er} F_i^{(s)} F_j F_i^{(r)} & \text{if } j \neq i, \end{cases}$$

$$T''_{i,-e}(E_j) = \begin{cases} -F_i K_i^{-e} & \text{if } j = i, \\ \sum_{r+s=-a_{i,j}} (-1)^r q_i^{er} E_i^{(s)} E_j E_i^{(s)} & \text{if } j \neq i, \end{cases}$$

$$T''_{i,-e}(F_j) = \begin{cases} -K_i^e E_i & \text{if } j = i, \\ \sum_{r+s=-a_{i,j}} (-1)^r q_i^{-er} F_i^{(r)} F_j F_i^{(s)} & \text{if } j \neq i, \end{cases}$$

$$T'_{i,e}(K_h) = T''_{i,-e}(K_h) = K_{s_i(h)}.$$

Moreover, the families $\{T'_{i,e} \mid i \in I\}$ and $\{T''_{i,e} \mid i \in I\}$ satisfy the braid group relation, i.e., we have for $i \neq j \in I$,

$$T'_{i,e} T'_{j,e} T'_{i,e} \cdots = T'_{j,e} T'_{i,e} T'_{j,e} \cdots \quad (\textit{both sides have } m_{i,j} \textit{ factors}),$$

and similar relations for $T''_{i,e}$'s.

In the sequel, we set $T_i := T'_{i,1}$. Thanks to the braid relation, for each reduced expression $w = s_{i_1} \cdots s_{i_l} \in W$, the composite

$$T_w := T_{i_1} \cdots T_{i_l}$$

is well-defined.

Let $w_0 \in W$ denote the longest element, and fix a reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N)$ for w_0 (i.e., $w_0 = s_{i_1} \cdots s_{i_N}$ is a reduced expression). For each $k = 2, \dots, N$, set

$$E(\mathbf{i})_1 := E_{i_1}, \quad E(\mathbf{i})_k := T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}).$$

Also, for each $\mathbf{c} = (c_1, c_2, \dots, c_N) \in \mathbb{Z}_{\geq 0}^N$, set

$$E(\mathbf{i}; \mathbf{c}) := E(\mathbf{i})_1^{(c_1)} \cdots E(\mathbf{i})_N^{(c_N)}.$$

Example 3.3.3. Suppose that our Cartan datum is the $\mathfrak{sl}_3$-type. In this case, there are only two reduced words for w_0; $\mathbf{i}_1 := (1, 2, 1)$ and $\mathbf{i}_2 := (2, 1, 2)$. We have

$$E(\mathbf{i}_1)_1 = E_1, \quad E(\mathbf{i}_1)_2 = E_2 E_1 - q E_1 E_2, \quad E(\mathbf{i}_1)_3 = E_2,$$
$$E(\mathbf{i}_2)_1 = E_2, \quad E(\mathbf{i}_2)_2 = E_1 E_2 - q E_2 E_1, \quad E(\mathbf{i}_2)_3 = E_1.$$

Note that $E(\mathbf{i}_1)_2$ and $E(\mathbf{i}_2)_2$ are not proportional, while they are in their classical limits.

Theorem 3.3.4 ([L90a, Proposition 2.3]). *Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for w_0.*

1. *$B(\mathbf{i}) := \{E(\mathbf{i}; \mathbf{c}) \mid \mathbf{c} \in \mathbb{Z}_{\geq 0}^N\}$ is an $\mathbf{A}$-basis of $U_q(\mathfrak{g})_{\mathbf{A}}^+$.*
2. *The $\mathbf{A}_0$-subspace $\mathcal{L}$ spanned by $B(\mathbf{i})$ is independent of the choice of $\mathbf{i}$.*
3. *The image $\mathcal{B}$ of $B(\mathbf{i})$ under the projection $\mathcal{L} \to \mathcal{L}/q\mathcal{L}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$, and it is independent of the choice of $\mathbf{i}$.*

Definition 3.3.5. We call $B(\mathbf{i})$ and $\mathcal{B}$ the Poincaré-Birkhoff-Witt-type basis and the crystal basis of $U_q(\mathfrak{g})_{\mathbf{A}}^+$, respectively.

Theorem 3.3.6 ([L90a, Theorem 3.2])**.** *The restriction of the projection* $\pi : \mathcal{L} \to \mathcal{L}/q\mathcal{L}$ *to the* $\mathbb{Q}$*-subspace* $\mathcal{L} \cap U_q(\mathfrak{g})^+_{\mathbf{A}} \cap \overline{\mathcal{L}}$ *is an isomorphism of* $\mathbb{Q}$*-vector spaces. In particular, the inverse image* $B := \pi^{-1}(\mathcal{B})$ *of the crystal basis* $\mathcal{B}$ *is an* $\mathbf{A}_0$*-basis of* $\mathcal{L}$*, an* $\mathbf{A}$*-basis of* $U_q(\mathfrak{g})^+_{\mathbf{A}}$*, and an* $\mathbf{A}_\infty$*-basis of* $\overline{\mathcal{L}}$*.*

Definition 3.3.7. B is called the canonical basis (or, lower global crystal basis) of $U_q(\mathfrak{g})^+$.

Let $\lambda \in P_+$, and consider the irreducible highest weight module $L(\lambda)$ with highest weight λ. Fix a lowest weight vector $v_{w_0(\lambda)} \in L(\lambda)_{w_0(\lambda)}$. Then, there exists a unique bar-involution on $L(\lambda)$ fixing $v_{w_0(\lambda)}$.

Theorem 3.3.8 ([L90a, Theorem 8.10])**.** *Let* $\lambda \in P_+$*.*

1. $B(\lambda) := \{bv_{w_0(\lambda)} \mid b \in B\} \setminus \{0\}$ *is an* $\mathbf{A}$*-basis of* $L(\lambda)_{\mathbf{A}} := U_q(\mathfrak{g})^+_{\mathbf{A}} v_{w_0(\lambda)}$*.*
2. *Let* $\mathcal{L}(\lambda)$ *be the* $\mathbf{A}_0$*-subspace of* $L(\lambda)$ *spanned by* $B(\lambda)$*. Then, the image* $\mathcal{B}(\lambda)$ *of* $B(\lambda)$ *under the projection* $\mathcal{L}(\lambda) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ *is a* $\mathbb{Q}$*-basis of* $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$*.*
3. *The restriction of* π *to the* $\mathbb{Q}$*-subspace* $\mathcal{L} \cap L(\lambda)_{\mathbf{A}} \cap \overline{\mathcal{L}}$ *is an isomorphism of* $\mathbb{Q}$*-vector spaces, and it maps* $B(\lambda)$ *to* $\mathcal{B}(\lambda)$ *bijectively.*

Definition 3.3.9. $B(\lambda)$ is called the canonical basis of $L(\lambda)$.

3.4. **Crystal bases.** Here, we introduce Kashiwara's crystal bases for modules in $\mathcal{O}_{\mathrm{int}}$. A key fact is that the subalgebra of $U_q(\mathfrak{g})$ generated by E_i, K_i, F_i is isomorphic to $U_q(\mathfrak{sl}_2)$ for all $i \in I$. Such a triple (E_i, K_i, F_i) is called an $\mathfrak{sl}_2$-triple.

Let $M \in \mathcal{O}_{\mathrm{int}}$. For each $i \in I$ and $m \in M_\lambda$ ($\lambda \in P$), there exist unique $N \in \mathbb{Z}_{\geq 0}$, $m_k \in M_{\lambda + k\alpha_i} \cap \operatorname{Ker} E_i$ ($k = 0, 1, \dots, N$) such that

$$m = \sum_{k=0}^{N} F_i^{(k)} m_k.$$

With this expression, we set

$$\widetilde{E}_i m := \sum_{k=1}^{N} F_i^{(k-1)} m_k, \quad \widetilde{F}_i m := \sum_{k=0}^{N} F_i^{(k+1)} m_k.$$

This defines linear maps $\widetilde{E}_i, \widetilde{F}_i : M \to M$, called the Kashiwara operators on M. The following is clear from the definition.

Proposition 3.4.1. *The Kashiwara operators commute with the* $U_q(\mathfrak{g})$*-homomorphisms.*

Definition 3.4.2. A crystal lattice of M is an $\mathbf{A}_0$-lattice $\mathcal{L}$ of M satisfying the following:

1. $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda := \mathcal{L} \cap M_\lambda$.
2. $\mathcal{L}$ is closed under the Kashiwara operators.

Let $\mathcal{L}$ be a crystal lattice of M. Since the Kashiwara operators preserve $\mathcal{L}$, it induces $\mathbb{Q}$-linear endomorphisms on $\mathcal{L}/q\mathcal{L}$; we denote them by the same symbols.

Definition 3.4.3. A crystal basis of M is a pair $(\mathcal{L}, \mathcal{B})$ of a crystal lattice $\mathcal{L}$ of M and a $\mathbb{Q}$-basis $\mathcal{B}$ of $\mathcal{L}/q\mathcal{L}$ satisfying the following:

1. $\mathcal{B} = \bigsqcup_{\lambda \in \Lambda} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda := \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$.
2. $\widetilde{E}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ and $\widetilde{F}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ for all $i \in I$.
3. For each $b, b' \in \mathcal{B}$ and $i \in I$, one has $\widetilde{f}_i b = b'$ if and only if $b = \widetilde{e}_i b'$.

Let $(\mathcal{L},\mathcal{B})$ be a crystal basis of M. We define maps $\varepsilon_i:\mathcal{B}\to\mathbb{Z}_{\geq 0}$, $\varphi_i:\mathcal{B}\to\mathbb{Z}_{\geq 0}$ $(i\in I)$, and $\mathrm{wt}:\mathcal{B}\to\Lambda$ by

$$\varepsilon_i(b):=\max\{k\mid \widetilde{E}_i^k b\neq 0\},\ \varphi_i(b):=\max\{k\mid \widetilde{F}_i^k b\neq 0\},\ \mathrm{wt}(b):=\lambda \text{ if } b\in\mathcal{B}_\lambda.$$

Remark 3.4.4. Let $(\mathcal{L},\mathcal{B})$ be a crystal basis. Then, $(\mathcal{B},\widetilde{E}_i,\widetilde{F}_i,\mathrm{wt})$ is a crystal of type our Cartan datum. In what follows, we always regard $\mathcal{B}$ as a crystal. Historically, the notion of crystals was introduced by axiomatizing the crystal bases.

Definition 3.4.5. Let $M,M'\in\mathcal{O}$, and $(\mathcal{L},\mathcal{B}),(\mathcal{L}',\mathcal{B}')$ crystal bases of M,M', respectively. We say $(\mathcal{L},\mathcal{B})$ and $(\mathcal{L}',\mathcal{B}')$ are isomorphic if there exists an isomorphism $M\to M'$ of $U_q(\mathfrak{g})$-modules which restricts to an isomorphism $\mathcal{L}\to\mathcal{L}'$ of $\mathbf{A}_0$-modules, and induces a bijection $\mathcal{B}\to\mathcal{B}'$ of sets.

Clearly, an isomorphism of crystal bases induces an isomorphism of crystals. The following is the existence and uniqueness theorem for crystal bases of finite-dimensional irreducible modules.

Theorem 3.4.6 ([K91, Theorem 2])**.** *Let $\lambda\in P_+$, $v_\lambda\in L(\lambda)$ a highest weight vector. Set*

$$\mathcal{L}(\lambda):=\mathrm{Span}_{\mathbf{A}_0}\{\widetilde{f}_{i_1}\cdots\widetilde{f}_{i_l}v_\lambda\mid l\in\mathbb{Z}_{\geq 0},\ i_1,\dots,i_l\in I\},$$
$$\mathcal{B}(\lambda):=\{\widetilde{f}_{i_1}\cdots\widetilde{f}_{i_l}v_\lambda+q\mathcal{L}(\lambda)\mid l\in\mathbb{Z}_{\geq 0},\ i_1,\dots,i_l\in I\}\setminus\{0\}.$$

Then, $(\mathcal{L}(\lambda),\mathcal{B}(\lambda))$ is a unique (up to isomorphism) crystal basis of $L(\lambda)$.

When our Cartan matrix is of type ABCD, there is a combinatorial realization of the crystal bases of irreducible modules in $\mathcal{O}_{\mathrm{int}}$ ([KN94]). In particular, we have the following.

Theorem 3.4.7. *When our Cartan datum is the $\mathfrak{sl}_n$-type (in this case, P_+ is identified with Par_{n-1}; see Example 1.1.10), we have an unique isomorphism $\mathcal{B}(\lambda)\simeq\mathrm{SST}_n(\lambda)$ of crystals.*

Remark 3.4.8. For $\lambda,\mu\in\mathrm{Par}_n$, $\mathrm{SST}_n(\lambda)\simeq\mathrm{SST}_n(\mu)$ if and only if $\lambda_i-\mu_i$ are constant (see Proposition 1.6.2 (4)). Hence, it is convenient to set $L(\lambda):=L(\hat{\lambda})$, $\mathcal{L}(\lambda):=\mathcal{L}(\hat{\lambda})$, $\mathcal{B}(\lambda):=\mathcal{B}(\hat{\lambda})$, where

$$\hat{\lambda}:=(\lambda_1-\lambda_n,\lambda_2-\lambda_n,\dots,\lambda_{n-1}-\lambda_n)\in\mathrm{Par}_{n-1}.$$

The following is the existence and uniqueness theorem for crystal bases of modules in $\mathcal{O}_{\mathrm{int}}$.

Theorem 3.4.9. *Let $M\in\mathcal{O}_{\mathrm{int}}$. By the complete reducibility, we have $M\simeq\bigoplus_{k=1}^N L(\lambda_k)$ for some $\lambda_1,\dots,\lambda_N\in P_+$.*

1. *Let $\psi:M\to\bigoplus_{k=1}^N L(\lambda_k)$ be an isomorphism of $U_q(\mathfrak{g})$-modules. Then,*
$$\left(\psi^{-1}\left(\bigoplus_{k=1}^N\mathcal{L}(\lambda_k)\right),\psi^{-1}\left(\bigsqcup_{k=1}^N\mathcal{B}(\lambda_k)\right)\right)$$
is a crystal basis of M.
2. *Let $(\mathcal{L},\mathcal{B})$ be a crystal basis of M. Then, it is isomorphic to $\left(\bigoplus_{k=1}^N\mathcal{L}(\lambda_k),\bigsqcup_{k=1}^N\mathcal{B}(\lambda_k)\right)$.*

Remark 3.4.10. Suppose that our Cartan datum is the $\mathfrak{sl}_{m+n}$-type. Let $M\in\mathcal{O}_{\mathrm{int}}$, and $(\mathcal{L},\mathcal{B})$ a crystal basis. Let $U_q(\mathfrak{l})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by E_i,F_i,K_h, $i\in[1,m+n-1]\setminus\{m\}$, $h\in P^\vee$. Then, $(\mathcal{L},\mathcal{B})$ is a crystal basis of M regarded as a $U_q(\mathfrak{l})$-module; the crystal structure is the same as $\mathrm{Res}^{m+n}_{m,n}(\mathcal{B})$.

3.5. **Global crystal bases.** In this subsection, we explain how to "globalize" crystal bases in order to get genuine bases of modules, and recall basic properties of global crystal bases.

Definition 3.5.1 ([K93a, Definition 2.1.2])**.** Let V be a $\mathbb{Q}(q)$-vector space, U_x an $\mathbf{A}_x$-lattice of V for $x \in \{0, \emptyset, \infty\}$. The triple (U_0, U, U_∞) is said to be balanced if the canonical map

$$U_0 \cap U \cap U_\infty \to U_0/qU_0; \ u \mapsto u + qU_0$$

is an isomorhism of $\mathbb{Q}$-vector spaces.

Let V be a $\mathbb{Q}(q)$-vector space with a balanced triple (U_0, U, U_∞). Take a $\mathbb{Q}$-basis $\mathcal{B}$ of U_0/qU_0. Since we have an isomorphism $G : U_0/qU_0 \to U_0 \cap U \cap U_\infty$ of $\mathbb{Q}$-vector spaces, which is the inverse of the canonical map $U_0 \cap U \cap U_\infty \to U_0/qU_0$, we obtain an $\mathbf{A}_x$-basis $G(\mathcal{B}) = \{G(b) \mid b \in \mathcal{B}\}$ of U_x for each $x \in \{0, \emptyset, \infty\}$. We call $G(\mathcal{B})$ the global basis of V associated to the balanced triple (U_0, U, U_∞) and the basis $\mathcal{B}$.

Lemma 3.5.2. *Let $V, U_0, U, U_\infty, \mathcal{B}, G$ be as above. Take a subset $\mathcal{B}' \subset \mathcal{B}$ and set U'_x to be the $\mathbf{A}_x$-span of $G(\mathcal{B}') := \{G(b) \mid b \in \mathcal{B}'\}$ for each $x \in \{0, \emptyset, \infty\}$. Also, let V' be the $\mathbb{Q}(q)$-span of $G(\mathcal{B}')$. Then, the following hold:*

1. *(U'_0, U', U'_∞) is a balanced triple with the global basis $G(\mathcal{B}')$.*
2. *$(U_0/U'_0, U/U', U_\infty/U'_\infty)$ is a balanced triple with the global basis $\{G(b) + V' \mid b \in \mathcal{B} \setminus \mathcal{B}'\}$.*

Let $V \in \mathcal{O}_{\mathrm{int}}$, and $(\mathcal{L}, \mathcal{B})$ its crystal basis. Suppose that V admits a bar-involution. Note that $\overline{\mathcal{L}}$ is an $\mathbf{A}_\infty$-lattice of V.

Definition 3.5.3. Let $V, \mathcal{L}, \mathcal{B}$ be as above. We say that V has a global crystal basis if there exists a $\mathbf{U}_\mathbf{A}$-submodule $V_\mathbf{A}$ of V which is an $\mathbf{A}$-lattice forming a balanced triple $(\mathcal{L}, V_\mathbf{A}, \overline{\mathcal{L}})$. The associated global basis $G(\mathcal{B})$ is called a global crystal basis of V.

Example 3.5.4. Let $\lambda \in P_+$. Then, the canonical basis $B(\lambda)$ of $L(\lambda)$ is the global crystal basis associated with the balanced triple $(\mathcal{L}(\lambda), L(\lambda)_\mathbf{A}, \overline{\mathcal{L}(\lambda)})$. When we emphasize the crystal structure of $\mathcal{B}(\lambda)$, we call $B(\lambda)$ the global crystal basis of $L(\lambda)$, rather than the canonical basis.

Let $M \in \mathcal{O}_{\mathrm{int}}$ be a $U_q(\mathfrak{g})$-module with a bar-involution, $(\mathcal{L}, \mathcal{B})$ a crystal basis of M, and $M_\mathbf{A}$ a $\mathbf{U}_\mathbf{A}$-submodule of M. Suppose that M has a global crystal basis $G(\mathcal{B})$ with the associated balanced triple $(\mathcal{L}, M_\mathbf{A}, \overline{\mathcal{L}})$.

Proposition 3.5.5 ([K93a])**.** *Let $i \in I$, $b \in B$ and $m \in \mathbb{Z}_{\geq 0}$. Then, we have the following.*

1. $\sum_{n \geq m} F_i^{(n)} M_\mathbf{A} = \bigoplus_{\substack{b' \in \mathcal{B} \\ \varepsilon_i(b') \geq m}} \mathbf{A} G^\jmath(b')$.
2. $\sum_{n \geq m} E_i^{(n)} M_\mathbf{A} = \bigoplus_{\substack{b' \in \mathcal{B} \\ \varphi_i(b') \geq m}} \mathbf{A} G^\jmath(b')$.
3. $F_i G^\jmath(b) = [\varepsilon_i(b)+1]_{d_i} G^\jmath(\widetilde{F}_i b) + \sum_{\substack{b' \in \mathcal{B} \\ \varepsilon_i(b') > \varepsilon_i(b)+1}} \varphi^{(i)}_{b',b} G^\jmath(b')$ *for some* $\varphi^{(i)}_{b',b} \in q_i^{2-\varepsilon_i(b')}\mathbb{Q}[q]$.
4. $E_i G^\jmath(b) = [\varphi_i(b)+1]_{d_i} G^\jmath(\widetilde{E}_i b) + \sum_{\substack{b' \in \mathcal{B} \\ \varphi_i(b') > \varphi_i(b)+1}} \varepsilon^{(i)}_{b',b} G^\jmath(b')$ *for some* $\varepsilon^{(i)}_{b',b} \in q_i^{2-\varphi_i(b')}\mathbb{Q}[q]$.

For $\lambda \in P_+$, set $I_\lambda(M)$ to be the sum of submodules of M isomorphic to $L(\lambda)$. Also, we set

$$\begin{aligned} W_{\succeq\lambda}(M) &:= \sum_{\mu \succeq \lambda} I_\mu(M), \\ W_{\succ\lambda}(M) &:= \sum_{\mu \succ \lambda} I_\mu(M), \\ W_\lambda(M) &:= W_{\succeq\lambda}(M)/W_{\succ\lambda}(M). \end{aligned}$$

Recall that $B = \bigsqcup_{k=1}^N \mathcal{B}_k$, $\mathcal{B}_k \simeq \mathcal{B}(\lambda_k)$ for some $\lambda_k \in P_+$. For $b \in \mathcal{B}$, set $I(b) := \lambda_k$ if $b \in \mathcal{B}_k$.

Theorem 3.5.6 ([K93a], [L93]). *Let $M, \mathcal{L}, \mathcal{B}, M_{\mathbf{A}}$ be as above. Then, for each $\lambda \in P_+$, the following hold:*

1. *$W_{\succeq\lambda}(M)$ has a global crystal basis $W_{\succeq\lambda}(G(\mathcal{B})) := \{G(b) \mid I(b) \succeq \lambda\}$ with the associated balanced triple $(W_{\succeq\lambda}(\mathcal{L}), W_{\succeq\lambda}(M_{\mathbf{A}}), W_{\succeq\lambda}(\overline{\mathcal{L}}))$, where $W_{\succeq\lambda}(\mathcal{L}) := W_{\succeq\lambda}(M) \cap \mathcal{L}$, and so on.*
2. *$W_{\succ\lambda}(M)$ has a global crystal basis $W_{\succ\lambda}(G(\mathcal{B})) := \{G(b) \mid I(b) \succ \lambda\}$ with the associated balanced triple $(W_{\succ\lambda}(\mathcal{L}), W_{\succ\lambda}(M_{\mathbf{A}}), W_{\succ\lambda}(\overline{\mathcal{L}}))$, where $W_{\succ\lambda}(\mathcal{L}) := W_{\succ\lambda}(M) \cap \mathcal{L}$, and so on.*
3. *$W_\lambda(M)$ has a global crystal basis $W_\lambda(G(\mathcal{B})) := \{G(b) + W_{\succ\lambda}(M) \mid I(b) = \lambda\}$ with the associated balanced triple $(W_\lambda(\mathcal{L}), W_\lambda(M_{\mathbf{A}}), W_\lambda(\overline{\mathcal{L}}))$, where $W_\lambda(\mathcal{L}) := W_{\succeq\lambda}(\mathcal{L})/W_{\succ\lambda}(\mathcal{L})$, and so on.*
4. *There exists a $\mathbf{U}$-module isomorphism $\xi : L(\lambda)^{\oplus m_\lambda} \to W_\lambda(M)$ which induces an isomorphism*
$$(\mathcal{L}(\lambda)^{\oplus m_\lambda}, (L(\lambda)_{\mathbf{A}})^{\oplus m_\lambda}, \overline{\mathcal{L}(\lambda)}^{\oplus m_\lambda}) \simeq (W_\lambda(\mathcal{L}), W_\lambda(M_{\mathbf{A}}), W_\lambda(\overline{\mathcal{L}})),$$
where $m_\lambda := \dim \mathrm{Hom}_{\mathbf{U}}(L(\lambda), M)$ denotes the multiplicity of $L(\lambda)$ in M.

Corollary 3.5.7. *For each $\lambda \in P_+$, the irreducible highest weight module $L(\lambda)$ has a unique (up to a scalar multiple) global crystal basis.*

3.6. **Tensor product modules.** The quantum group $U_q(\mathfrak{g})$ has a coassociative Hopf algebra structure with the structure maps defined as follows.

- The comultiplication $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$
$$\Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \ \Delta(F_i) = F \otimes 1 + K_i \otimes F_i, \ \Delta(K_h) = K_h \otimes K_h.$$
- The counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{Q}(q)$
$$\epsilon(E_i) = 0, \quad \epsilon(F_i) = 0, \quad \epsilon(K_h) = 1.$$
- The antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$.
$$S(E_i) = -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i, \quad S(K_i) = K_i^{-1}.$$

Given finitely many $U_q(\mathfrak{g})$-modules $M_1, \dots, M_m$, one can consider the tensor product module $M_1 \otimes \cdots \otimes M_m$ by means of Δ, which is well-defined by the coassociativity of Δ.

The following gives a representation theoretical meaning to the tensor product rule for crystals.

Proposition 3.6.1 ([K91, Theorem 1]). *Let $M, M' \in \mathcal{O}_{\mathrm{int}}$, and $(\mathcal{L}, \mathcal{B}), (\mathcal{L}', \mathcal{B}')$ crystal bases. Then, $(\mathcal{L} \otimes_{\mathbf{A}_0} \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$ is a crystal basis of $M \otimes M'$. (Note that $\mathcal{B} \otimes \mathcal{B}'$ is the tensor product of crystals $\mathcal{B}, \mathcal{B}'$; see Definition 1.3.4)*

Example 3.6.2. Suppose that our Cartan datum is the $\mathfrak{sl}_n$-type, and consider the vector representation $\mathbf{V}$; i.e., $\mathbf{V} = \bigoplus_{i=1}^n \mathbb{Q}(q)u_i$, on which $U_q(\mathfrak{sl}_n)$ acts as follows;

$$\begin{aligned} K_i u_j &= q^{\delta_{i,j}-\delta_{i+1,j}} u_j, \\ E_i u_j &= \delta_{i+1,j} u_i, \\ F_i u_j &= \delta_{i,j} u_{i+1}. \end{aligned}$$

Set $\mathbf{L} := \bigoplus_{i=1}^n \mathbf{A}_0 u_i$, and $\mathbf{B} := \{u_i + q\mathbf{L} \mid i \in [1,n]\}$. Then, it is easily checked that $(\mathbf{L}, \mathbf{B})$ is a crystal basis of $\mathbf{V}$. Also, the assignment $\mathbf{B} \to [1,n];\ u_i + q\mathbf{L} \mapsto i$ is an isomorphism of crystals. Consequently, we have an isomorphism

$$\mathbf{B}^{\otimes d} \to [1,n]^d;\ u_{i_1} \otimes \cdots \otimes u_{i_d} + q\mathbf{L}^{\otimes d} \mapsto (i_1, \dots, i_d)$$

of crystals.

Let M and N be $U_q(\mathfrak{g})$-modules with bar-involutions. Then, the involution $(\overline{\cdot} \otimes \overline{\cdot})$ on $M \otimes N$ is not a bar-involution on $M \otimes N$, in general. This is because $\Delta \circ \overline{\cdot} \neq (\overline{\cdot} \otimes \overline{\cdot}) \circ \Delta$ on $U_q(\mathfrak{g})$. The difference between these two maps is described by Lusztig's quasi-R-matrix.

Theorem 3.6.3 ([L93, Theorem 4.1.2])**.** *For each $\mu \in Q_+$, there exists a unique $\Theta_\mu \in U_q(\mathfrak{g})^+_\mu \otimes U_q(\mathfrak{g})^-_{-\mu}$ satisfying the following.*

1. $\Theta_0 = 1 \otimes 1$.
2. *Set $\Theta := \sum_{\mu \in Q_+} \Theta_\mu$. Then, we have*

$$\Theta \cdot (\overline{\cdot} \otimes \overline{\cdot}) \circ \Delta(x) = \Delta(\overline{x}) \cdot \Theta, \quad \textit{for all } x \in U_q(\mathfrak{g}).$$

Definition 3.6.4. We call Θ the quasi-R-matrix of $U_q(\mathfrak{g})$.

By using the quasi-R-matrix Θ, one can construct a bar-involution on $M \otimes N$.

Proposition 3.6.5 ([L93, Lemma 24.1.2])**.** *Let M, N be $U_q(\mathfrak{g})$-modules in $\mathcal{O}_{\mathrm{int}}$ with bar-involutions. Then, the automorphism $\overline{\cdot}$ on $M \otimes N$ defined by*

$$\overline{m \otimes n} := \Theta(\overline{m} \otimes \overline{n}), \quad m \in M,\ n \in N$$

is a bar-involution on the tensor product module $M \otimes N$.

Proposition 3.6.6. *Let M, N be as before. Assume that M and N have global crystal bases B_M and B_N whose underlying crystal basis are $(\mathcal{L}_M, \mathcal{B}_M)$ and $(\mathcal{L}_N, \mathcal{B}_N)$, respectively. Then, for each $b_1 \in B_M$ and $b_2 \in B_N$, there exists a unique $b_1 \diamond b_2 \in M \otimes N$ such that $\overline{b_1 \diamond b_2} = b_1 \diamond b_2$ and*

$$b_1 \diamond b_2 \in b_1 \otimes b_2 + \sum_{\substack{b_1' \in B_M,\ b_2' \in B_N \\ \mathrm{wt}(b_1') > \mathrm{wt}(b_1),\ \mathrm{wt}(b_2') < \mathrm{wt}(b_2)}} a_{b_1', b_2'; b_1, b_2} b_1' \otimes b_2' \quad \textit{for some } a_{b_1', b_2'; b_1, b_2} \in q\mathbb{Q}[q].$$

In particular, $B_M \diamond B_N := \{b_1 \diamond b_2 \mid b_1 \in B_M,\ b_2 \in B_N\}$ is a global crystal basis of $M \otimes N$ whose underlying crystal basis is $(\mathcal{L}_M \otimes \mathcal{L}_N, \mathcal{B}_M \otimes \mathcal{B}_N)$.

Proof. This is because $\Theta_0 = 1 \otimes 1$, and $\Theta_\mu \in U_q(\mathfrak{g})^+_\mu \otimes U_q(\mathfrak{g})^-_{-\mu}$ for all $\mu \in Q_+$. □

Remark 3.6.7. The quasi-R-matrix Θ is also used to construct the universal R-matrix.

3.7. **Classical limit.** In this subsection, we briefly review technique to take a classical limit. Set

$$\mathbf{A}_1 := \{f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 1\}.$$

For $a, n \in \mathbb{Z}$ and $h \in P$, set

$$(K_h; n)_a := \frac{K_h q^{an} - 1}{q^a - 1}.$$

Let $U_q(\mathfrak{g})_{\mathbf{A}_1}$ be the $\mathbf{A}_1$-subalgebra of $U_q(\mathfrak{g})$ generated by E_i, F_i, $i \in I$ and $(K_h; 0)_1$, $h \in P^\vee$. Then, $U_q(\mathfrak{g})_{\mathbf{A}_1} \otimes_{\mathbf{A}_1} \mathbb{Q}$ becomes a $\mathbb{Q}$-algebra, where $\mathbb{Q}$ is equipped with an $\mathbf{A}_1$-algebra structure via the homomorphism $\mathbf{A}_1 \to \mathbb{Q};\ f(q) \mapsto f(1)$.

Theorem 3.7.1 ([HK02, Theorem 3.4.9])**.** *There exists a unique isomorphism* $U_q(\mathfrak{g})_{\mathbf{A}_1} \otimes_{\mathbf{A}_1} \mathbb{Q} \to U_{\mathbb{Q}}(\mathfrak{g})$ *of Hopf algebras that sends* $E_i \otimes 1, F_i \otimes 1, (K_h; 0)_1 \otimes 1$ *to* e_i, f_i, h*, respectively for all* $i \in I$*,* $h \in P^\vee$*.*

Definition 3.7.2. Let B be a $\mathbb{Q}(q)$-subspace of $U_q(\mathfrak{g})$. The classical limit of B is the subspace $(B \cap U_q(\mathfrak{g})_{\mathbf{A}_1}) \otimes_{\mathbf{A}_1} \mathbb{Q}$ of $U_{\mathbb{Q}}(\mathfrak{g})$.

4. Quantum symmetric pairs

In this section, we "quantize" symmetric pairs to obtain quantum symmetric pairs. In Subsection 4.1, we define quantum symmetric pairs following [BW18a]. In Subsection 4.2, we define the intertwiners (or, the quasi-K-matrices). They are used to construct the bar-involution of $\mathbf{U}^\imath$-modules, and $\imath$-canonical bases. In Subsection 4.3, we construct the Poincaré-Birkhoff-Witt-type basis of $\mathbf{U}^\imath$. This kind of basis was first considered in [W17] in type AIII/AIV.

4.1. **Coideal subalgebra $\mathbf{U}^\imath$.** Let $(I_\bullet, \tau)$ be an admissible pair. The Lie algebra involution $\widetilde{\tau}$ on $\mathfrak{g}(A)$ induces a $\mathbb{Q}(q)$-algebra involution τ_q on $U_q(\mathfrak{g})$ in an obvious way;

$$\tau_q(E_i) = E_{\tau(i)},\ \tau_q(F_i) = F_{\tau(i)},\ \tau_q(K_h) = K_{\widetilde{\tau}(h)}, \quad i \in I,\ h \in P^\vee.$$

Set

$$\theta_q := T_{w_\bullet} \circ \tau_q \circ \omega \in \operatorname{Aut}(U_q(\mathfrak{g})).$$

Definition 4.1.1 ([BW18a, Definition 3.5])**.** Let $\zeta_i \in \pm q^{\mathbb{Z}}$, $\kappa_i \in \mathbb{Z}[q, q^{-1}]$ for $i \in I_\circ$ be such that

1. $\kappa_i = 0$ unless $\tau(i) = i$, $\langle h_i, \alpha_j \rangle = 0$ for all $j \in I_\bullet$, and $\langle h_k, \alpha_i \rangle \in 2\mathbb{Z}$ for all $k \in I_\circ$ such that $\tau(k) = k$ and $\langle h_k, \alpha_j \rangle = 0$ for all $j \in I_\bullet$.
2. $\overline{\kappa_i} = \kappa_i$.
3. $\zeta_i = \zeta_{\tau(i)}$ if $\langle \theta(h_i), \alpha_i \rangle = 0$.
4. $\zeta_i \zeta_{\tau(i)} = (-1)^{\langle 2\rho_\bullet^\vee, \alpha_i \rangle} q_i^{-\langle h_i, 2\rho_\bullet + w_\bullet(\alpha_{\tau i}) \rangle}$.

The algebra $\mathbf{U}^\imath$ with parameters ζ_i, κ_i, $i \in I_\circ$ is the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by the following elements:

$$E_i + \zeta_i T_{w_\bullet}(F_{\tau(i)}) K_i^{-1} + \kappa_i K_i^{-1},\ i \in I_\circ, \quad E_i, F_i,\ i \in I_\bullet, \quad K_h,\ h \in P_\imath^\vee.$$

For each $i \in I$, set

$$B_i := \begin{cases} E_i + \zeta_i T_{w_\bullet}(F_{\tau(i)}) K_i^{-1} + \kappa_i K_i^{-1} & \text{if } i \in I_\circ, \\ E_i & \text{if } i \in I_\bullet. \end{cases}$$

Remark 4.1.2. When $(I_\bullet, \tau) = (I, -w_0)$, the $\mathbf{U}^\imath$ is nothing but $U_q(\mathfrak{g})$. Hence, the quantum symmetric pairs are thought of as generalizations of the quantum groups.

Remark 4.1.3. The algebra $\mathbf{U}^\imath$ is a quantum analog of $U_{\mathbb{Q}}(\mathfrak{g}^\theta)$, i.e., the classical limit of $\mathbf{U}^\imath$ is isomorphic to $U_{\mathbb{Q}}(\mathfrak{g}^\theta)$ (see Theorem 4.1.5 below). However, $\mathbf{U}^\imath$ is not isomorphic to the quantum group $U_q(\mathfrak{g}^\theta)$, whose classical limit is also isomorphic to $U_{\mathbb{Q}}(\mathfrak{g}^\theta)$. A unified construction of such a coideal was first given by Letzter [Le99]. Later, Kolb [Ko14] generalized her construction to arbitrary Kac-Moody algebra (not necessarily of finite type). In Kolb's construction, the parameters ζ_i, κ_i can be chosen more freely. The constraints for ζ_i, κ_i in Bao-Wang's definition are needed to guarantee the existence of the bar-involution on $\mathbf{U}^\imath$ (see Lemma 4.1.8 below).

Lemma 4.1.4 ([Le99, Cor 4.2])**.** *The algebra $\mathbf{U}^\imath$ is a right coideal of $U_q(\mathfrak{g})$, that is, we have*

$$\Delta(\mathbf{U}^\imath) \subset \mathbf{U}^\imath \otimes U_q(\mathfrak{g}).$$

Consequently, for a $\mathbf{U}^\imath$-module M and $U_q(\mathfrak{g})$-modules $N_1, \dots, N_l$, the tensor product $M \otimes N_1 \otimes \cdots \otimes N_l$ has a well-defined $\mathbf{U}^\imath$-module structure via Δ.

Theorem 4.1.5 ([Le99, Theorem 4.8])**.** *If $\zeta_i|_{q=1} = s(\boldsymbol{I}_\bullet, \tau)(\alpha_{\tau(i)})$, then $\mathbf{U}^\imath$ is a subalgebra of $\mathbf{U}$ whose classical limit is $U_{\mathbb{Q}}(\mathfrak{g}^\theta)$, and it is maximal with this property.*

The following gives a presentation of $\mathbf{U}^\imath$ by generators and relations which resembles that of quantum groups.

Theorem 4.1.6 ([Ko14, Theorem 7.1])**.** *Let B be the free $\mathbb{Q}(q)$-algebra generated by F_i, $i \in \boldsymbol{I}_\bullet$, B_i, $i \in I$, and K_h, $h \in P_\imath^\vee$. Let $\Pi : B \to \mathbf{U}^\imath$ be the algebra surjection which sends F_i, B_i, K_h to F_i, B_i, K_h. Then, there exist $C_{i,j} \in B$, $i, j \in I$ such that* $\operatorname{Ker}\Pi$ *is the two-sided ideal of B generated by the following elements:*

$$\begin{aligned}
&K_0 - 1,\\
&K_h K_{h'} - K_{h+h'} \quad h, h' \in P_\imath^\vee,\\
&K_h F_i K_{-h} - q^{\langle h, -\overline{\alpha_i}\rangle} F_i \quad h \in P_\imath^\vee,\ i \in \boldsymbol{I}_\bullet,\\
&K_h B_i K_{-h} - q^{\langle h, \overline{\alpha_i}\rangle} B_i \quad h \in P_\imath^\vee,\ i \in I,\\
&F_i B_j - B_j F_i + \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad i \in \boldsymbol{I}_\bullet,\ j \in I,\\
&\sum_{s=0}^{1-a_{i,j}} (-1)^s F_i^{(s)} F_j F_i^{(1-a_{i,j}-s)}, \quad i \neq j \in \boldsymbol{I}_\bullet,\\
&\sum_{s=0}^{1-a_{i,j}} (-1)^s \begin{bmatrix} 1-a_{i,j} \\ s \end{bmatrix}_{q_i} B_i^s B_j B_i^{1-a_{i,j}-s} - C_{i,j}, \quad i \neq j \in I.
\end{aligned}$$

Remark 4.1.7. It is known that $C_{i,j} = 0$ if $i \notin \{j, \tau(i), \tau(j)\}$. Also, Kolb computed $C_{i,j}$ explicitly for i, j with $a_{i,j} = 0, -1, -2$. Recently, more explicit expressions in $\boldsymbol{I}_\bullet =$ case (without any assumption on $a_{i,j}$) were given in [CLW18].

$\mathbf{U}^\imath$ has also the bar-involution.

Lemma 4.1.8 ([BW18a, Lemma 3.15])**.** *There exists a unique $\mathbb{Q}$-algebra automorphism $\overline{\cdot}$ on $\mathbf{U}^\imath$ that sends E_j, B_i, K_h, q to E_j, B_i, K_{-h}, q^{-1}, respectively for all $j \in \boldsymbol{I}_\bullet$, $i \in I$, $h \in P_\imath^\vee$. We call this automorphism the bar-involution on $\mathbf{U}^\imath$.*

Definition 4.1.9. Let M be a $\mathbf{U}^\imath$-module. A $\mathbb{Q}$-linear automorphism $\overline{\cdot}$ on M is said to be a bar-involution on M if $\overline{\cdot}^2 = \mathrm{id}_M$, and $\overline{xm} = \overline{x}\,\overline{m}$ for all $x \in \mathbf{U}^\imath$ and $m \in M$.

4.2. **Intertwiners.** The bar-involution on $\mathbf{U}^\imath$ is not the restriction of the bar-involution on $U_q(\mathfrak{g})$. When a confusion is possible, we denote by ψ the one on $U_q(\mathfrak{g})$, and by $\psi^\imath$ on $\mathbf{U}^\imath$. Accordingly, we call a bar-involution of a $U_q(\mathfrak{g})$-module a ψ-involution, and a bar-involution of a $\mathbf{U}^\imath$-module a $\psi^\imath$-involution. The difference between these two bar-involutions are described by the intertwiner (also known as the quasi-K-matrix) Υ.

Theorem 4.2.1 ([BW18a, Theorem 4.8], see also [BaKo15b, Theorem 6.10])**.** *For each* $\nu \in Q_+$*, there exists a unique* $\Upsilon_\nu \in U_q(\mathfrak{g})^+_\nu$ *satisfying the following:*

(1) $\Upsilon_0 = 1$.

(2) *Set* $\Upsilon := \sum_{\nu \in Q_+} \Upsilon_\nu$*. Then, we have*

$$\psi^\imath(x)\Upsilon = \Upsilon\psi(x) \text{ for all } x \in \mathbf{U}^\imath.$$

Moreover, $\Upsilon_\nu = 0$ *unless* $\theta(\nu) = -\nu$.

By using Υ, one can make a ψ-involution into a $\psi^\imath$-involution.

Proposition 4.2.2. *Let* M *be a* $U_q(\mathfrak{g})$*-module in* $\mathcal{O}_{\mathrm{int}}$ *with a* ψ*-involution. Then,* $\Upsilon \circ \overline{\cdot}$ *is a* $\psi^\imath$*-involution on a* $\mathbf{U}^\imath$*-module* M.

Also, by combining the quasi-R-matrix Θ, one can make a $\psi^\imath$-involution from a ψ-involution and a $\psi^\imath$-involution. To explain this, set

$$\Theta^\imath := \Delta(\Upsilon) \circ \Theta \circ (\Upsilon^{-1} \otimes 1).$$

Then, we have the following.

Proposition 4.2.3. *Let* M *be a* $\mathbf{U}^\imath$*-module with a* $\psi^\imath$*-involution, and* N *a* $U_q(\mathfrak{g})$*-module with a* ψ*-involution. Then, the automorphism* $\overline{\cdot}$ *on* $M \otimes N$ *defined by*

$$\overline{m \otimes n} := \Theta^\imath(\overline{m} \otimes \overline{n}), \quad m \in M,\ n \in N$$

is a $\psi^\imath$*-involution on the* $\mathbf{U}^\imath$*-module* $M \otimes N$.

Remark 4.2.4. Given two $U_q(\mathfrak{g})$-modules M, N with bar-involutions, there are two ways to construct a bar-involution on the $\mathbf{U}^\imath$-module $M \otimes N$; one is to apply Proposition 4.2.2 to $M \otimes N$ after applying Proposition 3.6.3 to M, N, the other one is to apply Proposition 4.2.3 to M, N after applying Proposition 4.2.2 to M. Clearly, these two constructions coincide.

Theorem 4.2.5 ([BW18a, Theorem 5.7])**.** *Let* $M \in \mathcal{O}_{\mathrm{int}}$ *be a* $U_q(\mathfrak{g})$*-module with a global crystal basis* B *and a* ψ*-involution* ψ_M*. Let* $\psi^\imath_M := \Upsilon \circ \psi_M$*. Then, for each* $b \in B$*, there exists a unique* $b^\imath \in M$ *such that* $\psi^\imath_M(b^\imath) = b^\imath$ *and*

$$b^\imath \in b + \sum_{\substack{b' \in B \\ \mathrm{wt}(b') > \mathrm{wt}(b) \text{ and } \mathrm{wt}^\imath(b') = \mathrm{wt}^\imath(b)}} a_{b',b} b' \quad \text{for some } a_{b',b} \in q\mathbb{Q}[q].$$

In particular, $B^\imath := \{b^\imath \mid b \in B\}$ *is an* $\imath$*-canonical basis of* M.

Definition 4.2.6. We call $B^\imath$ the $\imath$-canonical basis of M associated with B.

Definition 4.2.7. Let M be a finite-dimensional $\mathbf{U}^\imath$-module with a given $\mathbb{Q}(q)$-basis $B^\imath$. Suppose that $M = \bigoplus_{\nu \in P_\imath} M_\nu$, where $M_\nu := \{m \in M \mid K_h m = q^{\langle h, \nu \rangle} m \text{ for all } h \in P_\imath\}$. The pair $(M, B^\imath)$ is called a based $\mathbf{U}^\imath$-module if the following hold:

(1) $B^\imath = \bigsqcup_{\nu \in P_\imath} B^\imath_\nu$, where $B^\imath_\nu := B^\imath \cap M_\nu$.

(2) The $\mathbf{A}$-submodule $M_{\mathbf{A}}$ of M generated by $B^\imath$ is a $\mathbf{U}^\imath_{\mathbf{A}}$-submodule of M.

(3) M has a $\psi^\imath$-involution.

(4) Let $\mathcal{L}$ be the $\mathbf{A}_0$-submodule of M generated by $B^\imath$. Then, the image of $B^\imath$ under the canonical map $\mathcal{L} \to \mathcal{L}/q\mathcal{L}$ forms a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$.

Theorem 4.2.8 ([BWW18, Theorem 4]). *Let $(M, B^\imath)$ be a based $\mathbf{U}^\imath$-module, and N a $U_q(\mathfrak{g})$-module with a global crystal basis B. Let $\psi^\imath_{M,N}$ denote the $\psi^\imath$-involution of $M \otimes N$ constructed as in Proposition 4.2.3. Then, for each $b_1 \in B^\imath$ and $b_2 \in B$, there exists a unique $b_1 \diamond_\imath b_2 \in M \otimes N$ such that $\psi^\imath_{M,N}(b_1 \diamond_\imath b_2) = b_1 \diamond_\imath b_2$ and*

$$b_1 \diamond_\imath b_2 \in b_1 \otimes b_2 + \sum_{\substack{b_1' \in B^\imath,\ b_2' \in B \\ \mathrm{wt}(b_2') > \mathrm{wt}(b_2)}} a^\imath_{b_1',b_2';b_1,b_2} b_1' \otimes b_2' \quad \textit{for some } a^\imath_{b_1',b_2';b_1,b_2} \in q\mathbb{Q}[q].$$

In particular, if we set $B^\imath \diamond_\imath B := \{b_1 \diamond_\imath b_2 \mid b_1 \in B^\imath,\ b_2 \in B\}$, then $(M \otimes N, B^\imath \diamond_\imath B)$ is a based $\mathbf{U}^\imath$-module.

Definition 4.2.9. We call $B^\imath \diamond_\imath B$ the $\imath$-canonical basis of the tensor product module $M \otimes N$ associated with $B^\imath, B$.

4.3. **PBW-basis of $\mathbf{U}^\imath$.** In this subsection, we construct a Poincaré-Birkhoff-Witt type basis of $\mathbf{U}^\imath$. First, we give a filtration on $\mathbf{U}^\imath$ by setting

$$\deg(F_i) = \deg(K_h) = 0, \quad \deg(B_j) = 1, \quad i \in I_\bullet,\ h \in P^\vee_\imath,\ j \in I.$$

Then, $\mathbf{U}^\imath$ becomes a filtered algebra; if we set $\mathbf{U}^\imath_m$, $m \in \mathbb{Z}_{\geq 0}$ to be the subspace spanned by $x_1 \cdots x_l$, $x_1, \dots, x_l \in \{F_i, K_h, B_j\}$, $\sum \deg(x_k) \leq m\}$, then we have $\mathbf{U}^\imath_{-1} := 0 \subset \mathbf{U}^\imath_0 \subset \mathbf{U}^\imath_1 \subset \cdots$, and $\mathbf{U}^\imath_m \mathbf{U}^\imath_{m'} \subset \mathbf{U}^\imath_{m+m'}$. Hence, one can consider its graded algebra $\mathrm{gr}\,\mathbf{U}^\imath := \sum_{m=0}^\infty \mathbf{U}^\imath_m/\mathbf{U}^\imath_{m-1}$, and the canonical isomorphism $\mathrm{gr} : \mathbf{U}^\imath \to \mathrm{gr}\,\mathbf{U}^\imath$ of vector spaces.

For $J = (j_1, \dots, j_n) \in I^n$, set $E_J := E_{j_1} \cdots E_{j_n}$. Clearly, $\{E_J \mid J \in \bigcup_{n=0}^\infty I^n\}$ spans $U_q(\mathfrak{g})^+$. Then, one can take a subset $\mathcal{J}$ of $\bigcup_{n=0}^\infty I^n$ in a way such that $\{E_J \mid J \in \mathcal{J}\}$ forms a basis of $U_q(\mathfrak{g})^+$. For $J = (j_1, \dots, j_n) \in \mathcal{J}$, set

$$B_J := B_{j_1} \cdots B_{j_n}, \quad \mathrm{wt}(J) := \alpha_{j_1} + \cdots + \alpha_{j_n}.$$

Note that $K_h B_J K_{-h} = q^{\langle h, \mathrm{wt}(J) \rangle} B_J$ for all $h \in P^\vee_\imath$ and $J \in \mathcal{J}$.

Proposition 4.3.1 ([Ko14, Proposition 6.2]). *Let $\mathbf{U}^\imath_{\mathcal{J}}$ denote the subspace of $\mathbf{U}^\imath$ spanned by B_J, $J \in \mathcal{J}$. Then, $\{B_J \mid J \in \mathcal{J}\}$ is a basis of $\mathbf{U}^\imath_{\mathcal{J}}$, and the multiplication map*

$$\mathbf{U}^\imath_{\mathcal{J}} \otimes \mathbf{U}^{\imath,0} \otimes U_q(\mathfrak{g}_\bullet)^- \to \mathbf{U}^\imath$$

is an isomorphism of vector spaces, where $\mathbf{U}^{\imath,0} := \mathbf{U}^\imath \cap U_q(\mathfrak{g})^0$.

Now, we are able to say more about $C_{i,j}$'s in Theorem 4.1.6. In fact, [Ko14, Theorem 7.1] also states that

$$\Pi(C_{i,j}) \in \sum_{\substack{J \in \mathcal{J} \\ \mathrm{wt}(J) < (1 - a_{i,j})\alpha_i + \alpha_j}} B_J \mathbf{U}^{\imath,0} U_q(\mathfrak{g}_\bullet)^-.$$

Proposition 4.3.2. *There exists a unique algebra homomorphism $U_q(\mathfrak{g})^+ \to \mathrm{gr}\,\mathbf{U}^\imath$ which sends E_j to B_j, $j \in I$. Moreover, it induces an isomorphism $p : U_q(\mathfrak{g})^+ \to \mathrm{gr}(\mathbf{U}^\imath_{\mathcal{J}})$.*

Proof. By the defining relation of $\mathbf{U}^\imath$, it is easy to verify the existence of the desired surjection $U_q(\mathfrak{g})^+ \to \mathrm{gr}\,\mathbf{U}^\imath$, and that its image is $\mathrm{gr}(\mathbf{U}^\imath_{\mathcal{J}})$. Let $p : U_q(\mathfrak{g})^+ \to \mathrm{gr}(\mathbf{U}^\imath_{\mathcal{J}})$ be the algebra surjection which sends E_j to B_j, $j \in I$. Recall that $U_q(\mathfrak{g})^+$ and $\mathbf{U}^\imath_{\mathcal{J}}$ have bases $\{E_J \mid J \in \mathcal{J}\}$ and $\{B_J \mid J \in \mathcal{J}\}$, respectively. Then, the composite $\mathrm{gr}^{-1} \circ p$ sends E_J to

B_J, $J \in \mathcal{J}$. This implies that $\mathrm{gr}^{-1} \circ p : U_q(\mathfrak{g})^+ \to \mathbf{U}^\imath_{\mathcal{J}}$ is an isomorphism of vector spaces, and hence, so is $p = \mathrm{gr} \circ \mathrm{gr}^{-1} \circ p$. This proves the proposition. □

Let $\mathbf{i} = (i_1, \dots, i_N) \in I^N$ be a reduced word for the longest element $w_0 \in W$. Recall that we have constructed the root vectors $E(\mathbf{i})_k$, $k = 1, \dots, N$, and the PBW basis $\{E(\mathbf{i}; \mathbf{c}) \mid \mathbf{c} \in \mathbb{Z}^N_{\geq 0}\}$ of $U_q(\mathfrak{g})^+$. Applying the $\mathbb{Q}(q)$-algebra involution ω to this PBW basis, we obtain a PBW basis $\{F(\mathbf{i}; \mathbf{c}) := \omega(E(\mathbf{i}; \mathbf{c})) \mid \mathbf{c} \in \mathbb{Z}^N_{\geq 0}\}$ of $U_q(\mathfrak{g})^-$. Set $B(\mathbf{i}; \mathbf{c}) := \mathrm{gr}^{-1} \circ p(E(\mathbf{i}; \mathbf{c}))$.

Theorem 4.3.3. *Let $\mathbf{i} \in I^N$ and $\mathbf{i}_\bullet \in I^{N_\bullet}$ be reduced words for $w_0 \in W$ and $w_\bullet \in W_\bullet$, respectively. Then,*

$$\{B(\mathbf{i}; \mathbf{c}) K_h F(\mathbf{i}_\bullet; \mathbf{c}_\bullet) \mid \mathbf{c} \in \mathbb{Z}^N_{\geq 0},\ h \in P^\vee_\imath,\ \mathbf{c}_\bullet \in \mathbb{Z}^{N_\bullet}_{\geq 0}\}$$

is a basis of $\mathbf{U}^\imath$.

5. KAZHDAN-LUSZTIG BASES

In this section, we formulate variants of the Kazhdan-Lusztig bases following [KL79], [Deo87], and [L03]. Throughout this section, we fix a generalized Cartan matrix A. Let $W = \langle s_i \mid i \in I \rangle$ denote the Weyl group. In Subsection 5.1, we introduce the Hecke algebra of W with unequal parameters, and some involutions on it. After formulating the Kazhdan-Lusztig bases in Subsection 5.2, we use them to recall the notion of left cell representations of W in Subsection 5.3. A combinatorial description of the left cells in type B with particular parameters will be important for us. In Subsection 5.4, we define a nondegenerate symmetric bilinear form on the Hecke algebra. Finally, in Subsection 5.5, we consider a parabolic analog of the constructions above.

5.1. Hecke algebras with unequal parameters.

Let Γ be an abelian group with a total order $<$ that is compatible with the group structure; if $g_1 < g_2$, then we have $gg_1 < gg_2$ for all $g \in \Gamma$. Set

$$\Gamma^+ := \{g \in \Gamma \mid g > 1\}.$$

Fix a map $I \to \Gamma^+ \sqcup \{1\}$; $i \mapsto q_i$ such that $q_i = q_j$ whenever s_i and s_j are W-conjugate. Such a map is called a weight function.

Example 5.1.1. (1) Let A be the indecomposable Cartan matrix whose Dynkin diagram is of type A_{d-1} (see Figure 1 in 111). As we have seen earlier, the Weyl group is the symmetric group $\mathfrak{S}_d = \langle s_1, \dots, s_{d-1} \rangle$. Let q be an indeterminate, and $\Gamma := \{q^n \mid n \in \mathbb{Z}^n\}$ equipped with the total order defined by

$$q^n \leq q^m \text{ if } n \leq m.$$

Then, we have

$$\mathbb{Z}[\Gamma] = \mathbb{Z}[q, q^{-1}], \quad \mathbb{Z}[\Gamma^+] = q\mathbb{Z}[q], \quad \mathbb{Z}[\Gamma^-] = q^{-1}\mathbb{Z}[q^{-1}].$$

The map $\{1, \dots, d-1\} \to \Gamma^+ \sqcup \{1\}; i \mapsto q$ is a weight function.

(2) Let A be the indecomposable Cartan matrix whose Dynkin diagram is of type B_d (see Figure 1 in 111). As we have seen earlier, the Weyl group is the group $W_d = \langle s_0, s_1, \dots, s_{d-1} \rangle$. Let p, q be indeterminates, and $\Gamma := \{p^k q^l \mid k, l \in \mathbb{Z}^n\}$ equipped with the total order defined by

$$p^k q^l \leq p^{k'} q^{l'} \text{ if } k < k' \text{ or } (k = k' \text{ and } l \leq l').$$

Then, we have

$$\mathbb{Z}[\Gamma] = \mathbb{Z}[p, p^{-1}, q, q^{-1}], \quad \mathbb{Z}[\Gamma^+] = p\mathbb{Z}[p, q, q^{-1}] \oplus q\mathbb{Z}[q], \quad \mathbb{Z}[\Gamma^-] = p^{-1}\mathbb{Z}[p^{-1}, q, q^{-1}] \oplus q^{-1}\mathbb{Z}[q^{-1}].$$

The map $\{0, 1, \dots, d-1\} \to \Gamma^+ \sqcup \{1\}; i \mapsto q_i$ defined by

$$q_i := \begin{cases} p & \text{if } i = 0, \\ q & \text{if } i \neq 0 \end{cases}$$

is a weight function.

Definition 5.1.2. The Hecke algebra $\mathcal{H} = \mathcal{H}(W, \{q_i\}_{i \in I})$ of W associated with the weight function $i \mapsto q_i$ is the associative algebra over $\mathbb{Z}[\Gamma]$ generated by $\{H_i \mid i \in I\}$ subject to the following relations:

- $(H_i - q_i^{-1})(H_i + q_i) = 0$ for all $i \in I$.
- $H_i H_j H_i \cdots = H_j H_i H_j \cdots$ (both sides have $m_{i,j}$ factors) for all $i \neq j \in I$.

For each $w \in W$ with a reduced expression $w = s_{i_1} \cdots s_{i_l}$, the product $H_{i_1} \cdots H_{i_l}$ is independent of the choice of a reduced expression of w; we denote it by H_w. Similarly, $q_w := q_{i_1} \cdots q_{i_l}$ is well-defined.

The ring $\mathbb{Z}[\Gamma]$ is equipped with a $\mathbb{Z}$-linear automorphism $\overline{\cdot}$ which sends g to g^{-1} for all $g \in \Gamma$. Let U, V be modules over $\mathbb{Z}[\Gamma]$. We say a $\mathbb{Z}$-linear map $f : U \to V$ is anti-linear if it satisfies $f(au) = \overline{a} f(u)$ for all $a \in \mathbb{Z}[\Gamma]$ and $u \in U$. In the sequel, we will often use the following automorphisms on $\mathcal{H}$, all of which are involutions.

Lemma 5.1.3.

(1) *There exists a unique anti-linear algebra automorphism* $\overline{\cdot}$ *of* $\mathcal{H}$ *such that* $\overline{H_w} = H_{w^{-1}}^{-1}$. *We call it the bar-involution on* $\mathcal{H}$.

(2) *There exists a unique anti-linear algebra automorphism* sgn *of* $\mathcal{H}$ *such that* $\mathrm{sgn}(H_w) = (-1)^{\ell(w)} H_w$. *Here,* $\ell : W \to \mathbb{Z}_{\geq 0}$ *denotes the length function on* W.

(3) *There exists a unique* $\mathbb{Z}[\Gamma]$*-algebra anti-automorphism* $(\cdot)^\flat$ *of* $\mathcal{H}$ *such that* $H_w^\flat = H_{w^{-1}}$.

Moreover, all of these automorphisms commute with each other.

For $y, w \in W$, define $r_{y,w} \in \mathbb{Z}[\Gamma]$ by

$$\overline{H_w} = \sum_{y \in W} r_{y,w} H_y.$$

It is well-known and easily proved that $r_{w,w} = 1$ for all $w \in W$ and $r_{y,w} = 0$ unless $y \leq w$.

5.2. **Kazhdan-Lusztig bases.** Let us formulate the Kazhdan-Lusztig basis and the dual Kazhdan-Lusztig basis.

Theorem 5.2.1 ([KL79, Theorem 1.1], [L03, Theorem 5.2])**.** *For each* $w \in W$, *there exists a unique* $C_w \in \mathcal{H}$ *such that*

(1) $\overline{C_w} = C_w$.

(2) $C_w = H_w + \sum_{y<w} c_{y,w} H_y$ *for some* $c_{y,w} \in \mathbb{Z}[\Gamma^+]$. *Here,* $<$ *denotes the Bruhat order on* W.

Remark 5.2.2. Set $\Gamma^- := \{g \in \Gamma \mid g < 1\}$. Note that we have $\overline{\mathbb{Z}[\Gamma^+]} = \mathbb{Z}[\Gamma^-]$. Replacing $\mathbb{Z}[\Gamma^+]$ with $\mathbb{Z}[\Gamma^-]$, we see the following: For each $w \in W$, there exists a unique $D_w \in \mathcal{H}$ such that

(1) $\overline{D_w} = D_w$.

(2) $D_w = H_w + \sum_{y<w} d_{y,w} H_y$ for some $d_{y,w} \in \mathbb{Z}[\Gamma^-]$.

Remark 5.2.3. Noting that the automorphisms $\overline{\cdot}$ and sgn commute with each other, it is easy to verify that $D_w = (-1)^{\ell(w)} \mathrm{sgn}(C_w)$.

It is obvious from the definitions that both $\{C_w \mid w \in W\}$ and $\{D_w \mid w \in W\}$ form $\mathbb{Z}[\Gamma]$-bases of $\mathcal{H}$. We call the former the Kazhdan-Lusztig basis, and the latter the dual Kazhdan-Lusztig basis of $\mathcal{H}$.

5.3. **Left cell representations.** Let us recall from [KL79] the notion of left cells of W and the associated left cell representations by means of Kazhdan-Lusztig bases.

Definition 5.3.1. Let $y, w \in W$.

(1) $y \to_L w$ if the coefficient of C_y in $C_{s_i} C_w$ expanded in the Kazhdan-Lusztig basis is nonzero for some $i \in I$.
(2) $y \leq_L w$ if there exist $y = y_0, y_1, \dots, y_l = w \in W$ such that $y_{i-1} \to_L y_i$.
(3) $y \underset{L}{\sim} w$ if $y \leq_L w$ and $w \leq_L y$.
(4) $y <_L w$ if $y \leq_L w$ and $y \underset{L}{\not\sim} w$.
(5) Each equivalence class of $W/\underset{L}{\sim}$ is called a left cell of W. We denote by $L(W)$ the set of left cells of W.
(6) For $X, Y \in L(W)$, $X \leq_L Y$ if $x \leq_L y$ for some (equivalently, all) $x \in X, y \in Y$.

Remark 5.3.2. By Remark 5.2.3, we obtain the same equivalence relation as $\underset{L}{\sim}$ if we replace C_w's by D_w's.

For each $X \in L(W)$ and $x \in X$, set

$$C_{\leq_L X} = \bigoplus_{y \leq_L x} \mathbf{A}_{\mathbb{Z}} C_y, \quad C_{<_L X} = \bigoplus_{y <_L x} \mathbf{A}_{\mathbb{Z}} C_y, \quad C^L_X = C_{\leq_L X}/C_{<_L X},$$
$$D_{\leq_L X} = \bigoplus_{y \leq_L x} \mathbf{A}_{\mathbb{Z}} D_y, \quad D_{<_L X} = \bigoplus_{y <_L x} \mathbf{A}_{\mathbb{Z}} D_y, \quad D^L_X = D_{\leq_L X}/D_{<_L X}.$$

Note that these are independent of the choice of $x \in X$. We denote the image of $m \in C_{\leq_L X}$ (resp., $m \in D_{\leq_L X}$) under the quotient map $C_{\leq_L X} \to C^L_X$ (resp., $D_{\leq_L X} \to D^L_X$) by $[m]_X$ (resp., $[m]'_X$).

Lemma 5.3.3. *Let $X \in L(W)$. Then, $C_{\leq_L X}$, $C_{<_L X}$, $D_{\leq_L X}$, and $D_{<_L X}$ are left ideals of $\mathcal{H}$, and consequently, C^L_X and D^L_X are left $\mathcal{H}$-modules. Moreover, C^L_X has a basis $\{[C_x]_X \mid x \in X\}$, while D^L_X has a basis $\{[D_x]'_X \mid x \in X\}$.*

Proof. The assertions are obvious from the definitions. □

We call C^L_X the left cell representation of $\mathcal{H}$ associated with $X \in L(W)$. Also, we call the bases $\{[C_x]_X \mid x \in X\}$ and $\{[D_x]'_X \mid x \in X\}$ the Kazhdan-Lusztig basis of C^L_X and the dual Kazhdan-Lusztig basis of D^L_X, respectively.

In general, it is difficult to describe the left cells explicitly. However, when our Hecke algebra is associated with a weight function in Example 5.1.1, a combinatorial description of the left cells is known.

Proposition 5.3.4. (1) *Suppose that our Hecke algebra is associated with the weight function given in Example 5.1.1* (1)*. Let $x, y \in \mathfrak{S}_d$. Then, x and y are in the same left cell of $\mathfrak{S}_d$ if and only if $Q(x) = Q(y)$.*
(2) *Suppose that our Hecke algebra is associated with the weight function given in Example 5.1.1* (2)*. Let $z, w \in W_d$. Then, z and w are in the same left cell of W_d if and only if $(Q^-(z), Q^+(z)) = (Q^-(w), Q^+(w))$.*

Proof. (1) is found in [BB05, Theorem 6.5.1]. Let us prove (2). Let $y \in W_n$, and write $y(i) = \epsilon_i y_i$ ($\epsilon_i \in \{+,-\}$, $y_i \in \{1,\dots,d\}$). Let $i_1 < \cdots < i_k$, $j_1 < \cdots < j_l$ be such that $\epsilon_i = +$ if and only if $i = i_m$ for some m, and $\epsilon_i = -$ if and only if $i = j_n$ for some n. Set $A^-(y)$ to be the insertion tableau of $(y_{j_1},\dots,y_{j_l})$, $B^-(y)$ the recording tableau of $(y_{j_1},\dots,y_{j_l})$ in letters $(j_1,\dots,j_l)$, $A^+(y) := P^+(y)$, and $B^+(y) := Q^+(y)$. Then, by [BI03, Theorem 7.7], z and w are in the same left cell of W_n if and only if $(B^+(z), B^-(z)) = (B^+(w), B^-(w))$. Hence, it suffices to show that we have $(B^+(z), B^-(z)) = (B^+(w), B^-(w))$ if and only if $(Q^-(z), Q^+(z)) = (Q^-(w), Q^+(w))$. However, it is clear that $A^-(y) = -P^-(y)$, and hence, $B^-(y) = Q^-(y)$. This proves the assertion.

□

Remark 5.3.5. Suppose that our Hecke algebra is associated with the weight function given in Example 5.1.1 (2). Let $X \in L(W_d)$ and $y \in X$. By Proposition 5.3.4 (2), the bitableau $(Q^-(y), Q^+(y))$ depends only on X, not on each $y \in X$. We call it the bitableau of X, and denote by $\mathbf{Q}(X)$. Also, we denote by $\mathrm{sh}(X)$ the shape of $\mathbf{Q}(X)$.

5.4. **Bilinear form on $\mathcal{H}$.** Let $\mathcal{H}^* := \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathcal{H}, \mathbb{Z}[\Gamma])$. $\mathcal{H}^*$ has a left $\mathcal{H}$-module structure given by

$$(Hf)(H') = f(H^\flat H'), \qquad \text{for all } f \in \mathcal{H}^*,\ H, H' \in \mathcal{H}.$$

Let $\{h_w \mid w \in W\} \subset \mathcal{H}^*$ be the dual basis of $\{H_w \mid w \in W\}$, that is, they are characterized by $h_y(H_w) = \delta_{y,w}$ for all $y, w \in W$.

Lemma 5.4.1. *For each $w \in W$ and $i \in I$, the following holds.*

$$H_i h_w = \begin{cases} h_{s_i w} & \text{if } w < s_i w, \\ h_{s_i w} + (q_i^{-1} - q_i) h_w & \text{if } s_i w < w. \end{cases}$$

Proof. For each $y \in W$, we compute as

$$\begin{aligned}(H_{s_i} h_w)(H_y) &= h_w(H_{s_i} H_y) \\ &= \begin{cases} h_w(H_{s_i y}) & \text{if } s_i y > y, \\ h_w(H_{s_i y} + (q_i^{-1} - q_i) H_y) & \text{if } s_i y < y \end{cases} \\ &= \begin{cases} 1 & \text{if } s_i y > y \text{ and } s_i y = w, \\ 1 & \text{if } s_i y < y \text{ and } s_i y = w, \\ q_i^{-1} - q_i & \text{if } s_i y < y \text{ and } y = w, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} h_{s_i w}(H_y) & \text{if } s_i w > w, \\ (h_{s_i w} + (q_i^{-1} - q_i) h_w)(H_y) & \text{if } s_i w < w. \end{cases}\end{aligned}$$

This implies

$$H_i h_w = \begin{cases} h_{s_i w} & \text{if } s_i w > w, \\ h_{s_i w} + (q_i^{-1} - q_i) h_w & \text{if } s_i w < w. \end{cases}$$

Thus, the proof completes. □

There exists an anti-linear automorphism $\overline{\cdot}$ of $\mathcal{H}^*$ defined by $\overline{f}(H) = \overline{f(\overline{H})}$ for $f \in \mathcal{H}^*$, $H \in \mathcal{H}$.

Lemma 5.4.2. *For each $w \in W$, we have*

$$\overline{h_w} = \sum_{y \geq w} \overline{r_{w,y}} h_y.$$

In particular, $\overline{h_{w_0}} = h_{w_0}$, where $w_0 \in W$ denotes the longest element.

Proof. Let $y \in W$. Then, we have

$$\overline{h_w}(H_y) = \overline{h_w(\overline{H_y})} = \overline{h_w(\sum_{z \leq y} r_{z,y} H_z)} = \overline{r_{w,y}}.$$

Since $\overline{h_w} = \sum_{y \in W} \overline{h_w}(H_y) h_y$, the assertion follows. □

Let $\{C_w^* \mid w \in W\} \subset \mathcal{H}^*$ denote the dual basis of $\{C_w \mid w \in W\}$.

Proposition 5.4.3. *C_w^* is characterized by the following two conditions:*

1. *$\overline{C_w^*} = C_w^*$.*
2. *$C_w^* = h_w + \sum_{z > w} c_{w,z}^* h_z$ for some $c_{w,z}^* \in \mathbb{Z}[\Gamma^+]$.*

Proof. Thanks to Lemma 5.4.2, one can prove that there exists a unique $C'_w \in \mathcal{H}^*$ such that $\overline{C'_w} = C'_w$ and $C'_w - h_w \in \bigoplus_{y > w} \mathbb{Z}[\Gamma^+] h_y$ in a similar way to Theorem 5.2.1. Hence, it suffices to show that C_w^* satisfies the two conditions.

The first condition is verified as follows. For each $y \in W$, we have

$$\overline{C_w^*}(C_y) = \overline{C_w^*(\overline{C_y})} = \overline{C_w^*(C_y)} = \overline{\delta_{y,w}} = \delta_{y,w} = C_w^*(C_y).$$

Since $\{C_y \mid y \in W\}$ is a basis of $\mathcal{H}$, we obtain $\overline{C_w^*} = C_w^*$.

Next, we prove the second condition. For each $y \in W$, we can write $H_y = C_y + \sum_{z < y} b_{z,y} C_z$ for some $b_{z,y} \in \mathbb{Z}[\Gamma^+]$. Then, we have

$$C_w^* = \sum_{y \in W} C_w^*(H_y) h_y = h_w + \sum_{y > w} b_{w,y} h_y.$$

This completes the proof. □

$\mathcal{H}$ and $\mathcal{H}^*$, and their bar-involutions are related as follows.

Lemma 5.4.4. *The linear map $d : \mathcal{H} \to \mathcal{H}^*$; $H \mapsto H \cdot h_{w_0}$ gives an isomorphism of left $\mathcal{H}$-modules. Moreover, we have*

1. *$d(\overline{H_y}) = h_{yw_0}$ for all $y \in W$.*
2. *$d(\overline{H}) = \overline{d(H)}$ for all $H \in \mathcal{H}$.*

Proof. By Lemma 5.4.1, the linear map $\varphi : \mathcal{H} \to \mathcal{H}^*$; $H_w \mapsto h_w$ is an isomorphism of left $\mathcal{H}$-modules. On the other hand, the map $\psi : \mathcal{H} \to \mathcal{H}$; $H \mapsto H \cdot H_{w_0}$ is clearly an isomorphism of left $\mathcal{H}$-modules. Thus, the composite map $d := \varphi \circ \psi : \mathcal{H} \to \mathcal{H}^*$ is an isomorphism of left $\mathcal{H}$-modules satisfying

$$d(H) = \varphi(H \cdot H_{w_0}) = H \cdot \varphi(H_{w_0}) = H \cdot h_{w_0} \quad \text{for all } H \in \mathcal{H}.$$

Also, we have, for all $y \in W$,

$$d(\overline{H_y}) = \varphi(\overline{H_y} \cdot H_{w_0}) = \varphi(H_{y^{-1}}^{-1} \cdot H_{y^{-1}} H_{yw_0}) = \varphi(H_{yw_0}) = h_{yw_0}.$$

Finally, for each $H, H' \in \mathcal{H}$, we have

$$\begin{aligned} d(\overline{H})(H') &= (\overline{H} \cdot h_{w_0})(H') = h_{w_0}\left(\left(\overline{H}\right)^{\flat} H' \right), \\ \overline{d(H)}(H') &= \overline{d(H)(\overline{H'})} = \overline{h_{w_0}(H^{\flat}\overline{H'})} = \overline{h_{w_0}}\left(\overline{H^{\flat}} H' \right). \end{aligned}$$

Then, the equality $d(\overline{H}) = \overline{d(H)}$ follows from the facts that $\overline{h_{w_0}} = h_{w_0}$ and $\left(\overline{H}\right)^{\flat} = \overline{H^{\flat}}$; the former is proved in Lemma 5.4.2, and the latter is in Lemma 5.1.3. □

Using this isomorphism, we define a bilinear form $\langle \cdot \mid \cdot \rangle$ on $\mathcal{H}$ by

$$\langle H \mid H' \rangle := d(H')(H), \quad (H, H' \in \mathcal{H}).$$

Clearly, this bilinear form satisfies $\langle H' \mid HH'' \rangle = \langle H^{\flat} H' \mid H'' \rangle$ for all $H, H', H'' \in \mathcal{H}$.

Lemma 5.4.5. *The bilinear from $\langle \cdot \mid \cdot \rangle$ is symmetric.*

Proof. Let $H_1, H_2 \in \mathcal{H}$. It suffices to show that $h_{w_0}(H_2^{\flat} H_1) = h_{w_0}(H_1^{\flat} H_2)$. Since $H_{w_0}^{\flat} = H_{w_0}$, it holds that $h_{w_0}(H^{\flat}) = h_{w_0}(H)$ for all $H \in \mathcal{H}$. Then, the assertion follows from an easy equation $(H_2^{\flat} H_1)^{\flat} = H_1^{\flat} H_2$. □

The next proposition justifies the name "dual Kazhdan-Lusztig basis".

Proposition 5.4.6. *The bases $\{C_w \mid w \in W\}$ and $\{D_{ww_0} \mid w \in W\}$ are dual to each other with respect to $\langle \cdot \mid \cdot \rangle$, that is, we have $\langle C_y \mid D_w \rangle = \delta_{y,ww_0}$ for all $y, w \in W$.*

Proof. Recall that $D_w = \sum_{y \leq w} d_{y,w} H_y$ with $d_{w,w} = 1$ and $d_{y,w} \in \mathbb{Z}[\Gamma^-]$ for all $y < w$. Then, we have

$$\begin{aligned} \overline{d(D_w)} &= d(\overline{D_w}) = d(D_w), \\ d(D_w) &= d(\overline{D_w}) = d(\sum_{y \leq w} \overline{d_{y,w} H_y}) = \sum_{y \leq w} \overline{d_{y,w}} h_{yw_0} = \sum_{z \geq ww_0} \overline{d_{zw_0,w}} h_z. \end{aligned}$$

This and Proposition 5.4.3 show that $d(D_w) = C^*_{ww_0}$. Hence, it holds that $\langle C_y \mid D_w \rangle = C^*_{ww_0}(C_y) = \delta_{y,ww_0}$, which proves the proposition. □

Next, we aim to describe the duality between C^L_X's and D^L_X's.

Lemma 5.4.7. *Let $y, w \in W$, $X \in L(W)$. Then, the following hold.*

1. *$y \to_L w$ if and only if $ww_0 \to_L yw_0$.*
2. *$y \leq_L w$ if and only if $ww_0 \leq_L yw_0$.*
3. *$Xw_0 := \{xw_0 \mid x \in X\} \in L(W)$.*

Proof. We first prove part (1). Suppose that $y \to_L w$. Then, there exists $i \in I$ such that $\langle C_{s_i} C_w \mid D_{yw_0} \rangle \neq 0$. This implies that $\langle C_w \mid C_{s_i} D_{yw_0} \rangle \neq 0$, and hence, we obtain $ww_0 \to_L yw_0$. Replacing y, w by yw_0, ww_0, we also have the opposite indication. This proves part (1). Assertion (2) is an immediate consequence of (1). We prove part (3). Let $x \in X$. Then, $X = \{y \in W \mid x \leq_L y \leq_L x\}$. By part (2), we have $x \leq_L y \leq_L x$ if and only if $xw_0 \leq_L yw_0 \leq_L xw_0$. This implies that $Xw_0 = \{z \in W \mid xw_0 \leq_L z \leq_L xw_0\}$, and it is a unique left cell of W containing xw_0. Thus, the proof completes. □

Lemma 5.4.8. *The bilinear from $\langle \cdot \mid \cdot \rangle$ induces a non-degenerate bilinear form on $C^L_X \times D^L_{Xw_0}$. Moreover, $\{[C_x]_X \mid x \in X\}$ and $\{[D_{xw_0}]'_{Xw_0} \mid x \in X\}$ form bases which are dual to each other.*

Proof. Let $x \in X$, $y, w \in W$ be such that $y <_L x$ and $ww_0 <_L xw_0$. It suffices to show that $\langle C_y \mid D_u \rangle = 0$ for all $u \leq_L xw_0$ and $\langle C_v \mid D_{ww_0} \rangle = 0$ for all $v \leq_L x$. Both are obvious from Lemma 5.4.7 (2). ☐

Proposition 5.4.9. *Let $X \in L(W)$. Then, we have an isomorphism $D^L_{Xw_0} \simeq C^L_X$ of $\mathcal{H}$-modules.*

Proof. It suffices to show that the characters $\mathrm{ch}_{D^L_{Xw_0}}$ of $D^L_{Xw_0}$ and $\mathrm{ch}_{C^L_X}$ of C^L_X coincide with each other. For each $w \in W$, we compute as

$$\begin{aligned}\mathrm{ch}_{C^L_X}(H_w) &= \sum_{x \in X} \langle H_w [C_x]_X \mid [D_{xw_0}]'_{Xw_0} \rangle \\ &= \sum_{x \in X} \langle [C_x]_X \mid H_{w^{-1}} [D_{xw_0}]'_{Xw_0} \rangle \\ &= \mathrm{ch}_{D^L_{Xw_0}}(H_{w^{-1}}) = \mathrm{ch}_{D^L_{Xw_0}}(H_w).\end{aligned}$$

Thus, the proof completes. ☐

5.5. **Parabolic Kazhdan-Lusztig bases.** Throughout this subsection, we fix a subset $J \subset I$ such that the parabolic subgroup W_J of W generated by $\{s_j \mid j \in J\}$ is finite. Let ${}^J W$ denote the set of minimal length coset representatives for $W_J \backslash W$, and $w_J \in W_J$ the longest element. Also, we set

$$x_J := q_{w_J} \sum_{w \in W_J} q_w^{-1} H_w \in \mathcal{H}.$$

Lemma 5.5.1. *Let $j \in J$. Then, the following hold.*

1. $x_J H_j = q_j^{-1} x_J$.
2. $x_J^\flat = x_J$.
3. $x_J = C_{w_J}$. *In particular,* $\overline{x_J} = x_J$.

Proof. The assertion (1) follows from a direct calculation and the fact that $W_J = \{w \in W_J \mid w < s_j w\} \sqcup \{w \in W_J \mid s_j w < w\}$. The assertion (2) follows from the definition of x_J and the facts that $W_J = \{w^{-1} \mid w \in W_J\}$, and $q_{w^{-1}} = q_w$ for all $w \in W$. The proof of (3) can be found in [X94, Proposition 1.17 (2)]. ☐

By Lemma 5.5.1 (1), the right ideal $x_J \mathcal{H}$ of $\mathcal{H}$ has a basis $\{x_J H_w \mid w \in {}^J W\}$. Also, by Lemma 5.5.1 (3), $x_J \mathcal{H}$ is closed under the involution $\overline{\cdot}$. Hence, we can construct analogs of the Kazhdan-Lusztig basis and the dual Kazhdan-Lusztig basis of $\mathcal{H}$ in the ideal $x_J \mathcal{H}$:

Theorem 5.5.2. [Deo87, Proposition 3.2]

1. *For each $w \in {}^J W$, there exists a unique ${}^J C_w \in x_J \mathcal{H}$ such that*
   - (a) $\overline{{}^J C_w} = {}^J C_w$.
   - (b) ${}^J C_w = x_J (H_w + \sum_{\substack{y \in {}^J W \\ y < w}} {}^J c_{y,w} H_y)$ *for some* ${}^J c_{y,w} \in \mathbb{Z}[\Gamma^+]$.
2. *For each $w \in {}^J W$, there exists a unique ${}^J D_w \in x_J \mathcal{H}$ such that*
   - (a) $\overline{{}^J D_w} = {}^J D_w$.
   - (b) ${}^J D_w = x_J (H_w + \sum_{\substack{y \in {}^J W \\ y < w}} {}^J d_{y,w} H_y)$ *for some* ${}^J d_{y,w} \in \mathbb{Z}[\Gamma^-]$.

Clearly, $\{{}^J C_w \mid w \in {}^J W\}$ and $\{{}^J D_w \mid w \in {}^J W\}$ are linear bases of $x_J \mathcal{H}$. We call them the parabolic Kazhdan-Lusztig basis and the dual parabolic Kazhdan-Lusztig basis of $x_J \mathcal{H}$, respectively. The following two propositions tell us how parabolic versions and usual ones relate to each other.

Proposition 5.5.3. [Deo87, Proposition 3.4] *Let* $w \in {}^J W$*. Then,* ${}^J C_w = C_{w_J w}$*.*

Proposition 5.5.4. *Let* $w \in {}^J W$*. Then,* ${}^J D_w = x_J D_w$*.*

Proof. For each $y \in W$, define $y_J \in W_J$ and ${}^J y \in {}^J W$ to be the unique elements satisfying $y = y_J {}^J y$ and $\ell(y) = \ell(y_J) + \ell({}^J y)$. Then, we have

$$\begin{aligned} x_J D_w &= x_J \sum_{y \leq w} d_{y,w} H_y \\ &= x_J \left(H_w + \sum_{y < w} d_{y,w} H_{y_J} H_{{}^J y} \right) \\ &= x_J \left(H_w + \sum_{y < w} q_{y_J}^{-1} d_{y,w} H_{{}^J y} \right) \quad \text{(by Lemma 5.5.1 (1))} \\ &= x_J \left(H_w + \sum_{\substack{y \in {}^J W \\ y < w}} \sum_{\substack{x \in W_J \\ xy < w}} q_x^{-1} d_{xy,w} H_y \right). \end{aligned}$$

This shows that $x_J D_w - x_J H_w \in \bigoplus_{\substack{y \in {}^J W \\ y < w}} \mathbb{Z}[\Gamma^-] x_J H_y$. Hence, by Theorem 5.5.2 (2), $x_J D_w$ coincides with ${}^J D_w$. ☐

For a later use, let us consider $x_J C_y$ and $x_J D_y$ for general $y \in W$.

Proposition 5.5.5. *Let* $y \in W$*. Then, we have*

$$x_J C_y = \sum_{\substack{w \in {}^J W \\ w_J w \leq_L y}} \alpha_w {}^J C_w,$$

for some $\alpha_w \in \mathbb{Z}[\Gamma]$*.*

Proof. Let us write

$$x_J C_y = \sum_{w \in {}^J W} \alpha_w {}^J C_w = \sum_{w \in {}^J W} \alpha_w C_{w_J w} \quad \text{for some } \alpha_w \in \mathbb{Z}[\Gamma].$$

Also, by the definition of $\leq_L$, we can write

$$x_J C_y = \sum_{z \leq_L y} \beta_z C_z \quad \text{for some } \beta_z \in \mathbb{Z}[\Gamma].$$

This shows $\alpha_w = 0$ unless $w_J w \leq_L y$. ☐

Lemma 5.5.6. [L03, Theorem 6.6 (*b*)] *Let* $w \in W$ *and* $i \in I$ *be such that* $s_i w < w$*. Then, it holds that* $H_i D_w = -q_i D_w$*.*

Proposition 5.5.7. *Let* $y \in W \setminus {}^J W$*. Then,* $x_J D_y = 0$*.*

Proof. Since $y \notin {}^J W$, there exists $j \in J$ such that $s_j y < y$. For such j, we have $x_J H_j = q_j^{-1} x_J$ (Lemma 5.5.1 (1)) and $H_j D_y = -q_j D_y$ (Lemma 5.5.6). Hence, we obtain

$$x_J D_y = q_j x_J H_j D_y = -q_j^2 x_J D,$$

which implies $x_J D_y = 0$, as desired. ☐

Set $P_J := q_{w_J} \sum_{x \in W_J} q_x^{-2} \in \mathbb{Z}[\Gamma]$. Note that, by Lemma 5.5.1 (1), it holds that $x_J^2 = P_J x_J$. Then, for each $H, H' \in \mathcal{H}$, we have

$$\langle x_J H \mid x_J H' \rangle = \langle x_J^2 H \mid H' \rangle = P_J \langle H \mid H' \rangle \in P_J \mathbb{Z}[\Gamma];$$

here, we use Lemma 5.5.1 (2). Hence, we can define a $\mathbb{Z}[\Gamma]$-valued bilinear form $\langle \cdot \mid \cdot \rangle_J$ on $x_J \mathcal{H}$ by $\langle \cdot \mid \cdot \rangle_J := \frac{1}{P_J} \langle \cdot \mid \cdot \rangle$.

Proposition 5.5.8. *The basis $\{{}^J C_w \mid w \in {}^J W\}$ and $\{{}^J D_{w_J w w_0} \mid w \in {}^J W\}$ are dual to each other with respect to $\langle \cdot \mid \cdot \rangle_J$, that is, we have $\langle {}^J C_y \mid {}^J D_w \rangle_J = \delta_{y, w_J w w_0}$ for all $y, w \in {}^J W$.*

Proof. Let $y, w \in {}^J W$. We compute as follows:

$$\begin{aligned} \langle {}^J C_y \mid {}^J D_w \rangle_J &= \frac{1}{P_J} \langle {}^J C_y \mid {}^J D_w \rangle \\ &= \frac{1}{P_J} \langle C_{w_J y} \mid x_J D_w \rangle \quad \text{(by Proposition 5.5.3 and 5.5.4)} \\ &= \langle C_{w_J y} \mid D_w \rangle \quad (\text{since } C_{w_J y} = {}^J C_y \in x_J \mathcal{H}) \\ &= \delta_{w_J y, w w_0} = \delta_{y, w_J w w_0} \quad \text{(by Proposition 5.4.6).} \end{aligned}$$

This proves the proposition. □

6. Generalized q-Schur algebras

In this section, we introduce variants of the q-Schur algebra as the centralizer algebra of a right $\mathcal{H}$-module. In Subsection 6.1, we define a right $\mathcal{H}$-module $\mathbb{T}(\pi)$ and its centralizer algebra $\mathbb{S}(\pi)$, and then list their fundamental properties. In Subsection 6.2, we upgrade the left cell representations to the $\mathbb{S}(\pi)$ setting. In Subsection 6.3, we describe the relations between the left cell representations of $\mathcal{H}$ and those of $\mathbb{S}(\pi)$.

6.1. Fundamental properties.

We follow ideas in [DDPW08, Chapter 9.1]. Let π be a finite index set. Suppose that we are given a map $\pi \to \{J \mid J \subset I \text{ and } W_J \text{ is finite}\}$. We denote by I_λ the image of $\lambda \in \pi$ under this map. For each $\lambda \in \pi$, we abbreviate W_{I_λ}, w_{I_λ}, x_{I_λ}, etc. to W_λ, w_λ, x_λ, etc..

Definition 6.1.1. Associated with π, we define a right $\mathcal{H}$-module $\mathbb{T}(\pi) := \bigoplus_{\lambda \in \pi} x_\lambda \mathcal{H}$, and its centralizer algebra $\mathbb{S}(\pi) := \mathrm{End}_{\mathcal{H}}(\mathbb{T}(\pi))$; we let $\mathbb{S}(\pi)$ act on $\mathbb{T}(\pi)$ from the left.

It is obvious that $\mathbb{T}(\pi)$ has two bases $\{{}^\lambda C_w \mid \lambda \in \pi,\ w \in {}^\lambda W\}$ and $\{{}^\lambda D_w \mid \lambda \in \pi,\ w \in {}^\lambda W\}$; we call them the Kazhdan-Lusztig basis and dual Kazhdan-Lusztig basis, respectively.

For each $m = \sum_{\lambda \in \pi} m_\lambda \in \mathbb{T}(\pi)$ with $m_\lambda \in x_\lambda \mathcal{H}$, we define $\overline{m} \in \mathbb{T}(\pi)$ to be $\sum_{\lambda \in \pi} \overline{m_\lambda}$. Also, for each $f \in \mathbb{S}(\pi)$, define $\overline{f} \in \mathbb{S}(\pi)$ by $\overline{f}(m) = \overline{f(\overline{m})}$ for all $m \in \mathbb{T}(\pi)$. This gives anti-linear automorphisms $\overline{\cdot}$ on $\mathbb{T}(\pi)$ and $\mathbb{S}(\pi)$.

For each $\lambda \in \pi$, define $p_\lambda \in \mathbb{S}(\pi)$ to be the composite

$$p_\lambda : \mathbb{T}(\pi) \twoheadrightarrow x_\lambda \mathcal{H} \hookrightarrow \mathbb{T}(\pi)$$

of the projection and the inclusion. Clearly, $\{p_\lambda \mid \lambda \in \pi\}$ is a family of orthogonal idempotents with $\sum_{\lambda \in \pi} p_\lambda = \mathrm{id}_{\mathbb{T}(\pi)}$. Hence, we have a decomposition

$$\mathbb{S}(\pi) = \bigoplus_{\lambda, \mu \in \pi} p_\lambda \mathbb{S}(\pi) p_\mu, \quad p_\lambda \mathbb{S}(\pi) p_\mu = \mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H}).$$

Take $f \in \mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$ arbitrarily. Since $x_\mu \mathcal{H}$ is generated (as a right $\mathcal{H}$-module) by x_μ, the f is determined by $f(x_\mu) \in x_\lambda \mathcal{H}$. Let us write

$$f(x_\mu) = \sum_{w \in {}^\lambda W} c_{\lambda,w,\mu}(f) x_\lambda H_w, \qquad \text{for some } c_{\lambda,w,\mu} \in \mathbb{Z}[\Gamma].$$

Lemma 6.1.2. *Let $w \in {}^\lambda W$ and $j \in I_\mu$ be such that $w < ws_j$. Then, we have*

$$c_{\lambda,w,\mu}(f) = q_j c_{\lambda,ws_j,\mu}(f).$$

Consequently, we have

$$f(x_\mu) = \sum_{w \in {}^\lambda W^\mu} \sum_{\substack{y \in W_\mu \\ wy \in {}^\lambda W}} q_y^{-1} c_{\lambda,w,\mu}(f) x_\lambda H_{wy},$$

and hence, f is determined by $(c_{\lambda,w,\mu}(f))_{w \in {}^\lambda W^\mu} \in \mathbb{Z}[\Gamma]^{{}^\lambda W^\mu}$, where ${}^\lambda W^\mu := {}^\lambda W \cap ({}^\mu W)^{-1}$.

Proof. We have

$$\begin{aligned} q_j^{-1} f(x_\mu) &= f(x_\mu H_j) \\ &= f(x_\mu) H_j \\ &= \sum_{\substack{w \in {}^\lambda W \\ ws_j < w}} c_{\lambda,w,\mu}(f) x_\lambda (H_{ws_j} + (q_j^{-1} - q_j) H_w) + \sum_{\substack{w \in {}^\lambda W \\ ws_j > w}} c_{\lambda,w,\mu}(f) x_\lambda H_{ws_j} \\ &= \sum_{\substack{w \in {}^\lambda W \\ ws_j < w}} (c_{\lambda,ws_j,\mu}(f) + (q_j^{-1} - q_j) c_{\lambda,w,\mu}(f)) x_\lambda H_w + \sum_{\substack{w \in {}^\lambda W \\ ws_j > w}} c_{\lambda,ws_j,\mu}(f) x_\lambda H_w. \end{aligned}$$

Comparing the coefficients of $x_\lambda H_w$, we obtain the assertion. □

Conversely, given $(c_{\lambda,w,\mu})_{w \in {}^\lambda W^\mu} \in \mathbb{Z}[\Gamma]^{{}^\lambda W^\mu}$, there exists a unique $g \in \mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$ such that $c_{\lambda,w,\mu}(g) = c_{\lambda,w,\mu}$ for all $w \in {}^\lambda W^\mu$. Thus, we obtain an $\mathbb{Z}[\Gamma]$-linear isomorphism between $\mathbb{Z}[\Gamma]^{{}^\lambda W^\mu}$ and $\mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$. We need the following result about the elements of ${}^\lambda W^\mu$.

Lemma 6.1.3 ([DDPW08, Theorem 4.18])**.** *Let $\lambda, \mu \in \pi$. For each $x \in {}^\lambda W^\mu$, there exists a unique $J_x \subset I$ such that W_{J_x} is finite, and the multiplication map*

$$W_\lambda \times \{x\} \times {}^{J_x} W_\mu \to W_\lambda x W_\mu;\ (u, x, v) \mapsto uxv$$

is a bijection, where ${}^{J_x} W_\mu := {}^{J_x} W \cap W_\mu$. Moreover, we have $\ell(uxv) = \ell(u) + \ell(x) + \ell(v)$ for all $u \in W_\lambda$ and $v \in {}^{J_x} W_\mu$. Also, J_x is characterized by the equality $W_{J_x} = x^{-1} W_\lambda x \cap W_\mu$.

For $\lambda, \mu \in \pi$ and $x \in {}^\lambda W^\mu$, define $\xi_{\lambda,x,\mu} \in \mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$ to be the one corresponding to $(\delta_{x,w} q_{x'})_{w \in {}^\lambda W^\mu} \in \mathbb{Z}[\Gamma]^{{}^\lambda W^\mu}$, where $x' \in W_\mu$ is the longest element in ${}^{J_x} W_\mu$ (J_x is as in Lemma 6.1.3). Then, the next proposition is clear.

Proposition 6.1.4. *$\{\xi_{\lambda,x,\mu} \mid \lambda, \mu \in \pi,\ x \in {}^\lambda W^\mu\}$ forms a basis of $\mathbb{S}(\pi)$.*

For each $\lambda, \mu \in \pi$, $x \in {}^\lambda W^\mu$, set

$$\eta_{\lambda,x,\mu} := q_{w_\lambda x x'} \sum_{w \in W_\lambda x W_\mu} q_w^{-1} H_w = x_\lambda H_x \sum_{v \in {}^{J_x} W_\mu} q_{x'} q_v^{-1} H_v.$$

Lemma 6.1.5. *Let $\lambda, \mu \in \pi$, $x \in {}^\lambda W^\mu$.*

(1) $\eta^\flat_{\lambda,x,\mu} = \eta_{\mu,x^{-1},\lambda}$.

(2) $\xi_{\lambda,x,\mu}(x_\mu) = \eta_{\lambda,x,\mu} = \frac{1}{P_\mu}\eta_{\lambda,x,\mu} \cdot x_\mu = \frac{1}{P_\lambda}x_\lambda \cdot \eta_{\lambda,x,\mu}$.

(3) $\overline{\xi_{\lambda,e,\mu}} = \xi_{\lambda,e,\mu}$, *where e denotes the identity element of W.*

Proof. (1) By the definition of x', we have $y := w_\lambda x x'$ is the longest element in $W_\lambda x W_\mu$. Also, it is easily checked that the map $W \to W$, $w \mapsto w^{-1}$ gives a bijection $W_\lambda x W_\mu \to W_\mu x^{-1} W_\lambda$. Since this bijection preserves the length, y^{-1} is the longest element in $W_\mu x^{-1} W_\lambda$. Then, we compute $\eta^\flat_{\lambda,x\mu}$ as follows:

$$\eta^\flat_{\lambda,x\mu} = q_y \sum_{w \in W_\lambda x W_\mu} q_w^{-1} H_{w^{-1}} = q_y \sum_{w \in W_\mu x^{-1} W_\lambda} q_{w^{-1}}^{-1} H_w = q_{y^{-1}} \sum_{w \in W_\mu x^{-1} W_\lambda} q_w^{-1} H_w = \eta_{\mu,x^{-1},\lambda}.$$

(2) By the definition of $\xi_{\lambda,x,\mu}$, we have

$$\begin{aligned} \xi_{\lambda,x,\mu}(x_\mu) &= \sum_{\substack{y \in W_\mu \\ xy \in {}^\lambda W}} q_{x'} q_y^{-1} x_\lambda H_{xy} \\ &= \sum_{\substack{y \in W_\mu \\ xy \in {}^\lambda W}} \sum_{z \in W_\lambda} q_{x'} q_y^{-1} q_{w_\lambda} q_z^{-1} H_{zxy} \quad \text{(by the definition of } x_\lambda) \\ &= \sum_{w \in W_\lambda x W_\mu} q_{w_\lambda} q_x q_{x'} q_w^{-1} H_w = \eta_{\lambda,x,\mu}. \end{aligned}$$

This proves the first equation. Next, we have

$$\eta_{\lambda,x,\mu} \cdot x_\mu = \xi_{\lambda,x,\mu}(x_\mu^2) = P_\mu \xi_{\lambda,x,\mu}(x_\mu) = P_\mu \eta_{\lambda,x,\mu},$$

which implies the second equality. Finally, the third equality follows from the fact that $\eta_{\lambda,x,\mu} = \xi_{\lambda,x,\mu}(x_\mu) \in x_\lambda \mathcal{H}$.

(3) It suffices to check that $\overline{\xi_{\lambda,e,\mu}}(x_\nu) = \xi_{\lambda,e,\mu}(x_\nu)$ for all $\nu \in \pi$. Only the non-trivial case is when $\nu = \mu$. Since we have

$$\overline{\xi_{\lambda,e,\mu}}(x_\mu) = \overline{\xi_{\lambda,e,\mu}(x_\mu)} = \overline{\eta_{\lambda,e,\mu}},$$

the problem is reduced to proving that $\eta_{\lambda,e,\mu}$ is fixed under the involution $\overline{\cdot}$. One can write

$$\eta_{\lambda,e,\mu} = \sum_{w \in W_\lambda W_\mu} q_{w_\lambda} q_{e'} q_w^{-1} H_w = x_\lambda \sum_{y \in {}^\lambda W_\mu} q_{e'} q_y^{-1} H_y.$$

On the other hand, we have

$$x_\lambda x_\mu = x_\lambda x_{I_\lambda \cap I_\mu} \sum_{y \in {}^\lambda W_\mu} q_{e'} q_y^{-1} H_y = P_{I_\lambda \cap I_\mu} x_\lambda \sum_{y \in {}^\lambda W_\mu} q_{e'} q_y^{-1} H_y.$$

Hence, we obtain

$$\eta_{\lambda,e,\mu} = \frac{1}{P_{I_\lambda \cap I_\mu}} x_\lambda x_\mu,$$

which is invariant under $\overline{\cdot}$. Thus, the proof completes.

□

The anti-automorphism on $\mathbb{S}(\pi)$ induced from the one $\flat$ on $\mathbb{T}(\pi)$ has the following good compatibility with the basis $\{\xi_{\lambda,x,\mu}\}$.

Proposition 6.1.6. *The linear map $\flat : \mathbb{S}(\pi) \to \mathbb{S}(\pi)$; $\xi_{\lambda,x,\mu} \mapsto \xi_{\mu,x^{-1},\lambda}$ defines an $\mathbb{Z}[\Gamma]$-algebra anti-automorphism on $\mathbb{S}(\pi)$.*

Proof. We have to verify that $(\xi_{\lambda,x,\mu}\cdot\xi_{\kappa,y,\nu})^\flat = \xi_{\nu,y^{-1},\kappa}\cdot\xi_{\mu,x^{-1},\lambda}$ for all λ,μ,ν,κ and $x \in {}^\lambda W^\mu$, $y \in {}^\kappa W^\nu$. Since the both sides are equal to zero unless $\kappa = \mu$, we may assume that $\kappa = \mu$. Let us write

$$\xi_{\lambda,x,\mu}\cdot\xi_{\mu,y,\nu} = \sum_{z\in {}^\lambda W^\nu} c_z \xi_{\lambda,z,\nu} \quad \text{for some } c_z \in \mathbb{Z}[\Gamma]. \tag{4}$$

Applying the both sides to $x_\nu \in \mathbb{T}(\pi)$, by Lemma 6.1.5 (2), we obtain

$$\frac{1}{P_\mu}\eta_{\lambda,x,\mu}\eta_{\mu,y,\nu} = \sum_{z\in {}^\lambda W^\mu} c_z \eta_{\lambda,z,\nu}. \tag{5}$$

To prove the assertion, we compute as follows:

$$\begin{aligned}
\xi_{\nu,y^{-1},\mu}\cdot\xi_{\mu,x^{-1},\lambda}(x_\lambda) &= \frac{1}{P_\mu}\eta_{\nu,y^{-1},\mu}\cdot\eta_{\mu,x^{-1},\lambda} && \text{(by Lemma 6.1.5 (2))}\\
&= (\frac{1}{P_\mu}\eta_{\lambda,x,\mu}\cdot\eta_{\mu,y,\nu})^\flat && \text{(by Lemma 6.1.5 (1))}\\
&= (\sum_{z\in {}^\lambda W^\nu} c_z\eta_{\lambda,z,\nu})^\flat && \text{(by equation (5))}\\
&= \sum_{z\in {}^\lambda W^\nu} c_z\eta_{\nu,z^{-1},\lambda} && \text{(by Lemma 6.1.5 (1))}\\
&= \sum_{z\in {}^\lambda W^\nu} c_z\xi_{\nu,z^{-1},\lambda}(x_\lambda) && \text{(by Lemma 6.1.5 (2))}\\
&= (\xi_{\lambda,x,\mu}\cdot\xi_{\mu,y,\nu})^\flat(x_\lambda) && \text{(by equation (4)).}
\end{aligned}$$

This shows that $\xi_{\nu,y^{-1},\mu}\cdot\xi_{\mu,x^{-1},\lambda} = (\xi_{\lambda,x,\mu}\cdot\xi_{\mu,y,\nu})^\flat$, and hence, the proof completes. ☐

Recall the bilinear form $\langle\cdot \mid \cdot\rangle_\lambda = \langle\cdot \mid \cdot\rangle_{I_\lambda}$ on $x_\lambda\mathcal{H}$ defined in Section 5.5.

Proposition 6.1.7. *Let $\lambda,\mu\in\pi$, $m \in x_\lambda\mathcal{H}$, and $n \in x_\mu\mathcal{H}$. Then, for each $w \in {}^\lambda W^\mu$, we have*

$$\langle m \mid \xi_{\lambda,w,\mu}(n)\rangle_\lambda = \langle \xi^\flat_{\lambda,w,\mu}(m) \mid n\rangle_\mu.$$

Proof. We compute as follows:

$$\begin{aligned}
\langle m \mid \xi_{\lambda,w,\mu}(n)\rangle_\lambda &= \frac{1}{P_\lambda}\langle m \mid \xi_{\lambda,w,\mu}(n)\rangle\\
&= \frac{1}{P_\lambda P_\mu}\langle m \mid \eta_{\lambda,w,\mu}n\rangle && \text{(by Lemma 6.1.5 (2))}\\
&= \frac{1}{P_\lambda P_\mu}\langle \eta_{\mu,w^{-1},\lambda}m \mid n\rangle && \text{(by Lemma 6.1.5 (1))}\\
&= \frac{1}{P_\mu}\langle \xi_{\mu,w^{-1},\lambda}(m) \mid n\rangle && \text{(by Lemma 6.1.5 (2))}\\
&= \langle \xi^\flat_{\lambda,w,\mu}(m) \mid n\rangle_\mu.
\end{aligned}$$

This proves the proposition. ☐

Define a bilinear form $\langle\cdot \mid \cdot\rangle_\pi$ on $\mathbb{T}(\pi)$ by $\langle m \mid n\rangle_\pi := \delta_{\lambda,\mu}\langle m \mid n\rangle_\lambda$ for all $\lambda,\mu\in\pi$, $m \in x_\lambda\mathcal{H}$, $n \in x_\mu\mathcal{H}$.

Corollary 6.1.8. *The two bases* $\{{}^\lambda C_w \mid \lambda \in \pi,\ w \in {}^\lambda W\}$ *and* $\{{}^\lambda D_{w_\lambda w w_0} \mid \lambda \in \pi,\ w \in {}^\lambda W\}$ *of* $\mathbb{T}(\pi)$ *are dual to each other with respect to the bilinear form* $\langle \cdot \mid \cdot \rangle_\pi$. *Moreover, for all* $m, n \in \mathbb{T}(\pi)$ *and* $x \in \mathbb{S}(\pi)$, *we have* $\langle m \mid xn\rangle_\pi = \langle x^\flat m \mid n\rangle_\pi$.

6.2. **Cell representations.** Let $X \in L(W)$ and $x \in X$. Set

$$C_{\leq_L X}(\pi) := \bigoplus_{\lambda\in\pi} \bigoplus_{\substack{w\in{}^\lambda W \\ w_\lambda w \leq_L x}} \mathbb{Z}[\Gamma]^\lambda C_w, \qquad D_{\leq_L X}(\pi) := \bigoplus_{\lambda\in\pi} \bigoplus_{\substack{w\in{}^\lambda W \\ w \leq_L x}} \mathbb{Z}[\Gamma]^\lambda D_w,$$

$$C_{<_L X}(\pi) := \bigoplus_{\lambda\in\pi} \bigoplus_{\substack{w\in{}^\lambda W \\ w_\lambda w <_L x}} \mathbb{Z}[\Gamma]^\lambda C_w, \qquad D_{<_L X}(\pi) := \bigoplus_{\lambda\in\pi} \bigoplus_{\substack{w\in{}^\lambda W \\ w <_L x}} \mathbb{Z}[\Gamma]^\lambda D_w,$$

$$C^L_X(\pi) := C_{\leq_L X}(\pi)/C_{<_L X}(\pi), \qquad D^L_X(\pi) := D_{\leq_L X}(\pi)/D_{<_L X}(\pi).$$

, Note that these objects are independent of the choice of $x \in X$. We denote the image of $m \in C_{\leq_L X}(\pi)$ (resp., $D_{\leq_L X}(\pi)$) under the quotient map $C_{\leq_L X}(\pi) \to C^L_X(\pi)$ (resp., $D_{\leq_L X}(\pi) \to D^L_X(\pi)$) by $[m]_X$ (resp., $[m]'_X$).,

Proposition 6.2.1. *Let* $X \in L(W)$.

(1) $C_{\leq_L X}(\pi)$ *is a* $\mathbb{S}(\pi)$*-submodule of* $\mathbb{T}(\pi)$.
(2) $C_{<_L X}(\pi)$ *is a* $\mathbb{S}(\pi)$*-submodule of* $\mathbb{T}(\pi)$.
(3) $C^L_X(\pi)$ *is a left* $\mathbb{S}(\pi)$*-module having a basis* $\{[{}^\lambda C_w]_X \mid \lambda \in \pi,\ w \in {}^\lambda W \cap w_\lambda X\}$. *Here,* $w_\lambda X := \{w_\lambda x \mid x \in X\}$.

Proof. We will prove only (1) since the proof of (2) is similar to that of (1), and (3) follows from (1) and (2). Fix $x \in X$. In order to show that $C_{\leq_L X}(\pi)$ is a $\mathbb{S}(\pi)$-submodule, it suffices to verify that $\xi_{\lambda,y,\mu}{}^\mu C_w \in C_{\leq_L X}(\pi)$ for all $\lambda, \mu \in \pi$, $y \in {}^\lambda W^\mu$, and $w \in {}^\mu W$ such that $w_\mu w \leq_L x$. By Proposition 5.5.3 and Lemma 6.1.5 (2), we have

$$\xi_{\lambda,y,\mu}{}^\mu C_w = \xi_{\lambda,y,\mu} C_{w_\mu w} = \frac{1}{P_\mu}\eta_{\lambda,y,\mu} C_{w_\mu w}.$$

Also, by Lemma 6.1.5 (2), we have $\eta_{\lambda,y,\mu} = \xi_{\lambda,y,\mu}(x_\mu) \in x_\lambda \mathcal{H}$; one can write $\eta_{\lambda,y,\mu} = x_\lambda H$ for some $H \in \mathcal{H}$. Then, $HC_{w_\mu w}$ is a linear combination of $C_{w'}$, $w' \leq_L w_\mu w (\leq_L x)$. Hence, by Proposition 5.5.5, $x_\lambda H C_{w_\mu w}$ is a linear combination of ${}^\lambda C_{w''}$ for $w'' \in {}^\lambda W$ with $w_\lambda w'' \leq_L w' (\leq_L x)$. Therefore, we have $\xi_{\lambda,y,\mu}{}^\mu C_w = \frac{1}{P_\mu}\eta_{\lambda,y,\mu} C_{w_\mu w} \in \frac{1}{P_\mu} C_{\leq_L X}(\pi)$. However, since $\xi_{\lambda,y,\mu}{}^\mu C_w \in x_\lambda \mathcal{H} = \bigoplus_{z\in{}^\lambda W} \mathbb{Z}[\Gamma]^\lambda C_z$, we conclude that $\xi_{\lambda,y,\mu}{}^\mu C_w \in C_{\leq_L X}(\pi)$. This completes the proof. □

Similarly, one can prove the following: $D_{\leq_L X}(\pi)$ and $D_{<_L X}(\pi)$ are $\mathbb{S}(\pi)$-submodules, and $D^L_X(\pi)$ is a left $\mathbb{S}(\pi)$-module having a basis $\{[{}^\lambda D_w]'_X \mid \lambda \in \pi,\ w \in {}^\lambda W \cap X\}$.

6.3. **Functor $\mathcal{F}_\pi$.** Let $\mathcal{F}_\pi : \mathcal{H}$-mod $\to \mathbb{S}(\pi)$-mod be the functor from the category $\mathcal{H}$-mod of finite-dimensional $\mathcal{H}$-modules to the category $\mathbb{S}(\pi)$-mod of finite-dimensional $\mathbb{S}(\pi)$-modules defined by

$$\mathcal{F}_\pi(M) := \mathbb{T}(\pi) \otimes_{\mathcal{H}} M.$$

Since $\mathbb{T}(\pi) \simeq \bigoplus_{\lambda\in\pi} x_\lambda \mathcal{H}(\pi)$ as right $\mathcal{H}(\pi)$-modules, we have

$$\mathcal{F}_\pi(M) = \bigoplus_{\lambda\in\pi} x_\lambda M.$$

Let $X \in L(W)$ and consider $\mathcal{F}_\pi(C_{\leq_L X})$. For each $\lambda \in \pi$, the subspace $x_\lambda C_{\leq_L X}$ is the subspace of $x_\lambda \mathcal{H}$ spanned by $x_\lambda C_w$, $w \leq_L x$ for some $x \in X$. Hence, it is spanned by $a_w{}^\lambda C_w$, $w \in {}^\lambda W$ for some $a_w \in \mathbb{Z}[\Gamma] \setminus \{0\}$. This proves that $\mathbb{Q}(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{F}_\pi(C_{\leq_L X}) \simeq$

$\mathbf{C}_{\leq_L X}(\pi)$ as $\mathbf{S}(\pi)$-modules, where $\mathbf{C}_{\leq_L X}(\pi) := \mathbb{Q}(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} C_{\leq_L X}(\pi)$, and so on. Similarly, we obtain the following.

Proposition 6.3.1. *Let $X \in L(W)$.*

(1) $\mathbb{Q}(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{F}_\pi(C_{\leq_L X}) \simeq \mathbf{C}_{\leq_L X}(\pi)$.
(2) $\mathbb{Q}(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{F}_\pi(C_{<_L X}) \simeq \mathbf{C}_{<_L X}(\pi)$.
(3) $\mathcal{F}_\pi(D_{\leq_L X}) \simeq D_{\leq_L X}(\pi)$.
(4) $\mathcal{F}_\pi(D_{<_L X}) \simeq D_{<_L X}(\pi)$.

Let $\mathbf{H} := \mathbb{Q}(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{H}$. By Proposition 5.4.9, we have $\mathbf{D}^L_{Xw_0} \simeq \mathbf{C}^L_X$ as a left $\mathbf{H}$-module. This proves the following:

Proposition 6.3.2. *Let $X \in L(W)$. Then, we have an isomorphism $\mathbf{D}^L_{Xw_0}(\pi) \simeq \mathbf{C}^L_X(\pi)$ of $\mathbf{S}(\pi)$-modules.*

Part 2. q-Schur duality

In this part, we apply the general theory developed so far, and show that the quantum symmetric pairs of type AIII/AIV with no black nodes and the Hecke algebra of type B with unequal parameters satisfy the quantum Schur duality. This result was obtained in [BWW16], generalizing the equal parameter version in [BW13] and the type D setting in [B17]. In Subsection 7.1, we define the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\jmath)$ as a particular quantum symmetric pair introduced in Section 4. In Subsection 7.2, we state the first half of the main results here, namely, the double centralizer property of $\mathbf{U}^\jmath$ and $\mathcal{H}$. After studying the relations of $\mathbf{U}^\jmath$ and the generalized q-Schur algebra in detail in Subsection 7.3, we state the second half of the main results, that is, the coincidence of the $\imath$-canonical basis and the Kazhdan-Lusztig basis in Subsection 7.4. Subsections 8.1 to 8.4 are the counterparts of the above for the other quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$.

7. $(\mathbf{U}^\jmath, \mathcal{H})$-DUALITY

Let $W = W_d = \langle s_0, s_1, \dots, s_{d-1} \rangle$ be the Weyl group of type B_d, and consider the Hecke algebra $\mathcal{H}$ associated with the datum given in Example 5.1.1 (2). Set $\mathbf{A}_\mathbb{Z} := \mathbb{Z}[p, p^{-1}, q, q^{-1}]$, and $\mathbf{H} := \mathbb{Q}(p, q) \otimes_{\mathbf{A}_\mathbb{Z}} \mathcal{H}$.

7.1. Quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\jmath)$.

Throughout this subsection, fix $r, d \in \mathbb{Z}_{>0}$. Set $\mathbb{I} := [-\underline{r}, \underline{r}]$, $\quad \mathbb{I}^\jmath := [1, r]$.

Let $(P, P^\vee, \Pi, \Pi^\vee, \langle \cdot, \cdot \rangle)$ be the root datum of $\mathfrak{sl}_{2r+1}$-type constructed in Example 1.1.7 (1). Here, we replace I with $\mathbb{I}$, and the index set of the standard basis of $\mathbb{R}^n$ with $[-r, r]$. Then, the quantum group $U_q(\mathfrak{sl}_{2r+1})$ associated with this Cartan datum is generated by $E_{\underline{i}}, F_{\underline{i}}, K_{\underline{i}}^{\pm 1}$, $\underline{i} \in \mathbb{I}$.

Set $\mathbf{U} = \mathbf{U}_{2r+1} := \mathbb{Q}(p, q) \otimes_{\mathbb{Q}(q)} U_q(\mathfrak{sl}_{2r+1})$. The bar-involution on $U_q(\mathfrak{sl}_{2r+1})$ can be extended on $\mathbf{U}$ by setting $\overline{p} := p^{-1}$.

Next, let us consider the quantum symmetric pair $\mathbf{U}^\jmath$ associated with the admissible pair given in Example 2.1.8. If we choose parameters appropriately, we have

$$B_{\underline{i}} = \begin{cases} E_{\underline{1}} + p^{-1} F_{-\underline{1}} K_{\underline{1}}^{-1} & \text{if } i = 1, \\ E_{-\underline{1}} + p K_{-\underline{1}}^{-1} F_{\underline{1}} & \text{if } i = -1, \\ E_{\underline{i}} + F_{-\underline{i}} K_{\underline{i}}^{-1} & \text{if } i \neq \pm 1. \end{cases}$$

Set

$$e_i := B_{\underline{i}},\ f_i := B_{-\underline{i}},\ k_i := K_{\underline{i}} K_{-\underline{i}}^{-1}, \quad i \in \mathbb{I}^\jmath.$$

Then, $\mathbf{U}^\jmath$ is the subalgebra of $\mathbf{U}$ generated by $e_i, f_i, k_i^{\pm 1}$, $i \in \mathbb{I}^\jmath$, and the defining relations become

$$\begin{aligned}
&k_i k_i^{-1} = 1 = k_i^{-1} k_i, \\
&k_i k_j = k_j k_i, \\
&k_i e_j k_i^{-1} = q^{\langle h_{\underline{i}} - h_{-\underline{i}}, \alpha_{\underline{j}} \rangle} e_j, \\
&k_i f_j k_i^{-1} = q^{-\langle h_{\underline{i}} - h_{-\underline{i}}, \alpha_{\underline{j}} \rangle} f_j, \\
&e_i e_j - e_j e_i = 0 = f_i f_j - f_j f_i, \quad |i-j| > 1, \\
&e_i^{(2)} e_j - e_i e_j e_i + e_j e_i^{(2)} = 0 = f_i^{(2)} f_j - f_i f_j f_i + f_j f_i^{(2)}, \quad |i-j| = 1, \\
&e_1^{(2)} f_1 - e_1 f_1 e_1 + f_1 e_1^{(2)} = -e_1 (pqk_1 + p^{-1} q^{-1} k_1^{-1}), \\
&f_1^{(2)} e_1 - f_1 e_1 f_1 + e_1 f_1^{(2)} = -(pqk_1 + p^{-1} q^{-1} k_1^{-1}) f_1.
\end{aligned}$$

Proposition 7.1.1. *There exists a unique $\mathbb{Q}(p,q)$-algebra anti-automorphism ϱ on $\mathbf{U}^\jmath$ that sends e_i, f_i, k_i to $p^{-\delta_{i,1}} q^{1+\delta_{i,1}} f_i k_i^{-1}, p^{\delta_{i,1}} q e_i k_i, k_i$, respectively for all $i \in \mathbb{I}^\jmath$.*

Proof. Recall the anti-automorphism ϱ on $\mathbf{U}$ from Lemma 3.1.3 (4). By restricting it to $\mathbf{U}^\imath$, we obtain the desired anti-automorphism on $\mathbf{U}^\jmath$. $\square$

Proposition 7.1.2. *There exists a unique $\mathbb{Q}(p,q)$-algebra anti-automorphism $\sigma^\jmath$ on $\mathbf{U}^\jmath$ that sends e_i, f_i, k_i to f_i, e_i, k_i, respectively for all $i \in \mathbb{I}^\jmath$.*

Proof. Clear from the presentation above. $\square$

7.2. **$(\mathbf{U}^\jmath, \mathcal{H})$-duality.** Let $\mathbf{V}$ denote the natural representation of $\mathbf{U}$. Namely, it is a $(2r+1)$-dimensional $\mathbb{Q}(p,q)$-vector space with a basis $u_{-r}, \dots, u_{-1}, u_0, u_1, \dots, u_r$ equipped with a $\mathbf{U}$-module structure defined by

$$\begin{aligned}
K_{\underline{i}} u_j &:= q^{\delta_{i-1,j} - \delta_{i,j}} u_j, \\
E_{\underline{i}} u_j &:= \delta_{i,j} u_{i-1}, \\
F_{\underline{i}} u_j &:= \delta_{i-1,j} u_i
\end{aligned}$$

This $\mathbf{U}$-module $\mathbf{V}$ has a unique bar-involution fixing each $u_{-r}, \dots, u_r$. Then, the d-th tensor product $\mathbf{V}^{\otimes d}$ of $\mathbf{V}$ is equipped with a $\mathbf{U}$-module structure, and a bar-involution, which is constructed from the one on $\mathbf{V}$ by means of the quasi-R-matrix Θ.

Let $\mathcal{I} := [-r, r]^d$ denote the set of all sequences $(i_1, \dots, i_d)$. For $\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{I}$, set

$$u_{\mathbf{i}} := u_{i_1} \otimes \cdots \otimes u_{i_d} \in \mathbf{V}^{\otimes d}.$$

Then, $\mathbf{V}^{\otimes d}$ has a basis $\{u_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{I}\}$.

Recall that the Weyl group W of type B_d acts naturally on $\mathcal{I}$; for $(i_1, \dots, i_d) \in \mathcal{I}$ and $j \in [1, d-1]$,

$$\begin{aligned}
(i_1, \dots, i_d) \cdot s_0 &:= (-i_1, i_2, \dots, i_d), \\
(i_1, \dots, i_d) \cdot s_j &:= (i_1, \dots, i_{j-1}, i_{j+1}, i_j, i_{j+2}, \dots, i_d).
\end{aligned}$$

This action can be lifted to the action of $\mathbf{H}$ on $\mathbf{V}^{\otimes d}$ as follows; for $\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{I}$ and $j \in [1, d-1]$,

$$u_{\mathbf{i}} \cdot H_0 := \begin{cases} p^{-1} u_{\mathbf{i}}, & \text{if } i_1 = 0, \\ u_{\mathbf{i} \cdot s_0}, & \text{if } i_1 > 0, \\ u_{\mathbf{i} \cdot s_0} + (p^{-1} - p) u_{\mathbf{i}}, & \text{if } i_1 < 0, \end{cases}$$

$$u_{\mathbf{i}} \cdot H_j := \begin{cases} q^{-1} u_{\mathbf{i}}, & \text{if } i_j = i_{j+1}, \\ u_{\mathbf{i} \cdot s_j}, & \text{if } i_j < i_{j+1}, \\ u_{\mathbf{i} \cdot s_j} + (q^{-1} - q) u_{\mathbf{i}}, & \text{if } i_j > i_{j+1}. \end{cases}$$

Theorem 7.2.1 ([BWW16, Theorem 4.4]). *Let us denote the structure map of* $\mathbf{U}^\jmath$*-module and* $\mathbf{H}$*-module structure on* $\mathbf{V}^{\otimes d}$ *by* Φ *and* Ψ*, respectively. Then, these two actions form double centralizer;*

$$\Phi(\mathbf{U}^\jmath) = \mathrm{End}_{\mathbf{H}}(\mathbf{V}^{\otimes d}), \quad \Psi(\mathbf{H}) = \mathrm{End}_{\mathbf{U}^\jmath}(\mathbf{V}^{\otimes d})^{\mathrm{op}}$$

7.3. **Surjection ξ.** As a right $\mathbf{H}$-module, $\mathbf{V}^{\otimes d}$ decomposes as

$$\mathbf{V}^{\otimes d} = \bigoplus_{\mathbf{i} \in \mathcal{I}_+} \mathbf{V}_{\mathbf{i}}, \quad \mathbf{V}_{\mathbf{i}} := u_{\mathbf{i}} \mathbf{H},$$

where $\mathcal{I}_+ := \{(i_1, \dots, i_d) \in \mathcal{I} \mid 0 \leq i_1 \leq \cdots \leq i_d\}$. It is straightforward to verify that for each $\mathbf{i} \in \mathcal{I}_+$, we have

$$\mathbf{V}_{\mathbf{i}}^{\otimes d} \simeq x_{I_{\mathbf{i}}} \mathbf{H},$$

where $I_{\mathbf{i}} := \{j \in [0, d-1] \mid \mathbf{i} \cdot s_j = \mathbf{i}\}$. Here, set $\pi^\jmath := \{\lambda = (\lambda_0, \dots, \lambda_r) \in \mathbb{Z}_{\geq 0}^{r+1} \mid \sum_{i=0}^r \lambda_i = d\}$. Note that the map $\lambda \mapsto \mathbf{i}_\lambda := (0^{\lambda_0}, \dots, r^{\lambda_r})$ gives a bijection $\pi^\jmath \to \mathcal{I}_+$; here, we abbreviate the sequence $(k, \dots, k)$ of l copies of k to k^l. Then, by considering the map $\lambda \mapsto I_{\mathbf{i}_\lambda}$, we obtain an isomorphism of right $\mathbf{H}$-modules

$$\eta : \mathbf{V}^{\otimes d} \simeq \mathbf{T}(\pi^\jmath); \ u_{\mathbf{i}_\lambda} h \mapsto x_\lambda h \quad (\mathbf{i} \in \pi^\jmath, h \in \mathbf{H}).$$

Combining this isomorphism, we obtain the surjection

$$\xi : \mathbf{U}^\jmath \to \mathrm{End}_{\mathbf{H}}(\mathbf{T}(\pi^\jmath)) = \mathbf{S}(\pi^\jmath); \ x \mapsto \eta \circ \Phi(x) \circ \eta^{-1}.$$

In this subsection, we describe this surjection explicitly.

For $i \in \mathbb{I}^\jmath$, define two maps $\widetilde{e}_i, \widetilde{f}_i : \pi^\jmath \to \pi^\jmath \sqcup \{0\}$, where 0 denotes a formal symbol, as follows. For $\lambda \in \pi^\jmath$, we set

$$\widetilde{e}_i \lambda := \begin{cases} (\lambda_0, \dots, \lambda_{i-2}, \lambda_{i-1} + 1, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_r) & \text{if } \lambda_i > 0, \\ 0 & \text{if } \lambda_i = 0, \end{cases}$$

and

$$\widetilde{f}_i \lambda := \begin{cases} (\lambda_0, \dots, \lambda_{i-2}, \lambda_{i-1} - 1, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_r) & \text{if } \lambda_{i-1} > 0, \\ 0 & \text{if } \lambda_{i-1} = 0. \end{cases}$$

Recall that $\mathbf{S}(\pi^\jmath)$ has a basis $\{\xi_{\lambda, x, \mu} \mid \lambda, \mu \in \pi^\jmath, \ x \in {}^\lambda W^\mu\}$. By convention, we set $\xi_{\lambda, x, \mu} = 0$ if $\lambda = 0$ or $\mu = 0$.

Proposition 7.3.1. *Let* $i \in \mathbb{I}^\jmath$*. Then, we have*

$$\xi(e_i) = \sum_{\lambda \in \pi^\jmath} \xi_{\widetilde{e}_i \lambda, e, \lambda}, \quad \xi(f_i) = \sum_{\lambda \in \pi^\jmath} \xi_{\widetilde{f}_i \lambda, e, \lambda}.$$

Proof. We prove only the statement for f_1; the proofs for f_i, $i \neq 1$ and for e_i are similar. It suffices to show that

$$\xi(f_1)(x_\lambda) = \xi_{\widetilde{f_1}\lambda, e, \lambda}(x_\lambda)$$

for all $\lambda \in \pi^\jmath$.

Recall the comultiplication Δ of $\mathbf{U}$; we have

$$\Delta^{(d-1)}(E_{\underline{i}}) = \sum_{k=1}^{d} 1^{\otimes k-1} \otimes E_{\underline{i}} \otimes (K_{\underline{i}}^{-1})^{\otimes d-k}, \ \Delta^{(d-1)}(F_{\underline{i}}) = \sum_{k=1}^{d} K_{\underline{i}}^{\otimes d-k} \otimes F_{\underline{i}} \otimes 1^{\otimes k-1}.$$

Then, we compute as

$$\begin{aligned} f_1 u_{\mathbf{i}_\lambda} &= (pK_{-\underline{1}}^{-1} F_{\underline{1}} + E_{-\underline{1}}) u_{\mathbf{i}_\lambda} \\ &= pq^{-1} q^{\lambda_0} \sum_{k=1}^{\lambda_0} q^{\lambda_0 - k} u_{0^{\lambda_0-k},1,0^{k-1},1^{\lambda_1},\dots,r^{\lambda_r}} \\ &\quad + \sum_{k=1}^{\lambda_0} q^{\lambda_0-k} u_{0^{k-1},-1,0^{\lambda_0-k},1^{\lambda_1},\dots,r^{\lambda_r}} \\ &= pq^{\lambda_0-1} \sum_{k=1}^{\lambda_0} q^{\lambda_0-k} u_{\widetilde{f_1}(\lambda)} H_{\lambda_0-1} \cdots H_{\lambda_0-(k-1)} \\ &\quad + \sum_{k=1}^{\lambda_0} q^{\lambda_0-k} u_{\widetilde{f_1}(\lambda)} H_{\lambda_0-1} \cdots H_1 H_0 H_1 \cdots H_{k-1} \\ &= pq^{2(\lambda_0-1)} \sum_{k=1}^{\lambda_0} q^{-k+1} u_{\widetilde{f_1}(\lambda)} H_{\lambda_0-1} \cdots H_{\lambda_0-(k-1)} \\ &\quad + pq^{2(\lambda_0-1)} \sum_{k=1}^{\lambda_0} p^{-1} q^{-(\lambda_0+k-2)} u_{\widetilde{f_1}(\lambda)} H_{\lambda_0-1} \cdots H_1 H_0 H_1 \cdots H_{k-1}. \end{aligned}$$

On the other hand, we have

$$\xi_{\widetilde{f_1}\lambda, e, \lambda}(x_\lambda) = \eta_{\widetilde{f_1}\lambda, e, \lambda} = x_{\widetilde{f_1}\lambda} \sum_{v \in {}^{\widetilde{f_1}\lambda} W_\lambda} q_{x'} q_v^{-1} H_v,$$

where x' is the longest element of ${}^{\widetilde{f_1}\lambda} W_\lambda$. It is easy to see that

$${}^{\widetilde{f_1}\lambda} W_\lambda = \{ s_{\lambda_0-1} \cdots s_{\lambda_0-(k-1)} \mid k = 1, \dots, \lambda_0 \} \sqcup \{ s_{\lambda_0-1} \cdots s_1 s_0 s_1 \cdots s_{k-1} \mid k = 1, \dots, \lambda_0 \},$$

and hence, $q_{x'} = pq^{2(\lambda_0-1)}$. Therefore, we have

$$\xi(f_1)(x_\lambda) = \eta(f_1 u_\lambda) = \xi_{\widetilde{f_1}\lambda, e, \lambda}(x_\lambda),$$

as desired. This completes the proof. □

From this result, we see the compatibility of automorphisms on $\mathbf{U}^\jmath$ and $\mathbb{S}(\pi^\jmath)$.

Corollary 7.3.2. *Let $x \in \mathbf{U}^\jmath$. Then, we have*

$$\xi(\sigma^\jmath(x)) = \xi(x)^\flat, \quad \xi(\psi^\jmath(x)) = \overline{\xi(x)}.$$

Proof. It suffices to prove the assertions for $x \in \{ e_i, f_i, k_i^{\pm 1} \mid i \in \mathbb{I}^\jmath \}$. When $x = e_i, f_i$, it follows from Proposition 8.3.1. When $x = k_i^{\pm 1}$, it is easily verified by a direct calculation. □

Corollary 7.3.3. *There exists a nondegenerate bilinear form* $\langle \cdot \mid \cdot \rangle$ *on* $\mathbf{V}^{\otimes d}$ *such that*

$$\langle xu \mid v \rangle = \langle u \mid \sigma^\jmath(x) v \rangle$$

for all $x \in \mathbf{U}^\jmath$ *and* $u, v \in \mathbf{V}^{\otimes d}$.

Proof. The bilinear form defined by

$$\langle u \mid v \rangle := \langle \eta(u) \mid \eta(v) \rangle_{\pi^\jmath} \quad (u, v \in \mathbf{V}^{\otimes d})$$

is clearly nondegenerate. Moreover, for each $x \in \mathbf{U}^\jmath$ and $u, v \in \mathbf{V}^{\otimes d}$, we have

$$\begin{aligned} \langle xu \mid v \rangle &= \langle \eta(xu) \mid \eta(v) \rangle_{\pi^\jmath} \\ &= \langle \xi(x)(\eta(u)) \mid \eta(v) \rangle_{\pi^\jmath} \\ &= \langle \eta(u) \mid \xi(x)^\flat(\eta(v)) \rangle_{\pi^\jmath} \\ &= \langle \eta(u) \mid \xi(\sigma^\jmath(x))(\eta(v)) \rangle_{\pi^\jmath} \\ &= \langle \eta(u) \mid \eta(\sigma^\jmath(x) v) \rangle_{\pi^\jmath} \\ &= \langle u \mid \sigma^\jmath(x) v \rangle. \end{aligned}$$

This proves the corollary. □

7.4. **$\imath$-canonical basis of $\mathbf{V}^{\otimes d}$ and KL-bases.** It is easy to see that the basis $B_1 := \{u_{-r}, \dots, u_r\}$ of $\mathbf{V}$ is the canonical basis of $\mathbf{V}$. Hence, we can construct the canonical basis $B := B_1^{\diamond d}$ of the tensor product module $\mathbf{V}^{\otimes d}$.

On the other hand, as we have seen in the previous subsection, we have an isomorphism $\eta : \mathbf{V}^{\otimes d} \simeq \mathbf{T}(\pi^\jmath)$ of right $\mathbf{H}$-modules. Hence, $\mathbf{V}^{\otimes d}$ has another basis; the parabolic KL-basis $\{\eta^{-1}({}^\lambda C_w) \mid \lambda \in \pi^\jmath,\ w \in {}^\lambda W\}$.

Theorem 7.4.1. *The $\imath$-canonical basis* $B^{\diamond d}$ *and the parabolic KL-basis* $\{\eta^{-1}({}^\lambda C_w) \mid \lambda \in \pi^\jmath,\ w \in {}^\lambda W\}$ *of* $\mathbf{V}^{\otimes d}$ *coincide with each other.*

Proof. By definitions of the $\imath$-canonical basis and the parabolic KL-basis, it suffices to show that the bar-involution $\psi^\imath$ on the tensor product module $\mathbf{V}^{\otimes d}$ is compatible with the bar-involution $\overline{\cdot}$ on $\mathbf{T}(\pi^\imath)$ under η, namely $\eta \circ \psi^\imath = \overline{\cdot} \circ \eta$. $\psi^\imath$ is characterized by the following two conditions:

(1) $\psi^\imath(u_\lambda) = u_\lambda$ for all $\lambda \in \pi^\jmath$.
(2) $\psi^\imath(x u_\lambda) = \psi^\imath(x) u_\lambda$ for all $\lambda \in \pi^\jmath$ and $x \in \mathbf{U}^\jmath$.

Hence, in order to show that $\eta \circ \psi^\imath = \overline{\cdot} \circ \eta$, it is enough to verify that $\eta^{-1} \circ \overline{\cdot}\, \eta(u_\lambda) = u_\lambda$, and $\eta^{-1} \circ \overline{\cdot} \circ \eta(x u_\lambda) = \psi^\jmath x u_\lambda$ for all $\lambda \in \pi^\jmath$ and $x \in \mathbf{U}^\jmath$. The first one follows from the fact that $\eta(u_\lambda) = x_\lambda$ is bar-invariant. The second one is an immediate consequence of Corollary 7.3.2. □

In what follows, we identify $\mathbf{V}^{\otimes d}$ with $\mathbf{T}(\pi^\jmath)$ via the isomorphism η regarding not only the module structures but also the bar-involutions, and the skew-invariant forms.

8. $(\mathbf{U}^\imath, \mathcal{H})$-DUALITY

8.1. **Quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$.** Throughout this subsection, fix $r, d \in \mathbb{Z}_{>0}$. Set $\mathbb{I} := [-r, r]$, $\quad \mathbb{I}^\imath := [\underline{1}, \underline{r}]$.

Let $(P, P^\vee, \Pi, \Pi^\vee, \langle \cdot, \cdot \rangle)$ be the root datum of $\mathfrak{sl}_{2r+2}$-type constructed in Example 1.1.7 (1). Here, we replace I with $\mathbb{I}$, and the index set of the standard basis of $\mathbb{R}^n$ with $[-\underline{r+1}, \underline{r+1}]$. Then, the quantum group $U_q(\mathfrak{sl}_{2r+2})$ associated with this Cartan datum is generated by $E_i, F_i, K_i^{\pm 1}$, $i \in \mathbb{I}$.

Set $\mathbf{U} = \mathbf{U}_{2r+2} := \mathbb{Q}(p,q) \otimes_{\mathbb{Q}(q)} U_q(\mathfrak{sl}_{2r+2})$. The bar-involution on $U_q(\mathfrak{sl}_{2r+2})$ can be extended on $\mathbf{U}$ by setting $\overline{p} := p^{-1}$.

Next, let us consider the quantum symmetric pair $\mathbf{U}^\imath$ associated with the admissible pair given in Example 2.1.8. If we choose parameters appropriately, we have

$$B_i = \begin{cases} E_i + F_{-i}K_i^{-1} & \text{if } i \neq 0, \\ E_0 + qF_0K_0^{-1} + \frac{p-p^{-1}}{q-q^{-1}}K_0^{-1} & \text{if } i = 0. \end{cases}$$

Set

$$t := B_0,\ e_{\underline{i}} := B_i,\ f_{\underline{i}} := B_{-i},\ k_{\underline{i}} := K_iK_{-i}^{-1}, \quad \underline{i} \in \mathbb{I}^\imath.$$

Then, $\mathbf{U}^\imath$ is the subalgebra of $\mathbf{U}$ generated by $t, e_{\underline{i}}, f_{\underline{i}}, k_{\underline{i}}^{\pm 1}$, $\underline{i} \in \mathbb{I}^\imath$, and the defining relations become

$$\begin{aligned}
&k_{\underline{i}}k_{\underline{i}}^{-1} = 1 = k_{\underline{i}}^{-1}k_{\underline{i}}, \\
&k_{\underline{i}}k_{\underline{j}} = k_{\underline{j}}k_{\underline{i}}, \\
&k_{\underline{i}}tk_{\underline{i}}^{-1} = t, \\
&k_{\underline{i}}e_{\underline{j}}k_{\underline{i}}^{-1} = q^{\langle h_i - h_{-i}, \alpha_j\rangle}e_{\underline{j}}, \\
&k_{\underline{i}}f_{\underline{j}}k_{\underline{i}}^{-1} = q^{-\langle h_i - h_{-i}, \alpha_j\rangle}e_{\underline{j}}, \\
&e_{\underline{i}}e_{\underline{j}} - e_{\underline{j}}e_{\underline{i}} = 0 = f_{\underline{i}}f_{\underline{j}} - f_{\underline{j}}f_{\underline{i}}, \quad |i-j| > 1, \\
&e_{\underline{i}}t - te_{\underline{i}} = 0 = f_{\underline{i}}t - tf_{\underline{i}}, \quad i \neq 1, \\
&e_{\underline{i}}^{(2)}e_{\underline{j}} - e_{\underline{i}}e_{\underline{j}}e_{\underline{i}} + e_{\underline{j}}e_{\underline{i}}^{(2)} = 0 = f_{\underline{i}}^{(2)}f_{\underline{j}} - f_{\underline{i}}f_{\underline{j}}f_{\underline{i}} + f_{\underline{j}}f_{\underline{i}}^{(2)}, \quad |i-j| = 1, \\
&e_{\underline{i}}^{(2)}t - e_{\underline{i}}te_{\underline{i}} + te_{\underline{i}}^{(2)} = 0 = f_{\underline{i}}^{(2)}t - f_{\underline{i}}tf_{\underline{i}} + tf_{\underline{i}}^{(2)}, \quad |i-j| > 1, \\
&t^{(2)}e_{\underline{1}} - te_{\underline{1}}t + e_{\underline{1}}t^{(2)} = e_{\underline{1}}, \\
&t^{(2)}f_{\underline{1}} - tf_{\underline{1}}t + f_{\underline{1}}t^{(2)} = f_{\underline{1}}.
\end{aligned}$$

Proposition 8.1.1. *There exists a unique $\mathbb{Q}(p,q)$-algebra anti-automorphism ϱ on $\mathbf{U}^\imath$ that sends $t, e_{\underline{i}}, f_{\underline{i}}, k_{\underline{i}}$ to $t, qf_{\underline{i}}k_{\underline{i}}^{-1}, qe_{\underline{i}}k_{\underline{i}}, k_{\underline{i}}$, respectively for all $\underline{i} \in \mathbb{I}^\imath$.*

Proposition 8.1.2. *There exists a unique $\mathbb{Q}(p,q)$-algebra anti-automorphism $\sigma^\imath$ on $\mathbf{U}^\jmath$ that sends $t, e_{\underline{i}}, f_{\underline{i}}, k_{\underline{i}}$ to $t, f_{\underline{i}}, e_{\underline{i}}, k_{\underline{i}}$, respectively for all $i \in \mathbb{I}^\jmath$.*

8.2. **$(\mathbf{U}^\imath, \mathcal{H})$-duality.** Let $\mathbf{V}$ denote the natural representation of $\mathbf{U}$. Namely, it is a $(2r+2)$-dimensional $\mathbb{Q}(p,q)$-vector space with a basis $u_{-\underline{r+1}}, \dots, u_{-\underline{1}}, u_{\underline{1}}, \dots, u_{\underline{r+1}}$ equipped with a $\mathbf{U}$-module structure defined by

$$\begin{aligned}
K_iu_{\underline{j}} &:= q^{\delta_{i,j} - \delta_{i+1,j}}u_{\underline{j}}, \\
E_iu_{\underline{j}} &:= \delta_{i+1,j}u_{\underline{i}}, \\
F_iu_{\underline{j}} &:= \delta_{i,j}u_{\underline{i+1}}.
\end{aligned}$$

This $\mathbf{U}$-module $\mathbf{V}$ has a unique bar-involution fixing $u_{-\underline{r+1}}, \dots, u_{\underline{r+1}}$. Then, the d-th tensor product $\mathbf{V}^{\otimes d}$ of $\mathbf{V}$ is equipped with a $\mathbf{U}$-module structure, and a bar-involution, which is constructed from the one on $\mathbf{V}$ by means of the quasi-R-matrix Θ (see Lemma 3.6.5).

Let $\mathcal{I} := [-\underline{r+1}, \underline{r+1}]^d$ denote the set of all sequences $(i_1, \dots, i_d)$. For $\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{I}$, set

$$u_{\mathbf{i}} := u_{i_1} \otimes \cdots \otimes u_{i_d} \in \mathbf{V}^{\otimes d}.$$

Then, $\mathbf{V}^{\otimes d}$ has a basis $\{u_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{I}\}$.

Recall that the Weyl group W of type B_d acts naturally on $\mathcal{I}$; for $(i_1,\dots,i_d)\in\mathcal{I}$ and $j\in[1,d-1]$,

$$\begin{aligned}(i_1,\dots,i_d)\cdot s_0 &:= (-i_1,i_2,\dots,i_d),\\ (i_1,\dots,i_d)\cdot s_j &:= (i_1,\dots,i_{j-1},i_{j+1},i_j,i_{j+2},\dots,i_d).\end{aligned}$$

This action can be lifted to the action of $\mathbf{H}$ on $\mathbf{V}^{\otimes d}$ as follows; for $\mathbf{i}=(i_1,\dots,i_d)\in\mathcal{I}$ and $j\in[1,d-1]$,

$$u_{\mathbf{i}}\cdot H_0 := \begin{cases} u_{\mathbf{i}\cdot s_0}, & \text{if } i_1>0,\\ u_{\mathbf{i}\cdot s_0}+(p^{-1}-p)u_{\mathbf{i}}, & \text{if } i_1<0,\end{cases}$$

$$u_{\mathbf{i}}\cdot H_j := \begin{cases} q^{-1}u_{\mathbf{i}}, & \text{if } i_j=i_{j+1},\\ u_{\mathbf{i}\cdot s_j}, & \text{if } i_j<i_{j+1},\\ u_{\mathbf{i}\cdot s_j}+(q^{-1}-q)u_{\mathbf{i}}, & \text{if } i_j>i_{j+1}.\end{cases}$$

Theorem 8.2.1 ([BWW16, Lem 2.6]). *Let us denote the structure map of* $\mathbf{U}^\imath$*-module and* $\mathbf{H}$*-module structure on* $\mathbf{V}^{\otimes d}$ *by* Φ *and* Ψ*, respectively. Then, these two actions form a double centralizer;*

$$\Phi(\mathbf{U}^\imath)=\mathrm{End}_{\mathbf{H}}(\mathbf{V}^{\otimes d}),\quad \Psi(\mathbf{H})=\mathrm{End}_{\mathbf{U}^\imath}(\mathbf{V}^{\otimes d})^{\mathrm{op}}$$

8.3. **Surjection ξ.** As a right $\mathbf{H}$-module, $\mathbf{V}^{\otimes d}$ decomposes as

$$\mathbf{V}^{\otimes d}=\bigoplus_{\mathbf{i}\in\mathcal{I}_+}\mathbf{V}_{\mathbf{i}},\quad \mathbf{V}_{\mathbf{i}}:=u_{\mathbf{i}}\mathbf{H},$$

where $\mathcal{I}_+ := \{(i_1,\dots,i_d)\in\mathcal{I}\mid 0<i_1\le\cdots\le i_d\}$. It is straightforward to verify that for each $\mathbf{i}\in\mathcal{I}_+$, we have

$$\mathbf{V}_{\mathbf{i}}\simeq x_{I_{\mathbf{i}}}\mathbf{H},$$

where $I_{\mathbf{i}} := \{j\in[1,d-1]\mid \mathbf{i}\cdot s_j=\mathbf{i}\}$. Here, set $\pi^\imath := \{\lambda=(\lambda_1,\dots,\lambda_{r+1})\in\mathbb{Z}_{\ge0}^{r+1}\mid \sum_{i=1}^{r+1}\lambda_i=d\}$. Note that the map $\lambda\mapsto\mathbf{i}_\lambda := (\underline{1}^{\lambda_1},\dots,\underline{r+1}^{\lambda_{r+1}})$ gives a bijection $\pi^\imath\to\mathcal{I}_+$. Then, by considering the map $\lambda\mapsto I_{\mathbf{i}_\lambda}$, we obtain an isomorphism of right $\mathbf{H}$-modules

$$\eta:\mathbf{V}^{\otimes d}\simeq\mathbf{T}(\pi^\imath);\ u_{\mathbf{i}_\lambda}h\mapsto x_\lambda h\quad(\mathbf{i}\in\pi^\imath,h\in\mathbf{H}).$$

Combining this isomorphism, we obtain the surjection

$$\xi:\mathbf{U}^\imath\to\mathrm{End}_{\mathbf{H}}(\mathbf{T}(\pi^\imath))=\mathbf{S}(\pi^\imath);\ x\mapsto\eta\circ\Phi(x)\circ\eta^{-1}.$$

In this subsection, we describe this surjection explicitly.

For $\underline{i}\in\mathbb{I}^\imath$, define two maps $\widetilde{e}_{\underline{i}},\widetilde{f}_{\underline{i}}:\pi^\imath\to\pi^\imath\sqcup\{0\}$, where 0 denotes a formal symbol, as follows. For $\lambda\in\pi^\imath$, we set

$$\widetilde{e}_{\underline{i}}\lambda := \begin{cases}(\lambda_1,\dots,\lambda_{i-1},\lambda_i+1,\lambda_{i+1}-1,\lambda_{i+2},\dots,\lambda_{r+1}) & \text{if } \lambda_{i+1}>0,\\ 0 & \text{if } \lambda_{i+1}=0,\end{cases}$$

and

$$\widetilde{f}_{\underline{i}}\lambda := \begin{cases}(\lambda_1,\dots,\lambda_{i-1},\lambda_i-1,\lambda_{i+1}+1,\lambda_{i+2},\dots,\lambda_{r+1}) & \text{if } \lambda_i>0,\\ 0 & \text{if } \lambda_i=0.\end{cases}$$

Recall that $\mathbf{S}(\pi^\imath)$ has a basis $\{\xi_{\lambda,x,\mu}\mid \lambda,\mu\in\pi^\imath,\ x\in{}^\lambda W^\mu\}$. By convention, we set $\xi_{\lambda,x,\mu}=0$ if $\lambda=0$ or $\mu=0$.

Proposition 8.3.1. *Let* $\underline{i} \in \mathbb{I}^\imath$. *Then, we have*

$$\xi(e_{\underline{i}}) = \sum_{\lambda \in \pi^\imath} \xi_{\widetilde{e_{\underline{i}}}\lambda, e, \lambda}, \quad \xi(f_{\underline{i}}) = \sum_{\lambda \in \pi^\imath} \xi_{\widetilde{f_{\underline{i}}}\lambda, e, \lambda}.$$

Proof. We prove only the statement for $f_{\underline{i}}$; the proof for $e_{\underline{i}}$ is similar. It suffices to show that

$$\xi(f_{\underline{i}})(x_\lambda) = \xi_{\widetilde{f_{\underline{i}}}\lambda, e, \lambda}(x_\lambda)$$

for all $\lambda \in \pi^\imath$.

Recall the comultiplication Δ of $\mathbf{U}$; we have for $j \in \mathbb{I}$,

$$\Delta^{(d-1)}(F_j) = \sum_{k=1}^{d} K_j^{\otimes(d-k)} \otimes F_j \otimes 1^{\otimes(k-1)}.$$

Then, we compute as follows:

$$\begin{aligned} f_i u_\lambda &= (E_{-i} + F_i K_{-i}^{-1}) u_\lambda \\ &= F_i u_\lambda \\ &= \sum_{k=1}^{\lambda_i} q^{\lambda_i - k} u_{\dots, \underline{i-1}^{\lambda_{i-1}}, \underline{i}^{\lambda_i - k}, \underline{i+1}, \underline{i}^{k-1}, \underline{i+1}^{\lambda_{i+1}}, \dots} \\ &= u_{\widetilde{f_{\underline{i}}}\lambda} \sum_{k=1}^{\lambda_i} q^{\lambda_i - k} H_{\lambda_{1,i}-1} H_{\lambda_{1,i}-2} \cdots H_{\lambda_{1,i}-(k-1)}, \end{aligned}$$

where $\lambda_{1,i} := \lambda_1 + \lambda_2 + \dots + \lambda_i$.

On the other hand, we have

$$\xi_{\widetilde{f_{\underline{i}}}\lambda, e, \lambda}(x_\lambda) = \eta_{\widetilde{f_{\underline{i}}}\lambda, e, \lambda} = x_{\widetilde{f_{\underline{i}}}} \sum_{v \in {}^{\widetilde{f_{\underline{i}}}\lambda}W_\lambda} q_{x'} q_v^{-1} H_v,$$

where x' is the longest element in ${}^{\widetilde{f_{\underline{i}}}\lambda}W_\lambda$. It is easy to see that

$${}^{\widetilde{f_{\underline{i}}}\lambda}W_\lambda = \{ s_{\lambda_{1,i}-1} s_{\lambda_{1,i}-2} \cdots s_{\lambda_{1,i}-(k-1)} \mid k = 1, \dots, \lambda_i \},$$

and hence, $q_{x'} = q^{\lambda_i - 1}$. Therefore, we have

$$\xi(f_{\underline{i}})(x_\lambda) = \eta(f_{\underline{i}} u_\lambda) = x_{\widetilde{f_{\underline{i}}}\lambda} \sum_{k=1}^{\lambda_i} q^{\lambda_i - k} H_{\lambda_{1,i}-1} H_{\lambda_{1,i}-2} \cdots H_{\lambda_{1,i}-(k-1)} = \xi_{\widetilde{f_{\underline{i}}}\lambda, e, \lambda}(x_\lambda),$$

as desired. This completes the proof. □

Proposition 8.3.2. *We have* $\xi(t) = \sum_{\lambda \in \pi^\imath} (\xi_{\lambda, s_0, \lambda} + q^{\lambda_1} \frac{p - p^{-1}}{q - q^{-1}} \xi_{\lambda, e, \lambda})$. *Moreover,* $\overline{\xi(t)} = \xi(t)$.

Proof. It suffices to show that

$$\xi(t)(x_\lambda) = (\xi_{\lambda, s_0, \lambda} + q^{\lambda_1} \frac{p - p^{-1}}{q - q^{-1}} \xi_{\lambda, e, \lambda})(x_\lambda)$$

for all $\lambda \in \pi^\imath$.

Recall the comultiplication Δ of $\mathbf{U}$; we have for $j \in \mathbb{I}$,

$$\Delta^{(d-1)}(E_j) = \sum_{k=1}^{d} 1^{\otimes(k-1)} \otimes E_j \otimes (K_j^{-1})^{\otimes(d-k)}.$$

Then, we compute as

$$\begin{aligned} tu_\lambda &= (E_0 + qF_0K_0^{-1} + \frac{p-p^{-1}}{q-q^{-1}}K_0^{-1})u_\lambda \\ &= E_0u_\lambda + q^{\lambda_1}\frac{p-p^{-1}}{q-q^{-1}}u_\lambda \\ &= \sum_{k=1}^{\lambda_1} q^{\lambda_1-k}u_{\underline{1}^{k-1},-\underline{1},\underline{1}^{\lambda_1-k},\underline{2}^{\lambda_2},\dots} + q^{\lambda_1}\frac{p-p^{-1}}{q-q^{-1}}u_\lambda \\ &= u_\lambda(\sum_{k=1}^{\lambda_1} q^{\lambda_1-k}H_0H_1\cdots H_{k-1} + q^{\lambda_1}\frac{p-p^{-1}}{q-q^{-1}}). \end{aligned}$$

On the other hand, we have

$$\xi_{\lambda,s_0,\lambda}(x_\lambda) = \eta_{\lambda,s_0,\lambda} = x_\lambda(\sum_{k=0}^{\lambda_1-1} q^{\lambda_1-1-k}H_0H_1\cdots H_k),$$

and

$$\xi_{\lambda,e,\lambda}(x_\lambda) = \eta_{\lambda,e,\lambda} = x_\lambda.$$

Thus, we obtain

$$\xi(t)(x_\lambda) = \eta(tu_\lambda) = (\xi_{\lambda,s_0,\lambda} + q^{\lambda_1}\frac{p-p^{-1}}{q-q^{-1}}\xi_{\lambda,e,\lambda})(x_\lambda),$$

which proves first equation of the proposition.

In order to prove the second equation, it suffices to show that $x_\lambda(\sum_{k=0}^{\lambda_1-1} q^{\lambda_1-1-k}H_0H_1\cdots H_k + q^{\lambda_1}\frac{p-p^{-1}}{q-q^{-1}})$ is bar-invariant. By induction on k, one can verify that

$$\overline{x_\lambda H_0H_1\cdots H_k} = x_\lambda(H_0H_1\cdots H_k + \sum_{l=0}^{k-1} q^{k-1-l}(q-q^{-1})H_0H_1\cdots H_l + (p-p^{-1})q^l).$$

Then,

$$\begin{aligned} &\overline{x_\lambda \sum_{k=0}^{\lambda_1-1} q^{\lambda_1-1-k}H_0H_1\cdots H_k} \\ &= x_\lambda \sum_{k=0}^{\lambda_1-1} q^{-\lambda+1+k}\left(H_0H_1\cdots H_k + \sum_{l=0}^{k-1} q^{k-1-l}(q-q^{-1})H_0H_1\cdots H_l + (p-p^{-1})q^l\right) \\ &= x_\lambda\left(\sum_{l=1}^{\lambda_1-1}(q^{-\lambda_1+1+l} + \sum_{k=l+1}^{\lambda_1-1} q^{-\lambda_1+1+k}q^{k-1-l}(q-q^{-1}))H_0H_1\cdots H_l + \sum_{k=0}^{\lambda_1-1}(p-p^{-1})q^{-\lambda_1+1+2k}\right) \\ &= x_\lambda\left(\sum_{l=1}^{\lambda_1-1} q^{\lambda_1-1-l}H_0H_1\cdots H_l + (q^{\lambda_1-q^{-\lambda_1}})\frac{p-p^{-1}}{q-q^{-1}}\right). \end{aligned}$$

Hence, it follows that $x_\lambda(\sum_{k=0}^{\lambda_1-1} q^{\lambda_1-1-k}H_0H_1\cdots H_k + q^{\lambda_1}\frac{p-p^{-1}}{q-q^{-1}})$ is bar-invariant, as desired. This completes the proof. □

Corollary 8.3.3. *Let $x \in \mathbf{U}^\imath$. Then, we have*

$$\xi(\sigma^\imath(x)) = \xi(x)^\flat, \quad \xi(\psi^\imath(x)) = \overline{\xi(x)}.$$

Corollary 8.3.4. *There exists a nondegenerate bilinear form* $\langle \cdot \mid \cdot \rangle$ *on* $\mathbf{V}^{\otimes d}$ *such that*

$$\langle xu \mid v \rangle = \langle u \mid \sigma^{\imath}(x)v \rangle$$

for all $x \in \mathbf{U}^{\imath}$ *and* $u, v \in \mathbf{V}^{\otimes d}$.

8.4. **$\imath$-canonical basis of $\mathbf{V}^{\otimes d}$ and KL-bases.** It is easy to see that the basis $B_1 := \{u_{-\underline{r+1}}, \dots, u_{-\underline{1}}, u_{\underline{1}}, \dots, u_{\underline{r+1}}\}$ of $\mathbf{V}$ is the canonical basis of $\mathbf{V}$. Hence, we can construct the canonical basis $B := B_1^{\diamond d}$ of the tensor product module $\mathbf{V}^{\otimes d}$.

On the other hand, as we have seen in the previous subsection, we have an isomorphism $\eta : \mathbf{V}^{\otimes d} \simeq \mathbf{T}(\pi^{\imath})$ of right $\mathbf{H}$-modules. Hence, $\mathbf{V}^{\otimes d}$ has another basis; the parabolic KL-basis $\{\eta^{-1}({}^{\lambda}C_w) \mid \lambda \in \pi^{\imath},\ w \in {}^{\lambda}W\}$.

Theorem 8.4.1. *The $\imath$-canonical basis* $B^{\diamond d}$ *and the parabolic KL-basis* $\{\eta^{-1}({}^{\lambda}C_w) \mid \lambda \in \pi^{\imath},\ w \in {}^{\lambda}W\}$ *of* $\mathbf{V}^{\otimes d}$ *coincide with each other.*

In what follows, we identify $\mathbf{V}^{\otimes d}$ with $\mathbf{T}(\pi^{\jmath})$ via the isomorphism η regarding not only the module structures but also the bar-involutions, and the skew-invariant forms.

Part 3. Representation theory of $\mathbf{U}^\jmath$

The aim of this part is to develop highest weight theory and crystal basis theory for $\mathbf{U}^\jmath$. The results here are contained in [W17] and [W18].

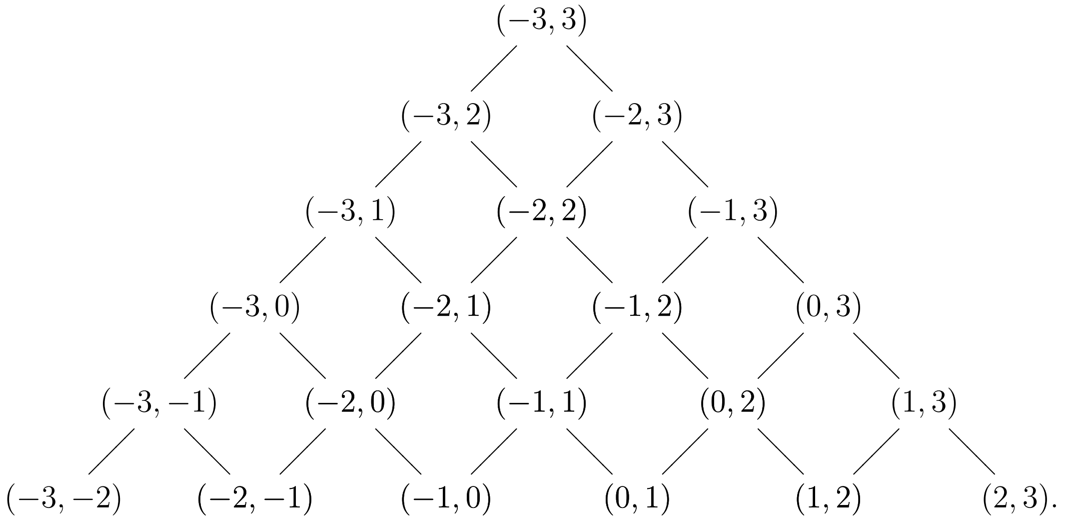
9. BASICS OF THE QUANTUM SYMMETRIC PAIR $(\mathbf{U}, \mathbf{U}^\jmath)$

The aim of this section is to formulate what we will study in this part. In Subsection 9.1, we decompose $\mathbf{U}^\jmath$ into three parts by means of PBW-type basis constructed in Subsection 4.3. This decomposition leads us to the highest weight theory for $\mathbf{U}^\jmath$. In Subsection 9.2, we define analogs of Verma modules and irreducible highest weight modules.

9.1. **Triangular decomposition of $\mathbf{U}^\jmath$.** Recall that $\Phi_+ = \{\epsilon_i - \epsilon_j \mid -r \leq i < j \leq r\}$ denotes the set of positive roots of Φ of $\mathbf{U}$ with respect to the simple roots $\Pi = \{\epsilon_i - \epsilon_{i+1} \mid -r \leq i < r\}$. We decompose Φ_+ into three parts as:

$$\begin{aligned} \Phi_+ &= \Phi_{<0} \sqcup \Phi_0 \sqcup \Phi_{>0}, \\ \Phi_{<0} &:= \{\epsilon_i - \epsilon_j \mid i + j < 0\}, \\ \Phi_0 &:= \{\epsilon_i - \epsilon_j \mid i + j = 0\}, \\ \Phi_{>0} &:= \{\epsilon_i - \epsilon_j \mid i + j > 0\}. \end{aligned}$$

For example, when $r = 3$, the positive roots are displayed as follows:



Here, (i, j) represents $\epsilon_i - \epsilon_j$. Then, the roots in Φ_0 lie on the vertical line through $(-3, 3)$, those in $\Phi_{<0}$ on the left to the line, and those in $\Phi_{>0}$ on the right.

Here, we recall the notion of reflection orders (or convex orders).

Definition 9.1.1. A total order $\preceq$ on Φ_+ is said to be a reflection order if it satisfies the following: for each $\alpha, \beta \in \Phi_+$ and $a, b \in \mathbb{R}_{>0}$, if $a\alpha + b\beta \in \Phi_+$ and $\alpha \prec \beta$, then $\alpha \prec a\alpha + b\beta \prec \beta$.

Proposition 9.1.2 ([D93, Proposition 2.13])**.** *Let $W(\mathbb{I})$ denote the Weyl group associated with our Cartan datum. Let $\mathbf{i} = (i_1, \ldots, i_N)$ be a reduced word for $w_0 \in W(\mathbb{I})$. Set $\alpha(\mathbf{i})_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then, the total order $\preceq$ on Φ_+ defined by $\alpha(\mathbf{i})_1 \prec \cdots \prec \alpha(\mathbf{i})_N$ is a reflection order. Moreover, this correspondence gives a bijection between the set of reduced words for $w_0 \in W(\mathbb{I})$ and the set of reflection orders on Φ_+.*

Lemma 9.1.3. *There exists a reflection order $\preceq$ on Φ_+ such that*

$$\Phi_{<0} \prec \Phi_0 \prec \Phi_{>0}. \tag{6}$$

Here, for subsets $A, B \subset \Phi_+$, $A \prec B$ means that $\alpha \prec \beta$ for all $\alpha \in A$ and $\beta \in B$.

Proof. It suffices to construct such a reflection order. For simplicity, we write (i,j) instead of $\epsilon_i - \epsilon_j$ for $i < j$. We decompose $\Phi_{<0}$ into $\Phi_{<0,-} := \{(i,j) \in \Phi_{<0} \mid j \leq 0\}$ and $\Phi_{<0,+} := \{(i,j) \in \Phi_{<0} \mid j > 0\}$. Similarly, we set $\Phi_{>0,-} := \{(i,j) \in \Phi_{>0} \mid i < 0\}$ and $\Phi_{>0,+} := \{(i,j) \in \Phi_{>0} \mid i \geq 0\}$. Let us define a total order $\preceq$ on Φ_+ by:

(1) $\Phi_{<0,-} \prec \Phi_{<0,+} \prec \Phi_0 \prec \Phi_{>0,-} \prec \Phi_{>0,+}$;
(2) for $(i,j), (i',j') \in \Phi_{<0,-}$, $(i,j) \prec (i',j')$ if and only if $i < i'$ or ($i = i'$ and $j < j'$);
(3) for $(i,j), (i',j') \in \Phi_{<0,+}$, $(i,j) \prec (i',j')$ if and only if $j < j'$ or ($j = j'$ and $i < i'$);
(4) for $(i,j), (i',j') \in \Phi_0$, $(i,j) \prec (i',j')$ if and only if $j < j'$;
(5) for $(i,j), (i',j') \in \Phi_{>0,-}$, $(i,j) \prec (i',j')$ if and only if $i < i'$ or ($i = i'$ and $j < j'$);
(6) for $(i,j), (i',j') \in \Phi_{>0,+}$, $(i,j) \prec (i',j')$ if and only if $j < j'$ or ($j = j'$ and $i < i'$).

Then, $\preceq$ is a reflection order on Φ_+ satisfying $\Phi_{<0} \prec \Phi_0 \prec \Phi_{>0}$; the proof is straightforward. □

Example 9.1.4. When $r = 3$, this total order is given as follows:

$$\begin{aligned}
&(-3,-2) \prec (-3,-1) \prec (-3,0) \prec (-2,-1) \prec (-2,0) \prec (-1,0) \\
\prec &(-3,1) \prec (-2,1) \prec (-3,2) \\
\prec &(-1,1) \prec (-2,2) \prec (-3,3) \\
\prec &(-2,3) \prec (-1,2) \prec (-1,3) \\
\prec &(0,1) \prec (0,2) \prec (1,2) \prec (0,3) \prec (1,3) \prec (2,3).
\end{aligned}$$

Fix a reflection order $\preceq$ satisfying condition (6) in Lemma 9.1.3. Let $\mathbf{i}$ be the reduced word for $w_0 \in W(\mathbb{I})$ corresponding to $\preceq$ under the bijection of Proposition 9.1.2. We set $E_{i,j} := E(\mathbf{i})_k$ for $-r \leq i < j \leq r$, where k is such that $\epsilon_i - \epsilon_j = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. For each i, j, define $E'_{i,j} := \mathrm{gr}^{-1} \circ p(E_{i,j})$, and set

$$f_{-j,-i} := E'_{i,j} \ \text{ if } i + j < 0, \quad h'_i := E'_{-i,i}, \quad e_{i,j} := E'_{i,j} \ \text{ if } i + j > 0.$$

Remark 9.1.5. By the construction, the classical limit of $E'_{i,j}$ becomes a scalar multiple of $B_{r+i+1,r+j+1}$ for all $-r \leq i < j \leq r$ (see Example 2.1.8 (1)).

Theorem 9.1.6. *The ordered monomials $\left(\prod_{i+j<0} f_{-j,-i}^{a_{i,j}}\right) \left(\prod_i (h'_i)^{b_i}\right) \left(\prod_{i=1}^r k_i^{d_i}\right) \left(\prod_{i+j>0} e_{i,j}^{c_{i,j}}\right)$, $a_{i,j}, b_i, c_{i,j} \in \mathbb{Z}_{\geq 0}$, $d_i \in \mathbb{Z}$ form a linear basis of $\mathbf{U}^\jmath$.*

Let us compute some of the root vectors. By [LS91, Lemma 1] (with a slight modification), we have

$$\begin{aligned}
E_{i-1,j} &= [E_{i,j}, E_{\underline{i}}]_1 \quad \text{if } (i-1,i) \prec (i,j), \\
E_{i,j+1} &= [E_{\underline{j+1}}, E_{i,j}]_1 \quad \text{if } (i,j) \prec (j,j+1).
\end{aligned}$$

In particular, it holds that

$$E_{-1,1} = [E_{\underline{1}}, E_{-\underline{1}}]_1, \quad E_{-(i+1),i+1} = \left[[E_{\underline{i+1}}, E_{-i,i}]_1, E_{-\underline{i+1}}\right]_1 \quad \text{for } 1 \leq i \leq r-1.$$

Thus, we obtain

$$h'_1 = [e_1, f_1]_1, \quad h'_{i+1} = \left[[e_{i+1}, h'_i]_1, f_{i+1}\right]_1. \tag{7}$$

This shows that the h_i''s are independent of the choice of a reflection order $\preceq$ satisfying condition (6) in Lemma 9.1.3.

Let $\mathbf{U}^\jmath_{<0}$ (resp., $\mathbf{U}^\jmath_0, \mathbf{U}^\jmath_{>0}$) denote the subspace of $\mathbf{U}^\jmath$ spanned by all ordered monomials in $f_{-j,-i}$ (resp., $h_i, e_{i,j}$). Then, we have an isomorphism of vector spaces

$$\mathbf{U}^\jmath \simeq \mathbf{U}^\jmath_{<0} \bigotimes \left(\mathbf{U}^\jmath_0 \bigotimes \mathbf{U}^{\jmath,0}\right) \bigotimes \mathbf{U}^\jmath_{>0}.$$

We call this linear isomorphism the triangular decomposition of $\mathbf{U}^\jmath$ associated with the reflection order $\preceq$, and $\mathbf{U}^\jmath_{<0}$ (resp., $\mathbf{U}^\jmath_0 \bigotimes \mathbf{U}^{\jmath,0}$, $\mathbf{U}^\jmath_{>0}$) the negative part (resp., Cartan part, positive part) of $\mathbf{U}^\jmath$. The triangular decomposition enables us to establish an analog of highest weight theory for the representation theory of $\mathbf{U}^\jmath$.

Remark 9.1.7. Unlike the ordinary triangular decomposition of a quantum group, the three parts of $\mathbf{U}^\jmath$ are just subspaces, not subalgebras. In addition, the negative part and the positive part may depend on the choice of a reflection order. However, by equation (7), the Cartan part is independent of such a choice.

9.2. **Verma modules and their irreducible quotients.** Recall $P_\imath$ and $P_\imath^\vee$ from page 22. When considering $\mathbf{U}^\jmath$, we denote them by $P_\jmath$ and $P_\jmath^\vee$, respectively. For $i \in \mathbb{I}^\jmath$, set

$$\beta_i := h_{\underline{i}} - h_{-\underline{i}} = \epsilon_{i-1} - \epsilon_i - \epsilon_{-i} + \epsilon_{-(i-1)} \in P_\jmath^\vee.$$

For each $i \in \mathbb{I}^\jmath$, there exists a unique $\delta_i \in \mathbb{R} \otimes_{\mathbb{Z}} P_\jmath$ such that $\langle \beta_j, \delta_i \rangle = \delta_{i,j}$ for all $j \in \mathbb{I}^\jmath$. Set

$$\Lambda^\jmath := \sum_{i \in \mathbb{I}^\jmath} \mathbb{Z}\delta_i, \text{ and } \gamma_i := \text{the image of } \alpha_{\underline{i}} \text{ in } P_\jmath^\vee.$$

By the definitions, we have

$$\langle \beta_i, \gamma_j \rangle = \langle h_{\underline{i}} - h_{-\underline{i}}, \alpha_{\underline{j}} \rangle = \begin{cases} 3 & \text{if } i = j = 1, \\ 2 & \text{if } i = j \neq 1, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

Set $Q^\jmath_+ := \sum_{i \in \mathbb{I}^\jmath} \mathbb{Z}_{\geq 0}\gamma_i$, and define a partial order $\leq$ on $\Lambda^\jmath$ by:

$$\mu \leq \lambda \text{ if and only if } \lambda - \mu \in Q^\jmath_+. \tag{8}$$

For a $\mathbf{U}^\jmath$-module M and $m \in M$, we say that m is of weight $\lambda \in \Lambda^\jmath$ if it satisfies

$$k_i m = q^{\langle \beta_i, \lambda \rangle} m$$

for all $i \in \mathbb{I}^\jmath$; we denote by M_λ the subspace consisting of all $m \in M$ of weight λ.

Lemma 9.2.1. *Let M be a $\mathbf{U}^\jmath$-module and $\lambda \in \Lambda^\jmath$. For each $i \in \mathbb{I}^\jmath$, we have*

$$f_i(M_\lambda) \subset M_{\lambda - \gamma_i}, \qquad e_i(M_\lambda) \subset M_{\lambda + \gamma_i}.$$

Proof. This follows immediately from the relations $k_i f_j k_i^{-1} = q^{\langle \beta_i, -\gamma_j \rangle} f_j$ and $k_i e_j k_i^{-1} = q^{\langle \beta_i, \gamma_j \rangle} e_j$. □

Recall the triangular decomposition of $\mathbf{U}^\jmath$

$$\mathbf{U}^\jmath \simeq \mathbf{U}^\jmath_{<0} \bigotimes \left(\mathbf{U}^\jmath_0 \bigotimes \mathbf{U}^{\jmath,0}\right) \bigotimes \mathbf{U}^\jmath_{>0},$$

and the root vectors $f_{-j,-i}, h_i', e_{i,j}$ associated with a reflection order satisfying condition (6) in Lemma 9.1.3. Let $(\mathbf{U}^\jmath_{>0})_+$ denote the subspace of $\mathbf{U}^\jmath_{>0}$ spanned by all ordered monomials in $e_{i,j}$'s other than 1.

Definition 9.2.2. Let $\lambda \in \Lambda^\imath$ and $H'_i \in \mathbb{Q}(p,q)$, $i = 1, 2, \dots, r$. The Verma module $V'(\lambda; \mathbf{H}')$ over $\mathbf{U}^\imath$ with highest weight λ associated with $\mathbf{H}' := (H'_1, \dots, H'_r) \in \mathbb{Q}(p,q)^r$ is defined to be

$$V'(\lambda; \mathbf{H}') := \mathbf{U}^\imath / I(\lambda; \mathbf{H}'),$$

where $I(\lambda; \mathbf{H}')$ denotes the left ideal of $\mathbf{U}^\imath$ generated by $(\mathbf{U}^\imath_{>0})_+$ and $k_i - q^{\langle \beta_i, \lambda \rangle}$, $h'_i - H'_i$ for $i \in \mathbb{I}^\imath$.

Remark 9.2.3. Verma modules $V'(\lambda; \mathbf{H}')$ can be 0.

By the triangular decomposition of $\mathbf{U}^\imath$, a nonzero Verma module $V'(\lambda; \mathbf{H}')$ has a unique maximal submodule, and hence, it has a unique irreducible quotient. We denote it by $L'(\lambda; \mathbf{H}')$ and call it the irreducible highest weight $\mathbf{U}^\imath$-module with highest weight λ associated with $\mathbf{H}'$, or simply, with highest weight $(\lambda; \mathbf{H}')$.

Definition 9.2.4. A nonzero $\mathbf{U}^\imath$-module M is called a highest weight module with highest weight $(\lambda; \mathbf{H}') \in \Lambda^\imath \times \mathbb{Q}(p,q)^r$ if there exists $m \in M_\lambda$ such that $(\mathbf{U}^\imath_{>0})_+ m = 0$, $h'_i m = H'_i m$ for $i \in \mathbb{I}^\imath$, and $M = \mathbf{U}^\imath m$. We call such an m a highest weight vector of M with highest weight $(\lambda; \mathbf{H}')$.

Our definition of highest weight modules over $\mathbf{U}^\imath$ depends on the choice of a reflection order satisfying condition (6) in Lemma 9.1.3. However, their $\mathbf{U}^\imath$-module structure is independent of such a choice, as we explain below.

Let M be a highest weight $\mathbf{U}^\imath$-module with highest weight $(\lambda; \mathbf{H}')$ associated with a reflection order $\preceq$. Let $v \in M$ be a highest weight vector. Take another reflection order $\preceq'$, and denote the corresponding root vectors by $f'_{i,j}, h''_i, e'_{i,j}$. Then, we see from equation (7) that $h''_i = h'_i$. Also, by the triangular decomposition associated with $\prec$, we have

$$e'_{i,j} \in \sum_{\substack{\nu, \mu \in Q^\imath_+ \\ \nu < \mu}} (\mathbf{U}^\imath_{<0})_{-\nu} \otimes (\mathbf{U}^\imath_0 \otimes \mathbf{U}^{\imath,0}) \otimes (\mathbf{U}^\imath_{>0})_\mu;$$

here, $(\mathbf{U}^\imath_{<0})_{-\nu} := \{x \in \mathbf{U}^\imath_{<0} \mid k_i x k_i^{-1} = q^{\langle \beta_i, -\nu \rangle} x \text{ for all } i \in \mathbb{I}^\imath\}$, and define $(\mathbf{U}^\imath_{>0})_\mu$ similarly. Therefore, it holds that $e'_{i,j} v = 0$ for all i, j. In addition, by expanding $f_{i,j}$ in ordered monomials in $f'_{i,j}, h''_i, e'_{i,j}$, we see that $f_{i,j} v$ is a linear combination of $f'_{i,j} v$'s. From these, we conclude that M is a highest weight module with highest weight $(\lambda; \mathbf{H}')$ associated with $\preceq'$. In particular, if we denote Verma modules and their irreducible quotients associated with $\preceq'$ by $V''(\cdot; \cdot)$ and $L''(\cdot; \cdot)$, respectively, then we have

$$V'(\lambda; \mathbf{H}') = V''(\lambda; \mathbf{H}'), \quad L'(\lambda; \mathbf{H}') = L''(\lambda; \mathbf{H}').$$

Hence, in what follows, we use only the reflection order given in the proof of Lemma 9.1.3.

Let $\mathcal{O}^\imath_{\mathrm{int}}$ denote the category of all $\mathbf{U}^\imath$-modules M satisfying the following:

($M1$) M is decomposed into weight spaces, i.e., $M = \bigoplus_{\lambda \in \Lambda^\imath} M_\lambda$.
($M2$) Each weight space is finite-dimensional.
($M3$) There exist finitely many weights $\mu_1, \dots \mu_n \in \Lambda^\imath$ such that each weight $\lambda \in \Lambda^\imath$ for which $M_\lambda \neq 0$ satisfies $\lambda \leq \mu_i$ for some $i = 1, \dots, n$.
($M4$) e_i and f_i act on M locally nilpotently, that is, for each $m \in M$, there exists $N \in \mathbb{N}$ such that $e_i^N m = 0 = f_i^N m$.

Note that Verma modules and their irreducible quotients are not necessarily objects of $\mathcal{O}^\imath_{\mathrm{int}}$, i.e., the actions of f_i on these modules are not always locally nilpotent. Also, $\mathcal{O}^\imath_{\mathrm{int}}$ has an infinite-dimensional object as we will see in the next section.

10. The case $r = 1$

In this section, we restrict our attention to $\mathbf{U}^\jmath_1$, the smallest $\mathbf{U}^\jmath$. In Subsection 10.1, we classify all the irreducible highest weight modules. Subsection 10.2 is devoted to proving the complete reducibility.

10.1. Classification of the irreducible modules in $\mathcal{O}^\jmath_{\mathrm{int}}$.

We introduce some more notation.

Definition 10.1.1.
(1) For $x \in \mathbf{U}$ and $n \in \mathbb{Z}_{>0}$, $x^{(n)} := \frac{x^n}{[n]!}$; we set $x^{(0)} := 1$, and $x^{(n)} := 0$ if $n < 0$.
(2) For $x, y \in \mathbf{U}$ and $a \in \mathbb{Z}$, $[x, y]_a := xy - q^a yx$.
(3) For an invertible element h, $\{h\} := h + h^{-1}$.
(4) For an integer $n \in \mathbb{Z}$, $\{n\} := \{pq^n\} = pq^n + p^{-1}q^{-n}$.

In the case $r = 1$, the root vectors are

$$f_{0,1} = f_1, \quad h'_1 = [e_1, f_1]_1, \quad e_{0,1} = e_1.$$

Then, the following are easy to verify.

Lemma 10.1.2. *In $\mathbf{U}^\jmath_1$, we have*

$$[h'_1, f_1]_{-1} = -[2]\{pqk_1\}f_1, \quad [e_1, h'_1]_{-1} = -[2]e_1\{pqk_1\}.$$

Lemma 10.1.3. *For each $n \in \mathbb{Z}_{\geq 0}$, we have*

$$e_1 f_1^{(n)} = f_1^{(n-1)}\left(h'_1 - [n-1]\{pq^{-n}k_1\}\right) + q^n f_1^{(n)} e_1.$$

Note that when $r = 1$, we have $\Lambda^\jmath = \mathbb{Z}\delta_1$ and $\gamma_1 = 3\delta_1$. Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$. By the definition of $\mathcal{O}^\jmath_{\mathrm{int}}$, there exists $a \in \mathbb{Z}$ such that $M_{a\delta_1} \neq \{0\}$ and $M_{(a+3)\delta_1} = \{0\}$. Since the action of h'_1 preserves the weights, it defines a linear endomorphism of $M_{a\delta_1}$. In order to consider the Jordan canonical form for the action of h'_1 on $M_{a\delta_1}$, we extend the base field $\mathbb{Q}(p, q)$ to its algebraic closure $\overline{\mathbb{Q}(p, q)}$ until the proof of Proposition 10.1.4. Let us write the Jordan canonical form as:

$$\begin{pmatrix} J_{d_1}(\mu_1) & & & \\ & J_{d_2}(\mu_2) & & \\ & & \ddots & \\ & & & J_{d_m}(\mu_m) \end{pmatrix},$$

where $J_{d_i}(\mu_i)$ denotes the Jordan block of size d_i whose eigenvalue is $\mu_i \in \overline{\mathbb{Q}(p, q)}$. We take a basis $\{v_{j,k} \mid j = 1, \dots, m,\ k = 1, \dots, d_j\}$ of $M_{a\delta_1}$ in such a way that

$$h'_1 v_{j,k} = \mu_j v_{j,k} + v_{j,k-1}$$

for all $j = 1, \dots, m,\ k = 1, \dots, d_j$, where $v_{j,0} := 0$. By Lemma 10.1.3, we have

$$e_1 f_1^{(n)} v_{j,k} = (\mu_j - [n-1]\{a - n\}) f_1^{(n-1)} v_{j,k} + f_1^{(n-1)} v_{j,k-1}. \tag{9}$$

Proposition 10.1.4. *We have $\mu_j = [N_j]\{a - N_j - 1\}$ for some $N_j \in \mathbb{Z}_{\geq 0}$. In particular, each μ_j belongs to $\mathbb{Q}(p, q)$.*

Proof. Consider the case $k = 1$. By the local nilpotency of f_1, there exists a unique nonnegative integer N_j such that

$$f_1^{(N_j)} v_{j,1} \neq 0 \text{ and } f_1^{(N_j+1)} v_{j,1} = 0.$$

Then, by equation (9), we have

$$0 = e_1 f_1^{(N_j+1)} v_{j,1} = (\mu_j - [N_j]\{a - N_j - 1\}) f_1^{(N_j)} v_{j,1}.$$

Since $f_1^{(N_j)} v_{j,1} \neq 0$, we conclude that $\mu_j = [N_j]\{a - N_j - 1\}$, as desired. □

Proposition 10.1.5. *Each d_j is equal to 1, that is, h'_1 is diagonalizable on $M_{a\delta_1}$.*

Proof. We use the notation N_j in the proof of Proposition 10.1.4. Assume, for a contradiction, that there exists $d_j > 1$. By equation (9), we have

$$e_1 f_1^{(n)} v_{j,2} = (\mu_j - [n-1]\{a - n\}) f_1^{(n-1)} v_{j,2} + f_1^{(n-1)} v_{j,1}$$

for all $n \geq 0$. Let N'_j denote the unique nonnegative integer such that

$$f_1^{(N'_j)} v_{j,2} \neq 0, \text{ and } f_1^{(N'_j+1)} v_{j,2} = 0.$$

When $N'_j > N_j$, we have

$$0 = (\mu_j - [N'_j]\{a - N'_j - 1\}) f_1^{(N'_j)} v_{j,2} + f_1^{(N'_j)} v_{j,1} = (\mu_j - [N'_j]\{a - N'_j - 1\}) f_1^{(N'_j)} v_{j,2}.$$

This implies that $\mu_j = [N'_j]\{a - N'_j - 1\} \neq [N_j]\{a - N_j - 1\}$, which causes a contradiction. When $N'_j = N_j$, we have

$$0 = (\mu_j - [N_j]\{a - N_j - 1\}) f_1^{(N_j)} v_{j,2} + f_1^{(N_j)} v_{j,1} = f_1^{(N_j)} v_{j,1}.$$

This contradicts the definition of N_j. When $N'_j < N_j$, we have

$$0 = (\mu_j - [N'_j]\{a - N'_j - 1\}) f_1^{(N'_j)} v_{j,2} + f_1^{(N'_j)} v_{j,1}.$$

Applying $e_1^{N'_j}$ on both sides, we obtain

$$0 = \prod_{l=1}^{N'_j+1} (\mu_j - [l-1]\{a - l\}) v_{j,2} + X v_{j,1} \quad \text{for some } X \in \mathbb{Q}(p,q).$$

Since the coefficient of $v_{j,2}$ is nonzero, this contradicts the linear independence of $v_{j,1}$ and $v_{j,2}$. This proves the proposition. □

Theorem 10.1.6. *For each $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, there exists a unique $(b+1)$-dimensional irreducible $\mathbf{U}^\imath_1$-module $L(a;b) \in \mathcal{O}^\imath_{\mathrm{int}}$ such that*

$$L(a;b) = \bigoplus_{n=0}^{b} \mathbb{Q}(p,q) v_n,$$

$$v_n = f_1^{(n)} v_0, \quad k_1 v_0 = q^a v_0, \quad h'_1 v_0 = [b]\{a - b - 1\} v_0.$$

Conversely, each irreducible $\mathbf{U}^\imath_1$-module in $\mathcal{O}^\imath_{\mathrm{int}}$ is isomorphic to $L(a;b)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$.

Proof. It is straightforward to show that $L(a;b)$ is a $(b+1)$-dimensional irreducible $\mathbf{U}^\imath_1$-module, and so we omit the details. Let $V \in \mathcal{O}^\imath_{\mathrm{int}}$ be an irreducible $\mathbf{U}^\imath_1$-module. By the definition of $\mathcal{O}^\imath_{\mathrm{int}}$, there exists an integer $a \in \mathbb{Z}$ such that $V_{a\delta_1} \neq 0$ and $e_1 V_{a\delta_1} = 0$. Also, by Propositions 10.1.4 and 10.1.5, there exist $b \in \mathbb{Z}_{\geq 0}$ and $v \in V_{a\delta_1} \setminus \{0\}$ such that $f_1^{(b)} v \neq 0$, $f_1^{(b+1)} v = 0$, and $h'_1 v = [b]\{a-b-1\} v$. Hence the $\mathbf{U}^\imath_1$-submodule generated by v is identical to $\bigoplus_{n=0}^{b} \mathbb{Q}(p,q) f_1^{(n)} v$, which is isomorphic to $L(a;b)$ by the definitions of v, a, and b. Since V is irreducible, we have $V = \mathbf{U}^\imath_1 v \simeq L(a;b)$. This proves the theorem. □

Note that $L(a;b)$ is the irreducible quotient $L'(\lambda;\mathbf{H}')$ of the Verma module $V'(\lambda;\mathbf{H}')$ with highest weight $(\lambda;\mathbf{H}')=(a\delta_1;[b]\{a-b-1\})$. Hence, Theorem 10.1.6 gives a necessary and sufficient condition for $L'(\lambda;\mathbf{H}')$ to be an object of $\mathcal{O}^{\jmath}_{\mathrm{int}}$.

Corollary 10.1.7. *Let $a\in\mathbb{Z}$ and $H'_1\in\mathbb{Q}(p,q)$. Then, the irreducible highest weight module $L'(a\delta_1;H'_1)$ belongs to $\mathcal{O}^{\jmath}_{\mathrm{int}}$ if and only if $H'_1=[b]\{a-b-1\}$ for some $b\in\mathbb{Z}_{\geq 0}$. Moreover, the assignment $(a,b)\mapsto[L(a;b)]$, where $[L(a;b)]$ denotes the isomorphism class of $L(a;b)$, gives a bijection from $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ to the set of isomorphism classes of irreducible $\mathbf{U}^{\jmath}_1$-modules in $\mathcal{O}^{\jmath}_{\mathrm{int}}$.*

10.2. **Complete reducibility.** Set $z_1:=h'_1+\frac{[2]pq}{1-q^2}k_1+\frac{[2]p^{-1}q^{-1}}{1-q^{-4}}k_1^{-1}\in\mathbf{U}^{\jmath}_1$.

Lemma 10.2.1. *In $\mathbf{U}^{\jmath}_1$, we have*

$$z_1f_1=q^{-1}f_1z_1,\quad z_1e_1=qe_1z_1.$$

Let $a\in\mathbb{Z}$ and $b\in\mathbb{Z}_{\geq 0}$, and take a highest weight vector $v\in L(a;b)$. Then we have

$$z_1f_1^{(n)}v=q^{-n}\left([b]\{a-b-1\}+\frac{[2]pq^{1+a}}{1-q^2}+\frac{[2]p^{-1}q^{-1-a}}{1-q^{-4}}\right)f_1^{(n)}v.$$

Let $z_1(a,b,n)$ denote the coefficient of $f_1^{(n)}v$ on the right-hand side, that is,

$$z_1(a,b,n):=-\frac{pq^{a-b-n}(q^{b+1}+q^{-b-1})}{q-q^{-1}}+\frac{p^{-1}q^{-a+2b-n+1}}{q-q^{-1}}.$$

It is easy to verify that the function $\mathbb{Z}\to\mathbb{Q}(p,q)$, $n\mapsto z_1(a,b,n)$, is injective for all $a,b\in\mathbb{Z}$.

Lemma 10.2.2. *Let $M\in\mathcal{O}^{\jmath}_{\mathrm{int}}$, $a,a'\in\mathbb{Z}$, and $b,b'\in\mathbb{Z}_{\geq 0}$. Then, each short exact sequence of the form*

$$0\to L(a;b)\xrightarrow{\iota}M\xrightarrow{\pi}L(a';b')\to 0 \tag{10}$$

splits.

Proof. Let $v\in L(a',b')$ be a highest weight vector, and take $u\in\pi^{-1}(v)$. Since $\mathbf{U}^{\jmath}_1$-module homomorphisms preserve generalized eigenspaces of z_1, we may assume that u is a generalized eigenvector of z_1 with eigenvalue $z_1(a',b',0)$. Then, e_1u is a generalized eigenvector of z_1 with eigenvalue $z_1(a',b',-1)$. Since $\pi(e_1u)=e_1\pi(u)=e_1v=0$, it follows that $e_1u\in\iota(L(a',b'))$. However, the eigenvalues of z_1 on $L(a,b)$ are $z_1(a,b,n)$, $0\leq n\leq b$. Therefore, $e_1u=0$, and hence we obtain a section $v\mapsto u$ of π. This proves the lemma. □

Now, the complete reducibility of $\mathbf{U}^{\jmath}$-modules in $\mathcal{O}^{\jmath}_{\mathrm{int}}$ follows from a standard argument; see, for example, [HK02, Section 3.5].

Theorem 10.2.3. *Every $\mathbf{U}^{\jmath}_1$-module in $\mathcal{O}^{\jmath}_{\mathrm{int}}$ is completely reducible.*

Corollary 10.2.4. *Let $M\in\mathcal{O}^{\jmath}_{\mathrm{int}}$. Then, M is decomposed into a direct sum of z_1-eigenspaces with possible eigenvalues $z_1(a,b,n)$, $a\in\mathbb{Z},0\leq n\leq b$. In particular, if $z_1m=z_1(a,b,0)m$ for some $m\in M$, then $e_1m=0$.*

11. Complete reducibility and the irreducible modules

The aim of this section is to upgrade the results in Section 10 to general r. In Subsection 11.1, we define a braid group action on $\mathbf{U}^{\jmath}$ and $\mathbf{U}^{\jmath}$-modules. In Subsection 11.2, we give a partial classification of the irreducible highest weight modules. The complete reducibility is proved in Subsection 11.3.

11.1. **Braid group action on $\mathbf{U}^\imath$ and $\mathbf{U}^\imath$-modules.** Throughout this subsection, we fix $e \in \{1, -1\}$.

Proposition 11.1.1 ([KP11, 4.5]). *For $i \in \mathbb{I}^\imath \setminus \{1\}$, there exist unique automorphisms $\tau'_{i,e}$ and $\tau''_{i,-e}$ on $\mathbf{U}^\imath$ satisfying the following:*

$$\tau'_{i,e}(e_j) = \begin{cases} -k_i^e f_i & \text{if } j = i, \\ e_j & \text{if } |i-j| > 1, \\ [e_j, e_i]_e & \text{if } |i-j| = 1, \end{cases} \qquad \tau'_{i,e}(f_j) = \begin{cases} -e_i k_i^{-e} & \text{if } j = i, \\ f_j & \text{if } |i-j| > 1, \\ [f_i, f_j]_{-e} & \text{if } |i-j| = 1, \end{cases}$$

$$\tau''_{i,-e}(e_j) = \begin{cases} -f_i k_i^{-e} & \text{if } j = i, \\ e_j & \text{if } |i-j| > 1, \\ [e_i, e_j]_e & \text{if } |i-j| = 1, \end{cases} \qquad \tau''_{i,-e}(f_j) = \begin{cases} -k_i^e e_i & \text{if } j = i, \\ f_j & \text{if } |i-j| > 1, \\ [f_j, f_i]_{-e} & \text{if } |i-j| = 1, \end{cases}$$

$$\tau'_{i,e}(k_j) = \tau''_{i,-e}(k_j) = \begin{cases} k_i^{-1} & \text{if } j = i, \\ k_j & \text{if } |i-j| > 1, \\ k_i k_j & \text{if } |i-j| = 1. \end{cases}$$

Moreover, $\{\tau'_{i,e}\}_{i \in \mathbb{I}^\imath \setminus \{1\}}$ and $\{\tau''_{i,-e}\}_{i \in \mathbb{I}^\imath \setminus \{1\}}$ satisfy the braid relation of type A_{r-1}.

Proof. Set $\tau_i := \tau'_{i,e}$ (resp., $\tau''_{i,-e}$), $i \in \mathbb{I}^\imath \setminus \{1\}$. We need to verify that the defining relations for $\mathbf{U}^\imath$ hold if we replace e_i, f_i, k_i by $\tau_j(e_i), \tau_j(f_i), \tau_j(k_i)$, respectively. One immediately finds that the nontrivial assertions are

$$\begin{aligned} &\tau_2(e_1)^2\tau_2(f_1) - (q+q^{-1})\tau_2(e_1)\tau_2(f_1)\tau_2(e_1) + \tau_2(f_1)\tau_2(e_1)^2 \\ &\qquad\qquad = -(q+q^{-1})\tau_2(e_1)(pq\tau_2(k_1) + p^{-1}q^{-1}\tau_2(k_1)^{-1}), \\ &\tau_2(f_1)^2\tau_2(e_1) - (q+q^{-1})\tau_2(f_1)\tau_2(e_1)\tau_2(f_1) + \tau_2(e_1)\tau_2(f_1)^2 \\ &\qquad\qquad = -(q+q^{-1})(pq\tau_2(k_1) + p^{-1}q^{-1}\tau_2(k_1)^{-1})\tau_2(f_1). \end{aligned}$$

These are checked by direct calculation, or by means of a computer program GAP [GAP16] with a package Quagroup (see [KP11, 4.5]). Also, one can verify the braid relation in the same way as for the braid group action on $\mathbf{U}$. This proves the proposition. □

Also, we define a braid group action on $\mathbf{U}^\imath$-modules in $\mathcal{O}^\imath_{\mathrm{int}}$. Since $\mathbf{U}^\imath$-contains $(r-1)$ $\mathfrak{sl}_2$-triples (e_i, k_i, f_i), $i \in \mathbb{I}^\imath \setminus \{1\}$, most parts of the propositions in this subsection follow form the ordinary quantum group theory. Hence, we omit the details.

Definition 11.1.2. Let $M \in \mathcal{O}^\imath_{\mathrm{int}}$. For each $i \in \mathbb{I}^\imath \setminus \{1\}$, we define two linear automorphisms $\tau'_{i,e}$ and $\tau''_{i,e}$ on M by:

$$\tau'_{i,e}(m) = \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ a-b+c=n}} (-q)^b q^{e(-ac+b)} f_i^{(a)} e_i^{(b)} f_i^{(c)} m,$$

$$\tau''_{i,e}(m) = \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ -a+b-c=n}} (-q)^b q^{e(-ac+b)} e_i^{(a)} f_i^{(b)} e_i^{(c)} m,$$

where $n \in \mathbb{Z}$, and $m \in M$ is such that $k_i m = q^n m$.

Proposition 11.1.3 (see [L93, Proposition 5.2.2]). *Let $M \in \mathcal{O}^\imath_{\mathrm{int}}$, $i \in \mathbb{I}^\imath \setminus \{1\}$, and let $\lambda \in \Lambda^\imath$ be such that $(\beta_i, \lambda) \geq 0$, $j \in \{0, 1, \dots, (\beta_i, \lambda)\}$; we set $h := (\beta_i, \lambda) - j$.*

1. *If $\eta \in M_\lambda$ is such that $e_i \eta = 0$, then $\tau'_{i,e}(f_i^{(j)}\eta) = (-1)^j q^{e(jh+j)} f_i^{(h)} \eta$.*
2. *If $\xi \in M_{-\lambda}$ is such that $f_i \xi = 0$, then $\tau''_{i,e}(e_i^{(j)}\xi) = (-1)^j q^{(e(jh+j))} e_i^{(h)} \xi$.*

Proposition 11.1.4 (see [L93, Proposition 5.2.3])**.** *Let* $M \in \mathcal{O}^\jmath_{\mathrm{int}}$*,* $i \in \mathbb{I}^\jmath \setminus \{1\}$*, and* $m \in M_\lambda$*.*

(1) *We have* $\tau'_{i,e}\tau''_{i,-e} = \mathrm{id}_M = \tau''_{i,-e}\tau'_{i,e}$*.*

(2) *We have* $\tau''_{i,e}(m) = (-1)^{(\beta_i,\lambda)}q^{e(\beta_i,\lambda)}\tau'_{i,e}(m)$*.*

Proposition 11.1.5 (see [L93, Proposition 37.1.2])**.** *Let* $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ *and* $i \in \mathbb{I}^\jmath \setminus \{1\}$*. Then, for each* $m \in M$ *and* $x \in \mathbf{U}^\jmath_r$*, we have*

$$\tau'_{i,e}(xm) = \tau'_{i,e}(x)\tau'_{i,e}(m), \qquad \tau''_{i,e}(xm) = \tau''_{i,e}(x)\tau''_{i,e}(m).$$

In what follows, we write $\tau_i = \tau''_{i,1}$ for $i \in \mathbb{I}^\jmath \setminus \{1\}$.

11.2. **Classification of the irreducible modules in $\mathcal{O}^\jmath_{\mathrm{int}}$.** Recall the triangular decomposition $\mathbf{U}^\jmath = \mathbf{U}^\jmath_{<0} \otimes (\mathbf{U}^\jmath_0 \otimes \mathbf{U}^{\jmath,0}) \otimes \mathbf{U}^\jmath_{>0}$ associated with the reflection order $\preceq$ defined in the proof of Lemma 9.1.3. Also, recall from (7) in Section 9.2, the explicit form of the root vectors $h'_i \in \mathbf{U}^\jmath_0$, $i \in \mathbb{I}^\jmath = [1, r]$. We remark that an irreducible highest weight module is determined by the eigenvalues of k_i's and h'_i's for a highest weight vector. However, h'_i's are sometimes difficult to deal with.

Proposition 11.2.1. *Let* $V'(\lambda; \mathbf{H}')$ *be the Verma module with highest weight* $(\lambda; \mathbf{H}')$*, and* $v \in V'(\lambda; \mathbf{H}')$ *a highest weight vector. Then,* $\mathbf{H}'$ *is determined by the* $\tau_i \cdots \tau_2(h'_1)$*-eigenvalue of* v *for* $i \in \mathbb{I}^\jmath$*.*

Proof. For each $i \in \mathbb{I}^\jmath$, set $\mathrm{ef}(i) := e_i \cdots e_2 e_1 f_1 f_2 \cdots f_i$. By equation (7), the h'_i is of the form

$$h'_i = \sum_{\sigma \in \mathfrak{S}_{2i}} a_i(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2i)},$$

where $\mathfrak{S}_{2i}$ denotes the $2i$-th symmetric group, $a_i(\sigma) \in \mathbb{Q}(q)$, $x_j = e_{i+1-j}$ for $1 \leq j \leq i$, and $x_j = f_{j-i}$ for $i+1 \leq j \leq 2i$. From this, noting that v is a highest weight vector, we deduce that $h'_i v$ is of the form

$$h'_i v = \left(\mathrm{ef}(i) + \sum_{1 \leq j < i} g_j \,\mathrm{ef}(j) \right) v,$$

where $g_j \in \mathbb{Q}(q)$. Therefore, the $\mathrm{ef}(j)$-eigenvalue of v for $j \leq i$ determine the h'_i-eigenvalue H'_i.

Also, $\tau_i \cdots \tau_2(h'_1)$ is of the form

$$\tau_i \cdots \tau_2(h'_1) = \sum_{\sigma \in \mathfrak{S}_{2i}} b_i(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2i)},$$

where $b_i(\sigma) \in \mathbb{Q}(q)$. In the same way as above, the $\tau_i \cdots \tau_2(h'_1)$-eigenvalue of v is determined by the $\mathrm{ef}(j)$-eigenvalue of v for $j \leq i$. Conversely, the $\tau_j \cdots \tau_2(h'_1)$-eigenvalue of v for $j \leq i$ altogether determine the $\mathrm{ef}(j)$-eigenvalue of v for $j \leq i$, which, in turn, determine the h'_i-eigenvalue H'_i of v. This proves the proposition. □

This proposition enables us to replace $h'_i v$ with $\tau_i \cdots \tau_2(h'_1)v$ for $i \in \mathbb{I}^\jmath$. Then, we define h_i, $i \in \mathbb{I}^\jmath$, by $h_1 := [e_1, f_1]_1$ and $h_i := \tau_i \cdots \tau_2(h_1)$. Also, we set $V(\lambda; \mathbf{H}) := V'(\lambda; \mathbf{H}')$ and $L(\lambda; \mathbf{H}) := L'(\lambda; \mathbf{H}')$, where $\mathbf{H} = (H_1, \dots, H_r)$ is uniquely determined by the equations $h_i v = H_i v$, $i \in \mathbb{I}^\jmath$, where $v \in V'(\lambda; \mathbf{H}')$ is a highest weight vector.

Remark 11.2.2. For each $k = 1, \dots, r$, the classical limit specializes $h_k \in \mathbf{U}^\jmath$ to $-B_{r-k+1,r+k+1} + \sum_{i=1}^k H_i \in \mathfrak{g}^\theta$ (see Example 2.1.8 (1)).

Let $L \in \mathcal{O}^\imath_{\mathrm{int}}$ be an irreducible $\mathbf{U}^\imath$-module. By condition (M3), there exists $\lambda \in \Lambda^\imath$ such that $L_\lambda \neq 0$ and $L_\mu = 0$ for all $\mu > \lambda$. By the case $r = 1$, h_1 acts on L_λ semisimply.

Lemma 11.2.3. *We have*

$$[h_1, h_2]_0 = [h_1, (q - q^{-1})(f_2[e_2, h_1]_1 - p^{-1}q^2 f_2 e_2 k_1^{-1})]_0 \in \mathbf{U}^\imath(e_2, e_2h_1, e_2h_1^2),$$

where $\mathbf{U}^\imath(e_2, e_2h_1, e_2h_1^2)$ *denotes the left ideal of* $\mathbf{U}^\imath$ *generated by* $e_2, e_2h_1, e_2h_1^2$.

Proof. By direct calculation (or by using GAP). □

This lemma implies that $[h_1, h_2]_0 L_\lambda = 0$, namely, the actions of h_1 and h_2 commute with each other on L_λ.

Lemma 11.2.4. *Let* $i, j \in \mathbb{I}^\imath$. *If* $j \neq i, i+1$, *then we have* $\tau_j(h_i) = h_i$.

Proof. The assertion in the case $j > i+1$ follows from the definitions of τ_j and h_i. When $j < i$, by the braid relation for the τ_j's, we see that

$$\begin{aligned}
\tau_j(h_i) &= \tau_j(\tau_i\tau_{i-1}\cdots\tau_2)(h_1) \\
&= \tau_i\cdots\tau_{j+2}\tau_j\tau_{j+1}\tau_j\cdots\tau_2(h_1) \\
&= \tau_i\cdots\tau_{j+2}\tau_{j+1}\tau_j\tau_{j+1}\tau_{j-1}\cdots\tau_2(h_1) \\
&= \tau_i\cdots\tau_{j+2}\tau_{j+1}\tau_j\cdots\tau_2\tau_{j+1}(h_1) \\
&= \tau_i\cdots\tau_2(h_1) = h_i.
\end{aligned}$$

This proves the lemma. □

Proposition 11.2.5. *Let* $L \in \mathcal{O}^\imath_{\mathrm{int}}$ *be an irreducible module. Take* $\lambda \in \Lambda^\imath$ *such that* $L_\lambda \neq 0$ *and* $L_\mu = 0$ *for all* $\mu > \lambda$. *Then, the actions of* $h_1, \dots, h_r$ *commute with each other on* L_λ.

Proof. Let $i, j \in \mathbb{I}^\imath$ be such that $j < i$. By Lemma 18.3.3,

$$[h_j, h_i]_0 = \tau_j\cdots\tau_2([h_1, h_i]_0) = \tau_j\cdots\tau_2\tau_i\cdots\tau_3([h_1, h_2]_0).$$

Also, by Lemma 18.3.2,

$$\tau_j\cdots\tau_2\tau_i\cdots\tau_3([h_1, h_2]_0) \in \mathbf{U}^\imath(\tau_{j,i}(e_2), \tau_{j,i}(e_2)h_j, \tau_{j,i}(e_2)h_j^2),$$

where $\tau_{j,i}$ denotes $\tau_j\cdots\tau_2\tau_i\cdots\tau_3$. Since $\tau_{j,i}(e_2) \in \mathbf{U}^\imath(\mathbf{U}^\imath_{>0})_+$, the vectors $\tau_{j,i}(e_2)h_j^l$, $l = 0, 1, 2$, act on L_λ by 0. This proves the proposition. □

As a corollary of this proposition, we can take a simultaneous eigenvector $v \in L_\lambda$ for $h_1, \dots, h_r$. Let $H_i \in \mathbb{Q}(p, q)$ denote the eigenvalue of h_i. Then the submodule generated by v is a highest weight module. Since L is irreducible, we conclude that L is isomorphic to $L(\lambda; \mathbf{H})$ for some $\lambda \in \Lambda^\imath$, $\mathbf{H} \in \mathbb{Q}(p, q)^r$.

Next, we investigate a necessary condition for $L(\lambda; \mathbf{H})$ being a nonzero object of $\mathcal{O}^\imath_{\mathrm{int}}$.

Lemma 11.2.6. *Let* $M \in \mathcal{O}^\imath_{\mathrm{int}}$, $v \in M$ *be such that* $e_1 v = 0$ *and* $h_1 v = [b]\{a - b - 1\}$ *for some* $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geq 0}$. *Let* $n \in \mathbb{Z}_{>0}$. *Then, there exist unique* $v_0, v_1, \dots, v_n \in M$ *satisfying the following:*

1. $f_2^{(n)} v = \sum_{k=0}^n v_k$.
2. $e_1 v_k = 0$, $h_1 v_k = [b + k]\{a + n - (b + k) - 1\} v_k$.

Proof. As the proof of this lemma needs some lengthy calculation, we put it in Subsection 11.4. □

Lemma 11.2.7. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ be a $\mathbf{U}^\jmath_2$-module, $v \in M$ a highest weight vector with $k_i v = q^{a_i} v$, $h_i v = H_i v$ for some $a_1 \in \mathbb{Z}$, $a_2 \in \mathbb{Z}_{\geq 0}$, $H_1, H_2 \in \mathbb{Q}(p,q)$. If $H_1 = [b_1]\{a_1 - b_1 - 1\}$ for some $b_1 \in \mathbb{Z}_{\geq 0}$, then $H_2 = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}$ for some $0 \leq b_2 \leq a_2$.*

Proof. By the representation theory for $U_q(\mathfrak{sl}_2)$, we have $f_2^{a_2+1} v = 0$, and $\tau_2^{-1}(v) = f_2^{(a_2)} v$. Set $u := \tau_2^{-1}(v)$. We claim that $e_1 u = 0$ and $h_1 u = H_2 u$. The former is true as we have $e_1 f_2 = f_2 e_1$. The latter follows from an easy calculation

$$h_1 u = \tau_2^{-1}(\tau_2(h_1) v) = \tau_2^{-1}(h_2 v) = H_2 u.$$

Then, by the case $r = 1$, H_2 must be of the form $H_2 = [b]\{a_1 + a_2 - b - 1\}$ for some $0 \leq b \leq a_1 + a_2$. Here, by Lemma 11.2.6, it must hold that $b = b_1 + b_2$ for some $0 \leq b_2 \leq a_2$. This proves the assertion. $\square$

Theorem 11.2.8. *Each irreducible module in $\mathcal{O}^\jmath_{\mathrm{int}}$ is isomorphic to $L(\lambda; \mathbf{H})$ for some $\lambda \in \Lambda^\jmath$ and $\mathbf{H} = (H_1, \dots, H_r) \in \mathbb{Q}(p,q)^r$ satisfying the following:*

1. *$a_i := (\beta_i, \lambda) \geq 0$ for each $i \in \mathbb{I}^\jmath \setminus \{1\}$.*
2. *For each $i \in \mathbb{I}^\jmath$, there exists $b_i \in \mathbb{Z}_{\geq 0}$ such that $0 \leq b_i \leq a_i$ for $i \in \mathbb{I}^\jmath \setminus \{1\}$ and $H_i = [b_1 + \cdots + b_i]\{a_1 + \cdots + a_i - (b_1 + \cdots + b_i) - 1\}$ for $i \in \mathbb{I}^\jmath$.*

Proof. We have shown that each irreducible module in $\mathcal{O}^\jmath_{\mathrm{int}}$ is isomorphic to $L(\lambda; \mathbf{H})$ for some $\lambda \in \Lambda^\jmath$ and $\mathbf{H} \in \mathbb{Q}(p,q)^r$. It is easy to verify that $L(\lambda; \mathbf{H})$ belongs to $\mathcal{O}^\jmath_{\mathrm{int}}$ if and only if $f_i^N v = 0$, $i \in \mathbb{I}^\jmath$, for a sufficiently large N, where $v \in L(\lambda; \mathbf{H})$ is a highest weight vector. By the case $r = 1$, the equality $f_1^N v = 0$ is equivalent to the existence of $b_1 \in \mathbb{Z}_{\geq 0}$ satisfying the equality $H_1 = [b_1]\{a_1 - b_1 - 1\}$. Also, by the representation theory of $U_q(\mathfrak{sl}_2)$, the condition $f_i^N v = 0$, $i \geq 2$, implies $a_i \geq 0$.

It remains to determine the possible values of $H_2, \dots, H_r$. By Lemma 11.2.7, there exists $b_2 \in \mathbb{Z}_{\geq 0}$ such that $b_2 \leq a_2$ and $H_2 = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}$. Let $i \geq 3$, and assume that for all $j < i$, $H_j = [b_1 + \cdots + b_j]\{a_1 + \cdots + a_j - (b_1 + \cdots + b_j) - 1\}$ for some $0 \leq b_j \leq a_j$. Set $T_i := (\tau_{i-1}\tau_i) \cdots (\tau_3\tau_4)(\tau_2\tau_3)$, and consider the subalgebra $T_i(\mathbf{U}^\jmath_2) \subset \mathbf{U}^\jmath$. We have $T_i(k_1) = k_1 \cdots k_{i-1}$, $T_i(k_2) = k_i$, $T_i(h_1) = h_{i-1}$, and $T_i(h_2) = h_i$. If we regard L as a $\mathbf{U}^\jmath_2$-module via the algebra homomorphism $T_i : \mathbf{U}^\jmath_2 \to \mathbf{U}^\jmath_r$, the v is a highest weight vector such that

$$k_1 v = q^{a_1 + \cdots + a_{i-1}} v, \ k_2 v = q^{a_i} v, \ h_1 v = H_{i-1} v, \ h_2 v = H_i v.$$

By lemma 11.2.7, H_i must be of the form $[b_1 + \cdots + b_{i-1} + b_i]\{a_1 + \cdots + a_{i-1} + a_i - (b_1 + \cdots + b_{i-1} + b_i) - 1\}$ for some $0 \leq b_i \leq a_i$. This proves the theorem. $\square$

From now on, we write $L(\mathbf{a}; \mathbf{b})$ instead of $L(\lambda; \mathbf{H})$, where $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_r)$ are such that $a_i = (\beta_i, \lambda)$, $H_i = [b_1 + \cdots + b_i]\{(a_1 + \cdots + a_i) - (b_1 + \cdots + b_i) - 1\}$. This causes no confusion since $\mathbf{a} \in \mathbb{Z}^r$, while $\lambda \in \Lambda^\jmath$. We call $L(\mathbf{a}; \mathbf{b})$ the irreducible highest weight $\mathbf{U}^\jmath$-module with highest weight $(\mathbf{a}; \mathbf{b})$.

Now, let us see the classical limit of highest weight modules. Let L be a highest weight $\mathbf{U}^\jmath$-module, and $v \in L$ a highest weight vector. Set $L_{\mathbf{A}_1} := \mathbf{U}^\jmath_{\mathbf{A}_1} v$.

Lemma 11.2.9. *$L_{\mathbf{A}_1} \otimes_{\mathbf{A}_1} \mathbb{Q}$ is a highest weight module over $U_{\mathbb{Q}}(\mathfrak{g}^\theta)$ with respect to the Cartan subalgebra described in Example 2.1.8* (1)*. Moreover, if $L \simeq L(\mathbf{a}; \mathbf{b})$, then for each $k = 1, \dots, r$, we have*

$$B_{r-k+1, r+k+1}(v \otimes 1) = \sum_{i=1}^{k} (a_i - 2b_i)(v \otimes 1).$$

Proof. By Remark 11.2.2, we have

$$(-B_{r-k_1,r+k+1} + \sum_{i=1}^{k} H_i)(v \otimes 1) = (h_k \otimes 1)(v \otimes 1) = 2\sum_{i=1}^{k} b_i.$$

Also, it is easy to see that

$$H_i(v \otimes 1) = ((K_{h_{\underline{i}} - h_{-\underline{i}}}; 0)_q \otimes 1)(v \otimes 1) = a_i v \otimes 1.$$

Then, the assertion follows immediately. □

Let $\lambda \in \Lambda^\jmath$ and $\mathbf{H} \in \mathbb{Q}(p,q)^r$ satisfy the conditions (1) and (2) in the theorem. Set $a^- := \max\{a_1 - (2\sum_{i=1}^r b_i - \sum_{i=2}^r a_i), 0\}$, $a^+ := -\min\{a_1 - (2\sum_{i=1}^r b_i - \sum_{i=2}^r a_i), 0\}$, and

$$\boldsymbol{\lambda}^- := \left(a^- + \sum_{i=1}^{r} b_i, a^- + \sum_{i=2}^{r} b_i, \dots, a^- + b_r, a^- \right),$$

$$\boldsymbol{\lambda}^+ := \left(a^+ + \sum_{i=2}^{r} (a_i - b_i), a^+ + \sum_{i=3}^{r} (a_i - b_i), \dots, a^+ + a_r^{-} b_r, a^+ \right).$$

This assignment gives a bijection from

$$\{(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^r \times \mathbb{Z}^r_{\geq 0} \mid 0 \leq b_i \leq a_i \text{ for all } i = 2, \dots, r\}$$

to

$$P^\jmath = P^\jmath_r := \{\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)} \mid \boldsymbol{\lambda}^-_{r+1} = 0 \text{ or } \boldsymbol{\lambda}^+_r = 0\}.$$

The inverse map π is given by the following formula; for $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}$ such that $\boldsymbol{\lambda}^-_{r+1} = 0$ or $\boldsymbol{\lambda}^+_r = 0$, if we write $\pi(\boldsymbol{\lambda}) = (\mathbf{a}, \mathbf{b})$, then we have

$$a_1 = 2\boldsymbol{\lambda}^-_1 - \boldsymbol{\lambda}^-_2 - \boldsymbol{\lambda}^+_1, \ a_i = \boldsymbol{\lambda}^-_i - \boldsymbol{\lambda}^-_{i+1} + \boldsymbol{\lambda}^+_{i-1} - \boldsymbol{\lambda}^+_i, \ b_i = \boldsymbol{\lambda}^-_i - \boldsymbol{\lambda}^-_{i+1}.$$

For $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}$, set $L(\boldsymbol{\lambda}) := L(\pi(\boldsymbol{\lambda}))$. Then, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Bip}_{(r+1,r)}$, we have $L(\boldsymbol{\lambda}) \simeq L(\boldsymbol{\mu})$ if and only if $\pi(\boldsymbol{\lambda}) = \pi(\boldsymbol{\mu})$, which is in turn equivalent to that $\boldsymbol{\lambda}^\pm_i - \boldsymbol{\mu}^\pm_i$ are constant.

11.3. **Complete reducibility.** In this subsection only, we set $A := \mathbf{U}^\jmath$, and write B for $\mathbf{U}^\jmath$ with p replaced by $p^{-1}q$. In order to avoid confusion, we denote by $e^A_i, f^A_i, k^{A,\pm 1}_i$ the generators of A, and by $e^B_i, f^B_i, k^{B,\pm 1}_i$ those of B. Consider the anti-algebra homomorphism $S : A \to B$ over $\mathbb{Q}(p,q)$ defined by:

$$S(e^A_i) = -e^B_i k^B_i, \quad S(f^A_i) = -k^{B,-1}_i f^B_i, \quad S(k^A_i) = k^{B,-1}_i.$$

It is straightforwardly checked that S is indeed an anti-algebra homomorphism. In addition, S is invertible:

$$S^{-1}(e^B_i) = -k^A_i e^A_i, \quad S^{-1}(f^B_i) = -f^A_i k^{A,-1}_i, \quad S^{-1}(k^B_i) = k^{A,-1}_i.$$

For an A-module $M \in \mathcal{O}^\jmath_{\mathrm{int}}$, define a B-module $S_*(M) := M^\vee$ by:

$$(x \cdot g)(m) = g(S^{-1}(x) \cdot m) \quad \text{for } x \in B, \ g \in M^\vee, \ m \in M,$$

where $M^\vee$ denotes the restricted dual of M, i.e., $M^\vee = \bigoplus_{\lambda \in \Lambda^\jmath} \mathrm{Hom}_{\mathbb{Q}(p,q)}(M_\lambda, \mathbb{Q}(p,q))$. Similarly, we associate an A-module $S^*(N)$ with each B-module N.

Lemma 11.3.1. *Let $L \in \mathcal{O}^\jmath_{\mathrm{int}}$ be the irreducible highest weight A-module with highest weight $(\lambda; \mathbf{H})$. Then, $S_*(L)$ is the irreducible lowest weight B-module with lowest weight $(-\lambda; \mathbf{H}')$ for some $\mathbf{H}' \in \mathbb{Q}(p,q)^r$.*

Proof. Let $v \in L$ be a highest weight vector, and let $g \in S_*(L)$ be a unique element satisfying $g(v) = 1$ and $g(u) = 0$ for all $u \in L_\mu$, $\mu < \lambda$. Then, we have

$$(k_i^B g)(v) = g(k_i^{A,-1} v) = q^{-(\beta_i, \lambda)} g(v),$$
$$(h_i^B g)(v) = g(S^{-1}(h_i^B) v),$$

where $h_i^B \in B$ is the elements corresponding to $h_i \in \mathbf{U}^\jmath$. Since $S^{-1}(h_i^B)v \in L_\lambda = \mathbb{Q}(p,q)v$, we have $S^{-1}(h_i^B)v = H_i' v$ for some $H_i' \in \mathbb{Q}(p,q)$, and hence $h_i^B g = H_i' g$. Therefore, Bg is a lowest weight module with lowest weight $(-\lambda; H_1', \dots, H_r')$.

Now, it remains to show that $S_*(L)$ is irreducible. Suppose that $N \subset S_*(L)$ is a B-submodule. Then $S^*(N)$ is a quotient of $S^*(S_*(L)) \simeq L$. Since L is irreducible, $S^*(N)$ is identical either to 0 or to L, and hence N is identical either to 0 or to $S_*(L)$. Thus, $S_*(L)$ is irreducible. This proves the lemma. □

Lemma 11.3.2. *Let M be an A-module in $\mathcal{O}^\jmath_{\mathrm{int}}$. Suppose that M contains an irreducible submodule $L \simeq L(\lambda; \mathbf{H})$ for some $\lambda \in \Lambda^\jmath$ and $\mathbf{H} \in \mathbb{Q}(p,q)^r$. Then, $M \simeq L \oplus (M/L)$.*

Proof. It suffices to show that the short exact sequence

$$0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} M/L \to 0$$

splits. By the previous lemma, $S_*(M)$ has an irreducible submodule $S_*(L)$. Applying S^* to the inclusion $S_*(L) \hookrightarrow S_*(M)$, we obtain a surjection $M \twoheadrightarrow L$ of A-modules. Since the composite map $L \xrightarrow{\iota} M \twoheadrightarrow L$ is nonzero, it follows from Schur's lemma that this composite map is an isomorphism of A-modules. By composing the inverse of this isomorphism with the surjection $M \twoheadrightarrow L$, we obtain a retraction of ι. This proves the lemma. □

Now, the complete reducibility of the $\mathbf{U}^\jmath$-modules in $\mathcal{O}^\jmath_{\mathrm{int}}$, and the consequences below, follow from a standard argument; see, for example, [HK02, Section 3.5].

Theorem 11.3.3. *Every $\mathbf{U}^\jmath$-module in $\mathcal{O}^\jmath_{\mathrm{int}}$ is completely reducible.*

Corollary 11.3.4. *Every highest weight module in $\mathcal{O}^\jmath_{\mathrm{int}}$ is irreducible.*

Theorem 11.3.5. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$. Irreducible decomposition of M is unique in the following sense. If we have two irreducible decompositions $M = \bigoplus_{j \in J} L_j = \bigoplus_{k \in K} L^k$ for some index sets J and K, then there exists a bijection $\phi : J \to K$ such that $L_j \simeq L^{\phi(j)}$ for all $j \in J$. Moreover, for each $j \in J$, the number of $j' \in J$ such that $L_{j'} \simeq L_j$ is finite.*

11.4. **Proof of Lemma 11.2.6.** Throughout this subsection, we fix a $\mathbf{U}^\jmath_2$-module $M \in \mathcal{O}^\jmath_{\mathrm{int}}$. Recall from the case $r = 1$ that M is decomposed as:

$$M = \bigoplus_{\substack{a \in \mathbb{Z} \\ b,n \in \mathbb{Z}_{\geq 0}}} M_{a,b,n},$$
$$M_{a,b,0} = \{u \in M \mid e_1 u = 0,\ k_1 u = q^a u,\ h_1 u = [b]\{a-b-1\}u\},$$
$$M_{a,b,n} = f_1^{(n)}(M_{a,b,0}).$$

Recall that $h_1 = [e_1, f_1]_1$ and $h_2 = \tau_2(h_1)$. Set $f_2' := q^{-2}\big[e_1, [f_1, f_2]_1\big]_1 - p^{-1}q^{-1} f_2 k_1^{-1}$. For each $a \in \mathbb{Z}$ and $b, n \in \mathbb{Z}_{\geq 0}$, we define $f_{2,i}'(a,b,n) \in \mathbf{U}^\jmath$, $i = 1,2,3$, by

$$f_{2,1}'(a,b,n) := q^{b-n-1}\overline{f_2'} + (pq^{a-b} - p^{-1}q^{-a+b}) f_2 - q^{-b+n+1} f_2',$$
$$f_{2,2}'(a,b,n) := pq^{a-b-n-2}\overline{f_2'} - (q^{b+1} + q^{-b-1}) f_2 + p^{-1}q^{-a+b+n+2} f_2',$$
$$f_{2,3}'(a,b,n) := q^{-n-2}\overline{f_2'} + (pq^{a-2b-1} - p^{-1}q^{-a+2b+1}) f_2 - q^{n+2} f_2'.$$

Also, we define three linear maps $f'_{2,i}$, $i = 1, 2, 3$, by

$$f'_{2,i}(m) := f'_{2,i}(a,b,n)m \quad \text{for } m \in M_{a,b,n}.$$

Set $h''_1 := h_1 + \frac{p^{-1}qk_1^{-1}}{q-q^{-1}}$.

Lemma 11.4.1. *We have the following:*

(1) $[h''_1, \overline{f_2}]_1 = q^2 f'_2$.
(2) $[h''_1, \overline{f'_2}]_1 = q^2 f_2$.
(3) $[h''_1, f'_2]_{-1} = -p\left(q^{-3}\overline{f'_2} - [2]f'_2 - f_2\left(q^{-1}(q-q^{-1})\overline{h''_1} + [2]p^{-1}q^{-1}k_1^{-1}\right)\right)k_1$.

Proof. This is easy and straightforward. □

Proposition 11.4.2. *Let $a \in \mathbb{Z}$, $b, n \in \mathbb{Z}_{\geq 0}$, and $m \in M_{a,b,n}$. Then, we have*

$$f'_{2,1}(m) \in M_{a+1,b+1,n}, \quad f'_{2,2}(m) \in M_{a+1,b,n}, \quad f'_{2,3}(m) \in M_{a-2,b-1,n-1}.$$

Proof. Since h_1 and k_1 act on m as scalar multiplication, so does h''_1; explicitly, we have $h''_1 m = h''_1(a,b,n)m$, where

$$h''_1(a,b,n) := [n+1][b-n]\{a-b-n-1\} - q[n][b-n+1]\{a-b-n\} + \frac{p^{-1}q^{-a+3n+1}}{q-q^{-1}}.$$

By Lemma 11.4.1, we have

$$\begin{aligned}
h''_1\overline{f'_2}m &= qh''_1(a,b,n)\overline{f'_2}m + q^2 f_2 m,\\
h''_1 f_2 m &= qh''_1(a,b,n)f_2 m + q^2 f'_2 m,\\
h''_1 f'_2 m &= q^{-1}h''_1(a,b,n)f'_2 m - p\left(q^{-3}\overline{f'_2} - [2]f'_2 - f_2\left(q^{-1}(q-q^{-1})\overline{h''_1(a,b,n)} + [2]p^{-1}q^{-1}q^{-a+3n}\right)\right)q^{a-3n}\\
&= -pq^{a-3n-3}\overline{f'_2}m + pq^{a-3n}\left(q^{-1}(q-q^{-1})\overline{h''_1(a,b,n)} + [2]p^{-1}q^{-a+3n-1}\right)f_2 m\\
&\quad + (q^{-1}h''_1(a,b,n) + pq^{a-3n}[2])f'_2 m.
\end{aligned}$$

Therefore, h''_1 defines a linear endomorphism on the vector space spanned by $\{\overline{f'_2}m, f_2 m, f'_2 m\}$ whose representation matrix is

$$\begin{pmatrix} qh''_1(a,b,n) & 0 & -pq^{a-3n-3} \\ q^2 & qh''_1(a,b,n) & pq^{a-3n-1}(q-q^{-1})\overline{h''_1(a,b,n)} + q^{-1}[2] \\ 0 & q^2 & q^{-1}h''_1(a,b,n) + pq^{a-3n}[2] \end{pmatrix}. \tag{11}$$

Hence, in order to prove Proposition 11.4.2, it suffices to show that the following three vectors

$$\begin{pmatrix} q^{b-n-1} \\ pq^{a-b} - p^{-1}q^{-a+b} \\ -q^{-b+n+1} \end{pmatrix}, \qquad \begin{pmatrix} pq^{a-b-n-2} \\ -(q^{b+1} + q^{-b-1}) \\ p^{-1}q^{-a+b+n+2} \end{pmatrix}, \qquad \begin{pmatrix} q^{-n-2} \\ pq^{a-2b-1} - p^{-1}q^{-a+2b+1} \\ -q^{n+2} \end{pmatrix}$$

are eigenvectors of the matrix (11) with eigenvalues $h''_1(a+1,b+1,n)$, $h''_1(a+1,b,n)$, and $h''_1(a-2,b-1,n-1)$, respectively. This can be checked by using a computer, or possibly by direct calculation. □

We normalize $f'_{2,i}$ as follows:

$$\begin{aligned} f_{2,1}(a,b,n) &:= \frac{1}{(q^{b+1}-q^{-b-1})\{a-2b-1\}} f'_{2,1}(a,b,n), \\ f_{2,2}(a,b,n) &:= -\frac{1}{\{a-b\}\{a-2b-1\}} f'_{2,2}(a,b,n), \\ f_{2,3}(a,b,n) &:= -\frac{1}{(q^{b+1}-q^{-b-1})\{a-b\}} f'_{2,3}(a,b,n), \end{aligned}$$

and define linear maps $f_{2,i}$, $i = 1,2,3$, by $f_{2,i}(m) = f_{2,i}(a,b,n)m$ for $m \in M_{a,b,n}$. Then, for each $m \in M_{a,b,n}$, we have $f_2 m = (f_{2,1} + f_{2,2} + f_{2,3})m$. Thanks to this equality and Proposition 11.4.2, in order to compute $f_{2,i}(m)$, it is enough to decompose $f_2 m$ into three h_1-eigenvectors with distinct eigenvalues. The computation becomes easier when $n = 0$ since in this case, $f_{2,3}(m) = 0$. Also, it follows that $f_2 m \in M_{a+1,b+1,0} \oplus M_{a+1,b,0}$ for $m \in M_{a,b,0}$. Repeating this, we have

$$f_2^{(n)} m \in \bigoplus_{k=0}^{n} M_{a+n,b+k,0} \quad \text{for } n \in \mathbb{Z}_{\geq 0},\ m \in M_{a,b,0}. \tag{12}$$

This completes the proof of Lemma 11.2.6.

12. QUASI-$\jmath$-CRYSTAL BASES

Now, we develop a new crystal basis theory. In Subsection 12.1, we define quasi-$\jmath$-crystal bases in a very straightforward manner. In Subsection 12.2, we prove that the tensor product module of a $\mathbf{U}^\jmath$-module with a quasi-$\jmath$-crystal basis, and a $\mathbf{U}$-module with a crystal basis, has a quasi-$\jmath$-crystal basis. This result gives the tensor product rule for quasi-$\jmath$-crystals in Subsection 12.3.

12.1. Quasi-$\jmath$-crystal bases.

Recall that $\mathbf{U}^\jmath = \mathbf{U}^\jmath_r$ has $(r-1)$ $\mathfrak{sl}_2$-triples: (e_i, k_i, f_i) for $i = 2, \dots, r$. Hence, one can define Kashiwara operators, $\widetilde{f}_i$ and $\widetilde{e}_i$, in the same way as in the crystal basis theory for quantum groups (see Subsection 3.4). Also, by the results from Section 10, we can define Kashiwara operators, $\widetilde{f}_1$ and $\widetilde{e}_1$. Let us give the precise definition of these operators.

Definition 12.1.1. Let M be a $\mathbf{U}^\jmath$-module. By the complete reducibility of $\mathbf{U}^\jmath_1$-modules in $\mathcal{O}^\jmath_{\mathrm{int}}$, one can uniquely write $M \simeq \bigoplus_{\boldsymbol{\lambda} \in P^\jmath_1} L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ for some $m_{\boldsymbol{\lambda}} \in \mathbb{N}$. Let $v_{\boldsymbol{\lambda},i}$, $1 \leq i \leq m_{\boldsymbol{\lambda}}$ be a basis of the weight space of $L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ of highest weight. We define linear operators $\widetilde{f}_1$ and $\widetilde{e}_1$ on M by

$$\widetilde{f}_1(f_1^{(n)} v_{\boldsymbol{\lambda},i}) = f_1^{(n+1)} v_{\boldsymbol{\lambda},i}, \qquad \widetilde{e}_i(f_1^{(n)} v_{\boldsymbol{\lambda},i}) = f_1^{(n-1)} v_{\boldsymbol{\lambda},i}.$$

Note that this definition is independent of the choice of $v_{\boldsymbol{\lambda},i}$'s.

Set $\mathbf{A}_0 := \{f/g \in \mathbb{Q}(p,q) \mid f, g \in p\mathbb{Q}[p,q,q^{-1}] + \mathbb{Q}[q],\ g \notin p\mathbb{Q}[p,q,q^{-1}] + q\mathbb{Q}[q]\}$; namely, $\mathbf{A}_0$ consists of all those $h \in \mathbb{Q}(p,q)$ for which $\lim_{q\to 0}(\lim_{p \to 0} h)$ exists. (Recall that p and q are independent.)

Definition 12.1.2. Let M be a $\mathbf{U}^\jmath$-module and $\mathcal{L}$ an $\mathbf{A}_0$-submodule of M. We say that $\mathcal{L}$ is a quasi-$\jmath$-crystal lattice of M if

(qL1) $\mathcal{L}$ is a free $\mathbf{A}_0$-module of rank $\dim_{\mathbb{Q}(p,q)} M$, and $\mathbb{Q}(p,q) \bigotimes_{\mathbf{A}_0} \mathcal{L} = M$,
(qL2) $\mathcal{L} = \bigoplus_{\lambda \in \Lambda^\jmath} \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda := \mathcal{L} \cap M_\lambda$,
(qL3) $\widetilde{f}_i(\mathcal{L}) \subset \mathcal{L}$ and $\widetilde{e}_i(\mathcal{L}) \subset \mathcal{L}$ for all $i \in \mathbb{I}^\jmath$.

If $\mathcal{L}$ is a quasi-$\jmath$-crystal lattice of M, then the Kashiwara operators induce $\mathbb{Q}$-linear maps, denoted by the same symbols, on $\mathcal{L}/q\mathcal{L}$.

Definition 12.1.3. Let M be a $\mathbf{U}^\jmath$-module, $\mathcal{L}$ an $\mathbf{A}_0$-submodule of M, and $\mathcal{B}$ a subset of $\mathcal{L}/q\mathcal{L}$. We say that $(\mathcal{L}, \mathcal{B})$ is a quasi-$\jmath$-crystal basis if

(qB1) $\mathcal{L}$ is a quasi-$\jmath$-crystal lattice of M,
(qB2) $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$,
(qB3) $\mathcal{B} = \bigsqcup_{\lambda \in \Lambda^\jmath} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda := \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$,
(qB4) $\widetilde{f}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ and $\widetilde{e}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ for all $i \in \mathbb{I}^\jmath$,
(qB5) for each $b, b' \in \mathcal{B}$ and $i \in \mathbb{I}^\jmath$, one has $\widetilde{f}_i(b) = b'$ if and only if $b = \widetilde{e}_i(b')$.

Definition 12.1.4. For a quasi-$\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$ and $i \in \mathbb{I}^\jmath$, we define three maps $\varphi_i : \mathcal{B} \to \mathbb{Z}_{\geq 0}$, $\varepsilon_i : \mathcal{B} \to \mathbb{Z}_{\geq 0}$, and $\mathrm{wt}^\jmath : \mathcal{B} \to \Lambda^\jmath$ by

$$\varphi_i(b) := \max\{n \mid \widetilde{f}_i^n(b) \neq 0\}, \ \varepsilon_i(b) := \max\{n \mid \widetilde{e}_i^n(b) \neq 0\}, \ \mathrm{wt}^\jmath(b) := \lambda \text{ if } b \in \mathcal{B}_\lambda.$$

Example 12.1.5. Let $r = 1$. For each $\boldsymbol{\lambda} \in P^\jmath_1$, the irreducible $\mathbf{U}^\jmath_1$-module $L(\boldsymbol{\lambda})$ has the following quasi-$\jmath$-crystal basis. Fix a highest weight vector $v \in L(\boldsymbol{\lambda})$. Let $\mathcal{L}(\boldsymbol{\lambda})$ denote the $\mathbf{A}_0$-lattice spanned by $\{f_1^{(n)} v \mid 0 \leq n \leq \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1}\}$, and set $\mathcal{B}(\boldsymbol{\lambda}) := \{f_1^{(n)} v + \mathcal{L}(\boldsymbol{\lambda})/q\mathcal{L}(\boldsymbol{\lambda}) \mid 0 \leq n \leq \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1}\}$. Then, the Kashiwara operators $\widetilde{f}_1$ and $\widetilde{e}_1$ act on $\mathcal{L}(\boldsymbol{\lambda})$ by:

$$\widetilde{f}_1(f_1^{(n)} v) = f_1^{(n+1)} v, \quad \widetilde{e}_1(f_1^{(n)} v) = f_1^{(n-1)} v.$$

It is straightforward to check that $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$ is indeed a quasi-$\jmath$-crystal basis of $L(\boldsymbol{\lambda})$. In addition, one has $\varphi_1(f_1^{(n)} v + q\mathcal{L}) = \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1} - n$, $\varepsilon_1(f_1^{(n)} v + q\mathcal{L}) = n$, and $\mathrm{wt}^\jmath(f_1^{(n)} v + q\mathcal{L}) = (2\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1} - \boldsymbol{\lambda}_1 - 3n)\delta_1$.

Definition 12.1.6. Let M be a $\mathbf{U}^\jmath$-module and $(\mathcal{L}, \mathcal{B})$ a quasi-$\jmath$-crystal basis of M. The quasi-$\jmath$-crystal graph associated with $(\mathcal{L}, \mathcal{B})$ is the colored directed graph with vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} b'$, where $b, b' \in \mathcal{B}$, $i \in \mathbb{I}^\jmath$ are such that $\widetilde{f}_i b = b'$.

We often identify $\mathcal{B}$ with its quasi-$\jmath$-crystal graph.

Proposition 12.1.7. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ be a $\mathbf{U}^\jmath$-module with a quasi-$\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. For each $i \in \mathbb{I}^\jmath$ and $m \in \mathcal{L}_\lambda$, consider the expression $m = \sum_{j=0}^N f_i^{(j)} m_j$, where $m_j \in M_{\lambda + j\gamma_i} \cap \operatorname{Ker} e_i$. Then, the following hold:*

1. *$m_j \in \mathcal{L}$ for all $j = 0, \dots, N$.*
2. *If $m + q\mathcal{L} \in \mathcal{B}$, then there exists a unique j_0 such that $u_j \in q\mathcal{L}$ for all $j \neq j_0$, and $m + q\mathcal{L} = m_{j_0} + q\mathcal{L}$.*

Proof. The assertion follows from the same argument as the ordinary crystal basis theory. □

Proposition 12.1.8. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ be a $\mathbf{U}^\jmath$-module with a quasi-$\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. Let $\lambda \in \Lambda^\jmath$ and set $a := (\lambda, \beta_1)$. For each $u \in \mathcal{L}_\lambda \cap \operatorname{Ker} e_1$, consider the unique expression $u = \sum_{b=0}^N u_b$, where $u_b \in M_\lambda$ is a $\mathbf{U}^\jmath_1$-highest weight vector of such that $k_1 u_b = q^a u_b$ and $h_1 u_b = [b]\{a - b - 1\} u_b$. Then, the following hold:*

1. *$u_b \in \mathcal{L}$ for all $b = 0, \dots, N$.*
2. *If $u + q\mathcal{L} \in \mathcal{B}$, then there exists a unique b_0 such that $u_b \in q\mathcal{L}$ for all $b \neq b_0$, and $u + q\mathcal{L} = u_{b_0} + q\mathcal{L}$.*

Proof. We prove the assertion by induction on N. When $N = 0$, there is nothing to prove. When $N > 0$, consider $\widetilde{f}_1^N u \in \mathcal{L}$. Since we have $\widetilde{f}_1^N u = f_1^{(N)} u = f_1^{(N)} u_N$, it holds that $f_1^{(N)} u_N \in \mathcal{L}$. Hence we have $u_N = \widetilde{e}_1^N f_1^{(N)} u_N \in \mathcal{L}$. This implies that $u - u_N = \sum_{b=0}^{N-1} u_b$ and $u - u_N \in \mathcal{L} \cap \operatorname{Ker} e_1$, and hence, by induction hypothesis, we have $u_b \in \mathcal{L}$ for all b. Now, let us assume that $u + q\mathcal{L} \in \mathcal{B}$. Set $b_0 := \varphi_1(u + q\mathcal{L})$. Since $0 \neq \widetilde{f}_1^{b_0} u = \sum_{b=b_0}^{N} f_1^{(b_0)} u_b$, it holds that $0 \leq b_0 \leq N$. Then, we have $\sum_{b=b_0}^{N} f_1^{(b_0)} u_b \in \mathcal{L} \backslash q\mathcal{L}$, and $\sum_{b=b_0+1}^{N} f_1^{(b_0+1)} u_b = \widetilde{f}_1^{b_0+1} u \in q\mathcal{L}$. Thus, we have $u_b \in q\mathcal{L}$ for all $b > b_0$, and $f_1^{(b_0)} u_{b_0} + q\mathcal{L} = \widetilde{f}_1^{b_0}(u + q\mathcal{L})$ (equivalently, $u_{b_0} + q\mathcal{L} = u + q\mathcal{L}$). Then, we have $u - u_{b_0} \in q\mathcal{L}$, and hence, $u_b \in q\mathcal{L}$ for all $b \neq b_0$. This completes the proof. ☐

Now, the following theorem can be proved in a similar way to the ordinary crystal basis theory.

Theorem 12.1.9. *Let $M \in \mathcal{O}^{\jmath}_{\mathrm{int}}$ be a $\mathbf{U}^{\jmath}_1$-module. Then, M has a quasi-$\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. Moreover, if $M \simeq \bigoplus_{\boldsymbol{\lambda} \in P^{\jmath}_1} L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ for some $m_{\boldsymbol{\lambda}} \in \mathbb{Z}_{\geq 0}$, then there exists an isomorphism $M \to \bigoplus_{\boldsymbol{\lambda} \in P^{\jmath}_1} L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ of $\mathbf{U}^{\jmath}_1$-modules which induces an isomorphism*

$$(\mathcal{L}, \mathcal{B}) \to \left(\bigoplus_{\boldsymbol{\lambda} \in P^{\jmath}_1} \mathcal{L}(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}, \bigsqcup_{\boldsymbol{\lambda} \in P^{\jmath}_1} \mathcal{B}(\boldsymbol{\lambda})^{m_{\boldsymbol{\lambda}}} \right).$$

12.2. **Tensor product rule.** Recall that $\mathbf{U}^{\jmath}$ is a right coideal of $\mathbf{U}$, i.e., $\Delta(\mathbf{U}^{\jmath}) \subset \mathbf{U}^{\jmath} \otimes \mathbf{U}$. Hence, we are interested in the $\mathbf{U}^{\jmath}$-module structure of the tensor product of a $\mathbf{U}^{\jmath}$-module and a $\mathbf{U}$-module. Let $\mathbf{V} = \mathbf{V}_r$ denote the vector representation of $\mathbf{U}$. If we set $\mathbf{L} = \mathbf{L}_r := \bigoplus_{i \in [-r,r]} \mathbf{A}_0 u_i$, $\mathbf{B} = \mathbf{B}_r := \{u_i + q\mathbf{L} \mid i \in [-r, r]\}$, then, $(\mathbf{L}, \mathbf{B})$ is an ordinary crystal basis of $\mathbf{V}$.

We first consider the case $r = 1$. Recall that the irreducible $\mathbf{U}^{\jmath}_1$-module $L(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in P^{\jmath}_1$ has a quasi-$\jmath$-crystal basis $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$. If $L(\boldsymbol{\lambda}) = L(a; b)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq}0$, then we write $\mathcal{L}(a; b) = \mathcal{L}(\boldsymbol{\lambda})$, $\mathcal{B}(a; b) = \mathcal{B}(\boldsymbol{\lambda})$.

Proposition 12.2.1. *Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geq 0}$. Then we have an isomorphism*

$$L(a; b) \otimes \mathbf{V} \simeq L(a + 2; b + 1) \oplus L(a - 1; b) \oplus L(a - 1; b - 1)$$

of $\mathbf{U}^{\jmath}_1$-modules. Moreover, $(\mathcal{L}(a; b) \otimes \mathbf{L}, \mathcal{B}(a; b) \otimes \mathbf{B})$ is a quasi-$\jmath$-crystal basis of $L(a; b) \otimes \mathbf{V}$.

Proof. Let $v \in L(a; b)$ be a highest weight vector, and set

$$\begin{aligned} v\boxed{0} &:= v \otimes u_0, \\ v\boxed{1} &:= v \otimes u_1 - \frac{q^{-b+1}(q - q^{-1})}{\{a - b - 1\}} f_1 v \otimes u_0 - pq^{a-2b} v \otimes u_{-1}, \\ v\boxed{-1} &:= f_1 v \otimes u_0 - q^b [b] v \otimes u_{-1} - pq^{a-b-2} [b] v \otimes u_1. \end{aligned}$$

Then, by direct calculation, we obtain

$$\begin{aligned} h_1 v\boxed{0} &= [b + 1]\{(a + 2) - (b + 1) - 1\} v\boxed{0}, \\ h_1 v\boxed{1} &= [b]\{(a - 1) - b - 1\} v\boxed{1}, \\ h_1 v\boxed{-1} &= [b - 1]\{(a - 1) - (b - 1) - 1\} v\boxed{-1}. \end{aligned}$$

These equations, together with Corollary 10.2.4 and Theorem 10.1.6, show that $\mathbf{U}^{\jmath}_1 v\boxed{0} \simeq L(a+2; b+1)$, $\mathbf{U}^{\jmath}_1 v\boxed{1} \simeq L(a-1; b)$, and $\mathbf{U}^{\jmath}_1 v\boxed{-1} \simeq L(a-1; b-1)$. Since $\dim(L(a; b) \otimes \mathbf{V}) =$

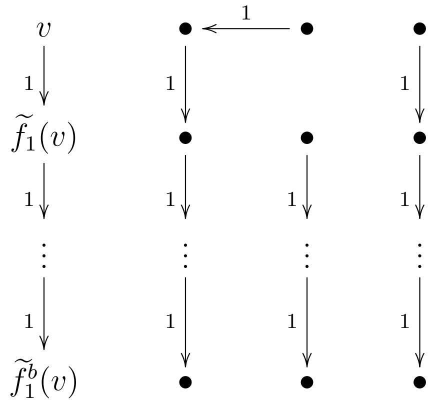
$3b = (b+1)+b+(b-1) = \sum_{k=-1}^{1} \dim \mathbf{U}^\jmath_1 v\boxed{k}$, we see that $L(a;b)\otimes\mathbf{V} = \mathbf{U}^\jmath_1 v\boxed{0} \oplus \mathbf{U}^\jmath_1 v\boxed{-1} \oplus \mathbf{U}^\jmath_1 v\boxed{1}$. Also, we calculate as:

$$\begin{aligned}
f_1^{(n)}(v\boxed{0}) &= f_1^{(n-1)}v\otimes u_{-1} + q^n f_1^{(n)} v\otimes u_0 + pq^{a-n+1} f_1^{(n-1)} v\otimes u_1 \\
&\in \begin{cases} v\otimes u_0 + q\mathcal{L}(a;b)\otimes\mathbf{L} & \text{if } n=0, \\ f_1^{(n-1)}v\otimes u_{-1} + q\mathcal{L}(a;b)\otimes\mathbf{L} & \text{if } 0\leq n\leq b+1, \end{cases} \\
f_1^{(n)}(v\boxed{1}) &= \frac{q^{-n}\{a-b-n-1\}}{\{a-b-1\}} f_1^{(n)} v\otimes u_1 - \frac{q^{-b+n+1}(q^{n+1}-q^{-n-1})}{\{a-b-1\}} f_1^{(n+1)} v\otimes u_0 \\
&\qquad - \frac{pq^{a-2b}\{a-b-n-1\}}{\{a-b-1\}} f_1^{(n)} v\otimes u_{-1} \\
&\in f_1^{(n)}v\otimes u_1 + q\mathcal{L}(a;b)\otimes\mathbf{L} \quad \text{if } 0\leq n\leq b, \\
f_1^{(n)}(v\boxed{-1}) &= q^n[n+1]f_1^{(n+1)}v\otimes u_0 - q^b[b-n]f_1^{(n)}v\otimes u_{-1} - pq^{a-b-n-2}[b-n]f_1(n)v\otimes u_1 \\
&\in f_1^{(n+1)}v\otimes u_0 + q\mathcal{L}(a;b)\otimes\mathbf{L} \quad \text{if } 0\leq n\leq b-1.
\end{aligned}$$

Since $\widetilde{f}_1^n(v\boxed{k}) = f_1^{(n)}(v\boxed{k})$, $k\in\{0,\pm1\}$, these equations imply that the $\mathbf{A}_0$-span of $\{\widetilde{f}_1^n(v\boxed{k}) \mid k\in\{0,\pm1\},\ n\in\mathbb{Z}_{\geq 0}\}$ coincides with $\mathcal{L}(a;b)\otimes\mathbf{L}$, and that $\{\widetilde{f}_1^n(v\boxed{k}) + q\mathcal{L}(a;b)\otimes\mathbf{L} \mid k\in\{0,\pm1\},\ n\in\mathbb{Z}_{\geq 0}\}\setminus\{0\}$ is identical to $\mathcal{B}(a;b)\otimes\mathbf{B}$. Now, it is easy to verify that $(\mathcal{L}(a;b)\otimes\mathbf{L}, \mathcal{B}(a;b)\otimes\mathbf{B})$ is a quasi-$\jmath$-crystal basis of $L(a;b)\otimes\mathbf{V}$. This proves the proposition. ☐

We give the quasi-$\jmath$-crystal graph of $\mathcal{B}(a;b)\otimes\mathbf{B}$:

$$u_{-1} \xrightarrow{-1} u_0 \qquad u_1\ .$$



More generally, we obtain the following theorem. As in the ordinary crystal basis theory, the proof is given by embedding the crystal basis of a $\mathbf{U}_3$-module into $(\mathbf{L}^{\otimes N}, \mathbf{B}^{\otimes N})$ for a suitable N.

Theorem 12.2.2. *Let M be a $\mathbf{U}^\jmath_1$-module with a quasi-$\jmath$-crystal basis $(\mathcal{L}_1, \mathcal{B}_1)$, and N a $\mathbf{U}_3$-module with a crystal basis $(\mathcal{L}_2, \mathcal{B}_2)$. Then, $M\otimes N$ has a quasi-$\jmath$-crystal basis*

$(\mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$*, on which the Kashiwara operators act as follows:*

$$\widetilde{f}_1(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{E}_{-\underline{1}}(b_2) & \textit{if } \varepsilon_1(b_1) < \varepsilon_{-\underline{1}}(b_2), \\ \widetilde{f}_1(b_1) \otimes b_2 & \textit{if } \varepsilon_1(b_1) \geq \varepsilon_{-\underline{1}}(b_2), \end{cases}$$

$$\widetilde{e}_1(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{F}_{-\underline{1}}(b_2) & \textit{if } \varepsilon_1(b_1) \leq \varepsilon_{-\underline{1}}(b_2), \\ \widetilde{e}_1(b_1) \otimes b_2 & \textit{if } \varepsilon_1(b_1) > \varepsilon_{-\underline{1}}(b_2). \end{cases}$$

Now, we turn to the case of a general r. Recall that Kashiwara operators $\widetilde{f}_i$ and $\widetilde{e}_i$ for $i \neq 1$ are defined by means of the $\mathfrak{sl}_2$-triple (f_i, k_i, e_i). Therefore, the next proposition follows from a standard argument; see, for example, [HK02, Section 4.4].

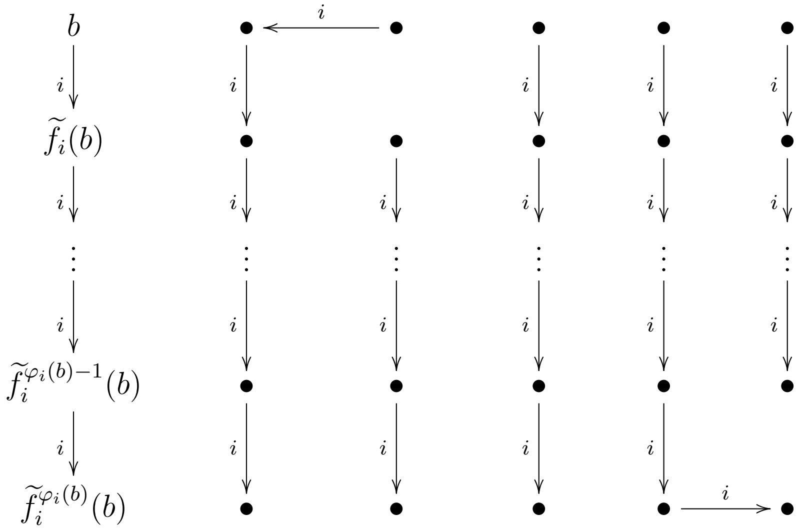
Proposition 12.2.3. *Let M be a $\mathbf{U}^\jmath$-module having a quasi-$\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. Then $(\mathcal{L} \otimes \mathbf{L}, \mathcal{B} \otimes \mathbf{B})$ is a quasi-$\jmath$-crystal basis of $M \otimes \mathbf{V}$, on which the Kashiwara operators act as follows: $\widetilde{f}_1$ and $\widetilde{e}_1$ acts as described in Theorem 12.2.2; for $i \in \mathbb{I}^\jmath \setminus \{1\}, b \in \mathcal{B}, j \in [-r, r]$,*

$$\widetilde{f}_i(b \otimes u_j) = \begin{cases} 0 & \textit{if } j = i \textit{ and } \widetilde{f}_i^2(b) = 0, \\ b \otimes u_i & \textit{if } j = i-1 \textit{ and } \widetilde{f}_i(b) = 0, \\ b \otimes u_{-i} & \textit{if } j = -(i-1) \textit{ and } \widetilde{e}_i(b) = 0, \\ \widetilde{f}_i(b) \otimes u_j & \textit{otherwise}, \end{cases}$$

$$\widetilde{e}_i(b \otimes u_j) = \begin{cases} b \otimes u_{i-1} & \textit{if } j = i \textit{ and } \widetilde{f}_i(b) = 0, \\ 0 & \textit{if } j = -(i-1) \textit{ and } \widetilde{e}_i^2(b) = 0, \\ b \otimes u_{-(i-1)} & \textit{if } j = -i \textit{ and } \widetilde{e}_i(b) = 0, \\ \widetilde{e}_i(b) \otimes u_j & \textit{otherwise}. \end{cases}$$

The action of $\widetilde{f}_i$ for $i \neq 1$ is visualized as:

$$u_{-i} \xrightarrow{-\underline{i}} u_{-(i-1)} \qquad u_j \qquad u_{i-1} \xrightarrow{\underline{i}} u_i \ .$$



The following theorem describes the tensor product rule for the Kashiwara operators $\widetilde{f}$'s and $\widetilde{e}$'s in full generality. The proof is given by embedding the crystal basis of a $\mathbf{U}$-module into $(\mathbf{L}^{\otimes N}, \mathbf{B}^{\otimes N})$ for a suitable N.

Theorem 12.2.4. *Let M be a $\mathbf{U}^{\imath}$-module having a quasi-$\jmath$-crystal basis $(\mathcal{L}_1, \mathcal{B}_1)$, and N a $\mathbf{U}$-module having a crystal basis $(\mathcal{L}_2, \mathcal{B}_2)$. Then, $M \otimes N$ has a quasi-$\jmath$-crystal basis $(\mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$, on which the Kashiwara operators act as follows: for $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$,*

$$\widetilde{f}_1(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{E}_{-\underline{1}}(b_2) & \text{if } \varepsilon_1(b_1) < \varepsilon_{-\underline{1}}(b_2), \\ \widetilde{f}_1(b_1) \otimes b_2 & \text{if } \varepsilon_1(b_1) \geq \varepsilon_{-\underline{1}}(b_2), \end{cases}$$

$$\widetilde{e}_1(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{F}_{-\underline{1}}(b_2) & \text{if } \varepsilon_1(b_1) \leq \varepsilon_{-\underline{1}}(b_2), \\ \widetilde{e}_1(b_1) \otimes b_2 & \text{if } \varepsilon_1(b_1) > \varepsilon_{-\underline{1}}(b_2), \end{cases}$$

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{E}_{-\underline{i}}(b_2) & \text{if } \varepsilon_{\underline{i}}(b_2) < \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) < \varepsilon_{-\underline{i}}(b_2), \text{ or} \\ & \text{if } \varepsilon_{\underline{i}}(b_2) \geq \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) + \varepsilon_{\underline{i}}(b_2) - \varphi_i(b_1) < \varepsilon_{-\underline{i}}(b_2), \\ \widetilde{f}_i(b_1) \otimes b_2 & \text{if } \varepsilon_{\underline{i}}(b_2) < \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) \geq \varepsilon_{-\underline{i}}(b_2), \\ b_1 \otimes \widetilde{F}_{\underline{i}}(b_2) & \text{if } \varepsilon_{\underline{i}}(b_2) \geq \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) + \varepsilon_{\underline{i}}(b_2) - \varphi_i(b_1) \geq \varepsilon_{-\underline{i}}(b_2), \end{cases}$$

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{F}_{-\underline{i}}(b_2) & \text{if } \varepsilon_{\underline{i}}(b_2) \leq \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) \leq \varepsilon_{-\underline{i}}(b_2), \text{ or} \\ & \text{if } \varepsilon_{\underline{i}}(b_2) > \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) + \varepsilon_{\underline{i}}(b_2) - \varphi_i(b_1) \leq \varepsilon_{-\underline{i}}(b_2), \\ \widetilde{e}_i(b_1) \otimes b_2 & \text{if } \varepsilon_{\underline{i}}(b_2) \leq \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) > \varepsilon_{-\underline{i}}(b_2), \\ b_1 \otimes \widetilde{E}_{\underline{i}}(b_2) & \text{if } \varepsilon_{\underline{i}}(b_2) > \varphi_i(b_1) \text{ and } \varepsilon_i(b_1) + \varepsilon_{\underline{i}}(b_2) - \varphi_i(b_1) > \varepsilon_{-\underline{i}}(b_2). \end{cases}$$

In order to memorize this tensor product rule for quasi-$\jmath$-crystal bases, we introduce the notion of $\imath$-signatures for quasi-$\jmath$-crystal bases. For $b \in \mathcal{B}$ and $\imath \in \mathbb{I}^{\imath}$, the $\imath$-signature $\mathrm{sgn}_{\imath}(b)$ of b is the sequence of $\varepsilon_i(b)$ $-$'s followed by $\varphi_i(b)$ $+$'s. For $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$, Theorem 12.2.4 tells us that $\mathrm{sgn}_{\imath}(b_1 \otimes b_2)$ is obtained by deleting adjacent pair $(+,-)$ in the concatenation of $\mathrm{sgn}^*_{-\underline{1}}(b_2)$ and $\mathrm{sgn}_1(b_1)$ when $i = 1$, and in the concatenation of $\mathrm{sgn}^*_{-\underline{i}}(b_2)$, $\mathrm{sgn}_{\imath}(b_1)$, and $\mathrm{sgn}_{\underline{u}}(b_2)$ until there are no such pairs, where $\mathrm{sgn}^*_{-\underline{i}}(b_2)$ is obtained from $\mathrm{sgn}_{-\underline{i}}(b_2)$ by reversing the sequence, and switching $+$ and $-$. Then,

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{E}_{-\underline{i}} b_2 & \text{if the leftmost } + \text{ in } \mathrm{sgn}_{\imath}(b_1 \otimes b_2) \text{ is originally in } \mathrm{sgn}^*_{-\underline{i}}(b_2), \\ \widetilde{f}_i b_1 \otimes b_2 & \text{if the leftmost } + \text{ in } \mathrm{sgn}_{\imath}(b_1 \otimes b_2) \text{ is originally in } \mathrm{sgn}_{\imath}(b_1), \\ b_1 \otimes \widetilde{F}_{\underline{i}} b_2 & \text{if the leftmost } + \text{ in } \mathrm{sgn}_{\imath}(b_1 \otimes b_2) \text{ is originally in } \mathrm{sgn}_{\underline{i}}(b_2), \end{cases}$$

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \widetilde{F}_{-\underline{i}} b_2 & \text{if the rightmost } - \text{ in } \mathrm{sgn}_{\imath}(b_1 \otimes b_2) \text{ is originally in } \mathrm{sgn}^*_{-\underline{i}}(b_2), \\ \widetilde{e}_i b_1 \otimes b_2 & \text{if the rightmost } - \text{ in } \mathrm{sgn}_{\imath}(b_1 \otimes b_2) \text{ is originally in } \mathrm{sgn}_{\imath}(b_1), \\ b_1 \otimes \widetilde{E}_{\underline{i}} b_2 & \text{if the rightmost } - \text{ in } \mathrm{sgn}_{\imath}(b_1 \otimes b_2) \text{ is originally in } \mathrm{sgn}_{\underline{i}}(b_2). \end{cases}$$

This description is useful particularly when we consider the tensor product of many crystals.

Corollary 12.2.5. *Let $N \in \mathcal{O}_{\mathrm{int}}$ be a $\mathbf{U}$-module with a crystal basis $(\mathcal{L}', \mathcal{B}')$. Then, $(\mathcal{L}', \mathcal{B}')$ is also a quasi-$\jmath$-crystal basis of N. Furthermore, for each $b \in \mathcal{B}$ and $i \in \mathbb{I}^{\jmath}$, we* *have the following:*

$$\begin{aligned}
&\widetilde{f}_1(b) = \widetilde{E}_{-\underline{1}}(b),\\
&\widetilde{e}_1(b) = \widetilde{F}_{-\underline{1}}(b),\\
&\widetilde{f}_i(b) = \begin{cases} \widetilde{E}_{-\underline{i}}(b) & \textit{if } \varepsilon_{\underline{i}}(b) < \varepsilon_{-\underline{i}}(b),\\ \widetilde{F}_{\underline{i}}(b) & \textit{if } \varepsilon_{\underline{i}}(b) \geq \varepsilon_{-\underline{i}}(b),\end{cases}\\
&\widetilde{e}_i(b) = \begin{cases} \widetilde{F}_{-\underline{i}}(b) & \textit{if } \varepsilon_{\underline{i}}(b) \leq \varepsilon_{-\underline{i}}(b),\\ \widetilde{E}_{\underline{i}}(b) & \textit{if } \varepsilon_{\underline{i}}(b) > \varepsilon_{-\underline{i}}(b).\end{cases}
\end{aligned}$$

Proof. Apply Theorem 12.2.4 for the trivial $\mathbf{U}^\jmath$-module $M = \mathbb{C}$. □

12.3. **Abstract quasi-$\jmath$-crystals.** As in the ordinary crystal basis theory, we define the notion of (abstract) quasi-$\jmath$-crystals as sets with structure maps $\widetilde{e}_i, \widetilde{f}_i$, $i \in \mathbb{I}^\jmath$, and $\mathrm{wt}^\jmath$.

By Corollary 12.2.5, we can define a functor from the category of crystals of $\mathfrak{sl}_{2r+1}$-type to the category of quasi-$\jmath$-crystals. Since $\widetilde{E}_{\underline{1}}$ and $\widetilde{F}_{\underline{1}}$ do not appear in the formula of Corollary 12.2.5, we can regard this functor to factor through the forgetting functor $\mathrm{Res}^{2r+1}_{r+1,r}$.

Lemma 12.3.1. *Let $\mathcal{B}$ be a crystal of $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_r)$-type. Then, $\mathcal{B}$ is equipped with a quasi-$\jmath$-crystal structure such that the structure maps $\widetilde{e}_i, \widetilde{f}_i$ are defined by the formula in Corollary 12.2.5.*

Recall that we have given to $[-r,r]^d$ and $\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}$ a crystal structure of $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_r)$-type. By Lemma 12.3.1, they are also equipped with quasi-$\jmath$-crystal structures. Let us describe these structures in a purely combinatorial way.

Let $\mathbf{s} = (s_1, \dots, s_d) \in [-r,r]^d$. First, set $\mathbf{s}_1$ to be $\mathbf{s}^{\mathrm{op}}_{-\underline{1}}$, where $\mathbf{s}^{\mathrm{op}}_{-\underline{1}}$ is obtained by reversing the sequence $\mathbf{s}_{-\underline{1}}$ (see Subsection 1.6). Then, $\widetilde{e}_1(\mathbf{s})$ is obtained from $\mathbf{s}$ by replacing s_k with 0, where s_k is the rightmost -1 in $\mathbf{s}_1$; if there is no -1 in $\mathbf{s}_1$, then $\widetilde{e}_1(\mathbf{s}) = 0$. Also, $\widetilde{f}_1(\mathbf{s})$ is obtained from $\mathbf{s}$ by replacing s_l with -1, where s_l is the leftmost 0 in $\mathbf{s}_1$; if there is no 0 in $\mathbf{s}_1$, then $\widetilde{f}_1(\mathbf{s}) = 0$.

Next, for $i \in \mathbb{I}^\jmath \setminus \{1\}$, set $\mathbf{s}_i$ to be the sequence obtained by deleting the adjacent pair $(-(i-1), i)$ in the concatenation of $\mathbf{s}^{\mathrm{op}}_{-\underline{i}}$ followed by $\mathbf{s}_{\underline{i}}$ until there are no such pairs. Then, $\widetilde{e}_i(\mathbf{s})$ is obtained from $\mathbf{s}$ by replacing s_k with $-(i-1)$, where s_k is the rightmost $-i$ in $\mathbf{s}_i$; if there is no $-i$ in $\mathbf{s}_i$, then $\widetilde{e}_i(\mathbf{s})$ is obtained from $\mathbf{s}$ by replacing s_k with $i-1$, where s_k is the rightmost i in $\mathbf{s}_i$; if there is no $\pm i$, then $\widetilde{e}_i(\mathbf{s}) = 0$. Also, $\widetilde{f}_i(\mathbf{s})$ is obtained from $\mathbf{s}$ by replacing s_l with $-i$, where s_l is the leftmost $-(i-1)$ in $\mathbf{s}_i$; if there is no $-(i-1)$ in $\mathbf{s}_i$, then $\widetilde{f}_i(\mathbf{s})$ is obtained from $\mathbf{s}$ by replacing s_l with i, where s_l is the leftmost $i-1$ in $\mathbf{s}_i$; if there is no $\pm(i-1)$, then $\widetilde{f}_i(\mathbf{s}) = 0$.

Finally, for $T \in \mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$, $\widetilde{x}_i(T)$ is the unique $T' \in \mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$ such that $(\mathrm{ME}((T')^-), \mathrm{ME}((T')^+)) = \widetilde{x}_i(\mathrm{ME}(T^-), \mathrm{ME}(T^+))$ if $\widetilde{x}_i(T) \neq 0$ $(x \in \{e, f\})$.

13. Quasi-$\jmath$-crystal bases of irreducible $\mathbf{U}^\jmath$-modules

The aim of this Section is to prove the existence of quasi-$\jmath$-crystal basis of an irreducible $\mathbf{U}^\jmath$-module. In Subsection 13.1, we apply the results obtained in Sections 5 to 6 in order to prove that a quasi-$\jmath$-crystal basis of $\mathbf{V}^{\otimes d}$ is compatible with an irreducible decomposition. In Subsection 13.2, we show that certain quasi-$\jmath$-crystal basis of an irreducible $\mathbf{U}^\jmath$-module is identified with the set of semistandard Young bitableaux. In Subsection 13.3, we finally define $\jmath$-crystal bases as a generalization of quasi-$\jmath$-crystal bases, and prove the existence and uniqueness.

13.1. **Quasi-$\jmath$-crystal basis of $\mathbf{V}^{\otimes d}$.**

In this section, we study the quasi-$\jmath$-crystal basis $(\mathbf{L}^{\otimes d}, \mathbf{B}^{\otimes d})$ of $\mathbf{V}^{\otimes d}$ in detail. One of the key facts is the following.

Theorem 13.1.1 ([BI03, Theorem 7.7]). *Let $X \in L(W)$. Then, C^L_X is an irreducible left $\mathcal{H}$-module.*

Recall the functor $\mathcal{F}_{\pi^\jmath} : \mathcal{H}\text{-mod} \to \mathbb{S}(\pi^\jmath)\text{-mod}$ defined by $\mathcal{F}_{\pi^\jmath}(M) := \mathbb{T}(\pi^\jmath) \otimes_{\mathcal{H}} M$. This functor induces another functor

$$\mathcal{F}^\jmath : \mathbf{H}\text{-mod} \to \mathbf{U}^\jmath\text{-mod};\ M \mapsto \mathbf{V}^{\otimes d} \otimes_{\mathbf{H}} M.$$

By the $(\mathbf{U}^\jmath, \mathbf{H})$-duality, this functor sends an irreducible module to an irreducible module or 0. Set $L(W)_0 := \{X \in L(W) \mid \mathcal{F}^\jmath(\mathbf{C}^L_X) \neq 0\}$. Then, for each $X \in L(W)_0$, there exists a unique $\boldsymbol{\lambda}(X) \in P^\jmath$ such that $\mathcal{F}^\jmath(\mathbf{C}^L_X) \simeq L(\boldsymbol{\lambda}(X))$. With this notation, we have

$$\mathbf{V}^{\otimes d} \simeq \bigoplus_{X \in L(W)_0} L(\boldsymbol{\lambda}(X)).$$

On the other hand, By Proposition 6.3.1, we have $\mathcal{F}^\jmath(\mathbf{C}^L_X) \simeq \mathbf{C}^L_X(\pi^\jmath)$ for all $X \in L(W)$. Hence, we obtain $L(\boldsymbol{\lambda}(X)) \simeq \mathbf{C}^L_X(\pi^\jmath)$ for all $X \in L(W)_0$.

Proposition 13.1.2. *Let $X \in L(W)_0$. Then, $\mathbf{C}_{\leq_L X}(\pi^\jmath)$, $\mathbf{C}_{<_L X}(\pi^\jmath)$, and $\mathbf{C}^L_X(\pi^\jmath)$ have quasi-$\jmath$-crystal bases.*

Proof. Let $x \in X$. Recall that the three modules have the parabolic KL-bases

$$\{{}^\lambda C_w \mid \lambda \in \pi^\jmath,\ w \in {}^\lambda W,\ \text{and } w_\lambda w \leq_L x\},$$
$$\{{}^\lambda C_w \mid \lambda \in \pi^\jmath,\ w \in {}^\lambda W,\ \text{and } w_\lambda w <_L x\},$$
$$\{[{}^\lambda C_w]_X \mid \lambda \in \pi^\jmath,\ w \in {}^\lambda W,\ \text{and } w_\lambda w \in X\},$$

respectively. Set $\mathcal{L}_{\leq_L X}$, $\mathcal{L}_{<_L X}$, $\mathcal{L}(X)$ to be the $\mathbf{A}_0$-span of these bases, and $\mathcal{B}_{\leq_L X}$, $\mathcal{B}_{<_L X}$, $\mathcal{B}(X)$ to be the image of these bases under the projections $\mathcal{L}_{\leq_L X} \to \mathcal{L}_{\leq_L X}/q\mathcal{L}_{\leq_L X}$, $\mathcal{L}_{<_L X} \to \mathcal{L}_{<_L X}/q\mathcal{L}_{<_L X}$, $\mathcal{L}(X) \to \mathcal{L}(X)/q\mathcal{L}(X)$, respectively.

Since $\mathcal{L}_{\leq_L X} \subset \mathbf{L}^{\otimes d}$, we have $\widetilde{x}_i(\mathcal{L}_{\leq_L X}) \subset \mathbf{L}^{\otimes d}$ for all $x \in \{e, f\}$, $i \in \mathbb{I}^\jmath$. Moreover, since these Kashiwara operator preserves the submodules, we have $\widetilde{x}_i(\mathcal{L}_{\leq_L X}) \subset \mathbf{L}^{\otimes d} \cap \mathbf{C}_{\leq_L X}(\pi^\jmath) = \mathcal{L}_{\leq_L X}$. Now, it is straightforward and easy to verify that $(\mathcal{L}_{\leq_L X}, \mathcal{B}_{\leq_L X})$ is a quasi-$\jmath$-crystal basis of $\mathbf{C}_{\leq_L X}(\pi^\jmath)$. Similarly, we see that $(\mathcal{L}_{<_L X}, \mathcal{B}_{<_L X})$ is a quasi-$\jmath$-crystal basis of $\mathbf{C}_{<_L X}(\pi^\jmath)$.

Since the Kashiwara operators preserve both $\mathcal{L}_{\leq_L X}$ and $\mathcal{L}_{<_L X}$, they also preserve $\mathcal{L}(X)$. Hence, we conclude that $(\mathcal{L}(X), \mathcal{B}(X))$ is a quasi-$\jmath$-crystal basis of $\mathbf{C}^L_X(\pi^\jmath)$. □

Proposition 13.1.3. *For each $X \in L(W)_0$, there exists a highest weight vector $v_X \in \mathbf{V}^{\otimes d}$ satisfying the following:*

1. *$L_X := \mathbf{U}^\jmath v_X \simeq \mathbf{C}^L_X(\pi^\jmath)$.*
2. *$(\mathcal{L}_X, \mathcal{B}_X)$ is a quasi-$\jmath$-crystal basis of L_X isomorphic to $(\mathcal{L}(X), \mathcal{B}(X))$, where $\mathcal{L}_X := \mathbf{L}^{\otimes d} \cap L_X$, $\mathcal{B}_X := \mathbf{B}^{\otimes d} \cap (\mathcal{L}_X / q\mathcal{L}_X)$.*
3. *$\mathbf{L}^{\otimes d} = \bigoplus_{X \in L(W)_0} \mathcal{L}_X$, and $\mathbf{B}^{\otimes d} = \bigsqcup_{X \in L(W)_0} \mathcal{B}_X$.*

Proof. Recall the bilinear form $\langle \cdot \mid \cdot \rangle$ on $\mathbf{V}^{\otimes d}$ in Corollary 7.3.3. We prove that there exist highest weight vectors v_X satisfying the conditions (1) and (2) by induction on X with respect to the partial order $\leq_L$ on $L(W)_0$. Assume that X is minimal. Then, we have $\mathbf{C}_{\leq_L X}(\pi^\jmath) = \mathbf{C}^L_X(\pi^\jmath) \simeq L(\boldsymbol{\lambda}(X))$. Hence, there exists a highest weight vector $v_X \in \mathbf{C}_{\leq_L X}(\pi^\jmath)$. Now, it is clear that v_X satisfies the condition (1) and (2).

Next, assume that we have chosen highest weight vectors v_Y satisfying the conditions (1) and (2) for all $Y <_L X$. Let C' denote the orthogonal complement of $\mathbf{C}_{<_L X}(\pi^\jmath)$ in $\mathbf{C}_{\leq_L X}(\pi^\jmath)$ with respect to $\langle \cdot \mid \cdot \rangle$. By the skew invariance of $\langle \cdot \mid \cdot \rangle$, C' is a $\mathbf{U}^\jmath$-submodule. Moreover, since $\langle \cdot \mid \cdot \rangle$ is nondegenerate, we have $\mathbf{C}\leq_L X(\pi^\jmath) = \mathbf{C}_{<_L X}(\pi^\jmath) \oplus C'$, and hence, $C' \simeq \mathbf{C}^L_X(\pi^\jmath) \simeq L(\boldsymbol{\lambda}(X))$. Since $\mathbf{C}_{\leq_L X}(\pi^\jmath)$ is spanned by the parabolic KL-basis, C' is spanned by the vectors of the form ${}^{\mathbf{i}}C_w + r_{\mathbf{i},w}$ with $\mathbf{i} \in \pi^\jmath$, $w \in {}^{\mathbf{i}}W$, $w_{\mathbf{i}}w <_L x$ for some $x \in X$, $r_{\mathbf{i},w} \in \bigoplus_{Y <_L X} q\mathcal{L}_Y$. Therefore, each highest weight vector v_X of C' satisfies the conditions (1) and (2). □

13.2. **Combinatorial description of $\mathcal{B}(X)$.** By Propositions 20.1.1 and 20.1.2, we have a decomposition

$$\mathbf{B}^{\otimes d} = \bigsqcup_{X \in L(W)_0} \mathcal{B}(X), \quad \mathcal{B}(X) = \{u_{\mathbf{i}_\lambda w} \mid \lambda \in \pi^\jmath,\ w \in {}^\lambda W, \text{ and } w_\lambda w \in X\} \tag{13}$$

of quasi-$\jmath$-crystals. Also, by Theorem 5.3.4 (2), $u_{\mathbf{i}}$ and $u_{\mathbf{j}}$ ($\mathbf{i}, \mathbf{j} \in [-r, r]^d$) belong to $\mathcal{B}(X)$ for some $X \in L(W)_0$ if and only if $Q^\pm(\mathbf{i}) = Q^\pm(\mathbf{j})$. Hence, each $\mathcal{B}(X)$ is of the form

$$\mathcal{B}(X) = \{u_{\mathbf{i}} \mid \mathbf{i} \in [-r, r]^d,\ (Q^-(\mathbf{i}), Q^+(\mathbf{i})) = \mathbf{Q}\}, \quad \mathbf{Q} \in \mathrm{ST}(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d).$$

On the other hand, we have a decomposition

$$\mathbf{B}^{\otimes d} \to \bigsqcup_{\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d)} \bigsqcup_{\mathbf{Q} \in \mathrm{ST}(\boldsymbol{\lambda})} \big(\mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda}) \times \{\mathbf{Q}\}\big) \tag{14}$$

of an $\mathfrak{s}(\mathfrak{gl}_{r+1} \otimes \mathfrak{gl}_r)$-crystal. By Lemma 12.3.1 and arguments in Subsection 12.3, this decomposition is also a decomposition of a quasi-$\jmath$-crystal.

Theorem 13.2.1. *We have $L(W)_0 = \{X \in L(W) \mid \mathrm{sh}\, X \in \mathrm{Bip}_{(r+1,r)}(d)\}$. Also, for each $X \in L(W)_0$ with $\mathrm{sh}\, X = \boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d)$, we have $\mathcal{B}(X) = \{(\mathbf{T}, \mathbf{Q}) \mid \mathbf{T} \in \mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})\}$, where $\mathbf{Q} := \mathbf{Q}(X)$, and $L_X \simeq L(\boldsymbol{\lambda})$.*

Proof. The first statement is obtained by comparing the decompositions (13) and (14). Let $X \in L(W)_0$, $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d)$ and $(Q^-; Q^+) \in \mathrm{ST}(\boldsymbol{\lambda})$ be such that $\mathrm{SST}(\boldsymbol{\lambda}) \times \{(Q^-; Q^+)\} = \mathcal{B}_X$. Then, $\mathrm{SST}(\boldsymbol{\lambda}) \times \{(Q^-; Q^+)\}$ contains a unique element $b_{\boldsymbol{\lambda}}$ such that

$$\widetilde{F}_{-\underline{i}} b_{\boldsymbol{\lambda}} = \widetilde{E}_{\underline{j}} b_{\boldsymbol{\lambda}} = 0 \text{ for all } i = 1, \dots, r \text{ and } j = 2, \dots, r.$$

Under the isomorphism $\mathrm{SST}(\boldsymbol{\lambda}) \times \{(Q^-; Q^+)\} \simeq \mathrm{SST}(\boldsymbol{\lambda})$ of crystals of type $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_r)$, the element $b_{\boldsymbol{\lambda}}$ is identified with $T_{\boldsymbol{\lambda}} = (T^-_{\boldsymbol{\lambda}}; T^+_{\boldsymbol{\lambda}}) \in \mathrm{SST}(\boldsymbol{\lambda})$. Since $\mathrm{wt}^\jmath(b_{\boldsymbol{\lambda}})$ is maximal among $\mathrm{wt}^\jmath(\mathcal{B}_t)$, we have $b_{\boldsymbol{\lambda}} := v_X + q\mathcal{L}_X$, where $v_X \in L_X$ is the $\mathbf{U}^\jmath$-highest weight vector. Suppose that $L_X = L(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \in P^\jmath$. Then, we have $\varphi_1(b_{\boldsymbol{\lambda}}) = \boldsymbol{\mu}_0 - \boldsymbol{\mu}_{-1}$. Recall from the proof of Lemma 11.2.7 that $\tau_2^{-1}(v_X)$ is a $\mathbf{U}^\jmath_1$-highest weight vector of highest weight $(\boldsymbol{\mu}_0, \boldsymbol{\mu}_{-2}; \boldsymbol{\mu}_2)$. This implies that $\varphi_1(\widetilde{f}_2^{\max)} b_{\boldsymbol{\lambda}}) = \boldsymbol{\mu}_0 - \boldsymbol{\mu}_{-2}$. Also, from the proof of Theorem 18.3.7, $T_i^{-1}(v_X)$ is a $\mathbf{U}^\jmath_2$-highest weight vector of highest weight $(\boldsymbol{\mu}_0, \boldsymbol{\mu}_{-(i-1)}, \boldsymbol{\mu}_{-i}; \boldsymbol{\mu}_{i-1}, \boldsymbol{\mu}_i)$ for all $i \geq 3$. Hence, we have

$$\varphi_1(\widetilde{f}_2^{\max}(\widetilde{f}_3^{\max}\widetilde{f}_2^{\max})(\widetilde{f}_4^{\max}\widetilde{f}_3^{\max}) \cdots (\widetilde{f}_i^{\max}\widetilde{f}_{i-1}^{\max}) b_{\boldsymbol{\lambda}}) = \boldsymbol{\mu}_0 - \boldsymbol{\mu}_{-i}.$$

Applying the same argument to $T_{\boldsymbol{\lambda}}$, we obtain the following:

$$\mathrm{wt}^\jmath(b_{\boldsymbol{\lambda}}) = \mathrm{wt}^\jmath(T_{\boldsymbol{\lambda}}), \quad \boldsymbol{\mu}_0 - \boldsymbol{\mu}_{-i} = \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-i} \text{ for all } i = 1, \dots, r.$$

Solving this system of equations, we conclude that $\boldsymbol{\mu} \sim_\pi \boldsymbol{\lambda}$, and hence, $L_X \simeq L(\boldsymbol{\lambda})$. □

Corollary 13.2.2. *For each* $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d)$*, we have* $0 \neq L(\boldsymbol{\lambda}) \in \mathcal{O}^{\jmath}_{\mathrm{int}}$.

This completes the classification of irreducible modules in $\mathcal{O}^{\jmath}_{\mathrm{int}}$.

13.3. **$\jmath$-crystal bases.** In general, a quasi-$\jmath$-crystal graph of an irreducible $\mathbf{U}^{\jmath}$-module is neither connected nor unique. In this subsection, we introduce the notion of $\jmath$-crystal bases as quasi-$\jmath$-crystal bases satisfying some additional conditions. And we prove the existence and uniqueness theorem for $\jmath$-crystal bases, and that they are connected.

Let $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}(d)$, and take a left cell $X \in L(W)$ satisfying $\mathbf{C}^L_X \simeq L(\boldsymbol{\lambda})$. Recall that $\mathbf{C}^L_X$ has a basis $\{[{}^{\lambda}C_w]_X \mid \lambda \in \pi^{\jmath},\ w \in {}^{\lambda}W, \text{ and } w_{\lambda}w \in X\}$, and it is in one-to-one correspondence with $\mathrm{SST}(\boldsymbol{\lambda})$; we denote by b_T the basis element corresponding to $T \in \mathrm{SST}(\boldsymbol{\lambda})$. For each $i \in \{2, \dots, r\}$, we define linear endomorphisms $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$ on $L(\boldsymbol{\lambda})$ by

$$\widetilde{e}_{i'}(b_T) = \begin{cases} b_{\widetilde{E}_{i-\frac{1}{2}}T} & \text{if } \widetilde{e}_j T = 0 \text{ for all } j = 1, \dots, i-1 \text{ and } \widetilde{E}_{j-\frac{1}{2}}T = 0 \text{ for all } j = 2, \dots, i-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widetilde{f}_{i'}(b_T) = \begin{cases} b_{T'} & \text{if } \widetilde{e}_{i'} b_{T'} = b_T, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the condition $\widetilde{e}_j T = 0$ for all $j = 1, \dots, i-1$ and $\widetilde{E}_{j-\frac{1}{2}}T = 0$ for all $j = 2, \dots, i-1$ is equivalent to $\widetilde{e}_j T = 0$ for all $j = 1, \dots, i-1$ and $\widetilde{e}_{j'} T = 0$ for all $j = 2, \dots, i-1$.

Let $X' \in L(W)$ be such that $C^L_{X'} \simeq C^L_X$. Since the linear map $[C_{w'}]_{X'} \mapsto [C_w]_X$, $w' \in X'$, $w \in X$ with $P^{\pm}(w') = P^{\pm}(w)$ gives an isomorphism $C^L_{X'} \to C^L_X$ of $\mathcal{H}$-modules, the definition of $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$ are independent of the choice of X as long as we have $\mathbf{C}^L_X \simeq L(\boldsymbol{\lambda})$.

Also, we define linear endomorphisms $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$, $i \in \{2, \dots, r\}$ on each $\mathbf{U}^{\jmath}$-module in $\mathcal{O}^{\jmath}_{\mathrm{int}}$ by the complete reducibility.

Remark 13.3.1. Later, we will give more intrinsic definitions of $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$.

Definition 13.3.2. Let $M \in \mathcal{O}^{\jmath}_{\mathrm{int}}$ be a $\mathbf{U}^{\jmath}$-module with a quasi-$\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. We say that $(\mathcal{L}, \mathcal{B})$ is a $\jmath$-crystal basis if it satisfies the following:

($\jmath$C 1) $\mathcal{L}$ is preserved by the operators $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$, $i \in [2, r]$.
($\jmath$C 2) We have $\widetilde{e}_{i'}(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ and $\widetilde{f}_{i'}(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ for all $i \in [2, r]$.

Let $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}$, and $v \in L(\boldsymbol{\lambda})$ be a highest weight vector. Set

$$\mathcal{L}(\boldsymbol{\lambda}) := \mathrm{Span}_{\mathbf{A}_0}\{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} v \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_l \in \mathbb{I}^{\jmath} \sqcup \{2', \dots, r'\}\},$$

$$\mathcal{B}(\boldsymbol{\lambda}) := \{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} v + q\mathcal{L}(\boldsymbol{\lambda}) \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_l \in \mathbb{I}^{\jmath} \sqcup \{2', \dots, r'\}\} \setminus \{0\}.$$

Theorem 13.3.3. *Let* $\boldsymbol{\lambda} \in P^{\jmath}$*. Then,* $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$ *is a unique* $\jmath$*-crystal basis of* $L(\boldsymbol{\lambda})$.

Proof. Let $X \in L(W)$ be such that $\mathbf{C}^L_X \simeq L(\boldsymbol{\lambda})$. By the definition of $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$, it is clear that they preserve $\mathcal{L}(X)$, and induce maps $\mathcal{B}(X) \to \mathcal{B}(X) \sqcup \{0\}$. Therefore, $(\mathcal{L}(X), \mathcal{B}(X))$ is a $\jmath$-crystal basis of $\mathbf{C}^L_X$.

Next, we show that $\mathcal{B}(X)$ is connected as a $\jmath$-crystal basis. To do so, it is convenient to identify $\mathcal{B}(X)$ with $\mathrm{SST}(\boldsymbol{\lambda})$. Let T_0 denote the unique highest weight vector of $\mathrm{SST}(\boldsymbol{\lambda})$. For $T \in \mathrm{SST}(\boldsymbol{\lambda})$, set

$$d(T) := \sum_{i,j} (|T(i,j)| - |T_0(i,j)|),$$

where $T(i,j)$ denotes the integer in the (i,j)-box of T. Then, we have $d(T) \geq 0$, and $d(T) = 0$ if and only if $T = T_0$. We prove that T is connected to T_0 by induction on $d(T)$. When $d(T) = 0$, we have $T = T_0$, and there is nothing to prove. When $d(T) > 0$, there exists $i \in \mathbb{I}^\jmath$ such that $\widetilde{e}_i T \neq 0$ or $\widetilde{e}_{i'} T \neq 0$. Hence, T is connected to either $\widetilde{e}_i T$ or $\widetilde{e}_{i'} T$. Since $d(\widetilde{e}_i T) = d(\widetilde{e}_{i'} T) = d(T) - 1$, the induction proceeds. This proves that $\mathrm{SST}(\boldsymbol{\lambda})$ is connected. Moreover, we obtain

$$\mathrm{SST}(\boldsymbol{\lambda}) = \{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} T_0 \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_l \in \mathbb{I}^\jmath \sqcup \{2', \dots, r'\}\} \setminus \{0\}.$$

Let $v \in \mathbf{C}_X^L$ be the unique highest weight vector satisfying $v + q\mathcal{L}(X) = T_0$. By above argument, for each $T \in \mathrm{SST}(\boldsymbol{\lambda})$, there exist $i_1, \dots, i_l \in \mathbb{I}^\jmath \sqcup \{2', \dots, r'\}$ such that $T = \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} T_0$. This implies that $\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} v + q\mathcal{L}(X) = T$, and therefore, $\mathcal{L}(X)$ is spanned by such vectors. Thus, the proof completes. □

Now, the existence and uniqueness theorem for $\jmath$-crystal basis can be proved in the same way as the ordinary crystal basis theory.

Theorem 13.3.4. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ be a $\mathbf{U}^\jmath$-module. Then, M has a $\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. If $M \simeq \bigoplus_{\boldsymbol{\lambda} \in P^\jmath_\pi} L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ for some $m_{\boldsymbol{\lambda}} \in \mathbb{Z}_{\geq 0}$, then there exists an isomorphism $M \to \bigoplus_{\boldsymbol{\lambda} \in P^\jmath_\pi} L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ inducing an isomorphism $(\mathcal{L}, \mathcal{B}) \to (\bigoplus_{\boldsymbol{\lambda} \in P^\jmath_\pi} \mathcal{L}(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}, \bigoplus_{\boldsymbol{\lambda} \in P^\jmath_\pi} \mathcal{B}(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}})$.*

14. GLOBAL $\jmath$-CRYSTAL BASES

This section is devoted to globalizing $\jmath$-crystal bases of irreducible $\mathbf{U}^\jmath$-modules. In Subsection 14.1, we introduce the notion of global $\jmath$-crystal bases, and give some examples. In Subsection 14.2, we prove the existence of global $\jmath$-crystal basis of an irreducible $\mathbf{U}^\jmath$-module.

14.1. **Global $\jmath$-crystal bases.** Let $\mathbf{U}^\jmath_{\mathbf{A}}$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}^\jmath$ generated by $e_i^{(n)}, f_i^{(n)}, k_i^{\pm 1}$, $i \in \mathbb{I}^\jmath$, $n \in \mathbb{Z}_{>0}$.

Lemma 14.1.1 ([L93, 1.3.5])**.** *Let A be a $\mathbb{Q}(q)$-algebra, $x, y \in A$ such that $xy = q^2 yx$. Then, for each $n \in \mathbb{Z}_{>0}$, we have*

$$(x+y)^n = \sum_{t=0}^{n} q^{t(n-t)} \begin{bmatrix} n \\ t \end{bmatrix} y^t x^{n-t}.$$

Lemma 14.1.2. *We have $\mathbf{U}^\jmath_{\mathbf{A}} \subset \mathbf{U}_{\mathbf{A}}$.*

Proof. It suffices to show that $e_i^{(n)}, f_i^{(n)} \in \mathbf{U}_{\mathbf{A}}$ for all $i \in \mathbb{I}^\jmath$, $n \in \mathbb{Z}_{>0}$. We prove $e_i^{(n)} \in \mathbf{U}_{\mathbf{A}}$; the proof for $f_i^{(n)} \in \mathbf{U}_{\mathbf{A}}$ is similar. Setting $x := E_{\underline{i}}$ and $y := p^{-\delta_{i,1}} F_{-\underline{i}} K_{\underline{i}}^{-1}$, we see that

$$e_i = x + y, \quad xy = q^2 yx.$$

Then, we can apply Lemma 14.1.1, and obtain

$$e_i^{(n)} = \sum_{t=0}^{n} q^{t(n-t)} y^{(t)} x^{(n-t)}.$$

It is easy to see that $y^{(t)} = p^{-\delta_{i,1} t} q^{-\delta_{i,1} \frac{t(t-1)}{2}} F_{-\underline{i}}^{(t)} K_{\underline{i}}^t \in \mathbf{U}_{\mathbf{A}}$. Hence, the assertion follows. □

Let V be a $\mathbf{U}^\jmath$-module in $\mathcal{O}^\jmath_{\mathrm{int}}$ with a $\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. Assume that V admits a $\psi^\jmath$-involution on V.

Definition 14.1.3. Let $V, \mathcal{L}, \mathcal{B}, \overline{\cdot}$ be as above. V is said to have a global $\jmath$-crystal basis if there exists a $\mathbf{U}^\jmath_{\mathbf{A}}$-submodule $V_{\mathbf{A}}$ of V which is an $\mathbf{A}$-lattice forming a balanced triple $(\mathcal{L}, V_{\mathbf{A}}, \overline{\mathcal{L}})$. The associated global basis $G^\jmath(\mathcal{B})$ is called a global $\jmath$-crystal basis of V.

Example 14.1.4. Let $\boldsymbol{\lambda} \in P^\jmath_1$ and consider the irreducible $\mathbf{U}^\jmath_1$-module $L(\boldsymbol{\lambda})$. Recall that $L(\boldsymbol{\lambda})$ is $(\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1} + 1)$-dimensional with a basis $G^\jmath(\boldsymbol{\lambda}) := \{f_1^{(n)} v \mid 0 \leq n \leq \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1}\}$, where v denotes a highest weight vector. Also, $L(\boldsymbol{\lambda})$ has a $\jmath$-crystal basis $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$, where $\mathcal{L}(\boldsymbol{\lambda})$ is the $\mathbf{A}_0$-span of $G^\jmath(\boldsymbol{\lambda})$, and $\mathcal{B}(\boldsymbol{\lambda}) = \{f_1^{(n)} v + q\mathcal{L}(\boldsymbol{\lambda}) \mid 0 \leq n \leq \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1}\}$. Set $L(\boldsymbol{\lambda})_{\mathbf{A}}$ to be the $\mathbf{A}$-span of $G^\jmath(\boldsymbol{\lambda})$. Note that there exists a unique $\psi^\jmath$-involution $\overline{\cdot}$ on $L(\boldsymbol{\lambda})$ fixing v. Then, $(\mathcal{L}(\boldsymbol{\lambda}), L(\boldsymbol{\lambda})_{\mathbf{A}}, \overline{\mathcal{L}(\boldsymbol{\lambda})})$ is a balanced triple, and $G^\jmath(\boldsymbol{\lambda})$ is a global $\jmath$-crystal basis of $L(\boldsymbol{\lambda})$.

Proposition 14.1.5. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ with a global $\jmath$-crystal basis $G^\jmath(\mathcal{B}_M)$, and $N \in \mathcal{O}_{\mathrm{int}}$ with a global crystal basis $G(\mathcal{B}_N)$ Then, $(G^\jmath(\mathcal{B}_M) \diamond_\imath G(\mathcal{B}_N))$ is a global $\jmath$-crystal basis of $M \otimes N$.*

Proposition 14.1.6. *Let $M \in \mathcal{O}_{\mathrm{int}}$ be a $\mathbf{U}$-module with a global crystal basis $G(\mathcal{B})$ associated to a crystal basis $(\mathcal{L}, \mathcal{B})$, a ψ-involution ψ_M, and a $\mathbf{U}_{\mathbf{A}}$-submodule $M_{\mathbf{A}}$. Then, $(\mathcal{L}, \mathcal{B})$ is a $\jmath$-crystal basis, $(\mathcal{L}, M_{\mathbf{A}}, \psi^\jmath_M(\mathcal{L}))$ is a balanced triple, and $G^\jmath(\mathcal{B}) := G(\mathcal{B})_\imath$ is the global $\jmath$-crystal basis associated to the balanced triple $(\mathcal{L}, M_{\mathbf{A}}, \psi^\jmath_M(\mathcal{L}))$ and the basis $\mathcal{B}$.*

Proof. That $(\mathcal{L}, \mathcal{B})$ is a $\jmath$-crystal basis has already been stated in Theorem **??**. Let us prove the rest. By Theorem 4.2.5, it is clear that $\mathcal{L}$ (resp., $M_{\mathbf{A}}$) is spanned by $G^\jmath(\mathcal{B})$ over $\mathbf{A}_0$ (resp., $\mathbf{A}$), and that $\psi^\jmath_M(\mathcal{L})$ is spanned by $G^\jmath(\mathcal{B})$ over $\mathbf{A}_\infty$. Hence, the canonical homomorphism $\mathcal{L} \cap M_{\mathbf{A}} \cap \psi^\jmath_M(\mathcal{L}) \to \mathcal{L}/q\mathcal{L}$ is an isomorphism, and therefore, $(\mathcal{L}, M_{\mathbf{A}}, \psi^\jmath_M(\mathcal{L}))$ is balanced. Finally, by Lemma 14.1.2, the $\mathbf{U}_{\mathbf{A}}$-module $M_{\mathbf{A}}$ is also a $\mathbf{U}^\jmath_{\mathbf{A}}$-module. This proves the proposition. ☐

14.2. **Global $\jmath$-crystal bases of irreducible $\mathbf{U}^\jmath$-modules.**

Lemma 14.2.1. *Let M, N be $\mathbf{U}$-modules. Suppose that they have bilinear forms $(\cdot, \cdot)$ such that $(xu, v) = (u, \varrho(x)v)$ for all $x \in \mathbf{U}$ and $u, v \in M$ or $u, v \in N$. Then, the bilinear form $(\cdot, \cdot)$ on $M \otimes N$ defined by $(m \otimes n, m' \otimes n') := (m, m')(n, n')$ also satisfies $(xu, v) = (u, \varrho(x)v)$ for all $x \in \mathbf{U}$ and $u, v \in M \otimes N$.*

Proof. It suffices to show that

$$(\varrho \otimes \varrho) \circ \Delta = \Delta \circ \varrho$$

on $\mathbf{U}$. This is easily achieved by applying the both sides on the generators of $\mathbf{U}$. ☐

Proposition 14.2.2. *Let $(\cdot, \cdot)$ be the bilinear form on $\mathbf{V}^{\otimes d}$ defined by $(u_{\mathbf{i}}, u_{\mathbf{j}}) := \delta_{\mathbf{i},\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathcal{I}_+$. Then, we have $(xu, v) = (u, \varrho(x)v)$ for all $x \in \mathbf{U}$ and $u, v \in \mathbf{V}^{\otimes d}$.*

Proof. By Lemma 14.2.1, it suffices to prove the assertion when $d = 1$, which is easy. ☐

Let $X \in L(W)$. Then, $\mathbf{C}^L_X(\pi^\jmath) \simeq L(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in P^\jmath$. Since $L(\boldsymbol{\lambda})$ is a highest weight module, there exists a unique $\lambda \in \pi^\jmath$ and $w \in {}^\lambda W$ such that $[{}^\lambda C_w]_X \in \mathbf{C}^L_X(\pi^\jmath)$ is a highest weight vector.

Recall the isomorphism $D^L_{Xw_0} \simeq C^L_X$ of left $\mathcal{H}$-modules from Proposition 5.4.9. Set $\mathbf{C}^L_X := \mathbb{Q}(p, q) \otimes_{\mathbf{A}_\mathbb{Z}} C^L_X$, and define $\mathbf{D}^L_{Xw_0}$ and $\mathbf{D}^L_{Xw_0}(\pi^\jmath)$ similarly. Then, we have

$$\mathbf{D}^L_{Xw_0}(\pi^\jmath) \simeq \mathbf{T}(\pi^\jmath) \otimes_{\mathbf{H}} \mathbf{D}^L_{Xw_0} \simeq \mathbf{T}(\pi^\jmath) \otimes_{\mathbf{H}} \mathbf{C}^L_X \simeq \mathbf{C}^L_X(\pi^\jmath)$$

as left $\mathbf{U}^\jmath$-modules. Hence, $[{}^\lambda D_{w_\lambda w w_0}]'_{Xw_0} \in \mathbf{D}^L_{Xw_0}(\pi^\jmath)$ is also a highest weight vector. Thus, we obtain two isomorphisms

$$\varphi_C : L(\boldsymbol{\lambda}) \to \mathbf{C}^L_X(\pi^\jmath);\ v_{\boldsymbol{\lambda}} \mapsto [{}^\lambda C_w]_X,$$
$$\varphi_D : L(\boldsymbol{\lambda}) \to \mathbf{D}^L_{Xw_0}(\pi^\jmath);\ v_{\boldsymbol{\lambda}} \mapsto [{}^\lambda D_{w_\lambda w w_0}]'_{Xw_0}$$

of $\mathbf{U}^\jmath$-modules, where $v_{\boldsymbol{\lambda}} \in L(\boldsymbol{\lambda})$ is a fixed highest weight vector.

Definition 14.2.3. Let $\boldsymbol{\lambda} \in P^\jmath$ and $v_{\boldsymbol{\lambda}} \in L(\boldsymbol{\lambda})$ be a highest weight vector. Define the bilinear form $(\cdot,\cdot)_1$ on $L(\boldsymbol{\lambda})$ by $(v_{\boldsymbol{\lambda}}, v_{\boldsymbol{\lambda}})_1 = 1$ and $(xm, n)_1 = (n, \sigma^\jmath(x)n)_1$ for all $x \in \mathbf{U}^\jmath$, $m, n \in L(\boldsymbol{\lambda})$.

Proposition 14.2.4. *Let $\boldsymbol{\lambda} \in P^\jmath$. Then, the bilinear form $(\cdot,\cdot)_1$ is nondegenerate.*

Proof. For $m, n \in L(\boldsymbol{\lambda})$, set $(m,n) := \langle \varphi_C(m) \mid \varphi_D(n) \rangle_{\pi^\jmath}$. Then, we have

$$(v_{\boldsymbol{\lambda}}, v_{\boldsymbol{\lambda}}) = \langle [{}^\lambda C_w]_X \mid [{}^\lambda D_{w_\lambda w w_0}]'_{Xw_0} \rangle_{\pi^\jmath} = 1,$$

and

$$(xm, n) = \langle x\varphi_C(m) \mid \varphi_D(m) \rangle_{\pi^\jmath} = \langle \varphi_C(m) \mid \sigma^\jmath(x)\varphi_D(n) \rangle_{\pi^\jmath} = (m, \sigma^\jmath(x)n).$$

Hence, we have $(\cdot,\cdot) = (\cdot,\cdot)_1$. Then, it is clear that $\{\varphi_C^{-1}([{}^\mu C_y]_X) \mid \mu \in \pi^\jmath,\ y \in {}^\mu W \cap w_\mu X\}$ and $\{\varphi_D^{-1}([{}^\mu D_{w_\mu y w_0}]'_{Xw_0}) \mid \mu \in \pi^\jmath,\ y \in {}^\mu W \cap Xw_0\}$ form bases which are dual to each other with respect to $(\cdot,\cdot)_1$. This proves the proposition. □

Recall that the set $\{(\mu, y) \mid \mu \in \pi^\jmath,\ y \in {}^\mu W \cap w_\mu X\}$ is identical to $\mathcal{B}(\boldsymbol{\lambda})$. For each $b \in \mathcal{B}(\boldsymbol{\lambda})$, set

$$G^\jmath_{\mathrm{low}}(b) := \varphi_C^{-1}([{}^\mu C_y]_X), \quad G^\jmath_{\mathrm{up}}(b) := \varphi_D^{-1}([{}^\mu D_{w_\mu y w_0}]'_{Xw_0}),$$

where (μ, y) is the pair corresponding to b. Then, $G^\jmath_{\mathrm{low}}(\boldsymbol{\lambda}) := \{G^\jmath_{\mathrm{low}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$ and $G^\jmath_{\mathrm{up}}(\boldsymbol{\lambda}) := \{G^\jmath_{\mathrm{up}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$ are bases of $L(\boldsymbol{\lambda})$.

Definition 14.2.5. Let $\boldsymbol{\lambda} \in P^\jmath(d)$, and $v_{\boldsymbol{\lambda}} \in L(\boldsymbol{\lambda})$ be a highest weight vector. Define a bilinear form $(\cdot,\cdot)_2$ on $L(\boldsymbol{\lambda})$, and a $\psi^\jmath$-involution $\psi^\jmath_{\boldsymbol{\lambda}}$ on $L(\boldsymbol{\lambda})$ by

$$(v_{\boldsymbol{\lambda}}, v_{\boldsymbol{\lambda}})_2 = 1,\ (xm, n)_2 = (m, \varrho(x)n)_2 \quad \text{for all } x \in \mathbf{U}^\jmath,\ m, n \in L(\boldsymbol{\lambda}),$$
$$\psi^\jmath_{\boldsymbol{\lambda}}(v_{\boldsymbol{\lambda}}) = v_{\boldsymbol{\lambda}}.$$

Let $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$ be the unique $\jmath$-crystal basis of $L(\boldsymbol{\lambda})$ such that $v_{\boldsymbol{\lambda}} + q\mathcal{L}(\boldsymbol{\lambda}) \in \mathcal{B}(\boldsymbol{\lambda})$.

Theorem 14.2.6. *Let $\boldsymbol{\lambda} \in P^\jmath(d)$. Then, the following hold.*

1. $\psi^\jmath_{\boldsymbol{\lambda}}(G^\jmath_{\mathrm{low}}(b)) = G^\jmath_{\mathrm{low}}(b)$ *for all* $b \in \mathcal{B}(\boldsymbol{\lambda})$.
2. $\psi^\jmath_{\boldsymbol{\lambda}}(G^\jmath_{\mathrm{up}}(b)) = G^\jmath_{\mathrm{up}}(b)$ *for all* $b \in \mathcal{B}(\boldsymbol{\lambda})$.
3. $G^\jmath_{\mathrm{low}}(\boldsymbol{\lambda})$ *and* $G^\jmath_{\mathrm{up}}(\boldsymbol{\lambda})$ *are dual bases with respect to* $(\cdot,\cdot)_1$.
4. $\mathcal{L}(\boldsymbol{\lambda}) = \{m \in L(\boldsymbol{\lambda}) \mid (m,m)_2 \in \mathbf{A}_0\}$. *Consequently,* $(\cdot,\cdot)_2$ *induces the bilinear form* $(\cdot,\cdot)_0$ *on* $\mathcal{L}(\boldsymbol{\lambda})/q\mathcal{L}(\boldsymbol{\lambda})$ *defined by* $(m+q\mathcal{L}(\boldsymbol{\lambda}), n+q\mathcal{L}(\boldsymbol{\lambda}))_0 := \lim_{q\to 0}(\lim_{p\to 0}(m,n)_2)$.
5. $\{G^\jmath_{\mathrm{low}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$ *forms an almost orthonormal basis with respect to* $(\cdot,\cdot)_2$, *i.e., we have* $(G^\jmath_{\mathrm{low}}(b), G^\jmath_{\mathrm{low}}(b'))_2 \in \delta_{b,b'} + q\mathbf{A}_0$ *for all* $b, b' \in \mathcal{B}(\boldsymbol{\lambda})$.
6. $(b, b')_0 = \delta_{b,b'}$ *for all* $b, b' \in \mathcal{B}(\boldsymbol{\lambda})$.
7. *Let* $L(\boldsymbol{\lambda})_{\mathbf{A}}$ *be the* $\mathbf{A}$*-span of* $G^\jmath_{\mathrm{low}}(\boldsymbol{\lambda})$. *Then,* $(\mathcal{L}(\boldsymbol{\lambda}), L(\boldsymbol{\lambda})_{\mathbf{A}}, \psi^\jmath_{\boldsymbol{\lambda}}(\mathcal{L}(\boldsymbol{\lambda})))$ *is balanced. Moreover, the global basis associated to* $\mathcal{B}(\boldsymbol{\lambda})$ *is* $\{G^\jmath_{\mathrm{low}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$. *In particular,* $L(\boldsymbol{\lambda})$ *has a global* $\jmath$*-crystal basis.*

Proof. Items (1) and (2) are obvious from the definition of $G^\jmath_{\mathrm{low}}(b)$ and $G^\jmath_{\mathrm{up}}(b)$. Item (3) follows from the proof of Proposition 14.2.4.

To prove the rest, observe that $L(\boldsymbol{\lambda})$ is realized as a subquotient of $\mathbf{V}^{\otimes d}$ by using Kazhdan-Lusztig basis elements. To be precise, let $X \in P^\jmath$ be such that $L(\boldsymbol{\lambda}) \simeq \mathbf{C}^L_X(\pi^\jmath)$ and $x \in X$. Then,

$$\mathbf{C}^L_X(\pi^\jmath) = \frac{\mathrm{Span}_{\mathbb{Q}(p,q)}\{{}^\lambda C_w \mid \lambda \in \pi^\jmath,\ w \in {}^\lambda W,\ w_\lambda w \leq_L x\}}{\mathrm{Span}_{\mathbb{Q}(p,q)}\{{}^\mu C_y \mid \mu \in \pi^\jmath,\ y \in {}^\mu W,\ w_\mu y <_L x\}}.$$

Then, items (4)-(6) follows from the definition of the Kazhdan-Lusztig basis, and the bilinear form $(\cdot,\cdot)$ on $\mathbf{V}^{\otimes d}$ constructed in Proposition 14.2.2. To prove item (7), it suffices to show that $L(\boldsymbol{\lambda})_{\mathbf{A}}$ is a $\mathbf{U}^\jmath_{\mathbf{A}}$-module. It follows from the fact that the $\mathbf{A}$-submodule of $\mathbf{V}^{\otimes d}$ spanned by the Kazhdan-Lusztig basis is a $\mathbf{U}_{\mathbf{A}}$-module, and that $\mathbf{U}^\jmath_{\mathbf{A}} \subset \mathbf{U}_{\mathbf{A}}$. □

15. Basic properties of global crystal bases

In this section, we study $\jmath$-crystal bases mainly for $\mathbf{U}$-modules. In Subsection 15.1, we state basis properties of global $\jmath$-crystal bases arising as $\jmath$-canonical bases of $\mathbf{U}$-modules. In Subsection 15.2, we prove some general facts concerning global $\jmath$-crystal bases, and state the main result of this section; its proof is given in the next section. In Subsection 15.3, we define two families of operators on $\mathbf{U}^\jmath$-modules, which give an intrinsic meaning of Kashiwara operators for $\jmath$-crystal bases.

15.1. **$\jmath$-canonical bases.** For partitions $\mu \subset \lambda$, define the skew partition λ/μ in a usual way. For bipartitions $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$, define the skew bipartition $\boldsymbol{\lambda}/\boldsymbol{\mu}$ to be $(\boldsymbol{\lambda}^-/\boldsymbol{\mu}^-; \boldsymbol{\lambda}^+/\boldsymbol{\mu}^+)$. A skew partition λ/μ is said to be a horizontal strip if each column of λ/μ contains at most one box. We say that $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a horizontal strip if $\lambda^\pm/\boldsymbol{\mu}^\pm$ are.

For $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r)}$, we refer the i-th row of $\boldsymbol{\lambda}^-$ to as the $-(i-1)$-th row of $\boldsymbol{\lambda}$, and the j-th row of $\boldsymbol{\lambda}^+$ to as the j-th row of $\boldsymbol{\lambda}$. Also, for i, set $\boldsymbol{\lambda}_i$ to be the length of the i-th row of $\boldsymbol{\lambda}$, i.e.,

$$\boldsymbol{\lambda}_i := \begin{cases} \boldsymbol{\lambda}^-_{-i+1} & \text{if } i \leq 0, \\ \boldsymbol{\lambda}^+_i & \text{if } i > 0. \end{cases}$$

For $i \in \mathbb{I}^\jmath$, set $\boldsymbol{\lambda} \downarrow_i := (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_{-1}, \dots, \boldsymbol{\lambda}_{-i}; \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_i) \in \mathrm{Bip}_{i+1,i}$. For $\mathbf{T} \in \mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$ and $i \in \mathbb{I}^\jmath$, set $\mathbf{T} \downarrow_i$ to be the semistandard tableau obtained from $\mathbf{T}$ by deleting the boxes whose entries are less than $-i$ or greater than i. Note that we have

$$\mathbf{T}_{\boldsymbol{\lambda}} \downarrow_i = \mathbf{T}_{\boldsymbol{\lambda}\downarrow_i}.$$

For $\mathbf{T} \in \mathrm{SST}_{(r+1,r)}(\boldsymbol{\lambda})$ and $i \in [-r, r]$, set $\mathbf{T}(i)$ to be the number of boxes of $\mathbf{T}$ whose entries are i.

Let $M \in \mathcal{O}_{\mathrm{int}}$ be a $\mathbf{U}$-module with a crystal basis $(\mathcal{L}, \mathcal{B})$. Then, we have a unique irreducible decomposition $\mathcal{B} = \bigsqcup_{i=1}^l \mathcal{B}_i$, where $\mathcal{B}_i \simeq \mathcal{B}(\lambda_i)$ for some $\lambda_i \in \mathrm{Par}_{2r+1}$. By retaking λ_i's if necessary, we may assume that $|\lambda_i| - |\lambda_j| < 2r+1$ for all i, j, and that there exists i such that $(\lambda_i)_{2r+1} = 0$. Then, λ's are uniquely determined; we set $P(M) = P_r(M) := \{\lambda_1, \dots, \lambda_l\}$. For $b \in \mathcal{B}$, we define $I(b) = I_r(b) \in P(M)$ to be λ_i if $b \in \mathcal{B}_i$. Also, set $C(b) = C_r(b) \subset \mathcal{B}$ denote the connected component of $\mathcal{B}$ containing b. Furthermore, if we write $b = \widetilde{F}_{i_1} \cdots \widetilde{F}_{i_l} b_0$ for some $i_1, \dots, i_l \in \mathbb{I}$, where b_0 denotes the highest weight vector in $C(b)$, then define $T_b \in \mathrm{SST}_{(r+1,r)}(I(b))$ by $T_b := \widetilde{F}_{i_1} \cdots \widetilde{F}_{i_l} T_0$, where $T_0 \in \mathrm{SST}_{(r+1,r)}(I(b))$ corresponding to $b_0 \in C(b) \in \mathcal{B}(I(b))$.

For $M \in \mathcal{O}^{\jmath}_{\mathrm{int}}$ with a $\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$ and $b \in \mathcal{B}$, we define $P^{\jmath}(M)$, $I^{\jmath}(b)$, $C^{\jmath}(b)$, and $T^{\jmath}_b$ in a similar way.

Let $M \in \mathcal{O}_{\mathrm{int}}$ be a based $\mathbf{U}$-module with a crystal basis $(\mathcal{L}, \mathcal{B})$, a global crystal basis $G(\mathcal{B})$, a ψ-involution ψ_M, and a balanced triple $(\mathcal{L}, M_{\mathbf{A}}, \psi_M(\mathcal{L}))$. Set $\psi^{\jmath}_M := \Upsilon \circ \psi_M$. We denote by $G^{\jmath}(\mathcal{B})$ the associated $\jmath$-canonical basis. Recall that $\psi^{\jmath}_M$ is a $\psi^{\jmath}$-involution on M, and $(\mathcal{L}, M_{\mathbf{A}}, \psi^{\jmath}_M(\mathcal{L}))$ is a balanced triple with the associated global basis $G^{\jmath}(\mathcal{B})$.

Lemma 15.1.1. *Let $b \in \mathcal{B}$. Let us write as*

$$G^{\jmath}(b) = G(b) + \sum_{\substack{b' \in \mathcal{B} \\ \mathrm{wt}^{\jmath}(b') = \mathrm{wt}^{\jmath}(b) \text{ and } \mathrm{wt}(b') < \mathrm{wt}(b)}} c_{b',b} G(b')$$

for some $c_{b',b} \in q\mathbf{A}_0 \cap \mathbf{A}$. Then, we have $c_{b',b} = 0$ unless

$$I^{\jmath}(b) \trianglelefteq I^{\jmath}(b') \text{ or } |I^{\jmath}(b')^-| < |I^{\jmath}(b)^-|. \tag{15}$$

Proof. By the construction of $G^{\jmath}(b)$, it suffices to show that $\psi^{\jmath}_M(G(b))$ is a linear combination of $G(b')$ with b' satisfying (15). Since $\psi^{\jmath}_M(G(b)) = \Upsilon G(b) \in \mathbf{U}^- G(b)$, it suffices to show that for each $l \in \mathbb{Z}_{\geq 0}$ and $i_1, \dots, i_l \in \mathbb{I}$, we have

$$F_{i_l} \cdots F_{i_1} G(b) \in \mathrm{Span}_{\mathbb{Q}(p,q)}\{G(b') \mid b' \text{ satisfies condition (15)}\}.$$

We prove it by induction on l. When $l = 0$, there are nothing to prove. So, assume that $l > 0$ and that $F_{i_{l-1}} \cdots F_{i_1} G(b) \in \mathrm{Span}_{\mathbb{Q}(p,q)}\{G(b') \mid b' \text{ satisfies condition (15)}\}$ for all $i_1, \dots, i_{l-1} \in \mathbb{I}$. If $i_l \neq \underline{1}$, then, by Proposition 3.5.6, we have

$$F_{i_l} G(b') \in \mathrm{Span}_{\mathbb{Q}(p,q)}\{G(b'') \mid I^{\jmath}(b') \trianglelefteq I^{\jmath}(b'')\}$$

for all b' satisfying condition (15). Since $|I^{\jmath}(b'')^-| = |I^{\jmath}(b')^-|$ for all b'' with $I^{\jmath}(b') \trianglelefteq I^{\jmath}(b'')$, b'' satisfies condition (15).

If $i_l = \underline{1}$, then $\mathrm{wt}(F_{i_l} G(b')) = \mathrm{wt}(G(b')) - \alpha_{\underline{1}}$. This immediately implies that $F_{i_l} G(b') \in \mathrm{Span}_{\mathbb{Q}(p,q)}\{G(b'') \mid |I^{\jmath}(b'')^-| < |I^{\jmath}(b')^-|\}$. Therefore, $F_{i_l} \cdots F_{i_1} G(b)$ is a linear combination of $G(b')$ with $|I^{\jmath}(b')^-| < |I^{\jmath}(b)^-|$. Thus, the proof completes. $\square$

Proposition 15.1.2. *Let $b \in \mathcal{B}$ and $i \in \mathbb{I}^{\jmath} \setminus \{1\}$. Then, we have*

$$e_i G^{\jmath}(b) = [\varphi_{\underline{i}}(b) + 1] G^{\jmath}(\widetilde{E}_{\underline{i}} b) + \sum_{\substack{b' \in \mathcal{B} \setminus \{\widetilde{E}_{\underline{i}} b\} \\ \mathrm{wt}^{\jmath}(b') = \mathrm{wt}^{\jmath}(b) + \gamma_i \text{ and } \mathrm{wt}(b') \leq \mathrm{wt}(b) + \alpha_{\underline{i}}}} e^{(i)}_{b',b} G^{\jmath}(b'),$$

$$f_i G^{\jmath}(b) = [\varphi_{-\underline{i}}(b) + 1] G^{\jmath}(\widetilde{E}_{-\underline{i}} b) + \sum_{\substack{b' \in \mathcal{B} \setminus \{\widetilde{E}_{-\underline{i}} b\} \\ \mathrm{wt}^{\jmath}(b') = \mathrm{wt}^{\jmath}(b) - \gamma_i \text{ and } \mathrm{wt}(b') \leq \mathrm{wt}(b) + \alpha_{-\underline{i}}}} f^{(i)}_{b',b} G^{\jmath}(b')$$

for some $e^{(i)}_{b',b}, f^{(i)}_{b',b} \in \mathbf{A}$. Moreover, $e^{(i)}_{b',b} = f^{(i)}_{b',b} = 0$ unless $I^{\jmath}(b) \trianglelefteq I^{\jmath}(b')$ or $|I^{\jmath}(b')^-| < |I^{\jmath}(b)^-|$.

Proof. We prove the assertion only for e_i; the proof for f_i is similar. By Lemma 15.1.1, we can write

$$G^{\jmath}(b) = G(b) + \sum_{b' \in \mathcal{B} \setminus \{b\}} c_{b',b} G(b')$$

for some $c_{b',b} \in \mathbf{A}$ such that $c_{b',b} = 0$ unless $I^{\jmath}(b) \trianglelefteq I^{\jmath}(b')$ or $|I^{\jmath}(b')^-| < |I^{\jmath}(b)^-|$. Since $e_i \in U_q(\mathfrak{l})$, it holds that

$$e_i G^{\jmath}(b) \in \mathrm{Span}_{\mathbf{A}}\{G(b'') \mid I^{\jmath}(b) \trianglelefteq I^{\jmath}(b') \text{ or } |I^{\jmath}(b')^-| < |I^{\jmath}(b)^-|\}.$$

Hence, it suffices to show that $[e_i G^\jmath(b) : G^\jmath(\widetilde{E}_{\underline{i}} b)] = [\varphi_{\underline{i}}(b) + 1]$. By the definitions of e_i and $G^\jmath(b)$, $e_i G^\jmath(b)$ is the sum of $E_{\underline{i}} G(b)$ and a linear combination of weight vectors of M of weight lower than $\mathrm{wt}(b) + \alpha_{\underline{i}}$. We know from Proposition 3.5.5 (4) that $[E_{\underline{i}} G^\jmath(b) : G(\widetilde{E}_{\underline{i}} b)] = [\varphi_{\underline{i}}(b) + 1]$. Hence, we have $[e_i G^\jmath(b) : G^\jmath(\widetilde{E}_{\underline{i}} b)] = [\varphi_{\underline{i}}(b) + 1]$. This proves the assertion. □

15.2. **Global $\jmath$-crystal bases.** Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$, $(\mathcal{L}, \mathcal{B})$ a $\jmath$-crystal basis of M, $\psi^\jmath_M$ a $\psi^\jmath$-involution, and $M_{\mathbf{A}}$ a $\mathbf{U}^\jmath_{\mathbf{A}}$-submodule of M. Suppose that M has a $\jmath$-global basis $G^\jmath(\mathcal{B})$ with the associated balanced triple $(\mathcal{L}, M_{\mathbf{A}}, \psi^\jmath_M(\mathcal{L}))$.

Following [K02], let us introduce modified Kashiwara operators:

Definition 15.2.1. *For $n \in \mathbb{Z}$, set*

$$\widetilde{f}_i^{(n)} := \sum_{t \geq 0, -n} f_i^{(n+t)} e_i^{(t)} A_n(t; k_i),$$

$$\widetilde{f}_1^{(n)} := \sum_{t \geq 0, -n} f_1^{(n+t)} e_1^{(t)} a_n(t; k_1),$$

where

$$A_n(t; x) := (-1)^t q^{t(1-n)} x^t \prod_{s=0}^{t-1} (1 - q^{n+2s}),$$

$$a_n(t; x) := (-1)^t p^t q^{t(1-n)} x^t \prod_{s=0}^{t-1} q^s (1 - q^{n+2s}).$$

Lemma 15.2.2. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ with the $\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. For $n \in \mathbb{Z}$, we have $\widetilde{f}_i^{(n)} \mathcal{L} \subset \mathcal{L}$, and $\widetilde{f}_i^{(n)} \mathcal{L} = \widetilde{f}_i^n \mathcal{L}$ modulo $q\mathcal{L}$.*

Proof. If $i \neq 1$, then the statement follows from [K02, Proposition 6.1]. Hence, we prove the case when $i = 1$. It suffices to prove the following: For each $u \in \mathcal{L}$ such that $e_1 u = 0$, $k_1 u = q^a u$, $e_1 f_1 u = [b]\{a - b - 1\} u$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, we have $\widetilde{f}_1^{(n)} f_1^{(m)} u = c f_1^{(m+n)}$ for some $c \in 1 + q\mathbf{A}_0 \cap \mathbf{A}$. First of all, we have

$$\widetilde{f}_1^{(n)} f_1^{(m)} u = \sum_{t \geq 0, -n} a_n(t; q^{a-3m}) \begin{bmatrix} m+n \\ m-t \end{bmatrix} \begin{bmatrix} b-m+t \\ t \end{bmatrix} \prod_{s=0}^{t-1} \{a - b - m + s\} f_1^{(m+n)} u.$$

We compute the coefficient, say A, of the right-hand side as follows.

$$\begin{aligned} A &= \sum_{t \geq 0, -n} A_n(t; q^{b-2m}) \begin{bmatrix} m+n \\ m-t \end{bmatrix} \begin{bmatrix} b-m+t \\ t \end{bmatrix} \prod_{s=0}^{t-1} (1 + p^2 q^{2(a-b-m+s)}) \\ &= \sum_{t \geq 0, -n} B_t + p^2 \sum_{t \geq 0, -n} B_t g_t, \end{aligned}$$

where $B_t := A_n(t; q^{b-2m}) \begin{bmatrix} m+n \\ m-t \end{bmatrix} \begin{bmatrix} b-m+t \\ t \end{bmatrix}$, and $g_t \in \mathbb{Z}[p, q, q^{-1}]$ with $\prod_{s=0}^{t-1} (1 + p^2 q^{2(a-b-m+s)}) = 1 + p^2 g_t$. By the proof of [K02, Proposition 6.1], we have $B_t \in 1 + q\mathbb{Z}[q]$. Also, it is clear that $p^2 \sum_{t \geq 0, -n} B_t g_t \in p^2 \mathbb{Z}[p, q, q^{-1}]$. Thus, we have $\widetilde{f}_1^{(n)} f_1^{(m)} u \in \mathcal{L}$ and $\widetilde{f}_1^{(n)} f_1^{(m)} u = f_1^{(m+n)} u = \widetilde{f}_1^n f_1^{(m)} u$ modulo $q\mathcal{L}$. This proves the lemma. □

Proposition 15.2.3. *Let $i \in \mathbb{I}^\jmath$, $b \in \mathcal{B}$ and $m \in \mathbb{Z}_{\geq 0}$. Then, we have the following.*

(1) $\sum_{n \geq m} f_i^{(n)} M_{\mathbf{A}} = \bigoplus_{\substack{b' \in \mathcal{B} \\ \varepsilon_i(b') \geq m}} \mathbf{A} G^\jmath(b')$.

(2) $\sum_{n\geq m} e_i^{(n)} M_{\mathbf{A}} = \bigoplus_{\substack{b'\in\mathcal{B} \\ \varphi_i(b')\geq m}} \mathbf{A}G^{\jmath}(b')$ *if* $i \neq 1$.

(3) $f_i G^{\jmath}(b) = [\varepsilon_i(b)+1]G^{\jmath}(\widetilde{f}_i b) + \sum_{\substack{b'\in\mathcal{B} \\ \varepsilon_i(b')>\varepsilon_i(b)+1}} \varphi^{(i)}_{b',b} G^{\jmath}(b')$ *for some* $\varphi^{(i)}_{b',b} \in q^{2-\varepsilon_i(b')}\mathbb{Q}[q]$.

(4) $e_i G^{\jmath}(b) = [\varphi_i(b)+1]G^{\jmath}(\widetilde{e}_i b) + \sum_{\substack{b'\in\mathcal{B} \\ \varphi_i(b')>\varphi_i(b)+1}} \varepsilon^{(i)}_{b',b} G^{\jmath}(b')$ *for some* $\varepsilon^{(i)}_{b',b} \in q^{2-\varphi_i(b')}\mathbb{Q}[q]$ *if* $i \neq 1$.

Proof. Since (e_i, k_i, f_i), $i \neq 1$ forms an $\mathfrak{sl}_2$-triple, most of the assertions follows from Proposition 3.5.5. What we have to prove are assertions (1) and (3) for $i = 1$. First, we prove part (1) by induction on m. When $m = 0$, the both sides of the equation to be proved are 0. Assume that assertion (1) holds for all $m' > m$. Let $b' \in \mathcal{B}$ be such that $\varepsilon_1(b') = m$. Set $b'_0 := \widetilde{e}_1^m b$, and consider $u := \widetilde{f}_1^{(m)} G^{\jmath}(b'_0)$. By the definition of $\widetilde{f}_1^{(m)}$ and Lemma 15.2.2, we have

$$u - f_1^{(m)} G^{\jmath}(b'_0) \in \sum_{n>m} f_1^{(n)} M_{\mathbf{A}} \text{ and } u + q\mathcal{L} = b'.$$

By our inductive hypothesis, we can write

$$u - f_1^{(m)} G^{\jmath}(b'_0) = \sum_{\substack{b''\in\mathcal{B} \\ \varepsilon_1(b'')>m}} a_{b''} G^{\jmath}(b'')$$

for some $a_{b''} \in \mathbf{A}$. Then, we can take $a'_{b''} \in q\mathbb{Q}[q]$ in a way such that $a_{b''} - \overline{a_{b''}} = a'_{b''} - \overline{a'_{b'''}}$. Set $v := u - \sum_{b''} a'_{b''} G^{\jmath}(b'') = f_1^{(m)} G^{\jmath}(b'_0) + \sum_{b''}(a_{b'''} - a'_{b''})G^{\jmath}(b'')$. Then, we have $v \in M_{\mathbf{A}} \cap \mathcal{L}$, $\psi^{\jmath}_M(v) = v$, and $v + q\mathcal{L} = u + q\mathcal{L} = b'$. These implies that $v = G^{\jmath}(b')$, and therefore, $G^{\jmath}(b') \in \sum_{n\geq m} f_1^{(n)} M_{\mathbf{A}}$. Hence, we obtain $\sum_{n\geq m} f_i^{(n)} M_{\mathbf{A}} \supset \bigoplus_{\substack{b'\in\mathcal{B} \\ \varepsilon_i(b')\geq m}} \mathbf{A}G^{\jmath}(b')$.

We prove the opposite inclusion. For each $\lambda \in \Lambda^{\jmath}$, we have

$$\begin{aligned}
(M_{\mathbf{A}})_\lambda &\subset \sum_{b\in\mathcal{B}_\lambda} \mathbf{A}G^{\jmath}(b) \\
&= \sum_{\substack{b\in\mathcal{B}_\lambda \\ \varepsilon_1(b)=0}} \mathbf{A}G^{\jmath}(b) + \sum_{\substack{b'\in\mathcal{B}_\lambda \\ \varepsilon_1(b')\geq 1}} \mathbf{A}G^{\jmath}(b') \\
&\subset \sum_{\substack{b\in\mathcal{B}_\lambda \\ \varepsilon_1(b)=0}} \mathbf{A}G^{\jmath}(b) + \sum_{n\geq 1} f_1^{(n)} (M_{\mathbf{A}})_{\lambda+n\gamma_1}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
f_1^{(m)}(M_{\mathbf{A}})_\lambda &\subset \sum_{\substack{b\in\mathcal{B}_\lambda \\ \varepsilon_1(b)=0}} \mathbf{A}f_1^{(m)} G^{\jmath}(b) + \sum_{n\geq 1} f_1^{(m)} f_1^{(n)} (M_{\mathbf{A}})_{\lambda+n\gamma_1} \\
&\subset \sum_{\substack{b\in\mathcal{B}_\lambda \\ \varepsilon_1(b)=0}} \mathbf{A}f_1^{(m)} G^{\jmath}(b) + \sum_{n\geq 1} f_1^{(m+n)} (M_{\mathbf{A}})_{\lambda+n\gamma_1} \\
&= \sum_{\substack{b\in\mathcal{B}_\lambda \\ \varepsilon_1(b)=0}} \mathbf{A}f_1^{(m)} G^{\jmath}(b) + \sum_{\substack{b'\in\mathcal{B}_\lambda \\ \varepsilon_1(b')>m}} \mathbf{A}G^{\jmath}(b') \quad \text{(by induction hypothesis).}
\end{aligned}$$

Also, by the argument above, $f_1^{(m)} G^{\jmath}(b)$ with $\varepsilon_1(b) = 0$ is contained in $\sum_{\varepsilon(b')\geq m} \mathbf{A}G^{\jmath}(b')$. This completes the proof of part (1).

Next, we turn to prove assertion (3) for $i = 1$ by descending induction on $m := \varepsilon_1(b)$. When m is maximum among $\{\varepsilon_1(b') \mid b' \in \mathcal{B}\}$, we have by (1) that

$$f_1 G^\jmath(b) \in \sum_{n>m} f_1^{(n)} M_{\mathbf{A}} = \sum_{\varepsilon_1(b')>m} \mathbf{A} G^\jmath(b') = 0,$$

and the equation in (3) holds. Assume that (3) is true for all $m' > m$. As in the proof of (1), let us write

$$\begin{aligned} G^\jmath(b) &= f_1^{(m)} G^\jmath(\widetilde{e}_1^m b) + \sum_{\substack{b'\in\mathcal{B} \\ \varepsilon_1(b')>m}} c_{b'} G^\jmath(b'), \\ G^\jmath(\widetilde{f}_1 b) &= f_1^{(m+1)} G^\jmath(\widetilde{e}_1^m b) + \sum_{\substack{b''\in\mathcal{B} \\ \varepsilon_1(b'')>m+1}} d_{b''} G^\jmath(b'') \end{aligned}$$

for some $c_{b'}, d_{b''} \in \mathbf{A}$. Then, we have

$$\begin{aligned} f_1 G^\jmath(b) &= [m+1] f_1^{(m+1)} G^\jmath(\widetilde{e}_1^m b) + \sum_{\varepsilon_1(b')>m} c_{b'} f_1 G^\jmath(b') \\ &= [m+1] f_1^{(m+1)} G^\jmath(\widetilde{e}_1^m b) + \sum_{\varepsilon_1(b')>m} c_{b'} \Big([\varepsilon_1(b')+1] G^\jmath(\widetilde{f}_1 b') + \sum_{\varepsilon_1(b'')>\varepsilon_1(b')+1} \varphi^{(1)}_{b'',b'} G^\jmath(b'') \Big) \\ &= [m+1] G^\jmath(\widetilde{f}_1 b) + \sum_{\varepsilon_1(b')>m} c_{b'} \Big([\varepsilon_1(b')+1] G^\jmath(\widetilde{f}_1 b') + \sum_{\varepsilon_1(b'')>\varepsilon_1(b')+1} \varphi^{(1)}_{b'',b'} G^\jmath(b'') \Big) \\ &\qquad - \sum_{\substack{b''\in\mathcal{B} \\ \varepsilon_1(b'')>m+1}} [m+1] d_{b''} G^\jmath(b''). \end{aligned}$$

Thus, we obtain that $f_i G^\jmath(b) = [\varepsilon_i(b)+1] G^\jmath(\widetilde{f}_i b) + \sum_{\substack{b'\in\mathcal{B} \\ \varepsilon_i(b')>\varepsilon_i(b)+1}} \varphi^{(i)}_{b',b} G^\jmath(b')$ for some $\varphi^{(i)}_{b',b} \in \mathbf{A}$. It remains to prove that $\varphi^{(i)}_{b',b} \in q^{2-\varepsilon_1(b')}\mathbb{Q}[q]$. Let us write

$$G^\jmath(b) = \sum_{k \geq m} f_1^{(k)} u_k$$

for some $u_k \in \mathcal{L}_{\mathrm{wt}^\jmath(b)+k\gamma_1}$ such that $e_1 u_k = 0$. Note that $G^\jmath(b) + q\mathcal{L} = u_m + q\mathcal{L}$. Then, we have

$$f_1 G^\jmath(b) = [m+1] f_1^{(m+1)} u_m + \sum_{k>m} [k+1] f_1^{(k+1)} u_k,$$

and that $f_1^{(m+1)} u_m \in \mathcal{L}$, $f_1^{(m+1)} u_m + q\mathcal{L} = \widetilde{f}_1 b$. Hence, we have $f_1 G^\jmath(b) = [m+1] G^\jmath(\widetilde{f}_1 b) + \sum_{k>m} [k+1] f_1^{(k+1)} u_k$ modulo $q^{2-m}\mathcal{L}$. Then, rewriting $f_1^{(k+1)} u_k$ as a sum of $G^\jmath(b')$, $\varepsilon_1(b') \leq k+1$ with coefficients in $q\mathbf{A}_0$, we conclude that the coefficient of $G^\jmath(b')$ in $f_1 G^\jmath(b)$ lies in $q^{2-\varepsilon_1(b')}\mathbf{A}_0 \cap \mathbf{A} = q^{2-\varepsilon_1(b')}\mathbb{Q}[q]$. This completes the proof. □

For a bipartition $\boldsymbol{\lambda} \in P^\jmath(M)$, define $I_{\boldsymbol{\lambda}}(M)$, $W_{\succeq\boldsymbol{\lambda}}(M)$, $W_{\succ\boldsymbol{\lambda}}(M)$, and $W_{\boldsymbol{\lambda}}(M)$ in a similar way as I_λ, $W_{\succeq\lambda}$, $W_{\succ\lambda}$, and W_λ, respectively.

Definition 15.2.4. We say that M has the property $(*)$ if there exists a poset $(S, \leq)$ and a map $s : \mathcal{B} \to S$ satisfying the following:

1. The abelian group $Q := \sum_{i\in\mathbb{I}} \mathbb{Z}\alpha_i$ acts on S freely; the action is written additively.
2. $\sigma \leq \sigma + \lambda$ for all $\lambda \in Q_+$, $\sigma \in S$.
3. $\sigma + \lambda \leq \sigma' + \lambda$ for all $\lambda \in Q$, $\sigma \leq \sigma' \in S$.

(4) $s(b) = s(b')$ only if $\mathrm{wt}(b) = \mathrm{wt}(b')$ for all $b, b' \in \mathcal{B}$.
(5) For $b \in \mathcal{B}$ and $i \in \mathbb{I}^\jmath \setminus \{1\}$, $s(\widetilde{E}_{\underline{i}} b) = s(b) + \alpha_{\underline{i}}$ if $\widetilde{E}_{\underline{i}} b \neq 0$.
(6) For $i \in \mathbb{I}^\jmath \setminus \{1\}$,

$$e_i G^\jmath(b) = [\varphi_{\underline{i}}(b) + 1] G^\jmath(\widetilde{E}_{\underline{i}} b) + \sum_{\substack{b' \in \mathcal{B} \setminus \{\widetilde{E}_{\underline{i}} b\} \\ \mathrm{wt}^\jmath(b') = \mathrm{wt}^\jmath(b) + \gamma_i \text{ and } s(b') \leq s(b) + \alpha_{\underline{i}}}} e^{(i)}_{b',b} G^\jmath(b'),$$

$$f_i G^\jmath(b) = [\varphi_{-\underline{i}}(b) + 1] G^\jmath(\widetilde{E}_{-\underline{i}} b) + \sum_{\substack{b' \in \mathcal{B} \setminus \{\widetilde{E}_{-\underline{i}} b\} \\ \mathrm{wt}^\jmath(b') = \mathrm{wt}^\jmath(b) - \gamma_i \text{ and } s(b') \leq s(b) + \alpha_{-\underline{i}}}} f^{(i)}_{b',b} G^\jmath(b')$$

for some $e^{(i)}_{b',b}, f^{(i)}_{b',b} \in \mathbf{A}$.

Lemma 15.2.5. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$, and $\mathcal{L}, \mathcal{B}, \psi^\jmath_M, M_{\mathbf{A}}$ as above.*
(1) *If $r = 1$, then M has the property $(*)$.*
(2) *If $M \in \mathcal{O}_{\mathrm{int}}$ and the global $\jmath$-crystal basis is the $\jmath$-canonical basis, then M has the property $(*)$.*

Proof. Setting S and s to be Λ and wt, respectively, part (1) is obvious, and part (2) follows from Proposition 15.1.2. □

The main result in this paper is the following:

Theorem 15.2.6. *Suppose that M has the property $(*)$. Then, for each $\boldsymbol{\lambda} \in P^\jmath(M)$, the following hold:*
(1) *$W_{\succeq \boldsymbol{\lambda}}(M)$ has a global $\jmath$-crystal basis $W_{\succeq \boldsymbol{\lambda}}(G^\jmath(\mathcal{B})) := \{G^\jmath(b) \mid I(b) \succeq \boldsymbol{\lambda}\}$ with the associated balanced triple $(W_{\succeq \boldsymbol{\lambda}}(\mathcal{L}), W_{\succeq \boldsymbol{\lambda}}(M_{\mathbf{A}}), W_{\succeq \boldsymbol{\lambda}}(\psi^\jmath_M(\mathcal{L})))$, where $W_{\succeq \boldsymbol{\lambda}}(\mathcal{L}) := W_{\succeq \boldsymbol{\lambda}}(M) \cap \mathcal{L}$, and so on.*
(2) *$W_{\succ \boldsymbol{\lambda}}(M)$ has a global $\jmath$-crystal basis $W_{\succ \boldsymbol{\lambda}}(G^\jmath(\mathcal{B})) := \{G^\jmath(b) \mid I(b) \succ \boldsymbol{\lambda}\}$ with the associated balanced triple $(W_{\succ \boldsymbol{\lambda}}(\mathcal{L}), W_{\succ \boldsymbol{\lambda}}(M_{\mathbf{A}}), W_{\succ \boldsymbol{\lambda}}(\psi^\jmath_M(\mathcal{L})))$, where $W_{\succ \boldsymbol{\lambda}}(\mathcal{L}) := W_{\succ \boldsymbol{\lambda}}(M) \cap \mathcal{L}$, and so on.*
(3) *$W_{\boldsymbol{\lambda}}(M)$ has a global $\jmath$-crystal basis $W_{\boldsymbol{\lambda}}(G^\jmath(\mathcal{B})) := \{G^\jmath(b) + W_{\succ \boldsymbol{\lambda}}(M) \mid I(b) = \boldsymbol{\lambda}\}$ with the associated balanced triple $(W_{\boldsymbol{\lambda}}(\mathcal{L}), W_{\boldsymbol{\lambda}}(M_{\mathbf{A}}), W_{\boldsymbol{\lambda}}(\psi^\jmath_M(\mathcal{L})))$, where $W_{\boldsymbol{\lambda}}(\mathcal{L}) := W_{\succeq \boldsymbol{\lambda}}(\mathcal{L}) / W_{\succ \boldsymbol{\lambda}}(\mathcal{L})$, and so on.*
(4) *There exists a $\mathbf{U}^\jmath$-module isomorphism $\xi : L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}} \to W_{\boldsymbol{\lambda}}(M)$ which induces an isomorphism*

$$(\mathcal{L}(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}, (L(\boldsymbol{\lambda})_{\mathbf{A}})^{\oplus m_{\boldsymbol{\lambda}}}, \psi^\jmath_{\boldsymbol{\lambda}}(\mathcal{L}(\boldsymbol{\lambda}))^{\oplus m_{\boldsymbol{\lambda}}}) \simeq (W_{\boldsymbol{\lambda}}(\mathcal{L}), W_{\boldsymbol{\lambda}}(M_{\mathbf{A}}), W_{\boldsymbol{\lambda}}(\psi^\jmath_M(\mathcal{L}))),$$

where $m_{\boldsymbol{\lambda}} := \dim \mathrm{Hom}_{\mathbf{U}^\jmath}(L(\boldsymbol{\lambda}), M)$ denotes the multiplicity of $L(\boldsymbol{\lambda})$ in M.

The proof will be given in Section 16.

Corollary 15.2.7. *Let $\boldsymbol{\lambda} \in P^\jmath$. Then, $G^\jmath_{\mathrm{low}}(\boldsymbol{\lambda})$ is a unique global $\jmath$-crystal basis of $L(\boldsymbol{\lambda})$ satisfying the property $(*)$.*

15.3. **Operators $\widetilde{e}_{i^+}$ and $\widetilde{f}_{i^+}$.** The definitions of $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$ are artificial, namely, they are defined by means of a distinguished basis $G^\jmath_{\mathrm{low}}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in P^\jmath$ (in Subsection 13.3, it is denoted by $\{b_T \mid T \in \mathcal{B}(\boldsymbol{\lambda})\}$). Here, we define new operators $\widetilde{e}_{i^+}$ and $\widetilde{f}_{i^+}$ for $i \in \mathbb{I}^\jmath \setminus \{1\}$, and then, explain that the operators $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$ on $\jmath$-crystal bases are in fact intrinsic.

Lemma 15.3.1. *Let $r \geq 2$, $\boldsymbol{\lambda} \in P^\jmath$, and consider the irreducible highest weight module $L(\boldsymbol{\lambda})$. As a $\mathbf{U}^\jmath_{r-1}$-module, $L(\boldsymbol{\lambda})$ is multiplicity-free.*

Proof. Let $b \in \mathcal{B}(\boldsymbol{\lambda})$ be a $\mathbf{U}^{\jmath}_{r-1}$-highest weight vector with highest weight, say, $\boldsymbol{\mu} \in P^{\jmath}_{r-1}$. If we identify $\mathcal{B}(\boldsymbol{\lambda})$ with $\mathrm{SST}(\boldsymbol{\lambda})$, we have $T^{\jmath}_b \downarrow_{r-1} = T_{\boldsymbol{\mu}}$. Since the entries of the boxes of $T^{\jmath}_{b'}$ corresponding to $\boldsymbol{\lambda}/\boldsymbol{\mu}$ are either $-r$ or r, it must hold that $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a horizontal strip. Conversely, given $\boldsymbol{\mu} \in P^{\jmath}_{r-1}$ such that $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a horizontal strip, there exists a unique $b \in \mathcal{B}(\boldsymbol{\lambda})$ which is a $\mathbf{U}^{\jmath}_{r-1}$-highest weight vector with highest weight $\boldsymbol{\mu}$. This proves the lemma. □

Lemma 15.3.2. *Let $r \geq 2$, $\boldsymbol{\lambda} \in P^{\jmath}$. Let $b \in \mathcal{B}(\boldsymbol{\lambda})$ be such that $\widetilde{e}_{r'} b \neq 0$. Then, there exist unique $b' \in \mathcal{B}(\boldsymbol{\lambda})$ and $j \in \mathbb{I}^{\jmath} \setminus \{1\}$ satisfying the following:*

- *b' is a $\mathbf{U}^{\jmath}_{r-1}$-highest weight vector.*
- *There exist unique $\varepsilon_i \in \{\emptyset, \prime\}$ for each $j \leq i \leq r-1$ such that $b = \widetilde{f}_{r'} \widetilde{f}_{(r-1)^{\varepsilon_{r-1}}} \cdots \widetilde{f}_{j^{\varepsilon_j}} b'$.*

Proof. By the definition of $\widetilde{e}_{r'}$, b is a $\mathbf{U}^{\jmath}_{r-1}$-highest weight vector with highest weight, say, $\boldsymbol{\mu} \in P^{\jmath}_{r-1}$ such that $(T^{\jmath}_b)^- = T^-_{\boldsymbol{\lambda}}$. Then, $T^{\jmath}_{\widetilde{e}_{r'} b} \downarrow_{r-1}$ is obtained from $T_{\boldsymbol{\mu}}$ by adding a box $\boxed{r-1}$ to the $(j-1)$-th row for some uniquely determined $j \in \mathbb{I}^{\jmath} \setminus \{1\}$. Set $b_{r-1} := \widetilde{e}_{r'} b$. Now, we have exactly one of the following; $\widetilde{e}_{r-1} b_{r-1} \neq 0$ or $\widetilde{e}_{(r-1)'} b_{r-1} \neq 0$. Choose a unique $\varepsilon_{r-1} \in \{\emptyset, \prime\}$ in a way such that $b_{r-2} := \widetilde{e}_{(r-1)^{\varepsilon_{r-1}}} b_{r-1} \neq 0$. Then, $T^{\jmath}_{b_{r-2}} \downarrow_{r-1}$ is obtained from $T_{\boldsymbol{\mu}}$ by adding a box $\boxed{r-2}$ to the $(j-1)$-th row. Repeating this procedure, we obtain $\varepsilon_i \in \{\emptyset, \prime\}$ and $b_{i-1} \in \mathcal{B}(\boldsymbol{\lambda})$ for $j \leq i \leq r-1$. By the construction, $T^{\jmath}_{b_{j-1}} \downarrow_{r-1}$ is obtained from $T_{\boldsymbol{\mu}}$ by adding a box $\boxed{j-1}$ to the $(j-1)$-th row, which turned out to be $T_{\boldsymbol{\mu}'}$, where $\boldsymbol{\mu}' \in P^{\jmath}_{r-1}$ such that $\boldsymbol{\mu}'_k = \boldsymbol{\mu}_k + \delta_{k,j-1}$, $k \in \{-(r-1), \dots, r-1\}$. Hence, b_{j-1} is a $\mathbf{U}^{\jmath}_{r-1}$-highest weight vector, and we have $b = \widetilde{f}_{r'} \widetilde{f}_{(r-1)^{\varepsilon_{r-1}}} \cdots \widetilde{f}_{j^{\varepsilon_j}} b_{j-1}$. This proves the assertion. □

Set $E_r(\boldsymbol{\lambda}) := \{\boldsymbol{\mu} \in P^{\jmath}_{r-1} \mid \boldsymbol{\mu}^- = \boldsymbol{\lambda}^- \downarrow_{r-1} \text{ and } \boldsymbol{\lambda}^+/\boldsymbol{\mu}^+ \text{ is a horizontal strip}\}$. Then, the assignment

$$\{b \in \mathcal{B}(\boldsymbol{\lambda}) \mid \widetilde{e}_{r'} b \neq 0\} \to E_r(\boldsymbol{\lambda});\ b \mapsto I^{\jmath}_{r-1}(b)$$

is bijective. To each $\boldsymbol{\mu} \in E_r(\boldsymbol{\lambda})$, we associate $b, b' \in \mathcal{B}(\boldsymbol{\lambda})$, $j \in \mathbb{I}^{\jmath} \setminus \{1\}$, and $\varepsilon_i \in \{\emptyset, \prime\}$, $j \leq i \leq r-1$ as in Lemma 15.3.2.

Let $r \geq 2$. We define operators $\widetilde{e}_{l^+}$ and $\widetilde{f}_{l^+}$ on every $\mathbf{U}^{\jmath}$-modules in $\mathcal{O}^{\jmath}_{\mathrm{int}}$ inductively for all $2 \leq l < r$. Let $\boldsymbol{\lambda} \in P^{\jmath}$. We define the linear operator $\widetilde{e}_{r^+}$ on $L(\boldsymbol{\lambda})$ by

$$\widetilde{e}_{r^+} := \bigoplus_{\boldsymbol{\mu} \in E_r(\boldsymbol{\lambda})} p_2(\boldsymbol{\mu}) \circ \frac{1}{[\varphi_{\underline{r}}(b_{\boldsymbol{\mu}}) + 1]} e_r \circ p_1(\boldsymbol{\mu}),$$

where $b_{\boldsymbol{\mu}} \in \mathcal{B}(\boldsymbol{\lambda})$ is the corresponding element to $\boldsymbol{\mu} \in E_r(\boldsymbol{\lambda})$, $p_1(\boldsymbol{\mu})$ is the projection from $L(\boldsymbol{\lambda})$ to the one-dimensional subspace $L(\boldsymbol{\mu})_{\mathrm{wt}^{\jmath}(\boldsymbol{\mu})}$;

$$L(\boldsymbol{\mu})_{\mathrm{wt}^{\jmath}(\boldsymbol{\mu})} \subset L(\boldsymbol{\mu}) \underset{\text{multiplicity free}}{\hookrightarrow} L(\boldsymbol{\lambda}),$$

and $p_2(\boldsymbol{\mu})$ is the projection from $L(\boldsymbol{\lambda})$ to the one-dimensional subspace $\widetilde{f}_{(r-1)^{\delta_{r-1}}} \cdots \widetilde{f}_{j^{\delta_j}} L(\boldsymbol{\mu}')_{\mathrm{wt}^{\jmath}(\boldsymbol{\mu}')}$;

$$\widetilde{f}_{(r-1)^{\delta_{r-1}}} \cdots \widetilde{f}_{j^{\delta_j}} L(\boldsymbol{\mu}')_{\mathrm{wt}^{\jmath}(\boldsymbol{\mu}')} \subset L(\boldsymbol{\mu}') \underset{\text{multiplicity free}}{\hookrightarrow} L(\boldsymbol{\lambda}),$$

where $\delta_l = \emptyset$ if $\varepsilon_l = \emptyset$, and $\delta_l = +$ if $\varepsilon_l = \prime$ for $l = j, \dots, r-1$. Also, we define $\widetilde{f}_{r^+}$ by

$$\widetilde{f}_{r^+} = \bigoplus_{\boldsymbol{\mu} \in E_r(\boldsymbol{\lambda})} \widetilde{e}^{-1}_{r^+} \circ p_2(\boldsymbol{\mu}),$$

where $\widetilde{e}_{r^+}^{-1}$ is the inverse of the linear isomorphism $\widetilde{e}_{r^+} : L(\boldsymbol{\mu})_{\mathrm{wt}^\jmath(\boldsymbol{\mu})} \to \widetilde{f}_{(r-1)^{\delta_{r-1}}} \cdots \widetilde{f}_{j^{\delta_j}} L(\boldsymbol{\mu}')_{\mathrm{wt}^\jmath(\boldsymbol{\mu}')}$. Finally, we extend the definitions of $\widetilde{e}_{r^+}$ and $\widetilde{f}_{r^+}$ to a general $\mathbf{U}^\jmath$-module $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ by the complete reducibility of M.

Proposition 15.3.3. *Let $\boldsymbol{\lambda} \in P^\jmath$ and $v \in L(\boldsymbol{\lambda})$ a highest weight vector. Then, we have*

$$\mathcal{L}(\boldsymbol{\lambda}) = \mathrm{Span}_{\mathbf{A}_0}\{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i,l} v \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_l \in \mathbb{I}^\jmath \sqcup \{2^+, \dots, r^+\}\},$$
$$\mathcal{B}(\boldsymbol{\lambda}) = \{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i,l} v + q\mathcal{L}(\boldsymbol{\lambda}) \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_l \in \mathbb{I}^\jmath \sqcup \{2^+, \dots, r^+\}\} \setminus \{0\}.$$

Moreover, on $\mathcal{B}(\boldsymbol{\lambda})$, we have $\widetilde{e}_{i'} = \widetilde{e}_{i^+}$ and $\widetilde{f}_{i'} = \widetilde{f}_{i^+}$ for all $i \in \mathbb{I}^\jmath \setminus \{1\}$.

Proof. We proceed by induction on r. Assume that the assertion holds for all $2 \leq l < r$ (we assume nothing when $r = 2$). Let $\boldsymbol{\mu} \in E_r(\boldsymbol{\lambda})$ and $b_{\boldsymbol{\mu}}$, b', $\boldsymbol{\mu}'$ be as above. By the uniqueness of the $\jmath$-crystal bases for $\mathbf{U}^\jmath_{r-1}$-modules, there exists a unique $v_{\boldsymbol{\mu}} \in \mathcal{L}(\boldsymbol{\lambda})$ such that $\mathbf{U}^\jmath_{r-1} v_{\boldsymbol{\mu}} = L(\boldsymbol{\mu})$, $v_{\boldsymbol{\mu}} + q\mathcal{L}(\boldsymbol{\lambda}) = b_{\boldsymbol{\mu}}$. Then, we can write

$$v_{\boldsymbol{\mu}} = G^\jmath_{\mathrm{low}}(b_{\boldsymbol{\mu}}) + \sum_{b' \in \mathcal{B}(\boldsymbol{\lambda}) \setminus \{b_{\boldsymbol{\mu}}\}} a_{b'} G^\jmath_{\mathrm{low}}(b')$$

for some $a_{b'} \in q\mathbf{A}_0$. Note that this equation implies that $\widetilde{e}_{r'}(v_{\boldsymbol{\mu}}) \in G^\jmath_{\mathrm{low}}(\widetilde{e}_{r'} b_{\boldsymbol{\mu}}) + q\mathcal{L}(\boldsymbol{\lambda})$. Also, we have

$$\frac{1}{[\varphi_{\underline{r}}(b_{\boldsymbol{\mu}}) + 1]} e_r v_{\boldsymbol{\mu}} = G^\jmath_{\mathrm{low}}(\widetilde{e}_{r'} b_{\boldsymbol{\mu}}) + \sum_{b' \in \mathcal{B}(\boldsymbol{\lambda})} c_{b'} G^\jmath_{\mathrm{low}}(b') \quad (\text{since } \widetilde{e}_{r'} b_{\boldsymbol{\mu}} = \widetilde{E}_{\underline{r}} b_{\boldsymbol{\mu}}.)$$

for some $c_{b'} \in \mathbf{A}$. Again, by the complete reducibility of the $\mathbf{U}^\jmath_{r-1}$-crystal bases, there exists a unique $v_{\boldsymbol{\mu}'} \in \mathcal{L}(\boldsymbol{\lambda})$ such that $\mathbf{U}^\jmath_{r-1} v_{\boldsymbol{\mu}'} = L(\boldsymbol{\mu}')$, $v_{\boldsymbol{\mu}'} + q\mathcal{L}(\boldsymbol{\lambda}) = b'$. By our induction hypothesis, we have $u := \widetilde{f}_{(r-1)^{\delta_{r-1}}} \cdots \widetilde{f}_{j^{\delta_j}}(v_{\boldsymbol{\mu}'}) \in \mathcal{L}(\boldsymbol{\lambda}) \cap \mathbf{U}^\jmath_{r-1} v_{\boldsymbol{\mu}'}$ and $u + q\mathcal{L}(\boldsymbol{\lambda}) = \widetilde{e}_{r'} b_{\boldsymbol{\mu}}$. Then, we can write

$$u = G^\jmath_{\mathrm{low}}(\widetilde{e}_{r'} b_{\boldsymbol{\mu}}) + \sum_{b' \in \mathcal{B}(\boldsymbol{\lambda})} d_{b'} G^\jmath_{\mathrm{low}}(b')$$

for some $d_{b'} \in q\mathbf{A}_0$. Hence, we have

$$\widetilde{e}_{r^+}(v_{\boldsymbol{\mu}}) \in G^\jmath_{\mathrm{low}}(\widetilde{e}_{r'} b_{\boldsymbol{\mu}}) + q\mathcal{L}(\boldsymbol{\lambda}).$$

Since we took $\boldsymbol{\mu} \in E_r(\boldsymbol{\lambda})$ arbitrarily, this equation ensures that $\widetilde{e}_{r^+}$ preserves $\mathcal{L}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda}) \sqcup \{0\}$, and that $\widetilde{e}_{r^+} = \widetilde{e}_{r'}$ on $\mathcal{B}(\boldsymbol{\lambda})$. By the definition of $\widetilde{f}_{r^+}$, it also preserves $\mathcal{L}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda}) \sqcup \{0\}$, and coincides with $\widetilde{f}_{r'}$ on $\mathcal{B}(\boldsymbol{\lambda})$. Now, the assertions are clear by the definition of $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$. □

Corollary 15.3.4. *Let $M \in \mathcal{O}^\jmath_{\mathrm{int}}$ be a $\mathbf{U}^\jmath$-module with a $\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. Then $\widetilde{e}_{i'} = \widetilde{e}_{i^+}$ and $\widetilde{f}_{i'} = \widetilde{f}_{i^+}$ on $\mathcal{B}$ for all $i \in \mathbb{I}^\jmath \setminus \{1\}$.*

16. PROOF OF THEOREM 15.2.6

The aim of this section is to complete the proof of Theorem 15.2.6. We first give a proof for the $r = 1$ case in Subsection 16.1, and then, for a general r in Subsection 16.2.

For a $\mathbf{U}^\jmath$-module M with a global $\jmath$-crystal basis $G^\jmath(\mathcal{B})$, and for $m \in M$, $b \in \mathcal{B}$, let $[m : G^\jmath(b)]$ denote the coefficient of $G^\jmath(b)$ in m.

16.1. **The case $r = 1$.** In this subsection, we prove Theorem 15.2.6 for $r = 1$.

Proof of Theorem 15.2.6. We proceed by descending induction on $\boldsymbol{\lambda}$ with respect to $\preceq$. Assume that the statement holds for all $\boldsymbol{\lambda}' \succ \boldsymbol{\lambda}$. Replacing M with $M/W_{\succ\boldsymbol{\lambda}}(M)$, we may assume that $\boldsymbol{\lambda}$ is maximal among $P^{\jmath}(M)$. Let $b_1, \dots, b_{m_{\boldsymbol{\lambda}}} \in \mathcal{B}$ and $u_1, \dots, u_{m_{\boldsymbol{\lambda}}} \in \mathcal{L}$ be distinct highest weight vectors of type $\boldsymbol{\lambda}$ with $u_i + q\mathcal{L} = b_i$, $i = 1, \dots, m_{\boldsymbol{\lambda}}$. By retaking the u_i's if necessary, we may assume that $[u_i : G^{\jmath}(b_j)] = \delta_{i,j}$ for all i, j. Fix i arbitrarily, and set $b := b_i$, $u := u_i$. Then, we can write

$$u = G^{\jmath}(b) + \sum_{\substack{b' \\ I^{\jmath}(b') \not\preceq \boldsymbol{\lambda}}} c_{b'} G^{\jmath}(b'), \quad c_{b'} \in q\mathbf{A}_0.$$

We first prove that $c_{b'} = 0$ for all b' with $\varepsilon_1(b') = 0$. Assume contrary, and take $b' \in \mathcal{B} \setminus \{b\}$ such that $c_{b'} \neq 0$, $\varepsilon_1(b') = 0$, and $\varphi_1(b')$ is minimal among $\{\varphi_1(b'') \mid c_{b''} \neq 0,\ \varepsilon_1(b'') = 0\}$. Set $\boldsymbol{\mu} := I^{\jmath}(b')$. Then, we have $\mathrm{wt}^{\jmath}(\boldsymbol{\mu}) = \mathrm{wt}^{\jmath}(\boldsymbol{\lambda})$, in particular, $\boldsymbol{\mu}_0 = \boldsymbol{\lambda}_0$. Since $\boldsymbol{\mu} \not\preceq \boldsymbol{\lambda}$, we have $\varphi_1(b') = \boldsymbol{\mu}_0 - \boldsymbol{\mu}_{-1} > \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1} = \varphi_1(b)$. Hence, we have

$$\begin{aligned} -f_1^{(\varphi_1(b)+1)} G^{\jmath}(b) = c_{b'} \big(G^{\jmath}(\widetilde{f}_1^{\varphi_1(b)+1} b') + &\sum_{\substack{b'' \\ \varepsilon_1(b'') > \varphi_1(b)+1}} d_{b'',b'} G^{\jmath}(b'') \big) \\ &+ \sum_{b''' \neq b'} \sum_{\substack{b'''' \\ \varepsilon_1(b'''') \geq \varepsilon_1(b''') + \varphi_1(b)+1}} d_{b'''',b'''} G^{\jmath}(b''''), \end{aligned}$$

for some $d_{b_1,b_2} \in \mathbf{A}$. By our assumption, the coefficient of $G^{\jmath}(\widetilde{f}_1^{\varphi_1(b)+1} b')$ in the right-hand side is equal to $c_{b'}$. On the other hand, the left-hand side is fixed by $\psi^{\jmath}_M$, and it belongs to $M_{\mathbf{A}}$. Therefore, we have $c_{b'} \in q\mathbf{A}_0 \cap \mathbf{A}$ and $\overline{c_{b'}} = c_{b'}$, which implies $c_{b'} = 0$.

Next, we prove that $c_{b'} = 0$ for all b' with $\varepsilon_1(b') > 0$. Assume contrary that $c_{b'} \neq 0$ for some such b'. Set $\boldsymbol{\mu} := I^{\jmath}(b')$. Since $\boldsymbol{\lambda}$ is maximal, we have $\boldsymbol{\mu}_0 + \boldsymbol{\mu}_{-1} < \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_{-1}$. Substituting $\boldsymbol{\mu}_0 = \boldsymbol{\lambda}_0 + \varepsilon_1(b')$, $\boldsymbol{\mu}_{-1} = \boldsymbol{\lambda}_0 - \varphi_1(b')$, and $\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1} = \varphi_1(b)$, we obtain $\varphi_1(b') > \varphi_1(b) + \varepsilon_1(b')$. We may assume that $(\varepsilon_1(b'), \varphi_1(b'))$ is minimal (with respect to the lexigographical order) among such b''s. Then, for all $t = 1, \dots, \varepsilon_1(b') + 1$, we have

$$-f_1^{(\varphi_1(b)+t)} G^{\jmath}(b) = c_{b'} \begin{bmatrix} \varepsilon_1(b') + \varphi_1(b) + t \\ \varepsilon_1(b') \end{bmatrix} G^{\jmath}(\widetilde{f}_1^{\varphi_1(b)+t} b') + (\text{other terms}).$$

This implies that $c_{b'} \in q\mathbf{A}_0$, $\overline{c_{b'}} = c_{b'}$, and $c_{b'} \left[\begin{smallmatrix} \varepsilon_1(b') + \varphi_1(b) + t \\ \varepsilon_1(b') \end{smallmatrix}\right] \in \mathbf{A}$ for all $t = 1, \dots, \varepsilon_1(b') + 1$. Now, it suffices to show that $c_{b'} \in \mathbf{A}$, which follows from next lemma.

This far, we have proved that $G^{\jmath}(b) = u$, and hence, we have $e_1 G^{\jmath}(b) = 0$ and $\mathbf{U}^{\jmath}_1 G^{\jmath}(b) \simeq L(\boldsymbol{\lambda})$. Then, for all $n = 1, \dots, \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{-1}$, we have

$$f_1^{(n)} G^{\jmath}(b) = f_1^{(n)} u = \widetilde{f}_1^n u.$$

The left-hand side belongs to $M_{\mathbf{A}}$, while the right-hand side belongs to $\mathcal{L}$. Moreover, we have $\psi^{\jmath}_M(f_1^{(n)} G^{\jmath}(b)) = f_1^{(n)} G^{\jmath}(b)$, and $\widetilde{f}_1^n u + q\mathcal{L} = \widetilde{f}_1^n b$. This implies that $f_1^{(n)} G^{\jmath}(b) = G^{\jmath}(\widetilde{f}_1^n b)$. Thus, the proof completes. ☐

Lemma 16.1.1. *Let $A \in \mathbb{Q}(p, q)$, $m \geq n \in \mathbb{Z}_{\geq 0}$. Suppose that $A \left[\begin{smallmatrix} m+t \\ n \end{smallmatrix}\right] \in \mathbf{A}$ for all $t = 1, \dots, n+1$. Then, we have $A \in \mathbf{A}$.*

Proof. Let us write $A = B/C$ for some $B, C \in \mathbf{A}_0 \cap \mathbf{A}$ that are coprime. By the hypothesis, C is a common devisor of $\left[\begin{smallmatrix} m+t \\ n \end{smallmatrix}\right]$, $t = 1, \dots, n+1$. Hence, it suffices to show that the greatest common divisor of them in $\mathbb{Z}[q]$ is equal to 1. This is equivalent to say that the greatest common divisor of $a_t := [m+t][m+t-1] \cdots [m+t-n+1]$, $t = 1, \dots, n+1$ is equal to

$[n]!$. Since $[l] = q^{-l}\prod_{1\neq d|l}\Phi_d$, where $\Phi_d = \Phi_d(q^2)$ denotes the d-th cyclotomic polynomial in variable q^2, we have

$$b_t := q^{n(m+t)-\frac{n(n-1)}{2}}a_t = \prod_{l=0}^{n-1}\prod_{1\neq d|(m+t-l)}\Phi_d,$$

which is the irreducible decomposition of b_t in $\mathbb{Z}[q^2]$. Then, we have

$$b_t = \prod_{d\geq 2}\Phi_d^{m_{d,t}}, \text{ where } m_{d,t} := |\{0 \leq l \leq n-1 \mid d|(m+t-l)\}|,$$

and hence,

$$\underset{1\leq t\leq n+1}{\gcd}(b_t) = \prod_{d\geq 2}\Phi_d^{\min_{1\leq t\leq n+1}(m_{d,t})}.$$

We prove that $\min_{1\leq t\leq n+1}(m_{d,t}) = \lfloor\frac{n}{d}\rfloor$ for all d. It is clear that $m_{d,t} \geq \lfloor\frac{n}{d}\rfloor$ for all t since $\{m+t, m+t-1, \dots, m+t-(\lfloor\frac{n}{d}\rfloor d-1)\}$ contains exactly $\lfloor\frac{n}{d}\rfloor$ integers divisible by d. If $\min_{1\leq t\leq n+1}(m_{d,t}) > \lfloor\frac{n}{d}\rfloor$, then $\{m+t-\lfloor\frac{n}{d}\rfloor d, m+t-(\lfloor\frac{n}{d}\rfloor d+1), \dots, m+t-(n-1)\}$ contains at least one multiple of d for all t. Then, for $t=1$, there exists $l_1 \in \{\lfloor\frac{n}{d}\rfloor d, \lfloor\frac{n}{d}\rfloor d+1, \dots, n-1\}$ such that $m+1-(\lfloor\frac{n}{d}\rfloor d+l_1) \in d\mathbb{Z}$. Set $t' := n-l_1+1$, and consider the integers

$$m+t'-\lfloor\frac{n}{d}\rfloor d, m+t'-(\lfloor\frac{n}{d}\rfloor d+1), \dots, m+t'-(n-1) = (m+1-l_1)+1.$$

These are $(n-\lfloor\frac{n}{d}\rfloor d)$ consecutive integers with $(m+1-l_1)+1 = 1$ modulo d. Since $n-\lfloor\frac{n}{d}\rfloor d < d$, they have no multiples of d. Hence, we have $\min_{1\leq t\leq n+1}(m_{d,t}) = \lfloor\frac{n}{d}\rfloor$ for all $d \geq 2$. Thus, we obtain

$$\underset{1\leq t\leq n+1}{\gcd}(b_t) = \prod_{d\geq 2}\Phi_d^{\lfloor\frac{n}{d}\rfloor} = \prod_{d=2}^{n}\Phi_d^{\lfloor\frac{n}{d}\rfloor} = \prod_{l=2}^{n}\left(\prod_{1\neq d'|l}\Phi_{d'}\right) = \prod_{l=2}^{n}[l] = [n]!.$$

This proves the lemma. □

16.2. **The case** $r \geq 2$**.** Now, we are ready to prove Theorem 15.2.6 by induction on r.

When $r = 1$, we have already completed the proof. Let $r \geq 2$ and assume that the assertions hold for all $r' < r$.

Lemma 16.2.1. *Let $\boldsymbol{\lambda} \in P^{\jmath}(M)$ be a maximal element, $b \in \mathcal{B}$ such that $I^{\jmath}(b) = \boldsymbol{\lambda}$ and $\widetilde{e}_i b = 0$ for all $i \in \mathbb{I}^{\jmath}$. Suppose the following:*

1. *There exists a homomorphism $\xi : L(\boldsymbol{\lambda}) \to M$ of $\mathbf{U}^{\jmath}$-modules such that $\xi(G^{\jmath}_{\mathrm{low}}(T^{\jmath}_{b'})) = G^{\jmath}(b')$ for all $b' \in C^{\jmath}(b)$ which is strongly connected to some $b'' \in C^{\jmath}(b)$ with $\mathrm{wt}^{\jmath}(b) <^{\jmath} \mathrm{wt}^{\jmath}(b'')$.*
2. *ξ commutes with the $\psi^{\jmath}$-involutions on $L(\boldsymbol{\lambda})$ and M.*
3. *$[\xi(G^{\jmath}_{\mathrm{low}}(T^{\jmath}_b)) : G^{\jmath}(b)] = 1$.*

Then, we have

$$\xi(G^{\jmath}_{\mathrm{low}}(T^{\jmath}_b)) = G^{\jmath}(b) + \sum_{\substack{b'\in\mathcal{B}\setminus\{b\} \\ T^{\jmath}_{b'}=T^{\jmath}_b}} c_{b'}G^{\jmath}(b') + \sum_{\substack{b''\in C^{\jmath}(b'),\ c_{b'}\neq 0 \\ s(b'')<s(b')}} c_{b''}G^{\jmath}(b'').$$

for some $c_{b'}, c_{b''} \in \mathbf{A}_0$.

Proof. Since $\mathbf{U}^\jmath$-module homomorphisms preserve $\jmath$-crystal lattices, we have $\xi(G^\jmath_{\mathrm{low}}(T^\jmath_b)) \in \mathcal{L}$, and $\xi(G^\jmath_{\mathrm{low}}(T^\jmath_b)) + q\mathcal{L} = b$. Let us write

$$\xi(G^\jmath_{\mathrm{low}}(T^\jmath_b)) = G^\jmath(b) + \sum_{b' \in \mathcal{B} \setminus \{b\}} c_{b'} G^\jmath(b')$$

for some $c_{b'} \in q\mathbf{A}_0$. Also, since ξ commutes with $\psi^\jmath$-involutions, we have $\overline{c_b} = c_b$, $\overline{c_{b'}} = c_{b'}$. We claim the following: if $b' \in \mathcal{B} \setminus \{b\}$ satisfies

(†) $\quad c_{b'} \neq 0$ and $s(b')$ is maximal among $\{s(b'') \mid b'' \in \mathcal{B} \setminus \{b\}$ and $c_{b''} \neq 0\}$,

then $T^\jmath_{b'}(-i) \geq \boldsymbol{\lambda}_{-i}$ for all $i = 0, 1, \dots, r$. By the case $r = 1$, we have $I^\jmath_1(b') \succeq I^\jmath_1(b)$, which implies $T^\jmath_{b'}(0) = T^\jmath_b(0) = \boldsymbol{\lambda}_0$, and $T^\jmath_{b'}(-1) \geq T^\jmath_b(-1) = \boldsymbol{\lambda}_{-1}$. We proceed by induction on i. Assume that $i \geq 2$, and that $T^\jmath_{b'}(-(i-1)) \geq \boldsymbol{\lambda}_{-(i-1)}$ for all b' satisfying (†). Suppose that there exists b' satisfying (†) such that $T^\jmath_{b'}(-i) < \boldsymbol{\lambda}_{-i}$. Let $b'' \in \mathcal{B} \setminus \{b\}$ be such that $s(b'') = s(b')$ and $\varphi_{-\underline{b''}}$ is minimal among such elements. Recall that $s(b'') = s(b')$ implies $\mathrm{wt}(b'') = \mathrm{wt}(b')$, and hence, $T^\jmath_{b''}(-i) = T^\jmath_{b'}(-i) < \boldsymbol{\lambda}_{-i}$. Then, we have

$$\varepsilon_{-\underline{i}}(b'') = \varphi_{-\underline{i}}(b'') + T^\jmath_{b''}(-(i-1)) - T^\jmath_{b''}(-i) > T^\jmath_{b''}(-(i-1)) - \boldsymbol{\lambda}_{-i} + \varphi_{-\underline{i}}(b'').$$

By the minimality of $\varphi_{-\underline{i}}(b'')$, it holds that

$$[f_i^{(t)} \sum_{b' \in \mathcal{B} \setminus \{b\}} c_{b''} G^\jmath(b') : G^\jmath(\widetilde{f}_i^t b'')] = c_{b'} \begin{bmatrix} t \\ \varphi_{-\underline{i}}(b'') \end{bmatrix} \neq 0$$

for all $T^\jmath_{b''}(-(i-1)) - \boldsymbol{\lambda}_{-i} + 1 \leq t \leq T^\jmath_{b''}(-(i-1)) - \boldsymbol{\lambda}_{-i} + \varphi_{-\underline{i}}(b'') + 1$. On the other hand, $f_i^{(t)} G^\jmath_{\mathrm{low}}(T^\jmath_b)$ is the sum of $G^\jmath_{\mathrm{low}}(T^\jmath_{\widetilde{f}_i^t b})$ and an $\mathbf{A}$-linear combination of $G^\jmath_{\mathrm{low}}(T^\jmath_{\widehat{b}})$ such that $\widehat{b} \in C^\jmath(b_t)$ is strongly connected to $b''' \in C^\jmath(b_t)$ with $\mathrm{wt}^\jmath(b) <^\jmath \mathrm{wt}^\jmath(b''')$. Hence, we have

$$\begin{aligned} \xi(G^\jmath_{\mathrm{low}}(T^\jmath_{\widetilde{f}_i^t b})) &= f_i^{(t)} \xi(G^\jmath_{\mathrm{low}}(T^\jmath_b)) + \sum_{\widehat{b}} a_{\widehat{b}} G^\jmath(\widehat{b}) \\ &= f_i^{(t)} G^\jmath(b) + f_i^{(t)} \sum_{b' \in \mathcal{B} \setminus \{b\}} c_{b'} G^\jmath(b') + \sum_{\widehat{b}} a_{\widehat{b}} G^\jmath(\widehat{b}) \end{aligned}$$

for some $a_{\widehat{b}} \in \mathbf{A}$. Here, note that we have $\widetilde{e}_j \widetilde{f}_i^t b = 0$ for all $j = 1, \dots, i-1$, $[\xi(G^\jmath_{\mathrm{low}}(T^\jmath_{\widetilde{f}_i^t b})) : G^\jmath(\widetilde{f}_i^t b)] = 1$, and $s(\widetilde{f}_i^t b'')$ is maximal among $\{s(b''') \mid b''' \neq \widetilde{f}_i^t b$ and $[\xi(G^\jmath_{\mathrm{low}}(T^\jmath_{\widetilde{f}_i^t b})) : G^\jmath(b''')] \neq 0\}$. Then, by our induction hypothesis on i, we obtain that $T^\jmath_{\widetilde{f}_i^t b''}(-(i-1)) \geq T^\jmath_{\widetilde{f}_i^t b}(-(i-1)) = \boldsymbol{\lambda}_{-i}$, which is a contradiction since $t \geq T^\jmath_{b''}(-(i-1)) - \boldsymbol{\lambda}_{-i} + 1$. Hence we must have $[\xi(G^\jmath_{\mathrm{low}}(T^\jmath_{\widetilde{f}_i^t b})) : G^\jmath(\widetilde{f}_i^t b'')] = 0$. Since

$$[\xi(G^\jmath_{\mathrm{low}}(T^\jmath_{\widetilde{f}_i^t b})) : G^\jmath(\widetilde{f}_i^t b'')] = c_{b''} \begin{bmatrix} t \\ \varphi_{-\underline{i}}(b'') \end{bmatrix} + [f_i^{(t)} G^\jmath(b) : G^\jmath(\widetilde{f}_i^t b'')] + a_{\widetilde{f}_i^t b''},$$

and the second and the third term of the right-hand side lies in $\mathbf{A}$, we obtain

$$c_{b''} \begin{bmatrix} t \\ \varphi_{-\underline{i}}(b'') \end{bmatrix} \in \mathbf{A}$$

for all $T^\jmath_{b''}(-(i-1)) - \boldsymbol{\lambda}_{-i} + 1 \leq t \leq T^\jmath_{b''}(-(i-1)) - \boldsymbol{\lambda}_{-i} + \varphi_{-\underline{i}}(b'') + 1$. By Lemma 16.1.1, this implies $c_{b''} = 0$.

This far, we have proved that if $b' \in \mathcal{B} \setminus \{b\}$ satisfies (†), then we have $T^\jmath_{b'}(-i) \geq \boldsymbol{\lambda}_{-i}$ for all $i \in \{0, 1, \dots, r\}$. In particular, we have $\mathbb{I}^\jmath(b') = \boldsymbol{\lambda}$ for such b' (since $\boldsymbol{\lambda}$ is maximal in $P^\jmath(M)$). In this case, the condition $T^\jmath_{b'}(-i) \geq \boldsymbol{\lambda}_{-i}$ for all i forces b' to satisfy that $T^\jmath_{b'} = T^\jmath_b$. Hence, we have

$$\xi(G^\jmath_{\mathrm{low}}(T^\jmath_b)) = G^\jmath(b) + \sum_{\substack{b' \in \mathcal{B} \setminus \{b\} \\ T^\jmath_{b'} = T^\jmath_b}} c_{b'} G^\jmath(b') + \sum_{\substack{b'' \in C^\jmath(b'),\ c_{b'} \neq 0 \\ s(b'') < s(b')}} c_{b''} G^\jmath(b''),$$

as desired. □

Lemma 16.2.2. *Let $\boldsymbol{\lambda} \in P^\jmath(M)$ be a maximal element, $j \in \mathbb{I}^\jmath \setminus \{1\}$, $b \in \mathcal{B}$ such that $I^\jmath(b) = \boldsymbol{\lambda}$, $\widetilde{e}_i b = 0$ for all $i \in \mathbb{I}^\jmath$, and $\widetilde{e}_{j'}(b) \neq 0$. Suppose the following:*

1. *There exists a homomorphism $\xi : L(\boldsymbol{\lambda}) \to M$ of $\mathbf{U}^\jmath$-modules such that $\xi(G^\jmath_{\mathrm{low}}(T^\jmath_{b'})) = G^\jmath(b')$ for all $b' \in C^\jmath(b)$ which is strongly connected to some $b'' \in C^\jmath(b)$ with $\mathrm{wt}^\jmath(b) <^\jmath \mathrm{wt}^\jmath(b'')$.*
2. *ξ commutes with the $\psi^\jmath$-involutions on $L(\boldsymbol{\lambda})$ and M.*

Then, we have

$$\xi(G^\jmath_{\mathrm{low}}(T^\jmath_b)) = G^\jmath(b) + \sum_{\substack{b' \in \mathcal{B} \setminus \{b\} \\ T^\jmath_{b'} = T^\jmath_b}} c_{b'} G^\jmath(b') + \sum_{\substack{b'' \in C^\jmath(b'),\ c_{b'} \neq 0 \\ s(b'') < s(b')}} c_{b''} G^\jmath(b'').$$

for some $c_{b'}, c_{b''} \in \mathbf{A}_0$.

Proof. If we can prove that $c_b := [\xi(G^\jmath_{\mathrm{low}}(T^\jmath_b)) : G^\jmath(b)] = 1$, then the assertion follows from the previous lemma. Hence, we aim to show $c_b = 1$.

By the same argument as before, we have

$$[\xi(G^\jmath_{\mathrm{low}}(T^\jmath_{\widetilde{f}_i^t b})) : G^\jmath(\widetilde{f}_i^t b'')] = c_{b''} \begin{bmatrix} t \\ \varphi_{-\underline{i}}(b'') \end{bmatrix} + c_b [f_i^{(t)} G^\jmath(b) : G^\jmath(\widetilde{f}_i^t b'')] + a_{\widetilde{f}_i^t b''},$$

for all $b'' \in \mathcal{B} \setminus \{b\}$ satisfying (†). Here, let us assume further that $s(b') > s(b)$. Then, we have $[f_i^{(t)} G^\jmath(b) : G^\jmath(\widetilde{f}_i^t b'')] = 0$ since $f_i^{(t)} G^\jmath(b)$ is a linear combination of $G^\jmath(\widetilde{b})$ with $s(\widetilde{b}) \leq s(b) + t\alpha_{-\underline{i}} < s(b'') + t\alpha_{-\underline{i}} = s(\widetilde{f}_i^t b'')$. Hence, we have $c_{b''} \begin{bmatrix} t \\ \varphi_{-\underline{i}}(b'') \end{bmatrix} \in \mathbf{A}$, and therefore, $c_{b''} = 0$ by Lemma 16.1.1. In particular, we obtain that $s(b)$ is maximal. Then, we have

$$[e_j(c_b G^\jmath(b) + \sum c_{b'} G^\jmath(b')) : G^\jmath(\widetilde{e}_{j'} b)] = c_b [\varphi_{\underline{j}}(b) + 1].$$

On the other hand, since $[e_j G^\jmath_{\mathrm{low}}(T^\jmath_b) : G^\jmath_{\mathrm{low}}(\widetilde{e}_{j'} b)] = [\varphi_{\underline{j}}(b) + 1]$, we have

$$[e_j(c_b G^\jmath(b) + \sum c_{b'} G^\jmath(b')) : G^\jmath(\widetilde{e}_{j'} b)] = [\varphi_{\underline{j}}(b) + 1],$$

and hence, $c_b = 1$, as desired. □

We prove Theorem 15.2.6 by descending induction (with respect to $\preceq$) on $\boldsymbol{\lambda}$. As in the $r = 1$ case, we may assume that $\boldsymbol{\lambda}$ is maximal among $P^\jmath(M)$. Then, in order to complete the proof, we have to show the following:

1. $I_{\boldsymbol{\lambda}}(M)$ has a basis $\{G^\jmath(b) \mid I^\jmath(b) = \boldsymbol{\lambda}\}$.
2. There exists an isomorphism $\xi : L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}} \to I_{\boldsymbol{\lambda}}(M)$ of $\mathbf{U}^\jmath$-modules which sends the $\jmath$-global basis elements of $L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ to those of $I_{\boldsymbol{\lambda}}(M)$, where $m_{\boldsymbol{\lambda}}$ denotes the multiplicity of $L(\boldsymbol{\lambda})$ in M.

Let $b_1, \ldots, b_{m_{\boldsymbol{\lambda}}} \in \mathcal{B}$ and $u_1, \ldots, u_{m_{\boldsymbol{\lambda}}} \in \mathcal{L}$ be distinct highest weight vectors of type $\boldsymbol{\lambda}$ with $u_t + q\mathcal{L} = b_t$, $t = 1, \ldots, m_{\boldsymbol{\lambda}}$. By retaking the u_t's if necessary, we may assume that $[u_t : G^\jmath(b_u)] = \delta_{t,u}$ for all t, u. Let $\xi_t : L(\boldsymbol{\lambda}) \to M$ be the $\mathbf{U}^\jmath$-homomorphism which sends $v_{\boldsymbol{\lambda}}$ to u_t.

Lemma 16.2.3. *We have* $\xi_t(G^\jmath_{\mathrm{low}}(T^\jmath_{b_t})) = G^\jmath(b_t)$ *for all* $t = 1, \ldots, m_{\boldsymbol{\lambda}}$.

Proof. By the setting above, we can write

$$\xi_t(G^\jmath_{\mathrm{low}}(T^\jmath_{b_t})) = u_t = G^\jmath(b_t) + \sum_{\substack{b' \\ I^\jmath(b') \not\succeq \boldsymbol{\lambda}}} c_{b'} G^\jmath(b'), \quad c_{b'} \in q\mathbf{A}_0.$$

Then, we can apply Lemma 16.2.1 to obtain $\xi_t(G^\jmath_{\mathrm{low}}(T^\jmath_{b_t})) = G^\jmath(b_t)$ as desired. □

In order to complete the proof, it suffices to prove the following: For each $t = 1, \ldots, m_{\boldsymbol{\lambda}}$ and $b \in C^\jmath(b_t)$, we have $\xi_t(G^\jmath_{\mathrm{low}}(T^\jmath_b)) = G^\jmath(b)$. We prove this statement by descending induction on $\mathrm{wt}^\jmath(b)$ and $I^\jmath_{r-1}(b)$. When $\mathrm{wt}^\jmath(b)$ is maximal, it must hold that $b = b_t$, and in this case, we have already shown that $\xi_t(G^\jmath_{\mathrm{low}}(T^\jmath_{b_t})) = G^\jmath(b_t)$. Suppose that $\mathrm{wt}^\jmath(b) <^\jmath \mathrm{wt}^\jmath(b_t)$, and the statement holds for all $b' \in \bigsqcup_{t=1}^{m_{\boldsymbol{\lambda}}} C^\jmath(b_t)$ such that $\mathrm{wt}^\jmath(b')^\jmath > \mathrm{wt}^\jmath(b)$ or $\mathrm{wt}^\jmath(b') = \mathrm{wt}^\jmath(b)$ and $I^\jmath_{r-1}(b') \succ I^\jmath_{r-1}(b)$. In this case, since b is not a $\mathbf{U}^\jmath$-highest weight vector, the exists $i \in \overline{\mathbb{I}^\jmath}$ such that $\widetilde{e}_i b \neq 0$.

Lemma 16.2.4. *Suppose there exists* $i \in \mathbb{I}^\jmath$ *such that* $\widetilde{e}_i b \neq 0$. *Then, the statement holds.*

Proof. Set $b' := \widetilde{e}_i^{\varepsilon_i(b)} b$. We prove the lemma by descending induction on $\varepsilon_i(b')$. Since $\mathrm{wt}^\jmath(b') > \mathrm{wt}^\jmath(b)$, we have $G^\jmath(b') = \xi_t(G^\jmath_{\mathrm{low}}(T^\jmath_{b'})) \in \mathbf{U}^\jmath G^\jmath(b_i)$. We know that $G^\jmath(b)$ (resp., $G^\jmath_{\mathrm{low}}(T^\jmath_b)$) is the sum of $\widetilde{f}_i^{(\varepsilon_i(b))} G^\jmath(b')$ (resp., $\widetilde{f}_i^{(\varepsilon_i(b))} G^\jmath_{\mathrm{low}}(T^\jmath_{b'})$) and a $q\mathbb{Q}[q]$-linear combination of $G^\jmath(b'')$ (resp., $G^\jmath_{\mathrm{low}}(T^\jmath_{b''})$) with $\mathrm{wt}^\jmath(b'') = \mathrm{wt}^\jmath(b)$ and $\varepsilon_i(b'') > \varepsilon_i(b)$. By our induction hypothesis, $G^\jmath(b) - \xi_t(G^\jmath_{\mathrm{low}}(b))$ is a $q\mathbb{Q}[q]$-linear combination of $G^\jmath(b'')$'s, and is $\psi^\jmath_M$-invariant. Such a vector must be zero, and hence, we obtain $G^\jmath(b) = \xi_t(G^\jmath_{\mathrm{low}}(b))$. □

Lemma 16.2.5. *Suppose there exists* $j \in \mathbb{I}^\jmath \setminus \{1\}$ *such that* $\widetilde{e}_{j'} b \neq 0$ *and* $\widetilde{e}_i b = 0$ *for all* $i \in \mathbb{I}^\jmath$. *Then, the statement holds.*

Proof. Apply Lemma 16.2.2. □

Now, one can complete the proof by combining Lemma 16.2.3-16.2.5 since each $b \in \mathcal{B}$ with $I^\jmath(b) = \boldsymbol{\lambda}$ is connected to b_t for some $t = 1, \ldots, m_{\boldsymbol{\lambda}}$.

Part 4. Representation theory of $\mathbf{U}^\imath$

This part is the counterpart of Part 3 for $\mathbf{U}^\imath$.

17. Basics of the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$

17.1. **Triangular decomposition of $\mathbf{U}^\imath$.** Recall that $\Phi_+ = \{\epsilon_{\underline{i}} - \epsilon_{\underline{j}} \mid -r \leq i < j \leq r+1\}$, and divide it into three parts as:

$$\begin{aligned} \Phi_+ &= \Phi_{<0} \sqcup \Phi_0 \sqcup \Phi_{>0}, \\ \Phi_{<0} &:= \{\epsilon_{\underline{i}} - \epsilon_{\underline{j}} \mid \underline{i} + \underline{j} < 0\}, \\ \Phi_0 &:= \{\epsilon_{\underline{i}} - \epsilon_{\underline{j}} \mid \underline{i} + \underline{j} = 0\}, \\ \Phi_{>0} &:= \{\epsilon_{\underline{i}} - \epsilon_{\underline{j}} \mid \underline{i} + \underline{j} > 0\}. \end{aligned}$$

Lemma 17.1.1. *There exists a reflection order $\preceq$ on Φ_+ such that*

$$\Phi_{<0} \prec \Phi_0 \prec \Phi_{>0} \tag{16}$$

Fix a reflection order $\preceq$ satisfying condition (16) in Lemma 17.1.1. Let $\mathbf{i}$ be the reduced word for w_0 corresponding to $\preceq$. We set $E_{\underline{i},\underline{j}} := E_k(\mathbf{i})$ for $-\underline{r+1} \leq \underline{i} < \underline{j} \leq \underline{r+1}$, where k is such that $\epsilon_{\underline{i}} - \epsilon_{\underline{j}} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. For each i, j, define $E'_{\underline{i},\underline{j}} := \mathrm{gr}^{-1} \circ p(E_{\underline{i},\underline{j}})$, and set

$$f_{-\underline{j},-\underline{i}} := E'_{\underline{i},\underline{j}} \text{ if } \underline{i} + \underline{j} < 0, \quad t'_i := E'_{-\underline{i+1},\underline{i+1}}, \quad e_{\underline{i},\underline{j}} := E'_{\underline{i},\underline{j}} \text{ if } \underline{i} + \underline{j} > 0.$$

Corollary 17.1.2. *The ordered monomials* $\left(\prod_{\underline{i}+\underline{j}<0} f_{-\underline{j},-\underline{i}}^{a_{\underline{i},\underline{j}}}\right) \left(\prod_{i=0}^{r} (t'_i)^{b_{\underline{i}}}\right) \left(\prod_{\underline{i}} k_{\underline{i}}^{d_{\underline{i}}}\right) \left(\prod_{\underline{i}+\underline{j}>0} e_{\underline{i},\underline{j}}^{c_{\underline{i},\underline{j}}}\right)$, $a_{\underline{i},\underline{j}}, b_{\underline{i}}, c_{\underline{i},\underline{j}} \in \mathbb{Z}_{\geq 0}$, $d_{\underline{i}} \in \mathbb{Z}$ *form a linear basis of* $\mathbf{U}^\imath$.

We have

$$t'_0 = t, \quad t'_{i+1} = [[e_{\underline{i}}, t'_i]_1, f_{-\underline{i}}]_1 \, .$$

This shows that the t'_i's are independent of the choice of a reflection order $\preceq$ satisfying condition (16) in Lemma 17.1.1.

Let $\mathbf{U}^\imath_{<0}$ (resp., $\mathbf{U}^\imath_0, \mathbf{U}^\imath_{>0}$) denote the subspace of $\mathbf{U}^\imath$ spanned by all ordered monomials in $f_{-\underline{j},-\underline{i}}$ (resp., $t'_i, e_{\underline{i},\underline{j}}$). Then, we have an isomorphism of vector spaces

$$\mathbf{U}^\imath \simeq \mathbf{U}^\imath_{<0} \bigotimes \left(\mathbf{U}^\imath_0 \bigotimes \mathbf{U}^{\imath,0}\right) \bigotimes \mathbf{U}^\imath_{>0}.$$

We call this linear isomorphism the triangular decomposition of $\mathbf{U}^\imath$ associated with the reflection order $\preceq$, and $\mathbf{U}^\imath_{<0}$ (resp., $\mathbf{U}^\imath_0 \bigotimes \mathbf{U}^{\imath,0}$, $\mathbf{U}^\imath_{>0}$) the negative part (resp., Cartan part, positive part) of $\mathbf{U}^\imath$. The triangular decomposition enables us to establish an analog of highest weight theory for the representation theory of $\mathbf{U}^\imath$.

17.2. **Verma modules and their irreducible quotients.** Recall $P_\imath$ and $P_\imath^\vee$ from page 22. For $\underline{i} \in \mathbb{I}^\imath$, set

$$\beta_{\underline{i}} := h_i - h_{-i} = \epsilon_{\underline{i}} - \epsilon_{\underline{i+1}} - (\epsilon_{-\underline{i+1}} - \epsilon_{-\underline{i}}) \in P_\imath^\vee.$$

For each $\underline{i} \in \mathbb{I}^\imath$, there exists a unique $\delta_{\underline{i}} \in \mathbb{R} \otimes_{\mathbb{Z}} P_\imath$ such that $\langle \beta_{\underline{j}}, \delta_{\underline{i}} \rangle = \delta_{i,j}$ for all $\underline{j} \in \mathbb{I}^\imath$. Set

$$\Lambda^\imath := \sum_{\underline{i} \in \mathbb{I}^\imath} \mathbb{Z}\delta_{\underline{i}}, \text{ and } \gamma_i := \text{the image of } \alpha_i \text{ in } \Lambda^\imath.$$

By the definitions, we have

$$\langle \beta_{\underline{i}}, \gamma_{\underline{j}} \rangle = \langle h_i - h_{-i}, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

Set $Q^\imath_+ := \sum_{i \in \mathbb{I}^\imath} \mathbb{Z}_{\geq 0} \gamma_{\underline{i}}$, and define a partial order $\leq$ on $\Lambda^\imath$ by:

$$\mu \leq \lambda \text{ if and only if } \lambda - \mu \in Q^\imath_+. \tag{17}$$

For a $\mathbf{U}^\imath$-module M and $m \in M$, we say that m is of weight $\lambda \in \Lambda^\imath$ if it satisfies

$$k_{\underline{i}} m = q^{\langle \beta_{\underline{i}}, \lambda \rangle} m$$

for all $\underline{i} \in \mathbb{I}^\imath$; we denote by M_λ the subspace consisting of all $m \in M$ of weight λ.

Lemma 17.2.1. *Let M be a $\mathbf{U}^\imath$-module and $\lambda \in \Lambda^\imath$. For each $i \in \mathbb{I}^\imath$, we have*

$$f_{\underline{i}}(M_\lambda) \subset M_{\lambda - \gamma_{\underline{i}}}, \qquad e_{\underline{i}}(M_\lambda) \subset M_{\lambda + \gamma_{\underline{i}}}, \qquad t(M_\lambda) \subset M_\lambda.$$

Recall the triangular decomposition of $\mathbf{U}^\imath$

$$\mathbf{U}^\imath \simeq \mathbf{U}^\imath_{<0} \bigotimes \left(\mathbf{U}^\imath_0 \bigotimes \mathbf{U}^{\imath,0} \right) \bigotimes \mathbf{U}^\imath_{>0},$$

and the root vectors $f_{-\underline{j},-\underline{i}}, t'_i, e_{\underline{i},\underline{j}}$ associated with the reflection order $\preceq$. Let $(\mathbf{U}^\imath_{>0})_+$ denote the subspace of $\mathbf{U}^\imath_{>0}$ spanned by all ordered monomials in $e_{\underline{i},\underline{j}}$'s other than 1.

Definition 17.2.2. Let $\lambda \in \Lambda^\imath$ and $T'_i \in \mathbb{Q}(p,q)$, $i = 0, \dots, r$. The Verma module $V'(\lambda; \mathbf{T}')$ over $\mathbf{U}^\imath$ with highest weight λ associated with $\mathbf{T}' := (T'_0, \dots, T'_r) \in \mathbb{Q}(p,q)^{r+1}$ is defined to be

$$V'(\lambda; \mathbf{T}') := \mathbf{U}^\imath / I(\lambda; \mathbf{T}'),$$

where $I(\lambda; \mathbf{T}')$ denotes the left ideal of $\mathbf{U}^\imath$ generated by $(\mathbf{U}^\imath_{>0})_+$, and $k_{\underline{i}} - q^{\langle \beta_{\underline{i}}, \lambda \rangle}$ for $\underline{i} \in \mathbb{I}^\imath$, and $t'_i - T'_i$ for $i = 0, \dots, r$.

By the triangular decomposition of $\mathbf{U}^\imath$, a nonzero Verma module $V'(\lambda; \mathbf{T}')$ has a unique maximal submodule, and hence, it has a unique irreducible quotient. We denote it by $L'(\lambda; \mathbf{T}')$ and call it the irreducible highest weight $\mathbf{U}^\imath$-module with highest weight λ associated with $\mathbf{T}'$, or simply, with highest weight $(\lambda; \mathbf{T}')$.

Definition 17.2.3. A nonzero $\mathbf{U}^\imath$-module M is called a highest weight module with highest weight $(\lambda; \mathbf{T}') \in \Lambda^\imath \times \mathbb{Q}(p,q)^{r+1}$ if there exists $m \in M_\lambda$ such that $(\mathbf{U}^\imath_{>0})_+ m = 0$, $t'_i m = T'_i m$ for $i = 0, \dots, r$, and $M = \mathbf{U}^\imath m$. We call such an m a highest weight vector of M with highest weight $(\lambda; \mathbf{T}')$.

Though our definition of highest weight modules over $\mathbf{U}^\imath$ depends on the choice of a reflection order, their $\mathbf{U}^\imath$-module structure is independent of such a choice.

Let $\mathcal{O}^\imath_{\mathrm{int}}$ denote the category of $\mathbf{U}^\imath$-modules M satisfying the following

- $(M1)$ M is decomposed into weight spaces, i.e., $M = \bigoplus_{\lambda \in \Lambda^\imath} M_\lambda$.
- $(M2)$ Each weight space is finite-dimensional.
- $(M3)$ There exists finitely many weights $\mu_1, \dots, \mu_n \in \Lambda^\imath$ such that the set of weights $\lambda \in \Lambda^\imath$ for which $M_\lambda \neq 0$ satisfies $\lambda \leq \mu_i$ for some $i = 1, \dots, n$.
- $(M4)$ $e_{\underline{i}}$ and $f_{\underline{i}}$, $\underline{i} \in \mathbb{I}^\imath$ act on M locally nilpotently.
- $(M5)$ t acts on M diagonally, and its possible eigenvalues are $\langle n \rangle$ for some $n \in \mathbb{Z}$, where we set $\langle n \rangle := \frac{pq^n - p^{-1}q^{-n}}{q - q^{-1}}$.

Remark 17.2.4. Compared with the definition of $\mathcal{O}^{\jmath}_{\mathrm{int}}$, there is an additional axiom in the definition of $\mathcal{O}^{\imath}_{\mathrm{int}}$. Without this axiom, we would obtain too many finite-dimensional irreducible modules. In what follows, we see that this additional axiom leads us to beautiful results.

18. COMPLETE REDUCIBILITY AND THE IRREDUCIBLE MODULES

Throughout this section, we fix $e \in \{1, -1\}$.

18.1. Braid group action on $\mathbf{U}^{\imath}$.

Proposition 18.1.1 ([KP11, 4.5]). *For $\underline{i} \in \mathbb{I}^{\imath}$, there exist unique automorphisms $\tau'_{\underline{i},e}$ and $\tau''_{\underline{i},-e}$ on $\mathbf{U}^{\imath}$ satisfying the following:*

$$\tau'_{\underline{i},e}(e_{\underline{j}}) = \begin{cases} -k_{\underline{i}}^{e} f_{\underline{i}} & \text{if } j = i, \\ e_{\underline{j}} & \text{if } |i-j| > 1, \\ [e_{\underline{j}}, e_{\underline{i}}]_e & \text{if } |i-j| = 1, \end{cases} \qquad \tau'_{\underline{i},e}(f_{\underline{j}}) = \begin{cases} -e_{\underline{i}} k_{\underline{i}}^{-e} & \text{if } j = i, \\ f_{\underline{j}} & \text{if } |i-j| > 1, \\ [f_{\underline{i}}, f_{\underline{j}}]_{-e} & \text{if } |i-j| = 1, \end{cases}$$

$$\tau''_{\underline{i},-e}(e_{\underline{j}}) = \begin{cases} -f_{\underline{i}} k_{\underline{i}}^{-e} & \text{if } j = i, \\ e_{\underline{j}} & \text{if } |i-j| > 1, \\ [e_{\underline{i}}, e_{\underline{j}}]_e & \text{if } |i-j| = 1, \end{cases} \qquad \tau''_{\underline{i},-e}(f_{\underline{j}}) = \begin{cases} -k_{\underline{i}}^{e} e_{\underline{i}} & \text{if } j = i, \\ f_{\underline{j}} & \text{if } |i-j| > 1, \\ [f_{\underline{j}}, f_{\underline{i}}]_{-e} & \text{if } |i-j| = 1, \end{cases}$$

$$\tau'_{\underline{i},e}(k_{\underline{j}}) = \tau''_{\underline{i},-e}(k_{\underline{j}}) = \begin{cases} k_{\underline{i}}^{-1} & \text{if } j = i, \\ k_{\underline{j}} & \text{if } |i-j| > 1, \\ k_{\underline{i}} k_{\underline{j}} & \text{if } |i-j| = 1. \end{cases}$$

$$\tau'_{\underline{i},e}(t) = \begin{cases} [e_{\underline{1}}, [t, f_{\underline{1}}]_1]_{-1} + t k_{\underline{1}} & \text{if } i = 1, \\ t & \text{if } i \neq 1, \end{cases} \qquad \tau''_{\underline{i},-e}(t) = \begin{cases} [e_{\underline{1}}, [t, f_{\underline{1}}]_{-1}]_1 + t k_{\underline{1}}^{-1} & \text{if } i = 1, \\ t & \text{if } i \neq 1. \end{cases}$$

Moreover, $\{\tau'_{\underline{i},e}\}_{\underline{i} \in \mathbb{I}^{\imath}}$ and $\{\tau''_{\underline{i},-e}\}_{\underline{i} \in \mathbb{I}^{\imath}}$ satisfy the braid relation of type A_r.

18.2. Braid group action on $\mathbf{U}^{\imath}$-modules.

In this subsection, we define a braid group action on $\mathbf{U}^{\imath}$-modules in $\mathcal{O}^{\imath}_{\mathrm{int}}$. Since the proofs of the propositions in this subsection are almost the same as those in the ordinary quantum group theory, we omit the details.

Definition 18.2.1. Let $M \in \mathcal{O}^{\imath}_{\mathrm{int}}$. For each $\underline{i} \in \mathbb{I}^{\imath}$, we define two automorphisms $\tau'_{\underline{i},e}$ and $\tau''_{\underline{i},e}$ on M by:

$$\tau'_{\underline{i},e}(m) = \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ a-b+c=n}} (-q)^b q^{e(-ac+b)} f_{\underline{i}}^{(a)} e_{\underline{i}}^{(b)} f_{\underline{i}}^{(c)} m,$$

$$\tau''_{\underline{i},e}(m) = \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ -a+b-c=n}} (-q)^b q^{e(-ac+b)} e_{\underline{i}}^{(a)} f_{\underline{i}}^{(b)} e_{\underline{i}}^{(c)} m,$$

where $n \in \mathbb{Z}$, and $m \in M$ is such that $k_{\underline{i}} m = q^n m$.

Proposition 18.2.2 (see [L93, Proposition 5.2.2]). *Let $M \in \mathcal{O}^{\imath}_{\mathrm{int}}$, $\underline{i} \in \mathbb{I}^{\imath}$, and let $\lambda \in \Lambda^{\imath}$ be such that $(\beta_{\underline{i}}, \lambda) \geq 0$, $j \in \{0, 1, \dots, (\beta_{\underline{i}}, \lambda)\}$; we set $h := (\beta_{\underline{i}}, \lambda) - j$.*

1. *If $\eta \in M_\lambda$ is such that $e_{\underline{i}} \eta = 0$, then $\tau'_{\underline{i},e}(f_{\underline{i}}^{(j)} \eta) = (-1)^j q^{e(jh+j)} f_{\underline{i}}^{(h)} \eta$.*
2. *If $\xi \in M_{-\lambda}$ is such that $f_{\underline{i}} \xi = 0$, then $\tau''_{\underline{i},e}(e_{\underline{i}}^{(j)} \xi) = (-1)^j q^{(e(jh+j))} e_{\underline{i}}^{(h)} \xi$.*

Proposition 18.2.3 (see [L93, Proposition 5.2.3]). *Let $M \in \mathcal{O}^{\imath}_{\mathrm{int}}$, $\underline{i} \in \mathbb{I}^{\imath}$, and $m \in M_\lambda$.*

1. *We have* $\tau'_{\underline{i},e}\tau''_{\underline{i},-e} = \mathrm{id}_M = \tau''_{\underline{i},-e}\tau'_{\underline{i},e}$.
2. *We have* $\tau''_{\underline{i},e}(m) = (-1)^{(\beta_{\underline{i}},\lambda)}q^{e(\beta_{\underline{i}},\lambda)}\tau'_{\underline{i},e}(m)$.

In what follows, we write $\tau_{\underline{i}} = \tau''_{\underline{i},1}$ for $\underline{i} \in \mathbb{I}^\imath$.

18.3. **Classification of the irreducible modules in $\mathcal{O}^\imath_{\mathrm{int}}$.** Recall the triangular decomposition $\mathbf{U}^\imath = \mathbf{U}^\imath_{<0} \otimes (\mathbf{U}^\imath_0 \otimes \mathbf{U}^{\imath,0}) \otimes \mathbf{U}^\imath_{>0}$. We remark that an irreducible highest weight module is determined by the eigenvalues of $k_{\underline{i}}$'s and t'_i's for a highest weight vector. However, t'_i's are sometimes difficult to deal with.

Proposition 18.3.1. *Let* $V'(\lambda;\mathbf{T}')$ *be the Verma module with highest weight* $(\lambda;\mathbf{T}')$. *Then,* $\mathbf{T}'$ *is determined by the* $\tau_{\underline{i}}\cdots\tau_{\underline{1}}(t'_0)$*-eigenvalue of* v *for* $i = 0,\dots,r$.

This proposition enables us to replace t'_i with $\tau_{\underline{i}}\cdots\tau_{\underline{1}}(t'_0)$ for $i \in \{0,\dots,r\}$. Then, we define t_i by $t_0 = t$, $t_i = \tau_{\underline{i}}\cdots\tau_{\underline{1}}(t_0)$ for $i \in \{1,\dots,r\}$. Also, we set $V(\lambda;\mathbf{T}) := V'(\lambda;\mathbf{T}')$ and $L(\lambda;\mathbf{T}) := L'(\lambda;\mathbf{T}')$, where $\mathbf{T} = (T_0,\dots,T_r)$ is uniquely determined by the equations $t_i v = T_i v$, $i = 0,\dots,r$ for a highest weight vector $v \in V'(\lambda;\mathbf{T}')$.

Let $L \in \mathcal{O}^\imath_{\mathrm{int}}$ be an irreducible $\mathbf{U}^\imath$-module. By condition (M3), there exists $\lambda \in \Lambda^\imath$ such that $L_\lambda \neq 0$ and $L_\mu = 0$ for all $\mu > \lambda$.

Lemma 18.3.2. *We have*

$$[t_0,t_1]_0 = (q-q^{-1})[t_0, f_{-\underline{1},\underline{2}}e_{\underline{1}}]_0 \in \mathbf{U}^\imath(e_{\underline{1}}, e_{\underline{1}}t_0),$$

where $\mathbf{U}^\imath(e_{\underline{1}}, e_{\underline{1}}t_0)$ *denotes the left ideal of* $\mathbf{U}^\imath$ *generated by* $e_{\underline{1}}$ *and* $e_{\underline{1}}t_0$.

This lemma implies that $[t_0,t_1]_0 L_\lambda = 0$; namely, the actions of t_0 and t_1 commute with each other on L_λ.

Lemma 18.3.3. *Let* $\underline{i},\underline{j} \in \mathbb{I}^\imath$. *If* $j \neq i, i+1$, *then we have* $\tau_{\underline{j}}(t_i) = t_i$.

Proposition 18.3.4. *Let* $L \in \mathcal{O}^\imath_{\mathrm{int}}$ *be an irreducible module. Take* $\lambda \in \Lambda^\imath$ *such that* $L_\lambda \neq 0$ *and* $L_\mu = 0$ *for all* $\mu > \lambda$. *Then, the actions of* $t_0,\dots,t_r$ *commute with each other on* L_λ.

As a corollary of this proposition, we can take a simultaneous eigenvector $v \in L_\lambda$ for $t_0,\dots,t_r$. Let $T_i \in \mathbb{Q}(p,q)$ denote the eigenvalue of t_i. Then the submodule generated by v is a highest weight module with highest weight $(\lambda;T_0,\dots,T_r)$. Since L is irreducible, we conclude that L is a highest weight module.

Lemma 18.3.5. *Let* $M \in \mathcal{O}^\imath_{\mathrm{int}}$, $v \in M$ *be such that* $e_{\underline{1}}v = 0$, $k_{\underline{1}}v = q^a v$ *for some* $a \in \mathbb{Z}_{\geq 0}$, *and* $t_0 v = \langle n\rangle$ *for some* $n \in \mathbb{Z}$. *Let* $d \in \mathbb{Z}_{>0}$. *Then, there exist unique* $v_0, v_1, \dots, v_d \in M$ *satisfying the following:*

1. $f_{\underline{1}}^{(m)}v = \sum_{k=0}^{m} v_k$.
2. $t_0 v_k = \langle n-m+2k\rangle v_k$ *for all* $k = 0,\dots,m$.

Proof. As the proof of this lemma needs some lengthy calcluation, we put it in the end of this section. □

Lemma 18.3.6. *Let* $M \in \mathcal{O}^\imath_{\mathrm{int}}$ *be a* $\mathbf{U}^\imath_1$*-module,* $v \in M$ *a highest weight vector with* $k_{\underline{1}}v = q^{a_{\underline{1}}}v$, $t_i = T_i v$ *for some* $a_{\underline{1}} \in \mathbb{Z}_{\geq 0}$, $T_0, T_1 \in \mathbb{Q}(p,q)$. *If* $T_0 = \langle n\rangle$ *for some* $n \in \mathbb{Z}$, *then* $T_1 = \langle n - a_{\underline{1}} + 2b_{\underline{1}}\rangle$ *for some* $0 \leq b_{\underline{1}} \leq a_{\underline{1}}$.

Theorem 18.3.7. *Each irreducible module in* $\mathcal{O}^\imath_{\mathrm{int}}$ *is a highest weight module with highest weight* $(\lambda;\mathbf{T})$ *for some* $\lambda \in \Lambda^\imath$ *and* $\mathbf{T} = (T_0,\dots,T_r) \in \mathbb{Q}(p,q)^{r+1}$ *satisfying the following:*

(1) $a_{\underline{i}} := (\beta_{\underline{i}}, \lambda) \geq 0$ *for each* $\underline{i} \in \mathbb{I}^\imath$.
(2) $T_0 = \langle n \rangle$ *for some* $n \in \mathbb{Z}$.
(3) *For each* $\underline{i} \in \mathbb{I}^\imath$, *there exists* $b_{\underline{i}} \in \mathbb{Z}_{\geq 0}$ *such that* $0 \leq b_{\underline{i}} \leq a_{\underline{i}}$ *and* $T_{i+1} = \langle n - (a_{\underline{1}} + \cdots + a_{\underline{i}}) + 2(b_{\underline{1}} + \cdots + b_{\underline{i}}) \rangle$.

From now on, we write $L(n; \mathbf{a}; \mathbf{b})$ instead of $L(\lambda; \mathbf{T})$, where $n \in \mathbb{Z}$, $\mathbf{a} = (a_{\underline{1}}, \dots, a_{\underline{r}})$ and $\mathbf{b} = (b_{\underline{1}}, \dots, b_{\underline{r}})$ are such that $a_{\underline{i}} = (\beta_{\underline{i}}, \lambda)$, $T_{i+1} = \langle n - (a_{\underline{1}} + \cdots + a_{\underline{i}}) + 2(b_{\underline{1}} + \cdots + b_{\underline{i}}) \rangle$.

As in the $\mathbf{U}^\jmath$-case, there exists a bijection from

$$\{(n.\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^r_{\geq 0} \times \mathbb{Z}^r_{\geq 0} \mid 0 \leq b_i \leq a_i.\}$$

to

$$P^\imath = P^\imath_r := \{\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r+1)} \mid \boldsymbol{\lambda}^-_{r+1} = 0 \text{ or } \boldsymbol{\lambda}^+_{r+1} = 0\}.$$

18.4. **Complete reducibility.** Consider the anti-algebra automorphism $S : \mathbf{U}^\imath \to \mathbf{U}^\imath$ over $\mathbb{Q}(p,q)$ defined by:

$$S(e_{\underline{i}}) = -e_{\underline{i}} k_{\underline{i}}, \quad S(f_{\underline{i}}) = -k_{\underline{i}}^{-1} f_{\underline{i}}, \quad S(k_{\underline{i}}) = k_{\underline{i}}^{-1}, \quad S(t) = t.$$

It is easily checked that S is an anti-algebra homomorphism. In addition, S has the inverse:

$$S^{-1}(e_{\underline{i}}) = -k_{\underline{i}} e_{\underline{i}}, \quad S^{-1}(f_{\underline{i}}) = -f_{\underline{i}} k_{\underline{i}}^{-1}, \quad S^{-1}(k_{\underline{i}}) = k_{\underline{i}}^{-1}, \quad S^{-1}(t) = t.$$

For a $\mathbf{U}^\imath$-module M, define a $\mathbf{U}^\imath$-module $S_*(M) := M^\vee$ by:

$$(x \cdot g)(m) = g(S^{-1}(x) \cdot m) \quad \text{for } x \in \mathbf{U}^\imath,\ g \in S(M),\ m \in M,$$

where $M^\vee$ denotes the restricted dual of M, i.e., $M^\vee = \bigoplus_{\lambda \in \Lambda^\imath} \mathrm{Hom}_{\mathbb{Q}(p,q)}(M_\lambda, \mathbb{Q}(p,q))$. Similarly, we define a $\mathbf{U}^\imath$-module $S^*(M)$ by replacing S^{-1} with S.

Lemma 18.4.1. *Let* $L \in \mathcal{O}^\imath_{\mathrm{int}}$ *be the irreducible highest weight* $\mathbf{U}^\imath$*-module with highest weight* $(\lambda; \mathbf{T})$. *Then,* $S_*(L)$ *is the irreducible lowest weight* $\mathbf{U}^\imath$*-module with lowest weight* $(-\lambda; \mathbf{T}')$ *for some* $\mathbf{T}' \in \mathbb{Q}(p,q)^{r+1}$.

Lemma 18.4.2. *Let* M *be a* $\mathbf{U}^\imath$*-module. Suppose that* M *contains an irreducible submodule* $L \simeq L(\lambda; \mathbf{T})$ *for some* $\lambda \in \Lambda^\imath$ *and* $\mathbf{T} \in \mathbb{Q}(p,q)^r$. *Then,* $M \simeq L \oplus (M/L)$.

Now, the complete reducibility of the $\mathbf{U}^\imath$-modules in $\mathcal{O}^\imath_{\mathrm{int}}$ follows from a standard argument; see, for example, [HK02, Section 3.5].

Theorem 18.4.3. *Every* $\mathbf{U}^\imath$*-module in* $\mathcal{O}^\imath_{\mathrm{int}}$ *is completely reducible.*

Corollary 18.4.4. *Every highest weight module in* $\mathcal{O}^\imath_{\mathrm{int}}$ *is irreducible.*

Theorem 18.4.5. *Let* $M \in \mathcal{O}^\imath_{\mathrm{int}}$. *Irreducible decomposition of* M *is unique in the following sense. If we have two irreducible decompositions* $M = \bigoplus_{j \in J} L_j = \bigoplus_{k \in K} L^k$ *for some index sets* J *and* K, *then there exists a bijection* $\phi : J \to K$ *such that* $L_j \simeq L^{\phi(j)}$ *for all* $j \in J$. *Moreover, for each* $j \in J$, *the number of* $j' \in J$ *such that* $L_{j'} \simeq L_j$ *is finite.*

18.5. **Proof of Lemma 18.3.5.** The root vectors of $\mathbf{U}^\imath_1$ are

$$\begin{aligned} &f_{\underline{1},\underline{2}} = f_{\underline{1}},\ f_{-\underline{1},\underline{2}} = [t, f_{\underline{1}}]_1, \\ &t'_0 = t,\ t'_1 = \big[e_{\underline{1}}, [t, f_{\underline{1}}]_1\big]_1, \\ &e_{-\underline{1},\underline{2}} = [e_{\underline{1}}, t]_1,\ e_{\underline{1},\underline{2}} = e_{\underline{1}}. \end{aligned}$$

Lemma 18.5.1. *For $n \in \mathbb{Z}_{\geq 0}$, the following hold.*

$$\begin{aligned} e_{\underline{1}} f_{\underline{1}}^{(n)} &= f_{\underline{1}}^{(n)} e_{\underline{1}} + f_{\underline{1}}^{(n-1)} \frac{q^{-n+1} k_{\underline{1}} - q^{n-1} k_{\underline{1}}^{-1}}{q - q^{-1}}, \\ f_{-\underline{1},\underline{2}} f_{\underline{1}}^{(n)} &= q^{-n} f_{\underline{1}}^{(n)} f_{-\underline{1},\underline{2}}, \\ t'_0 f_{\underline{1}}^{(n)} &= f_{\underline{1}}^{(n-1)} f_{-\underline{1},\underline{2}} + q^n f_{\underline{1}}^{(n)} t'_0, \\ t'_1 f_{\underline{1}}^{(n)} &= q^{-n} f_{\underline{1}}^{(n)} t'_1 - q^{(-2n+1)} [2] f_{\underline{1}}^{(n-1)} f_{-\underline{1},\underline{2}} k_{\underline{1}}, \\ e_{-\underline{1},\underline{2}} f_{\underline{1}}^{(n)} &= f_{\underline{1}}^{(n-1)} t'_1 - q^{(-n+1)} f_{\underline{1}}^{(n-2)} f_{-\underline{1},\underline{2}} k_{\underline{1}} + q^n f_{\underline{1}}^{(n)} e_{-\underline{1},\underline{2}}. \end{aligned}$$

Lemma 18.5.2. *Let $M \in \mathcal{O}^{\imath}_{\mathrm{int}}$. For $n \in \mathbb{Z}$, let M_n denote the t'_0-eigenspace of M with eigenvalue $\langle n \rangle$. Let $m \in M_n$. Then, $(f_{-\underline{1},\underline{2}} \pm (pq^n)^{\pm 1} f_{\underline{1}})m \in M_{n\pm 1}$ and $([e_{\underline{1}}, t]_{-1} \pm (pq^n)^{\pm 1} e_{\underline{1}})m \in M_{n\pm 1}$.*

Proof. We prove only the assertion $(f_{-\underline{1},\underline{2}} \pm (pq^n)^{\pm 1} f_{\underline{1}})m \in M_{n\pm 1}$ since the another is similar. By the defining relation of $\mathbf{U}^{\imath}$, we have

$$(t'_0)^2 f_{\underline{1}} m = ([2] t'_0 f_{\underline{1}} t'_0 - f_{\underline{1}} (t'_0)^2 + f_{\underline{1}}) m = [2]\langle n \rangle t'_0 f_{\underline{1}} m - (\langle n \rangle^2 - 1) f_{\underline{1}} m.$$

Also, it holds that

$$\langle n+1 \rangle + \langle n-1 \rangle = [2]\langle n \rangle, \quad \langle n \rangle^2 - 1 = \langle n+1 \rangle \cdot \langle n-1 \rangle.$$

From these equalities, it is easy to verify that $(t'_0 f_{\underline{1}} - \langle n \mp 1 \rangle f_{\underline{1}})m$ is an t'_0-eigenvector of eigenvalue $\langle n \pm 1 \rangle$. Recalling $f_{-\underline{1},\underline{2}} = [t'_0, f_{\underline{1}}]_1$, we have

$$(t'_0 f_{\underline{1}} - \langle n \mp 1 \rangle f_{\underline{1}})m = (f_{-\underline{1},\underline{2}} + q\langle n \rangle f_{\underline{1}} - \langle n \mp 1 \rangle f_{\underline{1}})m = (f_{-\underline{1},\underline{2}} \pm (pq^n)^{\pm 1} f_{\underline{1}})m.$$

This proves the lemma. □

Since $f_{\underline{1}} = \frac{(f_{-\underline{1},\underline{2}} + pq^n f_{\underline{1}}) - (f_{-\underline{1},\underline{2}} - p^{-1} q^{-n} f_{\underline{1}})}{pq^n + p^{-1} q^{-n}}$, we have

$$f_{\underline{1}} v \in M_{n-1} \oplus M_{n+1}$$

for all $v \in M_n$. Hence,

$$f_{\underline{1}}^{(m)} v \in \bigoplus_{k=0}^{m} M_{n-m+2k}$$

for all $m \in \mathbb{Z}_{\geq 0}$. This proves Lemma 18.3.5.

19. QUASI-$\imath$-CRYSTAL BASES

In the same way as the quasi-$\jmath$-crystal bases, we define the quasi-$\imath$-crystal bases for module in $\mathcal{O}^{\imath}_{\mathrm{int}}$. In this case, the Kashiwara operators are $\widetilde{e}_{\underline{i}}, \widetilde{f}_{\underline{i}}$, $\underline{i} \in \mathbb{I}^{\imath}$. Since $(e_{\underline{i}}, k_{\underline{i}}, f_{\underline{i}})$ is an $\mathfrak{sl}_2$-triple, these operators coincide with the ordinary Kashiwara operators.

19.1. **Tensor product rule.**

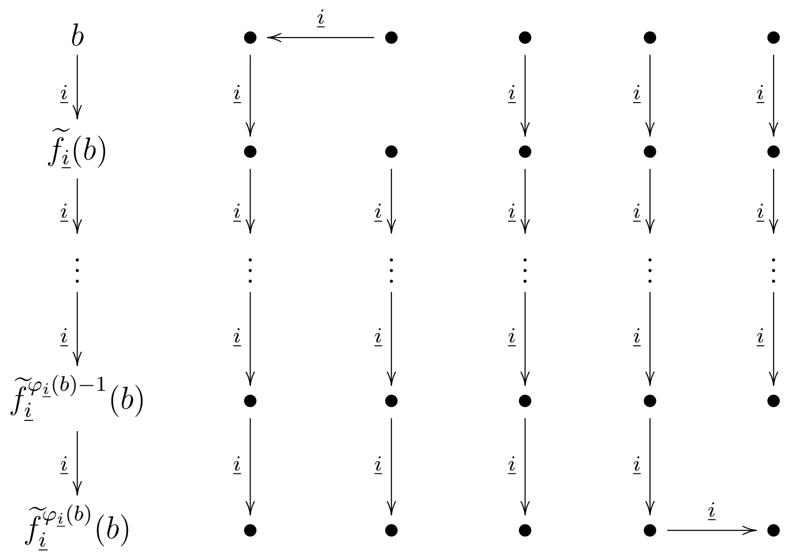
Proposition 19.1.1. *Let M be a $\mathbf{U}^{\imath}$-module having a quasi-$\jmath$-crystal basis $(\mathcal{L}, \mathcal{B})$. Then $(\mathcal{L} \otimes \mathbf{L}, \mathcal{B} \otimes \mathbf{B})$ is a quasi-$\jmath$-crystal basis of $M \otimes \mathbf{V}$, on which the Kashiwara operators act* *as follows: For* $\underline{i} \in \mathbb{I}^\imath, b \in \mathcal{B}, j \in \{-\underline{r+1}, \underline{r+1}\}$,

$$\widetilde{f}_{\underline{i}}(b \otimes u_{\underline{j}}) = \begin{cases} 0 & \textit{if } \underline{j} = \underline{i} \textit{ and } \widetilde{f}_{\underline{i}}^2(b) = 0, \\ b \otimes u_{\underline{i}} & \textit{if } \underline{j} = \underline{i} - 1 \textit{ and } \widetilde{f}_{\underline{i}}(b) = 0, \\ b \otimes u_{-\underline{i}} & \textit{if } \underline{j} = -\underline{i-1} \textit{ and } \widetilde{e}_{\underline{i}}(b) = 0, \\ \widetilde{f}_{\underline{i}}(b) \otimes u_{\underline{j}} & \textit{otherwise}, \end{cases}$$

$$\widetilde{e}_{\underline{i}}(b \otimes u_{\underline{j}}) = \begin{cases} b \otimes u_{\underline{i-1}} & \textit{if } \underline{j} = \underline{i} \textit{ and } \widetilde{f}_{\underline{i}}(b) = 0, \\ 0 & \textit{if } j = -\underline{i-1} \textit{ and } \widetilde{e}_{\underline{i}}^2(b) = 0, \\ b \otimes u_{-\underline{i-1}} & \textit{if } \underline{j} = -\underline{i} \textit{ and } \widetilde{e}_{\underline{i}}(b) = 0, \\ \widetilde{e}_{\underline{i}}(b) \otimes u_{\underline{j}} & \textit{otherwise}. \end{cases}$$

The action of $\widetilde{f}_{\underline{i}}$ for $\underline{i} \in \mathbb{I}^\imath$ is visualized as:

$$u_{-\underline{i}} \xrightarrow{-(i-1)} u_{-\underline{i-1}} \qquad u_{\underline{j}} \qquad u_{\underline{i-1}} \xrightarrow{i-1} u_{\underline{i}} \; .$$



Theorem 19.1.2. *Let* M *be a* $\mathbf{U}^\imath$*-module having a quasi-*$\imath$*-crystal basis* $(\mathcal{L}, \mathcal{B})$*, and* N *a* $\mathbf{U}$*-module having a crystal basis* $(\mathcal{L}', \mathcal{B}')$*. Then,* $M \otimes N$ *has a quasi-*$\imath$*-crystal basis* $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$*, on which the Kashiwara operators act as follows: for* $b \in \mathcal{B}$ *and* $b' \in \mathcal{B}'$,

$$\widetilde{f}_{\underline{i}}(b \otimes b') = \begin{cases} b \otimes \widetilde{E}_{-(i-1)}(b') & \textit{if } \varepsilon_{i-1}(b') < \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) < \varepsilon_{-(i-1)}(b'), \textit{ or} \\ & \textit{if } \varepsilon_{i-1}(b') \geq \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) + \varepsilon_{i-1}(b') - \varphi_{\underline{i}}(b) < \varepsilon_{-(i-1)}(b'), \\ \widetilde{f}_i(b) \otimes b' & \textit{if } \varepsilon_{i-1}(b') < \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) \geq \varepsilon_{-(i-1)}(b'), \\ b \otimes \widetilde{F}_{i-1}(b') & \textit{if } \varepsilon_{i-1}(b') \geq \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) + \varepsilon_{i-1}(b') - \varphi_{\underline{i}}(b) \geq \varepsilon_{-(i-1)}(b'), \end{cases}$$

$$\widetilde{e}_{\underline{i}}(b \otimes b') = \begin{cases} b \otimes \widetilde{F}_{-(i-1)}(b') & \textit{if } \varepsilon_{i-1}(b') \leq \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) \leq \varepsilon_{-(i-1)}(b'), \textit{ or} \\ & \textit{if } \varepsilon_{i-1}(b') > \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) + \varepsilon_{i-1}(b') - \varphi_{\underline{i}}(b) \leq \varepsilon_{-(i-1)}(b'), \\ \widetilde{e}_{\underline{i}}(b) \otimes b' & \textit{if } \varepsilon_{i-1}(b') \leq \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) > \varepsilon_{-(i-1)}(b'), \\ b \otimes \widetilde{E}_{i-1}(b') & \textit{if } \varepsilon_{i-1}(b') > \varphi_{\underline{i}}(b) \textit{ and } \varepsilon_{\underline{i}}(b) + \varepsilon_{i-1}(b') - \varphi_{\underline{i}}(b) > \varepsilon_{-(i-1)}(b'). \end{cases}$$

Corollary 19.1.3. *Let* $N \in \mathcal{O}_{\mathrm{int}}$ *be a* $\mathbf{U}$*-module with a crystal basis* $(\mathcal{L}', \mathcal{B}')$*. Then,* $(\mathcal{L}', \mathcal{B}')$ *is also a quasi-*$\imath$*-crystal basis of* N*. Furthermore, for each* $b \in \mathcal{B}$ *and* $i \in \mathbb{I}^\imath$*, we* *have the following:*

$$\widetilde{f}_{\underline{i}}(b) = \begin{cases} \widetilde{E}_{-(i-1)}(b) & \textit{if } \varepsilon_{i-1}(b) < \varepsilon_{-(i-1)}(b), \\ \widetilde{F}_{i-1}(b) & \textit{if } \varepsilon_{i-1}(b) \geq \varepsilon_{-(i-1)}(b), \end{cases}$$

$$\widetilde{e}_{\underline{i}}(b) = \begin{cases} \widetilde{F}_{-(i-1)}(b) & \textit{if } \varepsilon_{i-1}(b) \leq \varepsilon_{-(i-1)}(b), \\ \widetilde{E}_{i-1}(b) & \textit{if } \varepsilon_{i-1}(b) > \varepsilon_{-(i-1)}(b). \end{cases}$$

Proof. Apply Theorem 19.1.2 for $M = L(\emptyset;\emptyset)$, which is the trivial module of $\mathbf{U}^\jmath$. □

Recall that $\mathbf{B}^{\otimes N}$ is identified with $[-\underline{r+1}, \underline{r+1}]^N$, and its crystal structure is described in the beginning of this section. Applying Corollary 19.1.3 to the crystal basis $(\mathbf{L}^{\otimes N}, \mathbf{B}^{\otimes N})$, we obtain a quasi-$\jmath$-crystal structure of $\mathbf{B}^{\otimes N}$.

Corollary 19.1.4. *The quasi-$\jmath$-crystal basis $\mathbf{B}^{\otimes N} = [-\underline{r+1}, \underline{r+1}]^N$ obtained from Corollary 19.1.3 is described as follows: for $\mathbf{s} \in I^N$, $\widetilde{e}_1\mathbf{s}$ (resp., $\widetilde{f}_1\mathbf{s}$) is obtained from $\mathbf{s}$ by replacing the rightmost -1 in $\mathbf{s}_1$ with 0 (resp., the leftmost 0 in $\mathbf{s}_1$ with -1), and is 0 if there are no -1 (resp., 0) in $\mathbf{s}_1$. For $i \in \mathbb{I}^\jmath \setminus \{1\}$, $\widetilde{e}_i\mathbf{s}$ is obtained from $\mathbf{s}$ by replacing the rightmost i in $\mathbf{s}_i$with $i-1$ if $i \in \mathbf{s}_i$, or by replacing the rightmost $-i$ in $\mathbf{s}_i$ with $-(i-1)$ if $i \notin \mathbf{s}_i$, or is 0 if $i, -i \notin \mathbf{s}_i$. Finally, $\widetilde{f}_i\mathbf{s}$ is obtained from $\mathbf{s}$ by replacing the leftmost $-(i-1)$ in $\mathbf{s}_i$with $-i$ if $-(i-1) \in \mathbf{s}_i$, or by replacing the leftmost $i-1$ in $\mathbf{s}_i$ with i if $-(i-1) \notin \mathbf{s}_i$, or is 0 if $i-1, -(i-1) \notin \mathbf{s}_i$.*

19.2. **Abstract quasi-$\imath$-crystals.** As in the ordinary crystal basis theory, we define the notion of (abstract) quasi-$\imath$-crystals as sets with structure maps $\widetilde{e}_{\underline{i}}, \widetilde{f}_{\underline{i}}$, $\underline{i} \in \mathbb{I}^\imath$, and $\mathrm{wt}^\imath$.

By Corollary 19.1.3, we can define a functor from the category of crystals of $\mathfrak{sl}_{2r+2}$-type to the category of quasi-$\imath$-crystals. Since $\widetilde{E}_0$ and $\widetilde{F}_0$ do not appear in the formula of Corollary 19.1.3, we can regard this functor to factor through the forgetting functor $\mathrm{Res}^{2r+1}_{r+1,r+1}$.

Lemma 19.2.1. *Let $\mathcal{B}$ be a crystal of $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_{r+1})$-type. Then, $\mathcal{B}$ is equipped with a quasi-$\imath$-crystal structure such that the structure maps $\widetilde{e}_{\underline{i}}, \widetilde{f}_{\underline{i}}$ are defined by the formula in Corollary 19.1.3.*

Recall that we have given to $\mathrm{SST}_{(r+1,r+1)}(\boldsymbol{\lambda})$ a crystal structure of $\mathfrak{s}(\mathfrak{gl}_{r+1} \oplus \mathfrak{gl}_{r+1})$-type for each $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r+1)}$. Now, we can regard $\mathrm{SST}_{(r+1,r+1)}(\boldsymbol{\lambda})$ as a quasi-$\imath$-crystal.

20. QUASI-$\imath$-CRYSTAL BASES OF IRREDUCIBLE $\mathbf{U}^\imath$-MODULES

20.1. **Quasi-$\imath$-crystal basis of $\mathbf{V}^{\otimes d}$.** In this section, we study the quasi-$\imath$-crystal basis $(\mathbf{L}^{\otimes d}, \mathbf{B}^{\otimes d})$ of $\mathbf{V}^{\otimes d}$ in detail.

Recall the functor $\mathcal{F}_{\pi^\imath} : \mathcal{H}$-mod $\to \mathbb{S}(\pi^\imath)$-mod defined by $\mathcal{F}^\imath_\pi(M) := \mathbb{T}(\pi^\imath) \otimes_{\mathcal{H}} M$. This functor induces another functor

$$\mathcal{F}^\imath : \mathbf{H}\text{-mod} \to \mathbf{U}^\imath\text{-mod};\ M \mapsto \mathbf{V}^{\otimes d} \otimes_{\mathbf{H}} M.$$

By the $(\mathbf{U}^\imath, \mathbf{H})$-duality, this functor sends an irreducible module to an irreducible module or 0. Set $L(W)_0 := \{X \in L(W) \mid \mathcal{F}^\imath(\mathbf{C}^L_X) \neq 0\}$. Then, for each $X \in L(W)_0$, there exists a unique $\boldsymbol{\lambda}(X) \in P^\imath$ such that $\mathcal{F}^\imath(\mathbf{C}^L_X) \simeq L(\boldsymbol{\lambda}(X))$. With this notation, we have

$$\mathbf{V}^{\otimes d} \simeq \bigoplus_{X \in L(W)_0} L(\boldsymbol{\lambda}(X)).$$

On the other hand, By Proposition 6.3.1, we have $\mathcal{F}^\imath(\mathbf{C}^L_X) \simeq \mathbf{C}^L_X(\pi^\imath)$ for all $X \in L(W)$. Hence, we obtain $L(\boldsymbol{\lambda}(X)) \simeq \mathbf{C}^L_X(\pi^\imath)$ for all $X \in L(W)_0$.

Proposition 20.1.1. *Let $X \in L(W)_0$. Then, $\mathbf{C}_{\leq_L X}(\pi^\imath)$, $\mathbf{C}_{<_L X}(\pi^\imath)$, and $\mathbf{C}^L_X(\pi^\imath)$ have quasi-$\imath$-crystal bases.*

Proposition 20.1.2. *For each $X \in L(W)_0$, there exists a highest weight vector $v_X \in \mathbf{V}^{\otimes d}$ satisfying the following:*

1. *$L_X := \mathbf{U}^\imath v_X \simeq \mathbf{C}^L_X(\pi^\imath)$.*
2. *$(\mathcal{L}_X, \mathcal{B}_X)$ is a quasi-$\imath$-crystal basis of L_X isomorphic to $(\mathcal{L}(X), \mathcal{B}(X))$, where $\mathcal{L}_X := \mathbf{L}^{\otimes d} \cap L_X$, $\mathcal{B}_X := \mathbf{B}^{\otimes d} \cap (\mathcal{L}_X / q\mathcal{L}_X)$.*
3. *$\mathbf{L}^{\otimes d} = \bigoplus_{X \in L(W)_0} \mathcal{L}_X$, and $\mathbf{B}^{\otimes d} = \bigsqcup_{X \in L(W)_0} \mathcal{B}_X$.*

20.2. **Combinatorial description of $\mathcal{B}(X)$.** By Propositions 20.1.1 and 20.1.2, we have a decomposition

$$\mathbf{B}^{\otimes d} = \bigsqcup_{X \in L(W)_0} \mathcal{B}(X), \quad \mathcal{B}(X) = \{u_{\mathbf{i}w} \mid \mathbf{i} \in \pi^\imath,\ w \in {}^{\mathbf{i}}W, \text{ and } w_{\mathbf{i}}w \in X\} \tag{18}$$

of quasi-$\imath$-crystals. Also, by Theorem 5.3.4 (2), $u_{\mathbf{i}}$ and $u_{\mathbf{j}}$ ($\mathbf{i}, \mathbf{j} \in [-\underline{r+1}, \underline{r+1}]^d$) belong to $\mathcal{B}(X)$ for some $X \in L(W)_0$ if and only if $Q^{\pm}(\mathbf{i}) = Q^{\pm}(\mathbf{j})$. Hence, each $\mathcal{B}(X)$ is of the form

$$\mathcal{B}(X) = \{u_{\mathbf{i}} \mid \mathbf{i} \in [-\underline{r+1}, \underline{r+1}]^d,\ (Q^-(\mathbf{i}), Q^+(\mathbf{i})) = \mathbf{Q}\}, \quad \mathbf{Q} \in \mathrm{ST}(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r+1)}(d).$$

On the other hand, we have a decomposition

$$\mathbf{B}^{\otimes d} \to \bigsqcup_{\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r+1)}(d)} \bigsqcup_{\mathbf{Q} \in \mathrm{ST}(\boldsymbol{\lambda})} \big(\mathrm{SST}_{(r+1,r+1)}(\boldsymbol{\lambda}) \times \{\mathbf{Q}\}\big) \tag{19}$$

of an $\mathfrak{s}(\mathfrak{gl}_{r+1} \otimes \mathfrak{gl}_{r+1})$-crystal. By Lemma 19.2.1 and arguments in Subsection 19.2, this decomposition is also a decomposition of a quasi-$\imath$-crystal.

Theorem 20.2.1. *We have $L(W)_0 = \{X \in L(W) \mid \operatorname{sh} X \in \mathrm{Bip}_{(r+1,r+1)}(d)\}$. Also, for each $X \in L(W)_0$ with $\operatorname{sh} X = \boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r+1)}(d)$, we have $\mathcal{B}(X) = \{(\mathbf{T}, \mathbf{Q}) \mid \mathbf{T} \in \mathrm{SST}_{(r+1,r+1)}(\boldsymbol{\lambda})\}$, where $\mathbf{Q} := \mathbf{Q}(X)$, and $L_X \simeq L(\boldsymbol{\lambda})$.*

20.3. **$\jmath$-crystal bases.** In general, a quasi-$\imath$-crystal graph of an irreducible $\mathbf{U}^\imath$-module is neither connected nor unique. In this subsection, we introduce the notion of $\imath$-crystal bases as quasi-$\imath$-crystal bases satisfying some additional conditions. And we prove the existence and uniqueness theorem for $\imath$-crystal bases, and that they are connected.

Let $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r+1)}(d)$, and take a left cell $X \in L(W)$ satisfying $\mathbf{C}^L_X \simeq L(\boldsymbol{\lambda})$. Recall that $\mathbf{C}^L_X$ has a basis $\{[{}^{\mathbf{i}}C_w]_X \mid \mathbf{i} \in \pi^\imath,\ w \in {}^{\mathbf{i}}W, \text{ and } w_{\mathbf{i}}w \in X\}$, and it is in one-to-one correspondence with $\mathrm{SST}(\boldsymbol{\lambda})$; we denote by b_T the basis element corresponding to $T \in \mathrm{SST}(\boldsymbol{\lambda})$. For each $i \in [1, r]$, we define linear endomorphisms $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$ on $L(\boldsymbol{\lambda})$ by

$$\widetilde{e}_{i'}(b_T) = \begin{cases} b_{\widetilde{E}_{i-\frac{1}{2}}T} & \text{if } \widetilde{e}_j T = 0 \text{ for all } j = 1, \dots, i-1 \text{ and } \widetilde{E}_{j-\frac{1}{2}}T = 0 \text{ for all } j = 1, \dots, i-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widetilde{f}_{i'}(b_T) = \begin{cases} b_{T'} & \text{if } \widetilde{e}_{i'} b_{T'} = b_T, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the condition $\widetilde{e}_j T = 0$ for all $j = 1, \dots, i-1$ and $\widetilde{E}_{j-\frac{1}{2}}T = 0$ for all $j = 2, \dots, i-1$ is equivalent to $\widetilde{e}_j T = 0$ for all $j = 1, \dots, i-1$ and $\widetilde{e}_{j'} T = 0$ for all $j = 2, \dots, i-1$.

Let $X' \in L(W)$ be such that $C^L_{X'} \simeq C^L_X$. Since the linear map $[C_{w'}]_{X'} \mapsto [C_w]_X$, $w' \in X'$, $w \in X$ with $P^\pm(w') = P^\pm(w)$ gives an isomorphism $C^L_{X'} \to C^L_X$ of $\mathcal{H}$-modules, the definition of $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$ are independent of the choice of X as long as we have $\mathbf{C}^L_X \simeq L(\boldsymbol{\lambda})$.

Also, we define linear endomorphisms $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$, $i \in \{2, \dots, r\}$ on each $\mathbf{U}^\jmath$-module in $\mathcal{O}^\jmath_{\mathrm{int}}$ by the complete reducibility.

Remark 20.3.1. Later, we will give more intrinsic definitions of $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$.

Definition 20.3.2. Let $M \in \mathcal{O}^\imath_{\mathrm{int}}$ be a $\mathbf{U}^\imath$-module with a quasi-$\imath$-crystal basis $(\mathcal{L}, \mathcal{B})$. We say that $(\mathcal{L}, \mathcal{B})$ is an $\imath$-crystal basis if it satisfies the following:

($\imath$C 1) $\mathcal{L}$ is preserved by the operators $\widetilde{e}_{i'}$ and $\widetilde{f}_{i'}$, $i \in [1, r]$.
($\imath$C 2) We have $\widetilde{e}_{i'}(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ and $\widetilde{f}_{i'}(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ for all $i \in [1, r]$.

Let $\boldsymbol{\lambda} \in \mathrm{Bip}_{(r+1,r+1)}$, and $v \in L(\boldsymbol{\lambda})$ be a highest weight vector. Set

$$\mathcal{L}(\boldsymbol{\lambda}) := \mathrm{Span}_{\mathbf{A}_0}\{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} v \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_l \in \mathbb{I}^\jmath \sqcup \{2', \dots, r'\}\},$$
$$\mathcal{B}(\boldsymbol{\lambda}) := \{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_l} v + q\mathcal{L}(\boldsymbol{\lambda}) \mid l \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_l \in \mathbb{I}^\jmath \sqcup \{2', \dots, r'\}\} \setminus \{0\}.$$

Theorem 20.3.3. *Let $\boldsymbol{\lambda} \in P^\imath$. Then, $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$ is a unique $\imath$-crystal basis of $L(\boldsymbol{\lambda})$.*

Corollary 20.3.4. *$(\mathbf{L}^{\otimes d}, \mathbf{B}^{\otimes d})$ is an $\imath$-crystal basis of $\mathbf{V}^{\otimes d}$. Under the identification $\mathbf{B}^{\otimes d} = I^d$, the actions of $\widetilde{e}_i, \widetilde{f}_i$, $i \in \mathbb{I}^\imath$ are described by Corollary 19.1.4, and those of $\widetilde{e}_{i'}, \widetilde{f}_{i'}$, $i \in \mathbb{I}^\imath \setminus \{1\}$ are given as follows: $\widetilde{e}_{i'}\mathbf{s}$ is obtained from $\mathbf{s}$ by replacing the rightmost i in $\mathbf{s}_i$ with $i-1$ if $\widetilde{e}_j\mathbf{s} = 0$ for all $j = 1, \dots, i-1$ and $\widetilde{e}_{j'}\mathbf{s} = 0$ for all $j = 2, \dots, i-1$, and is 0 otherwise. Finally,*

$$\widetilde{f}_{i'}\mathbf{s} = \begin{cases} \mathbf{s}' & \text{if } \widetilde{e}_{i'}\mathbf{s}' = \mathbf{s}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, the existence and uniqueness theorem for $\imath$-crystal basis can be proved in the same way as the ordinary crystal basis theory.

Theorem 20.3.5. *Let $M \in \mathcal{O}^\imath_{\mathrm{int}}$ be a $\mathbf{U}^\imath$-module. Then, M has an $\imath$-crystal basis $(\mathcal{L}, \mathcal{B})$. If $M \simeq \bigoplus_{\boldsymbol{\lambda} \in P^\imath} L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ for some $m_{\boldsymbol{\lambda}} \in \mathbb{Z}_{\geq 0}$, then there exists an isomorphism $M \to \bigoplus_{\boldsymbol{\lambda} \in P^\imath} L(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}$ inducing an isomorphism $(\mathcal{L}, \mathcal{B}) \to (\bigoplus_{\boldsymbol{\lambda} \in P^\imath} \mathcal{L}(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}}, \bigoplus_{\boldsymbol{\lambda} \in P^\imath} \mathcal{B}(\boldsymbol{\lambda})^{\oplus m_{\boldsymbol{\lambda}}})$.*

21. Global $\imath$-crystal bases

21.1. **Global $\imath$-crystal bases.** Let $\mathbf{U}^\imath_{\mathbf{A}}$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}^\imath$ generated by $e_i^{(n)}, f_i^{(n)}, t, k_i^{\pm 1}$, $i \in \mathbb{I}^\jmath$, $n \in \mathbb{Z}_{>0}$.

Let V be a $\mathbf{U}^\imath$-module in $\mathcal{O}^\imath_{\mathrm{int}}$ with an $\imath$-crystal basis $(\mathcal{L}, \mathcal{B})$. Assume that V admits a $\psi^\imath$-involution on V.

Definition 21.1.1. Let $V, \mathcal{L}, \mathcal{B}, \overline{\cdot}$ be as above. V is said to have a global $\imath$-crystal basis if there exists a $\mathbf{U}^\imath_{\mathbf{A}}$-submodule $V_{\mathbf{A}}$ of V which is an $\mathbf{A}$-lattice forming a balanced triple $(\mathcal{L}, V_{\mathbf{A}}, \overline{\mathcal{L}})$. The associated global basis $G^\imath(\mathcal{B})$ is called a global $\imath$-crystal basis of V.

Proposition 21.1.2. *Let $M \in \mathcal{O}^\imath_{\mathrm{int}}$ with a global $\imath$-crystal basis $G^\imath(\mathcal{B}_M)$, and $N \in \mathcal{O}_{\mathrm{int}}$ with a global crystal basis $G(\mathcal{B}_N)$ Then, $(G^\jmath(\mathcal{B}_M) \diamond_\imath G(\mathcal{B}_N))$ is a global $\imath$-crystal basis of $M \otimes N$.*

Proposition 21.1.3. *Let $M \in \mathcal{O}_{\mathrm{int}}$ be a $\mathbf{U}$-module with a global crystal basis $G(\mathcal{B})$ associated to a crystal basis $(\mathcal{L}, \mathcal{B})$, a ψ-involution ψ_M, and a $\mathbf{U}_{\mathbf{A}}$-submodule $M_{\mathbf{A}}$. Then, $(\mathcal{L}, \mathcal{B})$ is an $\imath$-crystal basis, $(\mathcal{L}, M_{\mathbf{A}}, \psi^\imath_M(\mathcal{L}))$ is a balanced triple, and $G^\jmath(\mathcal{B}) := G(\mathcal{B})_\imath$ is the global $\imath$-crystal basis associated to the balanced triple $(\mathcal{L}, M_{\mathbf{A}}, \psi^\imath_M(\mathcal{L}))$ and the basis $\mathcal{B}$.*

21.2. **Global $\imath$-crystal basis of irreducible $\mathbf{U}^\imath$-module.** Let $X \in L(W)$. Then, $\mathbf{C}^L_X(\pi^\imath) \simeq L(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in P^\imath$. Since $L(\boldsymbol{\lambda})$ is a highest weight module, there exists a unique $\lambda \in \pi^\imath$ and $w \in {}^\lambda W$ such that $[{}^\lambda C_w]_X \in \mathbf{C}^L_X(\pi^\imath)$ is a highest weight vector.

Recall the isomorphism $D^L_{Xw_0} \simeq C^L_X$ of left $\mathcal{H}$-modules from Proposition 5.4.9. Set $\mathbf{C}^L_X := \mathbb{Q}(p,q) \otimes_{\mathbf{A}_\mathbb{Z}} C^L_X$, and define $\mathbf{D}^L_{Xw_0}$ and $\mathbf{D}^L_{Xw_0}(\pi^\jmath)$ similarly. Then, we have

$$\mathbf{D}^L_{Xw_0}(\pi^\jmath) \simeq \mathbf{T}(\pi^\imath) \otimes_{\mathbf{H}} \mathbf{D}^L_{Xw_0} \simeq \mathbf{T}(\pi^\imath) \otimes_{\mathbf{H}} \mathbf{C}^L_X \simeq \mathbf{C}^L_X(\pi^\imath)$$

as left $\mathbf{U}^\imath$-modules. Hence, $[{}^\lambda D_{w_\lambda w w_0}]'_{Xw_0} \in \mathbf{D}^L_{Xw_0}(\pi^\imath)$ is also a highest weight vector. Thus, we obtain two isomorphisms

$$\varphi_C : L(\boldsymbol{\lambda}) \to \mathbf{C}^L_X(\pi^\imath);\ v_{\boldsymbol{\lambda}} \mapsto [{}^\lambda C_w]_X,$$

$$\varphi_D : L(\boldsymbol{\lambda}) \to \mathbf{D}^L_{Xw_0}(\pi^\imath);\ v_{\boldsymbol{\lambda}} \mapsto [{}^\lambda D_{w_\lambda w w_0}]'_{Xw_0}$$

of $\mathbf{U}^\imath$-modules, where $v_{\boldsymbol{\lambda}} \in L(\boldsymbol{\lambda})$ is a fixed highest weight vector.

Definition 21.2.1. Let $\boldsymbol{\lambda} \in P^\imath$ and $v_{\boldsymbol{\lambda}} \in L(\boldsymbol{\lambda})$ be a highest weight vector. Define the bilinear form $(\cdot,\cdot)_1$ on $L(\boldsymbol{\lambda})$ by $(v_{\boldsymbol{\lambda}}, v_{\boldsymbol{\lambda}})_1 = 1$ and $(xm,n)_1 = (n, \sigma^\jmath(x)n)_1$ for all $x \in \mathbf{U}^\imath$, $m,n \in L(\boldsymbol{\lambda})$.

Proposition 21.2.2. *Let $\boldsymbol{\lambda} \in P^\imath$. Then, the bilinear form $(\cdot,\cdot)_1$ is nondegenerate.*

Recall that the set $\{(\mu,y) \mid \mu \in \pi^\imath,\ y \in {}^\mu W \cap w_\mu X\}$ is identical to $\mathcal{B}(\boldsymbol{\lambda})$. For each $b \in \mathcal{B}(\boldsymbol{\lambda})$, set

$$G^\imath_{\mathrm{low}}(b) := \varphi_C^{-1}([{}^\mu C_y]_X), \quad G^\imath_{\mathrm{up}}(b) := \varphi_D^{-1}([{}^\mu D_{w_\mu y w_0}]'_{Xw_0}),$$

where (μ,y) is the pair corresponding to b. Then, $G^\imath_{\mathrm{low}}(\boldsymbol{\lambda}) := \{G^\imath_{\mathrm{low}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$ and $G^\imath_{\mathrm{up}}(\boldsymbol{\lambda}) := \{G^\imath_{\mathrm{up}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$ are bases of $L(\boldsymbol{\lambda})$.

Definition 21.2.3. Let $\boldsymbol{\lambda} \in P^\imath(d)$, and $v_{\boldsymbol{\lambda}} \in L(\boldsymbol{\lambda})$ be a highest weight vector. Define a bilinear form $(\cdot,\cdot)_2$ on $L(\boldsymbol{\lambda})$, and a $\psi^\imath$-involution $\psi^\imath_{\boldsymbol{\lambda}}$ on $L(\boldsymbol{\lambda})$ by

$$(v_{\boldsymbol{\lambda}}, v_{\boldsymbol{\lambda}})_2 = 1,\ (xm,n)_2 = (m, \varrho(x)n)_2 \quad \text{for all } x \in \mathbf{U}^\imath,\ m,n \in L(\boldsymbol{\lambda}),$$

$$\psi^\imath_{\boldsymbol{\lambda}}(v_{\boldsymbol{\lambda}}) = v_{\boldsymbol{\lambda}}.$$

Let $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{B}(\boldsymbol{\lambda}))$ be the unique $\imath$-crystal basis of $L(\boldsymbol{\lambda})$ such that $v_{\boldsymbol{\lambda}} + q\mathcal{L}(\boldsymbol{\lambda}) \in \mathcal{B}(\boldsymbol{\lambda})$.

Theorem 21.2.4. *Let $\boldsymbol{\lambda} \in P^\imath(d)$. Then, the following hold.*

1. *$\psi^\imath_{\boldsymbol{\lambda}}(G^\imath_{\mathrm{low}}(b)) = G^\imath_{\mathrm{low}}(b)$ for all $b \in \mathcal{B}(\boldsymbol{\lambda})$.*
2. *$\psi^\imath_{\boldsymbol{\lambda}}(G^\imath_{\mathrm{up}}(b)) = G^\imath_{\mathrm{up}}(b)$ for all $b \in \mathcal{B}(\boldsymbol{\lambda})$.*
3. *$G^\imath_{\mathrm{low}}(\boldsymbol{\lambda})$ and $G^\imath_{\mathrm{up}}(\boldsymbol{\lambda})$ are dual bases with respect to $(\cdot,\cdot)_1$.*
4. *$\mathcal{L}(\boldsymbol{\lambda}) = \{m \in L(\boldsymbol{\lambda}) \mid (m,m)_2 \in \mathbf{A}_0\}$. Consequently, $(\cdot,\cdot)_2$ induces the bilinear form $(\cdot,\cdot)_0$ on $\mathcal{L}(\boldsymbol{\lambda})/q\mathcal{L}(\boldsymbol{\lambda})$ defined by $(m+q\mathcal{L}(\boldsymbol{\lambda}), n+q\mathcal{L}(\boldsymbol{\lambda}))_0 := \lim_{q\to 0}(\lim_{p\to 0}(m,n)_2)$.*
5. *$\{G^\imath_{\mathrm{low}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$ forms an almost orthonormal basis with respect to $(\cdot,\cdot)_2$, i.e., we have $(G^\imath_{\mathrm{low}}(b), G^\imath_{\mathrm{low}}(b'))_2 \in \delta_{b,b'} + q\mathbf{A}_0$ for all $b,b' \in \mathcal{B}(\boldsymbol{\lambda})$.*
6. *$(b,b')_0 = \delta_{b,b'}$ for all $b,b' \in \mathcal{B}(\boldsymbol{\lambda})$.*
7. *Let $L(\boldsymbol{\lambda})_{\mathbf{A}}$ be the $\mathbf{A}$-span of $G^\imath_{\mathrm{low}}(\boldsymbol{\lambda})$. Then, $(\mathcal{L}(\boldsymbol{\lambda}), L(\boldsymbol{\lambda})_{\mathbf{A}}, \psi^\jmath_{\boldsymbol{\lambda}}(\mathcal{L}(\boldsymbol{\lambda})))$ is balanced. Moreover, the global basis associated to $\mathcal{B}(\boldsymbol{\lambda})$ is $\{G^\imath_{\mathrm{low}}(b) \mid b \in \mathcal{B}(\boldsymbol{\lambda})\}$. In particular, $L(\boldsymbol{\lambda})$ has a global $\imath$-crystal basis.*

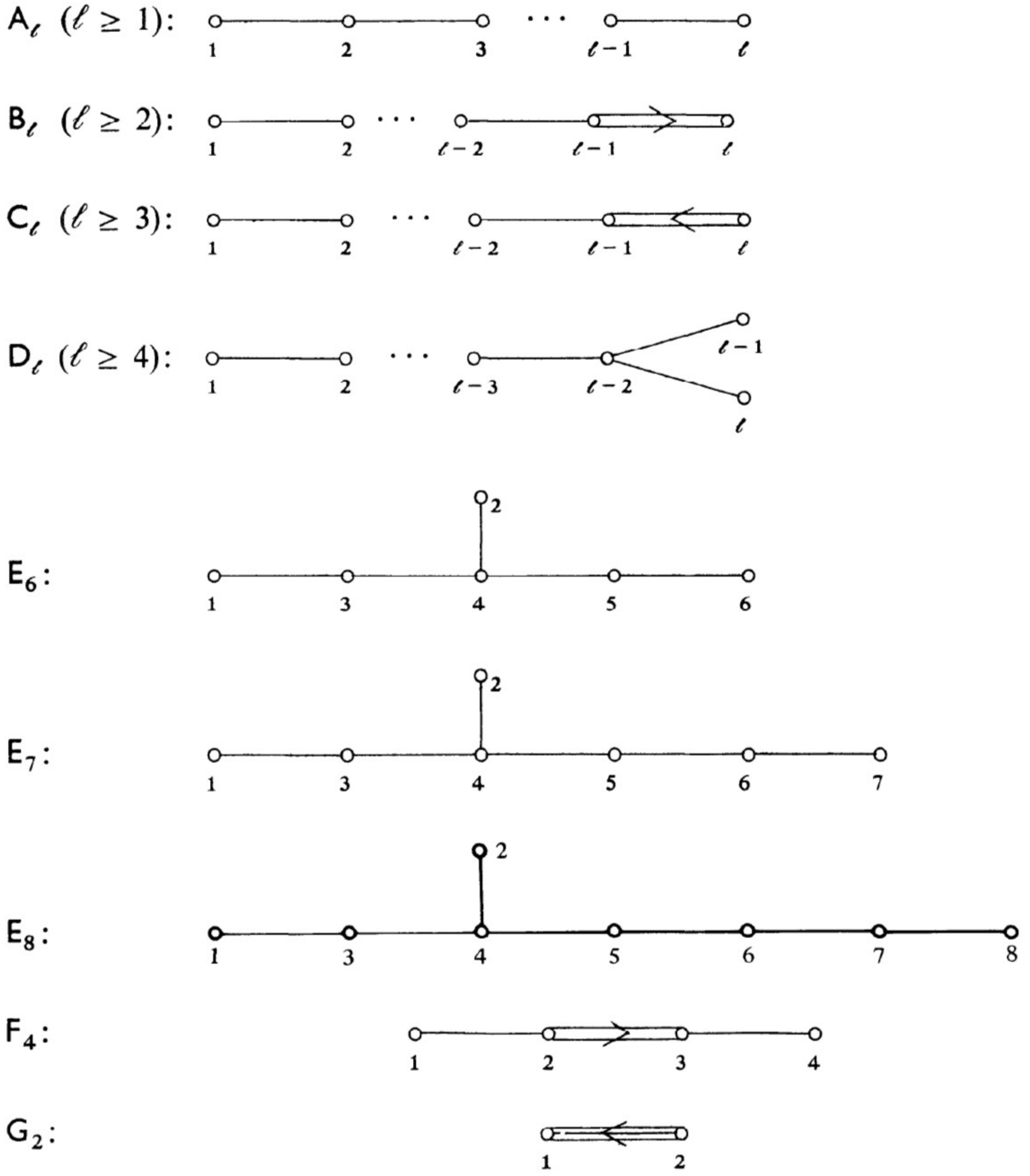


Figure 1. Dynkin diagrams of irreducible finite type, [H72]

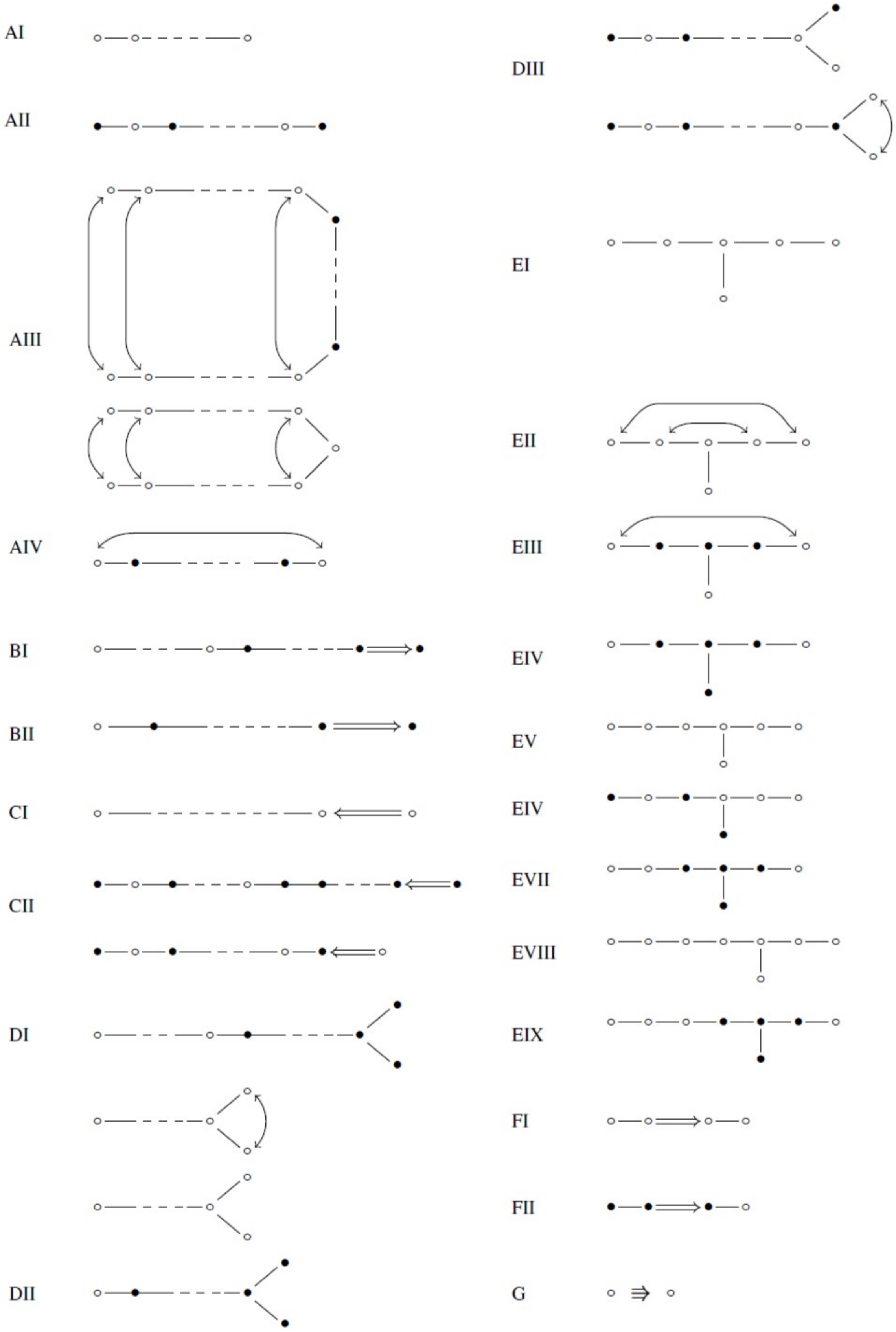


Figure 2. Satake diagrams of irreducible symmetric pairs, [BW18a]

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