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Probabilistic studies of the value-distributions of zeta and L-functions

by

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Contents

Acknowledgments v

Abstract vi

Notation ix

Chapter 1. Introduction 1
- 1. The Riemann zeta-function 1
- 2. Dirichlet L-functions 11
- 3. L-functions of higher degree 19
- 4. Other zeta-functions 29

Chapter 2. General L-functions 35
- 5. The Selberg class 35
- 6. Matsumoto zeta-functions 40
- 7. Setting in this thesis 44

Chapter 3. Random Euler products 48
- 8. Basic properties 48
- 9. Probability density functions 53
- 10. Comparisons with families of L-functions 62

Chapter 4. The value-distributions of L-functions 71
- 11. The first result: general L-functions 71
- 12. The second result: Artin L-functions 84
- 13. The third result: automorphic L-functions 99

Chapter 5. Further results on the Riemann zeta-function 114
- 14. Preliminaries on the random Euler product 114
- 15. Extreme values of the Riemann zeta-function 128

Chapter 6. Results for other zeta-functions 134
- 16. The fourth result: Hurwitz–Lerch zeta-functions 134
- 17. The fifth result: iterated integrals of $\log \zeta(s)$ 160

Appendix A. Probability measures 167
- A.1. Convergence of probability measures 168
- A.2. Supports of probability measures 171
- A.3. Esseen's inequalities 172

Appendix B. Complex functions 173
B.1. Logarithms and polylogarithms 173
B.2. The gamma function 175
B.3. Bessel functions 176

Bibliography 178

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Abstract

The theory of the value-distributions of zeta and L-functions has been developed since the classical work of Bohr and his collaborators in the early 20th century. The major modern method is to compare an L-function with some probabilistic model. For example, we study mean values of L-functions via the expected values of suitable random variables. Ihara and Matsumoto also introduced the notion of *M-functions* which are certain probability density functions closely related to the value-distributions of L-functions. The main terms of mean values of L-functions are represented by integrals involving the attached M-functions. Furthermore, many researchers studied error terms arising from the comparisons of L-functions with their probabilistic models. The upper bounds of such error terms are called the *discrepancy bounds*. This thesis contains five issues on the probabilistic studies of the value-distributions of zeta and L-functions.

First, we consider the value-distributions of L-functions as the complex variable $s = \sigma + it$ varies on vertical lines, namely in the t-aspect. The Riemann zeta-function is the most basic example of L-functions possessing Euler product representations, and the attached M-function was classically studied. Matsumoto further determined M-functions for Dedekind zeta-functions of Galois number fields. It should be noted that his method relied on a special property of the Euler products called the convexity, although we know many L-functions with non-convex Euler products, e.g. Dedekind zeta-functions of non-Galois number fields and automorphic L-functions of holomorphic primitive cusp forms. The first result of this thesis presents a general method of constructing M-functions by the axiomatic treatments of L-functions. Several axiomatic classes of L-functions were introduced by Selberg, Matsumoto, Steuding, and others. Then we prove that an L-function has a suitable M-function if it belongs to the intersection of Selberg's class and Steuding's class. It enables us to construct M-functions for L-functions with non-convex Euler products. In addition, we obtain sharp discrepancy bounds for such L-functions. We further prove an asymptotic formula related to the large deviations of the values of the Riemann zeta-function.

The results on the first issue are presented as Theorems I, II (on the Riemann zeta-function) and Theorems 11.1, 11.2 (on general L-functions).

It is also worth considering the value-distributions of L-functions in the other aspects. For example, the value-distribution of Dirichlet L-functions as the Dirichlet character varies has been studied. As the second issue, we consider Artin L-functions arising from non-Galois cubic fields. The main difference from the case of Dirichlet L-functions is that such Artin L-functions have Euler products of degree two, which causes combinatorial complexity of the treatment of Dirichlet coefficients. We overcome this difficulty by applying the results on counting cubic fields achieved by Bhargava–Shankar–Tsimerman and Taniguchi–Thorne in 2013. We show the existence of an M-function for the value-distribution of the Artin L-functions as the cubic field varies. Furthermore, we apply the result to derive a limit formula involving class numbers of cubic fields. It can be regarded as a cubic analogue of the Gauss–Siegel formula of class numbers of quadratic forms.

The result on the second issue is presented as Theorem III.

Automorphic L-functions of holomorphic primitive cusp forms also have Euler products of degree two. The third issue concerns the value-distributions of such automorphic L-functions as the cusp form varies. Recently, Matsumoto–Umegaki and Lebacque–Zykin tried to construct an M-function in this case, and their attempts were partially succeeded. The difficulty to complete their study was again combinatorial complexity of the treatment of Dirichlet coefficients. In this thesis, we construct the desired M-function by using the Eichler–Selberg trace formula which can be regarded as a relation between the Dirichlet coefficients and certain random variables. We further obtain discrepancy bounds in this case, which is similar to the result on the first issue.

The result on the third issue is presented as Theorem IV.

As the fourth issue, we study Hurwitz–Lerch zeta-functions with transcendental parameters, which do not have Euler product representations. It is classically known that such Hurwitz–Lerch zeta-functions have infinitely many zeros off the critical line, while the Riemann Hypothesis asserts that all non-trivial zeros of the Riemann zeta-function lie on the critical line. Furthermore, several inequalities on upper and lower bounds for the densities of these zeros of the Hurwitz–Lerch zeta-functions were proved by Garunkštis–Laurinčikas. Then we improve the inequalities by proving asymptotic formulas when the parameters are S-numbers in the sense of Mahler's classification of transcendental numbers.

The result on the fourth issue is presented as Theorem V.

Finally, we consider the value-distributions of certain functions represented by iterated integrals of the logarithm of the Riemann zeta-function as the fifth issue. Such functions were introduced by Inoue to study the value-distributions on the critical line precisely. Then we prove a probabilistic limit theorem of Inoue's functions. It refines the recent result of Endo–Inoue which ensures that the sets of the values of Inoue's functions on the critical line are dense in the complex plane.

The result on the fifth issue is presented as Theorem VI.

This thesis consists of six chapters. Chapter 1 provides statement of main results and a survey of the related studies of zeta and L-functions. In Chapter 2, we review axiomatic investigations of L-functions. Then we describe the setting for L-functions in this thesis. Chapter 3 is devoted to prepare probabilistic models attached to L-functions, which are called the *random Euler products*. After showing some preliminary results, we present general frameworks of comparing L-functions with the random Euler products. Then we proceed to the proof of the main results. In Chapter 4, we prove Theorems I, III, and IV. In Chapter 5, we refocus on the value-distribution of the Riemann zeta-function in the t-aspect. Theorem II is proved in this chapter. We need more effort for the proofs of the remaining main results. We prove some preliminary results by applying properties of S-numbers and polylogarithms for Theorems V and VI, respectively. The proofs of the theorems are completed in Chapter 6.

Notation

As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all positive integers, integers, rational numbers, real numbers, and complex numbers, respectively.

Let f and g be functions on $\mathbb{R}$ with $g(x) > 0$. The Landau–Vinogradov symbols are used to mean that the following relations hold between f and g.

notation	meaning
$f(x) = O(g(x))$	$\lvert f(x)\rvert \leq Cg(x)$ with a constant $C > 0$
$f(x) \ll g(x)$	$\lvert f(x)\rvert \leq Cg(x)$ with a constant $C > 0$
$f(x) \gg g(x)$	$\lvert f(x)\rvert \geq Cg(x)$ with a constant $C > 0$
$f(x) \asymp g(x)$	$f(x) \ll g(x)$ and $f(x) \gg g(x)$
$f(x) = o(g(x))$	$\lim f(x)/g(x) = 0$
$f(x) = \Omega(g(x))$	$\limsup \lvert f(x)\rvert/g(x) > 0$
$f(x) = \Omega_+(g(x))$	$\limsup f(x)/g(x) > 0$
$f(x) = \Omega_-(g(x))$	$\liminf f(x)/g(x) < 0$
$f(x) = \Omega_\pm(g(x))$	$f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$
$f(x) \sim g(x)$	$\lim f(x)/g(x) = 1$

The above inequalities are considered for $x \in X$, where $X \subset \mathbb{R}$ is sometimes implicitly given. The set X is usually an interval $[x_0, \infty)$ with some $x_0 > 0$. Similarly, the above limits are taken as x tends to some limit, usually $x \longrightarrow \infty$. The constant C in the above definition is called the implied constant of the Landau–Vinogradov symbol. We often use the notation such as $f(x) = O_\alpha(g(x))$ to indicate that the implied constant depends on some parameter α.

An arithmetic function is a function defined on $\mathbb{N}$. We say that an arithmetic function f is multiplicative if the condition $f(mn) = f(m)f(n)$ holds for all m, n with $(m, n) = 1$. Furthermore, we say that f is completely multiplicative if it holds for arbitrary m and n. We list several arithmetic functions used in this thesis.

$d_k(n)$	the number of representations of n as the product of k positive integers
$d(n)$	the number of positive divisors of n, i.e. $d_2(n) = d(n)$
$\omega(n)$	the number of distinct prime factors of n
$\Lambda(n)$	the von Mangoldt function: $\Lambda(n) = \log p$ if $n \neq 1$ is a power of prime p; and $\Lambda(n) = 0$ otherwise.

We see that $d_k(n)$ and $\omega(n)$ are multiplicative while $\Lambda(n)$ is not multiplicative.

Throughout this thesis, the notation log indicates the natural logarithm. Then we use the notation $\log_k$ to denote the k-th iterates of the natural logarithms:

$$\log_1 = \log \quad \text{and} \quad \log_{k+1} = \log(\log_k).$$

For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer not exceeding x. As usual, $\pi(x)$ counts the number of prime numbers not exceeding x. For $z \in \mathbb{C}$, we denote by $\operatorname{Log} z$ the principal branch of logarithm of z. Denote by $\Gamma(z)$, $J_\nu(z)$, and $I_\nu(z)$ the usual gamma function, the Bessel function of the first kind of order ν, and the modified Bessel function of the first kind of order ν, respectively. Several properties of these complex functions are collected in Appendix B.

Let f be a meromorphic function. Then we have at most finite zeros and poles of f in any compact subset of $\mathbb{C}$. Throughout this thesis, numbers of zeros or poles are counted with multiplicities. We use the letter ρ to denote a zero or pole of some meromorphic function.

Let S be a topological space. Denote by $\mathcal{B}(S)$ the class of Borel sets of S. A random element is a Borel measurable map $X : \Omega \to S$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Then $\mathbb{P}(\cdots)$ denotes the probability of an event "$\cdots$" but the space $(\Omega, \mathcal{F})$ is often implicitly given. The expected value and variance of X are denoted by $\mathbb{E}[X]$ and $\mathbb{V}[X]$, respectively.

We identify $\mathbb{C}$ with $\mathbb{R}^2$ by the map $x + iy \mapsto (x, y)$. Then we denote by meas A the usual Lebesgue measure of a measurable set A of $\mathbb{R}$ or $\mathbb{C}$. Furthermore, we write

$$\mathbb{P}_T(\cdots) = \frac{1}{T} \operatorname{meas} \left\{ t \in [0, T] \;\middle|\; \cdots \right\}$$

for $T > 0$, where "$\cdots$" means some condition to t. Denote the standard inner product of $\mathbb{C} = \mathbb{R}^2$ by $\langle z, w \rangle = \operatorname{Re} z \operatorname{Re} w + \operatorname{Im} z \operatorname{Im} w$. We put $\psi_w(z) = \exp(i\langle z, w \rangle)$, and thus $\psi_u(x) = \exp(ixu)$ for $x, u \in \mathbb{R}$. We denote the Fourier transforms by

$$\widetilde{f}(u) = \int_{\mathbb{R}} f(x)\psi_u(x)\, |dx| \quad \text{and} \quad \widetilde{g}(w) = \int_{\mathbb{C}} g(z)\psi_w(z)\, |dz|,$$

where $|dx| = (2\pi)^{-\frac{1}{2}} dx$ and $|dz| = (2\pi)^{-1} dxdy$ for $z = x + iy$. We use the notation $C^\infty(\mathbb{C})$ to denote the space of all functions f on $\mathbb{C}$ which are infinitely differentiable as functions on $\mathbb{R}^2$. Similarly, we denote the Schwartz space and the L^p-space by $\mathcal{S}(\mathbb{C})$ and $L^p(\mathbb{C})$, respectively.

Finally, the letter γ is used to denote the Euler–Mascheroni constant:

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} k^{-1} - \log n \right) = 0.5772...$$

if there exist no special descriptions. Remark that it is also popular to use the same letter to indicate the imaginary part of a zero of some zeta or L-function.

CHAPTER 1

Introduction

The study of the value-distributions of zeta and L-functions is a classical topic in analytic number theory. It began with the work of Bohr and his collaborators in the early 20th century. Their methods have been refined along with the development of the related areas of mathematics, especially probability theory. In this chapter, we survey the theory of the value-distributions of several zeta and L-functions.

1. The Riemann zeta-function

In his memoir *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Riemann introduced a complex function represented as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1} \tag{1.1}$$

for the purpose of studying the distribution of prime numbers. The series and product in (1.1) converge absolutely for $\operatorname{Re} s > 1$, where the notation $\prod_p$ stands for the infinite product over all prime numbers. Today, the function $\zeta(s)$ is called the Riemann zeta-function. Riemann himself proved two results on $\zeta(s)$.

(a) It can be analytically continued to the whole complex plane except only for the simple pole at $s = 1$ with residue 1.

(b) It satisfies the functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

He further presented the following statement without proof.

(c) There exist infinitely many zeros of $\zeta(s)$ in the critical strip $0 \leq \operatorname{Re} s \leq 1$.

(d) We have the asymptotic formula

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{1.2}$$

where $N(T)$ counts the number of zeros of $\zeta(s)$ in the critical strip which satisfy $0 < \operatorname{Im} \rho \leq T$.

(e) There exist constants A and B such that

$$\frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

where ρ runs through all zeros of $\zeta(s)$ in the critical strip.

(f) (the formulation by von Mangoldt) Let $\psi(x) = \sum_{n\leq x} \Lambda(n)$. Then we have the explicit formula

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2})$$

for $x > 1$ except for prime powers, where the sum over ρ is understood as

$$\sum_{\rho} \frac{x^{\rho}}{\rho} = \lim_{T\to\infty} \sum_{|\operatorname{Im}\rho|<T} \frac{x^{\rho}}{\rho}.$$

(g) All zeros of $\zeta(s)$ in the critical strip lie on the critical line $\operatorname{Re} s = 1/2$.

See Davenport [31, Section 8] for more information. The proofs of Riemann's statements were completed by the end of the 19th century except for (g), which is open until today and known as the Riemann Hypothesis.

1.1. The work of Bohr and his colleagues. At the beginning of 20th century, Bohr initiated the study of the distribution of values $\zeta(s)$ as $s = \sigma + it$ varies. The first result [13] was obtained for the values on the right-half plane $\operatorname{Re} s > 1$.

Theorem 1.1 (Bohr). *Let* $\epsilon > 0$. *Then* $\zeta(s)$ *takes any non-zero value infinitely many times in the strip* $1 < \operatorname{Re} s < 1 + \epsilon$.

Then he proceeded to research the value-distribution of $\zeta(s)$ for $1/2 < \operatorname{Re} s \leq 1$ which is more difficult due to the lack of representations (1.1). One of the key ideas to overcome this difficulty is approximating $\zeta(s)$ by the finite product

$$\zeta_N(s) = \prod_{n=1}^{N} \left(1 - p_n^{-s}\right)^{-1},$$

where p_n denotes the n-th prime number. Indeed, Bohr noticed that the following fact is deduced from a certain mean value theorem obtained by Bohr–Landau [18]. Let $1/2 < \sigma_1 < 3/4$, $\sigma_2 > 2$, and $\epsilon > 0$. Then the inequality

$$\frac{1}{T} \iint_{\substack{\sigma_1\leq\sigma\leq\sigma_2 \\ 1\leq t\leq T}} |\zeta(s)\zeta_N(s)^{-1} - 1|^2 \, d\sigma dt < \epsilon$$

holds if $T, N \geq 1$ are large enough. Applying a similar approximation lemma, Bohr–Courant [15] proved the following result.

Theorem 1.2 (Bohr–Courant). *Let* $1/2 < \sigma \leq 1$ *be a fixed real number. Then the set* $\{\zeta(\sigma + it) \mid t \in \mathbb{R}\}$ *is dense in* $\mathbb{C}$.

For some technical reasons, it is more natural to study the distribution of the logarithmic values of $\zeta(s)$. However, we should remark that $\log\zeta(s)$ can not be defined as a holomorphic function on the right-half plane $\operatorname{Re} s > 1/2$ due to the pole and possible zeros of $\zeta(s)$. Thus we define the region G as

$$G = \{\sigma + it \mid \sigma > 1/2\} \setminus \bigcup_{\zeta(\rho)=0 \text{ or } \infty} \{\sigma + i\operatorname{Im}\rho \mid 1/2 < \sigma \leq \operatorname{Re}\rho\} \tag{1.3}$$

and determine the branch of $\log\zeta(s)$ for $s=\sigma+it\in G$ as follows. First, we define $\log\zeta(s)$ for $\operatorname{Re}s>1$ by

$$\log\zeta(s)=-\sum_p \operatorname{Log}(1-p^{-s})=\sum_p\sum_{k=1}^{\infty}\frac{1}{k}p^{-ks} \tag{1.4}$$

according to (1.1). Then, we extend $\log\zeta(s)$ for $s\in G$ by the analytic continuation along the horizontal path from right. Bohr [**14**] proved the denseness of the values $\log\zeta(s)$ for $1/2<\operatorname{Re}s\leq 1$ similarly to Theorem 1.2.

THEOREM 1.3 (Bohr). *Let $1/2<\sigma\leq 1$ be a fixed real number. Then the set $\{\log\zeta(\sigma+it)\mid t\in\mathbb{R},\ \sigma+it\in G\}$ is dense in $\mathbb{C}$.*

Let $\mathcal{R}$ be a rectangle on the complex plane $\mathbb{C}$ with edges parallel to the axes. Then we consider the set

$$U_{\sigma,T}(\mathcal{R})=\{t\in[0,T]\mid \log\zeta(\sigma+it)\in\mathcal{R},\ \sigma+it\in G\}$$

with $\sigma>1/2$ and $T>0$. Theorem 1.3 implies that it is non-empty for $1/2<\sigma\leq 1$ if T is large enough. The next aim of Bohr's research was to determine the density of $U_{\sigma,T}(\mathcal{R})$ with respect to the interval $[0,T]$. In 1930s, Bohr–Jessen [**16**, **17**] arrived at the following limit theorem.

THEOREM 1.4 (Bohr–Jessen). *Let $\sigma>1/2$ be a fixed real number.*

(i) *There exists the limit value*

$$W_\sigma(\mathcal{R})=\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}U_{\sigma,T}(\mathcal{R}).$$

(ii) *There exists a continuous function $\mathcal{M}_\sigma:\mathbb{C}\to\mathbb{R}_{\geq 0}$ such that*

$$W_\sigma(\mathcal{R})=\int_{\mathcal{R}}\mathcal{M}_\sigma(w)\,|dw|.$$

An application of Theorem 1.4 is a limit formula for the density of a-points of $\log\zeta(s)$, namely, solutions of $\log\zeta(s)=a$. Denote by $N_a(T,\sigma_1,\sigma_2)$ the number of a-points of $\log\zeta(s)$ in the rectangle $\sigma_1<\operatorname{Re}s<\sigma_2$, $1<\operatorname{Im}s<T$.

THEOREM 1.5 (Bohr–Jessen). *Let $a\in\mathbb{C}$ and $1/2<\sigma_1<\sigma_2$. Then there exists the limit value*

$$C_a(\sigma_1,\sigma_2)=\lim_{T\to\infty}\frac{1}{T}N_a(T,\sigma_1,\sigma_2).$$

After Bohr's work, various modifications of Theorems 1.4 and 1.5 were made by many authors including notably Jessen–Wintner [**71**], Borchsenius–Jessen [**19**], Nikisin [**154**], and Laurinčikas [**96**, **98**]. In the following sections, we describe probabilistic interpretations of the limit values $W_\sigma(\mathcal{R})$ and $C_a(\sigma_1,\sigma_2)$ according to the work of Jessen–Wintner and Borchsenius–Jessen.

1.2. Probabilistic approaches. The study of the value-distribution of $\zeta(s)$ was developed by studying the limit value $W_\sigma(\mathcal{R})$ of Theorem 1.4 along with the methods of modern probability theory initiated by Kolmogorov. Jessen–Wintner [71] made an alternative proof of Theorem 1.4 and constructed $W_\sigma(\mathcal{R})$ explicitly in terms of probability measures. Denote by $S_{n,\sigma}$ the curve in the complex plane given by the parametric representation $z_{n,\sigma}(\theta) = -\log(1 - p_n^{-\sigma} e^{i\theta})$. Then we define

$$\mu_{n,\sigma}(A) = \frac{1}{2\pi} \operatorname{meas}\{\theta \in [0, 2\pi] \mid z_{n,\sigma}(\theta) \in A\} \tag{1.5}$$

for $A \in \mathcal{B}(\mathbb{C})$, which gives a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We further define $P_{N,\sigma} = \mu_{1,\sigma} * \cdots * \mu_{N,\sigma}$ for $N \geq 1$; see (A.1) for the definition of the convolution of probability measures.

Theorem 1.6 (Jessen–Wintner). *As $N \to \infty$, $P_{N,\sigma}$ converges weakly to a probability measure P_σ if and only if $\sigma > 1/2$. Moreover, the limit measure P_σ is absolutely continuous and satisfies*

$$P_\sigma(A) = \int_A \mathcal{M}_\sigma(w)\, |dw| \tag{1.6}$$

for all $A \in \mathcal{B}(\mathbb{C})$ with the function $\mathcal{M}_\sigma$ of Theorem 1.4. Furthermore, we have the following properties for $\mathcal{M}_\sigma$.

- (i) *The function $\mathcal{M}_\sigma$ belongs to the class $C^\infty(\mathbb{C})$.*
- (ii) *The support of $\mathcal{M}_\sigma$ is a compact subset of $\mathbb{C}$ if $\sigma > 1$. On the other hand, we have $\mathcal{M}_\sigma(w) > 0$ for all $w \in \mathbb{C}$ if $1/2 < \sigma \leq 1$.*
- (iii) *We have $\mathcal{M}_\sigma(w) \ll e^{-c|w|^2}$ for any $c > 0$, and every partial derivative has the same upper bound.*

We define a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as

$$P_{\sigma,T}(A) = \mathbb{P}_T(\log\zeta(\sigma + it) \in A)$$

for $\sigma > 1/2$ and $T > 0$. Here, we can omit the condition $\sigma + it \in G$ since the points t with $\sigma + it \notin G$ are countable at most. Then we have $P_{\sigma,T}(\mathcal{R}) = T^{-1} \operatorname{meas} U_{\sigma,T}(\mathcal{R})$ by definition. Jessen–Wintner [71] formulated the limit theorem in terms of the weakly convergence of probability measures.

Theorem 1.7 (Jessen–Wintner). *Let $\sigma > 1/2$. Then the probability measure $P_{\sigma,T}$ converges weakly to P_σ of Theorem 1.6 as $T \to \infty$.*

By equality (1.6), the rectangle $\mathcal{R}$ is a continuity set of P_σ. Hence Theorem 1.7 implies $P_{\sigma,T}(\mathcal{R}) \to P_\sigma(\mathcal{R})$ as $T \to \infty$. More generally, we see that it holds if $\mathcal{R}$ is any continuity set of Lebesgue measure. The method introduced by Jessen–Wintner was later developed by Borchsenius–Jessen [19]. Additionally, they studied a measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ defined as

$$\nu_{n,\sigma}(A) = \frac{1}{2\pi} \int_{\Omega(A)} |z'_{n,\sigma}(\theta)|^2 \, d\theta$$

for $n \in \mathbb{N}$ and $\sigma > 1/2$, where $\Omega(A) = \{\theta \in [0, 2\pi] \mid z_{n,\sigma}(\theta) \in A\}$. Remark that the convolution measure $Q_{N,\sigma} = \nu_{1,\sigma} * \cdots * \nu_{N,\sigma}$ can be defined similarly to $P_{N,\sigma}$ although $\nu_{n,\sigma}$ is not a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

THEOREM 1.8 (Borchsenius–Jessen). *As $N \to \infty$, $Q_{N,\sigma}$ converges weakly to a measure Q_σ if $\sigma > 1/2$. Furthermore, the limit measure Q_σ is absolutely continuous and satisfies*

$$Q_\sigma(A) = \int_A \mathcal{N}_\sigma(w)\, |dw|$$

for all $A \in \mathcal{B}(\mathbb{C})$, where $\mathcal{N}_\sigma : \mathbb{C} \to \mathbb{R}_{\geq 0}$ is a continuous function satisfying the following properties.

(i) *The function $\mathcal{N}_\sigma$ belongs to the class $C^\infty(\mathbb{C})$.*
(ii) *We have $\mathcal{N}_\sigma(w) > 0$ for all $w \in \mathbb{C}$ if $1/2 < \sigma \leq 1$.*
(iii) *We have $\mathcal{N}_\sigma(w) \ll e^{-c|w|^2}$ for any $c > 0$, and every partial derivative has the same upper bound.*

Then the limit measure Q_σ is associated with the distribution of a-points of $\log \zeta(s)$ as follows. For $a \in \mathbb{C}$ and $\sigma > 1/2$, we denote by $\phi_a(\sigma)$ the function

$$\phi_a(\sigma) = \lim_{T \to \infty} \frac{1}{T} \int_1^T \log |\log \zeta(\sigma + it) - a|\, dt$$

if the limit exists. The function $\phi_a(\sigma)$ is called the Jensen function.

THEOREM 1.9 (Borchsenius–Jessen). *Let $a \in \mathbb{C}$ and $\sigma > 1/2$. Then the Jensen function exists everywhere and is represented as*

$$\phi_a(\sigma) = \int_{\mathbb{C}} \log |w - a| \mathcal{M}_\sigma(w)\, |dw|,$$

where $\mathcal{M}_\sigma$ is the function of Theorem 1.6. Furthermore, the Jensen function ϕ_a is twice differentiable with the second derivative $\phi_a''(\sigma) = \mathcal{N}_\sigma(a)$, where $\mathcal{N}_\sigma$ is the function of Theorem 1.8. Finally, the limit value $C_a(\sigma_1, \sigma_2)$ of Theorem 1.5 is represented as

$$C_a(\sigma_1, \sigma_2) = \frac{1}{2\pi} \left(\phi_a'(\sigma_2) - \phi_a'(\sigma_1)\right) = \frac{1}{2\pi} \int_{\sigma_1}^{\sigma_2} \mathcal{N}_\sigma(a)\, d\sigma$$

for any $1/2 < \sigma_1 < \sigma_2$, which is positive if $\sigma_1 < 1$.

Furthermore, if we replace the function $\log \zeta(s)$ with $\zeta(s)$, some results similar to the above theorems hold after suitable modifications. We omit the statement of the results; see [**71**, Section 10] and [**19**, Section 44].

1.3. Discrepancy bounds. The major modern approach to study the value-distribution is to compare $\zeta(s)$ with some probabilistic model. With the notation of Sections 1.1 and 1.2, we define the quantity

$$D_\sigma(T, \mathcal{R}) = \left|P_{\sigma,T}(\mathcal{R}) - P_\sigma(\mathcal{R})\right|.$$

Then upper bounds for $D_\sigma(T, \mathcal{R})$ are called the discrepancy bounds for the value-distribution of $\zeta(s)$. Note that Theorem 1.4 deduces $D_\sigma(T, \mathcal{R}) = o(1)$ as $T \to \infty$.

The first improvement of the estimate of $D_\sigma(T,\mathcal{R})$ was achieved by Matsumoto [120–122] in 1980s.

THEOREM 1.10 (Matsumoto). *Let* $\sigma > 1/2$ *and* $\epsilon > 0$. *Then we have*

$$D_\sigma(T,\mathcal{R}) \ll \begin{cases} (\operatorname{meas}\mathcal{R}+1)(\log_2 T)^{-\frac{\sigma-1}{7}+\epsilon} & \text{for } \sigma > 1, \\ (\operatorname{meas}\mathcal{R}+1)(\log_2 T)^{-\frac{2\sigma-1}{15}+\epsilon} & \text{for } 1/2 < \sigma \leq 1, \end{cases} \tag{1.7}$$

where the implied constants depend on σ *and* ϵ.

Recall that the Bohr–Jessen limit theorem is applied to study the distribution of a-points of $\log\zeta(s)$. Matsumoto [122] also improved Theorem 1.5 by using his discrepancy bound.

THEOREM 1.11 (Matsumoto). *Let* $a \in \mathbb{C}$ *and* $1/2 < \sigma_1 < \sigma_2$. *Then there exist positive constants A and B which depend on a,* σ_1*, and* σ_2 *such that*

$$N_a(T,\sigma_1,\sigma_2) = C_a(\sigma_1,\sigma_2)T + \begin{cases} O\left((\log_2 T)^{-A}\right) & \text{for } \sigma_1 > 1, \\ O\left((\log_2 T)^{-B/\log_4 T}\right) & \text{for } \sigma_1 \leq 1, \end{cases}$$

where the implied constants depend on a, σ_1*, and* σ_2.

Harman–Matsumoto [56] refined estimates (1.7) in 1994. They proved

$$D_\sigma(T,\mathcal{R}) \ll \begin{cases} (\operatorname{meas}\mathcal{R}+1)(\log T)^{-\frac{\sigma-1}{2\sigma+3}+\epsilon} & \text{for } \sigma > 1, \\ (\operatorname{meas}\mathcal{R}+1)(\log T)^{-\frac{4\sigma-2}{8\sigma+21}+\epsilon} & \text{for } 1/2 < \sigma \leq 1, \end{cases} \tag{1.8}$$

where the implied constants depend on σ and ϵ. Over 20 years, estimates (1.8) were the best upper bounds for $D_\sigma(T,\mathcal{R})$. The recent progress was brought from the study of the distribution of extreme values of $\zeta(s)$; see Section 1.5. This enables us to evaluate the supremum value

$$D_\sigma(T) = \sup_{\mathcal{R}} |D_\sigma(T,\mathcal{R})|,$$

where $\mathcal{R}$ runs through all fixed rectangles with edges parallel to the axes. In 2019, Lamzouri–Lester–Radziwiłł [93] proved the upper bounds

$$D_\sigma(T) \ll \begin{cases} (\log T)^{-1}(\log_2 T)^2 & \text{for } \sigma = 1, \\ (\log T)^{-\sigma} & \text{for } 1/2 < \sigma < 1, \end{cases} \tag{1.9}$$

where the implied constants depend on σ. They also obtained an asymptotic formula for the density of a-points of $\zeta(s)$. See also Ha–Lee [54] for related results. The first main result is a sharp discrepancy bound which contains a slight improvement of (1.9). Note that a more general result is later presented as Theorem 11.2, whose proof is completed in Section 11.

THEOREM I. *Let* $\sigma > 1/2$. *Then we obtain*

$$D_\sigma(T) \ll \begin{cases} (\log T)^{-1}(\log_2 T) & \text{for } \sigma > 1, \\ (\log T)^{-1}(\log_2 T \log_3 T) & \text{for } \sigma = 1, \\ (\log T)^{-\sigma} & \text{for } 1/2 < \sigma < 1, \end{cases}$$

where the implied constants depend on σ.

1.4. Large deviations and extreme values (I). In the following two sections, we study the distribution of large values of $\log\zeta(s)$. For $\tau > 0$ and $\alpha \in \mathbb{R}$, we denote by $\mathcal{H}_\alpha(\tau)$ the inclined half plane $\{z \in \mathbb{C} \mid \operatorname{Re}(e^{-i\alpha}z) > \tau\}$. The ultimate goal is to describe the asymptotic behavior of the quantity

$$\Psi_{\sigma,T}(\tau;\alpha) = \mathbb{P}_T\left(\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma+it)) > \tau\right) = P_{\sigma,T}(\mathcal{H}_\alpha(\tau))$$

for large τ in some range depending on T. By the study of the discrepancy bounds, we expect that the quantity

$$\Psi_\sigma(\tau;\alpha) = P_\sigma(\mathcal{H}_\alpha(\tau))$$

gives a nice model to $\Psi_{\sigma,T}(\tau;\alpha)$. In this section, we study the behavior of $\Psi_\sigma(\tau;\alpha)$ as $\tau \to \infty$. Then we translate the results for $\Psi_\sigma(\tau;\alpha)$ to $\Psi_{\sigma,T}(\tau;\alpha)$ in Section 1.5. In the study of such large deviations, it is also natural to consider the set

$$\begin{aligned}\mathcal{T}(\tau) &= \mathcal{H}_0(\tau) \cup \mathcal{H}_{\pi/2}(\tau) \cup \mathcal{H}_\pi(\tau) \cup \mathcal{H}_{3\pi/2}(\tau)\\ &= \{z \in \mathbb{C} \mid |\operatorname{Re} z| > \tau \text{ or } |\operatorname{Im} z| > \tau\}\end{aligned}$$

and to study the behavior of $\Psi_\sigma(\tau) = P_\sigma(\mathcal{T}(\tau))$ as $\tau \to \infty$. Then we start with a simple observation derived from Theorem 1.6 (iii), that is, we have

$$\Psi_\sigma(\tau) \ll \exp(-c\tau^2) \tag{1.10}$$

for any $c > 0$. On the other hand, the results in [**19**,**71**] do not yield any lower bounds for $\Psi_\sigma(\tau)$. In 1986, Joyner [**72**] sharpened upper bound (1.10) for $1/2 < \sigma < 1$ and gave a lower bound in the same order.

THEOREM 1.12 (Joyner). *Let $1/2 < \sigma < 1$. Then there exist positive constants $c_1(\sigma)$ and $c_2(\sigma)$ such that*

$$\exp\left(-c_1(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\right) \le \Psi_\sigma(\tau) \le \exp\left(-c_2(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\right)$$

if τ is large enough.

The method of Joyner is related to the probabilistic studies by Montgomery [**147**] and Montgomery–Odlyzko [**148**]. Then, one may expect from Theorem 1.12 the existence of a suitable constant $A(\sigma) > 0$ such that an asymptotic formula of the form

$$\Psi_\sigma(\tau) = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}(1 + (\text{error term}))\right)$$

holds as $\tau \to \infty$. In 1999, Hattori–Matsumoto [**57**] showed the existence of such a constant and described it explicitly by using the modified Bessel function.

THEOREM 1.13 (Hattori–Matsumoto). *Let $1/2 < \sigma < 1$. Then we obtain*

$$\Psi_\sigma(\tau) = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}(1 + o(1))\right) \tag{1.11}$$

as $\tau \to \infty$, where $A(\sigma)$ is represented as

$$A(\sigma) = (1-\sigma)\left(\frac{1-\sigma}{\sigma}\int_0^\infty \log I_0(y^{-\sigma})\,dy\right)^{-\frac{\sigma}{1-\sigma}}. \tag{1.12}$$

In 2011, Lamzouri [**92**] proved similar asymptotic formulas for $\Psi_\sigma(\tau;\alpha)$ with the angles $\alpha = 0, \pi/2, \pi, 3\pi/2$.

Theorem 1.14 (Lamzouri). *Let* $1/2 < \sigma < 1$ *and* $\alpha = 0, \pi/2, \pi, 3\pi/2$. *If* $\tau > 0$ *is large enough, then we obtain*

$$\Psi_\sigma(\tau;\alpha) = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O((\log\tau)^{-1/2})\right)\right), \tag{1.13}$$

where $A(\sigma)$ *is the constant of* (1.12)*, and the implied constant depends only on* σ.

Note that formula (1.13) improves (1.11) since we obtain the inequalities

$$\Psi_\sigma(\tau;0) \le \Psi_\sigma(\tau) \le \sum_{j=0}^{3}\Psi_\sigma(\tau; j\pi/2), \tag{1.14}$$

and both sides of (1.14) save the same asymptotic formulas as (1.13). Inoue [**68**] recently announced a more general result on the value-distribution of a certain function $\widetilde{\eta}_m(s)$ which satisfies $\widetilde{\eta}_0(s) = \log\zeta(s)$; see Section 4.2.

1.5. Large deviations and extreme values (II). To begin with, we recall the study of the order of the magnitude of $\log\zeta(\sigma+it)$ as $t \to \infty$. Let $1/2 < \sigma < 1$. Titchmarsh [**180**, Theorem 14.5] proved the upper bound

$$\log\zeta(\sigma+it) = O\left(\frac{(\log t)^{2-2\sigma}}{\log_2 t}\right) \tag{1.15}$$

by assuming the Riemann Hypothesis, where t is large enough. On the other hand, Montgomery [**146**] proved the Ω-result

$$\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma+it)) = \Omega_\pm\left(\frac{(\log t)^{1-\sigma}}{(\log_2 t)^{\sigma}}\right) \tag{1.16}$$

as $t \to \infty$, which is independent of the truth of the Riemann Hypothesis. It is believed that upper bound (1.15) is not best possible, and more precisely, that the true order of the magnitude of $\log\zeta(\sigma+it)$ is close to (1.16). It motivates the investigation of the large deviation $\Psi_{\sigma,T}(\tau;\alpha)$ in the range $\tau \ll (\log T)^{1-\sigma+o(1)}$. The following result of Lamzouri [**92**] corresponds to Theorem 1.14.

Theorem 1.15 (Lamzouri). *Let* $1/2 < \sigma < 1$ *and* $\alpha = 0, \pi/2, \pi, 3\pi/2$. *Then there exists a positive constant* $b_1(\sigma)$ *such that*

$$\begin{aligned}\Psi_{\sigma,T}(\tau;\alpha)\\ = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O\left((\log\tau)^{-1/2} + r_1(\log T,\tau)\right)\right)\right)\end{aligned} \tag{1.17}$$

holds uniformly in the range $1 \ll \tau \le b_1(\sigma)(\log T)^{1-\sigma}(\log_2 T)^{-1}$, *where* $A(\sigma)$ *is the constant of* (1.12)*, and*

$$r_1(y,\tau) = \left(\frac{\tau}{y^{1-\sigma}(\log y)^{-1}}\right)^{\frac{2\sigma-1}{2-2\sigma}}.$$

Here, the implied constant in (1.17) *depends only on* σ.

In other words, we obtain the upper bound

$$\mathrm{Re}\left(e^{-i\alpha}\log\zeta(\sigma+it)\right) = O\left(\frac{(\log T)^{1-\sigma}}{\log_2 T}\right)$$

for all $t \in [0,T] \setminus \mathcal{E}$ with a small exceptional subset $\mathcal{E}$. In terms of comparisons of the probability measures $P_{\sigma,T}$ and P_σ, it is also worth considering asymptotic formulas of the form

$$\Psi_{\sigma,T}(\tau;\alpha) = \Psi_\sigma(\tau;\alpha)\cdot(1+(\text{error term}))\,. \tag{1.18}$$

Moreover, Theorem 1.15 is recovered from Theorem 1.14 if we obtain (1.18) in the same range $1 \ll \tau \leq b_1(\sigma)(\log T)^{1-\sigma}(\log_2 T)^{-1}$. In 2019, Lamzouri–Lester–Radziwiłł [**93**] proved the following result together with discrepancy bounds (1.9).

Theorem 1.16 (Lamzouri–Lester–Radziwiłł). *Let $1/2 < \sigma < 1$ and $\alpha = 0$ or π. Then there exists a positive constant $b_2(\sigma)$ such that*

$$\Psi_{\sigma,T}(\tau;\alpha) = \Psi_\sigma(\tau;\alpha)\cdot\left(1+O\left(r_2(\log T,\tau)\log_2 T\right)\right) \tag{1.19}$$

holds uniformly in the range[1] $3 \leq \tau \leq b_2(\sigma)(\log T)^{1-\sigma}(\log_2 T)^{-\frac{1}{\sigma}}$, *where*

$$r_2(y,\tau) = \left(\frac{\tau\log\tau}{y^{1-\sigma}}\right)^{\frac{\sigma}{1-\sigma}}. \tag{1.20}$$

The implied constant depends only on σ.

This result is not sufficient to recover Theorem 1.15 since the range of τ is narrower than that of Theorem 1.15. The second main result extends the range to fill the gap between Theorems 1.15 and 1.16.

Theorem II. *Let $1/2 < \sigma < 1$ and $\alpha \in \mathbb{R}$. For any $B \geq 1$, there exists a positive constant $b_3(\sigma, B)$ such that*

$$\Psi_{\sigma,T}(\tau;\alpha) = \Psi_\sigma(\tau;\alpha)\cdot\left(1+O\left((\log T)^{-B}+r_2(\log T,\tau)\right)\right)$$

uniformly in the range $1 \ll \tau \leq b_3(\sigma,B)(\log T)^{1-\sigma}(\log_2 T)^{-1}$, where $r_2(y,\tau)$ is defined as (1.20). The implied constant depends on σ, α, and B.

The proof of this result is presented in Section 15. Note that it is also a special case of the result in a recent joint work of the author, Endo, and Inoue [**38**]. More generally, they studied large deviations of values of Inoue's functions $\widetilde{\eta}_m(s)$ for $m \geq 0$.

1.6. Other topics. There still remain many issues on the theory of the value-distribution of $\zeta(s)$. We briefly introduce some of them to end Section 1.

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[1] The authors of [**93**] asserted that (1.19) holds in the range $3 \leq \tau \ll (\log T)^{1-\sigma}(\log_2 T)^{1-\frac{1}{\sigma}}$, which is wider than the range of τ in Theorem 1.15. However, their proof appears to be incorrect for $\tau \gg (\log T)^{1-\sigma}(\log_2 T)^{-\frac{1}{\sigma}}$.

1.6.1. *The value-distribution of* $(\zeta'/\zeta)(s)$. For $\operatorname{Re} s > 1$, the logarithmic derivative of $\zeta(s)$ is represented as

$$\frac{\zeta'}{\zeta}(s) = -\sum_p \sum_{k=1}^{\infty} (\log p) p^{-ks} \tag{1.21}$$

similarly to (1.4). Then it is expected that $(\zeta'/\zeta)(s)$ satisfies many properties similar to $\log \zeta(s)$ described above. Actually, Kershner–Wintner [**82**] proved a limit theorem for $(\zeta'/\zeta)(s)$ as an analogue of Theorem 1.4. A more precise result was obtained by Guo [**52**] in 1996. He proved the existence of a continuous function $M_\sigma : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that an asymptotic formula of the form

$$\frac{1}{T}\int_0^T \Phi\left(\frac{\zeta'}{\zeta}(\sigma+it)\right) dt = \int_{\mathbb{C}} \Phi(w) M_\sigma(w)\, |dw| + (\text{error term}) \tag{1.22}$$

holds for $\sigma > 1/2$, where the test function Φ is compactly supported and belongs to $C^\infty(\mathbb{C})$. Guo [**53**] also applied this result to study the distribution of zeros of $\zeta'(s)$. Furthermore, Suzuki [**176**] used the density function M_σ to study the distribution of zeros of the entire functions $A_\omega^{\pm}(s) = \xi(s+\omega) \pm \xi(s-\omega)$, where $\omega > 0$ and $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. Remark that (1.22) yields no discrepancy results directly since the indicator function $1_{\mathcal{R}}(w)$ does not belong to $C^\infty(\mathbb{C})$. In 2014, Lester [**105**, **106**] studied the value-distribution of $(\zeta'/\zeta)(s)$ where $s = \sigma + it$ varies near the critical line. He also obtained a certain discrepancy bound for $(\zeta'/\zeta)(s)$ by using a smooth approximation of the indicator function. Modifying the methods of Guo and Lester, the author [**134**, **136**] proved a similar result for $\sigma > 1/2$ in more general situations.

1.6.2. *Extreme values on the line* $\operatorname{Re} s = 1$. The distribution of values $\zeta(1+it)$ is more understood than $\zeta(\sigma+it)$ for $1/2 < \sigma < 1$. For example, Littlewood [**109**, **110**] proved that the Riemann Hypothesis implies the inequalities

$$\left(\frac{1}{2} + o(1)\right) \frac{\zeta(2)}{e^\gamma \log_2 t} \leq |\zeta(1+it)| \leq (2+o(1)) e^\gamma \log_2 t$$

as $t \to \infty$. See also Levinson [**107**] for Ω-results in this case. The behavior of the large deviations for values $\zeta(1+it)$ are significantly different from the results in Sections 1.4 and 1.5. In fact, Granville–Soundararajan [**51**] established certain asymptotic formulas for

$$\Psi_{1,T}^{+}(\tau) = \mathbb{P}_T(|\zeta(1+it)| > e^\gamma \tau) \quad \text{and} \quad \Psi_{1,T}^{-}(\tau) = \mathbb{P}_T\left(|\zeta(1+it)| < \frac{\zeta(2)}{e^\gamma \tau}\right)$$

as follows. Let A be the constant given by

$$A = \int_0^1 (\log I_0(y^{-1}) - y^{-1})\, dy + \int_1^\infty \log I_0(y^{-1})\, dy.$$

Then they proved the asymptotic formulas

$$\Psi_{1,T}^{\pm}(\tau) = \exp\left(-\frac{e^{\tau-A-1}}{\tau}\left(1 + O(\tau^{-1/2} + \sqrt{r_3(R(T),\tau)})\right)\right)$$

uniformly in the range $1 \ll \tau \leq R(T) := \log_2 T - 20$, where $r_3(y,\tau) = e^{\tau - y}$. For $\operatorname{Re} s = 1$, there are few results on the two-dimensional large deviations similar to Theorem 1.13. See Matsumoto [**125**] and Lamzouri [**89**] for results in this direction.

1.6.3. *The value-distribution on the line* $\operatorname{Re} s = 1/2$. The value-distribution of $\zeta(1/2+it)$ is quite different from the situation for $\sigma > 1/2$. Indeed, it is still an open problem whether the set $\{\zeta(1/2+it) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}$. For this problem, an approach based on random matrix theory was suggested by Kowalski–Nikeghbali [**88**] in 2012. Selberg studied an analogue of the Bohr–Jessen limit theorem for $\sigma = 1/2$ in his unpublished work. He obtained the limit formula

$$\lim_{T\to\infty} \mathbb{P}_T\left(\frac{\log \zeta|(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log_2 T}} > \tau \right) = \frac{1}{\sqrt{2\pi}} \int_\tau^\infty e^{-x^2}\, dx,$$

where τ is a fixed real number. The precise proof of this limit formula can be found in Laurinčikas [**97**]. Furthermore, several large deviation results were obtained by Jutila [**73**], Radziwiłł [**158**], and Inoue [**67**]. In Section 4.2, we revisit these issues, where we consider the value-distribution of Inoue's function $\widetilde{\eta}_m(1/2+it)$ with $m \geq 1$.

1.6.4. *The universality theorems.* As was described, the work of Bohr and his colleagues motivated various studies of the value-distribution of $\zeta(s)$. In a direction different from the results of the previous sections, Voronin presented several results in 1970s including notably the universality theorem. Let $0 < r < 1/4$. Suppose that $g(s)$ is a non-vanishing continuous function on the disk $|s| \leq r$ which is analytic in the interior $|s| < r$. Then Voronin [**182**, **183**] proved that

$$\liminf_{T\to\infty} \mathbb{P}_T\left(\max_{|s|\leq r} |\zeta(s+3/4+it) - g(s)| < \epsilon \right) > 0$$

holds for each $\epsilon > 0$. In other words, any function $g(s)$ with nice properties can be approximated by the Riemann zeta-function $\zeta(s+3/4+it)$, and furthermore, the set of such shifts t has a positive lower density. Note that Bagchi [**4**] and Gonek [**49**] obtained similar results independently of Voronin. For more information about the universality theorems, see the survey of Matsumoto [**129**] and the books of Karatsuba–Voronin [**81**], Laurinčikas [**99**], and Steuding [**175**].

2. Dirichlet L-functions

In Section 1, we introduced several results on the value-distribution of $\zeta(s)$ as the complex variable $s = \sigma + it$ varies on the vertical lines, namely in the t-aspect. In this section, we consider the value-distributions of Dirichlet L-functions as preparation for the main results in the latter sections. Let χ be an arithmetic function defined from a character $\widetilde{\chi}$ of the group $(\mathbb{Z}/m\mathbb{Z})^\times$ by $\chi(n) = \widetilde{\chi}(n \bmod m)$ if $(n,m) = 1$ and $\chi(n) = 0$ otherwise. Then χ is called a Dirichlet character to the modulus m, which is often identified with $\widetilde{\chi}$. The Dirichlet L-function attached to χ is defined as

$$L(s,\chi) = \sum_{n=1}^\infty \chi(n) n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

for $\operatorname{Re} s > 1$. The Dirichlet character χ_0 corresponding to the trivial character of $(\mathbb{Z}/m\mathbb{Z})^\times$ is called the principal character to the modulus m. In this case, the Dirichlet L-function $L(s, \chi_0)$ is essentially equal to the Riemann zeta-function since $L(s, \chi_0) = \zeta(s) \prod_{p|m}(1 - p^{-s})$. It is known that $L(s, \chi)$ is continued to an entire function if χ is non-principal. Furthermore, $L(s, \chi)$ satisfies several properties similar to (b)–(f) of Section 1 if χ is a primitive character; see Davenport [31]. Here we say that a Dirichlet character χ to the modulus m is primitive if it is not in the image of the natural injection

$$\iota : \widehat{(\mathbb{Z}/m'\mathbb{Z})^\times} \to \widehat{(\mathbb{Z}/m\mathbb{Z})^\times}$$

for all proper divisors m' of m. Note that any Dirichlet character χ is obtained by $\chi = \iota(\chi')$ with a unique primitive character χ' to the modulus m'. Such a modulus m' is called the conductor of the Dirichlet character χ. Then, the following generalized Riemann Hypothesis is still open.

(g′) Let χ be a primitive Dirichlet character. Then all zeros of $L(s, \chi)$ in the critical strip $0 \le \operatorname{Re} s \le 1$ lie on the critical line $\operatorname{Re} s = 1/2$.

Therefore $\log L(s, \chi)$ should be defined as a holomorphic function on the region

$$G_\chi = \{\sigma + it \mid \sigma > 1/2\} \setminus \bigcup_{L(\rho,\chi)=0} \{\sigma + i \operatorname{Im} \rho \mid 1/2 < \sigma \le \operatorname{Re} \rho\} \tag{2.1}$$

in a way similar to $\log \zeta(s)$.

It is natural to consider how the results on the value-distribution of $\zeta(s)$ in the t-aspect are modified for Dirichlet L-functions. Moreover, the investigation in the χ-aspect is rather interesting in this case, that is, we study the distribution of the values $L(s, \chi)$ as χ varies over the following set of characters. For a prime number $q \ge 3$, we define

$$\mathcal{X}_q = \{\chi \mid \chi \text{ is a primitive Dirichlet character of conductor } q\}.$$

It is also useful to restrict the situation to real characters $\chi_d = \left(\frac{d}{\cdot}\right)$, where $\left(\frac{a}{b}\right)$ denotes the Kronecker symbol. For this, we define three classes of discriminants d as

$$\begin{aligned} \mathcal{D} &= \{d \in \mathbb{Z} \mid d \text{ is a non-square integer with } d \equiv 0, 1 \bmod 4\}, \\ \mathcal{F} &= \{d \in \mathcal{D} \mid d \equiv 1 \bmod 4 \text{ or } d = 4m \text{ with } m \equiv 2, 3 \bmod 4\}, \\ \mathcal{P} &= \{d \in \mathcal{F} \mid d \text{ is a prime number}\}, \end{aligned}$$

and study the value-distributions of $L(s, \chi_d)$ as d varies over these classes. An integer $d \in \mathcal{F}$ is especially called a fundamental discriminant, and any real primitive character χ is given by $\chi = \chi_d$ with some $d \in \mathcal{F}$. In addition, we define

$$\mathcal{D}_X = \{d \in \mathcal{D} \mid |d| \le X\}$$

for every $X > 0$, and $\mathcal{F}_X$, $\mathcal{P}_X$ similarly. Then we also use the notation such as $\mathcal{D}_X^+ = \{d \in \mathcal{D}_X \mid d > 0\}$, $\mathcal{D}_X^- = \{d \in \mathcal{D}_X \mid d < 0\}$, and so on. In the following sections, we see several results on the value-distributions of $L(s, \chi)$ in the χ-aspect which can be regarded as analogues of results in Section 1. It is remarkable that some of them were obtained before the results for $\zeta(s)$.

2.1. Limit theorems and discrepancy estimates. Similarly to the notation $\mathbb{P}_T$, we define

$$\mathbb{P}_S(\cdots) = \frac{\#\{d \in S \mid \cdots\}}{\#S}$$

for any finite subset $S \subset \mathcal{D}$. Let $\sigma > 1/2$ be a fixed real number. Then we see that $P_{\sigma,S}(A) = \mathbb{P}_S(L(\sigma, \chi_d) \in A)$ gives a probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ since values $L(\sigma, \chi_d)$ are real. We denote by $F_{\sigma,S}(t) = \mathbb{P}_S(L(\sigma, \chi_d) \leq t)$ the distribution function of this probability measure. In 1951, Chowla–Erdős [26] studied the case $S = \mathcal{D}_X^+$ and proved a limit theorem for Dirichlet L-functions in χ-aspect.

Theorem 2.1 (Chowla–Erdős). *Let $\sigma > 3/4$. Then there exists a continuous distribution function $F_{\sigma,\mathcal{D}}$ such that $F_{\sigma,\mathcal{D}_X^+}(t) \to F_{\sigma,\mathcal{D}}(t)$ as $X \to \infty$ for all $t \in \mathbb{R}$.*

Hence $P_{\sigma,\mathcal{D}_X^+}$ converges weakly to a probability measure $P_{\sigma,\mathcal{D}}$ as $X \to \infty$ whose distribution is equal to $F_{\sigma,\mathcal{D}}$. Additionally, we obtain the limit formula

$$\lim_{X\to\infty} P_{\sigma,\mathcal{D}_X^+}(\mathcal{I}) = P_{\sigma,\mathcal{D}}(\mathcal{I})$$

for all intervals $\mathcal{I} = (a, b]$ by considering the difference $F_{\sigma,\mathcal{D}_X^+}(b) - F_{\sigma,\mathcal{D}_X^+}(a)$. Therefore Theorem 2.1 can be regarded as analogues of Theorems 1.4 and 1.7. In 1966, Barban [7] introduced another method to prove a similar limit theorem. He studied values $L(1, \chi_d)$ for negative discriminants d. To describe the limit distribution more precisely, he used the integral moments

$$m_k = \lim_{X\to\infty} \frac{1}{\#\mathcal{D}_X^-} \sum_{d\in\mathcal{D}_X^-} L(1, \chi_d)^k, \quad k = 0, 1, 2, \ldots. \tag{2.2}$$

The properties of m_k were studied by Barban [6] and later by Wolke [187], Lavrik [102], and others.

Theorem 2.2 (Barban). *Let $F_{\sigma,\mathcal{D}}$ be the distribution function of Theorem 2.1. Then we obtain $F_{1,\mathcal{D}_X^-}(t) \to F_{1,\mathcal{D}}(t)$ as $X \to \infty$ for all $t \in \mathbb{R}$. Furthermore, the characteristic function of $F_{1,\mathcal{D}}$ is represented as $\phi(y) = \sum_{k=0}^{\infty} \frac{m_k}{k!}(iy)^k$ by using the integral moment m_k of (2.2).*

The reason why Barban considered only the case $s = 1$ is probably because his motivation was to study the distribution of class numbers which are connected with the values $L(1, \chi_d)$. Let h_d denote the class number of a discriminant d in the narrow sense. We put $\epsilon_d = (u_d + v_d\sqrt{d})/2$ for $d > 0$, where (u_d, v_d) is the fundamental solution of the Pell equation $u^2 - dv^2 = 4$. Then we have the equality

$$\sqrt{|d|}L(1, \chi_d) = \begin{cases} h_d \log \epsilon_d & \text{for } d > 0, \\ \frac{\pi}{3} h_d & \text{for } d = -3, \\ \frac{\pi}{2} h_d & \text{for } d = -4, \\ \pi h_d & \text{for } d < -4 \end{cases}$$

which is known as Dirichlet's class number formula [31, Section 6]. Hence we deduce from Theorems 2.1 and 2.2 the limit formulas

$$\lim_{X\to\infty} \frac{1}{\#\mathcal{D}_X^+}\#\left\{d \in \mathcal{D}_X^+ \;\middle|\; h_d \log \epsilon_d \leq t\sqrt{d}\right\} = F_{1,\mathcal{D}}(t),$$

$$\lim_{X\to\infty} \frac{1}{\#\mathcal{D}_X^-}\#\left\{d \in \mathcal{D}_X^- \;\middle|\; h_d \leq \frac{t}{\pi}\sqrt{d}\right\} = F_{1,\mathcal{D}}(t).$$

Stankus [173,174] studied similar limit theorems for the probability measure defined as $P_{s,\mathcal{D}_X} = \mathbb{P}_{\mathcal{D}_X}(L(s,\chi_d) \in A)$ for $A \in \mathcal{B}(\mathbb{C})$. Then he proved that it converges weakly to some probability measure as $X \to \infty$ for any $s = \sigma + it$ with $\sigma > 1/2$. Furthermore, he obtained a similar result for another probability measure defined by using the set $\mathcal{X}_q$. These results were derived by adapting the method introduced by Elliott [33–36] in 1970s. This method is quite similar to Jessen–Wintner [71]. Moreover, Elliott applied Esseen's inequality [39] in probability theory to refine the limit theorems. Define a probability measure $P_{s,\mathcal{P}_X}(A) = \mathbb{P}_{\mathcal{P}_X}(|L(s,\chi_d)| \in A)$ for $A \in \mathcal{B}(\mathbb{R})$, where $s = \sigma + it$ with $\sigma > 1/2$. Then we denote by $F_{s,\mathcal{P}_X}(t) = \mathbb{P}_{\mathcal{P}_X}(|L(s,\chi_d)| \leq t)$ its distribution function.

Theorem 2.3 (Elliott). *Let $s = \sigma + it$ with $\sigma > 1/2$. Then there exists a continuous distribution function $F_{s,\mathcal{P}}$ such that*

$$F_{s,\mathcal{P}_X}(t) = F_{s,\mathcal{P}}(t) + O\left((\log_3 X)^{-2}\right)$$

uniformly for all $t \in \mathbb{R}$. The characteristic function of $F_{s,\mathcal{P}}$ is represented as

$$\phi(y) = \prod_p \frac{1}{2}\left\{ \exp\left(-iy \log|1 - p^{-s}|\right) + \exp\left(-iy \log|1 + p^{-s}|\right) \right\}.$$

Let $P_{s,\mathcal{P}}$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ attached to the distribution function $F_{s,\mathcal{P}}$. Then Theorem 2.3 yields the discrepancy bound

$$\begin{aligned} D_{s,\mathcal{P}}(X) &:= \sup_{\mathcal{I}} \left|P_{s,\mathcal{P}_X}(\mathcal{I}) - P_{s,\mathcal{P}}(\mathcal{I})\right| \\ &\ll (\log_3 X)^{-2}, \end{aligned} \tag{2.3}$$

where $\mathcal{I}$ runs through all intervals $(a,b]$ in $\mathbb{R}$. A modified result was also obtained by Fomenko [42] in 2006. Compared with the estimates in Section 1.3, there should be still room for improvement of (2.3). It is expected that one can derive sharper discrepancy bounds by adapting the method of the proof of Theorem I.

Finally, we recall the denseness results of Mishou–Nagoshi [141] in 2006. They proved the χ-universality theorem, i.e. the universality theorem for $L(s,\chi)$ in the χ-aspect. They also obtained the following conclusion as a consequence of the universality.

Theorem 2.4 (Mishou–Nagoshi). *Let $\epsilon > 0$ be a real number.*

(i) *Let $s = \sigma + it$ with $1/2 < \sigma \leq 1$ and $t \neq 0$. Then we have*

$$\liminf_{X\to\infty} \mathbb{P}_{\mathcal{F}_X^\pm}\left(|L(s,\chi_d) - z| < \epsilon\right) > 0,$$

where z is an arbitrary complex number.

(ii) *Let* $1/2 < \sigma \leq 1$. *Then we have*

$$\liminf_{X\to\infty} \mathbb{P}_{\mathcal{F}_X^\pm}\Big(|L(\sigma, \chi_d) - x| < \epsilon\Big) > 0.$$

where x *is an arbitrary real number.*

Therefore $\{L(s, \chi_d) \mid d \in \mathcal{F}^\pm\}$ is dense in $\mathbb{C}$ for $s = \sigma + it$ with $1/2 < \sigma \leq 1$ and $t \neq 0$, and $\{L(\sigma, \chi_d) \mid d \in \mathcal{F}^\pm\}$ is dense in $\mathbb{R}$ for $1/2 < \sigma \leq 1$. Mishou–Nagoshi [**142**, **143**] obtained similar results for prime discriminants. See also [**144**] for a recent result in this direction.

2.2. Large deviations and extreme values. In 1980, a notable result on the large deviation for values $L(1, \chi)$ was obtained by Monach [**145**]. Define a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as

$$P_{1,\mathcal{X}_q}(A) = \frac{\#\{\chi \in \mathcal{X}_q \mid \arg L(1, \chi) \in A\}}{\#\mathcal{X}_q},$$

where $\arg L(1, \chi)$ is understood as the imaginary part of $\log L(1, \chi)$. Adapting the method of Elliott, one can prove that it converges weakly to a probability measure P_1 as $q \to \infty$. Denote by $\mathcal{H}_\pm(\tau)$ the half lines $\mathcal{H}_+(\tau) = \{x \in \mathbb{R} \mid x > \tau\}$ and $\mathcal{H}_-(\tau) = \{x \in \mathbb{R} \mid x < -\tau\}$ for $\tau > 0$. Then Monach proved lower and upper bounds for the quantity $P_1(\mathcal{H}_\pm(\tau))$.

Theorem 2.5 (Monach). *There exist positive constants* c_1 *and* c_2 *such that*

$$\exp\left(-\exp(c_1 e^\tau)\right) \leq P_1(\mathcal{H}_\pm(\tau)) \leq \exp\left(-\exp(c_2 e^\tau)\right) \tag{2.4}$$

if $\tau > 0$ *is large enough.*

Note that (2.4) is similar to an estimate proved by Matsumoto [**125**] in 1991, which is about the large deviation for values $\log \zeta(1 + it)$ in the t-aspect. See also Lamzouri [**91**] for more precise results for $P_1(\mathcal{H}_\pm(\tau))$ and $P_{1,\mathcal{X}_q}(\mathcal{H}_\pm(\tau))$. The study of the large deviations of values $L(1, \chi)$ was significantly developed after Monach's work. In particular, we consider the probability measure

$$P_{1,\mathcal{F}_X}(A) = \mathbb{P}_{\mathcal{F}_X}\Big(\log L(1, \chi_d) \in A\Big).$$

One can show that it converges weakly to a probability measure $P_{1,\mathcal{F}}$ as $X \to \infty$. In this case, we modify the half lines as $\overline{\mathcal{H}}_+(\tau) = \mathcal{H}_+(\log \tau + \gamma)$ and $\overline{\mathcal{H}}_-(\tau) = \mathcal{H}_-(\log \tau + \gamma - \log \zeta(2))$ to study the large deviations

$$\Psi^+_{1,X}(\tau) = \mathbb{P}_{\mathcal{F}_X}\Big(|L(1, \chi_d)| > e^\gamma \tau\Big) = P_{1,\mathcal{F}_X}(\overline{\mathcal{H}}_+(\tau)),$$
$$\Psi^-_{1,X}(\tau) = \mathbb{P}_{\mathcal{F}_X}\left(|L(1, \chi_d)| < \frac{\zeta(2)}{e^\gamma \tau}\right) = P_{1,\mathcal{F}_X}(\overline{\mathcal{H}}_-(\tau)).$$

Put $\Psi^\pm_1(\tau) = P_{1,\mathcal{F}}(\overline{\mathcal{H}}_\pm(\tau))$. With the notation above, Granville–Soundararajan [**50**] achieved the following results in 2003.

THEOREM 2.6 (Granville–Soundararajan). *Let A be the constant given by*

$$A = \int_0^1 (\log\cosh(y^{-1}) - y^{-1})\, dy + \int_1^\infty \log\cosh(y^{-1})\, dy.$$

(i) *For large τ, we have*

$$\Psi_1^\pm(\tau) = \exp\left(-\frac{e^{\tau-A-1}}{\tau}(1+O(\tau^{-1}))\right).$$

(ii) *Let $R(X) = \log_2 X + \log_4 X - 20$, and take a real number r with $e \le r \le \log_2 X$. Then we have*

$$\Psi_{1,X}^\pm(\tau) = \exp\left(-\frac{e^{\tau-A-1}}{\tau}(1+O(\tau^{-1}+r^{-1}))\right)$$

uniformly in the range $\tau \le R(X) - \log_2 r$.

(iii) *We have*

$$\Psi_{1,X}^\pm(\tau) = \Psi_1^\pm(\tau)\cdot\left(1+O\left((\log X)^{-5} + r_3(R(X),\tau)\right)\right)$$

uniformly in the range $\tau \le R(X) - 2\log_3 X$, where $r_3(y,\tau) = e^{\tau-y}$.

Then, Theorems 1.14, 1.15, and 1.16 were obtained as analogues of these results for the value-distribution of $\zeta(\sigma+it)$ in the t-aspect. In 2007, Wu [**189**] presented an improved method for the proof of Theorem 2.6 (iii) by using the saddle-point method of Hildebrand–Tenenbaum [**61**]. The method of Lamzouri–Lester–Radziwiłł [**93**] appears to be based on Wu's approach, although they did not refer to his work.

2.3. M-functions.

In 2008, Ihara [**62**] introduced the notion of M-functions which are density function closely related to the value-distributions of L-functions. At first, he studied the value-distribution of $(L'/L)(s,\chi)$ as χ varies over the set $\mathcal{X}_q$. Note that his result was obtained for more general L-functions defined for global fields K, but we describe here the case $K = \mathbb{Q}$ for comparisons with the above results.

THEOREM 2.7 (Ihara). *Let $s = \sigma + it$ with $\sigma > 1$. Then there exists a continuous function $M_\sigma : \mathbb{C} \to \mathbb{R}_{\ge 0}$ such that*

$$\lim_{Q\to\infty} \frac{1}{\pi(Q)} \sum_{q\le Q} \frac{1}{\#\mathcal{X}_q} \sum_{\chi\in\mathcal{X}_q} \Phi\left(\frac{L'}{L}(s,\chi)\right) = \int_{\mathbb{C}} \Phi(w) M_\sigma(w)\, |dw| \tag{2.5}$$

for all continuous functions Φ on $\mathbb{C}$.

He also proved that a similar limit formula holds for $(\zeta'/\zeta)(\sigma+it)$ in the t-aspect with the same M-function M_σ. Hence one can see that the function M_σ agrees with Guo's one described in Section 1.6. Furthermore, Ihara constructed it explicitly in terms of Schwartz distributions. Let $\sigma > 1/2$. For a prime number p, we put

$$c_{\sigma,p} = \frac{-\log p}{p^{2\sigma}-1} \quad\text{and}\quad r_{\sigma,p} = \frac{p^\sigma \log p}{p^{2\sigma}-1}.$$

Then, we define a Schwartz distribution $M_{\sigma,p}$ on $\mathbb{C}$ as

$$M_{\sigma,p}(c_{\sigma,p}+re^{i\theta})=\frac{p^{2\sigma}-1}{|p^{\sigma}-e^{i\theta}|^2}\cdot\frac{\delta(r-r_{\sigma,p})}{r},$$

where δ is the Dirac delta distribution on $\mathbb{R}$. Denote by $M_{\sigma,P}$ the convolution $M_{\sigma,P}=*_{p\in P}M_{\sigma,p}$, where P is a finite set of prime numbers, and the convolutions are taken with respect to the measure $|dw|$. Let P_y be the set of prime numbers not exceeding y. Then Ihara [62, Theorem 2] showed that M_{σ,P_y} converges to M_σ as $y\to\infty$ for any $\sigma>1/2$. He further proved that the function M_σ belongs to $C^\infty(\mathbb{C})$. The Schwartz distribution $M_{\sigma,p}$ is associated with the Fourier transform $\widetilde{M}_{\sigma,p}$. We define the $\widetilde{M}$-functions $\widetilde{M}_{\sigma,P}$ and $\widetilde{M}_\sigma$ as the Fourier transforms of $M_{\sigma,P}$ and M_σ, respectively. Then we obtain $\widetilde{M}_{\sigma,P}=\prod_{p\in P}\widetilde{M}_{\sigma,p}$ and $\widetilde{M}_{\sigma,P_y}\to\widetilde{M}_\sigma$ as $y\to\infty$. Hence we have

$$\widetilde{M}_\sigma(z)=\prod_p\widetilde{M}_{\sigma,p}(z), \tag{2.6}$$

where p runs through all prime numbers. There are also two types of representations

$$\widetilde{M}_\sigma(z)=\sum_{n=1}^{\infty}\lambda_z(n)\lambda_{\overline{z}}(n)n^{-2\sigma}=\sum_{a,b=0}^{\infty}\left(\frac{-i}{2}\right)^{a+b}\mu_\sigma^{(a,b)}\frac{z^a\overline{z}^b}{a!b!}$$

with explicit coefficients $\lambda_z(n)$ and $\mu_\sigma^{(a,b)}$; see Ihara [62, Theorem 3]. Note that (2.5) yields the limit formulas

$$\lim_{Q\to\infty}\frac{1}{\pi(Q)}\sum_{q\le Q}\frac{1}{\#\mathcal{X}_q}\sum_{\chi\in\mathcal{X}_q}\psi_z\left(\frac{L'}{L}(s,\chi)\right)=\widetilde{M}_\sigma(z), \tag{2.7}$$

$$\lim_{Q\to\infty}\frac{1}{\pi(Q)}\sum_{q\le Q}\frac{1}{\#\mathcal{X}_q}\sum_{\chi\in\mathcal{X}_q}P^{(a,b)}\left(\frac{L'}{L}(s,\chi)\right)=(-1)^{a+b}\mu_\sigma^{(a,b)}, \tag{2.8}$$

where $P^{(a,b)}(w)=\overline{w}^aw^b$. Ihara proceeded to study formulas (2.5), (2.7), and (2.8) for $\sigma\le 1$ with his colleagues. In 2009, Ihara–Murty–Shimura [66] obtained an asymptotic formula similar to (2.8) at $s=1$.

Theorem 2.8 (Ihara–Murty–Shimura). *Let (a,b) be a pair of non-negative integers. For any $\epsilon>0$, we obtain*

$$\frac{1}{\#\mathcal{X}_q}\sum_{\chi\in\mathcal{X}_q}P^{(a,b)}\left(\frac{L'}{L}(1,\chi)\right)=(-1)^{a+b}\mu_1^{(a,b)}+O\left(q^{-1+\epsilon}\right).$$

If the generalized Riemann Hypothesis is true, the estimate of the error term can be improved to $O(q^{-1}(\log q)^d)$, where $d=a+b+1$ if $ab=0$, and $d=a+b+2$ otherwise.

Later Ihara's work joined up with Matsumoto. First, Ihara–Matsumoto [63] proved in 2011 a limit formula similar to (2.5) for $\sigma>1/2$, where $(L'/L)(s,\chi)$ is replaced by $\log L(s,\chi)$. Indeed, they showed the existence of a continuous function $\mathcal{M}_\sigma : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that

$$\lim_{Q\to\infty} \frac{1}{\pi(Q)} \sum_{q\leq Q} \frac{1}{\#\mathcal{X}_q} {\sum_{\chi\in\mathcal{X}_q}}' \Phi\left(\log L(s,\chi)\right) = \int_{\mathbb{C}} \Phi(w)\mathcal{M}_\sigma(w)\,|dw| \tag{2.9}$$

with some suitable test function Φ, where $\sum'$ stands for the sum over χ with $s \in G_\chi$. Here, the M-function $\mathcal{M}_\sigma$ is constructed similarly to M_σ, and it agrees with that of Theorem 1.4 (ii). Based on a similar method, Ihara–Matsumoto [**65**] also prove the following result.

Theorem 2.9 (Ihara–Matsumoto). *Let $s = \sigma + it$ with $\sigma > 1/2$ be a fixed complex number, and denote by $\sum'$ the sum over χ with $L(s,\chi) \neq 0$. Then we obtain*

$$\lim_{Q\to\infty} \frac{1}{\pi(Q)} \sum_{q\leq Q} \frac{1}{\#\mathcal{X}_q} {\sum_{\chi\in\mathcal{X}_q}}' \Phi\left(\frac{L'}{L}(s,\chi)\right) = \int_{\mathbb{C}} \Phi(w)M_\sigma(w)\,|dw|,$$

where the test function Φ is one of the following:

(i) *Φ is any bounded continuous function on $\mathbb{C}$;*
(ii) *Φ is the indicator function of either a compact set of $\mathbb{C}$ or the complement of such a set.*

Furthermore, Ihara–Matsumoto [**64**] introduced another method to prove the limit formula

$$\lim_{\substack{q\to\infty \\ q:\ \text{prime}}} \frac{1}{\#\mathcal{X}_q} {\sum_{\chi\in\mathcal{X}_q}}' \Phi\left(\frac{L'}{L}(s,\chi)\right) = \int_{\mathbb{C}} \Phi(w)M_\sigma(w)\,|dw| \tag{2.10}$$

with suitable test functions Φ. Assuming the generalized Riemann Hypothesis, they proved that (2.10) holds for all continuous functions Φ satisfying $\Phi(w) \ll e^{c|w|}$ with some $c > 0$. After the initial work of Ihara–Matsumoto, some analogous results were obtained by several researchers. See the survey of Matsumoto [**130**] for the results obtained by 2019. In particular, Mourtada–Murty [**150**] obtained the M-function for Dirichlet L-functions of real characters in 2015.

Theorem 2.10 (Mourtada–Murty). *Let $\sigma > 1/2$ and assume the generalized Riemann Hypothesis. Then there exists a continuous function $Q_\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that*

$$\lim_{X\to\infty} \frac{1}{\#\mathcal{F}_X} \sum_{d\in\mathcal{F}_X} \Phi\left(\frac{L'}{L}(\sigma,\chi_d)\right) = \int_{\mathbb{R}} \Phi(u)Q_\sigma(u)\,|du| \tag{2.11}$$

holds, where the test function Φ is one of the following:

(i′) *Φ is any bounded continuous function on $\mathbb{R}$;*
(ii′) *Φ is the indicator function of either a compact set of $\mathbb{R}$ or the complement of such a set.*

As Akbary–Hamieh [**1**] pointed out, the assumption of the generalized Riemann Hypothesis is removable in this result. Moreover, compared to Theorem 2.4, it remains to consider the M-function for the value-distribution of $L(s,\chi_d)$ with $s \notin \mathbb{R}$. We discuss related topics in Section 3.4.

3. L-functions of higher degree

Recall that the Riemann zeta-function and Dirichlet L-functions are represented as the infinite products $\zeta(s) = \prod_p \Phi_p(p^{-s})^{-1}$ and $L(s,\chi) = \prod_p \Phi_p(\chi(p)p^{-s})^{-1}$ with $\Phi_p(T) = 1 - T$. Furthermore, we know zeta and L-functions represented as $F(s) = \prod_p \Phi_p(p^{-s})^{-1}$ with polynomials $\Phi_p(T)$ of higher degree. Typical examples are the Dedekind zeta-function

$$\zeta_{\mathbb{Q}(i)}(s) = (1-2^{-s})^{-1} \prod_{p \equiv 1 \bmod 4} (1-p^{-s})^{-2} \prod_{p \equiv 3 \bmod 4} (1-p^{-2s})^{-1}$$

and the automorphic L-function

$$L(s,\Delta) = \prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1}$$

attached to the modular discriminant

$$\Delta(z) = e^{2\pi i z} \prod_{k=1}^{\infty} (1-e^{2\pi i k z})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}.$$

In the following sections, we see how results of Sections 1 and 2 are generalized to L-functions of higher degree, e.g. Dedekind zeta-functions, Artin L-functions, and automorphic L-functions of cusp forms. See Iwaniec–Kowalski [70, Section 5] for basic properties of these L-functions.

3.1. Dedekind zeta-functions.

Let K be a number field, and denote by $\mathcal{O}_K$ the ring of integers of K. Then the Dedekind zeta-function $\zeta_K(s)$ is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \mathrm{N}(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} (1-\mathrm{N}(\mathfrak{p})^{-s})^{-1},$$

where $\mathfrak{a}$ and $\mathfrak{p}$ run through all integral ideals and prime ideals of $\mathcal{O}_K$, respectively. Here, $\mathrm{N}(\mathfrak{a})$ stands for the ideal norm of $\mathfrak{a}$. Note that $\zeta_{\mathbb{Q}}(s)$ is equal to the Riemann zeta-function. In 1990s, Matsumoto [125, 126] generalized the Bohr–Jessen limit theorem to Dedekind zeta-functions $\zeta_K(s)$.

Theorem 3.1 (Matsumoto). *Let K be a number field of degree d. Then the following results hold for $\sigma > \max\{1-d^{-1}, 1/2\}$.*

(i) *There exists the limit value*

$$W_\sigma(\mathcal{R};\zeta_K) = \lim_{T\to\infty} \mathbb{P}_T(\log \zeta_K(\sigma+it) \in \mathcal{R}),$$

where $\mathcal{R}$ is a rectangle in $\mathbb{C}$ whose sides are parallel to the axes.

(ii) *Suppose that $K/\mathbb{Q}$ is a Galois extension. Then there exists a continuous function $\mathcal{M}_\sigma(\,\cdot\,;\zeta_K) : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that*

$$W_\sigma(\mathcal{R};\zeta_K) = \int_{\mathcal{R}} \mathcal{M}_\sigma(w;\zeta_K)\,|dw|.$$

Here, the branch of $\log \zeta_K(s)$ is determined similarly to $\log \zeta(s)$. Moreover, some results on the large deviation of values $\zeta_K(s)$ were obtained by Matsumoto [125] and Hattori–Matsumoto [57]. The discrepancy bound in this case was firstly obtained by Harman–Matsumoto [56] with the assumption that $K/\mathbb{Q}$ is Galois. In

2007, Matsumoto [**128**] refined the method of [**56**] to derive the discrepancy bound for an arbitrary number field K. Then, the work on $\mathcal{M}_\sigma(w;\zeta_K)$ for a non-Galois number field K remained. Modifying the method of Guo [**52**], the author [**134**,**136**] obtained a similar M-function $M_\sigma(w;\zeta_K)$ which is associated with the limit theorem for $(\zeta_K'/\zeta_K)(s)$. Finally, the existence of $\mathcal{M}_\sigma(w;\zeta_K)$ for an arbitrary number field K was proved in [**137**]. We prove a more general result in Chapter 4 as Theorem 11.1.

3.2. Hecke L-functions of ideal class characters. Matsumoto [**127**] further generalized the results in [**128**] to Hecke L-functions $L_K(s,\chi)$ attached to ideal class characters. Similarly to the case of Dirichlet L-functions, it is also worth studying the value-distributions of Hecke L-functions in the χ-aspect. Akbary–Hamieh [**1**] studied especially cubic characters defined on the ideal class group of $k=\mathbb{Q}(\sqrt{-3})$. Let

$$C=\{c\in\mathcal{O}_k \mid c\neq 1 \text{ is square-free and } c\equiv 1 \bmod 9\mathcal{O}_k\}$$

and $K=k(\sqrt[3]{c})$ with $c\in C$. Then there exists a cubic character χ_c such that the Dedekind zeta-function of K is factorized as $\zeta_K(s)=\zeta_k(s)L_k(s,\chi_c)L_k(s,\overline{\chi_c})$. Recalling that the formula $\zeta_{\mathbb{Q}(\sqrt{d})}(s)=\zeta(s)L(s,\chi_d)$ holds for $d\in\mathcal{F}$, we define

$$L_c(s)=L_k(s,\chi_c)L_k(s,\overline{\chi_c})$$

as a cubic analogue of the real quadratic Dirichlet L-function $L(s,\chi_d)$. Akbary–Hamieh showed in 2020 a limit theorem for the L-function $L_c(s)$ as c varies over the set

$$C_X=\{c\in C \mid \mathrm{N}(c)\leq X\}.$$

Theorem 3.2 (Akbary–Hamieh). *Let $\sigma>1/2$. Then there exists a continuous function $\mathscr{C}_\sigma:\mathbb{R}\to\mathbb{R}_{\geq 0}$ such that*

$$\lim_{X\to\infty}\frac{1}{\#C_X}\sum_{c\in C_X}\Phi\left(\frac{L_c'}{L_c}(\sigma)\right)=\int_{\mathbb{R}}\Phi(u)\mathscr{C}_\sigma(u)\,|du|$$

holds, where the test function Φ is one of (i′) *and* (ii′) *of Theorem 2.10.*

Here, we slightly modified the original statement by Akbary–Hamieh for the comparison with Theorem 2.10. They also proved a similar result for $\log L_c(s)$ and applied it to study the distribution of class numbers of $K=k(\sqrt[3]{c})$ as c varies over C. Furthermore, a quartic analogue of Theorem 3.2 was announced by Gao–Zhao [**44**].

3.3. Artin L-functions associated with S_n-fields. As seen in Section 3.2, the Dirichlet L-function $L(s,\chi_d)$ for $d\in\mathcal{F}$ is regarded as the L-function arising from the quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. Akbary–Hamieh introduced the L-function $L_c(s)$ of the cubic extension $k(\sqrt[3]{c})/k$ with $k=\mathbb{Q}(\sqrt{-3})$. Then, we consider a cubic extension $K/\mathbb{Q}$ as another cubic analogue of $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. If $K/\mathbb{Q}$ is Galois, then the situation is almost the same as $k(\sqrt[3]{c})/k$, that is, we have a Dirichlet character χ such that $\zeta_K(s)=\zeta(s)L(s,\chi)L(s,\overline{\chi})$. On the other hand, the case where $K/\mathbb{Q}$ is non-Galois is rather interesting. In that case we have $\mathrm{Gal}(\widehat{K}/\mathbb{Q})\simeq S_3$, where $\widehat{K}$ is the Galois closure of K over $\mathbb{Q}$. More generally, a number field K of degree n is said to be an S_n-field if $\mathrm{Gal}(\widehat{K}/\mathbb{Q}) \simeq S_n$. For an S_n-field K of degree $n \geq 2$, there exists an irreducible representation $\rho_K : \mathrm{Gal}(\widehat{K}/\mathbb{Q}) \to GL_{n-1}(\mathbb{C})$ such that

$$\zeta_K(s) = \zeta(s)L(s, \rho_K),$$

where $L(s, \rho_K)$ is the Artin L-function of the Galois representation ρ_K. In particular, the Artin L-function $L(s, \rho_K)$ for the quadratic field $K = \mathbb{Q}(\sqrt{d})$ is equal to the Dirichlet L-function $L(s, \chi_d)$. Recall that, if K is an S_n-field of signature (r_1, r_2), then $n = r_1 + 2r_2$. We define

$$L_n^{(r_2)}(X) = \{[K] \mid K \text{ is an } S_n\text{-field of signature } (r_1, r_2) \text{ with } |d_K| < X\},$$

where $[K]$ stands for the isomorphism class of S_n-fields containing K, and d_K is the discriminant of K. We write $K \in L_n^{(r_2)}(X)$ instead of $[K] \in L_n^{(r_2)}(X)$ for simplicity. Then Cho–Kim [**24**] generalized Theorem 2.4 to S_n-fields with $n \leq 5$ on the line $\mathrm{Re}\, s = 1$. For $n = 5$, this result is conditional since they assumed the strong Artin conjecture for $L(s, \rho_K)$, i.e. there exists a cuspidal representation π of $GL_{n-1}(\mathbb{A}_\mathbb{Q})$ such that $L(s, \rho_K) = L(s, \pi)$. In 2018, Cho–Kim [**25**] further studied a limit theorem by adapting the method of Barban [**7**]. They calculated the integral moments

$$m_k = \lim_{X\to\infty} \frac{1}{\# L_n^{(r_2)}(X)} \sum_{K \in L_n^{(r_2)}(X)} (\log L(1, \rho_K))^k \tag{3.1}$$

for all $k \geq 0$ similarly to (2.2) with the assumptions of the strong Artin conjecture and the "counting conjecture" [**25**, Conjecture 3.1]. The truths of these conjectures are elementary for $n = 2$. If $n = 3$, the strong Artin conjecture for $L(s, \rho_K)$ is true by the fact that the representation ρ_K is induced from a character of the cyclic group $\mathrm{Gal}(\widehat{K}/\mathbb{Q}(\sqrt{d_K}))$. Furthermore, the counting conjecture for $n = 3$ is also true by the work of Bhargava–Shankar–Tsimerman [**10**] and Taniguchi–Thorne [**177**] in 2013. For higher degree cases, little is known about the truths of these conjectures: the strong Artin conjecture for $n = 4$ [**22**] and the counting conjecture for $n = 4, 5$ [**9**, **23**, **169**, **190**].

Theorem 3.3 (Cho–Kim). *Assume the strong Artin conjecture and the counting conjecture. Furthermore, we assume that* $m_k \ll c^{k \log_2 k}$ *holds with some constant* $c > 1$. *Then there exists a distribution function* F *such that*

$$\lim_{X\to\infty} \frac{\#\{K \in L_n^{(r_2)}(X) \mid \log L(1, \rho_K) \leq t\}}{\# L_n^{(r_2)}(X)} = F(t)$$

holds at each point of continuity of F*, where the characteristic function of* F *is represented as* $\phi(y) = \sum_{k=0}^{\infty} \frac{m_k}{k!}(iy)^k$ *by using the integral moment* m_k *of* (3.1).

Remark that they gave an explicit formula of m_k, but it is too complicated to show that $m_k \ll c^{k \log_2 k}$ is satisfied. The author [**135**] refined Theorem 3.3 for $n = 3$ without any assumptions on the integral moments m_k. Furthermore, he obtained an M-function in this case. Recall that $\mathrm{sgn}(d_K) = (-1)^{r_2}$ holds in general. For simplicity, we then write $L_3^+(X) = L_3^{(0)}(X)$ and $L_3^-(X) = L_3^{(1)}(X)$. We define the region G_{ρ_K} and the branch of $\log L(s,\rho_K)$ similarly to G_χ and $\log L(s,\chi)$ as in Section 2. Then we prove the following result in Section 12.

Theorem III. *Choose a constant* $7/8<\sigma_0<1$ *arbitrarily. For* $\sigma>\sigma_0$*, there exists a continuous function* $C_\sigma:\mathbb{R}\to\mathbb{R}_{\geq 0}$ *such that*

$$\lim_{X\to\infty}\frac{1}{\#L_3^\pm(X)}\sum_{K\in L_3^\pm(X)}{}' \Phi\left(\log L(\sigma,\rho_K)\right)=\int_{\mathbb{R}}\Phi(u)C_\sigma(u)\,|du|$$

holds, where the test function Φ *is one of* (i′) *and* (ii′) *of Theorem 2.10. Here, the notation* $\sum'$ *stands for the sum over* K *with* $\sigma\in G_{\rho_K}$.

The density function C_σ is associated with the Fourier transform $\widetilde{C}_\sigma$. We prove that it is explicitly represented by the formula

(3.2)
$$\widetilde{C}_\sigma(x)=\prod_p\frac{1}{1+p^{-1}+p^{-2}}\left\{\frac{1}{6}\psi_x(-\log(1-p^{-2\sigma}))+\frac{1}{2}\psi_x(-2\log(1-p^{-\sigma}))\right.$$
$$\left.+\frac{1}{3}\psi_x(-\log(1+p^{-\sigma}+p^{-2\sigma}))+\frac{1}{p}\psi_x(-\log(1-p^{-\sigma}))+\frac{1}{p^2}\right\}$$

similarly to (2.6). The right-hand side of (3.2) contains much information on the local conditions of S_3-fields. Note that $L(s,\rho_K)$ is represented as

$$L(s,\rho_K)=\prod_{\substack{p\\(p)=\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}}(1-p^{-2s})^{-1}\prod_{\substack{p\\(p)=\mathfrak{p}_1\mathfrak{p}_2}}(1-p^{-s})^{-2}$$
$$\times\prod_{\substack{p\\(p)=\mathfrak{p}_1}}(1+p^{-s}+p^{-2s})^{-1}\prod_{\substack{p\\(p)=\mathfrak{p}_1^2\mathfrak{p}_2}}(1-p^{-s})^{-1}\prod_{\substack{p\\(p)=\mathfrak{p}_1^3}}1,\tag{3.3}$$

where we write the prime ideal decomposition in K as $(p)=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r}$ for a rational prime p. Furthermore, the densities of S_3-fields with each local conditions are calculated as

$$\lim_{X\to\infty}\frac{\#\{K\in L_3^\pm(X)\mid(p)=\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\}}{\#L_3^\pm(X)}=\frac{1}{1+p^{-1}+p^{-2}}\frac{1}{6},$$
$$\lim_{X\to\infty}\frac{\#\{K\in L_3^\pm(X)\mid(p)=\mathfrak{p}_1\mathfrak{p}_2\}}{\#L_3^\pm(X)}=\frac{1}{1+p^{-1}+p^{-2}}\frac{1}{2},$$
$$\lim_{X\to\infty}\frac{\#\{K\in L_3^\pm(X)\mid(p)=\mathfrak{p}_1\}}{\#L_3^\pm(X)}=\frac{1}{1+p^{-1}+p^{-2}}\frac{1}{3},$$
$$\lim_{X\to\infty}\frac{\#\{K\in L_3^\pm(X)\mid(p)=\mathfrak{p}_1^2\mathfrak{p}_2\}}{\#L_3^\pm(X)}=\frac{1}{1+p^{-1}+p^{-2}}\frac{1}{p},$$
$$\lim_{X\to\infty}\frac{\#\{K\in L_3^\pm(X)\mid(p)=\mathfrak{p}_1^3\}}{\#L_3^\pm(X)}=\frac{1}{1+p^{-1}+p^{-2}}\frac{1}{p^2}$$

by the work of Bhargava–Shankar–Tsimerman [10] and Taniguchi–Thorne [177]. These observations imply that $\widetilde{C}_\sigma(x)$ is associated with the expected value for some random element. We study such a random element in Chapter 3.

Then, we apply the class number formula [94, Proposition 13 of ch. XIV, §8] to see a result on the distribution of class numbers of S_3-fields. Let h_K and R_K be the class number and the regulator of an S_3-field K, respectively. Then we have

$$L(1,\rho_K) = D^{\pm}\frac{h_K R_K}{\sqrt{|d_K|}}, \tag{3.4}$$

where $D^+ = 4$ if $d_K > 0$ and $D^- = 2\pi$ if $d_K < 0$. From a result on the value-distribution of $L(s,\rho_K)$, the author [135] deduced the asymptotic formulas

$$\begin{aligned} \sum_{K\in L_3^+(X)} h_K R_K &= cX^{3/2} + O\left(X^{3/2}\exp\left(-\delta\frac{\log X}{\log_2 X}\right)\right), \\ \sum_{K\in L_3^-(X)} h_K R_K &= \frac{6}{\pi}cX^{3/2} + O\left(X^{3/2}\exp\left(-\delta\frac{\log X}{\log_2 X}\right)\right), \end{aligned} \tag{3.5}$$

where c and δ are positive constants. In particular, we obtain

$$c = \frac{\pi^2\zeta(3)}{432}\prod_p (1+p^{-2}-2p^{-3}-2p^{-4}+2p^{-6}+p^{-7}-p^{-8}).$$

See Corollary 12.16 for more information. These formulas are regarded as cubic analogues of Ayoub [2, 3], where the asymptotic formulas for quadratic fields were studied. They are also related to Gauss' formulas

$$\sum_{d\in\mathcal{D}_X^+} h_d \log\epsilon_d \sim \frac{\pi^2}{18\zeta(3)}X^{3/2} \quad\text{and}\quad \sum_{d\in\mathcal{D}_X^-} h_d \sim \frac{\pi}{18\zeta(3)}X^{3/2}$$

as $X \to \infty$, which were proved by Siegel [170] in 1944.

3.4. L-functions of cusp forms.

Let $S_k(N)$ be the space of holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$ of weight k with trivial nebentypus. The space $S_k(N)$ is equipped with the Petersson inner product

$$\langle f, g\rangle = \iint_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^{k-2}\,dxdy,$$

where $z = x + iy$. Note that every $f \in S_k(N)$ has the Fourier series expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{\frac{k-1}{2}}e^{2\pi i n z}.$$

Let $S_k^\flat(N)$ be the subspace of $S_k(N)$ spanned by oldforms, that is, cusp forms represented as $f(dz)$ for some $f \in S_k(N')$ and $d|N$, where N' is a proper divisor of N satisfying $dN'|N$. Denote by $S_k^\#(N)$ the orthogonal complement of $S_k^\flat(N)$ with respect to the Petersson inner product. We say that $f \in S_k^\#(N)$ is a newform if it is a non-zero eigenfunction of all Hecke operators. Note that the first Fourier coefficient $\lambda_f(1)$ does not vanish when f is a newform, and there exists an orthogonal basis of

$S_k^{\#}(N)$ consisting of newforms f such that $\lambda_f(1)=1$; see Iwaniec [69, Chapter 6]. We denote by $B_k(N)$ such a basis, whose elements are also called primitive cusp forms. The automorphic L-function attached to $f\in B_k(N)$ is defined as

$$L_f(s)=\sum_{n=1}^{\infty}\lambda_f(n)n^{-s}=\prod_p(1-\lambda_f(p)p^{-s}+\chi_0(p)p^{-2s})^{-1},$$

where χ_0 is the principal character to the modulus N. Matsumoto [123] proved a result on the value-distribution of $L_f(s)$ similar to Theorem 1.4 (i) in 1989. On the other hand, suitable analogues of Theorem 1.4 (ii) were not obtained until the work of Matsumoto–Umegaki [132] and a little later by the author [137]. See also [133] for a further generalization.

THEOREM 3.4 (Matsumoto–Umegaki). *Let f be a primitive cusp form in $S_k(N)$. Then the following results hold for $\sigma>1/2$.*

(i) *There exists the limit value*

$$W_\sigma(\mathcal{R};L_f)=\lim_{T\to\infty}\mathbb{P}_T(\log L_f(\sigma+it)\in\mathcal{R}),$$

where $\mathcal{R}$ is a rectangle in $\mathbb{C}$ whose sides are parallel to the axes.

(ii) *There exists a continuous function $\mathcal{M}_\sigma(\,\cdot\,;L_f):\mathbb{C}\to\mathbb{R}_{\geq0}$ such that*

$$W_\sigma(\mathcal{R};L_f)=\int_{\mathcal{R}}\mathcal{M}_\sigma(w;L_f)\,|dw|.$$

Here, we determine the branch of $\log L_f(s)$ in a way similar to $\log L(s,\chi)$. Note that automorphic L-functions are also defined from non-holomorphic cusp forms. Denote by $C(\Gamma\backslash\mathbb{H})$ the space spanned by the Maass cusp forms for the full modular group $\Gamma=SL_2(\mathbb{Z})$. Then there exists an orthonormal basis $\mathcal{U}=\{u_j\}_{j\geq1}$ of $C(\Gamma\backslash\mathbb{H})$ such that

$$\Delta u_j=\left(\frac{1}{4}+t_j^2\right)u_j\quad\text{and}\quad T_nu_j=\lambda_j(n)u_j$$

with $0<t_1\leq t_2\leq\cdots$, where Δ is the hyperbolic Laplace operator on $\mathbb{H}$, and T_n is the n-th Hecke operator. The automorphic L-function $L_{u_j}(s)$ is defined as

$$L_{u_j}(s)=\sum_{n=1}^{\infty}\lambda_j(n)n^{-s}=\prod_p(1-\lambda_j(p)p^{-s}+p^{-2s})^{-1}.$$

Let $s=\sigma+it$ be a fixed complex number with $\sigma>1/2$. We consider the value-distributions of $L_f(s)$ and $L_{u_j}(s)$ in the following aspects:

(a) f varies over $B_k(N)$, where k is fixed and $N\to\infty$;
(b) f varies over $B_k(N)$, where N is fixed and $k\to\infty$;
(c) u_j varies over $\mathcal{U}$, where $t_j\to\infty$.

Then analogues of limit theorems described in Section 2.1 were studied by many authors. The first one was obtained by Luo [117] in 1999. He studied the value-distribution of the symmetric square L-functions

$$L_{u_j}(s,\mathrm{sym}^2)=\prod_p\prod_{\ell=0}^{2}(1-\alpha_j(p)^{2-\ell}\beta_j(p)^{\ell}p^{-s})^{-1}$$

in case (c), where $\alpha_j(p)$ and $\beta_j(p)$ are determined from the equations

$$\alpha_j(p)+\beta_j(p)=\lambda_j(p) \quad\text{and}\quad \alpha_j(p)\beta_j(p)=1.$$

Then Luo proved an analogue of Theorem 2.2 for $L_{u_j}(1,\mathrm{sym}^2)$ by modifying the method of Barban [7]. Indeed, he calculated the integral moments

$$\lim_{T\to\infty}\frac{1}{r(T)}\sum_{t_j\le T}L_{u_j}(1,\mathrm{sym}^2)^k=m_k$$

for all $k\ge 0$, where $r(T)=\sum_{t_j\le T}1$. Similar results in case (a) were also obtained by Royer [160,161] and Fomenko [40]. Note that the studies of the integral moments of $L_f(s)$ and $L_{u_j}(s)$ are more difficult than those of Dirichlet L-functions since the coefficients $\lambda_f(n)$ and $\lambda_j(n)$ are not completely multiplicative. In fact, the lack of complete multiplicatively causes some combinatorial complexity in the calculations of integral moments. Habsieger–Royer [55] explained this complexity in terms of Dyck paths in combinatorics. In 2004, Cogdell–Michel [27] modified the method of Elliott [36] to derive a more precise result on the value-distribution of $L_f(s)$. Let q be a large prime number and $z\in\mathbb{C}$ with $|z|\ll(\log q)(\log_2 q\log_3 q)^{-1}$. Then they proved the asymptotic formula

$$\frac{1}{N(q)}\sum_{f\in B_2(q)}\omega(f)L_f(1,\mathrm{sym}^\nu)^z=\widetilde{M}_1(z;\mathrm{sym}^\nu)+O\left(\exp\left(-\delta\frac{\log q}{\log_2 q}\right)\right) \tag{3.6}$$

with some limit value $\widetilde{M}_1(z;\mathrm{sym}^\nu)$, where $\delta=\delta(\nu)$ is a positive constant, and we define $\omega(f)=(4\pi\langle f,f\rangle)^{-1}$ and $N(q)=\sum_{f\in B_2(q)}\omega(f)$. Here, $L_f(s,\mathrm{sym}^\nu)$ is the ν-th symmetric power L-function defined as

$$L_f(s,\mathrm{sym}^\nu)=\prod_{p|N}(1-\lambda_f(p)^\nu p^{-s})^{-1}\prod_{p\nmid N}\prod_{\ell=0}^{\nu}(1-\alpha_f(p)^{\nu-\ell}\beta_f(p)^\ell p^{-s})^{-1}, \tag{3.7}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are determined from the equations

$$\alpha_f(p)+\beta_f(p)=\lambda_f(p) \quad\text{and}\quad \alpha_f(p)\beta_f(p)=1$$

for $(p,N)=1$. Then Cogdell–Michel proved an analogue of Theorem 2.3 with a sharper estimate of the error term. A similar result in the case $\nu=1$ was obtained by Golubeva [48] independently. Fomenko [41] also studied the value-distribution of $L_f(1,\mathrm{sym}^2)$ in case (b). These results led to the study of large deviations of values of automorphic L-functions in the aspects of (a)–(c). Several results in this direction were seen in Liu–Royer–Wu [111], Lamzouri [90,92], and Wang–Xiao [184]. See also Lau–Wang [95] for a recent work, where they considered L-functions of cusp forms for GL_n.

In the remaining part of this section, we see the work on M-functions for the value-distributions of automorphic L-functions in case (a). The goal is to obtain a density function $\mathcal{A}_s$ for which a limit formula such as

$$\lim_{N\to\infty}\frac{1}{\#B_k(N)}\sum_{f\in B_k(N)}\Phi(\log L(s,f))=\int_{\mathbb{C}}\Phi(w)\mathcal{A}_s(w)\,|dw| \tag{3.8}$$

holds with some suitable test functions Φ. In 2018, Matsumoto–Umegaki [**131**] and Lebacque–Zykin [**103**] studied related topics toward formula (3.8). To state the results, we introduce two assumptions on $L_f(s,\mathrm{sym}^\nu)$.

Assumption A. *Let $f\in B_k(q^m)$, where k is an even integer with $2\le k<12$ or $k=14$, and q is a prime number. For a fixed integer $\nu\ge1$, the ν-th symmetric power L-function $L_f(s,\mathrm{sym}^\nu)$ is analytically continued to a holomorphic function for* $\operatorname{Re}s>1/2$. *Moreover, it satisfies the estimate*

$$L_f(\sigma+it,\mathrm{sym}^\nu)\ll q^m(|t|+2)$$

for $1/2<\sigma\le2$, *where the implied constant may depend on ν.*

Assumption B. *Let f be the same as in Assumption A. For $\nu\ge1$, the ν-th symmetric power L-function $L_f(s,\mathrm{sym}^\nu)$ has no zeros in the strip* $1/2<\operatorname{Re}s\le1$.

For $f\in B_k(q^m)$, we extend the definition of $\omega(f)$ as

$$\omega(f)=\frac{1}{C_k(1-C_q(m))\langle f,f\rangle}$$

according to Umegaki [**181**], where the constants C_k and $C_q(m)$ are given by

$$C_k=\frac{(4\pi)^{k-1}}{\Gamma(k-1)}\quad\text{and}\quad C_q(m)=\begin{cases}0 & \text{for } m=1,\\ q(q^2-1)^{-1} & \text{for } m=2,\\ q^{-1} & \text{for } m\ge3.\end{cases}$$

Furthermore, we define the partial symmetric power L-function $L_f^*(s,\mathrm{sym}^\nu)$ by forgetting the local factors for $p|N$ in (3.7), namely

$$L_f^*(s,\mathrm{sym}^\nu)=\prod_{p\nmid N}\prod_{\ell=0}^{\nu}(1-\alpha_f(p)^{\nu-\ell}\beta_f(p)^\ell p^{-s})^{-1}.$$

Theorem 3.5 (Matsumoto–Umegaki). *Let μ and ν be positive integers with $\mu-\nu=2$. Let k be a fixed even integer with $2\le k<12$ or $k=14$. Suppose that Assumptions A and B are satisfied for $L_f(s,\mathrm{sym}^\mu)$ and $L_f(s,\mathrm{sym}^\nu)$ with $f\in B_k(q^m)$ for all prime powers q^m. Then, for $\sigma>1/2$, there exists a continuous function $m_\sigma:\mathbb{R}\to\mathbb{R}_{\ge0}$ such that*

$$\lim_{\substack{q\to\infty\\ q:\ \text{prime}\\ m:\ \text{fixed}}}\sum_{f\in B_k(q^m)}\omega(f)\Phi\Big(\log L_f^*(\sigma,\mathrm{sym}^\mu)-\log L_f^*(\sigma,\mathrm{sym}^\nu)\Big)\tag{3.9}$$

$$=\int_{\mathbb{R}}\Phi(u)m_\sigma(u)\,|du|$$

holds, where the test function Φ is one of (i′) *and* (ii′) *of Theorem 2.10.*

In this result, the M-function m_σ is obtained as

$$m_\sigma(u)=\int_{\mathbb{C}}\mathcal{M}_\sigma(u+iv)\,|dv|,$$

where $\mathcal{M}_\sigma$ is the M-function in formula (2.9). Matsumoto–Umegaki also proved that a similar result holds with the same M-function m_σ when q is fixed and $m\to\infty$.

One may think that formula (3.9) is quite different from (3.8). Here, we explain the relation between $\log L_f^*(s, \mathrm{sym}^\mu) - \log L_f^*(s, \mathrm{sym}^\nu)$ and $\log L_f(s)$. Define

$$F_p(T, \mathrm{sym}^\nu) = (1 - \alpha_f(p)^\nu T)(1 - \beta_f(p)^\nu T)$$

for $p \neq q$ and $\nu \geq 1$. Recalling the condition $\alpha_f(p)\beta_f(p) = 1$, we have

$$\log L_f^*(s, \mathrm{sym}^\nu) - \log L_f^*(s, \mathrm{sym}^{\nu-2}) = -\sum_{p \neq q} \operatorname{Log} F_p(p^{-s}, \mathrm{sym}^\nu) \tag{3.10}$$

for $\operatorname{Re} s > 1$ and $\nu \geq 3$. On the other hand, the right-hand side of (3.10) is essentially equal to $\log L_f(s)$ if $\nu = 1$. Indeed, we obtain

$$\log L_f(s) = -\sum_{p \neq q} \operatorname{Log} F_p(p^{-s}, \mathrm{sym}^1) + E_q(s, f)$$

for $\operatorname{Re} s > 1$, where $E_q(s, f) = -\operatorname{Log}(1 - \lambda_f(q) q^{-s})$. Thus (3.9) can be regarded as an analogue of (3.8). However, the method of Matsumoto–Umegaki did not allow us to study the case $\nu < 3$.

Next, Lebacque–Zykin studied the M-function for $L_f(s)$ in another way. Let $f \in S_k(N)$ and $\chi \in \mathcal{X}_r$ with $(N, r) = 1$. Then we consider the twisted cusp form

$$(f \otimes \chi)(z) = \sum_{n=1}^{\infty} \lambda_f(n) \chi(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

It is known that if $f \in B_k(N)$, then $f \otimes \chi$ is again a primitive cusp form for $\Gamma_0(Nr^2)$ of weight k with nebentypus χ^2; see [**70**, Proposition 14.20]. Then they considered the value-distribution of the twisted L-functions $L_{f \otimes \chi}(s)$ in the χ-aspect. They supposed the following assumption on $L_{f \otimes \chi}(s)$.

Assumption C. *Let $N \geq 1$ and $k \geq 2$ be an even integer. Let r be a prime number with $(N, r) = 1$. Then all zeros of $L_{f \otimes \chi}(s)$ in the critical strip* $0 \leq \operatorname{Re} s \leq 1$ *lie on the critical line* $\operatorname{Re} s = 1/2$ *for any $f \in B_k(N)$ and $\chi \in \mathcal{X}_r$.*

This assumption is a generalization of the Riemann Hypothesis. Then the result of Lebacque–Zykin is as follows.

Theorem 3.6 (Lebacque–Zykin). *Let $N \geq 1$ and $k \geq 2$ be an even integer. We fix a primitive cusp form $f \in B_k(N)$ and a complex number $s = \sigma + it$ with $\sigma > 1/2$. Suppose that Assumption C is satisfied. Then there exists a continuous function $\mathscr{M}_\sigma(\,\cdot\,; f) : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that*

$$\lim_{\substack{r \to \infty \\ r:\ \text{prime} \\ (N,r)=1}} \frac{1}{\# \mathcal{X}_r} \sum_{\chi \in \mathcal{X} r} \Phi\left(\log L_{f \otimes \chi}(s)\right) = \int_{\mathbb{C}} \Phi(w) \mathscr{M}_\sigma(w; f)\, |dw|, \tag{3.11}$$

where the test function Φ is one of (i) *and* (ii) *of Theorem 2.9.*

By construction, one can see that the M-function $\mathscr{M}_\sigma(w; f)$ agrees with the function $\mathcal{M}_\sigma(w; L_f)$ of Theorem 3.4. Remark that (3.11) holds further continuous functions Φ satisfying $\Phi(w) \ll e^{c|w|}$ with some $c > 0$ since Lebacque–Zykin adapted the method of [64] for the proof of Theorem 3.6. They also proved the existence of the limit value

$$\lim_{\substack{q\to\infty\\ q\colon \text{prime}}} \sum_{f\in B_k(q)} \omega(f)\overline{L_f(s)^{\frac{iz}{2}}}L_f(s)^{\frac{iz'}{2}} = \widetilde{M}_s(z,z')$$

for any $z, z' \in \mathbb{C}$ under Assumption C. Note that this formula is similar to (3.6). Furthermore, they succeeded to represent $\widetilde{M}_s(z,z')$ in the form

$$M_s(z,z') = \sum_{m,n\in\mathbb{N}} m^{-\overline{s}}n^{-s} \sum_{x\in J(m)\cap J(n)} \overline{c_{z,x}(m)}c_{z',x}(n), \tag{3.12}$$

where the coefficient $c_{z,x}(n)$ and the subset $J(n)\subset\mathbb{N}$ can be explicitly calculated. However, they did not find a suitable M-function satisfying formula (3.8) due to the complexity of the right-hand side of (3.12).

The proof of limit formula (3.8) was completed by the author [139] in the case $k = 2$ and $N = q$ runs over prime numbers. To find the desired function $\mathcal{A}_s$, he applied the method used in the study of the M-function for the Artin L-function $L(s,\rho_K)$ of Section 3.3. Define the region G_f similarly to G_χ of (2.1). Then we prove the following result in Section 13.

Theorem IV. *Denote by $\sum'$ the sum over f with $s \in G_f$ if s is a fixed complex number.*

(i) *Let $s = \sigma + it$ with $\sigma > 1/2$ and $t \neq 0$. Then there exists a continuous function $\mathcal{A}_s : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that*

$$\lim_{\substack{q\to\infty\\ q\colon \text{prime}}} \frac{1}{\#B_2(q)} \sideset{}{'}\sum_{f\in B_2(q)} \Phi\left(\log L_f(s)\right) = \int_{\mathbb{C}} \Phi(w)\mathcal{A}_s(w)\,|dw|$$

holds, where the test function Φ is one of (i) *and* (ii) *of Theorem 2.9.*

(ii) *Let $\sigma > 1/2$. Then there exists a continuous function $\mathcal{A}_\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that*

$$\lim_{\substack{q\to\infty\\ q\colon \text{prime}}} \frac{1}{\#B_2(q)} \sideset{}{'}\sum_{f\in B_2(q)} \Phi\left(\log L_f(\sigma)\right) = \int_{\mathbb{R}} \Phi(u)\mathcal{A}_\sigma(u)\,|du|$$

holds, where the test function Φ is one of (i′) *and* (ii′) *of Theorem 2.10.*

Remark that the notation $\mathcal{A}_s$ indicates a function on $\mathbb{C}$ if s is not real, while the same notation indicates a function on $\mathbb{R}$ if s is real. The Fourier transform $\widetilde{\mathcal{A}}_s(z)$ is represented as

$$\widetilde{\mathcal{A}}_s(z) = \prod_p \int_0^\pi \psi_z\left(-\log(1-(2\cos\theta)p^{-s}+p^{-2s})\right)\,d\mu_p(\theta) \tag{3.13}$$

by using the p-adic Plancherel measure

$$d\mu_p(\theta) = \left(1+\frac{1}{p}\right)\left(1-\frac{2\cos 2\theta}{p}+\frac{1}{p^2}\right)^{-1}\frac{2}{\pi}\sin^2\theta\,d\theta. \tag{3.14}$$

Furthermore, $\widetilde{\mathcal{A}}_\sigma(x)$ is represented by the same formula by letting $s = \sigma$ and $z = x$. Thus we have $\widetilde{\mathcal{A}}_{\sigma+it}(x+iy) \to \widetilde{\mathcal{A}}_\sigma(x)$ as $t \to 0$ by noting that $\psi_z(u) = \psi_{\operatorname{Re} z}(u)$ for $u \in \mathbb{R}$. Equivalently, we obtain the relation

$$\lim_{t\to 0} \mathcal{A}_{\sigma+it}(u+iv) = \mathcal{A}_\sigma(u)\delta_1(v),$$

where $\delta_1(v) = (2\pi)^{\frac{1}{2}}\delta(v)$ for the Dirac delta distribution $\delta(v)$. Then, we see that the right-hand side of (3.13) reflects the equidistribution of $\lambda_f(p)$ as follows. Let $f \in B_k(N)$ and p be a prime number. By the achievement of Deligne, there exists a real number $\theta_p(f) \in [0, \pi]$ such that $\lambda_f(p) = 2\cos\theta_p(f)$ for $(p, N) = 1$. Then $L_f(s)$ is represented as

$$L_f(s) = \prod_{p|N}(1 - \lambda_f(p)p^{-1})^{-1} \prod_{p\nmid N}(1 - (2\cos\theta_p(f))p^{-s} + p^{-2s})^{-1}.$$

In 1997, Serre [**168**] proved the limit formula

$$\lim_{k+N\to\infty} \frac{\#\{f \in B_k(N) \mid a \le \theta_p(f) \le b\}}{\#B_k(N)} = \int_a^b d\mu_p(\theta) \tag{3.15}$$

for all intervals $[a, b] \subset [0, \pi]$, where p is a fixed prime number. This limit formula was also proved by Conrey–Duke–Farmer [**28**] in the case $N = 1$ and $k \to \infty$, and Sarnak [**164**] had proved a similar result for Maass cusp forms. Hence (3.13) is again associated with the expected value of some random variable; see Chapter 3. One may notice that (3.15) is similar to the Sato–Tate conjecture which asserts that

$$\lim_{x\to\infty} \frac{\#\{p \le x \mid a \le \theta_p(f) \le b\}}{\pi(x)} = \int_a^b \frac{2}{\pi}\sin^2\theta\, d\theta \tag{3.16}$$

holds for a fixed cusp form $f \in B_k(N)$ which is of non-CM type. Thus, limit formula (3.15) is sometimes called the vertical Sato–Tate conjecture while it was proved before the original Sato–Tate conjecture; the proof of (3.16) was achieved by Barnet-Lamb et al. [**8**] in 2011.

4. Other zeta-functions

4.1. Hurwitz–Lerch zeta-functions. In the previous sections, we studied zeta and L-functions represented as infinite products of the form $F(s) = \prod_p \Phi_p(p^{-s})^{-1}$ with some polynomials $\Phi_p(T)$. In this section, we consider the Lerch zeta-function defined as

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^s} \tag{4.1}$$

for $\lambda \in \mathbb{R}$ and $0 < \alpha \le 1$, where the series converges absolutely for $\operatorname{Re} s > 1$. Then the Lerch zeta-function does not have the above infinite product representation in general. For example, we have

$$L(1, 1/3, s) = 3^s \sum_{n=1}^{\infty} f(n)n^{-s}$$

for $f(n) = 1$ if $n \equiv 1 \bmod 3$ and $f(n) = 0$ otherwise. Since $f(n)$ is not multiplicative, we can not obtain the representation $\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p L_p(p^{-s})$ for any power series $L_p(T)$. Note that the Lerch zeta-function is a generalization of the Hurwitz zeta-function

$$\zeta(s, \alpha) = L(1, \alpha, s) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}.$$

Moreover we have $\zeta(s, 1) = \zeta(s)$. Then some of the results on the Riemann zeta-function are naturally generalized to Hurwitz–Lerch zeta-functions. See [100] for several results on $L(\lambda, \alpha, s)$. Especially, the Hurwitz zeta-function $\zeta(s, \alpha)$ is continued analytically over the whole complex plane except only for the simple pole at $s = 1$ with residue 1. The Lerch zeta-function $L(\lambda, \alpha, s)$ is continued to an entire function if $\lambda \notin \mathbb{Z}$. A significant difference between $\zeta(s)$ and $L(\lambda, \alpha, s)$ arises when we consider zeros off the critical line $\operatorname{Re} s = 1/2$. Since we have $L(1, 1, s) = \zeta(s)$, $L(1, 1/2, s) = (2^s - 1)\zeta(s)$, and $L(1/2, 1, s) = (1 - 2^{1-s})\zeta(s)$, they have no zeros in the right-half plane $\operatorname{Re} s > 1$. On the other hand, Davenport–Heilbronn [32] proved that there are infinitely many zeros of $\zeta(s, \alpha) = L(1, \alpha, s)$ in $\operatorname{Re} s > 1$ if $\alpha \neq 1, 1/2$ is a rational or transcendental number. Cassels [21] proved the result in the remaining case, i.e. $\zeta(s, \alpha)$ has infinitely many zeros in $\operatorname{Re} s > 1$ even if α is algebraic irrational. Furthermore, the distribution of zeros in the strip $1/2 < \operatorname{Re} s \leq 1$ were also studied. In contrast to the Riemann Hypothesis, $\zeta(s, \alpha)$ has infinitely many zeros in this strip if $\alpha \neq 1, 1/2$ is either rational or transcendental. This result is a consequence of a property of $\zeta(s, \alpha)$ called the strong universality which was proved by Bagchi [4] and Gonek [49]. The strong universality is not yet proved when α is algebraic irrational. Thus the question whether $\zeta(s, \alpha)$ has infinitely many zeros in $1/2 < \operatorname{Re} s \leq 1$ remains in this case. Recently, Sourmelidis–Steuding [172] announced a result close to the strong universality when α is algebraic irrational. However, their result is not enough to solve this problem. See also Worley [188], Mishou [140], and Lee–Mishou [104] for more results on the value-distribution of $\zeta(s, \alpha)$ with an algebraic irrational parameter α.

We denote by $N(T, \sigma_1, \sigma_2; \lambda, \alpha)$ the number of zeros of $L(\lambda, \alpha, s)$ such that $\sigma_1 < \operatorname{Re} \rho < \sigma_2$ and $0 < \operatorname{Im} \rho < T$. The following result is derived by combining the results of Garunkštis–Laurinčikas in [100, Chapter 8].

Theorem 4.1 (Garunkštis–Laurinčikas). *Let $0 < \lambda \leq 1$ and $0 < \alpha \leq 1$. Assume that α is either rational or transcendental. Then there exists a constant $\delta(\alpha) > 0$ such that, for any σ_1 and σ_2 with $1/2 < \sigma_1 < \sigma_2 < 1 + \delta(\alpha)$, we obtain*

$$C_1 T < N(T, \sigma_1, \sigma_2; \lambda, \alpha) < C_2 T$$

for sufficiently large T, where $C_1 = C_1(\sigma_1, \sigma_2; \lambda, \alpha)$ and $C_2 = C_2(\sigma_1, \sigma_2; \lambda, \alpha)$ are positive constants.

Then one may expect that $N(T, \sigma_1, \sigma_2; \lambda, \alpha) \sim CT$ as $T \to \infty$ for some positive constant $C = C(\sigma_1, \sigma_2; \lambda, \alpha)$ satisfying $C_1 \leq C \leq C_2$. If α is transcendental, the method of Borchsenius–Jessen [19] is available to show the existence of such a constant; see [138, Appendix].

THEOREM 4.2. *Let* $0 < \lambda \leq 1$ *and* $0 < \alpha \leq 1$. *Assume that* α *is transcendental. Then the limit value*

$$C(\sigma_1, \sigma_2; \alpha) = \lim_{T \to \infty} \frac{1}{T} N(T, \sigma_1, \sigma_2; \lambda, \alpha) \tag{4.2}$$

exists for any $1/2 < \sigma_1 < \sigma_2$.

It is notable that the limit value is determined only from σ_1, σ_2, α and does not depend on λ although the constants C_1 and C_2 of Theorem 4.1 may depend on λ.

The author [**138**] applied the methods of Guo [**52**, **53**] of Section 1.6 to refine limit formula (4.2). For this, we need a further restriction to the parameter α. To explain the restriction, we recall the classification of complex numbers introduced by Mahler [**118**, **119**]. According to the notation of the book of Baker [**5**], for a complex number ξ, and for positive integers n and h, let $P(T) \in \mathbb{Z}[T]$ be a polynomial of degree at most n and height at most h for which $|P(\xi)|$ takes the smallest non-zero value. Here, the height of a polynomial is the maximal value of the absolute values of all coefficients. Note that the number of polynomials of degree at most n and height at most h is finite, and therefore we can take at least one polynomial with the above condition. Then, we define a positive real number $\omega_{n,h}(\xi)$ by the equation

$$|P(\xi)| = h^{-n\omega_{n,h}(\xi)}.$$

Remark that the real number $\omega_{n,h}(\xi)$ is determined independently of the choice of the polynomial $P(T)$. We further define $\omega_n(\xi)$ and $\omega(\xi)$ as

$$\omega_n(\xi) = \limsup_{h \to \infty} \omega_{n,h}(\xi) \quad \text{and} \quad \omega(\xi) = \limsup_{n \to \infty} \omega_n(\xi),$$

and we put $\nu(\xi) = \inf\{n \mid \omega_n(\xi) = \infty\}$.

DEFINITION 4.3 (Mahler's classification). Let $\xi \in \mathbb{C}$. Then we define the notions of A, S, T, U-numbers as follows:

- ξ is an A-number if $\omega(\xi) = 0$ and $\nu(\xi) = \infty$;
- ξ is an S-number if $0 < \omega(\xi) < \infty$ and $\nu(\xi) = \infty$;
- ξ is a T-number if $\omega(\xi) = \infty$ and $\nu(\xi) = \infty$;
- ξ is a U-number if $\omega(\xi) = \infty$ and $\nu(\xi) < \infty$.

It is known that every class of A, S, T, U-numbers is not empty. Mahler himself proved that if two complex numbers are algebraically dependent, then they belong to the same class. Furthermore, a complex number is an A-number if and only if it is an algebraic number. Hence S, T, U-numbers are transcendental. Several results on A, S, T, U-numbers are seen in [**5**]. In particular, almost all complex numbers are S-numbers, that is, the complement of the set of S-numbers has Lebesgue measure zero. A typical example of S-numbers is Napier's constant e, and more generally, e^z is an S-number if z is a non-zero algebraic integer. As for π, we know just that it is an S- or T-number. There exist no examples of T-numbers constructed explicitly. Finally, Liouville numbers such as $\sum_n 10^{-n!}$ are known to be U-numbers.

Let $0 < \lambda \leq 1$ and $\sigma > 1/2$. Suppose that $0 < \alpha \leq 1$ is an S-number. Then the author proved that there exists a continuous function $M_\sigma(\,\cdot\,; \alpha) : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that an asymptotic formula of the form

$$\frac{1}{T}\int_0^T \Phi\left(L(\lambda,\alpha,\sigma+it)\right)\,dt = \int_{\mathbb{C}} \Phi(w)M_\sigma(w;\alpha)\,|dw| + (\text{error term}) \tag{4.3}$$

holds similarly to (1.22), where the test function Φ is compactly supported and belongs to $C^\infty(\mathbb{C})$. See Theorem 16.3 for the precise statement. Furthermore, by the construction of $M_\sigma(w;\alpha)$, its Fourier transform is represented as

$$\widetilde{M}_\sigma(z;\alpha) = \prod_{n=0}^{\infty} J_0\left(|z|(n+\alpha)^{-\sigma}\right).$$

Using asymptotic formula (4.3), we refine limit formula (4.2) as follows.

Theorem V. *Let $\lambda \in \mathbb{R}$ and $0 < \alpha \le 1$. Assume that α is an S-number. Then there exists an absolute constant $A > 0$ such that*

$$N(T,\sigma_1,\sigma_2;\lambda,\alpha) = C(\sigma_1,\sigma_2;\alpha)T + O\left(T(\log T)^{-A}\right)$$

holds for any $1/2 < \sigma_1 < \sigma_2$ if T is large. Here, the implied constant depends on λ, α, σ_1, and σ_2. Furthermore, the constant $C(\sigma_1,\sigma_2;\alpha)$ is represented as

$$C(\sigma_1,\sigma_2;\alpha) = \frac{1}{2\pi}\int_{\sigma_1}^{\sigma_2}\left(\int_{\mathbb{C}}(\log|w|)\frac{\partial^2}{\partial\sigma^2}M_\sigma(w;\alpha)\,|dw|\right)d\sigma.$$

The proof of this result is presented in Section 16.

4.2. Iterated integrals of $\log\zeta(s)$. Let $m \in \mathbb{Z}$. Then we define a holomorphic function $\mathcal{I}_m(s)$ as

$$\mathcal{I}_m(s) = (-1)^m \sum_p \sum_{k=1}^{\infty} \frac{1}{k^{m+1}(\log p)^m} p^{-ks}$$

for $\operatorname{Re} s > 1$. We have $\mathcal{I}_0(s) = \log\zeta(s)$ and $\mathcal{I}_{-1}(s) = (\zeta'/\zeta)(s)$ by formulas (1.4) and (1.21). Recall that $\mathcal{I}_{-1}(s) = (\zeta'/\zeta)(s)$ is holomorphic over $\mathbb{C}$ except for the zeros and poles of $\zeta(s)$. Then we define

$$\mathcal{G} = \mathbb{C} \setminus \bigcup_{\zeta(\rho)=0 \text{ or } \infty} \{\sigma + i\operatorname{Im}\rho \mid \sigma \le \operatorname{Re}\rho\}.$$

For $m \ge 0$, the functions $\mathcal{I}_m(s)$ are analytically continued to $\mathcal{G}$ recursively by the integrals

$$\mathcal{I}_m(\sigma+it) = \int_{2+it}^{\sigma+it} \mathcal{I}_{m-1}(z)\,dz + \mathcal{I}_m(2+it),$$

where the contour is the horizontal path from $2+it$ to $\sigma+it \in \mathcal{G}$. By definition, they are holomorphic functions on $\mathcal{G}$ satisfying the differential relation $\mathcal{I}_m'(s) = \mathcal{I}_{m-1}(s)$. The function $\mathcal{I}_m(s)$ is related to the eta-functions $\eta_m(s)$ and $\widetilde{\eta}_m(s)$ introduced by Inoue [67]. First, we put $\eta_0(s) = \widetilde{\eta}_0(s) = \log\zeta(s)$. Here, the definition of $\log\zeta(s)$ is a little different from that of Section 1.1 since we need to define it for $s \notin \mathcal{G}$. For this, we determine the values as

$$\log\zeta(\sigma) = \lim_{\epsilon\to+0} \log\zeta(\sigma+i\epsilon),$$

$$\log\zeta(\sigma+\gamma) = \lim_{\epsilon\to+0}\log\zeta(\sigma+i(\gamma-\operatorname{sgn}(\gamma)\epsilon)),$$

where γ indicates the ordinate of a non-real zero of $\zeta(s)$. Then the functions $\eta_m(s)$ are defined recursively by the vertical integrals

$$\eta_m(\sigma+it) = \int_{\sigma}^{\sigma+it}\eta_{m-1}(z)\,dz + c_m(\sigma)$$

for $m \geq 1$, where we put

$$c_m(\sigma) = \frac{i^m}{(m-1)!}\int_{\sigma}^{\infty}(x-\sigma)^{m-1}\log\zeta(x)dx.$$

Similarly, the functions $\widetilde{\eta}_m(s)$ are defined by the horizontal integrals

$$\widetilde{\eta}_m(\sigma+it) = \int_{\sigma+it}^{\infty+it}\widetilde{\eta}_{m-1}(z)\,dz$$

for $m \geq 1$. Remark that the functions $\eta_m(s)$ and $\widetilde{\eta}_m(s)$ are no longer holomorphic functions. Then, it was proved that $\eta_m(s) = i^m\widetilde{\eta}_m(s)$ holds for $\sigma \geq 1/2$ and $t > 0$ if the Riemann Hypothesis is true; see [37, Lemma 1]. The function $\eta_m(s)$ is associated with the famous function $S_m(t)$ defined as follows:

$$S_0(t) = \frac{1}{\pi}\operatorname{Im}\log\zeta(1/2+it),$$

and for $m \geq 1$,

$$S_m(t) = \int_0^t S_{m-1}(y)\,dy + b_m,$$

where $b_m = \pi^{-1}\operatorname{Im} c_m(1/2)$. The function $S(t) = S_0(t)$ especially has been paid attention by many researchers of analytic number theory since it is closely related to the distribution of zeros of $\zeta(s)$. For example, asymptotic formula (1.2) is refined by using $S(t)$ as

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}).$$

By the above definitions, we have $\operatorname{Im}\eta_m(1/2+it) = \pi S_m(t)$. As for $\widetilde{\eta}_m(s)$, we note that it is essentially equal to $\mathcal{J}_m(s)$. Indeed, the equality $\widetilde{\eta}_m(s) = (-1)^m\mathcal{J}_m(s)$ holds for all $s \in \mathcal{G}$.

As was described in Section 1.6, it is still open whether $\{\eta_0(1/2+it) \mid t \geq 0\}$ is dense in $\mathbb{C}$ or not. Then, one may ask whether $\{\eta_m(1/2+it) \mid t \geq 0\}$ is dense in $\mathbb{C}$ for $m \geq 1$. Endo–Inoue [37] answered this question.

Theorem 4.4 (Endo–Inoue). *Let $1/2 \leq \sigma < 1$ be a fixed real number.*

- (i) *The set $\{\eta_1(\sigma+it) \mid t \geq 0\}$ is dense in $\mathbb{C}$ if the number of zeros of $\zeta(s)$ with $\operatorname{Re}\rho > \sigma$ is finite.*
- (ii) *If $m \geq 2$, the following statements are equivalent.*
  - (A) *There exist no zeros of $\zeta(s)$ with $\operatorname{Re}\rho > \sigma$.*
  - (B) *The set $\{\eta_m(\sigma+it) \mid t \geq 0\}$ is dense in $\mathbb{C}$.*

In particular, $\{\eta_m(1/2+it) \mid t \geq 0\}$ is dense in $\mathbb{C}$ for all $m \geq 1$ if the Riemann Hypothesis is true. This result supports somewhat that $\{\eta_0(1/2+it) \mid t \geq 0\}$ is dense in $\mathbb{C}$. However, we should remark that $\{\mathscr{J}_{-1}(1/2+it) \mid t \in \mathbb{R}\}$ is NOT dense in $\mathbb{C}$, which follows from the argument of the proof of [**46**, Theorem 1]. For the function $\widetilde{\eta}_m(s)$, the denseness was proved for $m \geq 1$ unconditionally.

Theorem 4.5 (Endo–Inoue). *Let* $1/2 \leq \sigma < 1$ *and* $m \geq 1$. *Then the set* $\{\widetilde{\eta}_m(\sigma+it) \mid t \geq 0\}$ *is dense in* $\mathbb{C}$.

Motivated by their work, the author proceeded to study limit theorems for the function $\widetilde{\eta}_m(s)$ with $m \geq 1$. In his joint work with Endo and Inoue, they obtained analogues of Theorems 1.6 and 1.7. Let $\sigma \geq 1/2$ and $T > 0$. Then we define a probability measure on $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$ as

$$P_{\sigma,T,m}(A) = \mathbb{P}_T(\widetilde{\eta}_m(\sigma+it) \in A). \tag{4.4}$$

Furthermore, we denote by $S_{n,\sigma,m}$ the curve in $\mathbb{C}$ given by a parametric representation $z_{n,\sigma,m}(\theta) = \mathrm{Li}_{m+1}(p_n^{-\sigma}e^{i\theta})$, where $\mathrm{Li}_s(z)$ denotes the polylogarithm of order s. Then we define

$$\mu_{n,\sigma,m}(A) = \frac{1}{2\pi}\,\mathrm{meas}\{\theta \in [0,2\pi] \mid z_{n,\sigma,m}(\theta) \in A\} \tag{4.5}$$

for $A \in \mathscr{B}(\mathbb{C})$ similarly to (1.5). We further define $P_{N,\sigma,m} = \mu_{1,\sigma,m} * \cdots * \mu_{N,\sigma,m}$ for $N \geq 1$.

Theorem VI. *As* $N \to \infty$, $P_{N,\sigma,m}$ *converges weakly to a probability measure* $P_{\sigma,m}$ *if* $\sigma \geq 1/2$ *and* $m \geq 1$. *The limit measure* $P_{\sigma,m}$ *is absolutely continuous and satisfies*

$$P_{\sigma,m}(A) = \int_A \mathcal{M}_\sigma(w;\widetilde{\eta}_m)\,|dw|$$

for all $A \in \mathscr{B}(\mathbb{C})$, *where* $\mathcal{M}_\sigma(w;\widetilde{\eta}_m)$ *is a continuous function satisfying the following properties.*

(i) *The function* $\mathcal{M}_\sigma(w;\widetilde{\eta}_m)$ *belongs to the class* $C^\infty(\mathbb{C})$.

(ii) *The support of* $\mathcal{M}_\sigma(w;\widetilde{\eta}_m)$ *is a compact subset of* $\mathbb{C}$ *if* $\sigma \geq 1$. *On the other hand, we have* $\mathcal{M}_\sigma(w;\widetilde{\eta}_m) > 0$ *for all* $w \in \mathbb{C}$ *if* $1/2 \leq \sigma < 1$.

Finally, the probability measure $P_{\sigma,T,m}$ *converges weakly to* $P_{\sigma,m}$ *if* $\sigma \geq 1/2$ *and* $m \geq 1$.

The proof of this result is presented in Section 17. Note that it further refines Theorem 4.5 since we deduce

$$\lim_{T\to\infty} \mathbb{P}_T\Big(|\widetilde{\eta}_m(\sigma+it) - z| < \epsilon\Big) > 0$$

for any $z \in \mathbb{C}$ and $\epsilon > 0$ if $1/2 \leq \sigma < 1$. More results on the value-distributions of $\widetilde{\eta}_m(s)$ are contained in [**38**], e.g. discrepancy bounds, large deviations, and so on. See also Inoue [**68**] for some related results.

CHAPTER 2

General L-functions

In Chapter 1, we considered zeta and L-functions[1] associated with Dirichlet characters, number fields, and cusp forms. There are many L-functions arising from other mathematical objects. Several authors introduced classes of L-functions by regarding the common properties of such L-functions as the axioms of general L-functions. One of the most famous classes was introduced by Selberg [**167**] in 1992, which is now called the Selberg class. In this chapter, we present a summary of the axiomatic studies of general L-functions.

5. The Selberg class

5.1. Definition and basic properties. The Selberg class $\mathcal{S}$ is defined as the collection of all functions F for which the following axioms (S1)–(S5) are satisfied.

(S1) *Dirichlet series.* The function F is represented as

$$F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s} \tag{5.1}$$

in some right-half plane of $\mathbb{C}$, where $a_F(n) \in \mathbb{C}$.

(S2) *Ramanujan hypothesis.* The coefficient $a_F(n)$ satisfies $a_F(n) \ll n^{\epsilon}$ for each $\epsilon > 0$, where the implied constant may depend on ϵ.

(S3) *Analytic continuation.* There exists a non-negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order.

(S4) *Functional equation.* We define the complete L-function $\Lambda_F(s)$ as

$$\Lambda_F(s) = F(s)\gamma(s),$$

where the gamma-factor $\gamma(s)$ is represented as

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

for some constants $Q > 0$, $\lambda_j > 0$, and $\mu_j \in \mathbb{C}$ with $\operatorname{Re} \mu_j \geq 0$. Then it satisfies the functional equation

$$\Lambda_F(s) = \omega \overline{\Lambda_F(1-\overline{s})},$$

where ω is a complex number such that $|\omega| = 1$.

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[1] The difference between the terminologies of "zeta-function" and "L-function" comes from just conventional usages. In this thesis, we mainly use "L-function" when both terminologies appear.

(S5) *Euler product.* The function F is represented as the infinite product

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} b_F(p^k) p^{-ks} \right)$$

in some right-half plane, where $b_F(p^k) \in \mathbb{C}$ satisfies $b_F(p^k) \ll p^{k\theta}$ with some $\theta < 1/2$.

By the properties of Section 1, the Riemann zeta-function $\zeta(s)$ belongs to $\mathcal{S}$. Moreover, it contains many of the L-functions introduced in Chapter 1; see also Section 5.2. As an initial remark, we consider the relation between the coefficients $a_F(n)$ and $b_F(p^k)$. By Axiom (S5), we see that $a_F(n)$ is multiplicative, and thus $a_F(1) = 1$. Furthermore, the equality

$$\sum_{m=0}^{\infty} a_F(p^m) p^{-ms} = \exp\left(\sum_{k=1}^{\infty} b_F(p^k) p^{-ks} \right) \tag{5.2}$$

holds for all prime numbers p. Differentiating both sides, we obtain

$$(\log p) \sum_{m=0}^{\infty} m a_F(p^m) p^{-ms} = \left(\sum_{m=0}^{\infty} a_F(p^m) p^{-ms} \right) \left((\log p) \sum_{k=1}^{\infty} k b_F(p^k) p^{-ks} \right).$$

It yields $a_F(p) = b_F(p)$ and

$$a_F(p^m) = b_F(p^m) + \frac{1}{m} \sum_{k=1}^{m-1} k a_F(p^{m-k}) b_F(p^k)$$

for $m \geq 2$ by comparing coefficients. Note that the left-hand side of (5.2) converges absolutely for $\operatorname{Re} s > 0$ by Axiom (S2), and it is non-vanishing for $\operatorname{Re} s > \theta$ by the condition $b_F(p^k) \ll p^{k\theta}$. Furthermore, Axiom (S2) implies that (5.1) converges absolutely for $\operatorname{Re} s > 1$. From the above, we deduce that $F(s)$ has no zeros in $\operatorname{Re} s > 1$. Then the zeros of $F(s)$ in $\operatorname{Re} s < 0$ are easily calculated by Axiom (S4), which are called the trivial zeros. The non-trivial zeros lie on the critical strip $0 \leq \operatorname{Re} s \leq 1$. It is believed that the Selberg class is a wide class of L-functions such that analogues of the Riemann Hypothesis are valid.

Conjecture 5.1 (Grand Riemann Hypothesis, GRH). *Let $F \in \mathcal{S} \setminus \{1\}$. Then all non-trivial zeros of $F(s)$ lie on the critical line* $\operatorname{Re} s = 1/2$.

There are several invariants of $F \in \mathcal{S}$. First, the degree of F is defined as

$$d_F = 2 \sum_{j=1}^{r} \lambda_j.$$

Remark that the data r, Q, λ_j, and μ_j in Axiom (S4) are not uniquely determined from F. For example, we have two functional equations

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

$$\left(\frac{\pi}{2}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s}{4}\right) \Gamma\left(\frac{s}{4} + \frac{1}{2}\right) \zeta(s) = \left(\frac{\pi}{2}\right)^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{4}\right) \Gamma\left(\frac{1-s}{4} + \frac{1}{2}\right) \zeta(1-s)$$

for the Riemann zeta-function. Nevertheless, the degree d_F depends only on F by the following result of Selberg [167]. Let $N_F(T)$ denote the number of non-trivial zeros of F with $0 < \operatorname{Im}\rho \leq T$. Then we have

$$N_F(T) = \frac{d_F}{\pi} T \log T + c_F T + O(\log T), \tag{5.3}$$

where c_F is a constant depending only on F. Next, the conductor of F is defined as

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

which also depends only on F; see Kaczorowski–Perelli [77]. It is conjectured that $d_F \in \mathbb{Z}_{\geq 0}$ and $q_F \in \mathbb{Z}_{>0}$, and more precisely, that all λ_j in Axiom (S4) can be chosen to be equal to $1/2$. Define

$$\mathcal{S}_d = \{F \in \mathcal{S} \mid d_F = d\}.$$

Note that the datum r in Axiom (S4) can vanish. In that case, the gamma-factor is understood as $\gamma(s) = Q^s$, and we define $d_F = 0$. The structure of $\mathcal{S}_d$ was known for $0 \leq d < 2$. In particular, we have $\mathcal{S}_d = \emptyset$ for $0 < d < 1$ [29] and for $1 < d < 2$ [79] as expected. There exist no results on the structure of $\mathcal{S}_d$ for $d \geq 2$. Recently, Kaczorowski–Perelli [80] announced a result on the classification of L-functions in the "extended" Selberg class of degree 2 and conductor 1.

5.2. Examples. Here, we list several examples of L-functions in the class $\mathcal{S}_d$.

5.2.1. *L-functions of degree* 0. Trivially, the constant function 1 belongs to the class $\mathcal{S}_0$ and has the conductor 1. Conrey–Ghosh [29] further proved the converse theorem, i.e. we have $\mathcal{S}_0 = \{1\}$.

5.2.2. *L-functions of degree* 1. The Riemann zeta-function $\zeta(s)$ is a typical example of elements in $\mathcal{S}_1$. Furthermore, the Dirichlet L-function $L(s,\chi)$ belongs to $\mathcal{S}_1$ if χ is primitive. In general, if $F \in \mathcal{S}_d$ is an entire function and $\theta \in \mathbb{R}$, then $F_\theta(s) = F(s + i\theta)$ again belongs to $\mathcal{S}_d$. Hence we see that $L(s + i\theta, \chi)$ is a member of $\mathcal{S}_1$ if χ is a primitive Dirichlet character of conductor $q > 1$. The classification of L-functions in $\mathcal{S}_1$ was established by Kaczorowski–Perelli [75]. It depends on the conductor of $F \in \mathcal{S}_1$. If $q_F = 1$, then $F(s) = \zeta(s)$. If $q_F > 1$, then $F(s) = L(s + i\theta_F, \chi)$ with some primitive Dirichlet character χ of conductor q_F, where we put

$$\theta_F = 2 \operatorname{Im} \sum_{j=1}^{r} \left(\mu_j - \frac{1}{2}\right).$$

5.2.3. *L-functions of higher degree.* Let K be a number field of degree d. Then the Dedekind zeta-function $\zeta_K(s)$ belongs to $\mathcal{S}_d$ and has the conductor $|d_K|$. Let χ be a primitive character of ideal class group of K to the modulus $\mathfrak{f}$. Then the Hecke L-function $L_K(s,\chi)$ also belongs to $\mathcal{S}_d$ and has the conductor $\mathrm{N}(\mathfrak{f})|d_K|$. See, for example, [60, 94, 178].

Let f be a cusp form in $B_k(N)$. Then the automorphic L-function $L_f(s)$ belongs to $\mathcal{S}_2$ and has the conductor N. Furthermore, the ν-th symmetric power L-function $L_f(s, \mathrm{sym}^\nu)$ belongs to $\mathcal{S}_{\nu+1}$ for $\nu = 2$ by Gelbart–Jacquet [47] and for $\nu = 3, 4$ by

Kim–Shahidi [84, 85] and Kim [83]. The conductor of $L_f(s, \mathrm{sym}^\nu)$ is equal to N^ν. Recently, Newton–Thorne [152, 153] announced that it is valid for all $\nu \geq 1$.

5.2.4. *Conditional examples.* Let $K/\mathbb{Q}$ be a Galois extension of degree d, and let $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to GL_n(\mathbb{C})$ be an irreducible representation. The Artin conjecture asserts that the Artin L-function $L(s, \rho)$ is continued to an entire function if ρ is not a trivial representation. One of the most successful results is that $L(s, \rho)$ is continued to a meromorphic function on $\mathbb{C}$, which was proved by Brauer's theorem on induced representations. If the Artin conjecture is true, then $L(s, \rho)$ belongs to $\mathcal{S}_{dn}$. The conductor of $L(s, \rho)$ can be written in terms of the representation ρ; see [74, Section 3.3]. The Artin L-function $L(s, \rho_K)$ attached to an S_3-field K is a member of the class $\mathcal{S}_2$ unconditionally, whose conductor is equal to $|d_K|$.

Let $u_j \in \mathcal{U}$ be a Maass cusp form for $SL_2(\mathbb{Z})$ as in Section 1.3. Put $|\alpha_j(p)| = p^\theta$ with $\theta \geq 0$. The Ramanujan–Petersson conjecture asserts that θ equals to 0, which yields the bound $\lambda_j(n) \ll n^\epsilon$. The best result to date has been $\theta \leq 7/64$ obtained by Kim–Sarnak [83, Appendix 2] in 2003. If the Ramanujan–Petersson conjecture is true, then the automorphic L-function $L_{u_j}(s)$ belongs to $\mathcal{S}_2$ and possesses the conductor 1. More generally, if the generalized Ramanujan–Petersson conjecture is true for an irreducible cuspidal representation π of $GL_d(\mathbb{A}_\mathbb{Q})$, then the automorphic L-function $L(s, \pi)$ belongs to the class $\mathcal{S}_d$.

5.3. Further properties. Here, we describe several properties for $F \in \mathcal{S}$ which are used later. Denote by $N_F(\sigma, T)$ the number of non-trivial zeros of F such that $\mathrm{Re}\,\rho \geq \sigma$ and $|\mathrm{Im}\,\rho| \leq T$. If we assume GRH, then we know $N_F(\sigma, T) = 0$ for $\sigma > 1/2$ and $T > 0$. Hence it is reasonable to study the upper bounds for $N_F(\sigma, T)$ without GRH. The most general result was obtained by Kaczorowski–Perelli [78].

Lemma 5.2 (Kaczorowski–Perelli). *Let $F \in \mathcal{S}$ and $\epsilon > 0$. Then we obtain*

$$N_F(\sigma, T) \ll T^{(4d_F+12)(1-\sigma)+\epsilon} \tag{5.4}$$

uniformly for $1/2 \leq \sigma \leq 1$.

There exist sharper estimates in some restricted cases. In particular, an upper bound of the form

$$N_F(\sigma, T) \ll T^{d_F(1-\sigma)+\epsilon} \tag{5.5}$$

was proved by Heath-Brown [58] when F is the Dedekind zeta-function for a number field K of degree $d = d_F \geq 3$. Perelli [155] later generalized this result for general L-functions F of degree $d_F \geq 8/3$ in a subclass of $\mathcal{S}$. Recall that we have $N_F(1/2, T) \asymp T \log T$ as $T \to \infty$ by formula (5.3). Hence (5.4) and (5.5) are useful only for σ close to 1. On the other hand, the upper bound of the type

$$N_F(\sigma, T) \ll T^{1-c(\sigma-\frac{1}{2})+\epsilon} \tag{5.6}$$

holds with some $c > 0$ uniformly for $1/2 \leq \sigma \leq 1$ if F is one of the following:

- The Riemann zeta-function $\zeta(s)$ due to Selberg [166];
- Dirichlet L-functions $L(s, \chi)$ for all Dirichlet characters χ due to Fujii [43];
- The Dedekind zeta-function $\zeta_K(s)$ for an abelian extension $K/\mathbb{Q}$ since it is a finite product of Dirichlet L-functions;

- Automorphic L-functions $L_f(s)$ of normalized Hecke eigenforms for the full modular group $SL_2(\mathbb{Z})$ due to Luo [**116**].

These zero density estimates are associated with approximations of the function $\log F(s)$ by Dirichlet polynomials; see Section 11.1.

Next, we study the growth of $|F(\sigma+it)|$ as $|t| \to \infty$ for $F \in \mathcal{S}$. Let σ be a fixed real number. We define

$$\mu_F(\sigma) = \limsup_{|t|\to\infty} \frac{\log|F(\sigma+it)|}{\log|t|}.$$

Then we have $\mu_F(\sigma) = 0$ for $\sigma > 1$ since the series (5.1) converges for $\operatorname{Re} s > 1$. Furthermore, Axiom (S4) yields

$$|F(\sigma+it)| \asymp |t|^{(\frac{1}{2}-\sigma)d_F}|F(1-\sigma+it)|$$

by usual properties of $\Gamma(s)$; see Appendix B.2. Hence we obtain $\mu_F(\sigma) = (\frac{1}{2}-\sigma)d_F$ for $\sigma < 0$. The Phragmén–Lindelöf principle [**175**, Lemma 6.9] yields that $\mu_F(\sigma)$ is a convex function of σ. Thus, we obtain the convexity bound $\mu_F(\sigma) \le \frac{1-\sigma}{2}d_F$ for $0 \le \sigma \le 1$, or equivalently, we have

$$F(\sigma+it) \ll |t|^{\frac{1-\sigma}{2}d_F+\epsilon} \tag{5.7}$$

for each $\epsilon > 0$, where the implied constant depends only on F and ϵ. For more precise estimates of $\mu_F(\sigma)$ for $0 \le \sigma \le 1$, we see that the estimate for $\sigma = 1/2$ is essential by the functional equation. It is conjectured that $\mu_F(1/2) = 0$, i.e.

Conjecture 5.3 (Grand Lindelöf Hypothesis, GLH). *Let $F \in \mathcal{S}$. For every $\epsilon > 0$, we obtain*

$$F(1/2+it) \ll |t|^{\epsilon}$$

if $|t|$ is large, where the implied constant depends on F and ϵ.

For the Riemann zeta-function $\zeta(s)$, this conjecture was originally known as the Lindelöf Hypothesis. By upper bound (1.15), we see that the Riemann Hypothesis implies the Lindelöf Hypothesis. Furthermore, Conrey–Ghosh [**30**] discovered the relation between GRH and GLH as follows. Let $F \in \mathcal{S}$ be an entire function satisfying GRH. Suppose that all λ_j in Axiom (S4) can be chosen to be equal to $1/2$, and furthermore, that F is represented as the polynomial Euler product

$$F(s) = \prod_p \prod_{j=1}^{d} (1-\alpha_{p,j}p^{-s})^{-s}, \tag{5.8}$$

where $\alpha_{p,1}, \dots, \alpha_{p,d} \in \mathbb{C}$ with $|\alpha_{p,j}| \le 1$ for all j. Then it was proved that F satisfies GLH, and more precisely,

$$F(1/2+it) \ll \left(Q|t|^{d/2}\prod_{j=1}^{r}(1+|\mu_j|)\right)^{\epsilon}$$

for every $\epsilon > 0$, where the implied constant depends only on d and ϵ. Here, Q and μ_j are the data in Axiom (S4). See also Garunkštis [**45**] for an improvement of this result.

Finally, we see the work on the orthonormality conjecture of Selberg. Note that if $F, G \in \mathcal{S}$, then $FG \in \mathcal{S}$. We say that an element $F \in \mathcal{S}$ is primitive if $F = F_1 F_2$ with $F_1, F_2 \in \mathcal{S}$ implies $F_1 = 1$ or $F_2 = 1$. Clearly, $1 \in \mathcal{S}_0$ is primitive. Note further that (5.3) deduces $d_{FG} = d_F + d_G$. Hence L-functions in $\mathcal{S}_1$ are primitive by the facts $\mathcal{S}_0 = \{1\}$ and $\mathcal{S}_d = \emptyset$ for $0 < d < 1$. Selberg [167] made the following conjecture for primitive L-functions.

Conjecture 5.4 (Selberg Orthonormality Conjecture, SOC). *For any primitive L-functions $F, G \in \mathcal{S}$, we have*

$$\sum_{p \le x} \frac{a_F(p)\overline{a_G(p)}}{p} = \delta_{F,G} \log\log x + O(1),$$

where $\delta_{F,G} = 1$ if $F = G$, and $\delta_{F,G} = 0$ otherwise.

Note that every $F \in \mathcal{S}$ can be factorized into primitive L-functions. Denote by $F = F_1^{e_1} \cdots F_r^{e_r}$ a factorization of $F \in \mathcal{S}$ with $F \neq 1$ into distinct primitive L-functions. Then SOC deduces the asymptotic formula

$$\sum_{p \le x} \frac{|a_F(p)|^2}{p} = n_F \log\log x + O(1), \tag{5.9}$$

where $n_F = e_1^2 + \cdots + e_r^2$. Then, under SOC, an element F is primitive if and only if $n_F = 1$. Murty [151] showed that SOC is associated with several conjectures in number theory. In fact, he proved that SOC implies the Artin conjecture of all irreducible Galois representations. Furthermore, if $K/\mathbb{Q}$ is a solvable extension, then the strong Artin conjecture is true for irreducible representations of $\mathrm{Gal}(K/\mathbb{Q})$ under SOC. Another important application of SOC was obtained by Conrey–Ghosh [29]. They proved that SOC implies that the factorization into primitive L-functions in $\mathcal{S}$ is unique up to the order of factors.

See also the surveys [74, 76, 156, 157] for more information about the Selberg class. Several authors presented alternative classes of general L-functions. For example, Perelli [155] and Matsumoto [124] introduced a class of L-functions with polynomial Euler product independently of Selberg's work.

6. Matsumoto zeta-functions

6.1. Definition and the limit theorem. Let $\mathcal{M}$ be the collection of all functions F satisfying the following axioms (M1)–(M5). Since Laurinčikas [99], the elements of the class $\mathcal{M}$ are called the Matsumoto zeta-functions.

(M1) *Polynomial Euler product.* The function F is represented as

$$F(s) = \prod_{n=1}^{\infty} \Phi_n(p_n^{-s})^{-1} \tag{6.1}$$

in some right-half plane, where $\Phi_n(T)$ is a polynomial such that

$$\Phi_n(T) = \prod_{j=1}^{g(n)} (1 - a_n^{(j)} T^{f(j,n)})$$

with $g(n), f(j,n) \in \mathbb{Z}_{>0}$, and $a_n^{(j)} \in \mathbb{C}$.

(M2) *Convergence.* There exist non-negative constants α and β such that

$$|g(n)| \leq C p_n^{\alpha} \quad \text{and} \quad |a_n^{(j)}| \leq p_n^{\beta}$$

with some $C > 0$. Note that these conditions imply that (6.1) converges absolutely for $\operatorname{Re} s > \alpha + \beta + 1$.

(M3) *Analytic continuation.* The function F is meromorphically continued to the right-half plane $\operatorname{Re} s \geq \rho_F$, where $\alpha+\beta+\frac{1}{2} \leq \rho_F \leq \alpha+\beta+1$. Furthermore, all poles of F in this region are included in a compact set.

(M4) *Finite order.* There exists a constant $\delta \geq 0$ such that

$$F(\sigma + it) \ll |t|^{\delta}$$

for $\sigma \geq \rho_F$ if $|t|$ is large.

(M5) *Mean square.* There exist no poles of F on the vertical line $\operatorname{Re} s = \rho_F$. Furthermore, we obtain

$$\frac{1}{T} \int_0^T |F(\rho_F + it)|^2 \, dt \ll 1.$$

The class $\mathcal{M}$ contains many L-functions as well as the Selberg class $\mathcal{S}$. Denote by $\mathcal{S}^{\mathrm{poly}}$ the subclass of $\mathcal{S}$ which consists of L-functions represented as polynomial Euler products (5.8). It is believed that $\mathcal{S}^{\mathrm{poly}} = \mathcal{S}$ holds, and all known examples of elements of $\mathcal{S}$ belongs to the subclass $\mathcal{S}^{\mathrm{poly}}$. We see that $\mathcal{M}$ contains $\mathcal{S}^{\mathrm{poly}}$. Indeed, Axioms (M1)–(M3) are satisfied if $F \in \mathcal{S}^{\mathrm{poly}}$ by definition. Here, we can take $\alpha = \beta = 0$ in Axiom (M2). Axiom (M4) follows from convexity bound (5.7). Finally, the limit formula

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |F(\sigma + it)|^2 \, dt = \sum_{n=1}^{\infty} |a_F(n)|^2 n^{-2\sigma}$$

holds for $\sigma > \sigma_F := \max\{1 - d_F^{-1}, 1/2\}$ if $F \in \mathcal{S}$; see [**175**, Corollary 6.11]. Hence Axiom (M5) is satisfied in this case, where we can take $\rho_F = \sigma_F + \epsilon$ for any $\epsilon > 0$. Therefore, for example, the Riemann zeta-function, Dirichlet L-functions of primitive characters, and L-functions of holomorphic primitive cusp forms are all Matsumoto zeta-functions. Moreover, one can prove that Dirichlet L-functions of imprimitive characters are Matsumoto zeta-functions since we do not need functional equations in the axioms of $\mathcal{M}$. Remark further that we need just a weak upper bound on $|a_n^{(j)}|$ in Axiom (M2). It enables us to take more L-functions into consideration. For example, automorphic L-functions of Maass cusp forms $u_j \in \mathcal{U}$ are Matsumoto zeta-functions with $\alpha = 0, \beta = 7/64$, and $\rho_F = 39/64 + \epsilon$ by the Kim–Sarnak bound.

Let F be a Matsumoto zeta-function. Then we determine the branch of $\log F(s)$ as follows. Denote by G_F the region

$$G_F = \{\sigma + it \mid \sigma > \rho_F\} \setminus \bigcup_{F(\rho)=0 \text{ or } \infty} \{\sigma + i \operatorname{Im} \rho \mid \rho_F < \sigma \leq \operatorname{Re} \rho\}$$

similarly to G of (1.3). If $\operatorname{Re} s > \alpha + \beta + 1$, then we define

$$\log F(s) = -\sum_{n=1}^{\infty} \operatorname{Log} \Phi_n(p_n^{-s})$$

according to (6.1). We extend $\log F(s)$ for $s \in G_F$ by the analytic continuation along the horizontal line from right. In 1990, Matsumoto [**124**] generalized the Bohr–Jessen limit theorem for the class $\mathcal{M}$.

Theorem 6.1 (Matsumoto). *Let $F \in \mathcal{M}$ and $\sigma > \rho_F$. Then there exists the limit value*

$$W_\sigma(\mathcal{R}; F) = \lim_{T\to\infty} \mathbb{P}_T(\log F(\sigma + it) \in \mathcal{R}),$$

where $\mathcal{R}$ is a rectangle in $\mathbb{C}$ with edges parallel to the axes.

This is an analogue of Theorem 1.4 (i). To derive an analogue of assertion (ii), Matsumoto [**125**, **126**] presented a sufficient condition for the Euler product of F. The condition is called the convexity for the Euler product. For example, Dedekind zeta-functions $\zeta_K(s)$ of Galois extensions $K/\mathbb{Q}$ have convex Euler products. Hence we obtain analogues of Theorem 1.4 (ii) in these cases, as seen in Theorem 3.1. However, there exist many Matsumoto zeta-functions possessing non-convex Euler product, e.g. Dedekind zeta-functions of non-Galois extensions and L-functions of cusp forms. Therefore it remained worth considering the method to prove an analogue of Theorem 1.4 (ii) without the convexity.

6.2. The Steuding class. The class $\mathcal{M}$ was introduced as a wide class where analogues of the Bohr–Jessen limit theorem hold. Next, we consider subclasses of $\mathcal{M}$ for more precise studies of the value-distributions of L-functions. Steuding [**175**] introduced the class $\mathcal{S}'$ for the purpose of showing general universality theorems. It is defined as the collection of all functions F such that the following axioms (S1′)–(S6′) are satisfied.

(S1′) *Dirichlet series.* The function F is represented as

$$F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s}$$

in some right-half plane of $\mathbb{C}$, where $a_F(n) \in \mathbb{C}$.

(S2′) *Ramanujan hypothesis.* The coefficient $a_F(n)$ satisfies $a_F(n) \ll n^{\epsilon}$ for each $\epsilon > 0$, where the implied constant may depend on ϵ.

(S3′) *Analytic continuation.* There exists a real number $\sigma_F < 1$ such that F has an analytic continuation to the right-half plane $\operatorname{Re} s > \sigma_F$ except for at most a pole at $s = 1$.

(S4′) *Finite order.* There exists a constant $\mu_F \geq 0$ such that, for any fixed $\sigma > \sigma_F$, we have

$$F(\sigma + it) \ll |t|^{\mu_F + \epsilon}$$

for each $\epsilon > 0$, where the implied constant may depend on ϵ.

(S5′) *Polynomial Euler product.* The function F is represented as

$$F(s) = \prod_p \prod_{j=1}^d (1 - \alpha_{p,j} p^{-s})^{-1},$$

where $\alpha_{p,1}, \dots, \alpha_{p,d} \in \mathbb{C}$ with $|\alpha_{p,j}| \leq 1$.

(S6′) *Prime mean square.* There exists a positive constant κ such that

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a_F(p)|^2 = \kappa.$$

Note that the Steuding class $\mathcal{S}'$ is a subclass of $\mathcal{M}$. In fact, it is easily checked that Axioms (M1)–(M4) are satisfied with $\alpha = \beta = 0$ for $F \in \mathcal{S}'$. Furthermore, Steuding [**175**, Theorem 2.4] proved that the limit formula

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |F(\sigma + it)|^2 \, dt = \sum_{n=1}^\infty |a_F(n)|^2 n^{-2\sigma}$$

holds in the range

$$\sigma > \max \left\{ 1 - \frac{1 - \sigma_F}{1 + 2\mu_F}, \frac{1}{2} \right\}$$

if F satisfies Axioms (S1′)–(S4′). Hence Axiom (M5) is also satisfied for $F \in \mathcal{S}'$. As we have seen, Axioms (S1′)–(S5′) hold naturally for many L-functions. Then Axiom (S6′) seems to be a main ingredient of the Steuding class, which is closely related to formula (5.9) by the partial summation. Steuding [**175**, Theorem 5.14] achieved the universality theorem for L-functions in the class $\mathcal{S}'$. He explored some examples of elements of $\mathcal{S}'$; the Riemann zeta-function, Dirichlet L-functions, Dedekind zeta-functions, and L-functions of holomorphic primitive cusp forms. Furthermore, the author [**136**,**137**] noticed further that Axiom (S6′) is also useful to derive analogues of Theorem 1.4 (ii). He also obtained general discrepancy bounds for L-functions in the intersection of two classes $\mathcal{S}$ and $\mathcal{S}'$. See Theorem 11.2 for the precise statement of the result. Here, we prove the following lemma based on the positive-density method introduced by Laurinčikas–Matsumoto [**101**].

Lemma 6.2. *Let $F \in \mathcal{S}'$. Then the series $\sum_p |a_F(p)|/p$ diverges.*

Proof. First, we notice that the inequalities

$$|a_F(p)| \leq |\alpha_{p,1}| + \cdots + |\alpha_{p,d}| \leq d$$

hold by Axiom (S5′). For $0 < \mu < 1$, we define

$$\mathcal{P}_\mu = \{ p \mid p \text{ is a prime number such that } |a_F(p)| > d\mu \}$$

and denote by $\pi_\mu(x)$ the number of $p \in \mathcal{P}_\mu$ such that $p \leq x$. Then we obtain

$$\begin{aligned}\sum_{x/2 < p \leq y} |a_F(p)|^2 &\leq d^2 \sum_{\substack{x/2 < p \leq y \\ p \in \mathcal{P}_\mu}} 1 + d^2\mu^2 \sum_{\substack{x/2 < p \leq y \\ p \notin \mathcal{P}_\mu}} 1 \\ &= d^2(1 - \mu^2)(\pi_\mu(y) - \pi_\mu(x/2)) + d^2\mu^2(\pi(y) - \pi(x/2))\end{aligned}$$

for $x/2 < y \leq x$. Applying Axiom (S6′), we see that

$$\sum_{x/2<p\leq y} |a_F(p)|^2 \geq \frac{\kappa}{2}(\pi(y) - \pi(x/2))$$

if x is sufficiently large and $y > \frac{x}{2}(1+\delta)$ with some constant $0 < \delta < 1$. Therefore the inequality

$$\pi_\mu(y) - \pi_\mu(x/2) \geq \frac{\kappa - 2d^2\mu^2}{2d^2(1-\mu^2)}(\pi(y) - \pi(x/2))$$

is derived. Let μ be small enough to keep $\kappa - 2d^2\mu^2 > 0$. By the partial summation, we have

$$\begin{aligned}\sum_{\substack{x/2<p\leq x \\ p\in\mathcal{P}_\mu}} \frac{1}{p} &= (\pi_\mu(x) - \pi_\mu(x/2))\frac{1}{x} + \int_{x/2}^{x} (\pi_\mu(y) - \pi_\mu(x/2)) \frac{dy}{y^2} \\ &\geq \frac{\kappa - 2d^2\mu^2}{2d^2(1-\mu^2)} \left\{ (\pi(x) - \pi(x/2))\frac{1}{x} + \int_{\frac{x}{2}(1+\delta)}^{x} (\pi(y) - \pi(x/2)) \frac{dy}{y^2} \right\} \\ &\geq \frac{\kappa - 2d^2\mu^2}{4d^2(1-\mu^2)} \sum_{\frac{x}{2}(1+\delta)<p<x} \frac{1}{p}\end{aligned}$$

if δ is sufficiently small. Recall Mertens' formula

$$\sum_{p\leq x} \frac{1}{p} = \log_2 x + A + O\left((\log x)^{-1}\right),$$

where A is a certain constant; see [31, Section 7]. It yields the lower bound

$$\sum_{\substack{x/2<p\leq x \\ p\in\mathcal{P}_\mu}} \frac{1}{p} \gg \log_2 x,$$

and therefore, we conclude

$$\sum_{p\leq x} \frac{|a_F(p)|}{p} > d\mu \sum_{\substack{x/2<p\leq x \\ p\in\mathcal{P}_\mu}} \frac{1}{p} \to \infty \qquad (x \to \infty)$$

as desired. □

7. Setting in this thesis

The purpose of this thesis is to study the value-distributions of L-functions in various aspects including t-aspect, χ-aspect, and so on. For this, we define the notion of families for values of general L-functions.

Definition 7.1. Let $F \in \mathcal{S}'$ and $\sigma > \sigma_F$. Then the continuous family attached to the pair (F, σ) is the family of the values $\{F(\sigma + it)\}_{t\in\mathbb{R}}$.

Let $M_d^1(\mathbb{C})$ denote the set of all $d \times d$-matrices whose eigenvalues $\alpha_1, \ldots, \alpha_d$ satisfy $|\alpha_j| \leq 1$ for all j. For $F \in \mathcal{S}'$, it is often useful to represent its polynomial Euler product as

$$F(s) = \prod_p \det(I - p^{-s} A_p)^{-1}, \tag{7.1}$$

where A_p is a matrix in $M_d^1(\mathbb{C})$. We have also

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} b_F(p^k) p^{-ks} \right),$$

where $b_F(p^k) = \operatorname{tr}(A_p^k)/k$. Let $\{F(\sigma + it)\}_{t \in \mathbb{R}}$ be a continuous family attached to the pair (F, σ). Then we define a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as

$$P_{\sigma,T}(A; F) = \mathbb{P}_T(\log F(\sigma + it) \in A) \tag{7.2}$$

for $T > 0$. If $P_{\sigma,T}(\,\cdot\,; F)$ converges weakly to a probability measure $P_\sigma(\,\cdot\,; F)$ as $T \to \infty$, then we define

$$D_\sigma(T; F) = \sup_{\mathcal{R}} \left| P_{\sigma,T}(\mathcal{R}; F) - P_\sigma(\mathcal{R}; F) \right|, \tag{7.3}$$

where $\mathcal{R}$ runs through all rectangle with edges parallel to the axes. Furthermore, if there exists a continuous function $\mathcal{M}_\sigma(\,\cdot\,; F) : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that the limit formula

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\log F(\sigma + it))\, dt = \int_{\mathbb{C}} \Phi(w) \mathcal{M}_\sigma(w; F)\, |dw|$$

holds at least for all bounded continuous functions Φ, then we call $\mathcal{M}_\sigma(w; F)$ the M-function for the continuous family attached to the pair (F, σ).

Definition 7.2. Let $\mathcal{L}$ be the tuple of the following data:

- a discrete set $\mathscr{F}$;
- L-functions $L(s, f) \in \mathcal{S}^{\text{poly}}$ with $f \in \mathscr{F}$;
- an index set $\Lambda \subset \mathbb{R}$ such that $\sup \Lambda = +\infty$;
- finite sets $\mathscr{F}_q \subset \mathscr{F}$ for any $q \in \Lambda$;
- a weight function $\omega_q : \mathscr{F}_q \to \mathbb{R}_{\geq 0}$ for any $q \in \Lambda$.

Let $s = \sigma + it$ with $\sigma > 1/2$ be a fixed complex number. Then the discrete family attached to the pair $(\mathcal{L}, s)$ is the family of the values $\{L(s, f)\}_{f \in \mathscr{F}}$.

Furthermore, we say that a discrete family $\{L(s, f)\}_{f \in \mathscr{F}}$ is real if every $L(s, f)$ has the polynomial Euler product $L(s, f) = \prod_p \Phi_p(p^{-s})^{-1}$ with $\Phi_p(T) \in \mathbb{R}[T]$. Let $L(s, f) \in \mathcal{S}^{\text{poly}}$. Then we use the notation

$$\begin{aligned} L(s, f) &= \sum_{n=1}^{\infty} a(n, f) n^{-s} \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} b(p^k, f) p^{-ks} \right) = \prod_p \det(I - p^{-s} A_p(f))^{-1}, \end{aligned} \tag{7.4}$$

where $A_p(f)$ is a matrix in $M_d^1(\mathbb{C})$. Since $L(s,f)$ is also a Matsumoto zeta-function, one can define $\log L(s,f)$ as a holomorphic function on the region

$$G_f = \{\sigma + it \mid \sigma > 1/2\} \setminus \bigcup_{L(\rho,f)=0 \text{ or } \infty} \{\sigma + i\operatorname{Im}\rho \mid 1/2 < \sigma \leq \operatorname{Re}\rho\}.$$

Let $\{L(s,f)\}_{f\in\mathscr{F}}$ be a real discrete family attached to $(\mathcal{L}, s)$. Then we define

$$N(q;\mathcal{L}) = \sum_{f\in\mathscr{F}_q} \omega_q(f) \quad \text{and} \quad N'(q,s;\mathcal{L}) = {\sum_{f\in\mathscr{F}_q}}' \omega_q(f),$$

where the notation $\sum'$ stands for the sum over f such that $s \in G_f$. Here, we assume that there exists an absolute constant $\delta > 0$ such that the asymptotic formulas

$$N(q;\mathcal{L}) = 1 + O(q^{-\delta}) \quad \text{and} \quad N'(q,s;\mathcal{L}) = 1 + O(q^{-\delta}) \tag{7.5}$$

hold with the implied constants depending at most on the pair $(\mathcal{L}, s)$. Then we define a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as

$$P_{s,q}(A;\mathcal{L}) = \frac{1}{N'(q,s;\mathcal{L})} {\sum_{f\in\mathscr{F}_q}}' \omega_q(f) 1_A(\log L(s,f)). \tag{7.6}$$

Note that it is also a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if $s = \sigma$. Furthermore, if the probability measure $P_{s,q}(\,\cdot\,;\mathcal{L})$ converges weakly to a probability measure $P_s(\,\cdot\,;\mathcal{L})$ as $q \to \infty$, then we define

$$D_s(q;\mathcal{L}) = \begin{cases} \sup_{\mathcal{R}} \left| P_{s,q}(\mathcal{R};\mathcal{L}) - P_s(\mathcal{R};\mathcal{L}) \right| & \text{if } s = \sigma + it \text{ with } t \neq 0, \\ \sup_{\mathcal{I}} \left| P_{\sigma,q}(\mathcal{I};\mathcal{L}) - P_\sigma(\mathcal{I};\mathcal{L}) \right| & \text{if } s = \sigma, \end{cases} \tag{7.7}$$

where $\mathcal{I}$ runs through all intervals $[a,b]$ in $\mathbb{R}$. Let $s = \sigma + it$ with $\sigma > 1/2$ and $t \neq 0$. If there exist continuous functions $\mathcal{M}_s : \mathbb{C} \to \mathbb{R}_{\geq 0}$ and $\mathcal{M}_\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that the limit formulas

$$\begin{aligned} \lim_{\substack{q\to\infty \\ q\in\Lambda}} {\sum_{f\in\mathscr{F}_q}}' \omega_q(f)\Phi(\log L(s,f)) &= \int_{\mathbb{C}} \Phi(w)\mathcal{M}_s(w;\mathcal{L})\,|dw|, \\ \lim_{\substack{q\to\infty \\ q\in\Lambda}} {\sum_{f\in\mathscr{F}_q}}' \omega_q(f)\Phi(\log L(\sigma,f)) &= \int_{\mathbb{R}} \Phi(w)\mathcal{M}_\sigma(u;\mathcal{L})\,|du| \end{aligned} \tag{7.8}$$

hold at least for all bounded continuous functions Φ, then we call $\mathcal{M}_s(w;\mathcal{L})$ and $\mathcal{M}_\sigma(u;\mathcal{L})$ the M-functions for the real discrete family attached to the pair $(\mathcal{L}, s)$. We hereby describe some examples of real discrete families.

Example 7.3. With the notation of Section 2, let $\mathscr{F} = \mathcal{F}$, and take the index set as $\Lambda = \mathbb{R}_{>0}$. Define $\mathscr{F}_X = \mathcal{F}_X$ for $X \in \Lambda$ and $\omega_X(d) = \#\mathcal{F}_X^{-1}$ for any $d \in \mathcal{F}_X$. Then the family $\{L(s,\chi_d)\}_{d\in\mathscr{F}}$ is a typical example of real discrete families. By the zero density estimate of Heath–Brown [59], we see that condition (7.5) is satisfied. Under GRH, the probability measure is given by

$$P_{s,X}(A;\mathcal{L}) = \frac{\#\{d \in \mathcal{F}_X \mid \log L(s,\chi_d) \in A\}}{\#\mathcal{F}_X},$$

which is equal to $\mathbb{P}_{\mathcal{F}_X}(\log L(s,\chi_d) \in A)$ in the notation of Section 2. In this case, formula (7.8) is related to the left-hand side of (2.11).

Example 7.4. With the notation of Section 3.3, let $\mathscr{F} = \bigcup_{X>0} L_3^{\pm}(X)$, and take the index set as $\Lambda = \mathbb{R}_{>0}$. Define $\mathscr{F}_X = L_3^{\pm}(X)$ for $X \in \Lambda$ and $\omega_X(K) = \# L_3^{\pm}(X)^{-1}$ for any $K \in L_3^{\pm}(X)$. Then the family $\{L(s,\rho_K)\}_{K\in\mathscr{F}}$ is real by representation (3.3). For any $s = \sigma + it$ with $\sigma > 7/8 + \epsilon$, condition (7.5) is satisfied by the zero density estimate of Kowalski–Michel [**87**]; see Lemma 12.4. Note that formula (7.8) is associated with Theorem III.

Example 7.5. With the notation of Section 3.4, let $\mathscr{F} = \bigcup_q B_2(q)$, and take the index set Λ as the set of all prime numbers. Define $\mathscr{F}_q = B_2(q)$ for $q \in \Lambda$ and $\omega_q(f) = \# B_2(q)^{-1}$ for any $f \in B_2(q)$. Since the coefficients $\lambda_f(n)$ are always real, we see that $\{L(s,f)\}_{f\in\mathscr{F}}$ is a real discrete family. In this case, we obtain condition (7.5) by using the zero density estimate of Kowalski–Michel [**86**]. This family is associated with Theorem IV.

Example 7.6. With the notation of Section 3.4, let $\mathscr{F} = \bigcup_q B_2(q)$, and take the index set Λ as the set of all prime numbers. Define $\mathscr{F}_q = B_2(q)$ for $q \in \Lambda$. The difference from Example 7.5 is just to use the harmonic weight $\omega_q(f) = (4\pi\langle f,f\rangle)^{-1}$. Then $\{L(s,f)\}_{f\in\mathscr{F}}$ is again a real discrete family. In this case, condition (7.5) is derived by the zero density estimate along with the Petersson trace formula [**27**, Proposition 1.9]. Using this family, we obtain a variant of Theorem IV.

CHAPTER 3

Random Euler products

As seen in Section 1.2, Jessen–Wintner [71] explained the limit value $W_\sigma(\mathcal{R})$ of the Bohr–Jessen limit theorem by using probability measures associated with the curves $S_{n,\sigma}$ given by $z_{n,\sigma}(\theta) = -\log(1 - p_n^{-\sigma} e^{i\theta})$. Equivalently, it can be explained in terms of random variables. Let $X = (X_p)_p$ be a sequence of independent random variables indexed by prime numbers which are uniformly distributed on the unit circle $\mathcal{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then we define the random Euler product

$$\zeta(\sigma, X) = \prod_p (1 - p^{-\sigma} X_p)^{-1},$$

where p runs through all prime numbers. We see that this infinite product converges almost surely if $\sigma > 1/2$. Furthermore, we obtain

$$W_\sigma(\mathcal{R}) = \mathbb{P}(\log \zeta(\sigma, X) \in \mathcal{R})$$

for $\sigma > 1/2$. In this chapter, we study random Euler products arising from families of values of general L-functions introduced in Section 7.

8. Basic properties

8.1. Definition and examples. Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on $M_d(\mathbb{C})$. Then we define the Random Euler product attached to $\mathcal{X}$ as

$$L(s, \mathcal{X}) = \prod_p \det(I - p^{-s} \mathcal{X}_p)^{-1}, \tag{8.1}$$

where p runs through all prime numbers. To begin with, we consider the condition for the convergence of infinite product (8.1). It is elementary to see that (8.1) converges surely for $\operatorname{Re} s > 1$ if every $\mathcal{X}_p$ is supported on $M_d^1(\mathbb{C})$. With reasonable assumptions, we further prove that it converges almost surely beyond the line $\operatorname{Re} s = 1$.

Lemma 8.1. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices which are supported on $M_d^1(\mathbb{C})$. Suppose that there exists a real number $0 < \alpha \leq 1/2$ such that the condition*

$$\mathbb{E}[\operatorname{tr} \mathcal{X}_p] \ll_\epsilon p^{-\alpha+\epsilon} \tag{8.2}$$

is satisfied for each $\epsilon > 0$. Then infinite product (8.1) converges almost surely for $\operatorname{Re} s > 1 - \alpha$.

Proof. Let $s = \sigma + it$. The local factor of (8.1) can be calculated as

$$\begin{aligned}\det(I - p^{-s}\mathcal{X}_p)^{-1} &= \prod_{j=1}^{d}(1 - \alpha_j p^{-s})^{-1} \\ &= 1 + (\operatorname{tr}\mathcal{X}_p)p^{-s} + O(p^{-2\sigma})\end{aligned} \tag{8.3}$$

since the eigenvalues satisfy $|\alpha_j| \leq 1$. The series $\sum_p p^{-2\sigma}$ converges for $\sigma > 1/2$. We also obtain $\mathbb{E}[(\operatorname{tr}\mathcal{X}_p)p^{-s}] \ll p^{-\sigma-\alpha+\epsilon}$ by the assumption. Thus the series

$$\sum_p \mathbb{E}\left[(\operatorname{tr}\mathcal{X}_p)p^{-s}\right]$$

converges absolutely for $\sigma > 1 - \alpha$. Furthermore, the series

$$\sum_p \mathbb{V}\left[(\operatorname{tr}\mathcal{X}_p)p^{-s}\right]$$

also converges for $\sigma > 1 - \alpha$. Hence it follows from Kolmogorov's two-series theorem (Theorem A.1.3) that the random variable $\sum_p (\operatorname{tr}\mathcal{X}_p)p^{-s}$ converges almost surely for $\sigma > 1 - \alpha$. Then the result is derived from (8.3). ☐

With the assumptions as in Lemma 8.1, the random Euler product $L(s, \mathcal{X})$ is a $\mathbb{C}$-valued random variable for $\operatorname{Re} s > 1 - \alpha$. Additionally,

$$\log L(s, \mathcal{X}) = \sum_p \sum_{k=1}^{\infty} \frac{\operatorname{tr}(\mathcal{X}_p^k)}{k} p^{-ks}$$

is also a $\mathbb{C}$-valued random variable for $\operatorname{Re} s > 1 - \alpha$. We say that a sequence $\mathcal{X} = (\mathcal{X}_p)_p$ is real if the random polynomial $\det(1 - T\mathcal{X}_p)$ is supported on $\mathbb{R}[T]$ for every p. In this case, $L(\sigma, \mathcal{X})$ and $\log L(\sigma, \mathcal{X})$ are $\mathbb{R}$-valued random variables for $\sigma > 1 - \alpha$. As below, we present several examples of random Euler products satisfying the assumption of Lemma 8.1.

Example 8.2. Let $F \in \mathcal{S}'$ and denote its Euler product by (7.1). Let $X = (X_p)_p$ be a sequence of independent random variables uniformly distributed on $\mathcal{T}$. Then we consider the random Euler product

$$F(\sigma, X) = \prod_p \det(I - p^{-\sigma}X_pA_p)^{-1}, \tag{8.4}$$

which contains $\zeta(\sigma, X)$ as a special case $A_p \equiv 1 \in M_1^1(\mathbb{C})$. Since we have

$$\mathbb{E}[\operatorname{tr}(X_pA_p)] = \operatorname{tr} A_p \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}\, d\theta = 0,$$

infinite product (8.4) converges almost surely if $\sigma > 1/2$. The random Euler product $F(\sigma, X)$ is expected to be a nice model to the continuous family $\{F(\sigma + it)\}_{t\in\mathbb{R}}$.

Example 8.3. Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random variables supported on $\{\pm 1, 0\}$. Suppose that every $\mathcal{X}_p$ is distributed according to

$$\mathbb{P}(\mathcal{X}_p = a) = \frac{p}{p+1} \times \begin{cases} 1/2 & \text{if } a = +1, \\ 1/2 & \text{if } a = -1, \\ 1/p & \text{if } a = 0. \end{cases}$$

Then we see that the sequence $\mathcal{X}$ is real. Since we have

$$\mathbb{E}[\operatorname{tr} \mathcal{X}_p] = \frac{1 \cdot p}{2(p+1)} + \frac{(-1)p}{2(p+1)} + \frac{0 \cdot 1}{p+1} = 0,$$

infinite product (8.1) converges almost surely if $\operatorname{Re} s > 1/2$. In this case, the random Euler product $L(s, \mathcal{X})$ is expected to be a nice model to the discrete family of Example 7.3.

Example 8.4. Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on the set

$$\mathcal{A} = \{\operatorname{diag}(1,1), \operatorname{diag}(1,-1), \operatorname{diag}(\omega, \overline{\omega}), \operatorname{diag}(1,0), \operatorname{diag}(0,0)\}, \tag{8.5}$$

where ω is a primitive cube root of unity. Suppose that every $\mathcal{X}_p$ is distributed according to

$$\mathbb{P}(\mathcal{X}_p = A) = \frac{p^2}{p^2+p+1} \times \begin{cases} 1/6 & \text{if } A = \operatorname{diag}(1,1), \\ 1/2 & \text{if } A = \operatorname{diag}(1,-1), \\ 1/3 & \text{if } A = \operatorname{diag}(\omega, \overline{\omega}), \\ 1/p & \text{if } A = \operatorname{diag}(1,0), \\ 1/p^2 & \text{if } A = \operatorname{diag}(0,0). \end{cases}$$

Note that $\mathcal{X}$ is a real sequence. Since we have

$$\begin{aligned} \mathbb{E}[\operatorname{tr} \mathcal{X}_p] &= \frac{p^2}{p^2+p+1} \left(\frac{1+1}{6} + \frac{1+(-1)}{2} + \frac{\omega + \overline{\omega}}{3} + \frac{1+0}{p} + \frac{0+0}{p^2} \right) \\ &\ll p^{-1}, \end{aligned}$$

infinite product (8.1) converges almost surely if $\operatorname{Re} s > 1/2$. In this case, the random Euler product $L(s, \mathcal{X})$ is expected to be a nice model to the discrete family of Example 7.4.

Example 8.5. Let $(\Theta_p)_p$ be a sequence of independent $[0, \pi]$-valued random variable. Suppose that every Θ_p is distributed according to the p-adic Plancherel measure μ_p which is defined by (3.14). We define a sequence $\mathcal{X} = (\mathcal{X}_p)_p$ by putting $\mathcal{X}_p = \operatorname{diag}(e^{i\Theta_p}, e^{-i\Theta_p})$. Then we see that $\mathcal{X}$ is real. Since we have

$$\mathbb{E}[\operatorname{tr} \mathcal{X}_p] = \int_0^\pi (2 \cos \theta) \, d\mu_p(\theta) = 0,$$

infinite product (8.1) converges almost surely if $\operatorname{Re} s > 1/2$. In this case, the random Euler product $L(s, \mathcal{X})$ is expected to be a nice model to the discrete family of Example 7.5.

Example 8.6. Let $(\Theta_p)_p$ be a sequence of independent $[0,\pi]$-valued random variable. Suppose that every Θ_p is distributed according to the Sato–Tate measure μ_∞ which is defined as $d\mu_\infty(\theta) = \frac{2}{\pi}\sin^2\theta\, d\theta$. Similarly to Example 8.5, we define a sequence $\mathcal{X} = (\mathcal{X}_p)_p$ by putting $\mathcal{X}_p = \mathrm{diag}(e^{i\Theta_p}, e^{-i\Theta_p})$. Then the sequence $\mathcal{X}$ is real. Since we have

$$\mathbb{E}[\mathrm{tr}\,\mathcal{X}_p] = \int_0^\pi (2\cos\theta)\, d\mu_\infty(\theta) = 0,$$

infinite product (8.1) converges almost surely if $\mathrm{Re}\, s > 1/2$. In this case, the random Euler product $L(s,\mathcal{X})$ is expected to be a nice model to the discrete family of Example 7.6.

8.2. Supports of random Euler products. Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on $M_d^1(\mathbb{C})$ which satisfies (8.2) with some $0 < \alpha \leq 1/2$. Define $L_p(s,\mathcal{X}) = \det(I - p^{-s}\mathcal{X}_p)^{-1}$ for every prime number p. Then we denote the probability distributions of $\log L(s,\mathcal{X})$ and $\log L_p(s,\mathcal{X})$ by

$$P_s(A) = \mathbb{P}(\log L(s,\mathcal{X}) \in A) \quad \text{and} \quad \mu_{s,p}(A) = \mathbb{P}(\log L_p(s,\mathcal{X}) \in A), \tag{8.6}$$

where $A \in \mathcal{B}(\mathbb{C})$. For $\mathrm{Re}\, s > 1-\alpha$, they are probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Then, we show that P_s is represented as the infinite convolution of all $\mu_{s,p}$ similarly to Theorem 1.6.

Lemma 8.7. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on $M_d^1(\mathbb{C})$ which satisfies condition (8.2) with some $0 < \alpha \leq 1/2$. Let $s = \sigma + it$ with $\sigma > 1-\alpha$. Then the convolution measure $P_{y,s} = *_{p\leq y}\mu_{s,p}$ converges weakly to P_s as $y \to \infty$.*

Proof. Recall that the equality

$$\mathbb{P}\left(\sum_{p\leq y} \log L_p(s,\mathcal{X}) \in A\right) = P_{y,s}(A)$$

holds for all $A \in \mathcal{B}(\mathbb{C})$ since $\log L_p(s,\mathcal{X})$'s are independent. Furthermore, the random variable $\sum_{p\leq y} \log L_p(s,\mathcal{X})$ converges to $\log L(s,\mathcal{X})$ as $y \to \infty$ in law by Lemma 8.1. Hence we obtain the conclusion. □

We proceed to study the support of P_s. Some basic facts on supports of probability measures are collected in Appendix A.2. First, we deduce from Lemma 8.7 that the support of P_s is represented as

$$\mathrm{supp}(P_s) = \sum_p \mathrm{supp}(\mu_{s,p}) \tag{8.7}$$

for $\mathrm{Re}\, s > 1-\alpha$; see Theorem A.2.1. Applying this formula, we calculate the support of P_s as follows.

Proposition 8.8. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices which are supported on $M_d^1(\mathbb{C})$. Then* $\mathrm{supp}(P_s)$ *is compact for* $\mathrm{Re}\, s > 1$.

Proof. Let $s = \sigma + it$. By definition, we obtain

$$\log L_p(s, \mathcal{X}) = \sum_{k=1}^{\infty} \frac{\operatorname{tr}(\mathcal{X}_p^k)}{k} p^{-ks}$$

for $\sigma > 1$. Since we have $|\operatorname{tr}(\mathcal{X}_p^k)| \leq d$, the inequality

$$|\log L_p(s, \mathcal{X})| \leq 2dp^{-\sigma}$$

holds. Therefore every $\operatorname{supp}(\mu_{p,s})$ is included in the disk $|z| \leq 2dp^{-\sigma}$. The series $\sum_p p^{-\sigma}$ is finite for $\sigma > 1$, and thus the result follows from (8.7). □

Proposition 8.9. *Let $\sigma > 1/2$ and denote by $F(\sigma, X)$ the random Euler product of Example 8.2. Then we have* $\operatorname{supp}(P_\sigma) = \mathbb{C}$ *for* $1/2 < \sigma \leq 1$.

Proof. The proof is based on Lemma A.2.2 for $H = \mathbb{C}$ equipped with the inner product $(z, w) = z\overline{w}$. Put $z_n = a_F(p_n)p_n^{-\sigma}$. Then it is easily seen that $\sum_{n=1}^{\infty} |z_n|^2 < \infty$ holds. Furthermore, we have $\sum_{n=1}^{\infty} |(z_n, w)| = \infty$ for $w \neq 0$ by Lemma 6.2. Hence the assumptions of Lemma A.2.2 are satisfied. Let $z \in \mathbb{C}$ and $\epsilon > 0$ arbitrarily. Then we obtain

$$\sum_{n=m}^{N} \sum_{k=2}^{\infty} \frac{d}{k} p_n^{-k\sigma} \leq B_{\sigma,d} m^{1-2\sigma} \tag{8.8}$$

for any $N \geq m \geq 1$, where $B_{\sigma,d} > 0$ is a constant depending only on σ and d. One can take an integer $m = m(\epsilon, \sigma, d) \geq 2$ so that $B_{\sigma,d} m^{1-2\sigma} < \epsilon$ is satisfied. By Lemma A.2.2, there exist an integer $N = N(\epsilon, \sigma, d, z) \geq m$ and complex numbers $c_m, \ldots, c_N \in \mathcal{T}$ such that the inequality

$$\left| \left(z - \sum_{n=1}^{m-1} \sum_{k=1}^{\infty} \frac{\operatorname{tr}(A_p^k)}{k} p_n^{-k\sigma} \right) - \sum_{n=m}^{N} c_n a_F(p_n) p_n^{-\sigma} \right| < \epsilon \tag{8.9}$$

holds. Then we take $c_n = 1$ for $1 \leq n \leq N$. By (8.8) and (8.9), we obtain

$$\left| z - \sum_{n=1}^{N} \sum_{k=1}^{\infty} \frac{\operatorname{tr}(A_p^k)}{k} p_n^{-k\sigma} c_n^k \right| < 2\epsilon.$$

It implies $\sum_{n=1}^{\infty} \operatorname{supp}(\mu_{\sigma,p_n}) = \mathbb{C}$. Thus we obtain the desired result by (8.7). □

Proposition 8.10. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a real sequence of independent random matrices supported on $M_d^1(\mathbb{C})$ which satisfies (8.2) with some $0 < \alpha \leq 1/2$. If every* $\operatorname{supp}(\operatorname{tr} \mathcal{X}_p)$ *contains ± 1, then the following results hold.*

(i) *Let $s = \sigma + it$ with $1 - \alpha < \sigma \leq 1$ and $t \neq 0$. Then we have* $\operatorname{supp}(P_s) = \mathbb{C}$.
(ii) *Let $1 - \alpha < \sigma \leq 1$. Then we have* $\operatorname{supp}(P_\sigma) = \mathbb{R}$.

Proof. For this result, we use Lemma A.2.3. We firstly consider assertion (i). Let $s = \sigma + it$ with $1 - \alpha < \sigma \leq 1$ and $t \neq 0$. Then we take $H = \mathbb{C}$ equipped with the inner product $\langle z, w \rangle$. Put $z_n = p_n^{-s}$. In this case, the assumptions of Lemma A.2.3 are checked in Mishou–Nagoshi [**141**]. The remaining arguments are along the same lines as the proof of Proposition 8.9, and we omit them. When we consider assertion (ii), we take $H = \mathbb{R}$ equipped with the standard inner product, and we put

$z_n = p_n^{-\sigma}$. It is easily checked that the assumptions of Lemma A.2.3 are satisfied. Then one can derive the result similarly to (i). □

9. Probability density functions

The purpose of this section is to study probability density functions attached to random variable $\log L(s, \mathcal{X})$. For the random Euler product of Example 8.2, we prove the following result.

Theorem 9.1. *Let $\sigma > 1/2$ and denote by $F(\sigma, X)$ the random Euler product of Example 8.2. Then there exists a continuous function $\mathcal{M}_\sigma(\,\cdot\,; F) : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that*

$$\mathbb{P}(\log F(\sigma, X) \in A) = \int_A \mathcal{M}_\sigma(w; F)\,|dw|$$

holds for all $A \in \mathcal{B}(\mathbb{C})$.

For the random Euler products of Examples 8.3–8.6, we introduce the notion of admissible sequences as follows. Note that all sequences $\mathcal{X} = (\mathcal{X}_p)_p$ of the examples are admissible.

Definition 9.2. Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a real sequence of independent random matrices supported on $M_d^1(\mathbb{C})$. We say that it is admissible if every $\mathcal{X}_p$ satisfies condition (8.2) with some $0 < \alpha \leq 1/2$ and the inequality

$$\mathbb{E}[(\operatorname{tr} \mathcal{X}_p)^2] \geq \delta \tag{9.1}$$

with an absolute constant $\delta > 0$.

Theorem 9.3. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a real admissible sequence of independent random matrices.*

(i) *Let $s = \sigma + it$ be a complex number with $\sigma > 1 - \alpha$ and $t \neq 0$. Then there exists a continuous function $\mathcal{M}_s : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that*

$$\mathbb{P}(\log L(s, \mathcal{X}) \in A) = \int_A \mathcal{M}_s(w; \mathcal{X})\,|dw|$$

holds for all $A \in \mathcal{B}(\mathbb{C})$.

(ii) *Let $\sigma > 1 - \alpha$ be a real number. Then there exists a continuous function $\mathcal{M}_\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that*

$$\mathbb{P}(\log L(\sigma, \mathcal{X}) \in A) = \int_A \mathcal{M}_\sigma(u; \mathcal{X})\,|du|$$

holds for all $A \in \mathcal{B}(\mathbb{R})$.

The notation $\mathcal{M}_s(\,\cdot\,; \mathcal{X})$ of Theorem 9.3 indicates a function on $\mathbb{C}$ if $s \notin \mathbb{R}$, while the same notation indicates a function on $\mathbb{R}$ if $s \in \mathbb{R}$. We later find that these density functions are just equal to the M-functions described in Chapter 1.

9.1. Characteristic functions. The proofs of Theorems 9.1 and 9.3 begin with the study of the characteristic function defined as

$$\phi_s(z) = \mathbb{E}[\psi_z(\log L(s, \mathcal{X}))].$$

By Lemma 8.1, we see that $\sum_{p \le y} \log L_p(s, \mathcal{X})$ converges to $\log L(s, \mathcal{X})$ in law as $y \to \infty$. Hence the characteristic function is represented as

$$\phi_s(z) = \prod_p \mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))]. \tag{9.2}$$

Using this formula, we study the decay of $|\phi_s(z)|$ as $|z| \to \infty$. The method for the proof of the following result was originally introduced by Guo [52] who studied the case of the Riemann zeta-function. It was later generalized and developed by the author [134, 136, 137].

Proposition 9.4. *Let $\sigma > 1/2$ and denote by $F(\sigma, X)$ the random Euler product of Example 8.2. Then there exists a positive constant $c(\sigma, F)$ such that*

$$|\phi_\sigma(z)| \le \exp\left(-c(\sigma, F)\frac{|z|^{\frac{1}{\sigma}}}{\log |z|}\right)$$

for $\sigma > 1/2$ and $|z| \ge 3$.

Proof. Recall that $|\psi_z(w)| = 1$ for any $z, w \in \mathbb{C}$. Then the inequality

$$\left|\mathbb{E}[\psi_z(\log F_p(\sigma, X))]\right| \le 1$$

holds for every prime number p. Therefore, we deduce from (9.2) that

$$|\phi_\sigma(z)| \le \prod_{p > Q} \left|\mathbb{E}[\psi_z(\log F_p(\sigma, X))]\right|, \tag{9.3}$$

where $Q > 0$ is a large real number chosen later. Then we estimate the local factor $\mathbb{E}[\psi_z(\log F_p(\sigma, X))]$ for $p > Q$ as follows. By the Taylor expansion, we have the asymptotic formula

$$\psi_z(w) = 1 + i\langle z, w\rangle - \frac{1}{2}\langle z, w\rangle^2 + O(|z|^3|w|^3)$$

which is valid for $|z||w| < c$ with any positive constant c. We have also

$$\log F_p(\sigma, X) = \operatorname{tr}(A_p)X_p p^{-\sigma} + \frac{1}{2}\operatorname{tr}(A_p^2)X_p^2 p^{-2\sigma} + O(p^{-3\sigma}).$$

From the above formulas, we deduce

$$\begin{aligned} \mathbb{E}[\psi_z(\log F_p(\sigma, X))] = 1 + i\mathbb{E}[\langle z, \operatorname{tr}(A_p)X_p\rangle]p^{-\sigma} + \frac{i}{2}\mathbb{E}[\langle z, \operatorname{tr}(A_p^2)X_p^2\rangle]p^{-2\sigma} \\ - \frac{1}{2}\mathbb{E}[\langle z, \operatorname{tr}(A_p)X_p\rangle^2]p^{-2\sigma} + O\left(|z|^3 p^{-3\sigma}\right) \end{aligned} \tag{9.4}$$

if $|z|p^{-\sigma} < c$ is satisfied. Notice that the second and third terms vanish since we have $\mathbb{E}[X_p] = \mathbb{E}[X_p^2] = 0$. Furthermore, we obtain

$$\begin{aligned}\mathbb{E}[\langle z, \operatorname{tr}(A_p)X_p\rangle^2] &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{z\overline{a_F(p)}e^{-i\theta} + \overline{z}a_F(p)e^{i\theta}}{2}\right)^2 d\theta \\ &= \frac{1}{2}|z|^2|a_F(p)|^2.\end{aligned}$$

Hence there exists a small constant $c_1 > 0$ such that, if $0 < c < c_1$, then we have the inequality $\left|\mathbb{E}[\psi_z(\log F_p(\sigma, X))] - 1\right| < 1/2$ for all $p > Q(c, z, \sigma)$, where $Q(c, z, \sigma)$ is given by

$$Q(c, z, \sigma) = \left(\frac{|z|}{c}\right)^{1/\sigma}. \tag{9.5}$$

For $p > Q(c, z, \sigma)$, we derive

$$\operatorname{Log}\mathbb{E}[\psi_z(\log F_p(\sigma, X))] = -\frac{1}{4}|z|^2|a_F(p)|^2p^{-2\sigma} + O\left(|z|^3p^{-3\sigma}\right)$$

by the formula $\operatorname{Log}(1 + w) = w + O(|w|^2)$. Taking the real parts, we deduce

$$\log\left|\mathbb{E}[\psi_z(\log F_p(\sigma, X))]\right| = -\frac{1}{4}|z|^2|a_F(p)|^2p^{-2\sigma} + O\left(|z|^3p^{-3\sigma}\right).$$

By the partial summation along with Axiom (S6′) in the definition of the Steuding class, we obtain the lower bound

$$\sum_{p>Q}|z|^2|a_F(p)|^2p^{-2\sigma} \geq \frac{\kappa}{2(2\sigma-1)}\frac{|z|^2Q^{1-2\sigma}}{\log Q}$$

if Q is large enough, where κ is the positive constant of Axiom (S6′). On the other hand, we have the upper bound

$$\sum_{p>Q}|z|^3p^{-3\sigma} \leq c\sum_{p>Q}|z|^2p^{-2\sigma} \leq \frac{2c}{2\sigma-1}\frac{|z|^2Q^{1-2\sigma}}{\log Q}$$

by the prime number theorem. Thus, there exists a constant $c_2 = c_2(F)$ satisfying $0 < c_2 < c_1$ such that

$$\sum_{p>Q}\log\left|\mathbb{E}[\psi_z(\log F_p(\sigma, X))]\right| \leq -\frac{\kappa}{10(2\sigma-1)}\frac{|z|^2Q^{1-2\sigma}}{\log Q}$$

for $Q = Q(c_2, z, \sigma)$ as in (9.5). It yields the inequality

$$\begin{aligned}\prod_{p>Q}\left|\mathbb{E}[\psi_z(\log F_p(\sigma, X))]\right| &\leq \exp\left(-\frac{\kappa}{10(2\sigma-1)}\frac{|z|^2Q^{1-2\sigma}}{\log Q}\right) \\ &\leq \exp\left(-c(\sigma, F)\frac{|z|^{\frac{1}{\sigma}}}{\log|z|}\right)\end{aligned}$$

with a positive constant $c(\sigma, F)$ depending only on σ and F. Hence the desired result follows from (9.3). □

Proposition 9.5. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a real admissible sequence of independent random matrices.*

(i) *Let $s = \sigma + it$ be a complex number with $\sigma > 1 - \alpha$ and $t \neq 0$. There exists a positive constant $c(s, \mathcal{X})$ such that*
$$|\phi_s(z)| \leq \exp\left(-c(s, \mathcal{X}) \frac{|z|^{\frac{1}{\sigma}}}{\log |z|}\right)$$
for all $z = x + iy \in \mathbb{C}$ with $|z| \geq 3$.

(ii) *Let $\sigma > 1 - \alpha$ be a real number. There exists a positive constant $c(\sigma, \mathcal{X})$ such that*
$$|\phi_\sigma(z)| \leq \exp\left(-c(\sigma, \mathcal{X}) \frac{|x|^{\frac{1}{\sigma}}}{\log |x|}\right)$$
for all $z = x + iy \in \mathbb{C}$ with $|x| \geq 3$.

Proof. The results are obtained along the same lines as Proposition 9.4. In the present case, we remark that $\mathrm{tr}(\mathcal{X}_p^k)$ are $\mathbb{R}$-valued random variables. First, we consider assertion (i). We obtain

$$\begin{aligned}(9.6)\quad \mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))] = 1 + i\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)]\langle z, p^{-s}\rangle + \frac{i}{2}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p^2)]\langle z, p^{-2s}\rangle \\ - \frac{1}{2}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)^2]\langle z, p^{-s}\rangle^2 + O\left(|z|^3 p^{-3\sigma}\right)\end{aligned}$$

similarly to (9.4), if $|z|p^{-\sigma} < c$ is satisfied. Here, we have

$$\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)]\langle z, p^{-s}\rangle \ll |z|p^{-\sigma}, \qquad \mathbb{E}[\mathrm{tr}(\mathcal{X}_p^2)]\langle z, p^{-2s}\rangle \ll |z|p^{-2\sigma},$$
$$\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)^2]\langle z, p^{-s}\rangle^2 \ll |z|^2 p^{-2\sigma}.$$

Thus $\left|\mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))] - 1\right| < 1/2$ is satisfied for all $p > Q(c, z, \sigma)$ with some $0 < c < 1$, where $Q(c, z, \sigma)$ is given by (9.5). Hence we deduce from (9.6) that

$$\begin{aligned}&\mathrm{Log}\,\mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))] \\ &= i\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)]\langle z, p^{-s}\rangle + \frac{i}{2}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p^2)]\langle z, p^{-2s}\rangle \\ &\quad - \frac{1}{2}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)^2]\langle z, p^{-s}\rangle^2 + O_\epsilon\left(|z|^2 p^{-2\sigma-2\alpha+\epsilon} + |z|^3 p^{-3\sigma}\right)\end{aligned}$$

by using $\mathrm{Log}(1 + w) = w + O(|w|^2)$ along with (8.2). The first and second terms vanish if we take the real parts. Furthermore, we note that the equality $2\langle z, p^{-s}\rangle^2 = |z|^2 p^{-2\sigma} + \langle z^2, p^{-2s}\rangle$ holds. Therefore the asymptotic formula

$$\begin{aligned}\log\left|\mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))]\right| = -\frac{1}{4}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)^2]|z|^2 p^{-2\sigma} - \frac{1}{4}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)^2]\langle z^2, p^{-2s}\rangle \\ + O\left(|z|^2 p^{-2\sigma-\alpha} + |z|^3 p^{-3\sigma}\right)\end{aligned}$$

is derived for $p > Q(c, z, \sigma)$, where we choose ϵ as a sufficiently small real number such that $0 < \epsilon < \alpha$. By condition (9.1), the inequality

$$
\begin{aligned}
(9.7)\qquad & \log \left|\mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))]\right| \\
& \leq -\frac{\delta}{4}|z|^2 p^{-2\sigma} - \frac{\delta}{4}\langle z^2, p^{-2s}\rangle + B(|z|^2 p^{-2\sigma-\alpha} + |z|^3 p^{-3\sigma})
\end{aligned}
$$

is satisfied with positive constants δ and B. Then we have

$$
\begin{aligned}
\sum_{p>Q} |z|^2 p^{-2\sigma} &= |z|^2 \frac{1}{2\sigma - 1}\frac{Q^{1-2\sigma}}{\log Q} + O_\sigma\left(\frac{|z|^2 Q^{1-2\sigma}}{(\log Q)^2}\right), \\
\sum_{p>Q} \langle z^2, p^{-2s}\rangle &= \left\langle z^2, \frac{1}{2s-1}\frac{Q^{1-2s}}{\log Q}\right\rangle + O_s\left(\frac{|z|^2 Q^{1-2\sigma}}{(\log Q)^2}\right)
\end{aligned}
$$

by the prime number theorem, where Q is large enough. Since the inequality

$$
\left|\left\langle z^2, \frac{1}{2s-1}\frac{Q^{1-2s}}{\log Q}\right\rangle\right| \leq |z|^2 \frac{1}{|2s-1|}\frac{Q^{1-2\sigma}}{\log Q}
$$

holds, we obtain

$$
-\sum_{p>Q} |z|^2 p^{-2\sigma} - \sum_{p>Q}\langle z^2, p^{-2s}\rangle \leq -\frac{d(s)}{2\sigma-1}\frac{|z|^2 Q^{1-2\sigma}}{\log Q} + B_2(s)\frac{|z|^2 Q^{1-2\sigma}}{(\log Q)^2},
$$

where $d(s)$ is a constant given by $d(s) = 1 - (2\sigma - 1)|2s - 1|^{-1}$, and $B_2(s)$ is a positive constant. By the assumption that $s = \sigma + it$ with $t \neq 0$, we know $d(s) > 0$. As a result, we deduce from (9.7) that

$$
\begin{aligned}
\sum_{p>Q} \log \left|\mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))]\right| \leq{} & -\frac{d(s)\delta}{4(2\sigma-1)}\frac{|z|^2 Q^{1-2\sigma}}{\log Q} + \frac{B_2(s)\delta}{4\log Q}\frac{|z|^2 Q^{1-2\sigma}}{\log Q} \\
& + B_3(s)\left(Q^{-\alpha} + c\right)\frac{|z|^2 Q^{1-2\sigma}}{\log Q}
\end{aligned}
$$

for $Q = Q(c, z, \sigma)$, where $B_3(s)$ is a positive constant. If we choose $c > 0$ as a suitably small constant depending on s and δ, then it holds

$$
|\phi_s(z)| \leq \prod_{p>Q} \left|\mathbb{E}[\psi_z(\log L_p(s, \mathcal{X}))]\right| \leq \exp\left(-c(s, \mathcal{X})\frac{|z|^{1/\sigma}}{\log |z|}\right)
$$

with some constant $c(s, \mathcal{X}) > 0$ depending only on s and $\mathcal{X}$. Therefore we obtain the conclusion. As for assertion (ii), the difference from the proof of (i) is coming from the fact that $\mathbb{E}[\psi_z(\log L_p(\sigma, \mathcal{X}))]$ is represented as

$$
\mathbb{E}[\psi_z(\log L_p(\sigma, \mathcal{X}))] = \mathbb{E}[\exp(x \log L_p(\sigma, \mathcal{X}))]
$$

for $z = x + iy$, since we have $\psi_z(u) = \exp(ixu)$ for $u \in \mathbb{R}$. Thus we obtain

$$
\begin{aligned}
\mathbb{E}[\psi_z(\log L_p(\sigma, \mathcal{X}))] = 1 + i\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)]xp^{-\sigma} + \frac{i}{2}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p^2)]xp^{-2\sigma} \\
- \frac{1}{2}\mathbb{E}[\mathrm{tr}(\mathcal{X}_p)^2]x^2 p^{-2\sigma} + O\left(|x|^3 p^{-3\sigma}\right)
\end{aligned}
$$

for $|x|p^{-\sigma} < c$ in place of (9.6). This formula yields the inequality

$$|\phi_\sigma(z)| \leq \prod_{p>Q} \left|\mathbb{E}[\psi_z(\log L_p(\sigma, \mathcal{X}))]\right| \leq \exp\left(-c(\sigma, \mathcal{X})\frac{|x|^{1/\sigma}}{\log|x|}\right)$$

with some constant $c(\sigma, \mathcal{X}) > 0$ as desired. □

Proof of Theorem 9.1. By Proposition 9.4, the integral

$$\int_{\mathbb{C}} |\phi_\sigma(z)|\ |dz|$$

is finite. Hence Levy's inversion formula (A.5) yields that the function

$$\mathcal{M}_\sigma(w; F) = \int_{\mathbb{C}} \phi_\sigma(z)\psi_{-w}(z)\,|dz| \tag{9.8}$$

is a probability density function of P_σ with respect to $|dw|$. Furthermore, it is continuous by the dominated convergence theorem. □

Proof of Theorem 9.3. This is also obtained by Levy's inversion formula. Use formula (A.5) for assertion (i) and (A.4) for (ii). The density functions are given by

$$\mathcal{M}_s(w; \mathcal{X}) = \int_{\mathbb{C}} \phi_s(z)\psi_{-w}(z)\,|dz| \quad \text{and} \quad \mathcal{M}_\sigma(u; \mathcal{X}) = \int_{\mathbb{C}} \phi_\sigma(x)\psi_{-u}(x)\,|dx|$$

similarly to (9.8). □

Remark that Proposition 9.4 further yields the finiteness of

$$\int_{\mathbb{C}} |z|^k\,|\phi_\sigma(z)|\ |dz|$$

for any $k \geq 0$. Therefore, differentiating under the integral in (9.8), we see that the density function $\mathcal{M}_\sigma(w; F)$ of Theorem 9.1 belongs to the class $C^\infty(\mathbb{C})$, whose partial derivatives are represented as

$$\frac{\partial^{k+l}}{\partial u^k \partial v^l}\mathcal{M}_\sigma(w; F) = i^{k+l}\int_{\mathbb{C}} x^k y^l \phi_\sigma(z)\psi_w(z)\,|dz|.$$

Similarly, the density function $\mathcal{M}_s(w; \mathcal{X})$ of Theorem 9.3 (i) belongs to the class $C^\infty(\mathbb{C})$, and $\mathcal{M}_\sigma(u; \mathcal{X})$ of (ii) belongs to $C^\infty(\mathbb{R})$.

By Proposition 8.8, the density functions $\mathcal{M}_\sigma(w; F)$, $\mathcal{M}_s(w; \mathcal{X})$, and $\mathcal{M}_\sigma(u; \mathcal{X})$ are compactly supported if $s = \sigma + it$ with $\sigma > 1$. Let $s = \sigma + it$ with $1/2 < \sigma \leq 1$. Then we further obtain

$$\int_{|w-c|<\epsilon} \mathcal{M}_\sigma(w; F)\,|dw| > 0$$

for any $c \in \mathbb{C}$ and $\epsilon > 0$ by Proposition 8.9. If $\mathcal{X} = (\mathcal{X}_p)_p$ satisfies the assumptions of Proposition 8.10, then we have also

$$\int_{|w-c_1|<\epsilon} \mathcal{M}_s(w; \mathcal{X})\,|dw| > 0 \quad \text{and} \quad \int_{|u-c_2|<\epsilon} \mathcal{M}_\sigma(u; \mathcal{X})\,|du| > 0$$

for any $c_1 \in \mathbb{C}$, $c_2 \in \mathbb{R}$, and $\epsilon > 0$.

9.2. Moment-generating functions. Let $\sigma > 1/2$ and denote by $F(\sigma, X)$ the random Euler product of Example 8.2. Then the Fourier transform of the density function $\mathcal{M}_\sigma(w; F)$ equals to the characteristic function of $\log F(\sigma, X)$, that is,

$$\widetilde{\mathcal{M}}_\sigma(z; F) = \mathbb{E}[\psi_z(\log F(\sigma, X))].$$

Similarly, we obtain

$$\widetilde{\mathcal{M}}_s(z; \mathcal{X}) = \mathbb{E}[\psi_z(\log L(s, \mathcal{X}))] \quad \text{and} \quad \widetilde{\mathcal{M}}_\sigma(x; \mathcal{X}) = \mathbb{E}[\psi_x(\log L(\sigma, \mathcal{X}))],$$

where $\mathcal{X} = (\mathcal{X}_p)_p$ is a real admissible sequence of independent random matrices. More generally, let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on $M_d^1(\mathbb{C})$ which satisfies (8.2) with some $0 < \alpha \leq 1/2$. Then we define the moment-generating functions

$$\widetilde{\mathcal{M}}_s(z, z'; \mathcal{X}) = \mathbb{E}[\psi_{z,z'}(\log L(s, \mathcal{X}))], \tag{9.9}$$

where $\psi_{z,z'}$ is the quasi-character of the additive group $\mathbb{C}$ defined by

$$\psi_{z,z'}(w) = \exp\left(\frac{i}{2}(z\overline{w} + z'w)\right) \tag{9.10}$$

for $z, z' \in \mathbb{C}$ as in [**64**]. Note that we have $\psi_{z,\overline{z}}(w) = \psi_z(w)$. In addition, we have $\psi_{z,z}(w) = \exp(iz \operatorname{Re} w)$ and $\psi_{z,-z}(w) = \exp(z \operatorname{Im} w)$, which yield the equalities

$$\widetilde{\mathcal{M}}_s(z, z; \mathcal{X}) = \mathbb{E}[\exp(iz \operatorname{Re} \log L(s, \mathcal{X}))],$$
$$\widetilde{\mathcal{M}}_s(z, -z; \mathcal{X}) = \mathbb{E}[\exp(z \operatorname{Im} \log L(s, \mathcal{X}))].$$

The following lemmas are generalizations of the results in [**139**].

Lemma 9.6. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on $M_d^1(\mathbb{C})$ which satisfies condition (8.2) with some $0 < \alpha \leq 1/2$. Let $s = \sigma + it$ with $\sigma > 1 - \alpha$. Then the moment-generating functions $\widetilde{\mathcal{M}}_s(z, z; \mathcal{X})$ and $\widetilde{\mathcal{M}}_s(z, -z; \mathcal{X})$ exist for any $z \in \mathbb{C}$, which are entire functions in z.*

Proof. We show that the expected value $\mathbb{E}[\exp(a|\operatorname{Re} \log L(s, \mathcal{X})|)]$ is finite for any fixed real number $a > 0$. Since $\sum_{p \leq y} \log L_p(s, \mathcal{X})$ converges to $\log L(s, \mathcal{X})$ as $y \to \infty$ in law, we obtain

$$\mathbb{E}[\exp(b \operatorname{Re} \log L(s, \mathcal{X}))] \leq \liminf_{y \to \infty} \mathbb{E}\left[\exp\left(b \sum_{p \leq y} \operatorname{Re} \log L_p(s, \mathcal{X})\right)\right] \tag{9.11}$$

for each $b = \pm a$. From the independence of $\mathcal{X}_p$'s, we deduce

$$\mathbb{E}\left[\exp\left(b \sum_{p \leq y} \operatorname{Re} \log L_p(s, \mathcal{X})\right)\right] = \prod_{p \leq y} \mathbb{E}\left[\exp\left(b \operatorname{Re} \log L_p(s, \mathcal{X})\right)\right].$$

By the Taylor expansion, the asymptotic formula

$$\exp\left(b \operatorname{Re} \log L_p(s, \mathcal{X})\right) = 1 + b \operatorname{tr} X_p \operatorname{Re}(p^{-s}) + O(p^{-2\sigma})$$

holds for every p. Then, using (8.2), we obtain

$$\mathbb{E}\left[\exp\left(b \operatorname{Re} \log L_p(s, \mathcal{X})\right)\right] = 1 + O_\epsilon\left(p^{-(\sigma+\alpha)+\epsilon} + p^{-2\sigma}\right).$$

The series $\sum_p p^{-(\sigma+\alpha)+\epsilon}$ and $\sum_p p^{-2\sigma}$ converge for $\sigma > 1-\alpha$. Hence we conclude that $\mathbb{E}[\exp(b\operatorname{Re}\log L(s,\mathcal{X}))]$ is finite by (9.11). Then it yields

$$\begin{aligned}&\mathbb{E}[\exp(a|\operatorname{Re}\log L(s,\mathcal{X})|)]\\&\quad\leq \mathbb{E}[\exp(a\operatorname{Re}\log L(s,\mathcal{X}))]+\mathbb{E}[\exp(-a\operatorname{Re}\log L(s,\mathcal{X}))]<\infty\end{aligned}$$

as desired. This implies the existence of $\mathbb{E}[\exp(z\operatorname{Re}\log L(s,\mathcal{X}))]$ for any $z\in\mathbb{C}$. Moreover, it is an entire function in z by the standard method of analysis. The result for $\mathbb{E}[\exp(z\operatorname{Im}\log L(s,\mathcal{X}))]$ can be proved similarly. □

We have $|\psi_{z,z'}(w)|\leq\exp(R|\operatorname{Re}w|)\exp(R|\operatorname{Im}w|)$ with $R=\max\{|z|,|z'|\}$. By the Cauchy–Schwarz inequality, we then obtain

$$\begin{aligned}\left|\widetilde{\mathcal{M}}_s(z,z';\mathcal{X})\right| \leq &\left(\mathbb{E}\big[\exp(2R|\operatorname{Re}\log L(s,\mathcal{X})|)\big]\right)^{1/2}\\&\times\left(\mathbb{E}\big[\exp(2R|\operatorname{Im}\log L(s,\mathcal{X})|)\big]\right)^{1/2}.\end{aligned}$$

Then the right-hand side is finite by Lemma 9.6. As a result, the function $\widetilde{\mathcal{M}}_s(z,z;\mathcal{X})$ is defined for any $z,z'\in\mathbb{C}$.

Lemma 9.7. *Let $\mathcal{X}=(\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on $M_d^1(\mathbb{C})$ which satisfies condition (8.2) with some $0<\alpha\leq 1/2$. Let $s=\sigma+it$ with $\sigma>1-\alpha$. Then we have*

$$\widetilde{\mathcal{M}}_s(z,z';\mathcal{X})=\prod_p\widetilde{\mathcal{M}}_{s,p}(z,z';\mathcal{X}) \tag{9.12}$$

for any $z,z'\in\mathbb{C}$ satisfying $z'=\pm z$ or $\overline{z}$, where we put

$$\widetilde{\mathcal{M}}_{s,p}(z,z';\mathcal{X})=\mathbb{E}\left[\psi_{z,z'}(\log L_p(s,\mathcal{X}))\right].$$

Proof. Since $\psi_{z,\overline{z}}(w)=\psi_z(w)$, the functions $\widetilde{\mathcal{M}}_s(z,\overline{z};\mathcal{X})$ and $\widetilde{\mathcal{M}}_{s,p}(z,\overline{z};\mathcal{X})$ are equal to the characteristic functions of $\log L(s,\mathcal{X})$ and $\log L_p(s,\mathcal{X})$, respectively. Hence equality (9.12) is equivalent to (9.2) when $z'=\overline{z}$. By definition, we have

$$\begin{aligned}\widetilde{\mathcal{M}}_{s,p}(z,z;\mathcal{X})&=\mathbb{E}\left[\exp\left(iz\operatorname{Re}\log L_p(s,\mathcal{X})\right)\right],\\\widetilde{\mathcal{M}}_{s,p}(z,-z;\mathcal{X})&=\mathbb{E}\left[\exp\left(z\operatorname{Im}\log L_p(s,\mathcal{X})\right)\right]\end{aligned}$$

for every p, which are holomorphic in z. By an argument similar to the proof of Lemma 9.6, we see that the right-hand side of (9.12) is convergent uniformly on any compact subset of $\mathbb{C}$. Hence it gives a holomorphic function on $\mathbb{C}$ if $z'=\pm z$. On the other hand, Lemma 9.6 yields that the left-hand side is also holomorphic on $\mathbb{C}$. Then equalities (9.12) with $z'=\pm z$ are proved by the identity theorem together with the fact that it is valid for $z'=\overline{z}$ with $z\in\mathbb{R}\cup i\mathbb{R}$. □

Let $z,z'\in\mathbb{C}$. We define the function $E_s(z,z';\mathcal{X})$ as

$$E_s(z,z';\mathcal{X})=\mathbb{E}[|\psi_{z,z'}(\log L(s,\mathcal{X}))|]$$

similarly to (9.9). Here we notice that the equality $|\psi_{z,z'}(w)| = \psi_{\xi,\xi'}(w)$ holds with

$$(\xi, \xi') = \begin{cases} (i \operatorname{Im} z, i \operatorname{Im} z) & \text{if } z' = z, \\ (\operatorname{Re} z, -\operatorname{Re} z) & \text{if } z' = -z, \\ (0, 0) & \text{if } z' = \overline{z}. \end{cases} \tag{9.13}$$

Furthermore we obtain $|\psi_{\xi,\xi'}(w)| = \psi_{\xi,\xi'}(w)$. Then, we consider a finite truncation of the Dirichlet series for $\log L(s, \mathcal{X})$. For $y \geq 2$, we define

$$R_y(s, \mathcal{X}) = \sum_{\substack{p \\ p^k \leq y}} \sum_{k=1}^{\infty} \frac{\operatorname{tr}(\mathcal{X}_p^k)}{k} p^{-ks}$$

and denote its moment-generating function by $\widetilde{\mathcal{M}}_s^{(y)}(z, z'; \mathcal{X}) = \mathbb{E}\left[\psi_{z,z'}(R_y(s, \mathcal{X}))\right]$ with $z, z' \in \mathbb{C}$.

Lemma 9.8. *Let $\mathcal{X} = (\mathcal{X}_p)_p$ be a sequence of independent random matrices supported on $M_d^1(\mathbb{C})$ which satisfies condition (8.2) with some $0 < \alpha \leq 1/2$. Let $s = \sigma + it$ with $\sigma > 1 - \alpha$. Then we have*

$$\widetilde{\mathcal{M}}_s(z, z'; \mathcal{X}) = \widetilde{\mathcal{M}}_s^{(y)}(z, z'; \mathcal{X}) + O\left(E_s(z, z'; \mathcal{X})|z|y^{-\frac{1}{2}(\sigma-1+\alpha)}\right)$$

for $z, z' \in \mathbb{C}$ satisfying $z' = \pm z$ or $\overline{z}$ with $|z| \leq y^{\frac{1}{2}(\sigma-1+\alpha)}$. The implied constant depends only on σ and d.

Proof. By Lemma 9.7, we obtain

$$\widetilde{\mathcal{M}}_s(z, z'; \mathcal{X}) = \prod_{p \leq y} \widetilde{\mathcal{M}}_{s,p}(z, z'; \mathcal{X}) \cdot \prod_{p > y} \widetilde{\mathcal{M}}_{s,p}(z, z'; \mathcal{X}). \tag{9.14}$$

Each local factor can be represented as

$$\widetilde{\mathcal{M}}_{s,p}(z, z'; \mathcal{X}) = \mathbb{E}\left[\psi_{z,z'}\left(B_{p,y}^{\flat}(s, \mathcal{X}) + B_{p,y}^{\#}(s, \mathcal{X})\right)\right]$$

for any $p \leq y$, where

$$B_{p,y}^{\flat}(s, \mathcal{X}) = \sum_{k \leq \frac{\log y}{\log p}} \frac{\operatorname{tr}(\mathcal{X}_p^k)}{k} p^{-ks},$$

$$B_{p,y}^{\#}(s, \mathcal{X}) = \sum_{k > \frac{\log y}{\log p}} \frac{\operatorname{tr}(\mathcal{X}_p^k)}{k} p^{-ks} \ll_d \min\{y^{-\sigma}, p^{-2\sigma}\}.$$

Then, by the independence of $\mathcal{X}_p$'s, the first product of (9.14) is calculated as

$$\prod_{p \leq y} \widetilde{\mathcal{M}}_{s,p}(z, z'; \mathcal{X}) = \mathbb{E}\left[\psi_{z,z'}\left(R_y(s, \mathcal{X})\right) \psi_{z,z'}\left(\sum_{p \leq y} B_{p,y}^{\#}(s, \mathcal{X})\right)\right].$$

Since we have

$$\sum_{p \leq y} B_{p,y}^{\#}(s, \mathcal{X}) \ll_d \sum_{p \leq \sqrt{y}} y^{-\sigma} + \sum_{\sqrt{y} < p \leq y} p^{-2\sigma} \ll_\sigma y^{-(\sigma-\frac{1}{2})},$$

the asymptotic formula

$$\psi_{z,z'}\Big(\sum_{p\le y} B^{\#}_{p,y}(s,\mathcal{X})\Big) = 1 + O\left(|z|y^{-(\sigma-\frac{1}{2})}\right)$$

is valid in the range $|z| \le y^{\frac{1}{2}(\sigma-\frac{1}{2})}$. Therefore, we obtain

$$\prod_{p\le y}\widetilde{\mathcal{M}}_{s,p}(z,z';\mathcal{X}) = \widetilde{\mathcal{M}}^{(y)}_s(z,z';\mathcal{X}) + O\left(E^{(y)}_s(z,z';\mathcal{X})|z|y^{-(\sigma-\frac{1}{2})}\right), \tag{9.15}$$

where $E^{(y)}_s(z,z';\mathcal{X}) = \mathbb{E}\left[|\psi_{z,z'}(R_y(s,\mathcal{X}))|\right]$. Next, if $p > y$, the inequality

$$|z|\left|\log L_p(s,\mathcal{X})\right| < 1$$

is satisfied by the assumption $|z| \le y^{\frac{1}{2}(\sigma-1+\alpha)}$. Hence we have

$$\begin{aligned}\widetilde{\mathcal{M}}_{s,p}(z,z';\mathcal{X}) &= 1 + \frac{i}{2}(zp^{-\bar{s}} + z'p^{-s})\mathbb{E}[\operatorname{tr}\mathcal{X}_p] + O\left(|z|^2p^{-2\sigma}\right)\\ &= 1 + O_{\sigma,d}\left(|z|p^{-\frac{1}{2}(\sigma+1+\alpha)} + |z|^2p^{-2\sigma}\right)\end{aligned}$$

for $p > y$ by condition (8.2) with $\epsilon = \frac{1}{2}(\sigma - 1 + \alpha) > 0$. Then it yields

$$\begin{aligned}\prod_{p>y}\widetilde{\mathcal{M}}_{s,p}(z,z';\mathcal{X}) &= 1 + O\left(\sum_{p>y}|z|p^{-\frac{1}{2}(\sigma+1+\alpha)} + \sum_{p>y}|z|^2p^{-2\sigma}\right)\\ &= 1 + O\left(|z|y^{-\frac{1}{2}(\sigma-1+\alpha)}\right).\end{aligned} \tag{9.16}$$

The work of the estimate for $E^{(y)}_s(z,z';\mathcal{X}) = \mathcal{M}^{(y)}_s(\xi,\xi';\mathcal{X})$ is remaining, where (ξ,ξ') is the pair of (9.13). We apply (9.14), (9.15), and (9.16) with $(z,z') = (\xi,\xi')$ to deduce the bound $E^{(y)}_s(z,z';\mathcal{X}) \ll E_s(z,z';\mathcal{X})$. Thus, formula (9.15) yields

$$\prod_{p\le Y}\widetilde{\mathcal{M}}_{s,p}(z,z';\mathcal{X}) = \widetilde{\mathcal{M}}^{(y)}_s(z,z';\mathcal{X}) + O\left(E_s(z,z';\mathcal{X})|z|y^{-(\sigma-\frac{1}{2})}\right),$$

which completes the proof by (9.14) and (9.16). □

10. Comparisons with families of L-functions

10.1. Case of continuous families. Let $\{F(\sigma+it)\}_{t\in\mathbb{R}}$ be a continuous family attached to (F,σ) in the sense of Section 7. Then we expect that the value-distribution of values $F(\sigma+it)$ as t varies over $\mathbb{R}$ is related to the random Euler product $F(\sigma,X)$ of Example 8.2. For comparisons between $\{F(\sigma+it)\}_{t\in\mathbb{R}}$ and $F(\sigma,X)$, we begin by the mean values

$$\frac{1}{T}\int_0^T \Phi(\log F(\sigma+it))\,dt \quad \text{and} \quad \mathbb{E}\left[\Phi(\log F(\sigma,X))\right] \tag{10.1}$$

with $\Phi(w) = \psi_{z,z'}(w)$ defined by (9.10). Let $\mathcal{E}$ be a subset of $\mathbb{R}$. Then we define

$$\widetilde{\mathcal{M}}_{\sigma,T}(z,z';F)^{\mathcal{E}} = \frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \psi_{z,z'}(\log F(\sigma+it))\,dt \tag{10.2}$$

for any $z, z' \in \mathbb{C}$, and similarly,

$$\begin{aligned}\widetilde{\mathcal{M}}_\sigma(z, z'; F) &= \mathbb{E}\left[\psi_{z,z'}(\log F(\sigma, X))\right], \\ E_\sigma(z, z'; F) &= \mathbb{E}\left[|\psi_{z,z'}(\log F(\sigma, X))|\right].\end{aligned} \tag{10.3}$$

In this section, we prove several result under the following assumption on these moment-generating functions.

Assumption 10.1. *Let $F \in \mathcal{S}'$, and let $(\sigma_0, \mathcal{E}, R_\sigma)$ be the triple of the following data:*

- *a positive real number $1/2 \leq \sigma_0 < 1$;*
- *for any $\sigma > \sigma_0$ and $B \geq 1$, an exceptional subset $\mathcal{E} = \mathcal{E}_F(\sigma, B, T) \subset [0, T]$ such that*

$$\mathbb{P}_T(t \in \mathcal{E}_F(\sigma, B, T)) \ll \exp\left(-b \frac{\log T}{\log_2 T}\right) \tag{10.4}$$

  holds with some constant $b = b_F(\sigma, B) > 0$;
- *for any $\sigma > \sigma_0$, a function R_σ such that $R_\sigma(T) \to \infty$ as $T \to \infty$.*

Then, for any $\sigma > \sigma_0$ and $B \geq 1$, there exists a positive constant $a = a_F(\sigma, B)$ such that the asymptotic formula

$$\widetilde{\mathcal{M}}_{\sigma,T}(z, z'; F)^{\mathcal{E}} = \widetilde{\mathcal{M}}_\sigma(z, z'; F) + O\left(\frac{1 + E_\sigma(z, z'; F)}{(\log T)^B}\right) \tag{10.5}$$

holds for all $z, z' \in \mathbb{C}$ satisfying $z' = \pm z$ or $\overline{z}$ in the range $|z| \leq a R_\sigma(T)$. Here, the implied constants in (10.4) and (10.5) depend only on σ, B, and F.

In Section 11, we prove that Assumption 10.1 is satisfied with suitable data when the function F belongs to $\mathcal{S} \cap \mathcal{S}'$. The present section is devoted to prove that the density function $\mathcal{M}_\sigma(w; F)$ of Theorem 9.1 is the M-function for the continuous family $\{F(\sigma + it)\}_{t \in \mathbb{R}}$ under Assumption 10.1. To describe the precise statement, we prepare several classes of test functions. Let $S = \mathbb{R}$ or $\mathbb{C}$. We define the classes of continuous functions as

$$\begin{aligned}C^{\exp}(S) &= \{\Phi \in C(S) \mid \Phi(x) \ll e^{c|x|} \text{ for some } c > 0\}, \\ C^{\mathrm{poly}}(S) &= \{\Phi \in C(S) \mid \Phi(x) \ll |x|^c \text{ for some } c > 0\}, \\ C_b(S) &= \{\Phi \in C(S) \mid \Phi \text{ is bounded}\}.\end{aligned}$$

In addition, we define the class of indicator functions as

$$\begin{aligned}\mathcal{I}(S) = \{1_A \mid A \text{ is a continuity set of } S\} &\cup \{1_B \mid B \text{ is a compact subset of } S\} \\ &\cup \{1_C \mid S \setminus C \text{ is a compact subset of } S\}.\end{aligned}$$

Then we obtain the following mean value theorem.

Proposition 10.2. *Suppose that Assumption 10.1 is satisfied. Then the limit formula*

$$\lim_{T \to \infty} \frac{1}{T} \int_{[0,T] \setminus \mathcal{E}} \Phi\left(\log F(\sigma + it)\right) = \int_{\mathbb{C}} \Phi(w) \mathcal{M}_\sigma(w; F)\, |dw| \tag{10.6}$$

holds for all $\Phi \in C(\mathbb{C})$ *if* $\sigma > 1$, *and for all* $\Phi \in C^{\exp}(\mathbb{C})$ *if* $\sigma_0 < \sigma \leq 1$. *Moreover, the limit formula*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi\left(\log F(\sigma+it)\right) = \int_{\mathbb{C}} \Phi(w)\mathcal{M}_\sigma(w;F)\,|dw| \tag{10.7}$$

holds for all $\Phi \in C_b(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$ *if* $\sigma > \sigma_0$.

Proof. By the assumption, we know that (10.6) holds with $\Phi = \psi_z$ for any $z \in \mathbb{C}$ if $\sigma > \sigma_0$. Furthermore, the convergence is uniformly in any disk $|z| \leq R$. Hence Lemma A.1.5 is available. First, we obtain formula (10.6) for $\Phi \in C_b(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$ if $\sigma > \sigma_0$. The test function Φ is bounded in this case, and thus we derive (10.7) by recalling (10.4). Next, we consider unbounded cases. Let $\sigma > 1$ and $\Phi \in C(\mathbb{C})$. We define a continuous non-decreasing function ϕ_0 as

$$\phi_0(r) = \max_{|w|\leq r} |\Phi(w)|.$$

By Lemma A.1.5, it is sufficient to show that

$$\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \phi_0\left(|\log F(\sigma+it)|\right)^2\,dt \ll 1, \tag{10.8}$$

$$\int_{\mathbb{C}} \phi_0(|w|)\mathcal{M}_\sigma(w;F)\,|dw| < \infty. \tag{10.9}$$

Note that we have

$$|\log F(\sigma+it)| \leq \sum_p \sum_{k=1}^\infty \frac{|\operatorname{tr}(A_p^k)|}{k} p^{-k\sigma} \leq d\log\zeta(\sigma)$$

for $\sigma > 1$. Therefore $\phi_0(|\log F(\sigma+it)|) \leq \phi_0(d\log\zeta(\sigma))$ holds, which yields that condition (10.8) is satisfied. Furthermore, condition (10.9) is deduced from the fact that $\mathcal{M}_\sigma(w;F)$ is compactly supported for $\sigma > 1$. Thus we obtain the result if $\sigma > 1$. Let $\sigma_0 < \sigma \leq 1$. In this case, we take the function ϕ_0 as

$$\phi_0(r) = e^{cr}$$

with an arbitrary constant $c > 0$. Then we have

$$\phi_0(|\log F(\sigma+it)|)^2 \leq \exp(2c|\operatorname{Re}\log F(\sigma+it)|)\exp(2c|\operatorname{Im}\log F(\sigma+it)|).$$

By the Cauchy–Schwarz inequality, it is sufficient to prove

$$\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \exp(4c|\operatorname{Re}\log F(\sigma+it)|)\,dt \ll 1,$$
$$\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \exp(4c|\operatorname{Im}\log F(\sigma+it)|)\,dt \ll 1$$

for condition (10.8). Note that $e^{c|x|} \leq e^{cx} + e^{-cx}$ holds. Hence these bounds are deduced from (10.5) with $z' = \pm z$ along with the fact $|\widetilde{\mathcal{M}}_\sigma(z,z';F)| < \infty$ for any $z, z' \in \mathbb{C}$. We obtain (10.9) similarly. Therefore the proof is completed. □

Let $P_{\sigma,T}(A;F)=\mathbb{P}_T(\log F(\sigma+it)\in A)$ and $P_\sigma(A;F)=\mathbb{P}(\log F(\sigma,X)\in A)$ as in (7.2) and (8.6), respectively. Then formula (10.7) implies that the probability measure $P_{\sigma,T}(\,\cdot\,;F)$ converges weakly to $P_\sigma(\,\cdot\,;F)$ as $T\to\infty$. Moreover, we evaluate the quantity $D_\sigma(T;F)$ of (7.3) as follows.

Proposition 10.3. *Suppose that Assumption 10.1 is satisfied. Then we obtain the discrepancy bound*

$$D_\sigma(T;F)=O\left(R_\sigma(T)^{-1}+(\log T)^{-1}\right)$$

for $\sigma>\sigma_0$, where the implied constant depends on σ and F.

Proof. We define the probability measure $P_{\sigma,T}(\,\cdot\,;F)^{\mathcal{E}}$ as

$$P_{\sigma,T}(A;F)^{\mathcal{E}}=\frac{1}{\operatorname{meas}([0,T]\setminus\mathcal{E})}\int_{[0,T]\setminus\mathcal{E}}1_A(\log F(\sigma+it))\,dt.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we denote the distribution functions by

$$\begin{aligned}F(x,y)&=P_{\sigma,T}((-\infty,x]\times(-\infty,y];F)^{\mathcal{E}},\\ G(x,y)&=P_\sigma((-\infty,x]\times(-\infty,y];F)=\int_{\operatorname{Re}w\le x,\ \operatorname{Im}w\le y}\mathcal{M}_\sigma(w;F)\,|dw|.\end{aligned}$$

Then the function G is partially differentiable, and $A_1=\sup_{(x,y)\in\mathbb{R}^2}F_x(x,y)$ and $A_2=\sup_{(x,y)\in\mathbb{R}^2}F_y(x,y)$ are finite. Denote by $f(u,v)$ and $g(u,v)$ the characteristic functions of these distribution functions. For $z=u+iv$, they are represented as

$$\begin{aligned}f(u,v)&=\frac{1}{\operatorname{meas}([0,T]\setminus\mathcal{E})}\int_{[0,T]\setminus\mathcal{E}}\psi_{z,\bar{z}}(\log F(\sigma+it))\,dt\\ &=\widetilde{\mathcal{M}}_{\sigma,T}(z,\bar{z};F)^{\mathcal{E}}+O\left(\exp\left(-b\frac{\log T}{\log_2 T}\right)\right),\\ g(u,v)&=\widetilde{\mathcal{M}}_\sigma(z,\bar{z};F)\end{aligned}$$

by using (10.4). Furthermore, we derive from (10.5) the estimate

$$\begin{aligned}f(u,v)-g(u,v)&\ll\frac{1+E_\sigma(z,\bar{z};F)}{(\log T)^B}+\exp\left(-b\frac{\log T}{\log_2 T}\right)\\ &\ll(\log T)^{-B}\end{aligned}\tag{10.10}$$

if $z=u+iv$ satisfies the condition $|z|\le aR_\sigma(T)$. With the above preparations, we apply the two-dimensional Esseen inequality (Theorem A.3.2). It yields

$$\begin{aligned}\sup_{(x,y)\in\mathbb{R}^2}|F(x,y)-G(x,y)|&\ll\iint_{[-R,R]^2}\left|\frac{\hat{f}(u,v)-\hat{g}(u,v)}{uv}\right|\,dudv\\ &\quad+\int_{-R}^{R}\left|\frac{f(u,0)-g(u,0)}{u}\right|\,du+\int_{-R}^{R}\left|\frac{f(0,v)-g(0,v)}{v}\right|\,dv\\ &\quad+\frac{A_1+A_2}{R}\end{aligned}\tag{10.11}$$

for any $R > 0$, where we put

$$\hat{f}(u,v) = f(u,v) - f(u,0)f(0,v) \quad \text{and} \quad \hat{g}(u,v) = g(u,v) - g(u,0)g(0,v).$$

Put $r = (\log T)^{-2}$. We divide the first integral of the right-hand side of (10.11) as

$$\iint_{[-R,R]^2} \left| \frac{\hat{f}(u,v) - \hat{g}(u,v)}{uv} \right| dudv = \iint_{[-R,R]^2 \setminus C(r)} + \iint_{C(r)} = I_1 + I_2,$$

say, where we define

$$C(r) = \left\{ (u,v) \in [-R,R]^2 \;\middle|\; |u| \le r \text{ or } |v| \le r \right\}.$$

Then we have

$$I_1 \ll \left(\log \frac{R}{r} \right)^2 \sup_{(u,v) \in [-R,R]^2} \left| \hat{f}(u,v) - \hat{g}(u,v) \right|, \tag{10.12}$$

and moreover, the inequality

$$\left| \hat{f}(u,v) - \hat{g}(u,v) \right| \le |f(u,v) - g(u,v)| + |f(u,0) - g(u,0)| + |f(0,v) - g(0,v)|$$

holds. Here, we take $R = \frac{1}{\sqrt{2}} a R_\sigma(T)$ with the notation of Assumption 10.1. Remark that the condition $|z| \le a R_\sigma(T)$ holds for all $z = u + iv$ such that $(u,v) \in [-R,R]^2$. Then we deduce from (10.10) the upper bound

$$f(u,v) - g(u,v) \ll (\log T)^{-2} \tag{10.13}$$

for $(u,v) \in [-R,R]^2$, where the implied constant depends on σ and F. Hence (10.12) yields the estimate

$$I_1 \ll (\log T)^{-2} (\log_2 T). \tag{10.14}$$

Next, by the definition of $\hat{f}(u,v)$, we have

$$\begin{aligned} \hat{f}(u,v) &= (f(u,v) - f(u,0) - f(0,v) + 1) - (f(u,0) - 1)(f(0,v) - 1) \\ &= \iint_{\mathbb{R}^2} (e^{ixu} - 1)(e^{iyv} - 1) \, dF(x,y) \\ &\qquad - \iint_{\mathbb{R}^2} (e^{ixu} - 1) \, dF(x,y) \cdot \iint_{\mathbb{R}^2} (e^{iyv} - 1) \, dF(x,y). \end{aligned}$$

Since $e^{i\theta} - 1 \ll |\theta|$ for all $\theta \in \mathbb{R}$, we obtain

$$\begin{aligned} \hat{f}(u,v) &\ll |uv| \iint_{\mathbb{R}^2} |xy| \, dF(x,y) + |u| \iint_{\mathbb{R}^2} |x| \, dF(x,y) \cdot |v| \iint_{\mathbb{R}^2} |y| \, dF(x,y) \\ &\ll |uv| \iint_{\mathbb{R}^2} (x^2 + y^2) \, dF(x,y). \end{aligned}$$

The last integral is calculated as

$$\iint_{\mathbb{R}^2} (x^2 + y^2) \, dF(x,y) = \frac{1}{\operatorname{meas}([0,T] \setminus \mathcal{E})} \int_{[0,T] \setminus \mathcal{E}} |\log F(\sigma + it)|^2 .$$

Recall that $\operatorname{meas}([0,T]\setminus\mathcal{E})\sim T$ as $T\to\infty$ by (10.4). In addition, we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}}|\log F(\sigma+it)|^2=\int_{\mathbb{C}}|w|^2\mathcal{M}_\sigma(w;F)\,|dw|$$

by formula (10.6). Thus the estimate $\hat{f}(u,v)\ll|uv|$ follows. By a similar argument, we evaluate $\hat{g}(u,v)$ as

$$\hat{g}(u,v)\ll|uv|\iint_{\mathbb{R}^2}(x^2+y^2)\,dG(x,y)\ll|uv|,$$

where the implied constant depends on σ and F. Therefore, the integral I_2 is estimated as

$$I_2\ll\operatorname{meas}(C(r))\ll(\log T)^{-1}. \tag{10.15}$$

By (10.14) and (10.15), we obtain

$$\iint_{[-R,R]^2}\left|\frac{\hat{f}(u,v)-\hat{g}(u,v)}{uv}\right|\,dudv\ll(\log T)^{-1}.$$

Then, we proceed to the second integral of the right-hand side of (10.11). Here we divide it as

$$\int_{-R}^{R}\left|\frac{f(u,0)-g(u,0)}{u}\right|\,du=\int_{[-R,-r)\cup(r,R]}+\int_{[-r,r]}=I_3+I_4,$$

say. By upper bound (10.13), the integral I_3 is estimated as

$$I_3\ll\left(\log\frac{R}{r}\right)\sup_{u\in[-R,R]}|f(u,0)-g(u,0)|\ll(\log T)^{-2}\log_2 T.$$

Furthermore, we have

$$\begin{aligned}f(u,0)-g(u,0)&=\iint_{\mathbb{R}^2}(e^{ixu}-1)\,dF(x,y)-\iint_{\mathbb{R}^2}(e^{iyv}-1)\,dG(x,y)\\&\ll|u|\left(\iint_{\mathbb{R}^2}x^2\,dF(x,y)\right)^{1/2}+|u|\left(\iint_{\mathbb{R}^2}x^2\,dG(x,y)\right)^{1/2}\\&\ll_{\sigma,F}|u|,\end{aligned}$$

and therefore, we deduce $I_4\ll r=(\log T)^{-2}$. The third integral of the right-hand side of (10.11) is estimated along the same line. From the above, we finally arrive at

$$\sup_{(x,y)\in\mathbb{R}^2}|F(x,y)-G(x,y)|\ll(\log T)^{-1}+R_\sigma(T)^{-1}.$$

Note that the inequality

$$\begin{aligned}\left|P_{\sigma,T}(\mathcal{R};F)^{\mathcal{E}}-P_\sigma(\mathcal{R};F)\right|\le|F(b,d)-G(b,d)|-|F(a,d)-G(a,d)|\\-|F(b,c)-G(b,c)|+|F(a,c)-G(a,c)|\end{aligned}$$

holds if we write $\mathcal{R}=[a,b]\times[c,d]$. Hence

$$\sup_{\mathcal{R}}\left|P_{\sigma,T}(\mathcal{R};F)^{\mathcal{E}}-P_\sigma(\mathcal{R};F)\right|\ll(\log T)^{-1}+R_\sigma(q)^{-1} \tag{10.16}$$

follows. In addition, we see that

$$\sup_{\mathcal{R}} \left| P_{\sigma,T}(\mathcal{R}; F) - P_{\sigma,T}(\mathcal{R}; F)^{\mathcal{E}} \right| \leq \frac{\mathbb{P}_T(t \in \mathcal{E})}{1 - \mathbb{P}_T(t \in \mathcal{E})} + \mathbb{P}_T(t \in \mathcal{E}) \ll \exp\left(-b \frac{\log T}{\log_2 T} \right). \tag{10.17}$$

By (10.16) and (10.17), we obtain the desired discrepancy bound. □

10.2. Case of discrete families. Let $\{L(s, f)\}_{f \in \mathscr{F}}$ be a discrete family attached to $(\mathcal{L}, s)$ in the sense of Section 7. In this case, we expect that the value-distribution of values $L(s, f)$ as f varies over $\mathscr{F}$ is related to some suitable random Euler product $L(s, \mathcal{X})$, where $\mathcal{X} = (\mathcal{X}_p)_p$ is a sequence of independent random matrices satisfying the assumptions of Theorem 9.3. Similarly to (10.1), we consider the mean values

$$\sideset{}{'}\sum_{f \in \mathscr{F}_q} \omega_q(f) \Phi(\log L(s, f)) \quad \text{and} \quad \mathbb{E}\left[\Phi(\log L(s, \mathcal{X})) \right]$$

with $\Phi(w) = \psi_{z,z'}(w)$. For a subset of $\mathcal{E} \subset \mathscr{F}$, we define

$$\widetilde{\mathcal{M}}_{s,q}(z, z'; \mathcal{L})^{\mathcal{E}} = \sideset{}{'}\sum_{f \in \mathscr{F}_q \setminus \mathcal{E}} \omega_q(f) \psi_{z,z'}(\log L(s, f)) \tag{10.18}$$

for $z, z' \in \mathbb{C}$. Furthermore, we take $\widetilde{\mathcal{M}}_s(z, z'; \mathcal{X})$ and $E_s(z, z'; \mathcal{X})$ as in Section 9.2. Then the assumption in this case is as follows.

Assumption 10.4. *Let $\{L(s, f)\}_{f \in \mathscr{F}}$ be a real discrete family attached to $(\mathcal{L}, s)$, and let $\mathcal{X} = (\mathcal{X}_p)_p$ be a real admissible sequence of independent random matrices. Let $(\sigma_0, \mathcal{E}, R_s)$ be the triple of the following data:*

- *a positive real number $1/2 \leq \sigma_0 < 1$;*
- *for any $s = \sigma + it$ with $\sigma > \sigma_0$ and $B \geq 1$, an exceptional subset $\mathcal{E} = \mathcal{E}_{\mathcal{L}}(s, B, q) \subset \mathscr{F}_q$ such that*

$$\frac{\# \mathcal{E}_{\mathcal{L}}(s, B, q)}{\# \mathscr{F}_q} \ll \exp\left(-b \frac{\log q}{\log_2 q} \right) \tag{10.19}$$

  holds with some constant $b = b_{\mathcal{L}}(s, B) > 0$;
- *for any $s = \sigma + it$ with $\sigma > \sigma_0$, a function R_s such that $R_s(q) \to \infty$ as $q \to \infty$.*

Then, for any $s = \sigma + it$ with $\sigma > \sigma_0$ and $B \geq 1$, there exists a positive constant $a = a_{\mathcal{L}}(s, B)$ such that the asymptotic formula

$$\widetilde{\mathcal{M}}_{s,q}(z, z'; \mathcal{L})^{\mathcal{E}} = \widetilde{\mathcal{M}}_s(z, z'; \mathcal{X}) + O\left(\frac{1 + E_s(z, z'; \mathcal{X})}{(\log q)^B} \right) \tag{10.20}$$

holds for all $z, z' \in \mathbb{C}$ satisfying $z' = \pm z$ or $\overline{z}$ in the range $|z| \leq a R_s(q)$. Here, the implied constants in (10.19) and (10.20) depend only on s, B, and choices of $\mathcal{L}$ and $\mathcal{X} = (\mathcal{X}_p)_p$.

By the arguments similar to the proofs in Section 10.1, one can derive the following results. Recall that the density function $\mathcal{M}_s(\,\cdot\,;\mathcal{X})$ of Theorem 9.3 depends on whether s is real or not. It causes a significant difference between the results of Sections 10.1 and 10.2.

PROPOSITION 10.5. *Let $\{L(s,f)\}_{f\in\mathscr{F}}$ be a real discrete family attached to $(\mathcal{L},s)$, and let $\mathcal{X}=(\mathcal{X}_p)_p$ be a real admissible sequence of independent random matrices.*

(i) *Suppose that Assumption 10.4 is satisfied in the case $s=\sigma+it$ with $\sigma>\sigma_0$ and $t\neq 0$. Then the limit formula*

$$\lim_{\substack{q\to\infty\\ q\in\Lambda}}\sideset{}{'}\sum_{f\in\mathscr{F}_q\setminus\mathcal{E}}\omega_q(f)\Phi(\log L(s,f))=\int_{\mathbb{C}}\Phi(w)\mathcal{M}_s(w;\mathcal{X})\,|dw| \tag{10.21}$$

holds for all $\Phi\in C(\mathbb{C})$ if $\sigma>1$, and for all $\Phi\in C^{\exp}(\mathbb{C})$ if $\sigma_0<\sigma\leq 1$. Moreover, the limit formula

$$\lim_{\substack{q\to\infty\\ q\in\Lambda}}\sideset{}{'}\sum_{f\in\mathscr{F}_q}\omega_q(f)\Phi(\log L(s,f))=\int_{\mathbb{C}}\Phi(w)\mathcal{M}_s(w;\mathcal{X})\,|dw|$$

holds for all $\Phi\in C_b(\mathbb{C})\cup\mathcal{I}(\mathbb{C})$ if $\sigma>\sigma_0$.

(ii) *Suppose that Assumption 10.4 is satisfied in the case $s=\sigma$ with $\sigma>\sigma_0$. Then the limit formula*

$$\lim_{\substack{q\to\infty\\ q\in\Lambda}}\sideset{}{'}\sum_{f\in\mathscr{F}_q\setminus\mathcal{E}}\omega_q(f)\Phi(\log L(\sigma,f))=\int_{\mathbb{R}}\Phi(u)\mathcal{M}_\sigma(u;\mathcal{X})\,|du| \tag{10.22}$$

holds for all $\Phi\in C(\mathbb{R})$ if $\sigma>1$, and for all $\Phi\in C^{\exp}(\mathbb{R})$ if $\sigma_0<\sigma\leq 1$. Moreover, the limit formula

$$\lim_{\substack{q\to\infty\\ q\in\Lambda}}\sideset{}{'}\sum_{f\in\mathscr{F}_q}\omega_q(f)\Phi(\log L(\sigma,f))=\int_{\mathbb{R}}\Phi(u)\mathcal{M}_\sigma(u;\mathcal{X})\,|du|$$

holds for all $\Phi\in C_b(\mathbb{R})\cup\mathcal{I}(\mathbb{R})$ if $\sigma>\sigma_0$.

PROOF. The proof is similar to Proposition 10.2, and we omit the detail. Notice that (10.21) holds with $\Phi=\psi_z$ for any $z\in\mathbb{C}$ if $\sigma>\sigma_0$. Similarly, (10.22) holds with $\Phi=\psi_x$ for any $x\in\mathbb{R}$. Therefore Lemma A.1.4 is available instead of Lemma A.1.5. We take the function ϕ_0 as $\phi_0(r)=\max_{|w|\leq r}|\Phi(w)|$ if $s=\sigma+it$ with $\sigma>1$ and $t\neq 0$ and $\phi_0(r)=\max_{|u|\leq r}|\Phi(u)|$ if $s=\sigma>1$, where Φ is an arbitrary continuous function on $\mathbb{C}$ or $\mathbb{R}$. Furthermore, take $\phi_0(r)=e^{cr}$ if $\sigma_0<\operatorname{Re}s\leq 1$. In any case, the conditions

$$\sideset{}{'}\sum_{f\in\mathscr{F}_q}\omega_q(f)\phi_0\left(|\log L(s,f)|\right)^2\ll 1,$$

$$\int_{\mathbb{C}}\phi_0(|w|)\mathcal{M}_s(w;\mathcal{X})\,|dw|<\infty,\qquad \int_{\mathbb{R}}\phi_0(|u|)\mathcal{M}_\sigma(u;\mathcal{X})\,|du|<\infty$$

are checked by arguments similar to the case of Proposition 10.2. Hence we obtain the conclusion. ◻

Similarly to the case of continuous families, this result implies that the probability measure $P_{s,q}(\,\cdot\,;\mathcal{L})$ of (7.6) converges weakly to $P_s(\,\cdot\,;\mathcal{L})$ of (8.6) as $q \to \infty$. Denote by $D_s(q;\mathcal{L})$ and $D_\sigma(q;\mathcal{L})$ the quantities defined as (7.7). Then, we prove the following result.

Proposition 10.6. *Let $\{L(s,f)\}_{f\in\mathscr{F}}$ be a real discrete family attached to $(\mathcal{L},s)$, and let $\mathcal{X}=(\mathcal{X}_p)_p$ be a real admissible sequence of independent random matrices.*

(i) *Suppose that Assumption 10.4 is satisfied in the case $s=\sigma+it$ with $\sigma>\sigma_0$ and $t\neq 0$. Then we obtain*

$$D_s(q;\mathcal{L}) = O\left(R_s(q)^{-1} + (\log q)^{-1}\right),$$

where the implied constant depends on s and the choices of $\mathcal{L}$ and $\mathcal{X}$.

(ii) *Suppose that Assumption 10.4 is satisfied in the case $s=\sigma$ with $\sigma>\sigma_0$. Then we obtain*

$$D_\sigma(q;\mathcal{L}) = O\left(R_\sigma(q)^{-1} + (\log q)^{-1}\right),$$

where the implied constant depends on σ and the choices of $\mathcal{L}$ and $\mathcal{X}$.

Proof. The proof is based on Theorem A.3.2 for assertion (i), and on Theorem A.3.1 for (ii). We omit the details since they are obtained along the same lines of the proof of Proposition 10.3. □

CHAPTER 4

The value-distributions of L-functions

In this chapter, we complete the proofs of Theorems I, III, and IV presented in Chapter 1. By the preparations in Chapters 1 and 2, we derive them as some corollaries of more general results.

11. The first result: general L-functions

Let $F \in \mathcal{S}'$ and $\sigma > 1/2$, and denote by $\mathcal{M}_\sigma(w;F)$ the density function of Theorem 9.1. The first main target of this section is the following limit theorem.

Theorem 11.1. *Let $F \in \mathcal{S} \cap \mathcal{S}'$. Suppose that there exist positive constants δ, σ_0 with $1/2 \leq \sigma_0 < 1$ such that the zero density estimate $N_F(\sigma,T) \ll T^{1-\delta}$ holds for $\sigma > \sigma_0$. Then the limit formula*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi\left(\log F(\sigma+it)\right) = \int_{\mathbb{C}} \Phi(w) \mathcal{M}_\sigma(w;F)\,|dw|$$

holds in the following cases:

(a) $\sigma > 1$ *and* $\Phi \in C(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$*;*
(b) $\sigma_0 < \sigma \leq 1$ *and* $\Phi \in C_b(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$.

In this result, one can take at least $\sigma_0 = 1 - (4d_F + 12)^{-1}$ by Lemma 5.2. Furthermore, when estimate (5.6) holds, we extend the result for all $\sigma > 1/2$. The indicator function $1_{\mathcal{R}}$ particularly belongs to $\mathcal{I}(\mathbb{C})$ if $\mathcal{R}$ is any rectangle. Hence Theorem 11.1 generalizes the Bohr–Jessen limit theorem significantly. Indeed, it contains Theorems 1.4, 3.1, and 3.4. Secondary, we prove the discrepancy bounds for the functions in $\mathcal{S} \cap \mathcal{S}'$.

Theorem 11.2. *With the same assumptions as in Theorem 11.1, we obtain*

$$D_\sigma(T;F) \ll \begin{cases} (\log T)^{-1}(\log_2 T) & \text{for } \sigma > 1, \\ (\log T)^{-1}(\log_2 T \log_3 T) & \text{for } \sigma = 1, \\ (\log T)^{-\sigma} & \text{for } \sigma_0 < \sigma < 1, \end{cases}$$

where the implied constants depend on σ and F.

Since the zero density estimate $N_\zeta(\sigma,T) \ll T^{1-\delta}$ holds for all $\sigma > 1/2$, one can take $\sigma_0 = 1/2$ for the Riemann zeta-function. Thus, Theorem I is contained in this result. Let $\widetilde{\mathcal{M}}_{\sigma,T}(z,z';F)^{\mathcal{E}}$ and $\widetilde{\mathcal{M}}_\sigma(z,z';F)$ be the moment-generating functions of (10.2) and (10.3), respectively. By Propositions 10.2 and 10.3, the main step in the proofs of Theorems 11.1 and 11.2 is to check that Assumption 10.1 is satisfied with $F \in \mathcal{S} \cap \mathcal{S}'$.

PROPOSITION 11.3. *Let $F \in \mathcal{S} \cap \mathcal{S}'$. Suppose that there exist positive constants δ, σ_0 with $1/2 \leq \sigma_0 < 1$ such that $N_F(\sigma, T) \ll T^{1-\delta}$ holds for $\sigma > \sigma_0$. Then Assumption 10.1 is satisfied with the following triple $(\sigma_0, \mathcal{E}, R_\sigma)$:*

- *the above σ_0;*
- *the exceptional subset $\mathcal{E} = \mathcal{E}_F(\sigma, B, T)$ presented by (11.16) below;*
- *the function R_σ defined as*

$$R_\sigma(T) = \begin{cases} (\log T)(\log_2 T)^{-1} & \text{if } \sigma > 1, \\ (\log T)(\log_2 T \log_3 T)^{-1} & \text{if } \sigma = 1, \\ (\log T)^\sigma & \text{if } \sigma_0 < \sigma < 1. \end{cases} \tag{11.1}$$

11.1. Exceptional subsets. Before the proof of Proposition 11.3, we collect several lemmas on $F \in \mathcal{S}^{\mathrm{poly}}$. The logarithmic derivative of F is represented as

$$\frac{F'}{F}(s) = -\sum_{n=1}^{\infty} \Lambda_F(n) n^{-s} \tag{11.2}$$

for $\operatorname{Re} s > 1$, where $\Lambda_F(n)$ is the arithmetic function defined as

$$\Lambda_F(p^k) = (\alpha_{p,1}^k + \cdots + \alpha_{p,d}^k) \log p$$

if $n \neq 1$ is a prime power, and $\Lambda_F(n) = 0$ otherwise. Here, $\alpha_{p,1}, \ldots, \alpha_{p,d}$ are the complex numbers in polynomial Euler product (5.8). Remark that $\Lambda_\zeta(n) = \Lambda(n)$, and $|\Lambda_F(n)| \leq d\Lambda(n)$. Then, formula (11.2) yields

$$\log F(s) = \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{\log n} n^{-s} \tag{11.3}$$

for $\operatorname{Re} s > 1$. We consider its finite truncations as follows. Put

$$w_y(n) = \begin{cases} 1 & \text{if } 1 \leq n \leq y, \\ \dfrac{\log(y^2/n)}{\log y} & \text{if } y \leq n \leq y^2, \\ 0 & \text{if } n > y^2 \end{cases}$$

for $y > 1$. We define

$$R_y(s; F) = \sum_{n \leq y} \frac{\Lambda_F(n)}{\log n} n^{-s} \quad \text{and} \quad R_y^*(s; F) = \sum_{n \leq y^2} \frac{\Lambda_F(n)}{\log n} w_y(n) n^{-s} \tag{11.4}$$

for any $s \in \mathbb{C}$. The first lemma is an approximation formula of $\log F(s)$ by the Dirichlet polynomial $R_y^*(s; F)$. Note that Guo [**52**, Lemma 2.1.3] obtained a similar result in the case of $F = \zeta$.

LEMMA 11.4. *Let $F \in \mathcal{S}^{\mathrm{poly}}$, $1/2 \leq \sigma_0 < 1$, and $1 < y < h$. We define the subset $\mathscr{B} = \mathscr{B}_F(T, \sigma; y, h) \subset [2, T]$ as*

$$\mathscr{B}_F(T, \sigma; y, h) = \left\{ t \in [2, T] \;\middle|\; \begin{array}{l} \text{there exists a zero } \rho \text{ of } F \text{ such that} \\ \operatorname{Re} \rho \geq \frac{1}{2}(\sigma + \sigma_0), \ |\operatorname{Im} \rho - t| < h \end{array} \right\} \tag{11.5}$$

for $T > 2$ and $\sigma > \sigma_0$. Then we have

$$\log F(s) - R_y^*(s;F) \ll \frac{y}{|t|^2 \log y} + \frac{y \log h}{h \log y} \log T + \frac{y^{-\frac{1}{2}(\sigma-\sigma_0)}}{\log y} (\log T)^2 \tag{11.6}$$

for any $s = \sigma + it$ with $\sigma > \sigma_0$ and $t \in [2, T] \setminus \mathscr{B}$, where the implied constant depends on F and σ.

Proof. If $\sigma \geq 2$, then the set $\mathscr{B}$ is empty, and we obtain

$$\log F(s) - R_y^*(s;F) \ll_d \sum_{n>y} \frac{\Lambda(n)}{\log n} n^{-\sigma} \ll \frac{y^{1-\sigma}}{\log y}$$

since (11.3) is convergent. Therefore (11.6) holds in this case. Thus, we assume $\sigma < 2$ below. The following proof is based on a generalization of the formula of Selberg [**165**, Lemma 2]. First, we recall

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} v^z \frac{dz}{z^2} = \begin{cases} \log v & \text{if } v \geq 1, \\ 0 & \text{if } 0 < v \leq 1 \end{cases}$$

for any $c > 0$. Let ξ be any complex number on the vertical segment $\sigma \leq \operatorname{Re} \xi \leq 2$, $\operatorname{Im} \xi = t$. Then we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F'}{F}(z) \frac{y^{z-\xi} - y^{2(z-\xi)}}{(z-\xi)^2} dz \\ &= -\sum_{n=1}^{\infty} \Lambda_F(n) n^{-\xi} \frac{1}{2\pi i} \int_{(c-\operatorname{Re}\xi)-i\infty}^{(c-\operatorname{Re}\xi)+i\infty} \left\{ \left(\frac{y}{n}\right)^z - \left(\frac{y^2}{n}\right)^z \right\} \frac{dz}{z^2} \\ &= (\log y) \sum_{n \leq y^2} \Lambda_F(n) w_y(n) n^{-\xi}, \end{aligned}$$

where we take $c = 1 + \operatorname{Re} \xi$. Let $T_m \in (m, m+1]$ and $0 < \kappa < 1$. Denote by C_m the contour given by connecting the points $c - i\infty$, $c - iT_m$, $-\kappa - iT_m$, $-\kappa + iT_m$, $c + iT_m$, and $c + i\infty$, in order. By (5.3), one can choose real numbers T_m such that any point $z = u + iv$ on C_m has distance $\gg \log(|v| + 2)^{-1}$ from zeros and poles of F. Then we have

$$\frac{F'}{F}(z) \ll \log^2(|v| + 2) \tag{11.7}$$

for $z \in C_m$, which is deduced from [**78**, (2.1) in p.258] for $-1 \leq u \leq 2$, and is elementary for $u > 2$ by (11.2). Note that the implied constant in (11.7) depends only on F. While making a change of contours from $\operatorname{Re} z = c$ to C_m we come across poles of the integrand only when z is a zero or pole of F, or $z = \xi$. The integrals on the horizontal edges in C_m tend to 0 as $m \to \infty$ by (11.7). Similarly, the vertical integral from $-\kappa - iT_m$ to $-\kappa + iT_m$ is $O((\log y)^{-1} y^{-\sigma} (\log T)^2)$ as $m \to \infty$, where the implied constant depends on F and σ. Therefore we derive

$$\frac{F'}{F}(w) = -\sum_{n \leq y^2} \Lambda_F(n) w_y(n) n^{-\xi} - \frac{m_1}{\log y} \frac{y^{1-\xi} - y^{2(1-\xi)}}{(1-\xi)^2} + \frac{m_0}{\log y} \frac{y^{-\xi} - y^{-2\xi}}{\xi^2} +$$

$$+\frac{1}{\log y}\sum_{\rho}\frac{y^{\rho-\xi}-y^{2(\rho-\xi)}}{(\rho-\xi)^2}+O\left(\frac{y^{-\sigma}}{\log y}(\log T)^2\right),$$

where $m_1, m_0 \geq 0$ denote orders of the possible pole of F at $s=1$ and the possible zero at $s=0$, respectively, and $\sum_\rho$ indicates the sum over all zeros of F with $0<\operatorname{Re} s<1$. The second and third terms are estimated as

$$-\frac{m_1}{\log y}\frac{y^{1-\xi}-y^{2(1-\xi)}}{(1-\xi)^2}+\frac{m_0}{\log y}\frac{y^{-\xi}-y^{-2\xi}}{\xi^2} \ll \frac{y}{|t|^2\log y} \tag{11.8}$$

with implied constant depending on F and σ. As for the fourth term, we split the sum into the parts over ρ with $|\operatorname{Im}\rho - t| > h$ and the others. Remark that the assumption $t\in[2,T]\setminus\mathscr{B}$ implies $t\in[2,T]\setminus\mathscr{B}_F(T,\operatorname{Re}\xi;y,h)$ since $\operatorname{Re}\xi\geq\sigma$. Hence each part of the sum can be estimated in a similar way of Guo [52]. As a result, we obtain

$$\begin{aligned}\frac{1}{\log y}\sum_{\rho}\frac{y^{\rho-\xi}-y^{2(\rho-\xi)}}{(\rho-\xi)^2} &\ll \frac{y\log h}{h\log y}\log T+\frac{y^{-\frac{1}{2}(\operatorname{Re}\xi-\sigma_0)}}{(\operatorname{Re}\xi-\sigma_0)^2\log y}\log T\\ &\ll \frac{y\log h}{h\log y}\log T+\frac{y^{-\frac{1}{2}(\sigma-\sigma_0)}}{(\sigma-\sigma_0)^2\log y}\log T,\end{aligned} \tag{11.9}$$

where the implied constants depend on F and σ. By (11.8) and (11.9), we arrive at the estimate

$$\frac{F'}{F}(\xi)+\sum_{n\leq y^2}\Lambda_F(n)w_y(n)n^{-\xi} \ll \frac{y}{|t|^2\log y}+\frac{y\log h}{h\log y}\log T+\frac{y^{-\frac{1}{2}(\sigma-\sigma_0)}}{\log y}(\log T)^2$$

uniformly for ξ with $\sigma\leq\operatorname{Re}\xi\leq 2$ and $\operatorname{Im}\xi = t\in[2,T]\setminus\mathscr{B}$. Therefore we obtain the result by taking the integration along the horizontal path from $2+it$ to $\sigma+it$. □

LEMMA 11.5. *Let $F\in\mathcal{S}^{\mathrm{poly}}$. Suppose that there exist positive constants δ,σ_0 with $1/2\leq\sigma_0<1$ such that $N_F(\sigma,T)\ll T^{1-\delta}$ holds for $\sigma>\sigma_0$. Put $y=(\log T)^B$ with $B\geq 1$. Define the subset $\mathcal{A}_F(\sigma,T)\subset[0,T]$ as*

$$\begin{aligned}&\mathcal{A}_F(\sigma,B,T)\\ &=\left\{t\in[0,T]\ \middle|\ |\log F(\sigma+it)-R_y(\sigma+it;F)|>y^{-\frac{1}{2}(\sigma-\sigma_0)}(\log T)^2\right\}.\end{aligned}$$

Then there exists a positive constant $b=b_F(\sigma,B)$ such that

$$\mathbb{P}_T(t\in\mathcal{A}_F(\sigma,B,T))\ll\exp\left(-b\frac{\log T}{\log_2 T}\right)$$

for $\sigma>\sigma_0$, where the implied constant depends on F, σ, and B.

PROOF. We begin by the estimate of the size of the set

$$\begin{aligned}&\mathcal{A}_F^{(1)}(\sigma,B,T)\\ &=\left\{t\in[0,T]\ \middle|\ |\log F(\sigma+it)-R_y^*(\sigma+it;F)|>\frac{1}{2}y^{-\frac{1}{2}(\sigma-\sigma_0)}(\log T)^2\right\}.\end{aligned}$$

Put $h = y^{1+\frac{1}{2}(\sigma-\sigma_0)}$. By Lemma 11.4, we derive

$$\log F(s) - R_y^*(s;F) \ll \frac{y}{|t|^2 \log y} + \frac{y^{-\frac{1}{2}(\sigma-\sigma_0)}}{\log y}(\log T)^2$$

if $t \in [2,T] \setminus \mathscr{B}$, where $\mathscr{B} = \mathscr{B}_F(T,\sigma;y,h)$ is defined as (11.5). Assuming further $t \geq T^{1/2}$, then we evaluate the first term as

$$\frac{y}{|t|^2 \log y} \ll \frac{y}{T \log y} \ll \frac{y^{-\frac{1}{2}(\sigma-\sigma_0)}}{\log y}(\log T)^2$$

by $y = (\log T)^B$. Thus, the condition $t \in [T^{\frac{1}{2}}, T] \setminus \mathscr{B}$ implies $t \notin \mathcal{A}_F^{(1)}(\sigma, B, T)$ if T is large enough. Hence we obtain

$$\mathbb{P}_T(t \in \mathcal{A}_F^{(1)}(\sigma,B,T)) \leq \mathbb{P}_T(0 \leq t \leq T^{\frac{1}{2}}) + \mathbb{P}_T(t \in \mathscr{B}).$$

We have $\mathbb{P}_T(0 \leq t \leq T^{\frac{1}{2}}) = T^{-\frac{1}{2}}$. Furthermore, by the zero density estimate, the measure of $\mathscr{B}$ is evaluated as

$$\operatorname{meas} \mathscr{B} \leq 2h \cdot N\left(\frac{\sigma+\sigma_0}{2}, T\right) \ll T^{1-\frac{\delta}{2}}.$$

Hence we have $\mathbb{P}_T(t \in \mathscr{B}) = T^{-\frac{\delta}{2}}$. These estimates yield

$$\mathbb{P}_T(t \in \mathcal{A}_F^{(1)}(\sigma,B,T)) \ll \exp\left(-\frac{\log T}{\log_2 T}\right). \tag{11.10}$$

Next, we define another set $\mathcal{A}_F^{(2)}(\sigma,B,T)$ as

$$\begin{aligned}&\mathcal{A}_F^{(2)}(\sigma,B,T)\\ &= \left\{ t \in [0,T] \;\middle|\; |R_y^*(\sigma+it;F) - R_y(\sigma+it;F)| > \frac{1}{2}y^{-\frac{1}{2}(\sigma-\sigma_0)}(\log T)^2 \right\}.\end{aligned}$$

The difference between $R_y^*(s;F)$ and $R_y(s;F)$ is calculated as

$$R_y^*(\sigma+it;F) - R_y(\sigma+it;F) = \sum_{y<p\leq y^2} \frac{a_F(p)w_y(p)}{p^s} + O(dy^{1-2\sigma}) \tag{11.11}$$

for $\sigma > 1/2$ by (11.4) along with the inequality $|a_F(p)w_y(p)| \leq d$. The implied constant in (11.11) depends only on σ. Put $k = \lfloor (\log T)(4B\log_2 T)^{-1} \rfloor$. Then we consider the $2k$-th moment

$$M = \frac{1}{T}\int_0^T \left| \sum_{y<p\leq y^2} \frac{a_F(p)w_y(p)}{p^{\sigma+it}} \right|^{2k} dt.$$

We recall the inequality of Soundararajan [**171**, Lemma 3] which asserts

$$\frac{1}{T}\int_T^{2T} \left| \sum_{p\leq x} \frac{a(p)}{p^{1/2+it}} \right|^{2k} dt \ll k! \left(\sum_{p\leq x} \frac{|a(p)|^2}{p} \right)^k$$

for any $a(p) \in \mathbb{C}$ and $2 \leq x \leq T$ with $x^k \leq T(\log T)^{-1}$. Modifying slightly the proof in [**171**], one can prove a similar inequality

$$\frac{1}{T}\int_0^T \left| \sum_{x_1<p\leq x_2} \frac{a(p)}{p^{\sigma+it}} \right|^{2k} dt \ll k!\Bigg(\sum_{x_1<p\leq x_2} \frac{|a(p)|^2}{p^{2\sigma}} \Bigg)^k, \tag{11.12}$$

where $2 \leq x_1 < x_2 \leq T$ with $x_2^k \leq T(\log T)^{-1}$. Note that $y^{2k} \leq \sqrt{T}$ holds in our setting. Then we derive

$$M \ll k!\Bigg(\sum_{y<p\leq y^2} \frac{|a_F(p)w_y(p)|^2}{p^{2\sigma}} \Bigg)^k \ll \left(Cdy^{\frac{1}{2}-\sigma}\sqrt{\log T} \right)^{2k},$$

where $C = C(\sigma, B)$ is a positive constant. Then, asymptotic formula (11.11) yields

$$\begin{aligned} &\frac{1}{T}\int_0^T |R_y^*(\sigma+it;F) - R_y(\sigma+it;F)|^{2k}\, dt \\ &\leq 4^k M + 4^k \left(C'dy^{1-2\sigma} \right)^{2k} \ll \left(Kdy^{\frac{1}{2}-\sigma}\sqrt{\log T} \right)^{2k}, \end{aligned}$$

where $C' = C'(\sigma)$ and $K = K(\sigma, B)$ are positive constants. Hence we obtain

$$\begin{aligned} &\mathbb{P}_T(t \in \mathcal{A}_F^{(2)}(\sigma,B,T)) \\ &\leq \left(\frac{1}{2} y^{-\frac{1}{2}(\sigma-\sigma_0)} (\log T)^2 \right)^{-2k} \frac{1}{T}\int_0^T |R_y^*(\sigma+it;F) - R_y(\sigma+it;F)|^{2k}\, dt \\ &\ll \left(2Kdy^{-\frac{1}{2}(\sigma-\sigma_0)} \right)^{2k}. \end{aligned}$$

Inserting $k = \lfloor (\log T)(4B\log_2 T)^{-1} \rfloor$, we obtain

$$\mathbb{P}_T(t \in \mathcal{A}_F^{(2)}(\sigma,B,T)) \ll \exp\left(-b\frac{\log T}{\log_2 T} \right) \tag{11.13}$$

with a small positive constant $b = b_F(\sigma, T)$. By definition, the set $\mathcal{A}_F(\sigma, B, T)$ is included in $\mathcal{A}_F^{(1)}(\sigma,B,T) \cup \mathcal{A}_F^{(2)}(\sigma,B,T)$. Hence the result follows from (11.10) and (11.13). □

LEMMA 11.6. *Let* $F \in \mathcal{S}^{\text{poly}}$*, and put* $y = (\log T)^B$ *with* $B \geq 1$*. Define the subset* $\mathcal{B}_F(\sigma, B, T) \subset [0, T]$ *as*

$$\mathcal{B}_F(\sigma,B,T) = \left\{ t \in [0,T] \;\middle|\; |R_y(\sigma+it;F)| > (\log T)^{1-\sigma}(\log_2 T)^{-1} \right\}.$$

Then there exists a positive constant $b = b_F(\sigma, B)$ *such that*

$$\mathbb{P}_T(t \in \mathcal{B}_F(\sigma,B,T)) \ll \exp\left(-b\frac{\log T}{\log_2 T} \right)$$

for $1/2 < \sigma < 1$*, where the implied constant depends on* F*,* σ*, and* B*.*

PROOF. Let $k = \lfloor (\log T)(L\log_2 T)^{-1} \rfloor$, where $L = L_F(\sigma, B) > 0$ is a sufficiently large real number. Then we obtain

$$R_y(\sigma+it;F) = \sum_{p<k\log k} \frac{a_F(p)}{p^{\sigma+it}} + \sum_{k\log k\leq p\leq y} \frac{a_F(p)}{p^{\sigma+it}} + O(d\log\zeta(2\sigma)). \tag{11.14}$$

Similarly to the estimate for $\mathcal{A}_F^{(2)}(\sigma, B, T)$ in the proof of Lemma 11.5, we consider the $2k$-th moments

$$M_1 = \frac{1}{T}\int_0^T \Bigg|\sum_{p<k\log k} \frac{a_F(p)}{p^{\sigma+it}}\Bigg|^{2k} dt \quad \text{and} \quad M_2 = \frac{1}{T}\int_0^T \Bigg|\sum_{k\log k\leq p\leq y} \frac{a_F(p)}{p^{\sigma+it}}\Bigg|^{2k} dt.$$

Then we obtain the inequality

$$\frac{1}{T}\int_0^T |R_y(\sigma+it;F)|^{2k}\,dt \leq 9^k M_1 + 9^k M_2 + 9^k\left(Cd\log\zeta(2\sigma)\right)^{2k}, \tag{11.15}$$

where $C > 0$ is an absolute constant. The $2k$-th moment M_1 is evaluated as

$$M_1 \ll \left(\sum_{p<k\log k}\frac{d}{p^\sigma}\right)^{2k} \ll \left(\frac{2dk^{1-\sigma}}{(1-\sigma)(\log k)^\sigma}\right)^{2k}$$

by the prime number theorem. Using inequality (11.12), we also estimate M_2 as

$$M_2 \ll k!\left(\sum_{k\log k\leq p\leq y}\frac{d^2}{p^{2\sigma}}\right)^{k} \ll \left(\frac{2dk^{1-\sigma}}{\sqrt{2\sigma-1}(\log k)^\sigma}\right)^{2k}.$$

Thus, we deduce from (11.15) the upper bound

$$\frac{1}{T}\int_0^T |R_y(\sigma+it;F)|^{2k}\,dt \ll \left(K(\sigma,d)\frac{k^{1-\sigma}}{(\log k)^\sigma}\right)^{2k}$$

with some constant $K(\sigma, d) > 0$. It yields

$$\begin{aligned}\mathbb{P}_T(t\in\mathcal{B}_F(\sigma,B,T)) &\leq \left(\frac{(\log T)^{1-\sigma}}{\log_2 T}\right)^{-2k}\frac{1}{T}\int_0^T |R_y(\sigma+it;F)|^{2k}\,dt\\ &\ll \left(K(\sigma,d)\frac{k^{1-\sigma}\log_2 T}{(\log k)^\sigma(\log T)^{1-\sigma}}\right)^{2k}\\ &\ll \exp\left(-b\frac{\log T}{\log_2 T}\right)\end{aligned}$$

with a positive constant $b = b_F(\sigma, B)$, as desired. □

The exceptional subset $\mathcal{E} = \mathcal{E}_F(\sigma, B, T)$ of Proposition 11.3 can be presented by using the subsets of Lemmas 11.5 and 11.6. Indeed, we see later that

$$\mathcal{E} = \mathcal{A}_F(\sigma,B,T)\cup\mathcal{B}_F(\sigma,B,T) \tag{11.16}$$

gives a nice example of such exceptional subset, where $\mathcal{B}_F(\sigma, B, T)$ is understood as the empty set if $\sigma \geq 1$. Then, Lemmas 11.5 and 11.6 ensure that condition (10.4) is satisfied for this subset.

11.2. Completion of the proofs. Recall that the Dirichlet polynomial $R_y(s;F)$ of (11.4) is also represented as

$$R_y(s;F) = \sum_{\substack{p \\ p^k \le y}} \sum_{k=1}^{\infty} b_F(p^k) p^{-ks},$$

where $b_F(p^k)$ is the coefficient of Axiom (S5) in the Selberg class. Write

$$\sum_{\substack{p \\ p^k \le y}} \sum_{k=1}^{\infty} A(p^k) = \sum_{p^k \le y} A(p^k)$$

for any arithmetic function $A(n)$ for simplicity. Then we compare $R_y(\sigma+it;F)$ with the random variable

$$R_y(\sigma, X; F) = \sum_{p^k \le y} b_F(p^k) p^{-k\sigma} X_p^k.$$

The first step is to study the integral moments of $R_y(\sigma+it;F)$ and $R_y(\sigma,X;F)$. The following proposition is a generalization of [93, Lemma 3.5].

Proposition 11.7. *Let $F \in \mathcal{S}^{\mathrm{poly}}$ and $\sigma > 1/2$. If we put $y = (\log T)^B$ with $B \ge 1$, then we have*

$$\begin{aligned}&\frac{1}{T}\int_0^T \overline{R_y(\sigma+it;F)}^{\,j} R_y(\sigma+it;F)^{\ell}\, dt \\ &= \mathbb{E}\left[\overline{R_y(\sigma,X;F)}^{\,j} R_y(\sigma,X;F)^{\ell}\right] + O\left((dy)^{j+\ell} T^{-\frac{5}{6}}\right)\end{aligned}$$

for any non-negative integers j, ℓ with $j + \ell \le (\log T)(6B\log_2 T)^{-1}$, where the implied constant is absolute.

Proof. We calculate the integrand as

$$\begin{aligned}&\overline{R_y(\sigma+it;F)}^{\,j} R_y(\sigma+it;F)^{\ell} \\ &= \sum_{p_1^{m_1} \le y} \cdots \sum_{p_j^{m_j} \le y} \sum_{q_1^{n_1} \le y} \cdots \sum_{q_\ell^{n_\ell} \le y} \frac{b_F(p_1^{m_1}) \cdots b_F(p_j^{m_j})}{p_1^{m_1\sigma} \cdots p_j^{m_j\sigma}} \frac{b_F(q_1^{n_1}) \cdots b_F(q_\ell^{n_\ell})}{q_1^{n_1\sigma} \cdots q_\ell^{n_\ell\sigma}} \\ &\qquad \times \left(\frac{p_1^{m_1} \cdots p_j^{m_j}}{q_1^{n_1} \cdots q_\ell^{n_\ell}}\right)^{it},\end{aligned}$$

and similarly,

$$\begin{aligned}&\overline{R_y(\sigma,X;F)}^{\,j} R_y(\sigma,X;F)^{\ell} \\ &= \sum_{p_1^{m_1} \le y} \cdots \sum_{p_j^{m_j} \le y} \sum_{q_1^{n_1} \le y} \cdots \sum_{q_\ell^{n_\ell} \le y} \frac{b_F(p_1^{m_1}) \cdots b_F(p_j^{m_j})}{p_1^{m_1\sigma} \cdots p_j^{m_j\sigma}} \frac{b_F(q_1^{n_1}) \cdots b_F(q_\ell^{n_\ell})}{q_1^{n_1\sigma} \cdots q_\ell^{n_\ell\sigma}} \times\end{aligned}$$

$$\times \left(\frac{X_{p_1}^{m_1} \cdots X_{p_j}^{m_j}}{X_{q_1}^{n_1} \cdots X_{q_\ell}^{n_\ell}} \right).$$

Note that

$$\frac{1}{T} \int_0^T \left(\frac{p_1^{m_1} \cdots p_j^{m_j}}{q_1^{n_1} \cdots q_\ell^{n_\ell}} \right)^{it} dt = \begin{cases} 1 & \text{if } p_1^{m_1} \cdots p_j^{m_j} = q_1^{n_1} \cdots q_\ell^{n_\ell}, \\ O(T^{-\frac{5}{6}}) & \text{otherwise} \end{cases} \tag{11.17}$$

since we have

$$\left| \log \left(\frac{p_1^{m_1} \cdots p_j^{m_j}}{q_1^{n_1} \cdots q_\ell^{n_\ell}} \right) \right| \geq \frac{|p_1^{m_1} \cdots p_j^{m_j} - q_1^{n_1} \cdots q_\ell^{n_\ell}|}{\max\{p_1^{m_1} \cdots p_j^{m_j}, q_1^{n_1} \cdots q_\ell^{n_\ell}\}} \geq T^{-\frac{1}{6}}$$

for $p_1^{m_1} \cdots p_j^{m_j} \neq q_1^{n_1} \cdots q_\ell^{n_\ell}$. Furthermore, by the independence of X_p's, we deduce

$$\mathbb{E}\left[\frac{X_{p_1}^{m_1} \cdots X_{p_j}^{m_j}}{X_{q_1}^{n_1} \cdots X_{q_\ell}^{n_\ell}} \right] = \begin{cases} 1 & \text{if } p_1^{m_1} \cdots p_j^{m_j} = q_1^{n_1} \cdots q_\ell^{n_\ell}, \\ 0 & \text{otherwise.} \end{cases} \tag{11.18}$$

Comparing (11.17) and (11.18), we obtain

$$\frac{1}{T} \int_0^T \left(\frac{p_1^{m_1} \cdots p_j^{m_j}}{q_1^{n_1} \cdots q_\ell^{n_\ell}} \right)^{it} dt = \mathbb{E}\left[\frac{X_{p_1}^{m_1} \cdots X_{p_j}^{m_j}}{X_{q_1}^{n_1} \cdots X_{q_\ell}^{n_\ell}} \right] + O(T^{-\frac{5}{6}}).$$

Therefore the difference is estimated as

$$\begin{aligned} &\frac{1}{T} \int_0^T \overline{R_y(\sigma + it; F)}^j R_y(\sigma + it; F)^\ell \, dt - \mathbb{E}\left[\overline{R_y(\sigma, X; F)}^j R_y(\sigma, X; F)^\ell \right] \\ &\ll \sum_{p_1^{m_1} \leq y} \cdots \sum_{p_j^{m_j} \leq y} \sum_{q_1^{n_1} \leq y} \cdots \sum_{q_\ell^{n_\ell} \leq y} \frac{b_F(p_1^{m_1}) \cdots b_F(p_j^{m_j})}{p_1^{m_1\sigma} \cdots p_j^{m_j\sigma}} \\ &\qquad\qquad \times \frac{b_F(q_1^{n_1}) \cdots b_F(q_\ell^{n_\ell})}{q_1^{n_1\sigma} \cdots q_\ell^{n_\ell\sigma}} T^{-\frac{5}{6}} \\ &\ll (dy)^{j+\ell} T^{-\frac{5}{6}} \end{aligned}$$

by the inequality $|b_F(p^k)| \leq d/k$. □

COROLLARY 11.8. *Let $F \in \mathcal{S}^{\mathrm{poly}}$ and $\sigma > 1/2$. For $y = (\log T)^B$ with $B \geq 1$, there exist positive constants $K = K_F(\sigma, B)$ and $L = L_F(\sigma, B)$ such that*

$$\frac{1}{T} \int_0^T |R_y(\sigma + it; F)|^{2k} \, dt \ll \begin{cases} K^{2k} & \text{if } \sigma > 1, \\ (K \log_3 T)^{2k} & \text{if } \sigma = 1, \\ \left(K \dfrac{k^{1-\sigma}}{(\log 2k)^\sigma} \right)^{2k} & \text{if } 1/2 < \sigma < 1 \end{cases} \tag{11.19}$$

for any integer k such that $1 \leq k \leq (\log T)(L \log_2 T)^{-1}$. Furthermore,

$$\mathbb{E}\left[|R_y(\sigma, X; F)|^{2k}\right] \ll \begin{cases} K^{2k} & \text{if } \sigma > 1, \\ (K \log_3 T)^{2k} & \text{if } \sigma = 1, \\ \left(K \dfrac{k^{1-\sigma}}{(\log 2k)^{\sigma}}\right)^{2k} & \text{if } 1/2 < \sigma < 1 \end{cases} \tag{11.20}$$

for any integer $k \geq 1$. The implied constants depend only on σ, B, and F.

Proof. By Proposition 11.7, upper bounds (11.19) are deduced from (11.20) if we take L as a sufficiently large real number. Thus we evaluate $\mathbb{E}\left[|R_y(\sigma, X; F)|^{2k}\right]$. If $\sigma \geq 1$, then we have the inequalities

$$|R_y(\sigma, X; F)| \leq \sum_{p \leq y} \sum_{m=1}^{\infty} \frac{d}{m} p^{-m\sigma} \ll \begin{cases} \log \zeta(\sigma) & \text{for } \sigma > 1, \\ \log_2 y & \text{for } \sigma = 1 \end{cases}$$

by recalling $|b_F(p^k)| \leq d/k$. Therefore estimates (11.20) follow in this case. Let $1/2 < \sigma < 1$. If the inequality $y \leq Ck \log 2k$ holds with a constant $C \geq 2$, then we obtain

$$R_y(\sigma, X; F) = \sum_{p \leq y} \frac{a_F(p) X_p}{p^{\sigma}} + O(\log \zeta(2\sigma)) \ll \frac{C^{1-\sigma} k^{1-\sigma}}{(1-\sigma)(\log 2k)^{\sigma}}$$

by the prime number theorem, which yields upper bounds (11.20). Therefore we suppose the inequality $Ck \log 2k < y$ below. In that case, we have

$$\begin{aligned} &R_y(\sigma, X; F) \\ &= \sum_{p < Ck \log 2k} \frac{a_F(p) X_p}{p^{\sigma}} + \sum_{Ck \log 2k \leq p \leq y} \frac{a_F(p) X_p}{p^{\sigma}} + O(\log \zeta(2\sigma)) \end{aligned} \tag{11.21}$$

similarly to (11.14). We again evaluate the $2k$-th moment

$$M_1' = \mathbb{E}\left[\left|\sum_{p < Ck \log 2k} \frac{a_F(p) X_p}{p^{\sigma}}\right|^{2k}\right] \quad \text{and} \quad M_2' = \mathbb{E}\left[\left|\sum_{Ck \log 2k \leq p \leq y} \frac{a_F(p) X_p}{p^{\sigma}}\right|^{2k}\right].$$

By the prime number theorem, we have

$$M_1' \ll \left(\sum_{p < Ck \log 2k} \frac{d}{p^{\sigma}}\right)^{2k} \ll \left(\frac{2dC^{1-\sigma} k^{1-\sigma}}{(1-\sigma)(\log 2k)^{\sigma}}\right)^{2k}. \tag{11.22}$$

For the estimate of M_2', we use

$$\mathbb{E}\left[\left|\sum_{x_1 < p \leq x_2} \frac{a(p) X_p}{p^{\sigma}}\right|^{2k}\right] \ll k! \left(\sum_{x_1 < p \leq x_2} \frac{|a(p)|^2}{p^{2\sigma}}\right)^k$$

in place of (11.12), which can be easily proved by the independence of X_p's. Then we have

$$M_2' \ll k! \left(\sum_{Ck \log 2k \leq p \leq y} \frac{d^2}{p^{2\sigma}}\right)^k \ll \left(\frac{2dC^{\frac{1}{2}-\sigma} k^{1-\sigma}}{\sqrt{2\sigma - 1}(\log 2k)^{\sigma}}\right)^{2k}. \tag{11.23}$$

By (11.21), (11.22), and (11.23), we obtain the conclusion. $\square$

With the notation above, we further define the moment-generating functions $\widetilde{\mathcal{M}}_{\sigma,T}^{(y)}(z,z';F)^{\mathcal{E}}$ and $\widetilde{\mathcal{M}}_{\sigma}^{(y)}(z,z';F)$ as

$$\widetilde{\mathcal{M}}_{\sigma,T}^{(y)}(z,z';F)^{\mathcal{E}} = \frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \psi_{z,z'}(R_y(\sigma+it;F))\,dt,$$
$$\widetilde{\mathcal{M}}_{\sigma}^{(y)}(z,z';F) = \mathbb{E}\left[\psi_{z,z'}(R_y(\sigma,X;F))\right].$$

Then we deduce from Proposition 11.7 a finite truncated version of formula (10.5) as follows. Here, we note that a similar result was obtained by Lamzouri–Lester–Radziwiłł [**93**, Proposition 2.3] in the case $F=\zeta$, and we adapt their method for the proof of the following proposition.

Proposition 11.9. *Let $F\in\mathcal{S}^{\mathrm{poly}}$ and $\sigma>1/2$, and put $y=(\log T)^B$ with $B\geq 1$. Denote by $\mathcal{B}=\mathcal{B}_F(\sigma,B,T)$ the subset of Lemma 11.6. Then there exist positive constants $a=a_F(\sigma,B)$ and $b=b_F(\sigma,B)$ such that the asymptotic formula*

$$\widetilde{\mathcal{M}}_{\sigma,T}^{(y)}(z,z';F)^{\mathcal{B}} = \widetilde{\mathcal{M}}_{\sigma}^{(y)}(z,z';F) + O\left(\exp\left(-b\frac{\log T}{\log_2 T}\right)\right)$$

holds for all $z,z'\in\mathbb{C}$ in the range $\max\{|z|,|z'|\}\leq aR_\sigma(T)$, where $R_\sigma(T)$ is defined as (11.1), and the implied constant depends on σ, B, and F.

Proof. Let $N=\lfloor(\log T)(L\log_2 T)^{-1}\rfloor$, where $L=L_F(\sigma,B)>0$ is a sufficiently large real number. By the Taylor expansion, we have

$$\psi_{z,z'}(R_y(\sigma+it;F)) = \sum_{j+\ell\leq N}\frac{(\frac{i}{2})^{j+\ell}}{j!\ell!}z^j z'^{\ell}\overline{R_y(\sigma+it;F)}^{j}R_y(\sigma+it;F)^{\ell} + O\left(\sum_{n\geq N}\frac{(2aR_\sigma(T))^n}{n!}|R_y(\sigma+it;F)|^n\right)$$

If $t\in[0,T]\setminus\mathcal{B}$, then $R_\sigma(T)|R_y(\sigma+it;F)|\ll(\log T)(\log_2 T)^{-1}$ holds. Hence we derive

(11.24)
$$\widetilde{\mathcal{M}}_{\sigma,T}^{(y)}(z,z';F)^{\mathcal{B}} = \sum_{j+\ell\leq N}\frac{(\frac{i}{2})^{j+\ell}}{j!\ell!}z^j z'^{\ell}\frac{1}{T}\int_{[0,T]\setminus\mathcal{B}}\overline{R_y(\sigma+it;F)}^{j}R_y(\sigma+it;F)^{\ell}\,dt + O\left(\exp\left(-b\frac{\log T}{\log_2 T}\right)\right)$$

with some $b=b_F(\sigma,B)>0$ if $a=a_F(\sigma,B)$ is small enough. Then, in order to use Proposition 11.7, we evaluate the integral

$$\frac{1}{T}\int_{\mathcal{B}}\overline{R_y(\sigma+it;F)}^{j}R_y(\sigma+it;F)^{\ell}\,dt$$

with $j+\ell \leq N$. For $\sigma \geq 1$, there is nothing to do since $\mathcal{B}$ is empty. Let $1/2 < \sigma < 1$. By the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\frac{1}{T}\int_{\mathcal{B}} \overline{R_y(\sigma+it;F)}^{j} R_y(\sigma+it;F)^{\ell}\, dt\\
&\leq (\mathbb{P}_T(t\in\mathcal{B}))^{1/2}\left(\frac{1}{T}\int_0^T |R_y(\sigma+it;F)|^{2(j+\ell)}\, dt\right)^{1/2}\\
&\ll \exp\left(-b'\frac{\log T}{\log_2 T}\right)\left(K\frac{(j+\ell)^{1-\sigma}}{(\log 2(j+\ell))^{\sigma}}\right)^{j+\ell}
\end{aligned}$$

by Lemma 11.6 and Corollary 11.8, where $b' = b'_F(\sigma,B)$ and $K = K_F(\sigma,B)$ are positive constants. Furthermore, we have

$$\frac{(j+\ell)^{1-\sigma}}{(\log 2(j+\ell))^{\sigma}} \ll \frac{(\log T)^{1-\sigma}}{\log_2 T}$$

for $j+\ell \leq N$. As a result,

$$\begin{aligned}
&\sum_{j+\ell\leq N}\frac{(\frac{i}{2})^{j+\ell}}{j!\ell!} z^j z'^{\ell}\frac{1}{T}\int_{\mathcal{B}} \overline{R_y(\sigma+it;F)}^{j} R_y(\sigma+it;F)^{\ell}\, dt\\
&\ll \exp\left(-b'\frac{\log T}{\log_2 T}\right)\sum_{j+\ell\leq N}\frac{(a(\log T)^{\sigma})^{j+\ell}}{j!\ell!}\left(K'\frac{(\log T)^{1-\sigma}}{\log_2 T}\right)^{j+\ell}\\
&\ll \exp\left(-\frac{b'}{2}\frac{\log T}{\log_2 T}\right)
\end{aligned}$$

for $\max\{|z|,|z'|\} \leq aR_\sigma(T)$ if $a = a_F(\sigma,B)$ is small. Hence (11.24) deduces

$$\begin{aligned}
\widetilde{\mathcal{M}}_{\sigma,T}^{(y)}(z,z';F)^{\mathcal{B}} = &\sum_{j+\ell\leq N}\frac{(\frac{i}{2})^{j+\ell}}{j!\ell!} z^j z'^{\ell}\frac{1}{T}\int_0^T \overline{R_y(\sigma+it;F)}^{j} R_y(\sigma+it;F)^{\ell}\, dt\\
&+O\left(\exp\left(-b''\frac{\log T}{\log_2 T}\right)\right)
\end{aligned}$$

with some $b'' = b''_F(\sigma,B) > 0$. On the other hand, we obtain

$$\begin{aligned}
\widetilde{\mathcal{M}}_{\sigma}^{(y)}(z,z';F) = &\sum_{j+\ell\leq N}\frac{(\frac{i}{2})^{j+\ell}}{j!\ell!} z^j z'^{\ell}\mathbb{E}\left[\overline{R_y(\sigma+it;F)}^{j} R_y(\sigma+it;F)^{\ell}\right]\\
&+O\left(\exp\left(-b''\frac{\log T}{\log_2 T}\right)\right)
\end{aligned}$$

in a similar way. Then the desired result follows from Proposition 11.7. $\square$

Proof of Proposition 11.3. Let $B_1 = \max\left\{(2B+6)(\sigma-\sigma_0)^{-1},1\right\}$ and put $y = (\log T)^{B_1}$. Denote by $\mathcal{A} = \mathcal{A}_F(\sigma,B_1,T)$ and $\mathcal{B} = \mathcal{B}_F(\sigma,B_1,T)$ the subsets of Lemmas 11.5 and 11.6. Then we define $\mathcal{E} = \mathcal{A}\cup\mathcal{B}$ as was described. For $t\in[0,T]\setminus\mathcal{A}$, we have

$$\psi_{z,z'}\big(\log F(\sigma+it) - R_y(\sigma+it;F)\big) = 1 + O\left((\log T)^{-B}\right)$$

in the range $|z|, |z'| \leq aR_\sigma(T)$. Since the condition $t \notin \mathcal{E}$ implies $t \notin \mathcal{A}$, the left-hand side of (10.5) is calculated as

$$\begin{aligned}(11.25)\qquad &\widetilde{\mathcal{M}}_{\sigma,T}(z,z';F)^{\mathcal{E}} \\ &= \widetilde{\mathcal{M}}^{(y)}_{\sigma,T}(z,z';F)^{\mathcal{E}} + O\left(\frac{1}{(\log T)^B}\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}}\left|\psi_{z,z'}(R_y(\sigma+it;F))\right|\,dt\right).\end{aligned}$$

For $t \in [0,T] \setminus \mathcal{B}$, we have

$$|\psi_{z,z'}(R_y(\sigma+it;F))| \leq \exp\left(aK\frac{\log T}{\log_2 T}\right),$$

where $K = K_F(\sigma, B_1)$ is some positive constant. Hence we obtain

$$\begin{aligned}(11.26)\quad \widetilde{\mathcal{M}}^{(y)}_{\sigma,T}(z,z';F)^{\mathcal{E}} &= \widetilde{\mathcal{M}}^{(y)}_{\sigma,T}(z,z';F)^{\mathcal{B}} + O\left(\mathbb{P}_T(t\in\mathcal{A})\exp\left(aK\frac{\log T}{\log_2 T}\right)\right) \\ &= \widetilde{\mathcal{M}}^{(y)}_{\sigma,T}(z,z';F)^{\mathcal{B}} + O\left(\exp\left(-b\frac{\log qT}{\log_2 T}\right)\right)\end{aligned}$$

by Lemma 11.5 if the constant $a = a_F(\sigma, B_1)$ is sufficiently small. Furthermore, by Proposition 11.9 and Lemma 9.8, the moment-generating function $\widetilde{\mathcal{M}}^{(y)}_{\sigma,T}(z,z';F)^{\mathcal{B}}$ is calculated as

$$(11.27)\qquad \widetilde{\mathcal{M}}^{(y)}_{\sigma,T}(z,z';F)^{\mathcal{B}} = \widetilde{\mathcal{M}}_\sigma(z,z';F) + E,$$

where E is estimated as

$$\begin{aligned}E &\ll \exp\left(-b'\frac{\log T}{\log_2 T}\right) + E_\sigma(z,z';F)|z|(\log T)^{-B-3} \\ &\ll \frac{1+E_\sigma(z,z';F)}{(\log T)^B}.\end{aligned}$$

Note that the condition $t \notin \mathcal{E}$ implies $t \notin \mathcal{B}$. Then we again apply Proposition 11.9 and Lemma 9.8 to derive

$$\begin{aligned}\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}}\left|\psi_{z,z'}(R_y(\sigma+it;F))\right|\,dt &\leq \widetilde{\mathcal{M}}^{(y)}_{\sigma,T}(\xi,\xi';F)^{\mathcal{B}} \\ &\ll \frac{1+E_\sigma(z,z';F)}{(\log T)^B},\end{aligned}$$

where (ξ, ξ') is the pair of (9.13). Combining (11.25), (11.26), and (11.27), we deduce formula (10.5). □

Proof of Theorem 11.1. By Proposition 10.2, the remaining work is showing limit formula

$$(11.28)\qquad \lim_{T\to\infty}\frac{1}{T}\int_{\mathcal{E}}\Phi\left(\log F(\sigma+it)\right)\,dt = 0$$

for all $\Phi \in C(\mathbb{C})$ if $\sigma > 1$. We have

$$|F(\sigma+it)| \leq \sum_{n=1}^{\infty}|a_F(n)|n^{-\sigma}$$

by Axiom (S1). Hence the function $\Phi(\log F(\sigma+it))$ is bounded as a function in t. Therefore (11.28) holds by condition (10.4). □

Proof of Theorem 11.2. The result directly follows from Proposition 10.2. □

12. The second result: Artin L-functions

Let $\{L(s,\rho_K)\}_{K\in\mathscr{F}}$ be the family of Example 7.4, and let $\mathcal{X}=(\mathcal{X}_p)_p$ be the sequence of Example 8.4. Since we have

$$\mathbb{E}[(\operatorname{tr}\mathcal{X}_p)^2]=\frac{p^2}{p^2+p+1}\left(1+\frac{1}{p}\right)\geq\frac{3}{14},$$

the sequence $\mathcal{X}$ is admissible in the sense of Definition 9.2. Let $\sigma>1/2$ be a real number. By Theorem 9.3, we obtain a continuous function $\mathcal{M}_\sigma(\,\cdot\,;\mathcal{X}):\mathbb{R}\to\mathbb{R}_{\geq 0}$ such that

$$\mathbb{P}(\log L(\sigma,\mathcal{X})\in A)=\int_A\mathcal{M}_\sigma(u;\mathcal{X})\,|du|$$

for all $A\in\mathcal{B}(\mathbb{R})$. Then, we prove the following results in this section.

Theorem 12.1. *Suppose that there exist positive constants $1/2<\sigma_0<1$ and δ such that the zero density estimate*

$$\sum_{K\in L_3^{\pm}(X)}N(\sigma_0,(\log X)^3;\rho_K)\ll X^{1-\delta}\tag{12.1}$$

is valid, where the implied constant may depend on δ. Then the limit formula

$$\lim_{X\to\infty}\frac{1}{\#L_3^{\pm}(X)}\sideset{}{'}\sum_{K\in L_3^{\pm}(X)}\Phi(\log L(\sigma,\rho_K))=\int_{\mathbb{R}}\Phi(u)\mathcal{M}_\sigma(u;\mathcal{X})\,|du|$$

holds in the following cases:

(a) *$\sigma>1$ and $\Phi\in C(\mathbb{R})\cup\mathcal{I}(\mathbb{R})$;*
(b) *$\sigma=1$ and $\Phi\in C^{\exp}(\mathbb{R})\cup\mathcal{I}(\mathbb{R})$;*
(c) *$\sigma_0<\sigma<1$ and $\Phi\in C_b(\mathbb{R})\cup\mathcal{I}(\mathbb{R})$;*
(c′) *$\sigma_0<\sigma<1$ and $\Phi\in C^{\exp}(\mathbb{R})$ assuming GRH for $L(s,\rho_K)$ for all $K\in\mathscr{F}$.*

Theorem 12.2. *With the same assumptions as in Theorem 12.1, we obtain*

$$D_\sigma(X;\mathcal{L})\ll\begin{cases}(\log X)^{-1}(\log_2 X)^2 & \text{for } \sigma\geq 1,\\ (\log X)^{-\frac{\sigma-\sigma_0}{1-\sigma_0}}(\log_2 T)^2 & \text{for } \sigma_0<\sigma<1,\end{cases}$$

where the implied constants depend on σ.

We see later that zero density estimate (12.1) holds with arbitrary constant $7/8<\sigma_0<1$. Therefore, we deduce Theorem III from Theorem 12.1 by writing $\mathcal{M}_\sigma(u;\mathcal{X})=\mathcal{C}_\sigma(u)$. In order to prove Theorems 12.1 and 12.2, we prepare the following result.

Proposition 12.3. *Suppose that there exist positive constants* $1/2 < \sigma_0 < 1$ *and* δ *such that zero density estimate* (12.1) *holds. Then, for* $\sigma > \sigma_0$*, there exist positive constants* $a = a(\sigma)$ *and* $b = b(\sigma)$ *such that the asymptotic formula*

$$\begin{aligned}(12.2)\qquad &\frac{1}{\# L_3^{\pm}(X)} \sideset{}{'}\sum_{K \in L_3^{\pm}(X) \setminus \mathcal{E}} \exp\left(\frac{iz}{2} \log L(\sigma, \rho_K)\right) \\ &= \mathbb{E}\left[\exp\left(\frac{iz}{2} \log L(\sigma, \mathcal{X})\right)\right] + O\left(\exp\left(-b \frac{\log X}{\log_2 X}\right)\right)\end{aligned}$$

holds for all $z \in \mathbb{C}$ *in the range* $|z| \leq aR_\sigma(X)$ *with the following data:*

- *the exceptional subset* $\mathcal{E} = \mathcal{E}_{\mathcal{L}}(\sigma_0, X)$ *presented by* (12.5) *below;*
- *the function* R_σ *defined as*

$$R_\sigma(X) = \begin{cases} (\log X)(\log_2 X)^{-2} & \text{if } \sigma \geq 1, \\ (\log X)^{\frac{\sigma - \sigma_0}{1 - \sigma_0}} (\log_2 X)^{-2} & \text{if } \sigma_0 < \sigma < 1. \end{cases} \tag{12.3}$$

The implied constant in (12.2) *depends on* σ.

Let $\widetilde{\mathcal{M}}_{s,X}(z, z'; \mathcal{L})^{\mathcal{E}}$ and $\widetilde{\mathcal{M}}_s(z, z'; \mathcal{X})$ be the moment-generating functions of (10.18) and (9.9), respectively. Since we consider only the case $s = \sigma \in \mathbb{R}$, they are represented as

$$\begin{aligned}\widetilde{\mathcal{M}}_{\sigma,X}(z, z'; \mathcal{L})^{\mathcal{E}} &= \frac{1}{\# L_3^{\pm}(X)} \sideset{}{'}\sum_{K \in L_3^{\pm}(X) \setminus \mathcal{E}} \exp\left(\frac{i(z + z')}{2} \log L(\sigma, \rho_K)\right), \\ \widetilde{\mathcal{M}}_\sigma(z, z'; \mathcal{X}) &= \mathbb{E}\left[\exp\left(\frac{i(z + z')}{2} \log L(\sigma, \mathcal{X})\right)\right].\end{aligned}$$

Thus, Proposition 12.3 indicates that Assumption 10.4 is satisfied in the restricted case $s = \sigma \in \mathbb{R}$.

12.1. Preliminaries. We hereby list several preliminary lemmas used in the proof of Proposition 12.3.

12.1.1. *Counting cubic fields.* To begin with, we set up the notation for counting cubic fields. Based on [**159**, **177**], we introduce the notion of the local conditions of cubic fields as follows. Let

$$\mathscr{A} = \{(111), (21), (3), (1^2 1), (1^3)\}$$

be the set of symbols. Then we associate $\mathfrak{a} \in \mathscr{A}$ with a diagonal matrix in the set $\mathcal{A}$ of (8.5) by putting

$$A_{\mathfrak{a}} = \begin{cases} \operatorname{diag}(1, 1) & \text{if } \mathfrak{a} = (111), \\ \operatorname{diag}(1, -1) & \text{if } \mathfrak{a} = (21), \\ \operatorname{diag}(\omega, \overline{\omega}) & \text{if } \mathfrak{a} = (3), \\ \operatorname{diag}(1, 0) & \text{if } \mathfrak{a} = (1^2 1), \\ \operatorname{diag}(0, 0) & \text{if } \mathfrak{a} = (1^3). \end{cases}$$

We say that a cubic field K satisfies a local condition $\mathfrak{a} \in \mathscr{A}$ at p if

(a) for $\mathfrak{a} = (111)$, p splits totally in K, i.e. $(p) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$;
(b) for $\mathfrak{a} = (21)$, p splits partially in K, i.e. $(p) = \mathfrak{p}_1\mathfrak{p}_2$;
(c) for $\mathfrak{a} = (3)$, p remains inert in K, i.e. $(p) = \mathfrak{p}_1$;
(d) for $\mathfrak{a} = (1^2 1)$, p is partially ramified in K, i.e. $(p) = \mathfrak{p}_1^2\mathfrak{p}_2$;
(e) for $\mathfrak{a} = (1^3)$, p is totally ramified in K, i.e. $(p) = \mathfrak{p}_1^3$.

The notation $S = (S_p)_p$ is used to denote a collection of local conditions with the following data: (i) a finite set $\mathrm{supp}(S)$ consisting of prime numbers; (ii) an element $S_p \in \mathscr{A}$, for each $p \in \mathrm{supp}(S)$. We say that a cubic field K satisfies the local conditions $S = (S_p)_p$ if K satisfies S_p at p for every $p \in \mathrm{supp}(S)$. Then we define

$$L_3^{\pm}(X, S) = \left\{K \in L_3^{\pm}(X) \;\middle|\; K \text{ satisfies the local conditions } S\right\}.$$

Remark that the set $\mathrm{supp}(S)$ may be empty. We define $L_3^{\pm}(X, S) = L_3^{\pm}(X)$ in that case. Next, for a prime number p and $\mathfrak{a} \in \mathscr{A}$, we put

$$C_p(\mathfrak{a}) = \frac{p^2}{p^2+p+1} \times \begin{cases} 1/6 & \text{if } \mathfrak{a} = (111), \\ 1/2 & \text{if } \mathfrak{a} = (21), \\ 1/3 & \text{if } \mathfrak{a} = (3), \\ 1/p & \text{if } \mathfrak{a} = (1^2 1), \\ 1/p^2 & \text{if } \mathfrak{a} = (1^3), \end{cases}$$

$$K_p(\mathfrak{a}) = \frac{1 - p^{-\frac{1}{3}}}{(1 - p^{-\frac{5}{3}})(1 + p^{-1})} \times \begin{cases} 1/6 \cdot (1 + p^{-\frac{1}{3}})^3 & \text{if } \mathfrak{a} = (111), \\ 1/2 \cdot (1 + p^{-\frac{1}{3}})(1 + p^{-\frac{2}{3}}) & \text{if } \mathfrak{a} = (21), \\ 1/3 \cdot (1 + p^{-1}) & \text{if } \mathfrak{a} = (3), \\ 1/p \cdot (1 + p^{-\frac{1}{3}})^2 & \text{if } \mathfrak{a} = (1^2 1), \\ 1/p^2 \cdot (1 + p^{-\frac{1}{3}}) & \text{if } \mathfrak{a} = (1^3). \end{cases}$$

Note that both $\sum_{\mathfrak{a}\in\mathscr{A}} C_p(\mathfrak{a})$ and $\sum_{\mathfrak{a}\in\mathscr{A}} K_p(\mathfrak{a})$ are equal to 1. We further define

$$C(S) = \prod_{p \in \mathrm{supp}(S)} C_p(S_p) \quad \text{and} \quad K(S) = \prod_{p \in \mathrm{supp}(S)} K_p(S_p),$$

where the empty product is interpreted as the value 1. Put $C^+ = 1$, $C^- = 3$ and $K^+ = 1$, $K^- = \sqrt{3}$. Finally, we define $E^{\pm}(X, S)$ by the equation

$$\#L_3^{\pm}(X, S) = C^{\pm}\frac{1}{12\zeta(3)} X C(S) + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{\frac{5}{6}} K(S) + E^{\pm}(X, S).$$

Roberts [159] conjectured $E^{\pm}(X, S) = o(X^{\frac{5}{6}})$ as $X \longrightarrow \infty$. This conjecture was later proved to be true by Bhargava–Shankar–Tsimerman [10] and Taniguchi–Thorne [177] independently. More precisely, it was proved that

$$E^{\pm}(X, S) \ll_{\epsilon} X^{\frac{7}{9}+\epsilon} \prod_{p \in \mathrm{supp}(S)} p^{e_p(S_p)}, \tag{12.4}$$

where $e_p(\mathfrak{a}) = 8/9$ for $\mathfrak{a} = (111), (21), (3)$ and $e_p(\mathfrak{a}) = 16/9$ for $\mathfrak{a} = (1^2 1), (1^3)$.

12.1.2. *Zero density estimates and applications.* As mentioned in Section 3.3, the Artin L-function $L(s,\rho_K)$ has the polynomial Euler product

$$L(s,\rho_K)=\prod_p \det\left(I-p^{-s}A_p(K)\right)^{-1}$$

for $\operatorname{Re} s>1$, where $A_p(K)=A_{\mathfrak{a}}$ if K satisfies a local condition $\mathfrak{a}$ at p. Recall that the strong Artin conjecture is true for $L(s,\rho_K)$. Hence it is continued to an entire function. Denote by $\alpha_p(K)$ and $\beta_p(K)$ the eigenvalues of $A_p(K)$. Since $|\alpha_p(K)|\leq 1$ and $|\beta_p(K)|\leq 1$, the Ramanujan–Petersson conjecture is also true in this case. Finally, the functional equation of $L(s,\rho_K)$ is given by $\Lambda(s,\rho_K)=\Lambda(1-s,\rho_K)$, where $\Lambda(s,\rho_K)=|d_K|^{s/2}\gamma(s,\rho_K)L(s,\rho_K)$ with the gamma-factor

$$\gamma(s,\rho_K)=\begin{cases}\pi^{-s}\Gamma\left(\frac{s}{2}\right)^2 & \text{if } d_K>0,\\ \pi^{-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) & \text{if } d_K<0.\end{cases}$$

Note that $\gamma(s,\rho_K)$ are common over the set $L_3^+(X)$ or $L_3^-(X)$. From the above, the Artin L-function $L(s,\rho_K)$ is a member of $\mathcal{S}^{\mathrm{poly}}$. Then we obtain the following zero density estimate.

Lemma 12.4. *Let $X\geq 1$ and $T\geq 2$. For any $C_0>6$, we have*

$$\sum_{K\in L_3^\pm(X)} N(\sigma,T;\rho_K)\ll T^A X^{C_0\frac{1-\sigma}{2\sigma-1}}$$

for any $\sigma\geq 3/4$, where $A>0$ is an absolute constant. The implied constant depends only on the choice of C_0.

Proof. Let $S^\pm(X)$ be the set of all cuspidal representations π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ such that $L(s,\pi)=L(s,\rho_K)$ holds for some $K\in L_3^\pm(X)$. The conductor of $\pi\in S^\pm(X)$ satisfies $\mathrm{Cond}(\pi)\leq X$ since we have $\mathrm{Cond}(\pi)=|d_K|$ if $L(s,\pi)=L(s,\rho_K)$. We also obtain $\#S^\pm(X)\ll X$, which is deduced from the fact that there exists a one-to-one correspondence between $S^\pm(X)$ and $L_3^\pm(X)$. Finally, the gamma-factors in the functional equations of $L(s,\pi)$ are common in $\pi\in S^\pm(X)$. From the above, the desired estimate follows directly if we apply the zero density estimate of Kowalski–Michel [**87**, Theorem 2] to the family $S^\pm(X)$. ☐

We check that condition (12.1) holds for any $7/8<\sigma_0<1$. Taking $C_0=6+\delta$ with $\delta=(8\sigma_0-7)/2>0$, we obtain

$$\frac{C_0(1-\sigma)}{2\sigma-1}=1-\frac{(8+\delta)\sigma-(7+\delta)}{2\sigma-1}\leq 1-\delta$$

for $\sigma_0\leq\sigma\leq 1$. Note that $(\log X)^{3A}\ll X^{\delta/2}$. Hence Lemma 12.4 ensures the validity of (12.1). We define the subset $\mathcal{E}=\mathcal{E}(\sigma_0,X)\subset L_3^\pm(X)$ as

$$\mathcal{E}=\left\{K\in L_3^\pm(X)\;\middle|\;\begin{array}{c}\text{there exists a zero } \rho \text{ of } L(s,\rho_K) \text{ such that}\\ \operatorname{Re}\rho\geq\sigma_0 \text{ and } |\operatorname{Im}\rho|\leq(\log X)^3\end{array}\right\}. \tag{12.5}$$

Then condition (10.19) is satisfied with this subset. Next, we consider an upper bound of the value $\log L(s,\rho_K)$ with $K\notin\mathcal{E}$.

LEMMA 12.5. *Let* $s = \sigma + it$ *be a complex number with* $\sigma \geq \sigma_0 + 2(\log_2 X)^{-1}$ *and* $|t| \leq (\log X)^2$. *Then we have*

$$\log L(s, \rho_K) \ll (\log_2 X)(\log X)^{\frac{1-\sigma}{1-\sigma_0}} + \log_2 X \tag{12.6}$$

if $K \in L_3^{\pm}(X) \setminus \mathcal{E}$, *where the implied constant is absolute.*

PROOF. The proof is based on the method of Barban [6, Lemma 3]. We write $\kappa = (\log_2 X)^{-1}$ for simplicity. If $\sigma \geq 1 + \kappa/2$, then we obtain $\log L(s, \rho_K) \ll \log_2 X$ by Dirichlet series representation (11.3). Hence the result follows in this case. Let $\sigma_0 + 2\kappa \leq \sigma \leq 1 + \kappa/2$. Then we put $z_0 = \kappa^{-1} + \kappa + it$ and $R = \kappa^{-1} + \kappa - \sigma_0$. Remark that the function $(L'/L)(z, \rho_K)$ is holomorphic on $|z - z_0| < R$ since $L(z, \rho_K)$ has no zeros in this disk. We define

$$M(r) = \max_{|z - z_0| = r} \left| \frac{L'}{L}(z, \rho_K) \right|$$

for $0 < r < R$. Let $r_1 = \kappa^{-1} - 1$, $r_2 = \kappa^{-1} + \kappa - \sigma$, and $r_3 = \kappa^{-1} - \sigma_0$. Then we have $0 < r_1 < r_2 < r_3 < R$. As a consequence of the Hadamard three circles theorem, we obtain the inequality

$$\left| \frac{L'}{L}(s, \rho_K) \right| \leq M(r_1)^{1-a} M(r_3)^a, \tag{12.7}$$

where

$$a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} = \frac{1-\sigma}{1-\sigma_0} + O\left(\frac{1}{\log_2 X}\right).$$

Then we evaluate $M(r_1)$. Since we have $\operatorname{Re} z \geq 1 + \kappa$ on the circle $|z - z_0| = r_1$, the estimate $(L'/L)(s, \rho_K) \ll \log_2 X$ holds. Therefore we obtain

$$M(r_1) \leq A \log_2 X, \tag{12.8}$$

where $A \geq 1$ is an absolute constant. Next, let $z = x + iy$ be a complex number on the circle $|z - z_0| = r_3$. By [70, Proposition 5.7 (2)], we have

$$\frac{L'}{L}(z, \rho_K) = \sum_{|z-\rho|<1} \frac{1}{z - \rho} + O\left(\log\{|d_K|(|y| + 3)\}\right)$$

with an absolute implied constant, where ρ runs through zeros of $L(s, \rho_K)$. For $|z - z_0| = r_3$, the distance between z and ρ is at least κ since $K \notin \mathcal{E}$. We apply the result [70, Proposition 5.7 (1)] to estimate the number of zeros satisfying $|z - \rho| < 1$. Then the bound $(L'/L)(z, \rho_K) \ll \kappa^{-1} \log X$ follows. Therefore we obtain

$$M(r_3) \leq B(\log_2 X)(\log X), \tag{12.9}$$

where $B \geq 1$ is an absolute constant. Inserting (12.8) and (12.9) to (12.7), we obtain

$$\left| \frac{L'}{L}(s, \rho_K) \right| \leq A^{1-a} B^a (\log_2 X)(\log X)^a.$$

Note that the inequality $0 < a < 1$ holds since $r_1 < r_2 < r_3$. Thus we have absolutely $A^{1-a}B^a \ll 1$. Let $s_0 = 1 + \kappa/2 + it$. Then the relation

$$\log L(s, \rho_K) = \log L(s_0, \rho_K) - \int_\sigma^{1+\kappa/2} \frac{L'}{L}(x + it, \rho_K)\, dx$$

holds by the choice of the branch of $\log L(s, \rho_K)$. We have $\log L(s_0, \rho_K) \ll \log_2 X$, and it is deduced from the estimate on $(L'/L)(s, \rho_K)$ that

$$\int_\sigma^{1+\kappa/2} \frac{L'}{L}(x + it, \rho_K)\, dx \ll (\log_2 X)(\log X)^{\frac{1-\sigma}{1-\sigma_0}}.$$

Hence the proof is completed. □

If we further assume GRH, then upper bound (12.6) can be improved. The following result is a simple application of [**70**, Theorem 5.19].

Lemma 12.6. *Assume GRH for $L(s, \rho_K)$ for all $K \in \mathscr{F}$. For $s = \sigma + it$ with $\sigma > 1/2$ and $|t| \le (\log X)^2$, we have*

$$\log L(s, \rho_K) \ll \frac{(\log X)^{2-2\sigma}}{\log_2 X} + \log_2 X$$

if $K \in L_3^\pm(X)$, where the implied constant depends only on σ.

We also remark that this upper bound is valid at $s = 1$ unconditionally. Indeed, combining the result of Louboutin [**114**, **115**], we derive the following result.

Lemma 12.7. *For all $K \in L_3^\pm(X)$, we have*

$$\log L(1, \rho_K) \ll \log_2 X$$

with an absolute implied constant.

12.2. g-functions for Artin L-functions. For $z \in \mathbb{C}$, we define

$$g_z(s, \rho_K) = \exp\left(\frac{iz}{2} \log L(s, \rho_K)\right)$$

according to Ihara–Matsumoto [**64**], who studied similar g-functions for Dirichlet L-functions. In this subsection, we study several properties of $g_z(s, \rho_K)$. The z-th divisor function $d_z(n)$ is defined as the multiplicative function such that

$$d_z(p^k) = \frac{1}{k!} z(z+1) \cdots (z+k-1) =: H_k(z)$$

for a prime number p and an integer $k \ge 1$. Note that $d_r(n)$ is the usual divisor function if r is a positive integer. Then we obtain

$$\exp(z \log \zeta(s)) = \prod_p \exp\left(-z \log(1 - p^{-s})\right) = \sum_{n=1}^\infty d_z(n) n^{-s} \tag{12.10}$$

for $\operatorname{Re} s > 1$. We generalize (12.10) for $L(s, \rho_K)$ as follows. Let A be a diagonal matrix in $\mathscr{A}$. Then we define

$$H_k(z; A) = \sum_{j=0}^k \alpha^j \beta^{k-j} H_j(z) H_{k-j}(z), \tag{12.11}$$

where α and β are the eigenvalues of A. By this definition we obtain

$$\exp(-z\log\det(I-p^{-s}A))=\sum_{k=0}^{\infty}H_k(z;A)p^{-ks}. \tag{12.12}$$

Let $K\in L_3^{\pm}(X)$. We define $d_z(n;\rho_K)$ by extending $d_z(p^k;\rho_K)=H_k(z;A_p(K))$ multiplicatively. Since we obtain

$$g_z(s,\rho_K)=\prod_p\exp\left(-\frac{iz}{2}\log\det(I-p^{-s}A_p(K))\right)$$

for $\operatorname{Re}s>1$, we further derive the Dirichlet series representation

$$g_z(s,\rho_K)=\sum_{n=1}^{\infty}d_{iz/2}(n;\rho_K)n^{-s}. \tag{12.13}$$

Then, we proceed to define the function $\mathcal{F}_s(z)$ as

$$\mathcal{F}_s(z)=\prod_p\mathcal{F}_{s,p}(z) \tag{12.14}$$

for any $s,z\in\mathbb{C}$ with $\operatorname{Re}s>1/2$, where

$$\begin{aligned}\mathcal{F}_{s,p}(z)&=\mathbb{E}\left[\exp\left(-\frac{iz}{2}\log\det(I-p^{-s}\mathcal{X}_p)\right)\right]\\&=\sum_{\mathfrak{a}\in\mathscr{A}}C_p(\mathfrak{a})\exp\left(-\frac{iz}{2}\log\det(I-p^{-s}A_{\mathfrak{a}})\right).\end{aligned} \tag{12.15}$$

If $s=\sigma>1/2$, then we see that $\mathcal{F}_\sigma(z)=\widetilde{M}_\sigma(z,z;\mathcal{X})$ holds with the notation of Section 9.2. Therefore Lemma 9.7 implies

$$\mathcal{F}_\sigma(z)=\mathbb{E}\left[\exp\left(\frac{iz}{2}\log L(\sigma,\mathcal{X})\right)\right]. \tag{12.16}$$

Thus, the function $\mathcal{F}_\sigma(z)$ is associated with the mean value

$$\frac{1}{\#L_3^{\pm}(X)}\sideset{}{'}\sum_{K\in L_3^{\pm}(X)}g_z(\sigma,\rho_K).$$

Towards the proof of Proposition 12.3, we show several results on the function $\mathcal{F}_s(z)$.

Lemma 12.8. *If $z\in\mathbb{C}$ is fixed, then $\mathcal{F}_s(z)$ is a holomorphic function in s for* $\operatorname{Re}s>1/2$. *Similarly, if $s\in\mathbb{C}$ is fixed with* $\operatorname{Re}s>1/2$, *then $\mathcal{F}_s(z)$ is a holomorphic function in z on the whole complex plane.*

Proof. By definition, local factors $\mathcal{F}_{s,p}(z)$ are holomorphic functions in s for $\operatorname{Re}s>0$ if $z\in\mathbb{C}$ is fixed. Similarly, they are also entire functions in z if $s\in\mathbb{C}$ is fixed with $\operatorname{Re}s>0$. Thus it is sufficient to prove that (12.14) converges uniformly on for $\operatorname{Re}s\geq\sigma_0$ and $|z|\leq R$ with any $\sigma_0>1/2$ and $R>0$. In general, the inequality $|H_k(z;A)|\leq H_k(2|z|)$ holds for all $A\in\mathcal{A}$ by (12.11). Furthermore, we obtain $H_k(r)\leq 2^k(r+1)^k$ for $r>0$. Thus we truncate (12.12) as

$$\exp(-z\log\det(I-p^{-s}A))=\sum_{k<N}H_k(z;A)p^{-ks}+E_N \tag{12.17}$$

for any $N \geq 0$, where $|E_N| \leq 2(4|z|+2)^N p^{-N\sigma}$ if $p^{-\sigma} \leq (8|z|+4)^{-1}$ is satisfied. Applying this formula with $N = 2$, we derive

$$\exp\left(-\frac{iz}{2}\log\det(I - p^{-s}A)\right) = \sum_{k<2} H_k(iz/2; A)p^{-ks} + O_R\left(p^{-2\sigma_0}\right)$$

for $p > Q = (4R+4)^{1/\sigma_0}$. The coefficients $H_k(z;A)$ for $k = 0, 1$ are calculated as $H_0(z;A) = 1$ and $H_1(z;A) = z\operatorname{tr} A$. By (12.15), we obtain

$$\mathcal{F}_{s,p}(z) = 1 + O_R\left(p^{-\sigma_0-1} + p^{-2\sigma_0}\right)$$

for $p > Q$. If we have $\sigma_0 > 1/2$, then the series $\sum_p p^{-\sigma_0-1}$ and $\sum_p p^{-2\sigma_0}$ converse. Hence the result follows. □

Lemma 12.9. *Let $s = \sigma + it \in \mathbb{C}$ with $1/2 < \sigma < 1$ and $z \in \mathbb{C}$. Then there exists an absolute constant $c > 0$ such that*

$$|\mathcal{F}_s(z)| \leq \exp\left(\frac{c}{(2\sigma-1)(1-\sigma)}\frac{(|z|+3)^{\frac{1}{\sigma}}}{\log(|z|+3)}\right).$$

Proof. Let $Q = (4|z|+4)^{1/\sigma}$. We first consider the contributions of local parts $\mathcal{F}_{s,p}(z)$ for $p > Q$. Applying (12.17) again, we deduce

$$\exp\left(-\frac{iz}{2}\log\det(I - p^{-s}A)\right) = 1 + \frac{iz}{2}(\operatorname{tr} A)p^{-s} + E(A),$$

where $|E(A)| \leq 8(|z|+1)^2 p^{-2\sigma}$. Therefore we derive $\mathcal{F}_{s,p}(z) = 1 + \mu + E$, where

$$\mu = z\frac{p^2}{p^2+p+1}p^{-s-1} \quad \text{and} \quad |E| \leq 8(|z|+1)^2 p^{-2\sigma}.$$

Note that the inequalities $|\mu| \leq 1/8$ and $|E| \leq 1/2$ hold since $p > Q$ implies $(|z|+1)p^{-\sigma} \leq 1/4$. Then, we apply the uniform estimate $\operatorname{Log}(1+w) \ll |w|$ for $|w| \leq 2/3$ to deduce

$$\operatorname{Log}\mathcal{F}_{s,p}(z) \ll |z|p^{-\sigma-1} + (|z|+1)^2 p^{-2\sigma}.$$

By the prime number theorem, the estimate

$$\sum_{p>y} p^{-u} \ll \frac{1}{u-1}\frac{y^{1-u}}{\log y} \tag{12.18}$$

holds for any $1 < u < 2$ with an absolute implied constant. Then we use (12.18) with $u = \sigma + 1$ and 2σ. It is deduced that

$$\left|\prod_{p>Q}\mathcal{F}_{s,p}(z)\right| \leq \exp\left(\frac{c_1}{2\sigma-1}\frac{(|z|+3)^{\frac{1}{\sigma}}}{\log(|z|+3)}\right) \tag{12.19}$$

with an absolute constant $c_1 > 0$. The contributions of $\mathcal{F}_{s,p}(z)$ for $p \leq Q$ are estimated as follows. For all prime numbers p, we have

$$\operatorname{Log}\det(I - p^{-s}A) = \operatorname{Log}(1 - \alpha p^{-s}) + \operatorname{Log}(1 - \beta p^{-s}) \ll p^{-\sigma}.$$

Then, we again apply the prime number theorem to obtain

$$\sum_{p\le y} p^{-u} \ll \frac{1}{1-u}\frac{y^{1-u}}{\log y}$$

for $0 < u < 1$. This yields

$$\left|\prod_{p\le Q}\mathcal{F}_{s,p}(z)\right| \le \exp\left(\frac{c_2}{1-\sigma}\frac{(|z|+3)^{\frac{1}{\sigma}}}{\log(|z|+3)}\right), \tag{12.20}$$

where $c_2 > 0$ is an absolute constant. By (12.19) and (12.20), we deduce the desired upper bound for $|\mathcal{F}_s(z)|$. □

12.3. Approximations of g-functions. Let $\sigma > 1/2$, and put

$$c = \max\{1-\sigma, 0\} + \frac{1}{\log X} \quad \text{and} \quad \kappa = \frac{1}{\log_2 X}.$$

Then we define the functions $g_z^+(\sigma, \rho_K; Y)$ and $g_z^-(\sigma, \rho_K; Y)$ as

$$g_z^{\pm}(\sigma, \rho_K; Y) = \frac{1}{2\pi i}\int_{L^{\pm}} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w\, dw.$$

Here L^+ is the vertical line $\operatorname{Re} w = c$, and thus $g_z^+(\sigma, \rho_K; Y)$ is defined for any $\sigma > 1/2$ and $K \in L_3^{\pm}(X)$. On the other hand, L^- is given by $L_1 + \cdots + L_5$ as follows:

(1) L_1 is the vertical half-line from $c - i\infty$ to $c - i(\log X)^2$;
(2) L_2 is the horizontal segment from $c - i(\log X)^2$ to $-\kappa - i(\log X)^2$;
(3) L_3 is the vertical segment from $-\kappa - i(\log X)^2$ to $-\kappa + i(\log X)^2$;
(4) L_4 is the horizontal segment from $-\kappa + i(\log X)^2$ to $c + i(\log X)^2$;
(5) L_5 is the vertical half-line from $c + i(\log X)^2$ to $c + i\infty$.

The function $g_z^-(\sigma, \rho_K; Y)$ is defined for any $\sigma > \sigma_0 + (\log_2 X)^{-1}$ and $K \in L_3^{\pm}(X)\setminus\mathcal{E}$, where $\mathcal{E} = \mathcal{E}(\sigma_0, X)$ is the subset of (12.5). Then we prove several properties of the functions $g_z^{\pm}(\sigma, \rho_K; Y)$ listed as follows.

Lemma 12.10. *Suppose that* $(\log_2 X)^{-1} < \sigma - \sigma_0$ *is satisfied, and take a cubic field* K *belonging to* $L_3^{\pm}(X)\setminus\mathcal{E}$. *Then the equality*

$$g_z(\sigma, \rho_K) = g_z^+(\sigma, \rho_K; Y) - g_z^-(\sigma, \rho_K; Y)$$

holds for any $z \in \mathbb{C}$ *and* $Y \ge 1$.

Proof. We have $\operatorname{Re}(\sigma + w) > 1$ for $\operatorname{Re} w = c$ by the choice of c. Hence formula (12.13) is available to investigate the function $g_z(\sigma + w, \rho_K)$ on the line L^+, and we deduce that $|g_z(\sigma + w, \rho_K)|$ is bounded as $|\operatorname{Im} w| \to \infty$. Furthermore, we see that $|\Gamma(w)|$ is rapidly decreasing as $|\operatorname{Im} w| \to \infty$ by the estimate

$$\Gamma(w) \ll |v|^{u-1/2}\exp\left(-\frac{\pi}{2}|v|\right), \tag{12.21}$$

which is deduced from Stirling's formula (Theorem B.2.1). Therefore the function $g_z(\sigma + w, \rho_K)\Gamma(w)Y^w$ is absolutely integrable on the contour L^+. Then, we shift the contour to L^-. Remark that we have $\sigma - \kappa > \sigma_0$, and that the function $g_z(s, \rho_K)$ is holomorphic in the region $\operatorname{Re} s > \sigma_0$, $|\operatorname{Im} s| < (\log X)^3$ for $K \in L_3^{\pm}(X)\setminus\mathcal{E}$. Hence we do not come across any poles of the integrand except for the simple pole at $w = 0$ while shifting the contour. The residue at $w = 0$ is equal to $g_z(\sigma, \rho_K)$. Therefore, we obtain

$$\begin{aligned} &\frac{1}{2\pi i}\int_{L^+} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w\, dw \\ &= \frac{1}{2\pi i}\int_{L^-} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w\, dw + g_z(\sigma, \rho_K) \end{aligned}$$

as desired. ☐

Lemma 12.11. *Let $\sigma > 1/2$ be a real number, and take $K \in L_3^{\pm}(X)$ arbitrarily. Then the function $g_z^+(\sigma, \rho_K; Y)$ is represented as*

$$g_z^+(\sigma, \rho_K; Y) = \sum_{n=1}^{\infty} d_{iz/2}(n, \rho_K) n^{-\sigma} e^{-n/Y}$$

for any $z \in \mathbb{C}$ and $Y \geq 1$. If we let $Y = X^{\eta}$ with some $\eta > 0$, then we have

$$g_z^+(\sigma, \rho_K; Y) \ll X^{\eta} \tag{12.22}$$

for any $z \in \mathbb{C}$ with $|z| \leq R_\sigma(X)$, where $R_\sigma(X)$ is defined as (12.3). The implied constant in (12.22) depends only on η.

Proof. Recall that we have $\mathrm{Re}(\sigma + w) > 1$ on the vertical line $\mathrm{Re}\, w = c$. Hence Dirichlet series representation (12.13) yields the equality

$$\int_{\mathrm{Re}\, w = c} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w\, dw = \sum_{n=1}^{\infty} d_{iz/2}(n, \rho_K) n^{-\sigma} e^{-n/Y}.$$

In order to prove (12.22), we apply the inequality $|d_z(n)| \leq d_r(n)$ with $r = \lfloor |z| \rfloor + 1$. It yields

$$\left|g_z^+(\sigma, \rho_K; Y)\right| \leq \sum_{n=1}^{\infty} d_r(n) n^{-\sigma} e^{-n/Y} \ll \left(Y^{1-\sigma} + 1\right) (C \log Y)^{2(|z|+1)}. \tag{12.23}$$

By $Y = X^{\eta}$ and $|z| \leq R_\sigma(X)$, the desired result follows. ☐

Lemma 12.12. *Suppose that $3(\log_2 X)^{-1} \leq \sigma - \sigma_0$ is satisfied, and take a cubic field K belonging to $L_3^{\pm}(X) \setminus \mathcal{E}$. Let $Y = X^{\eta}$ with some $\eta > 0$. Then there exists a constant $b(\eta) > 0$ depending only on η such that*

$$g_z^-(\sigma, \rho_K; Y) \ll \exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right)$$

for any $z \in \mathbb{C}$ with $|z| \leq b(\eta) R_\sigma(X)$, where $R_\sigma(X)$ is defined as (12.3). The implied constant depends only on η.

Proof. We divide the integral contour L^- into $L_1, L_2, \dots, L_5$ as above. Then we have $\mathrm{Re}(\sigma + w) \geq 1 + (\log X)^{-1}$ on L_1 and L_5. Hence (12.13) is available to obtain the estimate

$$\log L(\sigma + w, \rho_K) \ll \log_2 X$$

for $w \in L_1 \cup L_5$. Next, if w lies on L_2, L_3, or L_4, then we have $\mathrm{Re}(\sigma + w) \geq \sigma_1 + 2\kappa$. By Lemma 12.5, the upper bound

$$g_z(\sigma + w, \rho_K) \ll \exp\left(Ab(\eta)\frac{\log X}{\log_2 X}\right)$$

is valid for any $w \in L^-$, where $A > 0$ is an absolute constant. Furthermore, the function Y^w satisfies

$$Y^w \ll \begin{cases} X^\eta & \text{if } w \text{ lies on } L_1, L_2, L_4, \text{ or } L_5, \\ \exp\left(-\eta \dfrac{\log X}{\log_2 X}\right) & \text{if } w \text{ lies on } L_3. \end{cases}$$

Finally, the integrals of $\Gamma(w)$ are estimated by (12.21). From the above, we obtain

$$\begin{aligned} \int_{L_1 \cup L_5} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w \, dw &\ll \exp\left(-(\log X)^2\right), \\ \int_{L_2 \cup L_4} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w \, dw &\ll \exp\left(-(\log X)^2\right), \\ \int_{L_3} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w \, dw &\ll \exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right) \end{aligned}$$

if the constant $b(\eta) > 0$ is small enough, where the implied constants depend only on η. □

Combining the above lemmas, we derive the following asymptotic formula for the function $g_z(\sigma, \rho_K)$.

COROLLARY 12.13. *Take a cubic field K belonging to $L_3^\pm(X) \setminus \mathcal{E}$. Let $Y = X^\eta$ with some $\eta > 0$. Then there exists a constant $0 < b(\eta) \leq 1$ depending only on η such that*

$$g_z(\sigma, \rho_K) = \sum_{n=1}^{\infty} \lambda_{iz/2}(n, \rho_K) n^{-\sigma} e^{-n/Y} + O\left(\exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right)\right)$$

for any $z \in \mathbb{C}$ with $|z| \leq b(\eta)R_\sigma(X)$, where $R_\sigma(X)$ is defined as (12.3). The implied constant depends on σ and η.

12.4. Completion of the proofs. We define arithmetic functions $\lambda_z(n)$, $\mu_z(n)$, and $\nu_z(n)$ with $z \in \mathbb{C}$ as the multiplicative functions satisfying

$$\begin{aligned} \lambda_z(p^k) &= \sum_{\mathfrak{a} \in \mathscr{A}} C_p(\mathfrak{a}) H_k(iz/2; A_\mathfrak{a}), \\ \mu_z(p^k) &= \sum_{\mathfrak{a} \in \mathscr{A}} K_p(\mathfrak{a}) H_k(iz/2; A_\mathfrak{a}), \\ \nu_z(p^k) &= \sum_{\mathfrak{a} \in \mathscr{A}} |H_k(iz/2; A_\mathfrak{a})| . \end{aligned}$$

Then we have $\lambda_z(n) = \mathbb{E}[d_{iz/2}(n, \mathcal{X})]$. The following lemma plays a key role in the proof of Proposition 12.3.

Lemma 12.14. *Suppose that there exist absolute constants $\alpha, \beta > 0$ such that the estimate $E^{\pm}(X, S) \ll_{\epsilon} X^{\alpha+\epsilon} \prod_{p \in \mathrm{supp}(S)} p^{\beta}$ holds for all local conditions $S = (S_p)_p$. Then, for every $\epsilon > 0$, we have*

$$\begin{aligned}&\sum_{K \in L_3^{\pm}(X)} d_{iz/2}(n, \rho_K)\\ &\quad = C^{\pm}\frac{1}{12\zeta(3)}X\lambda_z(n) + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6}\mu_z(n) + O\left(X^{\alpha+\epsilon}\nu_z(n)n^{\beta}\right),\end{aligned}$$

where the implied constant depends only on ϵ.

Proof. If $n = 1$, then the result is directly deduced from the assumption with $\mathrm{supp}(\mathcal{S}) = \emptyset$. Thus we consider the case where $n = p_1^{k_1} \cdots p_r^{k_r} > 1$. For $\mathfrak{A} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \in \mathscr{A}^r$, we denote by $\mathcal{S}(\mathfrak{A})$ the collection of all local conditions such that $\mathrm{supp}(\mathcal{S}(\mathfrak{A})) = \{p_1, \ldots, p_r\}$ and $\mathcal{S}(\mathfrak{A})_{p_j} = \mathfrak{a}_j$ for each j. If we suppose that K satisfies local conditions $\mathcal{S}(\mathfrak{A})$, then $d_{iz/2}(n, \rho_K)$ is calculated as

$$\lambda_{iz/2}(n, \rho_K) = H_{k_1}(iz/2; A_{\mathfrak{a}_1}) \cdots H_{k_r}(iz/2; A_{\mathfrak{a}_r}).$$

Hence we obtain

$$\begin{aligned}\sum_{K \in L_3^{\pm}(X)} d_{iz/2}(n, \rho_K) &= \sum_{\mathfrak{A} \in \mathscr{A}^r} \sum_{K \in L_3^{\pm}(X, \mathcal{S}(\mathfrak{A}))} d_{iz/2}(n, \rho_K)\\ &= \sum_{\mathfrak{A} \in \mathscr{A}^r} \#L_3^{\pm}(X, \mathcal{S}(\mathfrak{A})) \prod_{j=1}^{r} H_{k_j}(iz/2; A_{\mathfrak{a}_j}).\end{aligned} \tag{12.24}$$

Then we use $E^{\pm}(X, S) \ll_{\epsilon} X^{\alpha+\epsilon} \prod_{p \in \mathrm{supp}(S)} p^{\beta}$ to derive

$$\begin{aligned}\#L_3^{\pm}(X, \mathcal{S}(\mathfrak{A})) = C^{\pm}\frac{1}{12\zeta(3)}X\prod_{j=1}^{r} C_{p_j}(\mathfrak{a}_j) + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6}\prod_{j=1}^{r} K_{p_j}(\mathfrak{a}_j)\\ + O_{\epsilon}\left(X^{\alpha+\epsilon}\prod_{j=1}^{r} p_j^{\beta}\right).\end{aligned}$$

We insert this formula to (12.24). Note that the equality

$$\sum_{\mathfrak{A} \in \mathscr{A}^r} \prod_{j=1}^{r} C_{p_j}(\mathfrak{a}_j) H_{k_j}(z; A_{\mathfrak{a}_j}) = \lambda_z(n)$$

holds. Furthermore, similar equalities are obtained for $\mu_z(n)$ and $\nu_z(n)$. Hence we obtain the desired result. □

Put $Y = X^{\eta}$ with some $\eta > 0$. By Corollary 12.13, the left-hand side of (12.2) is calculated as

$$\frac{1}{\#L_3^{\pm}(X)} \sum_{K \in L_3^{\pm}(X) \setminus \mathcal{E}}{}' g_z(\sigma, \rho_K) = \frac{T_1 - T_2}{\#L_3^{\pm}(X)} + O\left(\exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right)\right) \tag{12.25}$$

for any $z \in \mathbb{C}$ with $|z| \leq b(\eta)R_{\sigma}(X)$, where

$$T_1 = \sum_{K \in L_3^{\pm}(X)} \sum_{n=1}^{\infty} d_{iz/2}(n, \rho_K) n^{-\sigma} e^{-n/Y},$$

$$T_2 = \sum_{K\in\mathcal{E}}{}' \sum_{n=1}^{\infty} d_{iz/2}(n,\rho_K)n^{-\sigma}e^{-n/Y}.$$

By zero density estimate (12.1), we have $\#\mathcal{E} \ll X^{1-\delta}$ with some $\delta > 0$. Hence the term T_2 is estimated as

$$T_2 \ll X^{1-\delta} \max_{K\in L_3^{\pm}(X)} |g_z^{+}(\sigma,\rho_K;Y)|. \tag{12.26}$$

The main term is obtained from T_1, which is calculated by Lemma 12.14 as follows.

Proposition 12.15. *Let $\sigma > 1/2$ be a real number, and denote by $\mathcal{F}_\sigma(z)$ the function defined as* (12.14). *If we let $Y = X^\eta$ with some $\eta > 0$, then there exists a constant $0 < b_\sigma(\eta) \le 1$ depending on σ and η such that*

$$T_1 = \frac{C^{\pm}X}{12\zeta(3)}\mathcal{F}_\sigma(z) + O\left(X\exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right) + X^{\frac{5}{6}+\eta} + X^{\frac{7}{9}+\frac{25}{9}\eta+\epsilon}\right)$$

for $z \in \mathbb{C}$ with $|z| \le b_\sigma(\eta)R_\sigma(X)$, where $R_\sigma(X)$ is defined as (12.3). *Here, the implied constant depends on σ and η.*

Proof. By Lemma 12.14, the term T_1 is calculated as

$$\begin{aligned} T_1 = {} & C^{\pm}\frac{1}{12\zeta(3)}X\left(\sum_{n=1}^{\infty}\lambda_z(n)n^{-\sigma}e^{-n/Y}\right) \\ & + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6}\left(\sum_{n=1}^{\infty}\mu_z(n)n^{-\sigma}e^{-n/Y}\right) \\ & + O_\epsilon\left(X^{\alpha+\epsilon}\sum_{n=1}^{\infty}\nu_z(n)n^{-\sigma+\beta}e^{-n/Y}\right). \end{aligned} \tag{12.27}$$

Note that we have

$$\sum_{k=0}^{\infty}\lambda_z(p^k)p^{-ks} = \sum_{\mathfrak{a}\in\mathscr{A}} C_p(\mathfrak{a})\sum_{k=0}^{\infty}H_k(iz/2;A_{\mathfrak{a}})p^{-ks} = \mathcal{F}_{s,p}(z),$$

where $\mathcal{F}_{s,p}(z)$ is defined as (12.15). Hence the equality

$$\sum_{n=1}^{\infty}\lambda_z(n)n^{-s} = \prod_p \mathcal{F}_{s,p}(z) = \mathcal{F}_s(z)$$

holds for any $s, z \in \mathbb{C}$ with $\operatorname{Re} s > 1/2$ by (12.14). It yields

$$\sum_{n=1}^{\infty}\lambda_z(n)n^{-\sigma}e^{-n/Y} = \int_{\operatorname{Re} w=c}\mathcal{F}_{\sigma+w}(z)\Gamma(w)Y^w\,dw$$

for any $\sigma > 1/2$ and $c > 0$. The function $\mathcal{F}_{\sigma+w}(z)$ is a holomorphic function on the half plane $\operatorname{Re} w > 1/2-\sigma$ by Lemma 12.8. Hence, shifting the integral contour to left, we obtain

$$\sum_{n=1}^{\infty}\lambda_z(n)n^{-\sigma}e^{-n/Y} = \mathcal{F}_\sigma(z) + \int_{\operatorname{Re} w=-\kappa_1}\mathcal{F}_{\sigma+w}(z)\Gamma(w)Y^w\,dw$$

for any $0 < \kappa_1 < \sigma - 1/2$. We take

$$\kappa_1 = \begin{cases} 2^{-1}\min\{\sigma - 1/2, 1 - \sigma\} & \text{if } 1/2 < \sigma < 1, \\ \sigma - 1 + (\log_2 X)^{-1} & \text{if } \sigma \geq 1 \end{cases}$$

so as to keep $1/2 < \operatorname{Re}(\sigma + w) < 1$ on the vertical line $\operatorname{Re} w = -\kappa_1$. Then we apply Lemma 12.9 to derive

$$\mathcal{F}_{\sigma+w}(z) \ll \exp\left(c_1(\sigma) b_\sigma(\eta) \frac{\log X}{\log_2 X}\right)$$

for $\operatorname{Re} w = -\kappa_1$ with some constant $c_1(\sigma) > 0$ depending only on σ. Furthermore, if the constant $b_\sigma(\eta) > 0$ is small enough, then

$$\int_{\operatorname{Re} w = -\kappa_1} \mathcal{F}_{\sigma+w}(z)\Gamma(w)Y^w \, dw \ll \exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right)$$

follows with the implied constant depending on σ and η. Therefore we obtain

$$\sum_{n=1}^{\infty} \lambda_z(n) n^{-\sigma} e^{-n/Y} = \mathcal{F}_\sigma(z) + O_{\sigma,\eta}\left(\exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right)\right). \tag{12.28}$$

Next, we consider the second and third terms of the right-hand side of (12.27). Recall that the inequality $|H_k(iz/2; A)| \leq H_k(r) = d_r(p^k)$ holds for any $A \in \mathcal{A}$, where we put $r = \lfloor |z| \rfloor + 1$. Hence we find $|\mu_z(p^k)|, |\nu_z(p^k)| \leq d_r(p^k)$. Furthermore, the inequalities $|\mu_z(n)|, |\nu_z(n)| \leq d_r(n)$ follow since they are multiplicative functions. Therefore, we derive the upper bounds

$$\sum_{n=1}^{\infty} \mu_z(n) n^{-\sigma} e^{-n/Y} \ll (Y^{1-\sigma} + 1)(C \log Y)^{2(|z|+1)} \ll_\eta X^\eta, \tag{12.29}$$

$$\sum_{n=1}^{\infty} \nu_z(n) n^{-\sigma+\beta} e^{-n/Y} \ll (Y^{1-\sigma+\beta} + 1)(C \log Y)^{2(|z|+1)} \ll_\eta X^{(1+\beta)\eta} \tag{12.30}$$

similarly to (12.23). Recall that one can take at least for $\alpha = 7/9$ and $\beta = 16/9$ by (12.4). Inserting (12.28), (12.29), and (12.30) into (12.27), we finally arrive at the desired conclusion. □

Proof of Proposition 12.3. We choose a real number $\eta > 0$ small enough to keep the estimates

$$X^{\frac{5}{6}+\eta} + X^{\frac{7}{9}+\frac{25}{9}\eta+\epsilon} \ll X \exp\left(-\frac{\eta}{2}\frac{\log X}{\log_2 X}\right) \quad \text{and} \quad X^\eta \ll X^{\frac{\delta}{2}}$$

with a constant $\delta > 0$ satisfying (12.26). Then we deduce the upper bound

$$T_2 \ll X^{1-\frac{\delta}{2}}$$

from (12.22) and (12.26). Inserting the estimates of T_1 and T_2 to (12.25), we obtain

$$\frac{1}{\# L_3^{\pm}(X)} \sideset{}{'}\sum_{K \in L_3^{\pm}(X) \setminus \mathcal{E}} \exp\left(\frac{iz}{2} L(\sigma, \rho_K)\right) = \mathcal{F}_\sigma(z) + O_{\sigma,\eta}\left(\exp\left(-\frac{\eta}{2}\frac{\log X}{\log\log X}\right)\right).$$

Recalling equality (12.16), we complete the proof. □

Proof of Theorem 12.1. The result in the case $\Phi \in C_b(\mathbb{R}) \cup \mathcal{I}(\mathbb{R})$ follows from Proposition 10.5. Then we prove the limit formula

$$\lim_{X\to\infty} \frac{1}{\# L_3^\pm(X)} \sideset{}{'}\sum_{K\in\mathcal{E}} \Phi(\log L(\sigma, \rho_K)) = 0 \tag{12.31}$$

in the following cases:

(a′) $\sigma > 1$ and $\Phi \in C(\mathbb{R})$;
(b′) $\sigma = 1$ and $\Phi \in C^{\exp}(\mathbb{R})$;
(c′) $\sigma_0 < \sigma < 1$ and $\Phi \in C^{\exp}(\mathbb{R})$ assuming GRH.

If $\sigma > 1$ is a fixed real number, then $\Phi(\log L(\sigma, \rho_K))$ is bounded for any $\Phi \in C(\mathbb{R})$ as $K \in L_3^\pm(X)$ varies. Hence the result in case (a′) follows from the upper bound $\mathcal{E} \ll X^{1-\delta}$. Let $\sigma = 1$ and $\Phi(u) \ll e^{c|u|}$. Then Lemma 12.7 yields

$$\Phi(\log L(1, \rho_K)) \ll \exp\left(Mc \log_2 X\right)$$

for any $K \in L_3^\pm(X)$, where $M > 0$ is an absolute constant. Hence we have

$$\sideset{}{'}\sum_{K\in\mathcal{E}} \Phi(\log L(\sigma, \rho_K)) \ll X^{1-\frac{\delta}{2}} \tag{12.32}$$

from which we deduce (12.31) in case (b′). Similarly, we further obtain the result in case (c′) by using Lemma 12.6. □

Proof of Theorem 12.2. The result directly follows from Proposition 10.6. □

Finally, we consider further applications of Proposition 12.3 obtained by the class number formula (3.4).

Corollary 12.16. *Let $r > -2$ be a fixed real number. Denote by h_K and R_K the class number and the regulator of a cubic field K. Then we obtain*

$$\sum_{K\in L_3^\pm(X)} (h_K R_K)^r = \frac{C^\pm \mathcal{F}_1(r)}{12\zeta(3)(D^\pm)^r} \cdot \frac{X^{\frac{r}{2}+1}}{\frac{r}{2}+1} + O\left(X^{\frac{r}{2}+1} \exp\left(-b\frac{\log X}{\log_2 X}\right)\right)$$

with an absolute implied constant.

Proof. By (12.2) and (12.32), we derive

$$\frac{1}{\# L_3^\pm(X)} \sum_{K\in L_3^\pm(X)} L(1, \rho_K)^r = \mathcal{F}_1(r) + O\left(\exp\left(-b\frac{\log X}{\log_2 X}\right)\right)$$

if we choose $z = -2ir$ with $r \in \mathbb{R}$. Using formula (3.4), we deduce

$$\sum_{K\in L_3^\pm(X)} \left(\frac{h_K R_K}{\sqrt{|d_K|}}\right)^r = \frac{C^\pm \mathcal{F}_1(r)}{12\zeta(3)(D^\pm)^r} X + O_r\left(X \exp\left(-b\frac{\log X}{\log_2 X}\right)\right). \tag{12.33}$$

For an integer $d > 0$, we put

$$f(d) = \sum_{\substack{K\in L_3^\pm(X)\\ |d_K|=d}} \left(\frac{h_K R_K}{\sqrt{|d_K|}}\right)^r.$$

Then, by the partial summation, we obtain

$$\sum_{K\in L_3^\pm(X)}(h_KR_K)^r=\sum_{0<d\le X}f(d)d^{r/2} \tag{12.34}$$

$$=F(X)X^{r/2}-\frac{r}{2}\int_1^X F(x)x^{r/2-1}\,dx$$

with $F(x)=\sum_{0<d\le x}f(d)$. Note that the function F is estimated as

$$F(x)=\frac{C^\pm\mathcal{F}_1(r)}{12\zeta(3)(D^\pm)^r}x+O_r\left(x\exp\left(-\delta\frac{\log x}{\log\log x}\right)\right)$$

by (12.33). Inserting it to (12.34), we obtain the result. □

Especially, we obtain formula (3.5) by putting $r=1$ and by calculating the constant $\mathcal{F}_1(1)=\prod_p\mathcal{F}_{1,p}(1)$ as

$$\begin{aligned}\mathcal{F}_{1,p}(1)=&\frac{p^2}{p^2+p+1}\left\{\frac{1}{6}(1-p^{-1})^{-2}+\frac{1}{2}(1-p^{-2})^{-1}\right.\\&\left.+\frac{1}{3}(1+p^{-1}+p^{-2})^{-1}+\frac{1}{p}(1-p^{-1})^{-1}+\frac{1}{p^2}\right\}\\=&(1-p^{-3})^{-2}(1-p^{-2})^{-1}(1+p^{-2}-2p^{-3}-2p^{-4}+2p^{-6}+p^{-7}-p^{-8}).\end{aligned}$$

13. The third result: automorphic L-functions

Let $\{L_f(s)\}_{f\in\mathcal{F}}$ be the family of Example 7.5, and let $\mathcal{X}=(\mathcal{X}_p)_p$ be the sequence of Example 8.5. Denote by $a(n,f)$ the Dirichlet coefficient in (7.4). Recall that there exists a real number $\theta_f(p)\in[0,\pi]$ such that

$$a(p^k,f)=U_k(\cos\theta_f(p)) \tag{13.1}$$

if $p\neq q$, where $U_k(T)$ is the k-th Chebyshev polynomial of the second kind. Furthermore, the random variable $\mathcal{X}_p$ is given by $\mathcal{X}_p=\operatorname{diag}(e^{i\Theta_p},e^{-i\Theta_p})$ with some $[0,\pi]$-valued random variable Θ_p. We define a random variable $a(n,\mathcal{X})$ by extending

$$a(p^k,\mathcal{X})=U_k(\cos\Theta_p) \tag{13.2}$$

multiplicatively in n. Then the random variable $L(s,\mathcal{X})$ is represented as

$$L(s,\mathcal{X})=\sum_{n=1}^\infty a(n,\mathcal{X})n^{-s}=\prod_p\left(1-a(p,\mathcal{X})p^{-s}+p^{-2s}\right)^{-1}.$$

Note that the expected value $\mathbb{E}[a(p^k,\mathcal{X})]$ is calculated as

$$\mathbb{E}[a(p^k,\mathcal{X})]=\int_0^\pi U_k(\cos\theta)\,d\mu_p(\theta)=\begin{cases}1 & \text{if } k \text{ is even,}\\ 0 & \text{if } k \text{ is odd.}\end{cases} \tag{13.3}$$

Furthermore, we obtain $\mathbb{E}[(\operatorname{tr}\mathcal{X}_p)^2]=1+p^{-1}\geq 1$. Therefore the sequence $\mathcal{X}$ is admissible. Let $s=\sigma+it$ be a complex number with $\sigma>1/2$. The purpose of this section is to prove the following results. Especially, Theorem IV is contained in Theorem 13.1.

Theorem 13.1. *Let $s = \sigma + it$ be a complex number with $\sigma > 1/2$.*

(i) *If $t \neq 0$, then the limit formula*

$$\lim_{\substack{q \to \infty \\ q\colon \text{prime}}} \frac{1}{\# B_2(q)} {\sum_{f \in B_2(q)}}' \Phi\left(\log L_f(s)\right) = \int_{\mathbb{C}} \Phi(w) \mathcal{M}_s(w; \mathcal{X})\, |dw|$$

holds in the following cases:

(a) $\sigma > 1$ *and* $\Phi \in C(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$*;*

(b) $\sigma = 1$ *and* $\Phi \in C^{\mathrm{poly}}(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$*;*

(c) $1/2 < \sigma < 1$ *and* $\Phi \in C_b(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$*;*

(c′) $1/2 < \sigma < 1$ *and* $\Phi \in C^{\exp}(\mathbb{C})$ *assuming GRH for* $L_f(s)$ *for all* $f \in \mathscr{F}$.

(ii) *If $t = 0$, then the limit formula*

$$\lim_{\substack{q \to \infty \\ q\colon \text{prime}}} \frac{1}{\# B_2(q)} {\sum_{f \in B_2(q)}}' \Phi\left(\log L_f(\sigma)\right) = \int_{\mathbb{R}} \Phi(w) \mathcal{M}_\sigma(u; \mathcal{X})\, |du|$$

holds in the following cases:

(a) $\sigma > 1$ *and* $\Phi \in C(\mathbb{R}) \cup \mathcal{I}(\mathbb{R})$*;*

(b) $\sigma = 1$ *and* $\Phi \in C^{\exp}(\mathbb{R}) \cup \mathcal{I}(\mathbb{R})$*;*

(c) $1/2 < \sigma < 1$ *and* $\Phi \in C_b(\mathbb{R}) \cup \mathcal{I}(\mathbb{R})$*;*

(c′) $1/2 < \sigma < 1$ *and* $\Phi \in C^{\exp}(\mathbb{R})$ *assuming GRH for* $L_f(s)$ *for all* $f \in \mathscr{F}$.

Theorem 13.2. *Let $s = \sigma + it$ be a complex number with $\sigma > 1/2$. Then we obtain the discrepancy bounds*

$$D_s(q; \mathcal{L}) \ll \begin{cases} (\log q)^{-1}(\log_2 q) & \text{for } \sigma > 1, \\ (\log q)^{-1}(\log_2 q \log_3 q) & \text{for } \sigma = 1, \\ (\log q)^{-\sigma} & \text{for } 1/2 < \sigma < 1 \end{cases}$$

in both cases $s \in \mathbb{R}$ and $s \notin \mathbb{R}$, where the implied constants depend on σ.

Similarly to the arguments in Sections 11 and 12, we deduce these theorems from the estimates on the moment-generating functions $\widetilde{\mathcal{M}}_{s,q}(z, z'; \mathcal{L})^{\mathcal{E}}$ and $\widetilde{\mathcal{M}}_s(z, z'; \mathcal{X})$ defined by (10.18) and (9.9), respectively.

Proposition 13.3. *With the notation above, Assumption 10.4 is satisfied with the following triple $(\sigma_0, \mathcal{E}, R_\sigma)$:*

- $\sigma_0 = 1/2$*;*
- *the exceptional subset $\mathcal{E} = \mathcal{E}_{\mathcal{L}}(s, B, q)$ presented by (13.7) below;*
- *the function $R_\sigma(q)$ defined as*

$$R_\sigma(q) = \begin{cases} (\log q)(\log_2 q)^{-1} & \text{if } \sigma > 1, \\ (\log q)(\log_2 q \log_3 q)^{-1} & \text{if } \sigma = 1, \\ (\log q)^{\sigma} & \text{if } 1/2 < \sigma < 1. \end{cases} \tag{13.4}$$

The method for proof of Proposition 13.3 is close to the method in Section 11 rather than that in Section 12. We define

$$R_y(s,f)=\sum_{p^k\le y}\frac{b(p^k,f)}{p^{ks}}\quad\text{and}\quad R_y(s,\mathcal{X})=\sum_{p^k\le y}\frac{b(p^k,\mathcal{X})}{p^{ks}}$$

for $y<q$, where the coefficients are given by

$$b(p^k,f)=\frac{2\cos(k\theta_p(f))}{k}\quad\text{and}\quad b(p^k,\mathcal{X})=\frac{2\cos(k\Theta_p)}{k}. \tag{13.5}$$

As approximations of $\widetilde{\mathcal{M}}_{s,q}(z,z';\mathcal{L})^{\mathcal{E}}$ and $\widetilde{\mathcal{M}}_s(z,z';\mathcal{X})$, we further define

$$\widetilde{\mathcal{M}}_{s,q}^{(y)}(z,z';\mathcal{L})^{\mathcal{E}}=\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{E}}\psi_{z,z'}(R_y(s,f)),$$

$$\widetilde{\mathcal{M}}_s^{(y)}(z,z';\mathcal{X})=\mathbb{E}\left[\psi_{z,z'}(R_y(s,\mathcal{X}))\right].$$

13.1. Exceptional subsets. Let $s=\sigma+it$ with $\sigma>1/2$, and put $y=(\log q)^B$ with $B\ge 1$. Then we define the subset $\mathcal{A}(s,B,q)\subset B_2(q)$ as

$$\mathcal{A}(s,B,q)=\left\{f\in B_2(q)\;\middle|\;|\log L(s,f)-R_y(s,f)|>y^{-\frac{1}{2}(\sigma-\frac{1}{2})}(\log q)^2\right\}.$$

Furthermore, we define $\mathcal{B}(s,B,q)=\emptyset$ if $\sigma\ge 1$ and

$$\mathcal{B}(s,B,q)=\left\{f\in B_2(q)\;\middle|\;|R_y(s,f)|>(\log q)^{1-\sigma}(\log_2 q)^{-1}\right\} \tag{13.6}$$

if $1/2<\sigma<1$. The exceptional subset $\mathcal{E}$ of Proposition 13.3 is defined as

$$\mathcal{E}=\mathcal{A}(s,B,q)\cup\mathcal{B}(s,B,q). \tag{13.7}$$

Then we prove that condition (10.19) is satisfied. For this, we introduce a large sieve type inequality.

LEMMA 13.4. *Let $s=\sigma+it$ with $\sigma>1/2$. Let $1\ll y\le z$ be large real numbers. Then we have*

$$\sum_{f\in B_2(q)}\frac{1}{\langle f,f\rangle}\left|\sum_{y\le p\le z}\frac{\lambda_f(p)}{p^s}\right|^{2k}\ll 2^{2k}\frac{(2k)!}{k!}\left(\sum_{y\le p\le z}\frac{1}{p^{2\sigma}}\right)^k+\frac{\log q}{\sqrt{q}}$$

for $k\in\mathbb{Z}$ with $1\le k\le(\log q)(2\log z)^{-1}$. The implied constant is absolute.

PROOF. Lamzouri [**92**, Lemma 6.5] proved the same result when s is a real number. We can show Lemma 13.4 in a similar way by noting that $\lambda_f(p)$ is always real. □

LEMMA 13.5. *Let $s=\sigma+it$ with $\sigma>1/2$. For $y=(\log q)^B$ with $B\ge 1$, there exists a positive constant $b=b(\sigma,B)$ such that*

$$\frac{\#\mathcal{A}(s,B,q)}{q}\ll\exp\left(-b\frac{\log q}{\log_2 q}\right)$$

with the implied constant depending on s and B.

Proof. By the asymptotic formula [27, Lemma 4.4], we have

$$\log L_f(s) = R_y(s,f) + O_\sigma\left(y^{-\frac{1}{2}(\sigma-\frac{1}{2})}(\log q)\right)$$

for $2q \geq |t|$ if there exist no zeros of $L_f(s)$ inside the rectangle

$$A_s(y) = \{z \in \mathbb{C} \mid \sigma_1 \leq \operatorname{Re} z \leq 1,\ |\operatorname{Im} z - t| \leq y+3\},$$

where $\sigma_1 = \frac{1}{2}(\sigma + \frac{1}{2})$. In other words, the condition $f \in \mathcal{A}(s,B,q)$ implies the existence of zeros of $L_f(s)$ inside the rectangle $A_s(y)$. Thus we have

$$\#\mathcal{A}(s,B,q) \leq \sum_{f \in B_2(q)} N(f;\sigma_0, t-y-3, t+y+3),$$

where $N(f;\alpha,t_1,t_2)$ counts the number of zeros ρ of $L_f(s)$ such that $\operatorname{Re}\rho \geq \alpha$ and $t_1 \leq \operatorname{Im}\rho \leq t_2$. Furthermore, we apply the zero density estimate of Kowalski–Michel [86, Theorem 4] to derive

$$\sum_{f \in B_2(q)} N(f;\sigma_0, t-y-3, t+y+3) \ll q^{1-\frac{1}{10}(\sigma-\frac{1}{2})}(\log q)^{K(B)}$$

for $q \geq e^{|t|}$ with a constant $K(B) > 0$. Hence we obtain the conclusion. □

Lemma 13.6. *Let $s = \sigma + it$ with $1/2 < \sigma < 1$. For $y = (\log q)^B$ with $B \geq 1$, there exists a positive constant $b = b(\sigma, B)$ such that*

$$\frac{\#\mathcal{B}(s,B,q)}{q} \ll \exp\left(-b\frac{\log q}{\log_2 q}\right),$$

where the implied constant depends on s and B.

Proof. The proof is based on a similar method in the proof of Lemma 11.6. Use Lemma 13.4 instead of (11.12). □

Recall the asymptotic formula

$$\#B_2(q) = \dim S_2(q) = \frac{q}{12} + O(1).$$

Then we deduce from Lemmas 13.5 and 13.6 the estimate

$$\begin{aligned}\frac{\#\mathcal{E}}{\#B_2(q)} &\ll \frac{\#\mathcal{A}(s,B,q) + \#\mathcal{B}(s,B,q)}{q}\\ &\ll \exp\left(-b\frac{\log q}{\log_2 q}\right)\end{aligned}$$

if we take $y = (\log q)^B$ with $B \geq 1$. Hence condition (10.19) is satisfied.

13.2. Results on Dirichlet coefficients. To begin with, we show the relations among the Dirichlet coefficients $a(p^k, f)$, $b(p^k, f)$, $a(p^k, \mathcal{X})$, and $b(p^k, \mathcal{X})$ defined as (13.1), (13.2), and (13.5).

Lemma 13.7. *We have*

$$b(p^k,f)=\sum_{j=0}^{k}c_k(j)a(p^j,f)\quad\textit{and}\quad b(p^k,\mathcal{X})=\sum_{j=0}^{k}c_k(j)a(p^j,\mathcal{X})$$

for all prime numbers $p\neq q$ *and* $k\geq 1$, *where*

$$c_k(j)=\begin{cases}1/k & \text{if } j=k,\\ -1/k & \text{if } j=k-2,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Recall that $\{U_j(\cos\theta)\}_{j\geq 0}$ is an orthonormal basis of $L^2([0,\pi])$ with respect to the Sato–Tate measure μ_∞. Hence we obtain

$$\frac{2\cos(k\theta)}{k}=\sum_{j=0}^{\infty}c_k(j)U_j(\cos\theta) \tag{13.8}$$

for any $\theta\in[0,\pi]$, where the coefficient $c_k(j)$ is determined by

$$\begin{aligned}c_k(j)&=\int_0^\pi\frac{2\cos(k\phi)}{k}U_j(\cos\phi)\,d\mu_\infty(\phi)\\&=\frac{4}{k\pi}\int_0^\pi\cos(k\phi)\sin((j+1)\phi)\sin\phi\,d\phi.\end{aligned}$$

The integral vanishes except for $j=k$ or $k-2$. We have also

$$c_k(k)=\frac{1}{k}\quad\text{and}\quad c_k(k-2)=-\frac{1}{k}.$$

Hence, putting $\theta=\theta_p(f)$ in formula (13.8), we obtain the former statement by the definitions of $a(p^k,f)$ and $b(p^k,f)$. Similarly, the latter one is proved by putting $\theta=\Theta_p$ in (13.8). □

Lemma 13.8. *There exist absolute constants* $\alpha,\beta>0$ *such that*

$$\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)}a(n,f)=\mathbb{E}\left[a(n,\mathcal{X})\right]+O\left(n^\alpha q^{-\beta}\right) \tag{13.9}$$

holds for $(n,q)=1$.

Proof. By equality (13.3), the expected value $\mathbb{E}\left[a(n,\mathcal{X})\right]$ equals to 1 if n is a perfect square, and to 0 otherwise. Hence the result is deduced from the Eichler–Selberg trace formula; see [**20**, Proposition 2.8]. □

Rudnick–Soundararajan [**162**] introduced a ring $\mathcal{H}$ generalized over the integers by symbols $x(1),x(2),\ldots$ with the Hecke relations

$$x(1)=1\quad\text{and}\quad x(m)x(n)=\sum_{d|(m,n)}x\left(\frac{mn}{d^2}\right).$$

For any $n\geq 1$, we regard $a(n,f)$ as a member of $\mathcal{H}$ and $a(n,\mathcal{X})$ as an $\mathcal{H}$-valued random variable since they satisfy the above Hecke relations. Several properties on the ring $\mathcal{H}$ are seen in [**92**, **162**]. In particular, it holds that

$$x(n_1)\cdots x(n_r)=\sum_{n\mid\prod_{m=1}^{r}n_m}b_n(n_1,\dots,n_r)x(n) \tag{13.10}$$

with a non-negative integer $b_n(n_1,\dots,n_r)$. We have the upper bound

$$b_n(n_1,\dots,n_r)\leq d(n_1)\cdots d(n_r) \tag{13.11}$$

for all $n,n_1,\cdots,n_r\geq 1$.

Lemma 13.9. *Let q be a large prime number. Let $p_1,\dots,p_r$ be prime numbers with $p_j\neq q$ for all j, and let $m_1,\dots,m_r\geq 1$. For each $\epsilon>0$, we obtain*

$$\begin{aligned}&\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)}b(p_1^{k_1},f)\cdots b(p_r^{k_r},f)\\&=\mathbb{E}[b(p_1^{k_1},\mathcal{X})\cdots b(p_r^{k_r},\mathcal{X})]+O\left((p_1^{k_1}\cdots p_r^{k_r})^{\alpha+\epsilon}q^{-\beta}\right),\end{aligned}$$

where α and β are positive absolute constants of (13.9), and the implied constant depends only on ϵ.

Proof. By Lemma 13.7 and formula (13.10), we have

$$\begin{aligned}&b(p_1^{k_1},f)\cdots b(p_r^{k_r},f)\\&=\sum_{j_1=0}^{k_1}\cdots\sum_{j_r=0}^{k_r}c_{k_1}(j_1)\cdots c_{k_r}(j_r)\sum_{n\mid\prod_{m=1}^{r}p_m^{j_m}}b_n(p_1^{j_1},\dots,p_r^{j_r})a(n,f).\end{aligned}$$

Then it is deduced from (13.9) that

$$\begin{aligned}&\sum_{f\in B_2(q)}b(p_1^{k_1},f)\cdots b(p_r^{k_r},f)\\&=\sum_{j_1=0}^{k_1}\cdots\sum_{j_r=0}^{k_r}c_{k_1}(j_1)\cdots c_{k_r}(j_r)\sum_{n\mid\prod_{m=1}^{r}p_m^{j_m}}b_n(p_1^{j_1},\dots,p_r^{j_r})\mathbb{E}[a(n,\mathcal{X})]+E,\end{aligned}$$

where the error term E is estimated as

$$\begin{aligned}E&\ll\sum_{j_1=0}^{k_1}\cdots\sum_{j_r=0}^{k_r}|c_{k_1}(j_1)|\cdots|c_{k_r}(j_r)|\sum_{n\mid\prod_{m=1}^{r}p_m^{j_m}}b_n(p_1^{j_1},\dots,p_r^{j_r})n^{\alpha}q^{-\beta}\\&\ll d(p_1^{k_1})^3\cdots d(p_r^{k_r})^3(p_1^{k_1}\cdots p_r^{k_r})^{\alpha}q^{-\beta}\end{aligned}$$

by using $|c_k(j)|\leq 1$ and (13.11). By an analogous argument, we obtain

$$\begin{aligned}&\mathbb{E}[b(p_1^{k_1},\mathcal{X})\cdots b(p_r^{k_r},\mathcal{X})]\\&=\sum_{j_1=0}^{k_1}\cdots\sum_{j_r=0}^{k_r}c_{k_1}(j_1)\cdots c_{k_r}(j_r)\sum_{n\mid\prod_{m=1}^{r}p_m^{j_m}}b_n(p_1^{j_1},\dots,p_r^{j_r})\mathbb{E}[a(n,\mathcal{X})].\end{aligned}$$

Therefore the desired result follows from the bound $d(n)\ll_{\epsilon}n^{\epsilon}$. □

We apply Lemma 13.9 to study the integral moments of $R_y(s,f)$. The following proposition is an analogue of Proposition 11.7.

Proposition 13.10. *Let $s = \sigma + it$ with $\sigma > 1/2$. We put $L = 2\alpha/\beta$, where α and β are as in (13.9). If $y < q$, then we have*

$$\begin{aligned}&\frac{1}{\#B_2(q)} \sum_{f \in B_2(q)} \overline{R_y(s,f)}^{\,j} R_y(s,f)^{\ell} \\ &= \mathbb{E}\left[\overline{R_y(s,\mathcal{X})}^{\,j} R_y(s,\mathcal{X})^{\ell}\right] + O\left(y^{j+\ell} q^{-\beta/2}\right)\end{aligned}$$

for any non-negative integers j, ℓ with $j+\ell \leq (\log q)(L \log_2 q)^{-1}$, where the implied constant is absolute.

Proof. Since $b(p^k, f)$ is real, we obtain

$$\begin{aligned}&\overline{R_y(s,f)}^{\,j} R_y(s,f)^{\ell} \\ &= \sum_{p_1^{m_1} \leq y} \cdots \sum_{p_j^{m_j} \leq y} \sum_{q_1^{n_1} \leq y} \cdots \sum_{q_\ell^{n_\ell} \leq y} \frac{b(p_1^{m_1}, f) \cdots b(p_j^{m_j}, f)}{p_1^{m_1 \bar{s}} \cdots p_j^{m_j \bar{s}}} \frac{b(q_1^{n_1}, f) \cdots b(q_\ell^{n_\ell}, f)}{q_1^{n_1 s} \cdots q_\ell^{n_\ell s}},\end{aligned}$$

which remains valid if we replace the symbol f with $\mathcal{X}$. By Lemma 13.9, the difference is estimated as

$$\begin{aligned}&\frac{1}{\#B_2(q)} \sum_{f \in B_2(q)} \overline{R_y(s,f)}^{\,j} R_y(s,f)^{\ell} - \mathbb{E}\left[\overline{R_y(s,\mathcal{X})}^{\,j} R_y(s,\mathcal{X})^{\ell}\right] \\ &\ll_{\epsilon} y^{(\alpha+\epsilon)(j+\ell)} q^{-\beta} \sum_{p_1^{m_1} \leq y} \cdots \sum_{p_j^{m_j} \leq y} \sum_{q_1^{n_1} \leq y} \cdots \sum_{q_\ell^{n_\ell} \leq y} \frac{1}{p_1^{m_1 \sigma} \cdots p_j^{m_j \sigma} q_1^{n_1 \sigma} \cdots q_\ell^{n_\ell \sigma}} \\ &\ll y^{(\alpha+1)(j+\ell)} q^{-\beta},\end{aligned}$$

where we take $\epsilon = 1/2$. Using the assumptions $y < q$ and $j+\ell \leq (\log q)(L \log_2 q)^{-1}$, we have $y^{\alpha(j+\ell)} < q^{\beta/2}$. Hence we obtain the conclusion. □

Corollary 13.11. *Let $s = \sigma + it$ with $\sigma > 1/2$. For $y = (\log q)^B$ with $B \geq 1$, there exist positive constants $K = K(\sigma, B)$ and $L = L(\sigma, B)$ such that*

$$\frac{1}{\#B_2(q)} \sum_{f \in B_2(q)} |R_y(s,f)|^{2k} \ll \begin{cases} K^{2k} & \text{if } \sigma > 1, \\ (K \log_3 q)^{2k} & \text{if } \sigma = 1, \\ \left(K \dfrac{k^{1-\sigma}}{(\log 2k)^{\sigma}}\right)^{2k} & \text{if } 1/2 < \sigma < 1 \end{cases} \tag{13.12}$$

for any integer k such that $1 \leq k \leq (\log q)(L \log_2 q)^{-1}$. Furthermore, we have

$$\mathbb{E}\left[|R_y(s,\mathcal{X})|^{2k}\right] \ll \begin{cases} K^{2k} & \text{if } \sigma > 1, \\ (K \log_3 q)^{2k} & \text{if } \sigma = 1, \\ \left(K \dfrac{k^{1-\sigma}}{(\log 2k)^{\sigma}}\right)^{2k} & \text{if } 1/2 < \sigma < 1 \end{cases} \tag{13.13}$$

for any integer $k \geq 1$. The implied constants depend on s and B.

Proof. By Proposition 13.10, estimate (13.12) is deduced from (13.13). Thus we evaluate $\mathbb{E}\left[|R_y(s,\mathcal{X})|^{2k}\right]$ below. The proof is similar to Corollary 11.8. Since the result is elementary for $\sigma \geq 1$ due to

$$|R_y(s,\mathcal{X})| \leq \sum_{p\leq y}\sum_{m=1}^{\infty}\frac{2}{m}p^{-m\sigma} \ll \begin{cases}\log\zeta(\sigma) & \text{for } \sigma > 1,\\ \log_2 y & \text{for } \sigma = 1,\end{cases}$$

we consider the case $1/2 < \sigma < 1$. Suppose that the inequality $y \leq Ck\log(2k)$ holds with a constant $C \geq 2$. In this case, the prime number theorem yields

$$R_y(s,\mathcal{X}) = \sum_{p\leq y}\frac{a(p,\mathcal{X})}{p^s} + O(\log\zeta(2\sigma)) \ll \frac{C^{1-\sigma}k^{1-\sigma}}{(1-\sigma)(\log 2k)^{\sigma}},$$

and the result follows. Therefore we suppose the inequality $Ck\log 2k < y$. In that case, we have

$$R_y(s,\mathcal{X}) = \sum_{p<Ck\log 2k}\frac{a(p,\mathcal{X})}{p^s} + \sum_{Ck\log 2k\leq p\leq y}\frac{a(p,\mathcal{X})}{p^s} + O(\log\zeta(2\sigma)). \tag{13.14}$$

The contribution of the first sum is estimated as

$$\mathbb{E}\left[\left|\sum_{p<Ck\log 2k}\frac{a(p,\mathcal{X})}{p^s}\right|^{2k}\right] \ll \left(\frac{4C^{1-\sigma}k^{1-\sigma}}{(1-\sigma)(\log 2k)^{\sigma}}\right)^{2k}. \tag{13.15}$$

Let $1 \ll x_1 < x_2$ be large real numbers. Then, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{x_1\leq p\leq x_2}\frac{a(p,\mathcal{X})}{p^s}\right|^{2k}\right]\\
&\ll \sum_{x_1\leq p_1\leq x_2}\cdots\sum_{x_1\leq p_{2k}\leq x_2}\frac{1}{(p_1\cdots p_{2k})^{\sigma}}\left|\mathbb{E}[a(p_1,\mathcal{X})\cdots a(p_{2k},\mathcal{X})]\right|\\
&= \sum_{n=1}^{2k}\sum_{\substack{x_1\leq p_1\leq x_2 \\ \omega(p_1\cdots p_{2k})=n}}\cdots\sum_{x_1\leq p_{2k}\leq x_2}\frac{1}{(p_1\cdots p_{2k})^{\sigma}}\left|\mathbb{E}[a(p_1,\mathcal{X})\cdots a(p_{2k},\mathcal{X})]\right|\\
&= \sum_{n=1}^{2k}\sum_{\substack{x_1\leq p_1\leq x_2 \\ p_j \text{ are distinct}}}\cdots\sum_{x_1\leq p_n\leq x_2}\\
&\qquad \sum_{\substack{m_1+\cdots+m_n=2k \\ \forall j,\ m_j\geq 1}}\binom{2k}{m_1,\ldots,m_n}\frac{|\mathbb{E}[a(p_1,\mathcal{X})^{m_1}]|\cdots|\mathbb{E}[a(p_n,\mathcal{X})^{m_n}]|}{(p_1^{m_1}\cdots p_n^{m_n})^{\sigma}}.
\end{aligned}$$

Recall that the Chebyshev polynomial $U_m(T)$ is an odd polynomial of degree m if m is odd. Furthermore, we have $\mathbb{E}[U_m(\cos\Theta_p)] = 0$ for odd m by (13.3). By induction, one can show that $\mathbb{E}[a(p,\mathcal{X})^m]$ vanishes unless m is even. Hence we deduce

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{x_1\le p\le x_2}\frac{a(p,\mathcal{X})}{p^s}\right|^{2k}\right]\\
&\ll \sum_{n=1}^{k}\sum_{\substack{x_1\le p_1\le x_2 \\ p_j \text{ are distinct}}}\cdots\sum_{x_1\le p_n\le x_2}\\
&\qquad\sum_{\substack{m_1+\cdots+m_n=k \\ \forall j,\ m_j\ge 1}}\binom{2k}{2m_1,\dots,2m_n}\frac{\left|\mathbb{E}[a(p_1,\mathcal{X})^{2m_1}]\right|\cdots\left|\mathbb{E}[a(p_n,\mathcal{X})^{2m_n}]\right|}{(p_1^{m_1}\cdots p_n^{m_n})^{2\sigma}}\\
&\ll 2^{2k}\frac{(2k)!}{k!}\sum_{n=1}^{k}\sum_{\substack{x_1\le p_1\le x_2 \\ p_j \text{ are distinct}}}\cdots\sum_{x_1\le p_n\le x_2}\sum_{\substack{m_1+\cdots+m_n=k \\ \forall j,\ m_j\ge 1}}\binom{k}{m_1,\dots,m_n}\frac{1}{(p_1^{m_1}\cdots p_n^{m_n})^{2\sigma}}\\
&= 2^{2k}\frac{(2k)!}{k!}\left(\sum_{x_1\le p\le x_2}\frac{1}{p^{2\sigma}}\right)^{k}
\end{aligned}$$

by using the inequalities $|a(p,\mathcal{X})|\le 2$ and $\binom{2k}{2m_1,\dots,2m_n}\le\frac{(2k)!}{k!}\binom{k}{m_1,\dots,m_n}$. Finally, taking $x_1=Ck\log 2k$ and $x_2=y$ with $C\ge 2$ sufficiently large, we arrive at

$$\mathbb{E}\left[\left|\sum_{Ck\log 2k\le p\le y}\frac{a(p,\mathcal{X})}{p^s}\right|^{2k}\right]\ll\left(\frac{4C^{\frac{1}{2}-\sigma}k^{1-\sigma}}{\sqrt{2\sigma-1}(\log 2k)^{\sigma}}\right)^{2k}. \tag{13.16}$$

By (13.14), (13.15), and (13.16), we obtain the conclusion. □

13.3. Treatment of g-functions. Similarly to the g-function $g_z(s,\rho_K)$ studied in Section 12.2, we define

$$g_z(s,f;y)=\exp\left(\frac{iz}{2}R_y(s,f)\right)\quad\text{and}\quad g_z(s,\mathcal{X};y)=\exp\left(\frac{iz}{2}R_y(s,\mathcal{X})\right)$$

for $y<q$ and $z\in\mathbb{C}$. In this section, we use the Taylor series

$$\begin{aligned}
g_z(s,f;y)&=\sum_{k\le N}\frac{1}{k!}\left(\frac{iz}{2}\right)^{k}R_y(s,f)^k+\sum_{k>N}\frac{1}{k!}\left(\frac{iz}{2}\right)^{k}R_y(s,f)^k\\
&=g_z^{\flat}(s,f;y,N)+g_z^{\#}(s,f;y,N),
\end{aligned}$$

say. Similarly, we obtain $g_z(s,\mathcal{X};y)=g_z^{\flat}(s,\mathcal{X};y,N)+g_z^{\#}(s,\mathcal{X};y,N)$ by replacing f with $\mathcal{X}$.

LEMMA 13.12. *With the same assumptions as in Corollary 13.11, we take an integer $N=\lfloor(\log q)(L\log_2 q)^{-1}\rfloor$. Then there exist positive constants $a=a(\sigma,B)$ and $b=b(\sigma,B)$ such that*

$$\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}}|g_z^{\#}(s,f;y,N)|^2\ll\exp\left(-b\frac{\log q}{\log_2 q}\right),$$

and

$$\mathbb{E}\left[|g_z^{\#}(s,\mathcal{X};y,N)|^2\right] \ll \exp\left(-b\frac{\log q}{\log_2 q}\right)$$

hold for all $z \in \mathbb{C}$ *with* $|z| \leq aR_\sigma(q)$, *where* $R_\sigma(q)$ *is defined as* (13.4), *and* $\mathcal{B} = \mathcal{B}(s,B,q)$ *is the subset of* (13.6). *The implied constants depend on* s *and* B.

Proof. Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}(13.17)\qquad &\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}}|g_z^{\#}(s,f;y,N)|^2\\ &\ll \sum_{j>N}\sum_{\ell>N}\frac{1}{j!\ell!}\left(\frac{|z|}{2}\right)^{j+\ell}\left(\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}}|R_y(s,f)|^{2(j+\ell)}\right)^{1/2}\\ &\ll \sum_{k>2N}\frac{|z|^k}{k!}\left(\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}}|R_y(s,f)|^{2k}\right)^{1/2}\end{aligned}$$

along with the inequality $\sum_{j+\ell=k} k!/(j!\ell!) \leq 2^k$. If $\sigma \geq 1$, then we know

$$|R_y(s,f)| \leq \sum_{p\leq y}\sum_{m=1}^{\infty}\frac{2}{m}p^{-m\sigma} \ll \begin{cases}\log\zeta(\sigma) & \text{for } \sigma > 1,\\ \log_2 y & \text{for } \sigma = 1.\end{cases}$$

Hence we deduce

$$(13.18)\qquad |z|^k\left(\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}}|R_y(s,f)|^{2k}\right)^{1/2} \ll \left(aK\frac{\log q}{\log_2 q}\right)^k$$

for $|z| \leq aR_\sigma(q)$, where $K = K(\sigma,B)$ is a positive constant, and the implied constant depends on s and B. Furthermore, (13.18) remains valid in the case of $1/2 < \sigma < 1$ by the definition of $\mathcal{B} = \mathcal{B}(s,B,q)$. Then, we use Stirling's formula to derive $k! \gg (N/2)^k$ for $k > 2N$. By (13.18), we obtain

$$\frac{|z|^k}{k!}\left(\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}}|R_y(s,f)|^{2k}\right)^{1/2} \ll 2^{-k}$$

by choosing $a = (10KL)^{-1}$. Thus we deduce from (13.17) that

$$\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}}|g_z^{\#}(s,f;y,N)|^2 \ll \sum_{k>2N}2^{-k} \ll \exp\left(-b\frac{\log q}{\log_2 q}\right)$$

with some constant $b = b(\sigma,B) > 0$. Next, we show the result for $g_z^{\#}(s,\mathcal{X};y,N)$. Similarly to (13.17), we obtain

$$\mathbb{E}\left[|g_z^{\#}(s,\mathcal{X};y,N)|^2\right] \ll \sum_{k>2N}\frac{|z|^k}{k!}\mathbb{E}\left[|R_y(s,\mathcal{X})|^{2k}\right]^{1/2}.$$

By Corollary 13.11, the upper bounds

$$|z|^k\mathbb{E}\left[|R_y(s,\mathcal{X})|^{2k}\right]^{1/2} \ll \begin{cases} \left(aK'\dfrac{\log q}{\log_2 q}\right)^k & \text{for } \sigma \geq 1, \\ \left(aK'\dfrac{k(\log q)^\sigma}{(k\log 2k)^\sigma}\right)^k & \text{for } 1/2 < \sigma < 1 \end{cases}$$

are valid with some positive constant $K' = K'(\sigma, B)$. Thus, for $k > 2N$, we deduce

$$\frac{|z|^k}{k!}\mathbb{E}\left[|R_y(s,\mathcal{X})|^{2k}\right]^{1/2} \ll 2^{-k}$$

if we take $a = (20K'L)^{-2}$. Therefore we obtain

$$\mathbb{E}\left[|g_z^\#(s,\mathcal{X};y,N)|^2\right] \ll \sum_{k>2N} 2^{-k} \ll \exp\left(-b'\frac{\log q}{\log_2 q}\right)$$

with some constant $b' = b'(\sigma, B) > 0$, which completes the proof. □

LEMMA 13.13. *With the same assumptions as in Corollary 13.11, we take an integer $N = \lfloor(\log q)(L\log_2 q)^{-1}\rfloor$. For any $c > 0$, there exists a positive constant $a = a(\sigma, B, c)$ such that*

$$\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)}|g_z^\flat(s,f;y,N)|^2 \ll \exp\left(c\frac{\log q}{\log_2 q}\right),$$

$$\mathbb{E}\left[|g_z^\flat(s,\mathcal{X};y,N)|^2\right] \ll \exp\left(c\frac{\log q}{\log_2 q}\right)$$

for all $z \in \mathbb{C}$ with $|z| \leq aR_\sigma(q)$, where $R_\sigma(q)$ is defined as (13.4). Here, the implied constants depend only on s and B.

PROOF. Applying Corollary 13.11, we obtain

$$\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)}|g_z^\flat(s,f;y,N)|^2 \ll \sum_{k\leq 2N}\frac{1}{k!}\left(aK\frac{\log q}{\log_2 q}\right)^k,$$

$$\mathbb{E}\left[|g_z^\flat(s,\mathcal{X};y,N)|^2\right] \ll \sum_{k\leq 2N}\frac{1}{k!}\left(aK\frac{\log q}{\log_2 q}\right)^k$$

for $|z| \leq aR_\sigma(q)$, where $K = K(\sigma, B)$ is a positive constant. We choose $a = c/K$, and the desired results follow directly from the Taylor series of exponential. □

13.4. Completion of the proofs. By the definitions of g-functions, we obtain

$$\psi_{z,z'}(R_y(s,f)) = g_z(\overline{s},f;y)g_{z'}(s,f;y),$$
$$\psi_{z,z'}(R_y(s,\mathcal{X})) = g_z(\overline{s},\mathcal{X};y)g_{z'}(s,\mathcal{X};y).$$

For the proof of Proposition 13.3, we estimate the difference

$$\begin{aligned}&\widetilde{\mathcal{M}}_{s,q}^{(y)}(z,z',\mathcal{L})^{\mathcal{B}} - \widetilde{\mathcal{M}}_s^{(y)}(z,z',\mathcal{X})\\ &= \frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}} g_z(\overline{s},f;y)g_{z'}(s,f;y) - \mathbb{E}\left[g_z(\overline{s},f;y)g_{z'}(s,f;y)\right].\end{aligned}$$

We begin with considering the contributions of $g_z^\flat(s,f;y,N)$ and $g_z^\flat(s,\mathcal{X};y,N)$.

PROPOSITION 13.14. *Let $s=\sigma+it$ with $\sigma>1/2$. For $y=(\log q)^B$ with $B\geq 1$, there exist positive constants $a=a(\sigma,B)$, $b=b(\sigma,B)$, and $L=L(\sigma,B)$ such that*

$$\begin{aligned}&\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)\setminus\mathcal{B}} g_z^\flat(\overline{s},f;y,N)g_{z'}^\flat(s,f;y,N)\\&=\mathbb{E}\left[g_z^\flat(\overline{s},\mathcal{X};y,N)g_{z'}^\flat(s,\mathcal{X};y,N)\right]+O\left(\exp\left(-b\frac{\log q}{\log_2 q}\right)\right)\end{aligned}$$

for $z,z'\in\mathbb{C}$ with $\max\{|z|,|z'|\}\leq aR_\sigma(q)$, where we take $N=\lfloor(\log q)(L\log_2 q)^{-1}\rfloor$. Here, $R_\sigma(q)$ is defined as (13.4), and $\mathcal{B}=\mathcal{B}(s,B,q)$ is the subset of (13.6). The implied constant depends on s and B.

PROOF. Applying Proposition 13.10, we have

$$\begin{aligned}&\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)} g_z^\flat(\overline{s},f;y,N)g_{z'}^\flat(s,f;y,N)\\&=\mathbb{E}\left[g_z^\flat(\overline{s},f;y,N)g_{z'}^\flat(s,f;y,N)\right]+E_1,\end{aligned}$$

where

$$E_1\ll\sum_{j\leq N}\sum_{\ell\leq N}\frac{1}{j!\ell!}\left(\frac{|z|}{2}\right)^j\left(\frac{|z'|}{2}\right)^\ell y^{j+\ell}q^{-\beta/2}\ll q^{-\beta/2}\max\{|z|,|z'|\}^{2N}y^{2N}.$$

Let $L=L(\sigma,B)$ be large enough to keep the bound $\max\{|z|,|z'|\}^{2N}y^{2N}\ll q^{\beta/4}$. Then we obtain $E_1\ll q^{-\beta/4}$. Hence, the remaining work is to give the bound of

$$E_2=\frac{1}{\#B_2(q)}\sum_{f\in\mathcal{B}} g_z^\flat(\overline{s},f;y,N)g_{z'}^\flat(s,f;y,N).$$

We just consider the case of $1/2<\sigma<1$, otherwise it is the empty sum. Then we see that

$$\begin{aligned}E_2&=\sum_{j\leq N}\sum_{\ell\leq N}\frac{1}{j!\ell!}\left(\frac{iz}{2}\right)^j\left(\frac{iz'}{2}\right)^\ell\frac{1}{\#B_2(q)}\sum_{f\in\mathcal{B}}R_y(\overline{s},f)^jR_y(s,f)^\ell\\&\ll\exp\left(-\frac{b}{2}\frac{\log q}{\log_2 q}\right)\sum_{k\leq 2N}\frac{|z|^k}{k!}\left(\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)}|R_y(s,f)|^{2k}\right)^{1/2}\end{aligned}$$

by using Lemma 13.6. Furthermore, we deduce from Corollary 13.11 the bound

$$\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)}|R_y(s,f)|^{2k}\ll\left(K'\frac{(\log q)^{1-\sigma}}{\log_2 q}\right)^{2k}$$

for $k\leq 2N$, where $K'=K'(\sigma,B)$ is a positive constant. It yields

$$\sum_{k\leq 2N}\frac{|z|^k}{k!}\left(\frac{1}{\#B_2(q)}\sum_{f\in B_2(q)}|R_y(s,f)|^{2k}\right)^{1/2}\ll\exp\left(\frac{b}{4}\frac{\log q}{\log_2 q}\right)$$

by choosing $a = a(\sigma, B)$ sufficiently small. From the above, we obtain

$$\begin{aligned}&\frac{1}{\#B_2(q)} \sum_{f\in B_2(q)\setminus\mathcal{B}} g_z^\flat(\bar{s},f;y,N)\, g_{z'}^\flat(s,f;y,N) \\ &\qquad - \mathbb{E}\left[g_z^\flat(\bar{s},\mathcal{X};y,N)\, g_{z'}^\flat(s,\mathcal{X};y,N)\right] \\ &\ll \exp\left(-\frac{b}{4}\frac{\log q}{\log_2 q}\right)\end{aligned}$$

as desired. □

Proposition 13.15. *Let $s = \sigma + it$ with $\sigma > 1/2$. For $y = (\log q)^B$ with $B \geq 1$, there exist positive constants $a = a(\sigma, B)$, $b = b(\sigma, B)$ such that*

$$\widetilde{\mathcal{M}}_{s,q}^{(y)}(z,z';\mathcal{L})^{\mathcal{B}} = \widetilde{\mathcal{M}}_{s}^{(y)}(z,z';\mathcal{X}) + O\left(\exp\left(-b\frac{\log q}{\log_2 q}\right)\right)$$

for $z, z' \in \mathbb{C}$ with $\max\{|z|, |z'|\} \leq aR_\sigma(q)$, where $R_\sigma(q)$ is defined as (13.4), and $\mathcal{B} = \mathcal{B}(s, B, q)$ is the subset defined as (13.6). The implied constant depends only on s and B.

Proof. First, we use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}&\frac{1}{\#B_2(q)}\widetilde{\mathcal{M}}_{s,q}^{(y)}(z,z';\mathcal{L})^{\mathcal{B}} - \frac{1}{\#B_2(q)} \sum_{f\in B_2(q)\setminus\mathcal{B}} g_z^\flat(\bar{s},f;y,N)\, g_{z'}^\flat(s,f;y,N) \\ &\ll \left(\frac{1}{\#B_2(q)} \sum_{f\in B_2(q)} |g_z^\flat(\bar{s},f;y,N)|^2\right)^{1/2} \left(\frac{1}{\#B_2(q)} \sum_{f\in B_2(q)\setminus\mathcal{B}} |g_{z'}^\#(s,f;y,N)|^2\right)^{1/2} \\ &+ \left(\frac{1}{\#B_2(q)} \sum_{f\in B_2(q)\setminus\mathcal{B}} |g_z^\#(\bar{s},f;y,N)|^2\right)^{1/2} \left(\frac{1}{\#B_2(q)} \sum_{f\in B_2(q)} |g_{z'}^\flat(s,f;y,N)|^2\right)^{1/2} \\ &+ \left(\frac{1}{\#B_2(q)} \sum_{f\in B_2(q)\setminus\mathcal{B}} |g_z^\#(\bar{s},f;y,N)|^2\right)^{1/2} \left(\frac{1}{\#B_2(q)} \sum_{f\in B_2(q)\setminus\mathcal{B}} |g_{z'}^\#(s,f;y,N)|^2\right)^{1/2}.\end{aligned}$$

By Lemmas 13.12 and 13.13, this is estimated as

$$\ll \exp\left(-\frac{b}{8}\frac{\log q}{\log_2 q}\right),$$

where the implied constant depends on s and B. Furthermore, by a similar argument, we have

$$\widetilde{\mathcal{M}}_{s}^{(y)}(z,z';\mathcal{X}) = \mathbb{E}\left[g_z^\flat(\bar{s},\mathcal{X};y,N)\, g_{z'}^\flat(s,\mathcal{X};y,N)\right] + O\left(\exp\left(-\frac{b}{8}\frac{\log q}{\log_2 q}\right)\right).$$

Hence the result is deduced from Proposition 13.14. □

Proof of Proposition 13.3. Along the almost same argument for the proof of Proposition 11.3, we deduce Proposition 13.3 from Proposition 13.15. □

Proof of Theorem 13.1. Assume GRH for $L_f(s)$ for all $f \in B_2(q)$. Then we obtain

$$\log L_f(s) \ll \frac{(\log q)^{2-2\sigma}}{\log_2 q} + \log_2 q$$

as an analogue of Lemma 12.6; see [**70**, Theorem 5.19]. Furthermore, we deduce from the estimates of Cogdell–Michel [**27**, Lemmas 4.1 and 4.2] the upper bound

$$\log L(1, f) \ll \log_2 q \tag{13.19}$$

similarly to Lemma 12.7. Therefore, assertion (ii) can be proved in a way similar to Theorem 12.1. Let $s = \sigma + it$ with $\sigma > 1/2$ and $t \neq 0$. Note that the result for $\Phi \in C_b(\mathbb{C}) \cup \mathcal{I}(\mathbb{C})$ follows from Proposition 10.5. The remaining work is to prove the limit formula

$$\lim_{\substack{q\to\infty \\ q\colon \text{prime}}} \frac{1}{\# B_2(q)} \sum_{f\in\mathcal{E}}{}' \Phi(\log L_f(s)) = 0 \tag{13.20}$$

in the following cases:

(a) $\sigma > 1$ and $\Phi \in C(\mathbb{R})$;
(b) $\sigma = 1$ and $\Phi \in C^{\text{poly}}(\mathbb{R})$;
(c′) $\sigma_0 < \sigma < 1$ and $\Phi \in C^{\exp}(\mathbb{R})$ assuming GRH for $L_f(s)$ for all $f \in B_2(q)$.

The proofs in cases (a) and (c′) are given similarly to the case of $s = \sigma \in \mathbb{R}$. The difference arises from only in case (b), where we just know upper bounds weaker than (13.19). In fact, by the standard argument along with the zero-free region of $L_f(s)$, there exists an absolute constant $c > 0$ such that we have

$$\frac{L'}{L}(s, f) \ll (\log q(|t| + 3))^2$$

for all $s = \sigma + it$ such that $\sigma \geq 1 - c/\log q(|t| + 2)$; see [**70**, Proposition 5.7]. It yields the upper bound

$$\log L(s, f) \ll (\log q(|t| + 3))^2$$

on the vertical line $\operatorname{Re} s = 1$. Using this upper bound instead of (13.19), we derive (13.20) in case (b′), which completes the proof. □

Proof of Theorem 13.2. The result directly follows from Proposition 10.6. □

Finally, we consider variants of Theorems 13.1 and 13.2. Let $\{L_f(s)\}_{f\in\mathscr{F}}$ be the family of Example 7.6, and let $\mathcal{Y} = (\mathcal{Y}_p)_p$ be the sequence of Example 8.6. Note that the sequence $\mathcal{Y}$ is again admissible. Then we obtain the following results.

Theorem 13.16. *Let $s = \sigma + it$ be a complex number with $\sigma > 1/2$.*

(i) *If $t \neq 0$, then the limit formula*

$$\lim_{\substack{q\to\infty \\ q\colon \text{prime}}} \sum_{f\in B_2(q)}{}' \frac{1}{4\pi\langle f, f\rangle} \Phi\left(\log L_f(s)\right) = \int_{\mathbb{C}} \Phi(w) \mathcal{M}_s(w; \mathcal{Y})\, |dw|$$

holds in cases (a)–(c′) *of Theorem 13.1* (i).

(ii) *If $t = 0$, then the limit formula*

$$\lim_{\substack{q\to\infty\\ q\text{: prime}}} \sum_{f\in B_2(q)}{}' \frac{1}{4\pi\langle f,f\rangle}\Phi\left(\log L_f(\sigma)\right) = \int_{\mathbb{R}} \Phi(w)\mathcal{M}_\sigma(u;\mathcal{Y})\,|du|$$

holds in cases (a)–(c′) *of Theorem 13.1* (ii).

THEOREM 13.17. *Let $s = \sigma + it$ be a complex number with $\sigma > 1/2$. Then we obtain the discrepancy bounds*

$$D_s(q;\mathcal{L}') \ll \begin{cases} (\log q)^{-1}(\log_2 q) & \textit{for } \sigma > 1,\\ (\log q)^{-1}(\log_2 q\log_3 q) & \textit{for } \sigma = 1,\\ (\log q)^{-\sigma} & \textit{for } 1/2 < \sigma < 1 \end{cases}$$

in both cases $s \in \mathbb{R}$ and $s \notin \mathbb{R}$, where the implied constants depend on σ.

The difference from the proofs of Theorems 13.1 and 13.2 is using the formula

$$\sum_{f\in B_2(q)} \frac{1}{4\pi\langle f,f\rangle} a(n,f) = \mathbb{E}\left[a(n,\mathcal{Y})\right] + O\left(n^\alpha q^{-\beta}\right) \tag{13.21}$$

in place of (13.9). Note that formula (13.21) is deduced from the Petersson trace formula [27, Proposition 1.9].

CHAPTER 5

Further results on the Riemann zeta-function

In this chapter, we focus on the distribution of extreme values of the Riemann zeta-function. We study the large deviations by comparing $\zeta(\sigma+it)$ with $\zeta(\sigma,X)$ as described in Section 1. The proof of Theorem II is completed in Section 15.

14. Preliminaries on the random Euler product

Let $X=(X_p)_p$ be a sequence of independent random variables which are uniformly distributed on the unit circle $\mathcal{T}$. Denote by $\zeta(\sigma,X)$ the random Euler product associated with $\zeta(\sigma+it)$. Then, we deduce from Theorem 9.1 that there exists a continuous function $\mathcal{M}_\sigma:\mathbb{C}\to\mathbb{R}_{\geq 0}$ satisfying

$$\mathbb{P}(\log\zeta(\sigma,X)\in A)=\int_A \mathcal{M}_\sigma(w)\,|dw|. \tag{14.1}$$

Let $\alpha\in\mathbb{R}$. The aim of this section is preparing several results on the large deviation

$$\Psi_\sigma(\tau;\alpha)=\mathbb{P}\left(\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma,X))>\tau\right). \tag{14.2}$$

14.1. Moment-generating functions. Let $\sigma>1/2$ and $\alpha\in\mathbb{R}$. We begin with studying the moment-generating function

$$\mathcal{F}_\sigma(s;\alpha)=\mathbb{E}\left[\exp\left(s\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma,X))\right)\right]$$

for $s=\kappa+it\in\mathbb{C}$. It is represented as $\mathcal{F}_\sigma(s;\alpha)=\widetilde{\mathcal{M}}_\sigma(-ie^{-i\alpha}s,-ie^{i\alpha}s;\zeta)$ with the notation of Section 9.2. For every prime number p, we define

$$\log\zeta_p(\sigma,X)=\sum_{k=1}^{\infty}\frac{1}{k}p^{-k\sigma}X_p^k.$$

Recall that the random variable $\sum_{p\leq y}\log\zeta_p(\sigma,X)$ converges to $\log\zeta(\sigma,X)$ in law as $y\to\infty$. The following result is a slight modification of Lemma 9.7, and we omit the proof.

Lemma 14.1. *Let $\sigma>1/2$ and $\alpha\in\mathbb{R}$. Then the moment-generating function $\mathcal{F}_\sigma(s;\alpha)$ exists for all $s\in\mathbb{C}$. Furthermore, it is an entire function represented as the infinite product*

$$\mathcal{F}_\sigma(s;\alpha)=\prod_p\mathcal{F}_{\sigma,p}(s;\alpha), \tag{14.3}$$

where $\mathcal{F}_{\sigma,p}(s;\alpha)=\mathbb{E}\left[\exp\left(s\operatorname{Re}(e^{-i\alpha}\log\zeta_p(\sigma,X))\right)\right]$.

Then, we prove the following lemma which was essentially proved by Lamzouri–Lester–Radziwiłł [**93**, Lemma 7.3]. Here, we present an alternative proof by the method similar to Proposition 9.4.

LEMMA 14.2. *Let $\sigma > 1/2$ and $\alpha \in \mathbb{R}$. Then there exists a positive constant $c(\sigma)$ such that*

$$\frac{|\mathcal{F}_\sigma(s;\alpha)|}{\mathcal{F}_\sigma(\kappa;\alpha)} \leq \exp\left(-c(\sigma)\frac{|t|^{\frac{1}{\sigma}}}{\log|t|}\right)$$

holds for $s = \kappa + it$ with $\kappa > 9$ and $|t| > \kappa/3$.

PROOF. Since we have $|\mathcal{F}_\sigma(s;\alpha)| \leq \mathcal{F}_\sigma(\kappa;\alpha)$ by definition, the inequality

$$\frac{|\mathcal{F}_\sigma(s;\alpha)|}{\mathcal{F}_\sigma(\kappa;\alpha)} \leq \prod_{p>Q}\frac{|\mathcal{F}_{\sigma,p}(s;\alpha)|}{\mathcal{F}_{\sigma,p}(\kappa;\alpha)} \tag{14.4}$$

follows from (14.3), where $Q > 0$ is a real number chosen later. We recall that $\mathbb{E}[\mathrm{Re}(e^{-i\alpha}X_p)] = \mathbb{E}[\mathrm{Re}(e^{-i\alpha}X_p^2)] = 0$. By the Taylor expansion, we have

$$\mathcal{F}_{\sigma,p}(s;\alpha) = 1 + \frac{s^2}{2}\mathbb{E}\left[\{\mathrm{Re}(e^{-i\alpha}X_p)\}^2\right]p^{-2\sigma} + O\left(|s|^3p^{-3\sigma}\right)$$

if $|s|p^{-\sigma} < c$ is satisfied with some constant $c > 0$. The remaining expected value is calculated as

$$\mathbb{E}\left[\{\mathrm{Re}(e^{-i\alpha}X_p)\}^2\right] = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{-i\alpha}\overline{e^{i\theta}} + \overline{e^{-i\alpha}}e^{i\theta}}{2}\right)^2 d\theta = \frac{1}{2}.$$

Therefore, we obtain

$$\mathcal{F}_{\sigma,p}(s;\alpha) = 1 + \frac{1}{4}s^2p^{-2\sigma} + O\left(|s|^3p^{-3\sigma}\right).$$

Let $Q = Q(c,s,\sigma)$ as in (9.5). Then there exists a small constant $c_1 > 0$ such that, for any $0 < c < c_1$, we have $|\mathcal{F}_{\sigma,p}(s;\alpha) - 1| < 1/2$ if $p > Q(c,s,\sigma)$. It deduces

$$\log|\mathcal{F}_{\sigma,p}(s;\alpha)| = \frac{1}{4}\mathrm{Re}(s^2)p^{-2\sigma} + O\left(|s|^3p^{-3\sigma}\right)$$

for $p > Q(c,s,\sigma)$. Furthermore, we derive

$$\log\frac{|\mathcal{F}_{\sigma,p}(s;\alpha)|}{F_{\sigma,p}(\kappa;\alpha)} = -\frac{1}{4}t^2p^{-2\sigma} + O\left(|s|^3p^{-3\sigma}\right)$$

since $\mathrm{Re}(s^2) - \kappa^2 = -t^2$. By the prime number theorem, we see that

$$\sum_{p>Q}t^2p^{-2\sigma} \geq \frac{1}{2(2\sigma-1)}\frac{t^2Q^{1-2\sigma}}{\log Q},$$

$$\sum_{p>Q}|s|^3p^{-3\sigma} \leq \frac{2c}{2\sigma-1}\frac{|s|^2Q^{1-2\sigma}}{\log Q}.$$

Note that $|s| \leq 4|t|$ holds by the assumption $|t| \geq \kappa/3$. Therefore, we have an absolute constant c_2 with $0 < c_2 < c_1$ such that

$$\sum_{p>Q} \log \frac{|\mathcal{F}_{\sigma,p}(s;\alpha)|}{\mathcal{F}_{\sigma,p}(\kappa;\alpha)} \leq -\frac{1}{4(2\sigma-1)} \frac{t^2 Q^{1-2\sigma}}{\log Q},$$

where $Q = Q(c_2, s, \sigma)$. Since $|t| \leq |s|$, we obtain $Q(c_2, s, \sigma) > Q(c_2, t, \sigma) =: Q'$. Therefore, we conclude

$$\prod_{p>Q} \frac{|\mathcal{F}_{\sigma,p}(s;\alpha)|}{\mathcal{F}_{\sigma,p}(\kappa;\alpha)} \leq \exp\left(-\frac{1}{4(2\sigma-1)} \frac{t^2 {Q'}^{1-2\sigma}}{\log Q'}\right)$$

from which we deduce the desired result by (14.4). □

Next, we study the behavior of the moment-generating function $\mathcal{F}_{\sigma,p}(s;\alpha)$ as $s = \kappa + it$ varies in the region

$$\Delta = \{s = \kappa + it \mid \kappa > 0 \text{ and } |t| < \kappa\}.$$

If the quantity $\kappa p^{-2\sigma}$ is small, $\mathcal{F}_{\sigma,p}(s;\alpha)$ is approximated by using the modified Bessel function $I_\nu(z)$.

PROPOSITION 14.3. *Let $\sigma > 1/2$ and $\alpha \in \mathbb{R}$. Let $s = \kappa + it \in \Delta$ with $\kappa > 0$ large enough. Suppose that a prime number p satisfies $\kappa p^{-2\sigma} \leq \delta$ with a small constant $\delta > 0$. Then we have*

$$\mathcal{F}_{\sigma,p}(s;\alpha) = I_0(sp^{-\sigma}) \left(1 + O(\kappa p^{-2\sigma})\right),$$

where the implied constant is absolute.

PROOF. We define a random variable $Y_{\sigma,p}$ as

$$Y_{\sigma,p} = \log \zeta_p(\sigma, X) - p^{-\sigma} X_p = \sum_{k=2}^{\infty} \frac{1}{k} p^{-k\sigma} X_p^k.$$

Remark that $|s| \leq 2\kappa$ holds if $s = \kappa + it \in \Delta$. Then we obtain

$$s \operatorname{Re}(e^{-i\alpha} Y_{\sigma,p}) \ll \kappa p^{-2\sigma} \leq \delta$$

by the assumption. Hence the asymptotic formula

$$\exp\left(s \operatorname{Re}(e^{-i\alpha} Y_{\sigma,p})\right) = 1 + O(\kappa p^{-2\sigma})$$

holds if δ is small enough. Therefore, the function $\mathcal{F}_{\sigma,p}(s;\alpha)$ is calculated as

$$\begin{aligned} \mathcal{F}_{\sigma,p}(s;\alpha) &= \mathbb{E}\left[\exp\left(sp^{-\sigma} \operatorname{Re}(e^{-i\alpha} X_p) + s \operatorname{Re}(e^{-i\alpha} Y_{\sigma,p})\right)\right] \\ &= I_0(sp^{-\sigma}) + O\left(I_0(\kappa p^{-\sigma}) \kappa p^{-2\sigma}\right). \end{aligned}$$

By the usual asymptotic formula for $I_0(z)$, one can prove $I_0(sp^{-\sigma}) \gg I_0(\kappa p^{-\sigma})$ for $s \in \Delta$; see Theorem B.3.2. Hence we obtain the conclusion. □

If the quantity $\kappa p^{-2\sigma}$ is not so small, then the above method is not available. In that case, we estimate

$$\mathcal{F}_{\sigma,p}(s;\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(s\lambda_{\sigma,p}(\theta;\alpha)\right)\, d\theta$$

by adapting the saddle-point method, where we put

$$\lambda_{\sigma,p}(\theta;\alpha) = -\operatorname{Re}(e^{-i\alpha}\operatorname{Log}(1-p^{-\sigma}e^{i\theta})) = \sum_{k=1}^{\infty}\frac{1}{k}p^{-k\sigma}\cos(k\theta-\alpha).$$

Some basic results on the function $\lambda_{\sigma,p}(\theta;\alpha)$ are listed in Appendix B.1.

Proposition 14.4. *Let $\sigma > 1/2$ and $\alpha \in \mathbb{R}$. Let $s = \kappa + it \in \Delta$ with $\kappa > 0$ large enough. Suppose that a prime number $p \geq 13$ satisfies $\kappa p^{-\sigma} \geq (\log\kappa)^5$. Then there exists a real number $\theta_1 = \theta_1(\sigma, p;\alpha)$ such that*

$$\mathcal{F}_{\sigma,p}(s;\alpha) = \frac{\exp\left(s\lambda_{\sigma,p}(\theta_1;\alpha)\right)}{\sqrt{2\pi s|\lambda''_{\sigma,p}(\theta_1;\alpha)|}}\left(1+O((\log\kappa)^{-5/2})\right),$$

where $\sqrt{s}$ is understood as $\exp(\frac{1}{2}\operatorname{Log} s)$. Here, the implied constant is absolute.

Proof. From the results in Appendix B.1, we first note that there exist real numbers $\theta_1 = \theta_1(\sigma, p;\alpha)$ and $\theta_2 = \theta_2(\sigma, p;\alpha)$ with $\theta_1 < \theta_2 < \theta_1 + 2\pi$ such that the function $\lambda_{\sigma,p}(\theta;\alpha)$ is decreasing for $\theta_1 \leq \theta \leq \theta_2$ and is increasing for $\theta_2 \leq \theta \leq \theta_1+2\pi$. We also find an absolute positive constant d such that $|\theta_2-\theta_1| \geq d$. Furthermore, if $p \geq 13$, then we obtain

$$\lambda^{(n)}_{\sigma,p}(\theta_1;\alpha) \ll n!p^{-\sigma} \quad\text{and}\quad \lambda''_{\sigma,p}(\theta_1;\alpha) \gg p^{-\sigma} \tag{14.5}$$

for any $n \geq 0$ with absolute implied constants. We represent $\mathcal{F}_{\sigma,p}(s;\alpha)$ as

$$\mathcal{F}_{\sigma,p}(s;\alpha) = \frac{1}{2\pi}\left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_1+2\pi}\right)\exp\left(s\lambda_{\sigma,p}(\theta;\alpha)\right)\, d\theta = I_1 + I_2,$$

say. We notice that $\lambda'_{\sigma,p}(\theta_1;\alpha) = 0$. Thus we have

$$\lambda_{\sigma,p}(\theta;\alpha) - \lambda_{\sigma,p}(\theta_1;\alpha) = \frac{\lambda''_{\sigma,p}(\theta_1;\alpha)}{2}(\theta-\theta_1)^2\mu(\theta),$$

where we put

$$\mu(\theta) = 1 + \sum_{k=1}^{\infty}\frac{2}{\lambda''_{\sigma,p}(\theta_1;\alpha)}\frac{\lambda^{(k+2)}_{\sigma,p}(\theta_1;\alpha)}{(k+2)!}(\theta-\theta_1)^k.$$

By (14.5), the coefficients are bounded absolutely. Hence, there exists an absolute constant δ with $0 < \delta < d$ such that $\mu(\theta) = 1 + O(|\theta-\theta_1|)$ is valid for $|\theta-\theta_1| \leq \delta$. Then we have

$$\begin{aligned}&\int_{\theta_1}^{\theta_1+\delta}\exp\left(s\lambda_{\sigma,p}(\theta;\alpha)\right)\, d\theta\\ &= \exp\left(s\lambda_{\sigma,p}(\theta_1;\alpha)\right)\int_{\theta_1}^{\theta_1+\delta}\exp\left(s\frac{\lambda''_{\sigma,p}(\theta_1;\alpha)}{2}(\theta-\theta_1)^2\mu(\theta)\right)\, d\theta =\end{aligned}$$

$$= \exp\left(s\lambda_{\sigma,p}(\theta_1;\alpha)\right)\int_0^{\delta'} \exp\left(s\frac{\lambda''_{\sigma,p}(\theta_1;\alpha)}{2}\phi\right)\frac{1+O(\sqrt{\phi})}{2\sqrt{\phi}}\,d\phi$$

by making change of variables as $\phi = (\theta-\theta_1)^2\mu(\theta)$, where $\delta' = \delta^2\mu(\theta_1+\delta) \asymp \delta^2$. Note that a simple calculation gives

$$\int_0^{\infty} \exp\left(s\frac{\lambda''_{\sigma,p}(\theta_1;\alpha)}{2}\phi\right)\frac{d\phi}{\sqrt{\phi}} = \sqrt{\frac{2\pi}{s|\lambda''_{\sigma,p}(\theta_1;\alpha)|}}.$$

Therefore we obtain

$$I_1 = \frac{1}{2}\exp\left(s\lambda_{\sigma,p}(\theta_1;\alpha)\right)\left(\frac{1}{\sqrt{2\pi s|\lambda''_{\sigma,p}(\theta_1;\alpha)|}} + E_1 + E_2 + E_3\right),$$

where the error terms E_j are estimated as

$$\begin{aligned} E_1 &\ll \int_{\delta'}^{\infty} \exp\left(-\kappa\frac{|\lambda''_{\sigma,p}(\theta_1;\alpha)|}{2}\phi\right)\frac{d\phi}{\sqrt{\phi}},\\ E_2 &\ll \int_0^{\delta'} \exp\left(-\kappa\frac{|\lambda''_{\sigma,p}(\theta_1;\alpha)|}{2}\phi\right)d\phi,\\ E_3 &\ll \int_{\theta_1+\delta}^{\theta_2} \exp\left(-\kappa\left|\lambda_{\sigma,p}(\theta;\alpha)-\lambda_{\sigma,p}(\theta_1;\alpha)\right|\right)d\theta \end{aligned}$$

by recalling $\lambda''_{\sigma,p}(\theta_1;\alpha) < 0$ and $\lambda_{\sigma,p}(\theta_1;\alpha) \geq \lambda_{\sigma,p}(\theta;\alpha)$. First, we evaluate E_1 as

$$E_1 \ll \frac{1}{\kappa|\lambda''_{\sigma,p}(\theta;\alpha)|}\exp\left(-\kappa\frac{|\lambda''_{\sigma,p}(\theta_1;\alpha)|}{2}\delta'\right) \ll \frac{1}{\kappa|\lambda''_{\sigma,p}(\theta_1;\alpha)|}$$

since we have $\sqrt{\phi} \geq \sqrt{\delta'} \gg 1$ for $\phi \geq \delta'$. By (14.5), we obtain

$$E_1 \ll \kappa^{-1}p^{\sigma}. \tag{14.6}$$

Similarly, we evaluate E_2 as

$$E_2 \ll \frac{1}{\kappa|\lambda''_{\sigma,p}(\theta_1;\alpha)|} \ll \kappa^{-1}p^{\sigma}. \tag{14.7}$$

As for the error term E_3, we remark that

$$\begin{aligned} \left|\lambda_{\sigma,p}(\theta;\alpha)-\lambda_{\sigma,p}(\theta_1;\alpha)\right| &\geq \lambda_{\sigma,p}(\theta_1;\alpha)-\lambda_{\sigma,p}(\theta_1+\delta;\alpha)\\ &= \frac{|\lambda''_{\sigma,p}(\theta_1;\alpha)|}{2}\delta' \end{aligned}$$

for $\theta_1+\delta \leq \theta \leq \theta_2$. Hence we derive

$$E_3 \ll \exp\left(-\kappa\frac{|\lambda''_{\sigma,p}(\theta_1;\alpha)|}{2}\delta'\right) \ll \exp\left(-C\kappa p^{-\sigma}\right) \tag{14.8}$$

with some absolute constant $C > 0$. Combining (14.6), (14.7), and (14.8), we deduce

$$I_1 = \frac{1}{2}\exp\left(s\lambda_{\sigma,p}(\theta_1;\alpha)\right)\left(\frac{1}{\sqrt{2\pi s|\lambda''_{\sigma,p}(\theta_1;\alpha)|}} + O(\kappa^{-1}p^{\sigma})\right) =$$

$$= \frac{1}{2}\frac{\exp\left(s\lambda_{\sigma,p}(\theta_1;\alpha)\right)}{\sqrt{2\pi s|\lambda''_{\sigma,p}(\theta_1;\alpha)|}}\left(1+O((\log\kappa)^{-5/2})\right)$$

by the assumption $\kappa p^{-\sigma} \geq (\log\kappa)^5$. A similar asymptotic formula is valid for the integral I_2, and hence the result follows. ☐

Then, it remains to consider the case of prime numbers $p \leq 11$. In this case, we obtain the following result as an analogue of Proposition 14.4.

Proposition 14.5. *Let $\sigma > 1/2$ and $\alpha \in \mathbb{R}$. Let $s = \kappa + it \in \Delta$ with $\kappa > 0$ large enough. For a prime number $p \leq 11$, there exist a real number $\theta_1 = \theta_1(\sigma, p; \alpha)$ and an even integer $n_1 = n_1(\sigma, p; \alpha) \geq 2$ such that*

$$\mathcal{F}_{\sigma,p}(s;\alpha) = \frac{\Gamma(\frac{1}{n_1})}{n_1\pi}\exp\left(s\lambda_{\sigma,p}(\theta_1;\alpha)\right)\left(\frac{n_1!}{s|\lambda^{(n_1)}_{\sigma,p}(\theta_1;\alpha)|}\right)^{\frac{1}{n_1}}\left(1+O(\kappa^{\frac{1}{n_1}-1})\right),$$

where $s^{\frac{1}{n_1}}$ is understood as $\exp(\frac{1}{n_1}\operatorname{Log} s)$. *Here, the implied constant depends only on σ and α.*

Proof. Let θ_1 and θ_2 be the same as in the proof Proposition 14.4. In the present case, the upper bound of (14.5) remains available. Hence we obtain

$$\lambda^{(n)}_{\sigma,p}(\theta_1;\alpha) \ll n! \tag{14.9}$$

for $p \leq 11$, where the implied constant is absolute. On the other hand, it seems hard to derive the lower bound of (14.5) for $p \leq 11$ by the method of Appendix B.1. We notice that there exists an even integer $n_1 = n_1(p, \sigma; \alpha) \geq 2$ such that $\lambda^{(n)}_{\sigma,p}(\theta_1;\alpha) = 0$ for $1 \leq n < n_1$ but $\lambda^{(n_1)}_{\sigma,p}(\theta_1;\alpha) < 0$. Then we have the lower bound

$$\lambda^{(n_1)}_{\sigma,p}(\theta_1;\alpha) \gg 1 \tag{14.10}$$

for $p \leq 11$, where the implied constant depends only on σ and α. Using (14.9) and (14.10) in place of (14.5), we obtain the result in a way similar to the proof of Proposition 14.4. ☐

14.2. Cumulant-generating functions. Let $\sigma > 1/2$ and $\alpha \in \mathbb{R}$. Note that the moment-generating functions $\mathcal{F}_\sigma(s;\alpha)$ and $\mathcal{F}_{\sigma,p}(s;\alpha)$ are positive on the real axis. Hence we define the cumulant-generating functions as

$$f_\sigma(\kappa;\alpha) = \log\mathcal{F}_\sigma(\kappa;\alpha) \quad \text{and} \quad f_{\sigma,p}(\kappa;\alpha) = \log\mathcal{F}_{\sigma,p}(\kappa;\alpha) \tag{14.11}$$

for $\kappa \in \mathbb{R}$. Let $s = \kappa + it \in \Delta$. By Propositions 14.3, 14.4, and 14.5, the function $\mathcal{F}_{\sigma,p}(s;\alpha)$ is non-zero for every prime number p if $\kappa > 0$ is large enough. Thus $\mathcal{F}_\sigma(s;\alpha)$ is also non-zero in the same region by Lemma 14.1. Hence, the cumulant-generating functions of (14.11) are continued to holomorphic functions in $\Delta_R = \Delta \cap \{\operatorname{Re} s > R\}$, where $R > 0$ is a large real number. They are connected by the formula

$$f_\sigma(s;\alpha) = \sum_p f_{\sigma,p}(s;\alpha). \tag{14.12}$$

Then, we recall that all zeros of the modified Bessel function $I_0(s)$ lie on the imaginary axis; see Theorem B.3.1. Furthermore, the values are positive on the real axis by definition. Then we define

$$g(s) = \log I_0(s),$$

which is a holomorphic function on Δ. Lamzouri [92] approximated the cumulant-generating function $f_\sigma(\kappa;\alpha)$ with $\alpha = 0, \pi$ by using the function $g(\kappa)$, where $\kappa > 0$ is a large real number. Furthermore, Lamzouri–Lester–Radziwiłł [93] presented similar statements for the derivatives $f_\sigma^{(n)}(\kappa;0)$ of orders $n \le 3$. In this section, we generalize the result for all $n \ge 0$ and $\alpha \in \mathbb{R}$.

Proposition 14.6. *Let $1/2 < \sigma < 1$ and $\alpha \in \mathbb{R}$. For any integer $n \ge 0$, we have*

$$f_\sigma^{(n)}(\kappa;\alpha) = g_n(\sigma)\frac{\kappa^{\frac{1}{\sigma}-n}}{\log\kappa}\left(1 + O\left(\frac{2^n n!}{\log\kappa}\right)\right) \tag{14.13}$$

if $\kappa > 0$ is large enough, where the constant $g_n(\sigma)$ is represented as

$$g_n(\sigma) = \sigma\prod_{j=1}^{n}\left(\frac{1}{\sigma} + 1 - j\right) \times \int_0^\infty g(y^{-\sigma})\,dy. \tag{14.14}$$

The implied constant in (14.13) depends only on σ and α, and the product in (14.14) is understood as the value 1 if $n = 0$.

Proof. By formula (14.12), we obtain

$$f_\sigma^{(n)}(\kappa;\alpha) = \sum_{p \le y_1} f_{\sigma,p}^{(n)}(\kappa;\alpha) + \sum_{y_1 < p \le y_2} f_{\sigma,p}^{(n)}(\kappa;\alpha) + \sum_{p > y_2} f_{\sigma,p}^{(n)}(\kappa;\alpha), \tag{14.15}$$

where the parameters y_1 and y_2 are determined by the equations

$$\frac{\kappa}{y_1^{2\sigma}} = \delta \quad\text{and}\quad \frac{\kappa}{y_2^{\sigma}} = \left(\frac{1}{\log\kappa}\right)^{\frac{\sigma}{2\sigma-1}}$$

with a constant $\delta > 0$ small enough. First, we consider the term for $p \le y_1$. In this case, the condition $\kappa p^{-\sigma} \ge (\log\kappa)^5$ is satisfied if $\kappa > 0$ is sufficiently large. Hence, by Proposition 14.4, we obtain

$$f_{\sigma,p}(s;\alpha) = s\lambda_{\sigma,p}(\theta_1;\alpha) - \operatorname{Log}(s|\lambda''_{\sigma,p}(\theta_1;\alpha)|) - \frac{1}{2}\log(2\pi) + h_{\sigma,p}(s;\alpha)$$

for $s = \kappa + it \in \Delta_R$ and $p \ge 13$, where $h_{\sigma,p}(s;\alpha) \ll (\log\kappa)^{-5/2}$ with an absolute constant. Note that $h_{\sigma,p}(s;\alpha)$ is holomorphic on Δ_R. From the Cauchy integral formula, we deduce

$$h_{\sigma,p}^{(n)}(\kappa;\alpha) = \frac{n!}{2\pi i}\int_{|z-\kappa|=\kappa/2}\frac{h_{\sigma,p}(z;\alpha)}{(z-\kappa)^{n+1}}\,dz \ll \frac{2^n n!}{\kappa^n(\log\kappa)^{5/2}}$$

for all $n \ge 0$. These imply the upper bounds

$$f_{\sigma,p}(\kappa;\alpha) \ll \kappa p^{-\sigma}, \qquad f'_{\sigma,p}(\kappa;\alpha) \ll p^{-\sigma},$$

and for $n \ge 2$,

$$f_{\sigma,p}^{(n)}(\kappa;\alpha) \ll 2^n n!\kappa^{-n}.$$

Therefore, we have $f_{\sigma,p}^{(n)}(\kappa;\alpha) \ll 2^n n! \kappa^{-n+1} p^{-\sigma}$ for all $n \geq 0$. Using Proposition 14.5, we obtain similar estimates for $p \leq 11$ with implied constants depending only on σ and α. Hence we evaluate the first sum of the right-hand side of (14.15) as

$$\sum_{p \leq y_1} f_{\sigma,p}^{(n)}(\kappa;\alpha) \ll 2^n n! \kappa^{-n+1} \sum_{p \leq y_1} p^{-\sigma} \ll 2^n n! \kappa^{-n+1} \frac{y_1^{1-\sigma}}{\log y_1}.$$

Since $y_1 \asymp \kappa^{\frac{1}{2\sigma}}$, we conclude

$$\sum_{p \leq y_1} f_{\sigma,p}^{(n)}(\kappa;\alpha) \ll 2^n n! \frac{\kappa^{\frac{1}{2\sigma}+\frac{1}{2}-n}}{\log \kappa} \leq 2^n n! \frac{\kappa^{\frac{1}{\sigma}-n}}{(\log \kappa)^2} \tag{14.16}$$

by recalling $1/2 < \sigma < 1$. Next, we let $p \geq y_1$. Then we have

$$f_{\sigma,p}(s;\alpha) = g(sp^{-\sigma}) + O(\kappa p^{-2\sigma})$$

for $s = \kappa + it \in \Delta_R$ by Proposition 14.3. By the Cauchy integral formula, we obtain

$$f_{\sigma,p}^{(n)}(\kappa;\alpha) = p^{-n\sigma} g^{(n)}(\kappa p^{-\sigma}) + O(2^n n! \kappa^{-n+1} p^{-2\sigma}). \tag{14.17}$$

If we further suppose $p > y_2$, then $0 < \kappa p^{-\sigma} < 1$ holds. Thus, the upper bounds

$$f_{\sigma,p}^{(n)}(\kappa;\alpha) \ll 2^n n! \kappa^{-n+2} p^{-2\sigma}$$

follow from the estimates of $g^{(n)}(u)$; see Lemma B.3.3. Hence we derive

$$\sum_{p > y_2} f_{\sigma,p}^{(n)}(\kappa;\alpha) \ll 2^n n! \kappa^{-n+2} \sum_{p > y_2} p^{-2\sigma} \ll 2^n n! \kappa^{-n+2} \frac{y_2^{1-2\sigma}}{\log y_2}.$$

Inserting $y_2 = \kappa^{\frac{1}{\sigma}} (\log \kappa)^{\frac{1}{2\sigma-1}}$, we deduce

$$\sum_{p > y_2} f_{\sigma,p}^{(n)}(\kappa;\alpha) \ll 2^n n! \frac{\kappa^{\frac{1}{\sigma}-n}}{(\log \kappa)^2}. \tag{14.18}$$

The main term comes from the term for $y_1 < p \leq y_2$. Put

$$u(y) = \kappa y^{-\sigma}.$$

By formula (14.17), we obtain

$$\sum_{y_1 < p \leq y_2} f_{\sigma,p}^{(n)}(\kappa;\alpha) = \kappa^{-n} \sum_{y_1 < p \leq y_2} u(p)^n g^{(n)}(u(p)) + E_1, \tag{14.19}$$

where E_1 is estimated as

$$E_1 \ll 2^n n! \kappa^{-n+1} \sum_{y_1 < p \leq y_2} p^{-2\sigma} \ll 2^n n! \frac{\kappa^{\frac{1}{\sigma}-n}}{(\log \kappa)^2}. \tag{14.20}$$

Recall the asymptotic formula $\pi(y) = \int_2^y \frac{dt}{\log t} + O(ye^{-8\sqrt{\log y}})$. Then, by the partial summation, we have

$$\sum_{y_1 < p \leq y_2} u(p)^n g^{(n)}(u(p)) = \int_{y_1}^{y_2} u(y)^n g^{(n)}(u(y)) \frac{dy}{\log y} + E_2, \tag{14.21}$$

where E_2 is estimated as

$$\begin{aligned}E_2 \ll& u(y_1)^n g^{(n)}(u(y_1))y_1 e^{-8\sqrt{\log y_1}} + u(y_2)^n g^{(n)}(u(y_2))y_2 e^{-8\sqrt{\log y_2}}\\ &+\int_{y_1}^{y_2} \frac{d}{dy}\left\{u(y)^n g^{(n)}(u(y))\right\} ye^{-8\sqrt{\log y}}\,dy\\ \ll& u(y_1)^n g^{(n)}(u(y_1))y_1 e^{-8\sqrt{\log y_1}} + u(y_2)^n g^{(n)}(u(y_2))y_2 e^{-8\sqrt{\log y_2}}\\ &+e^{-8\sqrt{\log y_1}}\int_{y_1}^{y_2} u(y)^n g^{(n)}(u(y))\,dy.\end{aligned}$$

Recall that we have $u(y_1) > 1$ and $0 < u(y_2) < 1$. Hence Lemma B.3.3 yields

$$\begin{aligned}u(y_1)^n g^{(n)}(u(y_1))y_1 e^{-8\sqrt{\log y_1}} &\ll 2^n n! u(y_1)y_1 e^{-8\sqrt{\log y_1}}\\ &\ll 2^n n! \kappa^{\frac{1}{2\sigma}+\frac{1}{2}} e^{-4\sqrt{\log\kappa}}\end{aligned}$$

and

$$\begin{aligned}u(y_2)^n g^{(n)}(u(y_2))y_2 e^{-8\sqrt{\log y_2}} &\ll n! u(y_2)^2 y_2 e^{-8\sqrt{\log y_2}}\\ &\ll n!\frac{\kappa^{\frac{1}{\sigma}}}{\log\kappa} e^{-8\sqrt{\log\kappa}}.\end{aligned}$$

Let y_3 be the parameter determined by $u(y_3) = 1$, i.e. we take $y_3 = \kappa^{\frac{1}{\sigma}}$. Then we have

$$\begin{aligned}&e^{-8\sqrt{\log y_1}}\int_{y_1}^{y_2} u(y)^n g^{(n)}(u(y))\,dy\\ &= e^{-8\sqrt{\log y_1}}\left(\int_{y_1}^{y_3}+\int_{y_3}^{y_2}\right) u(y)^n g^{(n)}(u(y))\,dy\\ &\ll 2^n n! \kappa^{\frac{1}{\sigma}} e^{-4\sqrt{\log\kappa}}.\end{aligned}$$

As a result, the error term E_2 is estimated as

$$E_2 \ll 2^n n! \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^2}. \tag{14.22}$$

Finally, we calculate the integral in (14.21). Making the change of variables, we obtain

$$\int_{y_1}^{y_2} u(y)^n g^{(n)}(u(y))\,\frac{dy}{\log y} = \kappa^{\frac{1}{\sigma}}\int_{u_2}^{u_1} u^n \frac{g^{(n)}(u)}{u^{\frac{1}{\sigma}}\log(\kappa/u)}\,\frac{du}{u},$$

where we put $u_1 = u(y_1)$ and $u_2 = u(y_2)$. Note that the asymptotic formula

$$\frac{1}{\log(\kappa/u)} = \frac{1}{\log\kappa} + O\left(\frac{|\log u|}{(\log\kappa)^2}\right)$$

holds for $u_1 \leq u \leq u_2$. Therefore the integral is calculated as

$$\int_{y_1}^{y_2} u(y)^n g^{(n)}(u(y))\,\frac{dy}{\log y} = \frac{\kappa^{\frac{1}{\sigma}}}{\log\kappa}\int_{u_2}^{u_1} \frac{g^{(n)}(u)}{u^{\frac{1}{\sigma}+1-n}}\,du + E_3, \tag{14.23}$$

where E_3 is estimated as

$$E_3 \ll \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^2}\int_{u_2}^{u_1}\frac{|g^{(n)}(u)|}{u^{\frac{1}{\sigma}+1-n}}|\log u|\,du \ll \frac{\kappa^{\frac{1}{\sigma}}}{(\log\kappa)^2} \tag{14.24}$$

since Lemma B.3.3 implies

$$\int_0^\infty \frac{|g^{(n)}(u)|}{u^{\frac{1}{\sigma}+1-n}}|\log u|\,du < \infty.$$

Furthermore, we obtain

$$\int_0^{u_2}\frac{g^{(n)}(u)}{u^{\frac{1}{\sigma}+1-n}}\,du \ll n!u_2^{2-\frac{1}{\sigma}} \ll \frac{n!}{\log\kappa},$$
$$\int_{u_1}^{\infty}\frac{g^{(n)}(u)}{u^{\frac{1}{\sigma}+1-n}}\,du \ll 2^n n!u_1^{1-\frac{1}{\sigma}} \ll 2^n n!\kappa^{\frac{1}{2}-\frac{1}{2\sigma}},$$

and therefore,

$$\int_{u_2}^{u_1}\frac{g^{(n)}(u)}{u^{\frac{1}{\sigma}+1-n}}\,du = \int_0^{\infty}\frac{g^{(n)}(u)}{u^{\frac{1}{\sigma}+1-n}}\,du + O\left(\frac{2^n n!}{\log\kappa}\right). \tag{14.25}$$

Here, we remark that the last integral is calculated as

$$\int_0^{\infty}\frac{g^{(n)}(u)}{u^{\frac{1}{\sigma}+1-n}}\,du = \prod_{j=1}^{n}\left(\frac{1}{\sigma}+1-j\right)\times\int_0^\infty\frac{g(u)}{u^{\frac{1}{\sigma}+1}}\,du$$

by the integration by parts. Moreover, it is equal to $g_n(\sigma)$ of (14.14) by making the change of variables. Combining (14.19), (14.21), (14.23), and (14.25) together with estimates (14.20), (14.22), and (14.24), we finally arrive at

$$\sum_{y_1<p\le y_2} f_{\sigma,p}^{(n)}(\kappa;\alpha) = g_n(\sigma)\frac{\kappa^{\frac{1}{\sigma}-n}}{\log\kappa}\left(1+O\left(\frac{2^n n!}{\log\kappa}\right)\right).$$

It completes the proof along with (14.16) and (14.18). □

Let $\tau > 0$ be a large real number. According to the method of Wu [**189**], we then consider the solution of the equation $f_\sigma'(\kappa;\alpha)=\tau$. To begin with, we check the existence and the uniqueness of such a solution.

LEMMA 14.7. *Let $\sigma > 1/2$ and $\alpha\in\mathbb{R}$. For each $\tau>0$, there exists a unique real number $\kappa=\kappa(\tau;\sigma,\alpha)>0$ such that*

$$f_\sigma'(\kappa;\alpha)=\tau.$$

Furthermore, we have $\kappa\to\infty$ as $\tau\to\infty$.

PROOF. First, we show the equality $f_\sigma'(0;\alpha)=0$. Since $f_\sigma(\kappa;\alpha)=\log\mathcal{F}_\sigma(\kappa;\alpha)$, we have $f_\sigma'(\kappa;\alpha)=\mathcal{F}_\sigma'(\kappa;\alpha)/\mathcal{F}_\sigma(\kappa;\alpha)$ for all $\kappa\in\mathbb{R}$. By definition, we obtain

$$\mathcal{F}_\sigma'(\kappa;\alpha)=\mathbb{E}\left[Y\exp(\kappa Y)\right],$$

where we put $Y=\mathrm{Re}(e^{-i\alpha}\log\zeta(\sigma,X))$. Since $\mathbb{E}[X_p^k]=0$ for all $k\ge 1$, we derive $\mathcal{F}_\sigma'(0;\alpha)=\mathbb{E}[Y]=0$. It is easy to see $\mathcal{F}_\sigma(0;\alpha)=1$, and thus we obtain

$f_\sigma'(0;\alpha)=0$. By this, it is sufficient to show that $f_\sigma''(\kappa;\alpha)>0$ if $\kappa>0$ for the proof of the lemma. We calculate the second derivative as

$$\begin{aligned} f_\sigma''(\kappa;\alpha) &= \frac{\mathcal{F}_\sigma''(\kappa;\alpha)\mathcal{F}_\sigma(\kappa;\alpha)-\mathcal{F}_\sigma'(\kappa;\alpha)^2}{\mathcal{F}_\sigma(\kappa;\alpha)^2} \\ &= \frac{1}{\mathcal{F}_\sigma(\kappa;\alpha)}\mathbb{E}\left[\left(Y-f_\sigma'(\kappa;\alpha)\right)^2\exp(\kappa Y)\right]>0. \end{aligned}$$

Hence the result follows. □

Then, we obtain an asymptotic formula of the solution $\kappa(\tau;\sigma,\alpha)$ as $\tau\to\infty$ by applying Proposition 14.6 with $n=1$. This was also proved by Lamzouri–Lester–Radziwiłł [**93**, Corollary 7.6] when $\alpha=0$.

Corollary 14.8. *Let $1/2<\sigma<1$ and $\alpha\in\mathbb{R}$. For $\tau>0$, we take the real number $\kappa(\tau;\sigma,\alpha)>0$ as in Lemma 14.7. Then we have*

$$\kappa(\tau;\sigma,\alpha)=C(\sigma)(\tau\log\tau)^{\frac{\sigma}{1-\sigma}}\left(1+O\left(\frac{\log_2\tau}{\log\tau}\right)\right)$$

if τ is large enough, where the implied constant depends on σ and α. Here, the constant $C(\sigma)$ is given by

$$C(\sigma)=\left(\frac{1-\sigma}{\sigma}g_1(\sigma)\right)^{-\frac{\sigma}{1-\sigma}}.$$

Proof. Put $\kappa=\kappa(\tau;\sigma,\alpha)$. By Proposition 14.6, we have

$$\tau=g_1(\sigma)\frac{\kappa^{\frac{1-\sigma}{\sigma}}}{\log\kappa}\left(1+O\left(\frac{1}{\log\kappa}\right)\right), \tag{14.26}$$

which yields

$$\kappa=\left(\frac{\tau\log\kappa}{g_1(\sigma)}\right)^{\frac{\sigma}{1-\sigma}}\left(1+O\left(\frac{1}{\log\kappa}\right)\right). \tag{14.27}$$

In addition, taking the logarithms in (14.26), we deduce

$$\log\kappa=\frac{\sigma}{1-\sigma}(\log\tau)\left(1+O\left(\frac{\log_2\tau}{\log\tau}\right)\right). \tag{14.28}$$

Hence we derive the desired result by inserting (14.28) into (14.27). □

14.3. Shifted M-functions. Let $\sigma>1/2$ and $\alpha\in\mathbb{R}$. By (14.1), we see that

$$\mathbb{P}(\mathrm{Re}(e^{-i\alpha}\log\zeta(\sigma,X))\in A)=\int_A m_\sigma(u;\alpha)\,|du|$$

for $A\in\mathcal{B}(\mathbb{R})$, where the function $m_\sigma(u;\alpha)$ is given by

$$m_\sigma(u;\alpha)=\int_{\mathbb{R}}\mathcal{M}_\sigma(e^{i\alpha}(u+iv))\,|dv|. \tag{14.29}$$

Then we obtain

$$\Psi(\tau;\alpha)=\int_\tau^\infty m_\sigma(u;\alpha)|du|=\int_0^\infty m_\sigma(u+\tau;\alpha)|du|.$$

The shifted M-function $m_\sigma^\tau(u;\alpha)$ is defined as

$$m_\sigma^\tau(u;\alpha) = \frac{e^{\kappa\tau}}{\mathcal{F}_\sigma(\kappa;\alpha)} e^{\kappa u} m_\sigma(u+\tau;\alpha), \tag{14.30}$$

where $\kappa = \kappa(\tau;\sigma,\alpha) > 0$ is the real number of Lemma 14.7. It is a probability density function on $\mathbb{R}$ since we have $\int_{\mathbb{R}} m_\sigma^\tau(u;\alpha)\,|du| = 1$. Moreover, the formula

$$\widetilde{m}_\sigma^\tau(t;\alpha) = \int_{\mathbb{R}} m_\sigma^\tau(u;\alpha) e^{itu}\,|du| = e^{-it\tau}\frac{\mathcal{F}_\sigma(\kappa+it;\alpha)}{\mathcal{F}_\sigma(\kappa;\alpha)} \tag{14.31}$$

holds for all $t\in\mathbb{R}$. By this and $f_\sigma'(\kappa;\alpha)=\tau$, we obtain

$$\int_{\mathbb{R}} m_\sigma^\tau(u;\alpha)u\,|du| = \frac{1}{i}\frac{d}{dt}\widetilde{m}_\sigma^\tau(t;\alpha)\Big|_{t=0} = 0.$$

We prove that $m_\sigma^\tau(u;\alpha)$ can be approximated by the Gaussian function.

Theorem 14.9. *Let $1/2<\sigma<1$ and $\alpha\in\mathbb{R}$. For $\tau>0$, we take the real number $\kappa=\kappa(\tau;\sigma,\alpha)>0$ as in Lemma 14.7. Then we obtain*

$$m_\sigma^\tau(u;\alpha) = \frac{1}{\sqrt{f_\sigma''(\kappa;\alpha)}}\left\{\exp\left(-\frac{u^2}{2f_\sigma''(\kappa;\alpha)}\right) + O\left(\kappa^{-\frac{1}{2\sigma}}\sqrt{\log\kappa}\right)\right\} \tag{14.32}$$

uniformly for $u\in\mathbb{R}$ if $\tau>0$ is large enough. Here, the implied constant depends only on σ and α.

Proof. By Lemma 14.2 and (14.31), the function $\widetilde{m}_\sigma^\tau(t;\alpha)$ is absolutely integrable. Hence we obtain the inversion formula

$$m_\sigma^\tau(u;\alpha) = \int_{\mathbb{R}} \widetilde{m}_\sigma^\tau(t;\alpha) e^{-itu}\,|dt|.$$

Using Lemma 14.2 again, we have

$$m_\sigma^\tau(u;\alpha) = \int_{-\kappa/3}^{\kappa/3} \widetilde{m}_\sigma^\tau(t;\alpha) e^{-itu}\,|dt| + E_1, \tag{14.33}$$

where $E_1 \ll \sqrt{\kappa}\exp(-\sqrt{\kappa/3})$. We define an entire function $G(z)$ as

$$G(z) = \exp\left(-\tau z - \frac{f_\sigma''(\kappa;\alpha)}{2}z^2\right)\frac{\mathcal{F}_\sigma(z+\kappa;\alpha)}{\mathcal{F}_\sigma(\kappa;\alpha)} = 1 + \sum_{n=3}^\infty \frac{a_n}{n!}z^n.$$

Here, the coefficients a_n are calculated as

$$a_n = \sum_{k=1}^{\lfloor n/3\rfloor}\frac{1}{k!}\sum_{\substack{n_1+\cdots n_k=n\\ \forall j,\ n_j\geq 3}}\binom{n}{n_1,\ldots,n_k} f_\sigma^{(n_1)}(\kappa;\alpha)\cdots f_\sigma^{(n_k)}(\kappa;\alpha)$$

since $G(z)$ is also represented as

$$\begin{aligned} G(z) &= \exp\left(f_\sigma(z+\kappa;\alpha) - f_\sigma(\kappa;\alpha) - f_\sigma'(\kappa;\alpha)z - \frac{f_\sigma''(\kappa;\alpha)}{2}z^2\right)\\ &= \exp\left(\sum_{n=3}^\infty \frac{f_\sigma^{(n)}(\kappa;\alpha)}{n!}z^n\right)\end{aligned}$$

at least near the origin. We have

$$\widetilde{m}^{\tau}_{\sigma}(t;\alpha) = \exp\left(-\frac{f''_{\sigma}(\kappa;\alpha)}{2}t^2\right) G(it)$$

by (14.31), and the integral of (14.33) is calculated as

$$\int_{-\kappa/3}^{\kappa/3} \widetilde{m}^{\tau}_{\sigma}(t;\alpha) e^{-itu}\,|dt| = \int_{-\kappa/3}^{\kappa/3} \exp\left(-\frac{f''_{\sigma}(\kappa;\alpha)}{2}t^2\right) e^{-itx}\,|dt| + E_2 \tag{14.34}$$

where

$$\begin{aligned} E_2 &= \int_{-\kappa/3}^{\kappa/3} \exp\left(-\frac{f''_{\sigma}(\kappa;\alpha)}{2}t^2\right) (G(it)-1)e^{-itx}\,|dt| \\ &\ll \int_{0}^{\kappa/3} \exp\left(-\frac{f''_{\sigma}(\kappa;\alpha)}{2}t^2\right) \left(\sum_{n=3}^{\infty} \frac{|a_n|}{n!}t^n\right) dt. \end{aligned}$$

We further evaluate the error term E_2 as follows. Notice that we have

$$\sum_{n=3}^{\infty} \frac{|a_n|}{n!}t^n \leq \exp\left(\sum_{n=3}^{\infty} \frac{|f^{(n)}_{\sigma}(\kappa;\alpha)|}{n!}t^n\right) - 1$$

for $0 \leq t \leq \kappa/3$. Furthermore, it is deduced from Proposition 14.6 that

$$\sum_{n=3}^{\infty} \frac{|f^{(n)}_{\sigma}(\kappa;\alpha)|}{n!}t^n \ll \frac{\kappa^{\frac{1}{\sigma}}}{\log\kappa} \sum_{n=3}^{\infty} \left(\frac{2t}{\kappa}\right)^n \ll \frac{\kappa^{\frac{1}{\sigma}-3}}{\log\kappa}t^3,$$

where the implied constant depends only on σ and α. Hence there exists a constant $B = B(\sigma,\alpha) > 0$ such that

$$\sum_{n=3}^{\infty} \frac{|a_n|}{n!}t^n \leq \sum_{n=1}^{\infty} \frac{1}{n!}\left(B\frac{\kappa^{\frac{1}{\sigma}-3}}{\log\kappa}t^3\right)^n,$$

which deduces

$$\begin{aligned} E_2 &\ll \sum_{n=1}^{\infty} \frac{1}{n!}\left(B\frac{\kappa^{\frac{1}{\sigma}-3}}{\log\kappa}\right)^n \int_{0}^{\kappa/3} \exp\left(-\frac{f''_{\sigma}(\kappa;\alpha)}{2}t^2\right) t^{3n}\,dt \\ &\ll \frac{1}{\sqrt{f''_{\sigma}(\kappa;\alpha)}} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{3n+1}{2})}{n!} \left(2\sqrt{2}B\frac{\kappa^{\frac{1}{\sigma}-3}}{\sqrt{f''_{\sigma}(\kappa;\alpha)}^{3}\log\kappa}\right)^n. \end{aligned}$$

By Proposition 14.6, we see that

$$\frac{\kappa^{\frac{1}{\sigma}-3}}{\sqrt{f''_{\sigma}(\kappa;\alpha)}^{3}\log\kappa} \ll \kappa^{-\frac{1}{2\sigma}}\sqrt{\log\kappa}$$

holds. Therefore we arrive at the estimate

$$E_2 \ll \frac{1}{\sqrt{f''_{\sigma}(\kappa;\alpha)}}\kappa^{-\frac{1}{2\sigma}}\sqrt{\log\kappa}$$

if $\kappa > 0$ is large enough. Finally, we obtain

$$\begin{aligned}\int_{-\kappa/3}^{\kappa/3} \exp\left(-\frac{f_\sigma''(\kappa;\alpha)}{2}t^2\right) e^{-itu}\,|dt| \\ = \frac{1}{\sqrt{f_\sigma''(\kappa;\alpha)}} \exp\left(-\frac{u^2}{2f_\sigma''(\kappa;\alpha)}\right) + E_3,\end{aligned} \tag{14.35}$$

where

$$E_3 \ll \int_{\kappa/3}^{\infty} \exp\left(-\frac{f_\sigma''(\kappa;\alpha)}{2}t^2\right) dt \ll \exp\left(-\frac{f_\sigma''(\kappa;\alpha)}{18}\kappa^2\right).$$

Since $f_\sigma''(\kappa;\alpha) \ll \kappa^{\frac{1}{\sigma}-2}(\log\kappa)^{-1}$ by Proposition 14.6, we obtain the conclusion by combining (14.33), (14.34), and (14.35) together with the estimates of the error terms. □

Remark that the main term of (14.32) dominates the error term only if $|u|$ is small. Nevertheless, Theorem 14.9 is sufficient for deducing the following corollary which is an improvement of [**93**, Proposition 7.1].

Corollary 14.10. *With the same assumptions as in Theorem 14.9, we obtain*

$$\Psi_\sigma(\tau;\alpha) = \frac{\mathcal{F}_\sigma(\kappa;\alpha)e^{-\kappa\tau}}{\kappa\sqrt{2\pi f_\sigma''(\kappa;\alpha)}}\left(1 + O\left(\kappa^{-\frac{1}{2\sigma}}\sqrt{\log\kappa}\right)\right)$$

if $\tau > 0$ is large enough, where the implied constant depends on σ and α.

Proof. By definition, we have

$$\Psi_\sigma(\tau;\alpha) = \int_0^\infty m_\sigma(u+\tau;\alpha)\,|du| = \mathcal{F}_\sigma(\kappa;\alpha)e^{-\kappa\tau}\int_0^\infty e^{-\kappa u} m_\sigma^\tau(u;\alpha)\,|du|.$$

Applying Theorem 14.9, we derive

$$\begin{aligned}&\int_0^\infty e^{-\kappa u} m_\sigma^\tau(u;\alpha)\,|du| \\ &= \frac{1}{\sqrt{2\pi f_\sigma''(\kappa;\alpha)}}\left\{\int_0^\infty \exp\left(-\kappa u - \frac{u^2}{2f_\sigma''(\kappa;\alpha)}\right) du \right.\\ &\qquad\qquad \left. + O\left(\kappa^{-\frac{1}{2\sigma}}\sqrt{\log\kappa}\int_0^\infty e^{-\kappa u}\,du\right)\right\} \\ &= \frac{1}{\kappa\sqrt{2\pi f_\sigma''(\kappa;\alpha)}}\left\{\int_0^\infty \exp\left(-u - \frac{u^2}{2\kappa^2 f_\sigma''(\kappa;\alpha)}\right) du + O\left(\kappa^{-\frac{1}{2\sigma}}\sqrt{\log\kappa}\right)\right\}.\end{aligned}$$

A simple calculation yields the formula

$$\int_0^\infty \exp\left(-u - \frac{u^2}{2\kappa^2 f_\sigma''(\kappa;\alpha)}\right) du = 1 + O\left(\kappa^{-\frac{1}{2\sigma}}\sqrt{\log\kappa}\right)$$

by Proposition 14.6. Hence the desired result follows. □

Moreover, we refine Theorem 1.14 from the above results: Proposition 14.6 and Corollaries 14.8, 14.10. The following result can be found in [**93**, Corollary 7.7] when $\alpha = 0$ with a weaker error term.

Corollary 14.11. *Let $1/2 < \sigma < 1$ and $\alpha \in \mathbb{R}$. Then we obtain*

$$\Psi_\sigma(\tau;\alpha) = \exp\left(-A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O\left(\frac{\log_2\tau}{\log\tau}\right)\right)\right)$$

if $\tau > 0$ is large enough, where $A(\sigma)$ is the constant of (1.12), and the implied constant depends on σ and α.

Proof. Take the real number $\kappa = \kappa(\tau;\sigma,\alpha)$ as in Lemma 14.7. Then, by Proposition 14.6 and Corollary 14.10, we have the asymptotic formula

$$\begin{aligned}\log\Psi_\sigma(\tau;\alpha) &= f_\sigma(\kappa;\alpha) - \kappa\tau + O(\log\kappa)\\ &= -(1-\sigma)g_1(\sigma)\frac{\kappa^{\frac{1}{\sigma}}}{\log\kappa}\left(1 + O\left(\frac{1}{\log\kappa}\right)\right)\end{aligned}$$

since $f_\sigma'(\kappa;\alpha) = \tau$ and $g_0(\sigma) = \sigma g_1(\sigma)$. Applying Corollary 14.8, we derive

$$\frac{\kappa^{\frac{1}{\sigma}}}{\log\kappa}\left(1 + O\left(\frac{1}{\log\kappa}\right)\right) = \frac{1-\sigma}{\sigma}C(\sigma)^{\frac{1}{\sigma}}\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O\left(\frac{\log_2\tau}{\log\tau}\right)\right).$$

Therefore we conclude

$$\log\Psi_\sigma(\tau;\alpha) = -A(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O\left(\frac{\log_2\tau}{\log\tau}\right)\right),$$

where $A(\sigma)$ is calculated as

$$\begin{aligned}A(\sigma) &= (1-\sigma)g_1(\sigma)\cdot\frac{1-\sigma}{\sigma}\left(\frac{1-\sigma}{\sigma}g_1(\sigma)\right)^{-\frac{1}{1-\sigma}}\\ &= (1-\sigma)\left(\frac{1-\sigma}{\sigma}\int_0^\infty g(y^{-\sigma})\,dy\right)^{-\frac{\sigma}{1-\sigma}}\end{aligned}$$

as desired. $\square$

15. Extreme values of the Riemann zeta-function

In Section 14, we studied the behavior of the large deviation $\Psi_\sigma(\tau;\alpha)$ defined as (14.2). To study further the distribution of extreme values of $\zeta(\sigma+it)$, we compare

$$\Psi_{\sigma,T}(\tau;\alpha) = \mathbb{P}_T\left(\mathrm{Re}(e^{-i\alpha}\log\zeta(\sigma+it)) > \tau\right)$$

with $\Psi_\sigma(\tau;\alpha)$. For the comparison, we use again the moment-generating functions $\widetilde{\mathcal{M}}_{\sigma,T}(z,z';\zeta)^{\mathcal{E}}$ and $\widetilde{\mathcal{M}}_\sigma(z,z';\zeta)$ defined as (10.2) and (10.3) with $F = \zeta$, respectively. Recall that we have

$$\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}}\exp\left(s\,\mathrm{Re}(e^{-i\alpha}\log\zeta(\sigma+it))\right)dt = \widetilde{\mathcal{M}}_{\sigma,T}(-ie^{-i\alpha}s, -ie^{i\alpha}s;\zeta)^{\mathcal{E}}$$

and $\mathcal{F}_\sigma(s;\alpha) = \widetilde{\mathcal{M}}_\sigma(-ie^{-i\alpha}s, -ie^{i\alpha}s;\zeta)$. Let $1/2 < \sigma < 1$ and $B \geq 1$. In a way similar to the proof of Proposition 11.3, we deduce from Proposition 11.9 the asymptotic formula

$$\begin{aligned}\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} &\exp\left(s\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma+it))\right)\,dt \\ &= \mathcal{F}_\sigma(s;\alpha) + O\left(\frac{\mathcal{F}_\sigma(\kappa;\alpha)}{(\log T)^{B+2}}\right)\end{aligned} \tag{15.1}$$

in the range $|s| \leq a(\log T)^\sigma$, where $a = a(\sigma, B)$ is a positive constant, and the exceptional subset $\mathcal{E} = \mathcal{E}_\zeta(\sigma, B, T)$ satisfies (10.4) with a constant $b = b(\sigma, B) > 0$.

Here, we prepare several results towards the proof of Theorem II. Write

$$\mathbb{P}_T(\cdots)^{\mathcal{E}} = \frac{1}{\operatorname{meas}([0,T]\setminus\mathcal{E})}\operatorname{meas}\{t \in [0,T]\setminus\mathcal{E} \mid \cdots\},$$

where "$\cdots$" means some condition to t. Then we define

$$\Psi_{\sigma,T}(\tau;\alpha)^{\mathcal{E}} = \mathbb{P}_T\left(\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma+it)) > \tau\right)^{\mathcal{E}}$$

similarly to $\Psi_{\sigma,T}(\tau;\alpha)$.

Lemma 15.1. *Let $1/2 < \sigma < 1$, $B \geq 1$, and $\alpha \in \mathbb{R}$. Then there exists a positive constant $c = c(\sigma, B)$ such that*

$$\Psi_{\sigma,T}(\tau;\alpha) - \Psi_{\sigma,T}(\tau;\alpha)^{\mathcal{E}} \ll \Psi_\sigma(\tau;\alpha)\exp\left(-\frac{b}{2}\frac{\log T}{\log_2 T}\right)$$

in the range $1 \ll \tau \leq c(\log T)^{1-\sigma}(\log T)^{-1}$, where $b = b(\sigma, B)$ is the constant of (10.4). The implied constant depends on σ, B, and α.

Proof. By definition, we have

$$\Psi_{\sigma,T}(\tau;\alpha) - \Psi_{\sigma,T}(\tau;\alpha)^{\mathcal{E}} \ll \mathbb{P}_T(t \in \mathcal{E}),$$

which is evaluated by (10.4). Furthermore, by Corollary 14.11, we obtain

$$\Psi_\sigma(\tau;\alpha) \gg \exp\left(-\frac{b}{2}\frac{\log T}{\log_2 T}\right)$$

for $1 \ll \tau \leq c(\log T)^{1-\sigma}(\log T)^{-1}$ if c is small enough. □

Then, we evaluate the difference $\Psi_{\sigma,T}(\tau;\alpha)^{\mathcal{E}} - \Psi_\sigma(\tau;\alpha)$. For this, we define distribution functions F and G on $\mathbb{R}$ as

$$\begin{aligned}F(t) &= \mathbb{P}_T\left(\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma+it)) \leq t\right)^{\mathcal{E}}, \\ G(t) &= \mathbb{P}\left(\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma,X)) \leq t\right) = \int_{-\infty}^{t} m_\sigma(u;\alpha)\,|du|,\end{aligned}$$

where $m_\sigma(u;\alpha)$ is the function of (14.29). Furthermore, we define

$$P(A) = \frac{\displaystyle\int_{\mathbb{R}} 1_{A+\tau}(t)e^{\kappa t}\,dF(t)}{\displaystyle\int_{\mathbb{R}} e^{\kappa t}\,dF(t)} \quad\text{and}\quad Q(A) = \frac{\displaystyle\int_{\mathbb{R}} 1_{A+\tau}(t)e^{\kappa t}\,dG(t)}{\displaystyle\int_{\mathbb{R}} e^{\kappa t}\,dG(t)}$$

for $A \in \mathcal{B}(\mathbb{R})$, where $\kappa = \kappa(\tau; \sigma, \alpha) > 0$ is the real number of Lemma 14.7. Then P and Q are probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Moreover, the shifted M-function $m_\sigma^\tau(u; \alpha)$ of (14.30) is a probability density function of Q, that is, the formula

$$Q(A) = \int_A m_\sigma^\tau(u; \alpha) \, |du|$$

holds for all $A \in \mathcal{B}(\mathbb{R})$. Denote by

$$\Phi(t) = P((-\infty, t]) \quad \text{and} \quad \Psi(t) = Q((-\infty, t])$$

the distribution functions of P and Q, respectively.

Lemma 15.2. *With the notation above, we obtain the equalities*

$$\Psi_{\sigma,T}(\tau; \alpha)^{\mathcal{E}} = e^{-\tau\kappa} \int_{\mathbb{R}} e^{\kappa t} \, dF(t) \cdot \int_0^\infty e^{-\kappa t} \, d\Phi(t), \tag{15.2}$$

$$\Psi_\sigma(\tau; \alpha) = e^{-\tau\kappa} \int_{\mathbb{R}} e^{\kappa t} \, dG(t) \cdot \int_0^\infty e^{-\kappa t} \, d\Psi(t). \tag{15.3}$$

Proof. By definition, we have

$$\int_{\mathbb{R}} f(t) \, d\Phi(t) = \frac{\displaystyle\int_{\mathbb{R}} f(t-\tau) e^{\kappa t} \, dF(t)}{\displaystyle\int_{\mathbb{R}} e^{\kappa t} \, dF(t)}$$

for any measurable function f on $\mathbb{R}$. Taking $f(t) = 1_{[0,\infty)}(t) e^{-\kappa t}$, we obtain

$$\int_0^\infty e^{-\kappa t} \, d\Phi(t) = \frac{\displaystyle\int_{\mathbb{R}} 1_{[\tau,\infty)}(t) e^{\kappa\tau} \, dF(t)}{\displaystyle\int_{\mathbb{R}} e^{\kappa t} \, dF(t)} = \frac{e^{\kappa\tau}}{\displaystyle\int_{\mathbb{R}} e^{\kappa t} \, dF(t)} \Psi_{\sigma,T}(\tau; \alpha)^{\mathcal{E}}.$$

Hence equality (15.2) follows. One can prove (15.3) in a similar way. □

Lemma 15.3. *Let* $1/2 < \sigma < 1$, $B \geq 1$, *and* $\alpha \in \mathbb{R}$. *Then there exists a positive constant* $c = c(\sigma, B)$ *such that*

$$\int_{\mathbb{R}} e^{\kappa t} \, d(F - G)(t) \cdot \int_0^\infty e^{-\kappa t} \, d\Psi(t) \ll \Psi_\sigma(\tau; \alpha) (\log T)^{-B}$$

in the range $1 \ll \tau \leq c (\log T)^{1-\sigma} (\log T)^{-1}$, *where the implied constant depends on* σ, B, *and* α.

Proof. Let $a = a(\sigma, B)$ be the constant described above. By Corollary 14.8, the condition $|\kappa| \leq a (\log T)^\sigma$ is satisfied if we choose $c = c(\sigma, B)$ small enough. Therefore we obtain

$$\begin{aligned} &\int_{\mathbb{R}} e^{\kappa t} \, d(F - G)(t) \\ &= \frac{1}{\operatorname{meas}([0,T] \setminus \mathcal{E})} \int_{[0,T] \setminus \mathcal{E}} \exp\left(\kappa \operatorname{Re}(e^{-i\alpha} \log \zeta(\sigma + it))\right) dt - \mathcal{F}_\sigma(\kappa; \alpha) = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \exp\left(\kappa \operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma+it))\right)\,dt - \mathcal{F}_\sigma(\kappa;\alpha) + O(\mathbb{P}_T(t\in\mathcal{E}))\\
&\ll \mathcal{F}_\sigma(\kappa;\alpha)(\log T)^{-B-2} + \exp\left(-b\frac{\log T}{\log_2 T}\right)\\
&\ll \mathcal{F}_\sigma(\kappa;\alpha)(\log T)^{-B-2}
\end{aligned}$$

by using (15.1) along with (10.4). Furthermore, Theorem 14.9 yields

$$\int_0^\infty e^{-\kappa t}\,d\Psi(t) = \int_0^\infty e^{-\kappa u} m_\sigma^\tau(u;\alpha)\,|du| \ll \frac{1}{\kappa\sqrt{f_\sigma''(\kappa;\alpha)}}.$$

Thus we deduce the result from the above estimates along with Corollary 14.10. □

Lemma 15.4. *Let $1/2<\sigma<1$, $B\geq 1$, and $\alpha\in\mathbb{R}$. Then there exists a positive constant $c=c(\sigma,B)$ such that*

$$\int_{\mathbb{R}} e^{\kappa t}\,dF(t)\cdot\int_0^\infty e^{-\kappa t}\,d(\Phi-\Psi)(t) \ll \Psi_\sigma(\tau;\alpha)\left\{(\log T)^{-B} + \frac{(\tau\log\tau)^{\frac{\sigma}{1-\sigma}}}{(\log T)^\sigma}\right\}$$

in the range $1\ll\tau\leq c(\log T)^{1-\sigma}(\log T)^{-1}$, where the implied constant depends on σ, B, and α.

Proof. Let again $a=a(\sigma,B)$ be the constant described above. Recall that the condition $|\kappa|\leq a(\log T)^\sigma$ is satisfied if we choose $c=c(\sigma,B)$ small enough. Then formula (15.1) yields

$$\begin{aligned}
\int_{\mathbb{R}} e^{\kappa t}\,dF(t) &= \frac{1}{\operatorname{meas}([0,T]\setminus\mathcal{E})}\int_{[0,T]\setminus\mathcal{E}} \exp\left(\kappa\operatorname{Re}(e^{-i\alpha}\log\zeta(\sigma+it))\right)\,dt\\
&\ll \mathcal{F}_\sigma(\kappa;\alpha).
\end{aligned}$$

By the integration by parts, we evaluate the second integral as

$$\int_0^\infty e^{-\kappa t}\,d(\Phi-\Psi)(t) \ll \sup_{t\in\mathbb{R}}|\Phi(t)-\Psi(t)|.$$

Then we estimate the difference $h(t)=\Phi(t)-\Psi(t)$ by applying Esseen's inequality (Theorem A.3.1). Recall that $\Psi'(t)=m_\sigma^\tau(t;\alpha)$ is bounded on $t\in\mathbb{R}$. Furthermore we have $\sup_{t\in\mathbb{R}}|\Psi'(t)|\ll 1/\sqrt{f_\sigma''(\kappa;\alpha)}$ by Theorem 14.9. Hence we obtain

$$\sup_{t\in\mathbb{R}}|h(t)| \ll \int_{-R}^{R}\left|\frac{\phi(u)-\psi(u)}{u}\right|\,du + \frac{1}{R\sqrt{f_\sigma''(\kappa;\alpha)}} \tag{15.4}$$

for any $R>0$, where $\phi(u)$ and $\psi(u)$ are the characteristic functions of P and Q. Note that they are calculated as

$$\phi(u) = \frac{\displaystyle\int_{\mathbb{R}} e^{(\kappa+iu)t}\,dF(t)}{\displaystyle\int_{\mathbb{R}} e^{\kappa t}\,dF(t)}e^{-iu\tau} \quad\text{and}\quad \psi(u) = \frac{\displaystyle\int_{\mathbb{R}} e^{(\kappa+iu)t}\,dG(t)}{\displaystyle\int_{\mathbb{R}} e^{\kappa t}\,dG(t)}e^{-iu\tau}.$$

Recall that we have

$$\int_{\mathbb{R}} e^{\kappa t}\,dF(t) \asymp \int_{\mathbb{R}} e^{\kappa t}\,dG(t) = \mathcal{F}_\sigma(\kappa;\alpha)$$

by formula (15.1). Then, we take $R = aR_\sigma(T)/2$. Remark that $|\kappa + iu| \leq aR_\sigma(T)$ holds for $|u| \leq R$ if $c = c(\sigma, B)$ is small enough. Hence we see that

$$\begin{aligned} \phi(u) - \psi(u) &\ll \frac{\left|\int_{\mathbb{R}} e^{\kappa t}\,d(F-G)(t)\right|}{\int_{\mathbb{R}} e^{\kappa t}\,dF(t)} + \frac{\left|\int_{\mathbb{R}} e^{(\kappa+iu)t}\,d(F-G)(t)\right|}{\int_{\mathbb{R}} e^{\kappa t}\,dG(t)} \\ &\ll (\log T)^{-B-2} + \mathbb{P}_T(t \in \mathcal{E}) \\ &\ll (\log T)^{-B-2} \end{aligned} \tag{15.5}$$

by using formula (15.1) with $s = \kappa$ and $\kappa + iu$. On the other hand, we have

$$\int_{\mathbb{R}} e^{(\kappa+iu)t}\,dF(t) = \int_{\mathbb{R}} e^{\kappa t}\,dF(t) + O\left(|u| \int_{\mathbb{R}} |t| e^{\kappa t}\,dF(t)\right)$$

since $e^{i\theta} = 1 + O(|\theta|)$ holds for any $\theta \in \mathbb{R}$. It deduces

$$\frac{\int_{\mathbb{R}} e^{(\kappa+iu)t}\,dF(t)}{\int_{\mathbb{R}} e^{\kappa t}\,dF(t)} = 1 + O\left(|u|\sqrt{M_T}\frac{\sqrt{\mathcal{F}_\sigma(2\kappa;\alpha)}}{\mathcal{F}_\sigma(\kappa;\alpha)}\right)$$

by the Cauchy–Schwarz inequality, where

$$\begin{aligned} M_T &= \int_{\mathbb{R}} |t|^2\,dF(t) \\ &= \frac{1}{\operatorname{meas}([0,T] \setminus \mathcal{E})} \int_{[0,T]\setminus\mathcal{E}} \left|\operatorname{Re}(e^{-i\alpha} \log \zeta(\sigma + it))\right|^2\,dt. \end{aligned}$$

We also derive a similar asymptotic formula

$$\frac{\int_{\mathbb{R}} e^{(\kappa+iu)t}\,dG(t)}{\int_{\mathbb{R}} e^{\kappa t}\,dG(t)} = 1 + O\left(|u|\sqrt{I}\frac{\sqrt{\mathcal{F}_\sigma(2\kappa;\alpha)}}{\mathcal{F}_\sigma(\kappa;\alpha)}\right),$$

where

$$I = \int_{\mathbb{R}} |t|^2\,dG(t) = \int_{\mathbb{R}} |u|^2 m_\sigma(u;\alpha)\,|du|.$$

Recall that we have $\lim_{T\to\infty} M_T = I < \infty$ by Proposition 10.3. In addition, it is deduced from Proposition 14.4 with $n = 0$ the inequality

$$\frac{\sqrt{\mathcal{F}_\sigma(2\kappa;\alpha)}}{\mathcal{F}_\sigma(\kappa;\alpha)} \leq \exp\left(\frac{1}{2}\frac{\log T}{\log_2 T}\right)$$

in the range $1 \ll \tau \leq c(\log T)^{1-\sigma}(\log_2 T)^{-1}$ if c is sufficiently small. Therefore we deduce

$$\phi(u) - \psi(u) \ll |u| \exp\left(\frac{1}{2}\frac{\log T}{\log_2 T}\right). \tag{15.6}$$

Put $r = \exp\left(-(\log T)(\log_2 T)^{-1}\right)$. By (15.5) and (15.6), we find

$$\begin{aligned}\int_{-R}^{R}\left|\frac{\phi(u)-\psi(u)}{u}\right| du &= \left(\int_{-R}^{-r} + \int_{-r}^{r} + \int_{r}^{R}\right)\left|\frac{\phi(u)-\psi(u)}{u}\right| du \\ &\ll \left(\log\frac{R}{r}\right)(\log T)^{-B-2} + r\exp\left(\frac{1}{2}\frac{\log T}{\log_2 T}\right) \\ &\ll (\log T)^{-B-1},\end{aligned}$$

which yields

$$\sup_{t\in\mathbb{R}}|h(t)| \ll (\log T)^{-B-1} + \frac{1}{(\log T)^{\sigma}\sqrt{f_\sigma''(\kappa;\alpha)}}$$

by estimate (15.4). From the above, we arrive at

$$\begin{aligned}&e^{-\tau\kappa}\int_{\mathbb{R}} e^{\kappa t}\, dF(t)\int_{0}^{\infty} e^{-\kappa t}\, d(\Phi-\Psi)(t) \\ &\ll \Psi_\sigma(\tau;\alpha)\cdot\left\{(\log T)^{-B-1}\kappa\sqrt{f_\sigma''(\kappa;\alpha)} + \frac{\kappa}{(\log T)^{\sigma}}\right\}\end{aligned}$$

by Corollary 14.10. Furthermore, Proposition 14.4 and Corollary 14.8 yield upper bounds $\kappa\sqrt{f_\sigma''(\kappa;\alpha)} \ll \log T$ and $\kappa \ll (\tau\log\tau)^{\frac{\sigma}{1-\sigma}}$. Hence we obtain the result. □

Proof of Theorem II. By (15.2) and (15.3), we derive

$$\begin{aligned}\left|\Psi_{\sigma,T}(\tau;\alpha)^{\mathcal{E}} - \Psi_\sigma(\tau;\alpha)\right| \leq{}& e^{-\tau\kappa}\int_{0}^{\infty} e^{-\kappa t}\, d\Psi(t)\left|\int_{\mathbb{R}} e^{-\kappa t}\, d(F-G)(t)\right| \\ &+ e^{-\tau\kappa}\int_{\mathbb{R}} e^{\kappa t}\, dF(t)\left|\int_{0}^{\infty} e^{-\kappa t}\, d(\Phi-\Psi)(t)\right|.\end{aligned}$$

Combining Lemmas 15.3 and 15.4, we obtain

$$\Psi_{\sigma,T}(\tau;\alpha)^{\mathcal{E}} - \Psi_\sigma(\tau;\alpha) \ll \Psi_\sigma(\tau;\alpha)\left\{(\log T)^{-B} + \frac{(\tau\log\tau)^{\frac{\sigma}{1-\sigma}}}{(\log T)^{\sigma}}\right\}.$$

Then the desired conclusion follows from Lemma 15.1 and this. □

CHAPTER 6

Results for other zeta-functions

In the previous chapters, we studied the value-distributions of L-functions equipped with Euler product representations. On the other hand, the Hurwitz–Lerch zeta-functions have no Euler products in general as seen in Section 4.1. In this chapter, we study the value-distribution of the Lerch zeta-function $L(\lambda, \alpha, \sigma+it)$ in the t-aspect. The proof of Theorem V is particularly completed in Section 16.5. In Section 17, we also study the value-distribution of the function $\widetilde{\eta}_m(\sigma+it)$ introduced in Section 4.2 and finally prove Theorem VI.

16. The fourth result: Hurwitz–Lerch zeta-functions

Let $\sigma > 0$ and $0 < \alpha \leq 1$. Denote by $S_{n,\sigma,\alpha}$ the circle given by the parametric representation $z_{n,\sigma,\alpha}(\theta) = (n+\alpha)^{-\sigma} e^{i\theta}$ for an integer $n \geq 0$. Then we define the probability measure $\mu_{n,\sigma,\alpha}$ as

$$\mu_{n,\sigma,\alpha}(A) = \frac{1}{2\pi} \operatorname{meas}\{\theta \in [0,2\pi] \mid z_{n,\sigma}(\theta;\alpha) \in A\}$$

for $A \in \mathcal{B}(\mathbb{C})$ similarly to $\mu_{n,\sigma}$ of Section 1.2. Put $P_{N,\sigma,\alpha} = \mu_{1,\sigma,\alpha} * \cdots * \mu_{N,\sigma,\alpha}$ for $N \geq 1$. As consequences of Jessen–Wintner [**71**, Section 5], we derive the following results.

Lemma 16.1. *Let* $0 < \alpha \leq 1$. *Then* $P_{N,\sigma,\alpha}$ *converges weakly to a probability measure* $P_{\sigma,\alpha}$ *as* $N \to \infty$ *if and only if* $\sigma > 1/2$. *The limit measure* $P_{\sigma,\alpha}$ *is absolutely continuous and satisfies*

$$P_{\sigma,\alpha}(A) = \int_A M_\sigma(w;\alpha)\,|dw|$$

for all $A \in \mathcal{B}(\mathbb{C})$, *where* $M_\sigma(\,\cdot\,;\alpha) : \mathbb{C} \to \mathbb{R}_{\geq 0}$ *is a continuous function satisfying the following properties:*

(i) *The function* $M_\sigma(w;\alpha)$ *belongs to the class* $C^\infty(\mathbb{C})$.

(ii) *The support of* $M_\sigma(w;\alpha)$ *is a compact subset of* $\mathbb{C}$ *if* $\sigma > 1$. *On the other hand, we have* $M_\sigma(w;\alpha) > 0$ *for all* $w \in \mathbb{C}$ *if* $1/2 < \sigma \leq 1$.

(iii) *We have* $M_\sigma(w;\alpha) \ll e^{-c|w|^2}$ *for any* $c > 0$, *and every partial derivative has the same upper bound.*

Note that the Fourier transform $\widetilde{M}_\sigma(z;\alpha)$ is represented as the infinite product

$$\widetilde{M}_\sigma(z;\alpha) = \prod_{n=0}^{\infty} J_0(|z|(n+\alpha)^{-\sigma})$$

by the arguments in [71, Section 4]. In the study of the value-distribution of the Lerch zeta-function $L(\lambda, \alpha, s)$, the function $M_\sigma(w; \alpha)$ plays a role similar to $\mathcal{M}_\sigma(w)$ defined for the Riemann zeta-function, if α satisfies adequate properties. We define a subset $\mathfrak{S} \subset (0, 1]$ as follows.

Definition 16.2. We define the set $\mathfrak{S}$ as the collection of all $\alpha \in (0, 1]$ that satisfy the following conditions.

(A1) *Independence.* The system $\{\log(n + \alpha)\}_{n \geq 0}$ is linearly independent over $\mathbb{Q}$.

(A2) *Spacing.* There exists a constant $\Omega(\alpha) > 0$ such that for large $X > 0$ and for any positive integer N, we have

$$\left| \sum_{j=1}^{N} \epsilon_j \log(n_j + \alpha) \right| \geq X^{-\Omega(\alpha) N^2}$$

provided $\sum_{j=1}^{N} \epsilon_j \log(n_j + \alpha) \neq 0$ for arbitrary integers $0 \leq n_1, \ldots, n_N \leq X$ and $\epsilon_1, \ldots, \epsilon_N \in \{\pm 1\}$.

Then, we introduce a class of test functions on $\mathbb{C}$ according to Ihara–Matsumoto [63, Section 9]. Denote by $\Lambda(\mathbb{C})$ the collection of all functions $f \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})$ with $\widetilde{f} \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})$ such that the inverse formula

$$f(z) = \int_{\mathbb{C}} \widetilde{f}(w) \psi_{-z}(w) \, |dw|$$

holds. Note that Parseval's identity

$$\int_{\mathbb{C}} \widetilde{f}(z) \overline{\widetilde{g}(z)} \, |dz| = \int_{\mathbb{C}} f(w) \overline{g(w)} \, |dw|$$

is valid for any $f, g \in \Lambda(\mathbb{C})$. The Schwartz space $\mathcal{S}(\mathbb{C})$ is included in $\Lambda(\mathbb{C})$, and especially, any compactly supported function in $C^\infty(\mathbb{C})$ belongs to the class $\Lambda(\mathbb{C})$. Moreover, the function $M_\sigma(w; \alpha)$ is a member of $\Lambda(\mathbb{C})$ by Lemma 16.1 (iii). The key tool for the study of the value-distribution of $L(\lambda, \alpha, s)$ is the following mean value theorem.

Theorem 16.3. *Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathfrak{S}$. Let σ_1 be a large fixed positive constant, and let $\theta, \delta > 0$ with $\theta + \delta < 1/4$. Then there exists $T_0 = T_0(\alpha, \sigma_1, \theta, \delta) > 0$ such that for all $T \geq T_0$ and for all $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, we have*

$$\frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it)) dt = \int_{\mathbb{C}} \Phi(w) M_\sigma(w; \alpha) \, |dw| + E,$$

where the test function Φ belongs to the class $\Lambda(\mathbb{C})$. Here, the error term E is estimated as

$$E \ll \exp\left(-\frac{1}{2} (\log T)^{\theta/2}\right) \int_{\Omega} |\widetilde{\Phi}(z)| \, |dz| + \int_{\mathbb{C} \setminus \Omega} |\widetilde{\Phi}(z)| \, |dz|$$

with the square Ω defined as

$$\Omega = \{x + iy \in \mathbb{C} \mid |x| \leq (\log T)^\delta, |y| \leq (\log T)^\delta\}. \tag{16.1}$$

The implied constant depends only on λ and σ_1.

16.1. Preliminaries. Definition 16.2 is motivated by Mahler's classification mentioned in Section 4.1. In fact, it is easy to see that any transcendental number satisfies condition (A1). As for condition (A2), we have the following result.

Lemma 16.4. *Every S-number in $(0,1]$ belongs to the set $\mathfrak{S}$.*

Proof. Since all S-numbers are transcendental, every S-number $\alpha \in (0,1]$ satisfies condition (A1). Then we show that it also satisfies condition (A2). By definition, if α is an S-number, then there exists a positive real number $\omega'(\alpha)$ such that we have

$$|P(\alpha)| \geq h^{-n\omega'(\alpha)} \tag{16.2}$$

for every non-zero polynomial $P(T) \in \mathbb{Z}[T]$ of degree at most n and height at most h. Let μ and ν be non-negative integers with $\mu + \nu = N$, where $N \geq 1$. Then we take N integers $0 \leq m_1, \dots, m_\mu, n_1, \dots, n_\nu \leq X$ which satisfy

$$(m_1 + \alpha) \cdots (m_\mu + \alpha) \neq (n_1 + \alpha) \cdots (n_\nu + \alpha).$$

Here, if one of μ and ν is equal to 0, the product is understood as the value 1. We have

$$\begin{aligned} \left| \sum_{j=1}^{\mu} \log(m_j + \alpha) - \sum_{k=1}^{\nu} \log(n_k + \alpha) \right| &= \left| \log \frac{(m_1 + \alpha) \cdots (m_\mu + \alpha)}{(n_1 + \alpha) \cdots (n_\nu + \alpha)} \right| \\ &\geq \frac{|(m_1 + \alpha) \cdots (m_\mu + \alpha) - (n_1 + \alpha) \cdots (n_\nu + \alpha)|}{\max\{(m_1 + \alpha) \cdots (m_\mu + \alpha), (n_1 + \alpha) \cdots (n_\nu + \alpha)\}} \\ &\geq \frac{|P(\alpha)|}{(2X)^N} \end{aligned} \tag{16.3}$$

with the polynomial $P(T) = (m_1 + T) \cdots (m_\mu + T) - (n_1 + T) \cdots (n_\nu + T)$. The degree of $P(T)$ is at most $\mu + \nu = N$, and the height is at most

$$\binom{N}{\lfloor N/2 \rfloor} X^N \leq (2X)^N.$$

Hence, by (16.2), we have

$$|P(\alpha)| \geq (2X)^{-\omega'(\alpha) N^2}.$$

Therefore we obtain the desired inequality by (16.3) when X is large. □

Almost all real numbers are S-numbers with respect to the one-dimensional Lebesgue measure; see [5, Theorem 8.2]. Hence we deduce the following corollary.

Corollary 16.5. *Almost all α in $(0,1]$ belong to $\mathfrak{S}$ with respect to the one-dimensional Lebesgue measure.*

In addition, we present two preliminary lemmas on $L(\lambda, \alpha, s)$ used later.

Lemma 16.6. *Let $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then we have*

$$L(\lambda, \alpha, s) = \sum_{0 \leq n \leq Y} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^s} + \delta_\lambda \frac{Y^{1-s}}{s-1} + O_\lambda(Y^{-\sigma})$$

for any $\sigma > 1/2$ and $2\pi \leq |t| \leq \pi \lambda Y$, where $\delta_\lambda = 1$ if $\lambda \in \mathbb{Z}$, and $\delta_\lambda = 0$ otherwise.

Proof. This is just the result of [**100**, Theorem 3.1.2] if $\lambda \notin \mathbb{Z}$. When $\lambda \in \mathbb{Z}$, the Lerch zeta-function is the Hurwitz zeta-function. Thus the lemma is deduced from [**100**, Theorem 3.1.3]. □

Lemma 16.7. *Let $\lambda \in \mathbb{R}$ and $0 < \alpha \le 1$. Then we obtain*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |L(\lambda,\alpha,\sigma+it)|^2\,dt = \zeta(2\sigma,\alpha)$$

for any $\sigma > 1/2$.

Proof. The result is elementary for $\sigma > 1$ since formula (4.1) is available. Furthermore, we deduce from [**100**, Theorem 3.3.3] the conclusion for $\sigma = 1$, and from [**100**, Theorem 3.3.1] for $1/2 < \sigma < 1$. □

16.2. Proof of Theorem 16.3. We prove Theorem 16.3 by modifying the method of Guo [**52**]. The key step to the proof is to show the following result.

Proposition 16.8. *Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathfrak{S}$. Let σ_1 be a large fixed positive constant, and let $\theta, \delta > 0$ with $\theta+\delta < 1/4$. Then there exists $T_0 = T_0(\alpha,\sigma_1,\theta,\delta) > 0$ such that for all $T \ge T_0$ and for all $\sigma \in [1/2+(\log T)^{-\theta}, \sigma_1]$, we have*

$$\frac{1}{T}\int_0^T \psi_z\left(L(\lambda,\alpha,\sigma+it)\right)\,dt = \widetilde{M}_\sigma(z;\alpha) + O\left(\exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)\right)$$

for any $z \in \Omega$ with the square Ω of (16.1). The implied constant depends only on λ and σ_1.

According to [**52**], we divide the proof of this proposition into four parts.

Proposition 16.9. *Let $\lambda \in \mathbb{R}$ and $0 < \alpha \le 1$. Let σ_1 be a large fixed positive constant, and let $\theta, \delta > 0$ with $\theta < 2/3$. Then there exists $T_0 = T_0(\theta) > 0$ such that for all $T \ge T_0$ and for all $\sigma \in [1/2+(\log T)^{-\theta}, \sigma_1]$, we have*

$$\frac{1}{T}\int_0^T \psi_z(L(\lambda,\alpha,\sigma+it))\,dt = \frac{1}{T}\int_0^T \psi_z\Bigg(\sum_{0\le n\le X}\frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}}\Bigg)\,dt + E_1$$

for any $z \in \Omega$, where $X = \exp((\log T)^{3\theta/2})$, and E_1 is estimated as

$$E_1 \ll_{\lambda,\sigma_1} \exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right).$$

Proof. Without loss of generality, we may assume $0 < \lambda \le 1$. By the definition of ψ_z, we have $|\psi_z(w)| = 1$ and $|\psi_z(w) - \psi_z(w')| \le |z||w-w'|$ for any $z, w, w' \in \mathbb{C}$. Hence we have

$$|E_1| \le \frac{2\pi}{T} + \frac{|z|}{T}\int_{2\pi}^T \Bigg|L(\lambda,\alpha,\sigma+it) - \sum_{0\le n\le X}\frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}}\Bigg|\,dt.$$

Let $Y = T/\lambda$. Then we have $X < Y$ by $\theta < 2/3$. We deduce from Lemma 16.6 that

$$\Bigg|L(\lambda,\alpha,\sigma+it) - \sum_{0\le n\le X}\frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}}\Bigg| \ll_\lambda \Bigg|\sum_{X<n\le Y}\frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}}\Bigg| + \frac{T^{1-\sigma}}{t} + T^{-\sigma}$$

for $\sigma > 1/2$ and $2\pi \leq t \leq T$. Applying the Cauchy–Schwarz inequality, we obtain

$$E_1 \ll_\lambda \frac{1}{T} + \frac{|z|}{T^{1/2}} \left(\int_0^T \left| \sum_{X<n\leq Y} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}} \right|^2 dt \right)^{1/2} + |z| T^{-\sigma} \log T. \tag{16.4}$$

By the inequality [**149**, Theorem 2], we derive

$$\int_0^T \left| \sum_{X<n\leq Y} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}} \right|^2 dt = \sum_{X<n\leq Y} \frac{T+O(n)}{(n+\alpha)^{2\sigma}}.$$

Hence we have

$$\int_0^T \left| \sum_{X<n\leq Y} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}} \right|^2 dt \ll_\lambda T \sum_{n>X} (n+\alpha)^{-2\sigma} \ll_{\sigma_1} \frac{T}{2\sigma-1} X^{1-2\sigma}.$$

Therefore, by (16.4), we obtain

$$E_1 \ll_{\lambda,\sigma_1} \frac{1}{T} + \frac{|z|}{\sqrt{2\sigma-1}} X^{1/2-\sigma} + |z| T^{-\sigma} \log T.$$

Since $X = \exp((\log T)^{3\theta/2})$, $|z| \leq 2(\log T)^\delta$, and $\sigma \geq 1/2 + (\log T)^{-\theta}$, we conclude

$$E_1 \ll_{\lambda,\sigma_1} \exp\left(-\frac{1}{2} (\log T)^{\theta/2} \right)$$

for any $T \geq T_0$, where $T_0 = T_0(\theta)$ is a positive constant. □

PROPOSITION 16.10. *Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathfrak{S}$. Let σ_1 be a large fixed positive constant, and let $\theta, \delta > 0$ with $\theta+\delta < 1/4$. Then there exists $T_0 = T_0(\alpha, \sigma_1, \theta, \delta) > 0$ such that for all $U \geq T \geq T_0$ and for all $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, we have*

$$\frac{1}{T} \int_0^T \psi_z\left(\sum_{0\leq n\leq X} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}} \right) dt = \frac{1}{U} \int_0^U \psi_z\left(\sum_{0\leq n\leq X} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+iu}} \right) du + E_2$$

for any $z \in \Omega$, where $X = \exp((\log T)^{3\theta/2})$, and E_2 is estimated as

$$E_2 \ll \exp\left(-\frac{1}{2} (\log T)^{\theta/2} \right).$$

PROOF. For simplicity, we write

$$F(t) = F(t, \sigma, X; \lambda, \alpha) = \sum_{0\leq n\leq X} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^{\sigma+it}}.$$

Then we have

$$\begin{aligned} \psi_z(F(t)) &= \exp\left(\frac{i\overline{z}}{2} F(t) + \frac{iz}{2} \overline{F(t)} \right) \\ &= \sum_{0\leq \mu+\nu<N} \frac{1}{\mu!\nu!} \left(\frac{i\overline{z}}{2} \right)^\mu \left(\frac{iz}{2} \right)^\nu F(t)^\mu \overline{F(t)}^\nu + O\left(\frac{|z|^N}{N!} |F(t)|^N \right) \end{aligned}$$

for a sufficiently large positive even integer N. Hence E_2 is estimated as

$$E_2 = \sum_{0\le\mu+\nu<N} \frac{1}{\mu!\nu!}\left(\frac{i\bar{z}}{2}\right)^{\mu}\left(\frac{iz}{2}\right)^{\nu} H(\mu,\nu) + O\left(\frac{|z|^N}{N!}\left(\frac{1}{T}\int_0^T |F(t)|^N\,dt + \frac{1}{U}\int_0^U |F(u)|^N\,du\right)\right), \tag{16.5}$$

where

$$H(\mu,\nu) = \frac{1}{T}\int_0^T F(t)^{\mu}\overline{F(t)}^{\nu}\,dt - \frac{1}{U}\int_0^U F(u)^{\mu}\overline{F(u)}^{\nu}\,du.$$

First, we estimate the first term of (16.5). Note that $H(\mu,\nu)$ is evaluated as

$$H(\mu,\nu) \ll \frac{1}{T}\underset{(m_1+\alpha)\cdots(m_\mu+\alpha)\neq(n_1+\alpha)\cdots(n_\nu+\alpha)}{\sum_{0\le m_1\le X}\cdots\sum_{0\le m_\mu\le X}\sum_{0\le n_1\le X}\cdots\sum_{0\le n_\nu\le X}} \frac{1}{(m_1+\alpha)^{\sigma}\cdots(m_\mu+\alpha)^{\sigma}} \times \frac{1}{(n_1+\alpha)^{\sigma}\cdots(n_\nu+\alpha)^{\sigma}}\left|\log\frac{(m_1+\alpha)\cdots(m_\mu+\alpha)}{(n_1+\alpha)\cdots(n_\nu+\alpha)}\right|^{-1}$$

for $(\mu,\nu)\neq(0,0)$. Hence, by condition (A2) of Definition 16.2, we have

$$H(\mu,\nu) \ll \frac{1}{T}\left(\sum_{0\le n\le X}\frac{1}{(n+\alpha)^{\sigma}}\right)^{N} X^{\Omega(\alpha)N^2} \le \frac{1}{T}X^{(\Omega(\alpha)+1)N^2}$$

for $X \ge X(\alpha,\sigma_1)$, where $X(\alpha,\sigma_1)$ is a positive constant . By this and $H(0,0)=0$, the first term of (16.5) is estimated as

$$\ll \frac{(1+|z|^2)^{N/2}}{T}X^{(\Omega(\alpha)+1)N^2}. \tag{16.6}$$

Then we consider the second term of (16.5). Putting $N=2M$, we have

$$\begin{aligned}
&\frac{1}{U}\int_0^U |F(u)|^{2M}du \\
&= \underset{(m_1+\alpha)\cdots(m_M+\alpha)=(n_1+\alpha)\cdots(n_M+\alpha)}{\sum_{0\le m_1\le X}\cdots\sum_{0\le m_M\le X}\sum_{0\le n_1\le X}\cdots\sum_{0\le n_M\le X}} \frac{e^{2\pi i\lambda m_1}\cdots e^{2\pi i\lambda m_M}e^{-2\pi i\lambda n_1}\cdots e^{-2\pi i\lambda n_M}}{\{(n_1+\alpha)\cdots(n_M+\alpha)\}^{2\sigma}} \\
&\quad + O\Bigg(\frac{1}{U}\underset{(m_1+\alpha)\cdots(m_M+\alpha)\neq(n_1+\alpha)\cdots(n_M+\alpha)}{\sum_{0\le m_1\le X}\cdots\sum_{0\le m_M\le X}\sum_{0\le n_1\le X}\cdots\sum_{0\le n_M\le X}} \frac{1}{(m_1+\alpha)^{\sigma}\cdots(m_M+\alpha)^{\sigma}} \\
&\qquad \times \frac{1}{(n_1+\alpha)^{\sigma}\cdots(n_M+\alpha)^{\sigma}}\left|\log\frac{(m_1+\alpha)\cdots(m_M+\alpha)}{(n_1+\alpha)\cdots(n_M+\alpha)}\right|^{-1}\Bigg).
\end{aligned}$$

By condition (A1) of Definition 16.2, the equation

$$(m_1+\alpha)\cdots(m_M+\alpha) = (n_1+\alpha)\cdots(n_M+\alpha)$$

reduces $\{m_1, \dots, m_M\} = \{n_1, \dots, n_M\}$. Thus the diagonal term is evaluated as

$$\ll M! \Bigg(\sum_{0 \le n \le X} \frac{1}{(n+\alpha)^{2\sigma}} \Bigg)^M \le M! \left(\frac{c(\alpha, \sigma_1)}{2\sigma - 1} \right)^M, \tag{16.7}$$

where $c(\alpha, \sigma_1)$ is a positive constant. The off-diagonal term is also estimated as

$$\ll \frac{1}{U} \Bigg(\sum_{0 \le n \le X} \frac{1}{(n+\alpha)^{\sigma}} \Bigg)^N X^{\Omega(\alpha) N^2} \ll \frac{1}{T} X^{(\Omega(\alpha)+1) N^2} \tag{16.8}$$

for $X \ge X(\alpha, \sigma_1)$ by applying condition (A2). Therefore we have

$$E_2 \ll \frac{(1+|z|^2)^{N/2}}{T} X^{(\Omega(\alpha)+1)N^2} + |z|^N \frac{(\frac{N}{2})!}{N!} \left(\frac{c(\alpha, \sigma_1)}{2\sigma - 1} \right)^{N/2}$$

by (16.6), (16.7), and (16.8). We have $\theta + 2\delta < \frac{1}{2} - \frac{3}{4}\theta$ since $\theta + \delta < \frac{1}{4}$. We take $\eta = \frac{1}{2}\{(\theta + 2\delta) + (\frac{1}{2} - \frac{3}{4}\theta)\}$ and $N = 2\lfloor (\log T)^{\eta} \rfloor$. Recalling $X = \exp((\log T)^{3\theta/2})$, $|z| \le 2(\log T)^{\delta}$, and $\sigma \ge 1/2 + (\log T)^{-\theta}$, we obtain

$$E_2 \ll \exp\left(-\frac{1}{2} (\log T)^{\theta/2} \right)$$

for $T \ge T_0$, where $T_0 = T_0(\alpha, \sigma_1, \theta, \delta)$ is a positive constant. □

Proposition 16.11. *Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathfrak{S}$. Then, for any $\sigma > 1/2$ and $X \ge 1$, we have*

$$\lim_{U \to \infty} \frac{1}{U} \int_0^U \psi_z \Bigg(\sum_{0 \le n \le X} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^{\sigma + iu}} \Bigg) du = \prod_{0 \le n \le X} J_0(|z|(n+\alpha)^{-\sigma}).$$

Proof. Let $\gamma_n = -\log(n+\alpha)$ for $n \ge 0$. Then $\{\gamma_n\}_{n \ge 0}$ is linearly independent over $\mathbb{Q}$ by condition (A1) of Definition 16.2. Hence the complex numbers $e^{i\gamma_n}$ are equidistributed on the unit circle. It yields

$$\begin{aligned} \frac{1}{U} \int_0^U \psi_z \Bigg(\sum_{0 \le n \le X} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^{\sigma + iu}} \Bigg) du &= \frac{1}{U} \int_0^U \prod_{0 \le n \le X} \psi_z \left(\frac{e^{2\pi i \lambda n}}{(n+\alpha)^{\sigma}} e^{i\gamma_n u} \right) du \\ &\to \prod_{0 \le n \le X} \frac{1}{2\pi} \int_0^{2\pi} \psi_z \left(\frac{e^{2\pi i \lambda n}}{(n+\alpha)^{\sigma}} e^{i\theta} \right) d\theta \end{aligned}$$

as $U \to \infty$. Moreover, we see that

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_z \left(\frac{e^{2\pi i \lambda n}}{(n+\alpha)^{\sigma}} e^{i\theta} \right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \exp(i|z|(n+\alpha)^{-\sigma} \cos\varphi)\, d\varphi$$

by the change of variables. The right-hand side is equal to $J_0(|z|(n+\alpha)^{-\sigma})$, and therefore the result follows. □

Proposition 16.12. *Let $\lambda \in \mathbb{R}$ and $0 < \alpha \le 1$. Let σ_1 be a large fixed positive constant, and let $\theta, \delta > 0$. Then there exists $T_0 = T_0(\theta) > 0$ such that for all $T \ge T_0$ and for all $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, we have*

$$\prod_{0 \le n \le X} J_0(|z|(n+\alpha)^{-\sigma}) = \widetilde{M}_{\sigma}(z; \alpha) + E_3$$

for any $z \in \Omega$, *where* $X = \exp((\log T)^{3\theta/2})$, *and* E_3 *is estimated as*

$$E_3 \ll_{\sigma_1} \exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right).$$

Proof. Let $n > X$ with $X = \exp((\log T)^{3\theta/2})$. Then, for sufficiently large T, we have

$$J_0(|z|(n+\alpha)^{-\sigma}) = 1 + O(|z|^2(n+\alpha)^{-2\sigma}),$$

and moreover, we see that

$$\prod_{n>X} J_0(|z|(n+\alpha)^{-\sigma}) = 1 + O\Bigg(|z|^2 \sum_{n>X} \frac{1}{(n+\alpha)^{2\sigma}}\Bigg).$$

Since we have

$$\sum_{n>X} \frac{1}{(n+\alpha)^{2\sigma}} \ll_{\sigma_1} \frac{X^{1-2\sigma}}{2\sigma - 1},$$

the error term E_3 is estimated as

$$|E_3| \le \prod_{0\le n\le X} \left|J_0(|z|(n+\alpha)^{-\sigma})\right| \left|\prod_{n>X} J_0(|z|(n+\alpha)^{-\sigma}) - 1\right| \ll_{\sigma_1} \frac{|z|^2 X^{1-2\sigma}}{2\sigma - 1}.$$

By $X = \exp((\log T)^{3\theta/2})$, $|z| \le 2(\log T)^{\delta}$, and $\sigma \ge 1/2 + (\log T)^{-\theta}$, we have

$$E_3 \ll_{\sigma_1} \exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)$$

for $T \ge T_0$, where $T_0 = T_0(\theta)$ is a positive constant. □

Proof of Proposition 16.8. It is directly deduced by combining Propositions 16.9–16.12. □

Proof of Theorem 16.3. By the assumption on the test function Φ, we obtain the inversion formula

$$\Phi(L(\lambda,\alpha,s)) = \int_{\mathbb{C}} \widetilde{\Phi}(z)\psi_{-z}(L(\lambda,\alpha,s))\,|dz|.$$

Then we use Proposition 16.8. For $T \ge T_0$ and $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, we have

$$\begin{aligned}\frac{1}{T}\int_0^T \Phi(L(\lambda,\alpha,\sigma+it))\,dt &= \int_{\mathbb{C}} \widetilde{\Phi}(z)\frac{1}{T}\int_0^T \psi_{-z}(L(\lambda,\alpha,\sigma+it))\,dt|dz| \\ &= \int_{\Omega} \widetilde{\Phi}(z)\frac{1}{T}\int_0^T \psi_{-z}(L(\lambda,\alpha,\sigma+it))\,dt|dz| + E_1 \\ &= \int_{\Omega} \widetilde{\Phi}(z)\widetilde{M}_{\sigma}(-z;\alpha)\,|dz| + E_1 + E_2 \\ &= \int_{\mathbb{C}} \widetilde{\Phi}(z)\widetilde{M}_{\sigma}(-z;\alpha)\,|dz| + E_1 + E_2 + E_3.\end{aligned}$$

The above error terms are estimated as

$$E_1,\ E_3 \ll \int_{\mathbb{C}\setminus\Omega} |\widetilde{\Phi}(z)|\,|dz|,$$

and

$$E_2 \ll_{\lambda,\sigma_1} \exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)\int_{\Omega}|\widetilde{\Phi}(z)|\,|dz|$$

by the inequalities $|\psi_z(w)| \leq 1$ and $|\widetilde{M}_\sigma(z;\alpha)| \leq 1$. Finally, we have

$$\int_{\mathbb{C}}\widetilde{\Phi}(z)\widetilde{M}_\sigma(-z;\alpha)\,|dz| = \int_{\mathbb{C}}\widetilde{\Phi}(z)\overline{\widetilde{M}_\sigma(z;\alpha)}\,|dz| = \int_{\mathbb{C}}\Phi(w)\overline{M_\sigma(w;\alpha)}\,|dw|$$

by Parseval's identity. Since $M_\sigma(w;\alpha)$ is real, we derive the conclusion. □

16.3. Discrepancy bounds. Let $\sigma > 1/2$ and $T > 0$. For $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$, we define the probability measure $P_{\sigma,T}(\,\cdot\,;\lambda,\alpha)$ as

$$P_{\sigma,T}(A;\lambda,\alpha) = \mathbb{P}_T(L(\lambda,\alpha,\sigma+it) \in A)$$

for $A \in \mathcal{B}(\mathbb{C})$. The characteristic function of $P_{\sigma,T}(\,\cdot\,;\lambda,\alpha)$ is equal to

$$\frac{1}{T}\int_0^T \psi_z\left(L(\lambda,\alpha,\sigma+it)\right)\,dt.$$

Note further that the characteristic function of $P_{\sigma,\alpha}$ presented in Lemma 16.1 is equal to the function $\widetilde{M}_\sigma(z;\alpha)$. Therefore, Proposition 16.8 implies that $P_{\sigma,T}(\,\cdot\,;\lambda,\alpha)$ converges weakly to $P_{\sigma,\alpha}$ as $T \to \infty$ if $\alpha \in \mathfrak{S}$; see Lemma A.1.1. Then, we evaluate the discrepancy

$$D_\sigma(T,\mathcal{R};\lambda,\alpha) = \left|P_{\sigma,T}(\mathcal{R};\lambda,\alpha) - P_{\sigma,\alpha}(\mathcal{R})\right|$$

as follows.

THEOREM 16.13. *Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathfrak{S}$. We take a large fixed real number $\sigma_1 > 0$ and a small fixed real number $\epsilon_1 > 0$. Then, for $1/2+\epsilon_1 \leq \sigma \leq \sigma_1$, we have*

$$D_\sigma(T,\mathcal{R};\lambda,\alpha) \ll (\operatorname{meas}\mathcal{R}+1)(\log T)^{-1/4+\epsilon},$$

where $\mathcal{R}$ is any rectangle in $\mathbb{C}$ whose edges are parallel to the axes and have length greater than $(\log T)^{-1/4+\epsilon}$. The implied constant depends on $\alpha, \lambda, \sigma_1$, and ϵ_1.

We deduce Theorem 16.13 from Theorem 16.3 by approximating the indicator function $1_{\mathcal{R}}$ by functions in $\Lambda(\mathbb{C})$. For this, we use the following function $F_{a,b,\omega}$ introduced by Lester [**106**, Lemma 4.1].

LEMMA 16.14 (Lester). *Let $K(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$. Then, for any $a, b, \omega \in \mathbb{R}$ with $a < b$ and $\omega > 0$, there exists a continuous function $F_{a,b,\omega} : \mathbb{R} \to \mathbb{R}$ such that the following conditions hold.*

(i) *We have $F_{a,b,\omega}(x) - 1_{[a,b]}(x) \ll K(\omega(x-a)) + K(\omega(x-b))$ for any $x \in \mathbb{R}$.*

(ii) *The function $F_{a,b,\omega}$ belongs to $L^1(\mathbb{R})$, and more precisely,*

$$\int_{\mathbb{R}}(F_{a,b,\omega}(x) - 1_{[a,b]}(x))\,dx \ll \omega^{-1}.$$

(iii) *If $|u| \geq \omega$, then $\widetilde{F}_{a,b,\omega}(u) = 0$.*

(iv) *We have $\widetilde{F}_{a,b,\omega}(u) \ll (b-a) + \omega^{-1}$ for any $u \in \mathbb{R}$.*

Note that the function $F_{a,b,\omega}$ is constructed by using the (refined) Beurling–Selberg function

$$H(x) = \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ \sum_{m=-\infty}^{\infty} \mathrm{sgn}(m)(x-m)^{-2} + 2x^{-1} \right\}$$

which approximate well the signum function $\mathrm{sgn}(x)$.

Proof of Theorem 16.13. We take the positive constants $\lambda, \alpha, \sigma_1, \theta, \delta$ as in the assumption of Theorem 16.8. For $T \geq T_0(\alpha, \sigma_1, \theta, \delta)$, we write the rectangle $\mathcal{R}$ as

$$\mathcal{R} = \{w \in \mathbb{C} \mid a \leq \mathrm{Re}\, w \leq b, c \leq \mathrm{Im}\, w \leq d\}$$

with $b-a, d-c \gg (\log T)^{-\delta}$. We also take $\omega = (2\pi)^{-1}(\log T)^{\delta}$. Note that condition (iv) of Lemma 16.14 reduces $\widetilde{F}_{a,b,\omega}(x) \ll (b-a)$ in this case. Then we define

$$\Phi(w) = F_{a,b,\omega}(u) F_{c,d,\omega}(v)$$

for $w = u + iv$. Since we have

$$\Lambda(\mathbb{C}) = \{f \in L^1(\mathbb{C}) \mid f \text{ is continuous and } \widetilde{f} \in L^1(\mathbb{C})\}$$

by [**64**, Section 5.1], it is deduced from Lemma 16.14 that Φ is a member of $\Lambda(\mathbb{C})$. Moreover, we have

$$\Phi(w) - 1_{\mathcal{R}}(w) \ll K(\omega(u-a)) + K(\omega(u-b)) + K(\omega(v-c)) + K(\omega(v-d)), \tag{16.9}$$

and $\widetilde{\Phi}(z) \ll (b-a)(d-c) = \mathrm{meas}\, \mathcal{R}$ by Lemma 16.14. In the above setting, we apply Theorem 16.3. Let $\epsilon_1 > 0$, and we take a large real number $T_1 > 0$ so that $(\log T_1)^{-\theta} < \epsilon_1$ holds. Then, for all $T \geq T_1$ and for all $1/2 + \epsilon_1 \leq \sigma \leq \sigma_1$, we have

$$\frac{1}{T}\int_0^T \Phi(L(\lambda, \alpha, \sigma + it))\, dt = \int_{\mathbb{C}} \Phi(w) M_\sigma(w; \alpha)\, |dw| + E, \tag{16.10}$$

where

$$E \ll_{\lambda,\sigma_1} \exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right) \int_{\Omega} |\widetilde{\Phi}(z)|\, |dz| + \int_{\mathbb{C}\setminus\Omega} |\widetilde{\Phi}(z)|\, |dz|.$$

By Lemma 16.14, we see that

$$E \ll_{\lambda,\sigma_1} \exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)(\log T)^{2\delta}(\mathrm{meas}\, \mathcal{R}) \ll (\mathrm{meas}\, \mathcal{R})(\log T)^{-\delta}. \tag{16.11}$$

We estimate the difference between the left-hand side of (16.10) and

$$P_{\sigma,T}(\mathcal{R}; \lambda, \alpha) = \frac{1}{T}\int_0^T 1_{\mathcal{R}}(L(\lambda, \alpha, \sigma + it))\, dt.$$

Applying inequality (16.9), we see that it is estimated as

$$\begin{aligned} &\ll \frac{1}{T}\int_0^T K\left(\omega(\mathrm{Re}\, L(\lambda, \alpha, \sigma + it) - a)\right)\, dt \\ &\quad + \frac{1}{T}\int_0^T K\left(\omega(\mathrm{Re}\, L(\lambda, \alpha, \sigma + it) - b)\right)\, dt + \end{aligned}$$

$$+\frac{1}{T}\int_0^T K\left(\omega(\operatorname{Im}L(\lambda,\alpha,\sigma+it)-c)\right)\,dt$$
$$+\frac{1}{T}\int_0^T K\left(\omega(\operatorname{Im}L(\lambda,\alpha,\sigma+it)-d)\right)\,dt.$$

Here, we consider just the first term

$$\frac{1}{T}\int_0^T K\left(\omega(\operatorname{Re}L(\lambda,\alpha,\sigma+it)-a)\right)\,dt \tag{16.12}$$

since the other terms are estimated in a similar way. We have

$$K(\omega x)=\frac{2}{\omega^2}\int_0^\omega(\omega-u)\cos(2\pi xu)\,du=\frac{2}{\omega^2}\operatorname{Re}\int_0^\omega(\omega-u)e^{2\pi ixu}\,du.$$

Hence (16.12) is estimated as

$$\begin{aligned}
&\ll\frac{1}{\omega^2}\int_0^\omega(\omega-u)\left|\frac{1}{T}\int_0^T\exp(2\pi iu(\operatorname{Re}L(\lambda,\alpha,\sigma+it)-a))dt\right|\,du\\
&=\frac{1}{\omega^2}\int_0^\omega(\omega-u)\left|\frac{1}{T}\int_0^T\exp(2\pi iu\operatorname{Re}L(\lambda,\alpha,\sigma+it))dt\right|du\\
&\ll_{\lambda,\sigma_1}\frac{1}{\omega^2}\int_0^\omega(\omega-u)\left|\widetilde{M}_\sigma(2\pi u;\alpha)+\exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)\right|\,du\\
&\ll_{\alpha,\epsilon_1}\frac{1}{\omega}+\exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)\ll(\log T)^{-\delta}
\end{aligned}$$

by using Proposition 16.8 and the fact that $\widetilde{M}_\sigma(z;\alpha)$ is rapidly decreasing. Therefore we have

$$\frac{1}{T}\int_0^T\Phi(L(\lambda,\alpha,\sigma+it))\,dt-P_{\sigma,T}(\mathcal{R};\lambda,\alpha)\ll_{\lambda,\alpha,\sigma_1,\epsilon_1}(\log T)^{-\delta}. \tag{16.13}$$

At the end of the proof, we estimate the difference

$$\begin{aligned}
&\int_{\mathbb{C}}\Phi(w)M_\sigma(w;\alpha)\,|dw|-\int_{\mathcal{R}}M_\sigma(w;\alpha)\,|dw|\\
&=\int_{\mathbb{C}}(\Phi(w)-1_{\mathcal{R}}(w))M_\sigma(w;\alpha)\,|dw|
\end{aligned}$$

coming from the right-hand side of (16.10). Note that we have

$$\int_{\mathbb{C}}K(\omega(u-a))M_\sigma(w;\alpha)\,|dw|=\int_{\mathbb{R}}K(\omega(u-a))m_\sigma(u;\alpha)\,|du|, \tag{16.14}$$

where

$$m_\sigma(u;\alpha)=\int_{\mathbb{R}}M_\sigma(u+iv;\alpha)\,|dv|.$$

By Lemma 16.1 (iii), the function $m_\sigma(u;\alpha)$ is estimated as

$$m_\sigma(u;\alpha)\ll_{\alpha,\epsilon_1}1.$$

Hence we obtain that (16.14) is evaluated as

$$\ll_{\alpha,\epsilon_1} \int_{\mathbb{R}} K(\omega(u-a))\,|du| \ll \omega^{-1} = (\log T)^{-\delta}.$$

Thus, applying (16.9) again, we derive

$$\int_{\mathbb{C}} \Phi(w) M_\sigma(w;\alpha)\,|dw| - \int_{\mathcal{R}} M_\sigma(w;\alpha)\,|dw| \ll_{\alpha,\epsilon_1} (\log T)^{-\delta} \tag{16.15}$$

since the remaining terms are evaluated similarly. Therefore, we conclude

$$D_\sigma(T,\mathcal{R};\lambda,\alpha) \ll (\operatorname{meas}\mathcal{R}+1)(\log T)^{-\delta}$$

by combining (16.11), (16.13), and (16.15). We take $\theta = \epsilon/2$ and $\delta = 1/4-\epsilon$ for each $0<\epsilon<1/4$. Then we have $\theta,\delta>0$ and $\theta+\delta<1/4$, and the desired result follows. □

16.4. Further results on the density function. We need more results on $\widetilde{M}_\sigma(z;\alpha)$ as a function of σ. The goal of this section is the following estimate.

PROPOSITION 16.15. *Let $0<\alpha\le 1$ and $\sigma>1/2$. Then there exists a positive constant $c(\sigma)$ such that for any integers $k,\ell,m\ge 0$, we have*

$$\frac{\partial^{k+\ell+m}}{\partial x^k \partial y^\ell \partial \sigma^m} \widetilde{M}_\sigma(z;\alpha) \ll_{k,\ell,m} \exp\left(-c(\sigma)|z|^{1/\sigma}\right)$$

if $|z|$ is large enough.

More generally, we consider the function $\widetilde{M}(s,z_1,z_2;\alpha)$ defined as

$$\widetilde{M}(s,z_1,z_2;\alpha) = \prod_{n=0}^{\infty} \widetilde{M}_n(s,z_1,z_2;\alpha), \tag{16.16}$$

where

$$\begin{aligned} \widetilde{M}_n&(s,z_1,z_2;\alpha) \\ &= \int_0^1 \exp\{iz_1(n+\alpha)^{-s}\cos(2\pi\theta) + iz_2(n+\alpha)^{-s}\sin(2\pi\theta)\}\,d\theta \\ &= 1 - \frac{z_1^2+z_2^2}{4}(n+\alpha)^{-2s} \\ &\qquad + \int_0^1 \sum_{k=3}^{\infty} \frac{i^k}{k!}(n+\alpha)^{-ks}(z_1\cos(2\pi\theta)+z_2\sin(2\pi\theta))^k\,d\theta \end{aligned} \tag{16.17}$$

for $s,z_1,z_2\in\mathbb{C}$. Note that $\widetilde{M}_n(\sigma,x,y;\alpha) = J_0(|x+iy|(n+\alpha)^{\sigma})$ if $\sigma,x,y\in\mathbb{R}$. Thus we obtain $\widetilde{M}(\sigma,x,y;\alpha) = \widetilde{M}_\sigma(x+iy;\alpha)$.

LEMMA 16.16. *Let $s,z_1,z_2\in\mathbb{C}$ with* $\operatorname{Re} s>1/2$. *If we fix two of the variables, the function $\widetilde{M}(s,z_1,z_2;\alpha)$ is holomorphic with respect to the remaining variable.*

Proof. Let K be any compact subset of the right-half plane $\operatorname{Re} s > 1/2$, and let K_1, K_2 be any compact subsets of $\mathbb{C}$. We prove that infinite product (16.16) uniformly converges on $K \times K_1 \times K_2$. Assume that (s, w_1, w_2) varies in $K \times K_1 \times K_2$, and let σ_0 be the smallest real parts of $s \in K$. Then there exists a sufficiently large constant $N = N(K, K_1, K_2; \alpha)$ such that

$$\widetilde{M}_n(s, z_1, z_2; \alpha) = 1 + O_{K_1,K_2}\left((n+\alpha)^{-2\sigma_0}\right)$$

for $n \geq N$ by (16.17). Since the series $\sum_n (n+\alpha)^{-2\sigma_0}$ converges for $\sigma_0 > 1/2$, we obtain the desired convergence. By definition, if we fix two of the variables, the functions $\widetilde{M}_n(s, z_1, z_2; \alpha)$ are holomorphic with respect to the remaining variable. Therefore we obtain the conclusion. □

Lemma 16.17. *Let $0 < \alpha \leq 1$ and $\sigma > 1/2$. Then there exist positive constants $K(\sigma; \alpha)$ and $c(\sigma)$ such that for any $x, y \in \mathbb{R}$ with $|x| + |y| \geq K(\sigma; \alpha)$, we have*

$$\left|\widetilde{M}(s, z_1, z_2; \alpha)\right| \leq \exp\left(-c(\sigma)(|x| + |y|)^{1/\operatorname{Re} s}\right)$$

for any $s, z_1, z_2 \in \mathbb{C}$ with $|s - \sigma| < 1/(x^2 + y^2)$, $|z_1 - x| < 1/2$, $|z_2 - y| < 1/2$.

Proof. Let n_0 be a real number satisfying

$$n_0 + \alpha = \left(\frac{|x| + |y|}{c_0}\right)^{1/\operatorname{Re} s}$$

where c_0 is a positive absolute constant chosen later. At first, we take $c_0 < 1/2$ and $K(\sigma; \alpha) \geq 1$. Then, for $n \geq n_0$, we have

$$(|x| + |y|)(n+\alpha)^{-\operatorname{Re} s} \leq (|x| + |y|)(n_0+\alpha)^{-\operatorname{Re} s} = c_0.$$

Therefore we have

$$\left|\frac{z_1^2 + z_2^2}{4}(n+\alpha)^{-2s}\right| \leq c_0^2,$$

$$\left|\int_0^1 \sum_{k=3}^{\infty} \frac{i^k}{k!}(n+\alpha)^{-ks}(z_1\cos(2\pi\theta) + z_2\sin(2\pi\theta))^k \, d\theta\right| \leq 4c_0^3$$

for $n \geq n_0$, $|z_1 - x| < 1/2$, and $|z_2 - y| < 1/2$. From these inequalities, we deduce

$$\begin{aligned} &\operatorname{Log} \widetilde{M}_n(s, z_1, z_2; \alpha) \\ &= -\frac{z_1^2 + z_2^2}{4}(n+\alpha)^{-2s} + O\left((|x| + |y|)^3 (n+\alpha)^{-3\operatorname{Re} s}\right). \end{aligned} \tag{16.18}$$

Since we have

$$\begin{aligned} &\left|\frac{z_1^2 + z_2^2}{4}(n+\alpha)^{-2s} - \frac{x^2 + y^2}{4}(n+\alpha)^{-2\operatorname{Re} s}\right| \\ &\leq \left|\frac{z_1^2 + z_2^2}{4} - \frac{x^2 + y^2}{4}\right|(n+\alpha)^{-2\operatorname{Re} s} + \frac{x^2 + y^2}{4}\left|(n+\alpha)^{-2s} - (n+\alpha)^{-2\operatorname{Re} s}\right| \\ &\ll (|x| + |y|)(n+\alpha)^{-2\operatorname{Re} s} + \log(n+\alpha)(n+\alpha)^{-2\operatorname{Re} s} \end{aligned}$$

for $|s-\sigma| < 1/(x^2+y^2)$, there exists an absolute constant $A>0$ such that

$$\begin{aligned}&\left|\operatorname{Log}\widetilde{M}_n(s,z_1,z_2;\alpha)+\frac{x^2+y^2}{4}(n+\alpha)^{-2\operatorname{Re}s}\right|\\ &\leq A\left((|x|+|y|)(n+\alpha)^{-\operatorname{Re}s}+(|x|+|y|)^{-1}\right)(|x|+|y|)^2(n+\alpha)^{-2\operatorname{Re}s}\\ &\quad + A\log(n+\alpha)(n+\alpha)^{-2\operatorname{Re}s}.\end{aligned}$$

Then, assuming further $c_0 < (32A)^{-1}$ and $K(\sigma,\alpha)\geq 32A$, we derive

$$A\left((|x|+|y|)(n+\alpha)^{-\operatorname{Re}s}+(|x|+|y|)^{-1}\right)\leq A(c_0+K^{-1})\leq\frac{1}{16}$$

for $|x|+|y|\geq K(\sigma;\alpha)$. Hence, by (16.18), we obtain

$$\log|\widetilde{M}_n(s,z_1,z_2;\alpha)|\leq-\frac{(|x|+|y|)^2}{16}(n+\alpha)^{-2\operatorname{Re}s}+A\log(n+\alpha)(n+\alpha)^{-2\operatorname{Re}s}$$

for $n\geq n_0$, and moreover,

$$\begin{aligned}\left|\prod_{n\geq n_0}\widetilde{M}_n(s,z_1,z_2;\alpha)\right|\leq\exp\Bigg(&-\frac{(|x|+|y|)^2}{16}\sum_{n\geq n_0}(n+\alpha)^{-2\operatorname{Re}s}\\ &+A\sum_{n\geq n_0}\log(n+\alpha)(n+\alpha)^{-2\operatorname{Re}s}\Bigg).\end{aligned}\tag{16.19}$$

Since we have

$$|\operatorname{Re}s-\sigma|<|s-\sigma|<\frac{1}{x^2+y^2}\leq\frac{4}{K(\sigma;\alpha)},$$

we deduce $2\operatorname{Re}s-1>\sigma-1/2>0$ if we take $K(\sigma;\alpha)>4/\sqrt{2\sigma-1}$. Then, for the first term of (16.19), we have

$$\begin{aligned}(|x|+|y|)^2\sum_{n\geq n_0}(n+\alpha)^{-2\operatorname{Re}s}&\geq\frac{1}{2\operatorname{Re}s-1}(n_0+\alpha)^{1-2\operatorname{Re}s}\\ &=\frac{c_0^{\frac{2\operatorname{Re}s-1}{\operatorname{Re}s}}}{2\operatorname{Re}s-1}(|x|+|y|)^{1/\operatorname{Re}s}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{n\geq n_0}\log(n+\alpha)(n+\alpha)^{-2\operatorname{Re}s}&\leq\frac{1}{2\operatorname{Re}s-1}\log(n_0+\alpha)(n_0+\alpha)^{1-2\operatorname{Re}s}\\ &\quad\times\left(1+\frac{2\operatorname{Re}s-1}{n_0+\alpha}+\frac{1}{(2\operatorname{Re}s-1)\log(n_0+\alpha)}\right).\end{aligned}$$

Therefore, we see that there exists $K(\sigma;\alpha)>0$ such that for any $|x|+|y|\geq K(\sigma;\alpha)$,

$$\sum_{n\geq n_0}\log(n+\alpha)(n+\alpha)^{-2\operatorname{Re}s}\leq\Theta\frac{c_0^{\frac{2\operatorname{Re}s-1}{\operatorname{Re}s}}}{2\operatorname{Re}s-1}(|x|+|y|)^{1/\operatorname{Re}s}$$

with a positive constant $\Theta < (16A)^{-1}$. Thus (16.19) deduces

$$\left|\prod_{n\geq n_0} \widetilde{M}_n(s, z_1, z_2; \alpha)\right| \leq \exp\left(-c(\sigma)(|x| + |y|)^{1/\operatorname{Re} s}\right), \tag{16.20}$$

where $c(\sigma)$ is a positive constant depending only on σ. Then, we estimate the contribution of the term for $n < n_0$. By (16.17), we have

$$\left|\widetilde{M}_n(s, z_1, z_2; \alpha)\right| \leq \exp\left((|\operatorname{Im} z_1| + |\operatorname{Im} z_2|)(n+\alpha)^{-\operatorname{Re} s}\right) \leq \exp\left((n+\alpha)^{-\operatorname{Re} s}\right)$$

since $|z_1 - x| < 1/2$ and $|z_2 - y| < 1/2$ with $x, y \in \mathbb{R}$. Therefore we derive

$$\begin{aligned}\left|\prod_{n<n_0} \widetilde{M}_n(s, z_1, z_2; \alpha)\right| &\leq \exp\left(\sum_{n<n_0} (n+\alpha)^{-\operatorname{Re} s}\right)\\ &\leq \exp\left(c'(\sigma;\alpha)(|x| + |y|)^{1/2\operatorname{Re} s}\right),\end{aligned} \tag{16.21}$$

where $c'(\sigma; \alpha)$ is a suitable positive constant. Hence we have the desired result from estimates (16.20) and (16.21). □

Proof of Proposition 16.15. By Lemma 16.16, we can apply Cauchy's integral formula for the function $\widetilde{M}(s, z_1, z_2; \alpha)$. Hence Proposition 16.15 is easily deduced from the upper bound of Lemma 16.17. □

Corollary 16.18. *Let* $0 < \alpha \leq 1$ *and* $\sigma > 1/2$. *Then the function*

$$\frac{\partial^m}{\partial\sigma^m} M_\sigma(z; \alpha) \tag{16.22}$$

belongs to $\mathcal{S}(\mathbb{C})$ *for any integer* $m \geq 0$.

Proof. By Proposition 16.15, the function

$$\frac{\partial^m}{\partial\sigma^m} \widetilde{M}_\sigma(z; \alpha) \tag{16.23}$$

belongs to $\mathcal{S}(\mathbb{C})$. Thus we have

$$\begin{aligned}\frac{\partial^m}{\partial\sigma^m} M_\sigma(z; \alpha) &= \frac{\partial^m}{\partial\sigma^m} \int_{\mathbb{C}} \widetilde{M}_\sigma(w; \alpha)\psi_{-z}(w)\,|dw|\\ &= \int_{\mathbb{C}} \frac{\partial^m}{\partial\sigma^m} \widetilde{M}_\sigma(w; \alpha)\psi_{-z}(w)\,|dw|.\end{aligned}$$

In other words, the Fourier inverse of function (16.23) is equal to function (16.22). Hence function (16.22) belongs to $\mathcal{S}(\mathbb{C})$. □

16.5. Distribution of zeros. With the above preparations, we proceed to the proof of Theorem V. Note that the result can be proved more generally in the case $\alpha \in \mathfrak{S}$.

16.5.1. *Mean square of the logarithm of $L(\lambda, \alpha, s)$.*

Proposition 16.19. *Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathfrak{S}$. We take a large fixed real number $\sigma_1 > 0$ and a small fixed real number $\epsilon_1 > 0$. For $1/2 + \epsilon_1 \leq \sigma \leq \sigma_1$, we have*

$$\begin{aligned}(16.24)\qquad &\frac{1}{T}\int_0^T \log|L(\lambda, \alpha, \sigma + it)|\,dt \\ &= \int_{\mathbb{C}} \log|w| M_\sigma(w; \alpha)\,|dw| + O\left((\log T)^{-A}\right)\end{aligned}$$

with an absolute constant $A > 0$, where the implied constant depends on λ, α, σ_1, and ϵ_1.

Remark that Proposition 16.19 is an analogue of [**53**, Theorem 1.1.3] for Lerch zeta-functions. It is also connected with Theorem V by well-known Littlewood's lemma [**100**, Lemma 8.4.9]; see also [**180**, p. 221]. Before the proof, we estimate the upper bound of the integral

$$\frac{1}{T}\int_0^T \log^2|L(\lambda, \alpha, \sigma + it)|\,dt$$

for $\sigma > 1/2$. The bound is used to evaluate the error term coming from the left-hand side of (16.24).

Proposition 16.20. *With the same assumptions as in Proposition 16.19, we have*

$$\frac{1}{T}\int_0^T \log^2|L(\lambda, \alpha, \sigma + it)|\,dt \ll 1$$

for all $1/2 + \epsilon_1 \leq \sigma \leq \sigma_1$, where the implied constant depends on λ, α, σ_1, and ϵ_1.

The proof is based on the method in [**93**, Section 5]. Let $f(z) = \alpha^z L(\lambda, \alpha, z)$. By [**100**, Lemma 8.5.2], it satisfies

$$(16.25)\qquad 1 - \frac{\alpha}{x-1} < |f(z)| < 1 + \frac{\alpha}{x-1}$$

for $z = x + iy$ with $x > 1 + \alpha$. Let $\sigma_0 = \max(\sigma_1, 3)$. Then we put

$$r = \sigma_0 - \frac{1}{2}\left(\sigma + \frac{1}{2}\right), \quad \delta = \frac{1}{\sigma_1 + 4}\left(\sigma - \frac{1}{2}\right), \quad r_1 = r - \delta, \quad r_2 = r - 2\delta.$$

Note that we have $r > r_1 > r_2 > 0$, $0 < \delta < 1$, and $r + 2\delta < \sigma_0$. We define $s_0(t) = \sigma_0 + it$. If $t \geq \sigma_0$, then f is holomorphic in $|z - s_0(t)| \leq r + 2\delta$ and $1/2 \leq |f(s_0(t))| \leq 3/2$ by inequality (16.25). Then, for $t \geq \sigma_0$ and $0 < u \leq r + 2\delta$, we define

$$\begin{aligned} M_u(t) &= \max_{\substack{z \in \mathbb{C} \\ |z - s_0(t)| \leq u}} \left|\frac{f(z)}{f(s_0(t))}\right| + 3, \\ N_u(t) &= \sum_{\substack{\rho \\ |\rho - s_0(t)| \leq u}} 1, \end{aligned}$$

where ρ runs through zeros of $L(\lambda, \alpha, s)$. We begin with the following formula.

Lemma 16.21. *With the above notation and assumptions, we have*

$$\log|L(\lambda,\alpha,\sigma+it)| = \sum_{|\rho-s_0(t)|\le r_1} \log|\sigma+it-\rho| + O\left(\log M_r(t)\right)$$

for all $1/2+\epsilon_1 \le \sigma \le \sigma_1$ *and* $t \ge \sigma_0$*, where the implied constant depends on* α*,* σ_1*, and* ϵ*.*

Proof. We apply [53, Lemma 2.2.1] to derive

$$\begin{aligned} &\left|\frac{f'}{f}(z) - \sum_{|\rho-s_0(t)|\le r_1} \frac{1}{z-\rho}\right| \\ &\le \frac{36r_1}{(r_1-r_2)^2}\left\{\log M_r(t) + N_{r_1}(t)\log\left(\frac{r_1}{r-r_1}+1\right)\right\} \\ &\le \frac{36\sigma_0}{\delta^2}\left(\log M_r(t) + N_{r_1}(t)\frac{\sigma_0}{\delta}\right) \end{aligned} \tag{16.26}$$

for $|z-s_0(t)| \le r_2$. Then Jensen's formula yields the equation

$$\int_0^r \frac{N_x(t)}{x}\,dx + \log|f(s_0(t))| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(s_0(t)+re^{i\theta})|\,d\theta. \tag{16.27}$$

The left-hand side of (16.27) is estimated as

$$\ge \int_{r_1}^r \frac{N_x(t)}{x}\,dx + \log|f(s_0(t))| \ge N_{r_1}(t)\frac{r-r_1}{r} + \log|f(s_0(t))|.$$

Furthermore, the right-hand side of (16.27) is

$$\le \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(s_0(t)+re^{i\theta})}{f(s_0(t))}\right| d\theta + \log|f(s_0(t))| \le \log M_r(t) + \log|f(s_0(t))|.$$

Hence we have

$$N_{r_1}(t) \le \frac{r}{r-r_1}\log M_r(t) \le \frac{\sigma_0}{\delta}\log M_r(t). \tag{16.28}$$

By (16.26) and (16.28), we obtain

$$\frac{f'}{f}(z) - \sum_{|\rho-s_0(t)|\le r_1} \frac{1}{z-\rho} \ll_{\sigma_1,\epsilon_1} \log M_r(t)$$

for all $|z-s_0(t)| \le r_2$, since $\sigma_0 \ll_{\sigma_1} 1$ and $\delta \gg_{\sigma_1,\epsilon_1} 1$. Note that $|\sigma+it-s_0(t)| = \sigma_0 - \sigma \le r_2$. Integrating from $s_0(t)$ to $\sigma+it$, we have

$$\begin{aligned} &\log|f(\sigma+it)| - \sum_{|\rho-s_0(t)|\le r_1} \log|\sigma+it-\rho| \\ &= \log|f(s_0(t))| - \sum_{|\rho-s_0(t)|\le r_1} \log|s_0(t)-\rho| \\ &\quad + \int_{s_0(t)}^{\sigma+it} \left(\frac{f'}{f}(z) - \sum_{|\rho-s_0(t)|\le r_1} \frac{1}{z-\rho}\right) dz \\ &= \log|f(s_0(t))| - \sum_{|\rho-s_0(t)|\le r_1} \log|s_0(t)-\rho| + O_{\sigma_1,\epsilon_1}\left(\log M_r(t)\right). \end{aligned}$$

Since $1/2 \le |f(s_0(t))| \le 3/2$, we have $\log |f(s_0(t))| \ll 1$. Furthermore, since $1 \le |s_0(t) - \rho| \le \sigma_0$ for all zeros ρ with $|s_0(t) - \rho| < r_1$, we have

$$\sum_{|\rho - s_0(t)| \le r_1} \log |s_0(t) - \rho| \ll_{\sigma_1} N_{r_1}(t) \ll_{\sigma_1, \epsilon_1} \log M_r(t)$$

by (16.28). By the definition of f, the desired result follows. □

By Lemma 16.21, we have

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \qquad\qquad\qquad\qquad\qquad\qquad (16.29) \\ \ll \frac{1}{T} \int_T^{2T} \left(\sum_{|\rho - s_0(t)| \le r_1} \log |\sigma + it - \rho| \right)^2 dt + \frac{1}{T} \int_T^{2T} \log^2 M_r(t) \, dt \end{aligned}$$

for $T \ge \sigma_0$. Then we estimate two integrals of the right-hand side of (16.29).

Lemma 16.22. *For $1/2 + \epsilon_1 \le \sigma \le \sigma_1$, we have*

$$\frac{1}{T} \int_T^{2T} \left(\sum_{|\rho - s_0(t)| \le r_1} \log |\sigma + it - \rho| \right)^2 dt \ll 1, \tag{16.30}$$

where the implied constant depends on λ, α, σ_1, and ϵ_1.

Proof. We have

$$\begin{aligned} &\int_T^{2T} \left(\sum_{|\rho - s_0(t)| \le r_1} \log |\sigma + it - \rho| \right)^2 dt \\ &\le \sum_{n = \lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \int_n^{n+1} \left(\sum_{|\rho - s_0(t)| \le r_1} \log |\sigma + it - \rho| \right)^2 dt. \end{aligned}$$

According to the method in [93], define

$$\mathcal{D}_n = \bigcup_{l=0}^{\lfloor 1/\sqrt{\delta} \rfloor + 1} D_r \left(\sigma_0 + i(n + l\sqrt{\delta}) \right),$$

where $D_r(c) = \{ z \in \mathbb{C} \mid |z - c| \le r \}$. Then we see that

$$\mathcal{D}_n \supset \bigcup_{n \le t \le n+1} D_{r_1}(s_0(t)),$$

and therefore,

$$\begin{aligned} \int_n^{n+1} \left(\sum_{|\rho - s_0(t)| \le r_1} \log |\sigma + it - \rho| \right)^2 dt &\le \int_n^{n+1} \left(\sum_{\rho \in \mathcal{D}_n} \log |\sigma + it - \rho| \right)^2 dt \\ &\le \left(\sum_{\rho \in \mathcal{D}_n} \left(\int_n^{n+1} \log^2 |\sigma + it - \rho| \, dt \right)^{1/2} \right)^2 \end{aligned}$$

by Minkowski's inequality. If $n \le t \le n+1$ and $\rho = \beta + i\gamma \in \mathcal{D}_n$, we have

$$|t - \gamma| \le |\sigma + it - \rho| \le c$$

for a constant $c = c(\sigma_1) > 1$. Thus,

$$\log^2 |\sigma + it - \rho| \le \log^2 |t - \gamma| + \log^2 c.$$

Moreover, we see that

$$\int_n^{n+1} \log^2 |t-\gamma|\, dt \le \int_{n-r-\gamma}^{n+r+2-\gamma} \log^2 x\, dx \le \int_{-2r-2}^{2r+2} \log^2 x\, dx \ll_{\sigma_1} 1$$

since $n - 1 \le \gamma \le n + 2 + r$ for $\rho = \beta + i\gamma \in \mathcal{D}_n$. From the above, we deduce that the left-hand side of (16.30) is evaluated as

$$\begin{aligned} &\int_T^{2T} \left(\sum_{|\rho - s_0(t)| \le r_1} \log |\sigma + it - \rho| \right)^2 dt \\ &\ll_{\sigma_1} \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \left(\sum_{\rho \in \mathcal{D}_n} 1 \right)^2 \le \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \left(\sum_{l=0}^{\lfloor 1/\sqrt{\delta} \rfloor + 1} N_r\left(n + l\sqrt{\delta}\right) \right)^2. \end{aligned} \tag{16.31}$$

By an argument similar to (16.28), we see that

$$N_r\left(n + l\sqrt{\delta}\right) \le \frac{\sigma_0}{\delta} \log M_{r+\delta}\left(n + l\sqrt{\delta}\right).$$

Applying this inequality, we obtain

$$\sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \left(\sum_{l=0}^{\lfloor 1/\sqrt{\delta} \rfloor + 1} N_r\left(n + l\sqrt{\delta}\right) \right)^2 \ll_{\sigma_1, \epsilon_1} \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \log^2 \left(2 \max_{z \in E_n} |f(z)| + 3 \right), \tag{16.32}$$

where

$$E_n = \bigcup_{l=0}^{\lfloor 1/\sqrt{\delta} \rfloor + 1} D_{r+\delta}\left(\sigma_0 + i\left(n + l\sqrt{\delta}\right)\right).$$

We take $s_n = \sigma_n + it_n \in E_n$ so that $|f(z)|$ takes the maximum value at s_n. The function $F(x) = \log^2(x + 3)$ is convex for $x \ge 0$. Hence we have

$$\begin{aligned} \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \log^2 \left(2 \max_{z \in E_n} |f(z)| + 3 \right) &= \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} F(2|f(s_n)|) \\ &\ll TF\left(\frac{1}{T} \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} 2|f(s_n)| \right) \\ &\le TF\left(\left(\frac{4}{T} \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} |f(s_n)|^2 \right)^{1/2} \right). \end{aligned} \tag{16.33}$$

Then we deduce from (16.31), (16.32), and (16.33) the estimate

$$\begin{aligned} &\frac{1}{T} \int_T^{2T} \left(\sum_{|\rho - s_0(t)| \le r_1} \log |\sigma + it - \rho| \right)^2 dt \\ &\ll_{\sigma_1, \epsilon_1} \log^2 \left(\left(\frac{4}{T} \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} |f(s_n)|^2 \right)^{1/2} + 3 \right). \end{aligned} \tag{16.34}$$

The remaining work is estimating $\sum_n |f(s_n)|^2$. For this, we define

$$S_j = \{s_n \mid n \equiv j \bmod (4\lfloor r+2\delta\rfloor + 6)\}.$$

Then we have

$$\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor+1} |f(s_n)|^2 \le \sum_{j=1}^{4\lfloor r+2\delta\rfloor+6} \sum_{\substack{s_n\in S_j \\ \lfloor T\rfloor\le n\le\lfloor 2T\rfloor+1}} |f(s_n)|^2. \tag{16.35}$$

Since the function f is holomorphic on the disk $|z-(\sigma_0+it_n)| \le r+2\delta$ and we have $|s_n-(\sigma_0+it_n)| \le r+\delta$, the inequality of Titchmarsh [**180**, Lemma in p. 256] yields

$$|f(s_n)|^2 \le \frac{1}{\pi\delta^2}\iint_{D_{r+2\delta}(\sigma_0+it_n)} |f(z)|^2\,dxdy.$$

Moreover, if $s_m, s_n \in S_j$ with $m > n$, then we have

$$|t_m - t_n| \ge \{m-(r+\delta)\} - \{n + (\lfloor \tfrac{1}{\sqrt{\delta}}\rfloor + 1)\sqrt{\delta} + (r+\delta)\} > 2r+4\delta.$$

Thus $D_{r+2\delta}(\sigma_0+it_m)$ and $D_{r+2\delta}(\sigma_0+it_n)$ are disjoint, and therefore,

$$\begin{aligned}\sum_{\substack{s_n\in S_j \\ \lfloor T\rfloor\le n\le\lfloor 2T\rfloor+1}} & |f(s_n)|^2 \\ &\le \frac{1}{\pi\delta^2}\int_{\sigma_0-(r+2\delta)}^{\sigma_0+(r+2\delta)}\int_{T-1-(2r+3\delta)}^{2T+3+(2r+3\delta)} |f(x+iy)|^2\,dydx \\ &\ll_{\alpha,\sigma_1,\epsilon_1} \int_{\frac{1}{2}+c_1\epsilon_1}^{2\sigma_0+2}\int_{T-1-(2r+3\delta)}^{2T+3+(2r+3\delta)} |L(\lambda,\alpha,x+iy)|^2\,dydx\end{aligned} \tag{16.36}$$

for a positive constant $c_1 = c_1(\sigma_1)$. Recall that, if $\sigma \ge \frac{1}{2}+\epsilon_1$, then

$$\int_0^T |L(\lambda,\alpha,\sigma+it)|^2\,dt \ll_{\lambda,\alpha,\epsilon_1} T$$

by Lemma 16.7. Hence we see that

$$\int_{\frac{1}{2}+c_1\epsilon_1}^{2\sigma_0+2}\int_{T-1-(2r+3\delta)}^{2T+3+(2r+3\delta)} |L(\lambda,\alpha,x+iy)|^2\,dydx \ll_{\lambda,\alpha,\sigma_1,\epsilon_1} T. \tag{16.37}$$

By (16.35), (16.36), and (16.37), we obtain

$$\frac{1}{T}\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor+1} |f(s_n)|^2 \ll_{\lambda,\alpha,\sigma_1,\epsilon_1} 1. \tag{16.38}$$

Combining (16.34) and (16.38), we conclude

$$\frac{1}{T}\int_T^{2T}\left(\sum_{|\rho-s_0(t)|\le r_1} \log|\sigma+it-\rho|\right)^2 dt \ll_{\lambda,\alpha,\sigma_1,\epsilon_1} 1$$

as desired. □

For the second integral of the right-hand side of (16.29), we obtain the following estimate.

LEMMA 16.23. *For* $1/2+\epsilon_1 \le \sigma \le \sigma_1$, *we have*

$$\frac{1}{T}\int_T^{2T} \log^2 M_r(t)\,dt \ll 1,$$

where the implied constant depends on λ, α, σ_1, *and* ϵ_1.

PROOF. Note that the function $G(x)=\log^2 x$ is convex for $x>e$ and $M_r(t)^2>e$. Then, applying Jensen's inequality, we have

$$\int_T^{2T} \log^2 M_r(t)\,dt \ll \int_T^{2T} \log^2 M_r(t)^2\,dt \le T\log^2\left(\frac{1}{T}\int_T^{2T} M_r(t)^2\,dt\right).$$

We also obtain

$$\int_T^{2T} M_r(t)^2\,dt \le \sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor+1}\int_n^{n+1} M_r(t)^2\,dt.$$

Let

$$F_n = \bigcup_{n\le t\le n+1} D_r(s_0(t)),$$

and denote by $s_n' \in F_n$ a point where $|f(z)|$ takes the maximum value. Then we have

$$\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor+1}\int_n^{n+1} M_r(t)^2\,dt \ll \sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor+1} |f(s_n')|^2 + T.$$

By a similar argument in the proof of Lemma 16.21, one can show

$$\sum_{n=\lfloor T\rfloor}^{\lfloor 2T\rfloor+1} |f(s_n')|^2 \ll_{\lambda,\alpha,\sigma_1,\epsilon_1} T.$$

Hence the result follows. □

By Lemmas 16.22 and 16.23, we have

$$\frac{1}{T}\int_T^{2T} \log^2|L(\lambda,\alpha,\sigma+it)|dt \ll 1.$$

To finish the proof of Proposition 16.20, we need an estimate when T is small.

LEMMA 16.24. *Let* $T_0>0$ *be a fixed real number. Then we have*

$$\int_0^{T_0} \log^2|L(\lambda,\alpha,\sigma+it)|dt \ll 1 \tag{16.39}$$

for $1/2+\epsilon_1 \le \sigma \le \sigma_1$, *where the implied constant depends on* λ, α, σ_1, ϵ_1, *and* T_0.

PROOF. First, we denote by $\rho_1,\dots,\rho_n$ all zeros of $L(\lambda,\alpha,s)$ in the rectangle $1/2+\epsilon_1 \le \operatorname{Re} s \le \sigma_1$ and $0\le \operatorname{Im} s \le T_0$. We take a positive real number r so that the disks $D_r(\rho_1), \dots, D_r(\rho_n)$ are distinct. Here n and r are determined only from λ, α, σ_1, ϵ_1, and T_0. Let Ω be the closure of the set

$$\left\{ \sigma + it \;\middle|\; \frac{1}{2} + \epsilon_1 \leq \sigma \leq \sigma_1,\ 0 \leq t \leq T_0 \right\} \setminus (D_r(\rho_1) \cup \cdots \cup D_r(\rho_n))$$

and define $M = \max_{s \in \Omega} \log^2 |L(\lambda, \alpha, s)|$. We divide the integral of (16.39) into

$$\int_{I_1} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \quad \text{and} \quad \int_{I_2} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt,$$

where

$$I_1 = \{t \in [0, T_0] \mid \sigma + it \in \Omega\} \quad \text{and} \quad I_2 = [0, T_0] \setminus I_1.$$

The integral over I_1 is estimated as

$$\int_{I_1} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \leq M T_0 \ll 1, \tag{16.40}$$

where the implied constant depends only on $\lambda, \alpha, \sigma_1, \epsilon_1$, and T_0. Let m_k be the order of the zero of $L(\lambda, \alpha, s)$ at $s = \rho_k$. Then we have

$$\log^2 |L(\lambda, \alpha, s)| \ll m_k^2 \log^2 |s - \rho_k| + \log^2 |L_k(\lambda, \alpha, s)|,$$

where $L(\lambda, \alpha, s) = (s - \rho_k)^{m_k} L_k(\lambda, \alpha, s)$. Therefore, the integral over I_2 is

$$\begin{aligned} \int_{I_2} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \leq & \sum_{k=1}^{n} m_k^2 \int_{\gamma_k - r}^{\gamma_k + r} \log^2 |\sigma + it - \rho_k| \, dt \\ & + \sum_{k=1}^{n} \int_{\gamma_k - r}^{\gamma_k + r} \log^2 |L_k(\lambda, \alpha, \sigma + it)| \, dt, \end{aligned} \tag{16.41}$$

where we write $\rho_k = \beta_k + i\gamma_k$. Then we obtain

$$\int_{\gamma_k - r}^{\gamma_k + r} \log^2 |\sigma + it - \rho_k| \, dt \leq 2 \int_0^r \log^2 (x + |\sigma - \beta_k|) \, dx \ll 1,$$

$$\int_{\gamma_k - r}^{\gamma_k + r} \log^2 |L_k(\lambda, \alpha, \sigma + it)| \, dt \ll 1$$

with the implied constant depending on $\lambda, \alpha, \sigma_1, \epsilon_1$, and T_0. The maximum value of m_k is determined only from $\lambda, \alpha, \sigma_1, \epsilon_1, T_0$, and therefore we derive

$$\int_{I_1} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \ll 1 \tag{16.42}$$

by equality (16.41). Lemma 16.24 is deduced from estimates (16.40) and (16.42). □

Proof of Proposition 16.20. By (16.29) and Lemmas 16.22 and 16.23, we have

$$\int_{2^{k-1} T_0}^{2^k T_0} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \ll 2^{k-1} T_0$$

for all $k \geq 1$. Together with Lemma 16.24, by summing up over k, we have the result. □

16.5.2. *Approximation of* $\log|w|$. The next work is to approximate the function $\log|w|$ by elements of the class $\Lambda(\mathbb{C})$. The following functions were originally used by Guo [53]. Let

$$f(u) = \begin{cases} \exp\left(-(b-a)\left(\dfrac{1}{u-a}+\dfrac{1}{b-u}\right)\right) & \text{if } a<u<b,\\ 0 & \text{otherwise} \end{cases}$$

for $a, b \in \mathbb{R}$ satisfying $0 < b-a < 1$. Then we define

$$F(x) = \frac{\displaystyle\int_{-\infty}^{x+b-a_1} f(u)\,du}{\displaystyle\int_{-\infty}^{\infty} f(u)\,du} \cdot \frac{\displaystyle\int_{x+a-b_1}^{\infty} f(u)\,du}{\displaystyle\int_{-\infty}^{\infty} f(u)\,du}$$

for $a_1, b_1 \in \mathbb{R}$ with $a_1 < b_1$. With the above setting, we define

$$\Phi(w) = F(|w|)\log|w|.$$

Note that it is infinitely differentiable and is supported on $a_2 \le |w| \le b_2$, where $a_2 = a_1-(b-a)$ and $b_2 = b_1+(b-a)$. Moreover, we take the above a, b, a_1, b_1, a_2, b_2 as the functions

$$\begin{aligned} a &= 1-(\log T)^{-\gamma}, & b &= 1,\\ a_1 &= 2(\log T)^{-\gamma}, & b_1 &= (\log T)^{\gamma},\\ a_2 &= (\log T)^{-\gamma}, & b_2 &= (\log T)^{\gamma}+(\log T)^{-\gamma} \end{aligned}$$

with a constant $0 < \gamma < 1$. Then the following lemma holds.

Lemma 16.25 (Guo). *The function Φ satisfies the following conditions.*

(i) $\Phi(w) = \log|w|$ *for* $a_1 \le |w| \le b_1$.
(ii) $|\Phi(w)| \le |\log|w||$ *for* $a_2 \le |w| \le a_1$ *and* $b_1 \le |w| \le b_2$.
(iii) *The Fourier transform is estimated as*

$$\widetilde{\Phi}(z) \ll (|\log a_2| + |\log b_2|)\min\left(b_2^2, \frac{b_2^2+a_2^{-2}}{(b-a)^2x^2}, \frac{b_2^2+a_2^{-2}}{(b-a)^2y^2}, \frac{b_2^2+a_2^{-2}}{(b-a)^4x^2y^2}\right).$$

Proof of Proposition 16.19. Since the function Φ belongs to the class $\Lambda(\mathbb{C})$, we have

$$\frac{1}{T}\int_0^T \Phi(L(\lambda,\alpha,\sigma+it))\,dt = \int_{\mathbb{C}} \Phi(w)M_\sigma(w;\alpha)\,|dw| + E \tag{16.43}$$

by Theorem 16.3, where

$$E \ll_{\lambda,\sigma_1} \exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)\int_\Omega |\widetilde{\Phi}(z)|\,|dz| + \int_{\mathbb{C}\setminus\Omega} |\widetilde{\Phi}(z)|\,|dz|.$$

By condition (iii) of Lemma 16.25, the first integral is estimated as

$$\int_\Omega |\widetilde{\Phi}(z)|\,|dz| \ll \int_{-(\log T)^\delta}^{(\log T)^\delta}\int_{-(\log T)^\delta}^{(\log T)^\delta} (\log T)^{2\gamma+\epsilon}\,dxdy \ll (\log T)^{2\gamma+2\delta+\epsilon}$$

with sufficiently small $\epsilon > 0$. For the estimate of the second integral, we consider a covering of the region $\mathbb{C} \setminus \Omega$ as follows:

$$\mathbb{C} \setminus \Omega \subset U_1 \cup U_2 \cup U_3,$$

where we put

$$\begin{aligned} U_1 &= \left\{ z = x + iy \;\middle|\; |x| \geq (\log T)^{\delta},\ |y| \leq (\log T)^{\delta/2} \right\}, \\ U_2 &= \left\{ z = x + iy \;\middle|\; |x| \leq (\log T)^{\delta/2},\ |y| \geq (\log T)^{\delta} \right\}, \\ U_3 &= \left\{ z = x + iy \;\middle|\; |x| \geq (\log T)^{\delta/2},\ |y| \geq (\log T)^{\delta/2} \right\}. \end{aligned}$$

Then we have

$$\int_{U_1} |\widetilde{\Phi}(z)|\,|dz| \ll \int_{-(\log T)^{\delta/2}}^{(\log T)^{\delta/2}} \int_{(\log T)^{\delta}}^{\infty} \frac{(\log T)^{4\gamma+\epsilon}}{x^2}\, dx dy \ll (\log T)^{4\gamma - \frac{\delta}{2} + \epsilon}$$

by condition (iii) of Lemma 16.25. In a similar way, we have

$$\int_{U_2} |\widetilde{\Phi}(z)|\,|dz| \ll (\log T)^{4\gamma - \frac{\delta}{2} + \epsilon},$$

and furthermore,

$$\int_{U_3} |\widetilde{\Phi}(z)|\,|dz| \ll \int_{(\log T)^{\delta/2}}^{\infty} \int_{(\log T)^{\delta/2}}^{\infty} \frac{(\log T)^{6\gamma+\epsilon}}{x^2 y^2}\, dx dy \ll (\log T)^{6\gamma - \delta + \epsilon}.$$

Assuming $\gamma < \frac{\delta}{4}$, we have $4\gamma - \frac{\delta}{2} > 6\gamma - \delta$. Thus we obtain

$$\begin{aligned} E &\ll_{\lambda,\sigma_1} \exp\left(-\frac{1}{2} (\log T)^{\theta/2} \right) (\log T)^{2\gamma+2\delta+\epsilon} + (\log T)^{4\gamma - \frac{\delta}{2} + \epsilon} \\ &\ll (\log T)^{4\gamma - \frac{\delta}{2} + \epsilon} \end{aligned} \tag{16.44}$$

for any $T \geq T_0$, where $T_0 = T_0(\alpha, \theta, \delta, \sigma_1, \gamma)$ is a positive constant. Next, we consider the error term

$$E_L = \frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it))\, dt - \frac{1}{T} \int_0^T \log |L(\lambda, \alpha, \sigma + it)|\, dt$$

coming from the left-hand side of (16.43). Let

$$\begin{aligned} I &= \{ t \in [0, T] \mid |L(\lambda, \alpha, \sigma + it)| \leq a_1 \}, \\ J &= \{ t \in [0, T] \mid |L(\lambda, \alpha, \sigma + it)| \geq b_1 \}. \end{aligned}$$

Then Lemma 16.25 gives

$$|E_L| \leq \frac{1}{T} \int_{I \cup J} |\log |L(\lambda, \alpha, \sigma + it)||\, dt.$$

Applying the Cauchy–Schwarz inequality, we see that

$$|E_L| \leq \left(\frac{\operatorname{meas} I + \operatorname{meas} J}{T} \right)^{1/2} \left(\frac{1}{T} \int_0^T \log |L(\lambda, \alpha, \sigma + it)|^2\, dt \right)^{1/2}. \tag{16.45}$$

Define two rectangles $\mathcal{R}$ and $\mathcal{R}'$ as

$$\begin{aligned}\mathcal{R} &= \{z = x + iy \mid |x| \leq (\log T)^{-\gamma},\ |y| \leq (\log T)^{-\gamma}\},\\ \mathcal{R}' &= \{z = x + iy \mid |x| \leq 2^{-1/2}(\log T)^{\gamma},\ |y| \leq 2^{-1/2}(\log T)^{\gamma}\}.\end{aligned}$$

Since we have $(\log T)^{-\gamma} \gg (\log T)^{-1/16}$, it is deduced from Theorem 16.13 that

$$\begin{aligned}\frac{\operatorname{meas} I}{T} &\ll_{\lambda,\alpha,\sigma_1,\epsilon_1} \int_{\mathcal{R}} M_\sigma(w;\alpha)\,|dw| + (\operatorname{meas}\mathcal{R} + 1)(\log T)^{-1/8}\\ &\ll \int_{\mathcal{R}} M_\sigma(w;\alpha)\,|dw| + (\log T)^{-1/8},\end{aligned} \tag{16.46}$$

$$\begin{aligned}\frac{\operatorname{meas} J}{T} &= 1 - \frac{\operatorname{meas}([0,T] \setminus J)}{T}\\ &\ll \int_{\mathbb{C}\setminus\mathcal{R}'} M_\sigma(w;\alpha)\,|dw| + (\log T)^{2\gamma-1/8}.\end{aligned} \tag{16.47}$$

The integrals in (16.46) and (16.47) are estimated as

$$\int_{\mathcal{R}} M_\sigma(w;\alpha)\,|dw| \ll_{\alpha,\epsilon_1} \int_{\mathcal{R}} 1\,du\,dv \ll (\log T)^{-2\gamma}, \tag{16.48}$$

$$\int_{\mathbb{C}\setminus\mathcal{R}'} M_\sigma(w;\alpha)\,|dw| \ll_{\alpha,\epsilon_1} \int_{2^{-1}(\log T)^{\gamma}}^{\infty} e^{-r^2} r\,dr \ll \exp\left(-\frac{1}{4}(\log T)^{2\gamma}\right) \tag{16.49}$$

by Lemma 16.1 (iii). Then we use Proposition 16.20 to deduce from estimates (16.45)–(16.48) and (16.49) the upper bound

$$E_L \ll_{\lambda,\alpha,\sigma_1,\epsilon_1} (\log T)^{2\gamma-1/8}. \tag{16.50}$$

Finally, we evaluate

$$E_R = \int_{\mathbb{C}} \Phi(w) M_\sigma(w;\alpha)\,|dw| - \int_{\mathbb{C}} \log|w| M_\sigma(w;\alpha)\,|dw|$$

which comes from the right-hand side of (16.43). We obtain

$$|E_R| \leq \int_{|w|\leq a_1} |\log|w||M_\sigma(w;\alpha)\,|dw| + \int_{|w|\geq b_1} |\log|w||M_\sigma(w;\alpha)\,|dw|$$

by Lemma 16.25. Applying Lemma 16.1 (iii) again, we see that the first integral is

$$\ll_{\alpha,\epsilon_1} |\log a_1| a_1^2 \ll (\log T)^{-2\gamma+\epsilon},$$

and the second integral is

$$\ll_{\alpha,\epsilon_1} \int_{b_1}^{\infty} (\log r) e^{-r^2} r\,dr \ll \int_{b_1}^{\infty} r^{-2} dr \ll (\log T)^{-\gamma}.$$

If we take $\epsilon < \gamma$, we have

$$E_R \ll_{\alpha,\epsilon_1} (\log T)^{-\gamma}. \tag{16.51}$$

We take $\gamma = \delta/12$ and fix $\theta, \delta > 0$ with $\theta + \delta < 1/4$. Combining estimates (16.44), (16.50), and (16.51), we conclude that

$$\frac{1}{T}\int_0^T \log|L(\lambda,\alpha,\sigma+it)|\,dt - \int_{\mathbb{C}} \log|w| M_\sigma(w;\alpha)\,|dw| \ll_{\lambda,\alpha,\sigma_1,\epsilon_1} (\log T)^{-\delta/12}.$$

Hence the result follows. □

16.5.3. *Completion of the proof.* We use the following lemma of Garunkštis–Laurinčikas [**100**, Lemma 8.4.11] to prove Theorem V.

LEMMA 16.26 (Garunkštis–Laurinčikas). *Let* $1/2 \le \sigma \le 1+\alpha$. *Then we have*

$$2\pi\int_\sigma^{1+\alpha} N(T,u;\lambda,\alpha)\,du = \sigma T\log\alpha + \int_0^T \log|L(\lambda,\alpha,\sigma+it)|\,dt + O(\log T),$$

where $N(T,u;\lambda,\alpha)$ *counts the number of zeros of* $L(\lambda,\alpha,s)$ *in the region* $\operatorname{Re} s > u$, $0 < \operatorname{Im} s < T$.

PROOF OF THEOREM V. By Proposition 16.19 and Lemma 16.26, we have

$$\begin{aligned} &2\pi\int_\sigma^{1+\alpha} N(T,u;\lambda,\alpha)\,du \\ &= \sigma T\log\alpha + T\int_{\mathbb{C}} \log|w| M_\sigma(w;\alpha)\,|dw| + O_{\lambda,\alpha,\sigma}\left(T(\log T)^{-A}\right). \end{aligned}$$

If a small positive real number h satisfies

$$\frac{1}{2} < \frac{1}{2}\left(\sigma+\frac{1}{2}\right) \le \sigma - h < \sigma < \sigma + h \le \sigma + 1,$$

then we have also

$$\begin{aligned} &2\pi\int_{\sigma\pm h}^{1+\alpha} N(T,u;\lambda,\alpha)\,du \\ &= (\sigma\pm h)T\log\alpha + T\int_{\mathbb{C}} \log|w| M_{\sigma\pm h}(w;\alpha)\,|dw| + O_{\lambda,\alpha,\sigma}\left(T(\log T)^{-A}\right). \end{aligned}$$

Hence the formula

$$\begin{aligned} &2\pi\int_\sigma^{\sigma+h} N(T,u;\lambda,\alpha)\,du \\ &= hT\log\alpha + T(\phi_\alpha(\sigma+h) - \phi_\alpha(\sigma)) + O_{\lambda,\alpha,\sigma}\left(T(\log T)^{-A}\right) \end{aligned} \tag{16.52}$$

holds, where $\phi_\alpha(\sigma)$ is the Jensen function

$$\phi_\alpha(\sigma) = \int_{\mathbb{C}} \log|w| M_\sigma(w;\alpha)\,|dw|.$$

By Corollary 16.18, we see that $\phi_\alpha(\sigma)$ is infinitely differentiable, and therefore

$$\frac{\partial^n}{\partial\sigma^n}\phi_\alpha(\sigma) = \int_{\mathbb{C}} \log|w| \frac{\partial^n}{\partial\sigma^n} M_\sigma(w;\alpha)\,|dw|.$$

Hence we have

$$\phi_\alpha(\sigma+h)-\phi_\alpha(\sigma)=h\int_{\mathbb{C}}\log|w|\frac{\partial}{\partial\sigma}M_\sigma(w;\alpha)\,|dw|+O_{\alpha,\sigma}(h^2).$$

Since the function $N(T,u;\lambda,\alpha)$ is decreasing in u, we have

$$\begin{aligned}N(T,\sigma;\lambda,\alpha)\geq T\frac{\log\alpha}{2\pi}&+\frac{T}{2\pi}\int_{\mathbb{C}}\log|w|\frac{\partial}{\partial\sigma}M_\sigma(w;\alpha)\,|dw|\\&+O_{\lambda,\alpha,\sigma}\left(h^{-1}T(\log T)^{-A}+hT\right)\end{aligned}$$

by (16.52). Similarly, we obtain

$$\begin{aligned}N(T,\sigma;\lambda,\alpha)\leq T\frac{\log\alpha}{2\pi}&+\frac{T}{2\pi}\int_{\mathbb{C}}\log|w|\frac{\partial}{\partial\sigma}M_\sigma(w;\alpha)\,|dw|\\&+O_{\lambda,\alpha,\sigma}\left(h^{-1}T(\log T)^{-A}+hT\right)\end{aligned}$$

by considering $\sigma-h$ instead of $\sigma+h$. We take $h=(\log T)^{-A/2}$. The above error terms are $\ll T(\log T)^{-A/2}$, and therefore we have

$$\begin{aligned}N(T,\sigma;\lambda,\alpha)=T\frac{\log\alpha}{2\pi}&+\frac{T}{2\pi}\int_{\mathbb{C}}\log|w|\frac{\partial}{\partial\sigma}M_\sigma(w;\alpha)\,|dw|\\&+O_{\lambda,\alpha,\sigma}\left(T(\log T)^{-A/2}\right).\end{aligned}$$

Thus, for any fixed real numbers $1/2<\sigma_1<\sigma_2$, we obtain

$$\begin{aligned}&N(T,\sigma_1,\sigma_2;\lambda,\alpha)\\&=N(T,\sigma_2;\lambda,\alpha)-N(T,\sigma_1;\lambda,\alpha)\\&=\frac{T}{2\pi}\int_{\mathbb{C}}\log|w|\left(\frac{\partial}{\partial\sigma}M_{\sigma_2}(w;\alpha)-\frac{\partial}{\partial\sigma}M_{\sigma_1}(w;\alpha)\right)|dw|\\&\quad+O\left(T(\log T)^{-A/2}\right)\\&=\frac{T}{2\pi}\int_{\sigma_1}^{\sigma_2}\int_{\mathbb{C}}\log|w|\frac{\partial^2}{\partial\sigma^2}M_\sigma(w;\alpha)\,|dw|d\sigma+O\left(T(\log T)^{-A/2}\right)\end{aligned}$$

as desired, where the implied constant depends on λ,α,σ_1 and σ_2. □

17. The fifth result: iterated integrals of $\log\zeta(s)$

17.1. Probability density functions. Let $\widetilde{\eta}_m(s)$ be the function defined by the m-th iterated integral of $\log\zeta(s)$ as in Section 4.2. Let $\sigma\geq 1/2$ and $m\geq 1$. For the probability measure $\mu_{n,\sigma,m}$ of (4.5), we define

$$c(\mu_{n,\sigma,m})=\int_{\mathbb{C}}z\,d\mu_{n,\sigma,m}(z)\quad\text{and}\quad M(\mu_{n,\sigma,m})=\int_{\mathbb{C}}|z|^2\,d\mu_{n,\sigma,m}(z).$$

Then, the weakly convergence of $P_{N,\sigma,m} = \mu_{1,\sigma,m} * \cdots * \mu_{N,\sigma,m}$ is derived from the convergences of two series

$$\sum_{n=1}^{\infty} |c(\mu_{n,\sigma,m})| \quad \text{and} \quad \sum_{n=1}^{\infty} M(\mu_{n,\sigma,m}) \tag{17.1}$$

by the result of Jessen–Wintner [**71**]; see also Lemma A.1.2. Note that

$$c(\mu_{n,\sigma,m}) = \frac{1}{2\pi}\int_0^{2\pi} z_{n,\sigma,m}(\theta)\, d\theta = 0,$$
$$M(\mu_{n,\sigma,m}) = \frac{1}{2\pi}\int_0^{2\pi} |z_{n,\sigma,m}(\theta)|^2\, d\theta \ll p_n^{-2\sigma}(\log p_n)^{-2m}$$

by definition. Therefore the convergences of (17.1) follow. Let $P_{\sigma,m}$ be the limit measure to which $P_{N,\sigma,m}$ converges weakly as $N \to \infty$. Then, we denote by $\phi_{\sigma,m}(z)$ the characteristic function of $P_{\sigma,m}$. We see that it is represented as

$$\phi_{\sigma,m}(z) = \prod_{n=1}^{\infty} \phi_{n,\sigma,m}(z), \tag{17.2}$$

where $\phi_{n,\sigma,m}(z)$ is the characteristic function of $\mu_{n,\sigma,m}$, that is,

$$\phi_{n,\sigma,m}(z) = \frac{1}{2\pi}\int_0^{2\pi} \psi_z(z_{n,\sigma,m}(\theta))\, d\theta.$$

Proposition 17.1. *Let $\sigma \geq 1/2$ and $m \geq 1$. Then the integral*

$$\int_{\mathbb{C}} |z|^k |\phi_{\sigma,m}(z)|\, |dz| \tag{17.3}$$

is finite for any $k \geq 0$.

Proof. We represent $\phi_{n,\sigma,m}(z)$ as the integral

$$\phi_{n,\sigma,m}(z) = \frac{1}{2\pi}\int_0^{2\pi} \exp(ig_\tau(\theta)|z|)\, d\theta,$$

where $z = |z|e^{i\tau}$ and $g_\tau(\theta) = \operatorname{Re} z_{n,\sigma,m}(\theta)\cos\tau + \operatorname{Im} z_{n,\sigma,m}(\theta)\sin\tau$. It was proved that the curve $S_{n,m}$ is convex by Lewis [**108**]. Furthermore, $\operatorname{Re} z_{n,\sigma,m}(\theta)$ and $\operatorname{Im} z_{n,\sigma,m}(\theta)$ possess continuous second derivatives with respect to θ by definition. One can further show that $g_\tau''(\theta)$ has exactly two zeros on $[0, 2\pi)$ for every fixed τ; see Lemma B.1.1. Thus, we apply the result of Jessen–Wintner [**71**, Theorem 12] to derive

$$\phi_{n,\sigma,m}(z) \ll |z|^{-1/2} \tag{17.4}$$

with the implied constant depending on σ, m, and n. Using this upper bound for $n \leq 2k + 5$, we obtain

$$|z|^k \phi_{\sigma,m}(z) \ll |z|^{-\frac{5}{2}}$$

by equality (17.2). Hence integral (17.3) is finite. □

Then Levy's inversion formula (A.5) yields that the function

$$\mathcal{M}_\sigma(w;\widetilde{\eta}_m)=\int_{\mathbb{C}}\phi_{\sigma,m}(z)\psi_{-w}(z)\,|dz| \tag{17.5}$$

is a probability density function of $P_{\sigma,m}$ with respect to the measure $|dw|$. Furthermore, differentiating under the integral in (17.5), we see that the density function $\mathcal{M}_\sigma(w;\widetilde{\eta}_m)$ belongs to the class $C^\infty(\mathbb{C})$. Let $\sigma\geq 1$. Remark that the series

$$\sum_{n=1}^{\infty}|z_{n,\sigma,m}(\theta)|$$

converges for all $\theta\in[0,2\pi)$. Then the support of the function $\mathcal{M}_\sigma(w;\widetilde{\eta}_m)$ is a compact subset of $\mathbb{C}$, which is proved by an argument similar to the proof of Proposition 8.8. Let $1/2\leq\sigma<1$. In this case, we see that the support is equal to $\mathbb{C}$ similarly to Proposition 8.9. Furthermore, we derive the positivity of $\mathcal{M}_\sigma(w;\widetilde{\eta}_m)$ as follows. We define two probabilistic measures

$$P^\flat_{N,\sigma,m}=\mu_{2,\sigma,m}*\mu_{p^\flat_1,\sigma,m}*\cdots*\mu_{p^\flat_N,\sigma,m},$$
$$P^\#_{N,\sigma,m}=\mu_{p^\#_1,\sigma,m}*\cdots*\mu_{p^\#_N,\sigma,m},$$

where $p^\flat_n$ is the n-th prime number congruent to 1 mod 4, and $p^\#_n$ is the n-th prime number congruent to -1 mod 4. Then $P^\flat_{N,\sigma,m}$ and $P^\#_{N,\sigma,m}$ converge weakly to some probability measures $P^\flat_{\sigma,m}$ and $P^\#_{\sigma,m}$ as $N\to\infty$, respectively. Furthermore, these limit measures satisfy many of the same properties as $P_{\sigma,m}$. Indeed, one can prove

$$\operatorname{supp}(P^\flat_{\sigma,m})=\operatorname{supp}(P^\#_{\sigma,m})=\mathbb{C}$$

for $1/2\leq\sigma<1$. Moreover, applying (17.4), we see that there exist non-negative continuous functions $\mathcal{M}^\flat_\sigma(w;\widetilde{\eta}_m)$ and $\mathcal{M}^\#_\sigma(w;\widetilde{\eta}_m)$ such that

$$P^\flat_{\sigma,m}(A)=\int_A\mathcal{M}^\flat_\sigma(w;\widetilde{\eta}_m)\,|dw| \quad\text{and}\quad P^\#_{\sigma,m}(A)=\int_A\mathcal{M}^\#_\sigma(w;\widetilde{\eta}_m)\,|dw|$$

for all $A\in\mathcal{B}(\mathbb{C})$. Since $P_{\sigma,m}=P^\flat_{\sigma,m}*P^\#_{\sigma,m}$, we derive the equality

$$\mathcal{M}^\#_\sigma(w;\widetilde{\eta}_m)=\int_{\mathbb{C}}\mathcal{M}^\flat_\sigma(w-\xi;\widetilde{\eta}_m)\mathcal{M}^\#_\sigma(\xi;\widetilde{\eta}_m)\,|d\xi|$$

for any $w\in\mathbb{C}$. Since the functions $\mathcal{M}^\flat_\sigma(w;\widetilde{\eta}_m)$ and $\mathcal{M}^\#_\sigma(w;\widetilde{\eta}_m)$ are continuous and are non-zeros on every disk on $\mathbb{C}$, we conclude that $\mathcal{M}_\sigma(w;\widetilde{\eta}_m)>0$ for any $w\in\mathbb{C}$.

17.2. Limit theorems. For the proof of Theorem VI, the remaining work is just proving that the probability measure $P_{\sigma,T,m}$ of (4.4) converges weakly to $P_{\sigma,m}$ as $T\to\infty$. By Lemma A.1.1, it is sufficient to show the following limit formula.

Proposition 17.2. *Let $\sigma\geq 1/2$ and $m\geq 1$. For any $R>0$, we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\psi_z(\widetilde{\eta}_m(\sigma+it))\,dt=\phi_{\sigma,m}(z)$$

uniformly in $|z|\leq R$.

Let $X = (X_p)_p$ be a sequence of independent random variables which are uniformly distributed on the unit circle. For $m \geq 1$, we define the random variable

$$\widetilde{\eta}_m(\sigma, X) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k^{m+1}(\log p)^m} p^{-k\sigma} X_p^k.$$

We see that the series converges almost surely if $\sigma \geq 1/2$. Furthermore, the characteristic function $\phi_{\sigma,m}(z)$ is represented as

$$\phi_{\sigma,m}(z) = \mathbb{E}[\psi_z(\widetilde{\eta}_m(\sigma, X))].$$

For $y \geq 2$, we also define

$$R_{y,m}(s) = \sum_{p^k \leq y} \frac{1}{k^{m+1}(\log p)^m} p^{-ks},$$
$$R_{y,m}(\sigma, X) = \sum_{p^k \leq y} \frac{1}{k^{m+1}(\log p)^m} p^{-k\sigma} X_p^k.$$

Then we prove the following preliminary lemmas.

Lemma 17.3. *Let $\sigma \geq 1/2$ and $m \geq 1$. If we put $y = (\log T)^B$ with $B \geq 1$, then we have*

$$\frac{1}{T}\int_{14}^{T} \left|\widetilde{\eta}_m(\sigma + it) - R_{y,m}(\sigma + it)\right|^2 dt \ll \frac{y^{-(2\sigma-1)}}{(\log y)^{2m}} + \frac{T^{-\frac{1}{135}(2\sigma-1)}}{(\log T)^{2m}},$$
$$\mathbb{E}\left[\left|\widetilde{\eta}_m(\sigma, X) - R_{y,m}(\sigma, X)\right|^2\right] \ll \frac{y^{-(2\sigma-1)}}{(\log y)^{2m}},$$

where the implied constants depend on σ, m, and B.

Proof. The former result is a consequence of Inoue [67, Theorem 5]. To prove the latter result, we calculate the expected value as

$$\begin{aligned}\mathbb{E}\left[\left|\widetilde{\eta}_m(\sigma, X) - R_{y,m}(\sigma, X)\right|^2\right] &= \mathbb{E}\left[\left|\sum_{p>y} \frac{X_p}{p^{\sigma}(\log p)^m}\right|^2\right] + O\left(\frac{y^{1-2\sigma}}{(\log y)^{2m}}\right)\\ &= \sum_{p>y} \frac{1}{p^{2\sigma}(\log p)^{2m}} + O\left(\frac{y^{1-2\sigma}}{(\log y)^{2m}}\right)\\ &\ll \frac{y^{1-2\sigma}}{(\log y)^{2m}}\end{aligned}$$

by using the independence of X_p's. □

Lemma 17.4. *Let $\sigma \geq 1/2$ and $m \geq 1$. If we put $y = (\log T)^B$ with $B \geq 1$, then we have*

$$\frac{1}{T}\int_0^T \overline{R_{y,m}(\sigma + it)}^{j} R_{y,m}(\sigma + it)^{\ell}\, dt$$
$$= \mathbb{E}\left[\overline{R_{y,m}(\sigma, X)}^{j} R_{y,m}(\sigma, X)^{\ell}\right] + O\left(y^{j+\ell}T^{-\frac{5}{6}}\right)$$

for any non-negative integers j, ℓ *with* $j+\ell \leq (\log T)(6B\log_2 T)^{-1}$*, where the implied constant is absolute.*

Proof. The result can be proved in a way similar to Proposition 11.7. □

Lemma 17.5. *Let* $\sigma \geq 1/2$ *and* $m \geq 1$*. If we put* $y=(\log T)^B$ *with* $B \geq 1$*, then there exists a positive constant* $K=K(\sigma,m)$ *such that*

$$\frac{1}{T}\int_0^T |R_{y,m}(\sigma+it)|^{2N}\,dt \ll \begin{cases} K^{2N} & \text{if } \sigma \geq 1, \\ \left(K\dfrac{N^{1-\sigma}}{(\log 2N)^{\sigma+m}}\right)^{2N} & \text{if } 1/2<\sigma\leq 1, \\ \left(K\dfrac{N^{1/2}}{(\log 2N)^{m}}\right)^{2N} & \text{if } \sigma=1/2, \end{cases} \tag{17.6}$$

$$\mathbb{E}\left[|R_{y,m}(\sigma,X)|^{2N}\right] \ll \begin{cases} K^{2N} & \text{if } \sigma \geq 1, \\ \left(K\dfrac{N^{1-\sigma}}{(\log 2N)^{\sigma+m}}\right)^{2N} & \text{if } 1/2<\sigma\leq 1, \\ \left(K\dfrac{N^{1/2}}{(\log 2N)^{m}}\right)^{2N} & \text{if } \sigma=1/2, \end{cases} \tag{17.7}$$

for any integer $N \geq 1$*. The implied constants depend on* σ*,* m*, and* B*.*

Proof. If $\sigma \geq 1$, then we have

$$|R_{y,m}(\sigma+it)| \leq \sum_{p\leq y}\sum_{k=1}^{\infty}\frac{1}{k^{m+1}(\log p)^m}p^{-k\sigma} \ll_\sigma 1.$$

Therefore (17.6) follows for $\sigma \geq 1$. Let $1/2 \leq \sigma < 1$. If $y \leq CN\log 2N$ holds with a constant $C \geq 2$, then we obtain

$$R_{y,m}(\sigma+it) = \sum_{p\leq y}\frac{1}{(\log p)^m p^{\sigma+it}} + O_\sigma(1) \ll_{\sigma,m} \frac{N^{1-\sigma}}{(\log 2N)^{\sigma+m}}$$

by the prime number theorem. Hence (17.6) is satisfied in this case. Thus we suppose the inequality $CN\log 2N < y$ below. We obtain

$$R_{y,m}(s) = \sum_{p<CN\log 2N}\frac{1}{(\log p)^m p^s} + \sum_{CN\log 2N\leq p\leq y}\frac{1}{(\log p)^m p^s} + O_\sigma(1).$$

Then we consider the $2N$-th moments

$$M_1 = \frac{1}{T}\int_0^T \left|\sum_{p<CN\log 2N}\frac{1}{(\log p)^m p^{\sigma+it}}\right|^{2N} dt,$$

$$M_2 = \frac{1}{T}\int_0^T \left|\sum_{CN\log 2N\leq p\leq y}\frac{1}{(\log p)^m p^{\sigma+it}}\right|^{2N} dt.$$

Applying the prime number theorem again, we derive

$$M_1 \ll \left(\sum_{p<CN\log 2N}\frac{1}{(\log p)^m p^{\sigma}}\right)^{2k} \ll \left(\frac{C_1 N^{1-\sigma}}{(\log 2N)^{\sigma+m}}\right)^{2N}, \tag{17.8}$$

where $C_1 = C_1(\sigma, m)$ is a positive constant. Furthermore, we use (11.12) to deduce

$$M_2 \ll N!\Bigg(\sum_{CN\log 2N \leq p \leq y} \frac{1}{(\log p)^{2m} p^{2\sigma}}\Bigg)^N \tag{17.9}$$
$$\ll \begin{cases} \left(\dfrac{C_2 N^{1-\sigma}}{(\log 2N)^{\sigma+m}}\right)^{2N} & \text{if } 1/2 < \sigma < 1, \\ \left(\dfrac{C_2 N^{1-\sigma}}{(\log 2N)^{m}}\right)^{2N} & \text{if } \sigma = 1/2, \end{cases}$$

where $C_2 = C_2(\sigma, m)$ is a positive constant. Since we obtain

$$\frac{1}{T}\int_0^T |R_{y,m}(\sigma+it)|^{2N}\, dt \leq 9^N M_1 + 9^N M_2 + 9^N C_3(\sigma),$$

with a constant $C_3(\sigma) > 0$, upper bounds (17.6) follow from (17.8) and (17.9). One can prove (17.7) in a similar way. □

Proof of Proposition 17.2. Recall that $|\psi_z(w) - \psi_z(w')| \leq |z||w - w'|$ for all $z, w, w' \in \mathbb{C}$. Then we have

$$\frac{1}{T}\int_0^T \psi_z(\widetilde{\eta}_m(\sigma+it))\, dt - \frac{1}{T}\int_0^T \psi_z(R_{m,y}(\sigma+it))\, dt$$
$$\ll_R \frac{1}{T} + \frac{1}{T}\int_{14}^T \left|\widetilde{\eta}_m(\sigma+it) - R_{y,m}(\sigma+it)\right|\, dt$$
$$\ll \frac{1}{T} + \left(\frac{1}{T}\int_{14}^T \left|\widetilde{\eta}_m(\sigma+it) - R_{y,m}(\sigma+it)\right|^2\, dt\right)^{1/2}$$

for $|z| \leq R$ by the Cauchy–Schwarz inequality. By Lemma 17.3, we further derive

$$\frac{1}{T}\int_0^T \psi_z(\widetilde{\eta}_m(\sigma+it))\, dt - \frac{1}{T}\int_0^T \psi_z(R_{m,y}(\sigma+it))\, dt \tag{17.10}$$
$$\ll \frac{1}{T} + \frac{y^{-\frac{1}{2}(2\sigma-1)}}{(\log y)^m} + \frac{T^{-\frac{1}{270}(2\sigma-1)}}{(\log T)^m} \ll (\log_2 T)^{-m}.$$

Along the same lines, we obtain

$$\phi_{\sigma,m}(z) - \mathbb{E}[\psi_z(R_{m,y}(\sigma, X))] \tag{17.11}$$
$$\ll \mathbb{E}\left[\left|\widetilde{\eta}_m(\sigma, X) - R_{y,m}(\sigma, X)\right|^2\right]^{1/2}$$
$$\ll \frac{y^{-\frac{1}{2}(2\sigma-1)}}{(\log y)^m} \ll (\log_2 T)^{-m}.$$

From the above, the remaining work is to evaluate the difference

$$\frac{1}{T}\int_0^T \psi_z(R_{m,y}(\sigma+it))\, dt - \mathbb{E}[\psi_z(R_{m,y}(\sigma, X))]. \tag{17.12}$$

By the Taylor expansion, we have

$$\begin{aligned}(17.13)\qquad & \frac{1}{T}\int_{0}^{T} \psi_{z}(R_{m,y}(\sigma+it))\,dt \\ &= \sum_{j+\ell<N} \frac{(\frac{i}{2})^{j+\ell}}{j!\ell!} z^{j}\overline{z}^{\ell} \frac{1}{T}\int_{0}^{T} \overline{R_{y,m}(\sigma+it)}^{j} R_{y,m}(\sigma+it)^{\ell}\,dt \\ &\quad + O\left(\frac{R^{N}}{N!}\frac{1}{T}\int_{0}^{T} |R_{y,m}(\sigma+it)|^{N}\,dt\right).\end{aligned}$$

Let $N = 2\lfloor(\log T)(\log_2 T)^{-1}\rfloor$ with a large real number T. Applying Lemma 17.5, we deduce

$$(17.14)\qquad \frac{R^{N}}{N!}\frac{1}{T}\int_{0}^{T} |R_{y,m}(\sigma+it)|^{N}\,dt \ll \left(\frac{3\sqrt{2}KR}{\sqrt{N}(\log N)^{m}}\right)^{N/2}.$$

By (17.13) and (17.14), we obtain the asymptotic formula

$$\begin{aligned}& \frac{1}{T}\int_{0}^{T} \psi_{z}(R_{m,y}(\sigma+it))\,dt \\ &= \sum_{j+\ell<N} \frac{(\frac{i}{2})^{j+\ell}}{j!\ell!} z^{j}\overline{z}^{\ell} \frac{1}{T}\int_{0}^{T} \overline{R_{y,m}(\sigma+it)}^{j} R_{y,m}(\sigma+it)^{\ell}\,dt + \left(\exp\left(-\frac{\log T}{\log_2 T}\right)\right),\end{aligned}$$

and similarly,

$$\begin{aligned}& \mathbb{E}[\psi_{z}(R_{m,y}(\sigma, X))] \\ &= \sum_{j+\ell<N} \frac{(\frac{i}{2})^{j+\ell}}{j!\ell!} z^{j}\overline{z}^{\ell} \mathbb{E}\left[\overline{R_{y,m}(\sigma+it)}^{j} R_{y,m}(\sigma+it)^{\ell}\right] + O\left(\exp\left(-\frac{\log T}{\log_2 T}\right)\right).\end{aligned}$$

Therefore, (17.12) is estimated as

$$(17.15)\qquad \ll \sum_{j+\ell<N} \frac{(\frac{R}{2})^{j+\ell}}{j!\ell!} y^{j+\ell} T^{-\frac{5}{6}} + \exp\left(-\frac{\log T}{\log_2 T}\right) \ll \exp\left(-\frac{\log T}{\log_2 T}\right)$$

by Lemma 17.4. Combining (17.10), (17.11), and (17.15), we obtain the desired result. □

APPENDIX A

Probability measures

Denote by $\mathcal{P}(\mathbb{R}^k)$ the set of all probability measures on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. A probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is often regarded as a member of $\mathcal{P}(\mathbb{R}^2)$ by identifying $\mathbb{C}$ with $\mathbb{R}^2$. We hereby list several notions and results on probability measures. The proofs of the results are mostly included in the books of Billingsley [11, 12], Loève [112, 113], and Tenenbaum [179]. We also refer to Jessen–Wintner [71] for some special results.

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^k)$. We define the convolution measure $\mu * \nu \in \mathcal{P}(\mathbb{R}^k)$ as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^k} \mu(A - x)\, d\nu(x), \tag{A.1}$$

where we write $A - x = \{a - x \mid a \in A\}$ for $x \in \mathbb{R}^k$. By definition, we have $\mu_1 * \mu_2 = \mu_2 * \mu_1$ and $\mu_1 * (\mu_2 * \mu_3) = (\mu_1 * \mu_2) * \mu_3$. Let $\nu = \mu_1 * \cdots * \mu_n$. In general, the equality

$$\int_{\mathbb{R}^k} f(x)\, d\nu(x) = \int_{\mathbb{R}^k} d\mu_n(x_n) \cdots \int_{\mathbb{R}^k} f(x + \cdots + x_n)\, d\mu_1(x_1)$$

holds for any non-negative measurable function f.

A probability measure $\mu \in \mathcal{P}(\mathbb{R}^k)$ is said to be absolutely continuous if there exists a measurable function D on $\mathbb{R}^k$ such that

$$\mu(A) = \int_A D(x)\, dx$$

for all $A \in \mathcal{B}(\mathbb{R}^k)$. The function D is determined uniquely from μ up to values on null sets, which is called a probability density function of μ. When $\mu \in \mathcal{P}(\mathbb{R})$ or $\mu \in \mathcal{P}(\mathbb{C})$, we often consider the condition

$$\mu(A) = \int_A D(x)\, |dx| \tag{A.2}$$

for $A \in \mathcal{B}(\mathbb{R})$ with $|dx| = (2\pi)^{-\frac{1}{2}} dx$, and

$$\mu(A) = \int_A D(z)\, |dz| \tag{A.3}$$

for $A \in \mathcal{B}(\mathbb{C})$ with $|dz| = (2\pi)^{-1} dx dy$.

The characteristic function of $\mu \in \mathcal{P}(\mathbb{R}^k)$ is defined as

$$\phi(y; \mu) = \int_{\mathbb{R}^k} \psi_x(y)\, d\mu(x),$$

where we put $\psi_x(y) = \exp(i\langle x, y\rangle)$ with the standard inner product $\langle x, y\rangle$ of $\mathbb{R}^k$. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^k)$. We have the equality

$$\phi(y; \mu * \nu) = \phi(y; \mu)\phi(y; \nu)$$

by definition. It is known that $\mu = \nu$ if and only if $\phi(y; \mu) = \phi(y; \nu)$. Then we introduce Levy's inversion formulas as follows. These formulas are essentially seen in Jessen–Wintner [71, Section 3]. Let $\mu \in \mathcal{P}(\mathbb{R})$. Suppose that the integral

$$\int_{\mathbb{R}} |\phi(y; \mu)| \, dy$$

is finite. Then μ is absolutely continuous, and the probability density function D in (A.2) is represented as

$$D(x) = \int_{\mathbb{R}} \phi(y; \mu)\psi_{-x}(y) \, |dy|, \tag{A.4}$$

which is a non-negative continuous function on $\mathbb{R}$. Let $\mu \in \mathcal{P}(\mathbb{C})$. In this case, we suppose that the integral

$$\int_{\mathbb{C}} |\phi(w; \mu)| \, dudv$$

is finite, where $w = u + iv$. Then μ is absolutely continuous, and the probability density function D in (A.3) is represented as

$$D(z) = \int_{\mathbb{C}} \phi(w; \mu)\psi_{-z}(w) \, |dw|, \tag{A.5}$$

which is a non-negative continuous function on $\mathbb{C}$.

Let $\mu \in \mathcal{P}(\mathbb{R}^k)$. Then we define its distribution function $F_\mu : \mathbb{R}^k \to \mathbb{R}$ as

$$F_\mu(x) = \mu\big((-\infty, x_1] \times \cdots \times (-\infty, x_k]\big)$$

for $x = (x_1, \ldots, x_k)$. Let $X : \Omega \to \mathbb{R}^k$ be a random element. Then

$$\mu_X(A) = \mathbb{P}(X(\omega) \in A)$$

gives a probability measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. The probability measure μ_X is said to be a probability distribution of X.

A.1. Convergence of probability measures

Let $(\mu_n)_n$ be a sequence of probability measures on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Then μ_n is said to be convergent weakly as $n \to \infty$ if there exists $\mu \in \mathcal{P}(\mathbb{R}^k)$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^k} f(x) \, d\mu_n(x) = \int_{\mathbb{R}^k} f(x) \, d\mu(x)$$

for all $f \in C_b(\mathbb{R}^k)$. In this case, the inequality

$$\int_{\mathbb{R}^k} f(x) \, d\mu(x) \leq \liminf_{n\to\infty} \int_{\mathbb{R}^k} f(x) \, d\mu_n(x)$$

is valid for any non-negative continuous function f. A Borel set A is called a continuity set of $\mu \in \mathcal{P}(\mathbb{R}^k)$ if $\mu(\partial A) = 0$ holds, where ∂A is the boundary of A. Furthermore, if $\mathrm{meas}(\partial A) = 0$ holds, then we simply say that A is a continuity set.

Lemma A.1.1. *Let* $(\mu_n)_n \subset \mathcal{P}(\mathbb{R}^k)$ *be a sequence and* $\mu \in \mathcal{P}(\mathbb{R}^k)$*. Then the following statements are equivalent.*

- (A) μ_n *converges weakly to* μ *as* $n \to \infty$*.*
- (B) $\mu_n(A) \to \mu(A)$ *as* $n \to \infty$ *for all continuity set* A *of* μ*.*
- (C) $F_{\mu_n}(x)$ *converges to* $F_\mu(x)$ *as* $n \to \infty$ *at each point of continuity of* F_μ*.*
- (D) $\phi(y;\mu_n)$ *converges uniformly to* $\phi(y;\mu)$ *as* $n \to \infty$ *in every sphere* $|y| \le R$*.*

Next, for $\mu \in \mathcal{P}(\mathbb{R}^k)$, we put

$$c(\mu) = \int_{\mathbb{R}^k} x \, d\mu(x) \quad \text{and} \quad M(\mu) = \int_{\mathbb{R}^k} |x|^2 \, d\mu(x).$$

Lemma A.1.2 (Jessen–Wintner). *Let* $(\mu_n)_n \subset \mathcal{P}(\mathbb{R}^k)$ *and* $\mu \in \mathcal{P}(\mathbb{R}^k)$*. Suppose that* $M(\mu_n)$ *is finite for all* n*. Then the convergences of*

$$\sum_{n=1}^{\infty} |c(\mu_n)| \quad \textit{and} \quad \sum_{n=1}^{\infty} M(\mu_n)$$

imply the weakly convergence of the convolution measure $\mu_1 * \cdots * \mu_n$ *as* $n \to \infty$*. Furthermore, the convergence is absolute in the sense that it is convergent in any order of terms of convolution.*

Let $(X_n)_n$ be a sequence of $\mathbb{C}$-valued random variables, and let X be an $\mathbb{C}$-valued random variable. Then X_n is said to be convergent surely as $n \to \infty$ if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \in \Omega$. Furthermore, X_n is said to be convergent almost surely as $n \to \infty$ if $\mathbb{P}\big(\lim_{n\to\infty} X_n = X\big) = 1$. Finally, X_n is said to be convergent in law as $n \to \infty$ if μ_{X_n} converges weakly to μ_X as $n \to \infty$. By definition, if X_n converges surely, then X_n converges almost surely. In addition, if X_n converges almost surely, then X_n converges in law.

Theorem A.1.3 (Kolmogorov's two-series theorem). *Let* $(X_n)_n$ *be a sequence of independent* $\mathbb{C}$*-valued random variables. Then the convergences of*

$$\sum_{n=1}^{\infty} \mathbb{E}[X_n] \quad \textit{and} \quad \sum_{n=1}^{\infty} \mathbb{V}[X_n]$$

imply that $X_1 + \cdots + X_n$ *converges almost surely as* $n \to \infty$*.*

Related to the convergences of probability measures, we recall a lemma of Ihara–Matsumoto [**64**, Lemma A]. Define

$$\Lambda(\mathbb{R}^k) = \{\Phi \in L^1(\mathbb{R}^k) \mid \Phi \text{ is continuous and } \widetilde{\Phi} \in L^1(\mathbb{R}^k)\}$$

Then a function $M : \mathbb{R}^k \to \mathbb{R}_{\ge 0}$ is called a good density function if it belongs to $\Lambda(\mathbb{R}^k)$ and satisfies

$$\int_{\mathbb{R}^k} M(x) \, |dx| = 1,$$

where we write $|dx| = (2\pi)^{-k/2} dx_1 \cdots dx_k$ for $x = (x_1, \ldots, x_k)$.

LEMMA A.1.4. *Let M be a good density function on $\mathbb{R}^k$. Let $(S_n)_n$ be a sequence of finite sets equipped with functions $\omega_n : S_n \to \mathbb{R}_{\geq 0}$ and $f_n : S_n \to \mathbb{R}^k$ such that $\sum_{\chi \in S_n} \omega_n(\chi) = 1$. Then we consider the condition*

$$\lim_{n\to\infty} \sum_{\chi \in S_n} \omega_n(\chi)\Phi\left(f_n(\chi)\right) = \int_{\mathbb{R}^k} \Phi(x)M(x)\,|dx| \tag{A.1}$$

for a test function $\Phi : \mathbb{R}^k \to \mathbb{C}$. Suppose that condition (A.1) holds for any additive character $\Phi = \psi_y$ with $y \in \mathbb{R}^k$, and that the convergence is uniform in every sphere $|y| \leq R$. Then we have the following results.

(a) (A.1) *holds for any $\Phi \in C_b(\mathbb{R}^k)$.*

(b) (A.1) *holds for any $\Phi \in C(\mathbb{R}^k)$ such that $\Phi(x) \ll \phi_0(|x|)$, where $\phi_0(r)$ is a continuous non-decreasing function on $[0, \infty)$ which satisfies $\phi_0(r) > 0$, $\phi_0(r) \to \infty$ as $r \to \infty$, and*

$$\sum_{\chi \in S_n} \omega_n(\chi)\phi_0\left(|f_n(\chi)|\right)^2 \ll 1,$$

$$\int_{\mathbb{R}^k} \phi_0(|x|)M(x)\,|dx| < \infty.$$

(c) (A.1) *holds for the indicator function of either a compact subset of $\mathbb{R}^k$ or the complement of such a subset.*

(c′) (A.1) *holds for the indicator function of any continuity set of $\mathbb{R}^k$.*

Assertion (c′) is not contained in the original statement by Ihara–Matsumoto. It is easily deduced since the assumption implies that the probability measure

$$P_n(A) = \sum_{\chi \in S_n} \omega_n(\chi)1_A\left(f_n(\chi)\right)$$

converges weakly to $Q(A) = \int_A M(x)\,|dx|$ as $n \to \infty$. Along the same lines of the proof of [**64**, Lemma A], one can derive the following result.

LEMMA A.1.5. *Let M be a good density function on $\mathbb{R}^k$. Let $\mathcal{E} = \mathcal{E}(T)$ be a subset of $\mathbb{R}$ satisfying $\mathbb{P}_T(t \in \mathcal{E}(T)) \to 0$ as $T \to \infty$. Let $f : \mathbb{R} \to \mathbb{R}^k$ be continuous except for at most countable points of $\mathbb{R}$. Then we consider the condition*

$$\lim_{T\to\infty} \frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \Phi(f(t))\,dt = \int_{\mathbb{R}^k} \Phi(x)M(x)\,|dx| \tag{A.2}$$

for a test function $\Phi : \mathbb{R}^k \to \mathbb{C}$. Suppose that condition (A.2) holds for any additive character $\Phi = \psi_y$ with $y \in \mathbb{R}^k$, and that the convergence is uniform in every sphere $|y| \leq R$. Then we have the following results.

(a) (A.2) *holds for any $\Phi \in C_b(\mathbb{R}^k)$.*

(b) (A.2) *holds for any $\Phi \in C(\mathbb{R}^k)$ such that $\Phi(x) \ll \phi_0(|x|)$, where $\phi_0(r)$ is a continuous non-decreasing function on $[0, \infty)$ which satisfies $\phi_0(r) > 0$, $\phi_0(r) \to \infty$ as $r \to \infty$, and*

$$\frac{1}{T}\int_{[0,T]\setminus\mathcal{E}} \phi_0\left(|f(t)|\right)^2\,dt \ll 1,$$

$$\int_{\mathbb{R}^k} \phi_0(|x|)M(x)\,|dx| < \infty.$$

(c) (A.2) *holds for the indicator function of either a compact subset of* $\mathbb{R}^k$ *or the complement of such a subset.*

(c′) (A.2) *holds for the indicator function of any continuity set of* $\mathbb{R}^k$.

A.2. Supports of probability measures

Let $\mu \in \mathcal{P}(\mathbb{R}^k)$. The support of μ is defined as

$$\operatorname{supp}(\mu) = \{x \in \mathbb{R}^k \mid \mu(A) > 0 \text{ holds for any set } A \text{ with } x \in A^i\},$$

where A^i denotes the interior of A. We see that $\operatorname{supp}(\mu)$ is non-empty closed set for any $\mu \in \mathcal{P}(\mathbb{R}^k)$. For a random element $X : \Omega \to \mathbb{R}^k$, we define $\operatorname{supp}(X)$ as $\operatorname{supp}(\mu_X)$. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^k)$. Then the equality

$$\operatorname{supp}(\mu * \nu) = \operatorname{supp}(\mu) + \operatorname{supp}(\nu)$$

holds by definition, where we denote by $A + B$ the set of all points $a + b$ with $a \in A$ and $b \in B$. Furthermore, if $S_n = A_1 + \cdots + A_n$, then we denote by $\lim_{n\to\infty} S_n$ the set of all points in $\mathbb{R}^k$ which are represented in at least one way as the limit of $a_1 + \cdots + a_n$ with $a_j \in A_j$ for all j.

Theorem A.2.1 (Jessen–Wintner). *Let* $(\mu_n)_n \subset \mathcal{P}(\mathbb{R}^k)$ *and write* $\nu_n = \mu_1 * \cdots * \mu_n$. *Suppose that* ν_n *converges weakly to* $\nu \in \mathcal{P}(\mathbb{R}^k)$. *Then we obtain*

$$\operatorname{supp}(\nu) = \lim_{n\to\infty} (\operatorname{supp}(\mu_1) + \cdots + \operatorname{supp}(\mu_n)).$$

The following lemmas are also useful to show that the support of a probability measure $\mu \in \mathcal{P}(\mathbb{R})$ or $\mu \in \mathcal{P}(\mathbb{C})$ is equal to $\mathbb{R}$ or $\mathbb{C}$, respectively. The proof of the first one is contained in Laurinčikas [99, Theorem 6.1.16]. The second one was proved by Mishou–Nagoshi [141, Lemma 2.2]

Lemma A.2.2. *Let* H *be a complex Hilbert space with inner product* $(\cdot,\cdot)$, *and take a sequence* $(u_n)_n \subset H$ *satisfying*

(i) $\sum_n \|u_n\|^2 < \infty$

(ii) $\sum_n |(u_n, u)| = \infty$ *for any* $u \in H$ *with* $u \neq 0$.

Then, for any $v \in H$, $k \geq 1$, *and* $\epsilon > 0$, *there exist an integer* $N = N(v, k, \epsilon) \geq k$ *and complex numbers* $c_k, \ldots, c_N \in H$ *with* $|c_n| = 1$ *such that*

$$\left\| v - \sum_{n=k}^{N} c_n u_n \right\| < \epsilon.$$

Lemma A.2.3. *Let* H *be a real Hilbert space with inner product* $(\cdot,\cdot)$, *and take a sequence* $(u_n)_n \subset H$ *satisfying*

(i) $\sum_n \|u_n\|^2 < \infty$

(ii) $\sum_n |(u_n, u)| = \infty$ *for any* $u \in H$ *with* $\|u\| = 1$.

Then, for any $v \in H$, $k \geq 1$, *and* $\epsilon > 0$, *there exist an integer* $N = N(v, k, \epsilon) \geq k$ *and numbers* $c_k, \ldots, c_N \in \{1, -1\}$ *such that*

$$\left\| v - \sum_{n=k}^{N} c_n u_n \right\| < \epsilon.$$

A.3. Esseen's inequalities

Finally, we recall inequalities proved by Esseen [39] which evaluates the difference of two probability measures. Here, we refer to the result of Loève [112] for $\mathcal{P}(\mathbb{R})$, and to the result of Sadikova [163] for $\mathcal{P}(\mathbb{R}^2)$.

THEOREM A.3.1. *Let* $\mu, \nu \in \mathcal{P}(\mathbb{R})$ *with distribution functions*

$$F(x) = \mu\left((-\infty, x]\right) \quad \text{and} \quad G(x) = \nu\left((-\infty, x]\right).$$

Suppose that G *is differentiable on* $\mathbb{R}$, *and put* $A = \sup_{x\in\mathbb{R}} G'(x)$. *Denote by* $f(y) = \phi(y; \mu)$ *and* $g(y) = \phi(y; \nu)$ *the characteristic functions. Then we have*

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \leq \frac{1}{\pi} \int_{-R}^{R} \left| \frac{f(y) - g(y)}{y} \right| dy + \frac{24}{\pi} \frac{A}{R}$$

for any $R > 0$.

THEOREM A.3.2. *Let* $\mu, \nu \in \mathcal{P}(\mathbb{R}^2)$ *with distribution functions*

$$F(x, y) = \mu\left((-\infty, x] \times (-\infty, y]\right) \quad \text{and} \quad G(x, y) = \nu\left((-\infty, x] \times (-\infty, y]\right).$$

Suppose that G *is partially differentiable on* $\mathbb{R}^2$, *and put* $A_1 = \sup_{(x,y)\in\mathbb{R}^2} G_x(x, y)$ *and* $A_2 = \sup_{(x,y)\in\mathbb{R}^2} G_y(x, y)$. *Denote by* $f(u, v) = \phi(u, v; \mu)$ *and* $g(u, v) = \phi(u, v; \nu)$ *the characteristic functions, and we define*

$$\hat{f}(u, v) = f(u, v) - f(u, 0)f(0, v) \quad \text{and} \quad \hat{g}(u, v) = g(u, v) - g(u, 0)g(0, v).$$

Then we have

$$\begin{aligned} \sup_{(x,y)\in\mathbb{R}^2} |F(x, y) - G(x, y)| &\leq \frac{2}{(2\pi)^2} \iint_{[-R,R]^2} \left| \frac{\hat{f}(u, v) - \hat{g}(u, v)}{uv} \right| dudv \\ &+ \frac{2}{\pi} \int_{-R}^{R} \left| \frac{f(u, 0) - g(u, 0)}{u} \right| du + \frac{2}{\pi} \int_{-R}^{R} \left| \frac{f(0, v) - g(0, v)}{v} \right| dv \\ &+ \left(3\sqrt{2} + 4\sqrt{3} + \frac{24}{\pi} \right) \frac{2(A_1 + A_2)}{R} \end{aligned}$$

for any $R > 0$.

APPENDIX B

Complex functions

B.1. Logarithms and polylogarithms

Let $m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. For $0 < r < 1$, we define

$$\lambda_r(\theta; \alpha, m) = \operatorname{Re}(e^{-i\alpha} \operatorname{Li}_{m+1}(re^{i\theta})) = \sum_{k=1}^{\infty} \frac{r^k}{k^{m+1}} \cos(k\theta - \alpha)$$

for $\theta \in \mathbb{R}$, where $\operatorname{Li}_s(w)$ is the polylogarithm of order s defined as

$$Li_s(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^s}.$$

Since $Li_1(w) = -\operatorname{Log}(1 - w)$, we have $\lambda_{p^{-\sigma}}(\theta; \alpha, 0) = \lambda_{p,\sigma}(\theta; \alpha)$ with the notation in Section 14.1. By definition, the differential relation

$$\lambda_r'(\theta; \alpha, m) = \lambda_r(\theta; \alpha - \pi/2, m - 1) \tag{B.1}$$

holds. We prove several results on the function $\lambda_r(\theta; \alpha, m)$.

Lemma B.1.1. *Let $m \geq -1$ and $\alpha \in \mathbb{R}$. For any fixed real number $0 < r < 1$, the function $\lambda_r(\theta; \alpha, m)$ has exactly two zeros in the interval $\theta \in [0, 2\pi)$.*

Proof. We prove this lemma by induction on m. Recall that $\operatorname{Li}_0(w) = w/(1-w)$ maps a circle to a circle. In particular, the interior of the circle $\mathscr{C}_0 = \operatorname{Li}_0(C_r)$ contains the origin, where C_r describes the circle $\{|w| = r\}$. Therefore the number of the intersections of $\mathscr{C}_0$ with any straight line passing through the origin is exactly two, which is equal to the number of zeros of $\lambda_r(\theta; \alpha, -1)$ in the interval $[0, 2\pi)$. Let $m \geq 0$. We have $\lambda_r(\theta; \alpha, m) = \lambda_r(\theta; \alpha + \pi/2, m + 1)$ by relation (B.1). Note that the function $\lambda_r(\theta; \alpha + \pi/2, m + 1)$ is smooth and periodic. Thus $\lambda_r'(\theta; \alpha + \pi/2, m + 1)$ vanishes at least twice in the period. Hence, there exist at least two zeros of $\lambda_r(\theta; \alpha, m)$ in $[0, 2\pi)$. If there are three zeros of $\lambda_r(\theta; \alpha, m)$ in $[0, 2\pi)$, then we see that $\lambda_r'(\theta; \alpha, m)$ has also three zeros in $[0, 2\pi)$ by Rolle's theorem. However, it implies that the function $\lambda_r(\theta; \alpha - \pi/2, m - 1)$ has three zeros in $[0, 2\pi)$ by (B.1), which contradicts the assumption of induction. □

Let $m \geq 0$ and $\alpha \in \mathbb{R}$. Denote by θ_1 and θ_2 the zeros of $\lambda_r'(\theta; \alpha, m)$ with $0 \leq \theta_1 < \theta_2 < 2\pi$. Then we have $\lambda_r(\theta_1; \alpha, m) \neq \lambda_r(\theta_2; \alpha, m)$; otherwise we have the third zero of $\lambda_r'(\theta; \alpha, m)$ between θ_1 and θ_2. Furthermore, we obtain the following result as a consequence of Lemma B.1.1.

Lemma B.1.2. *Let $m \geq 0$ and $\alpha \in \mathbb{R}$. Then there exist real numbers θ_1 and θ_2 with $\theta_1 < \theta_2 < \theta_1 + 2\pi$ such that the function $\lambda_r(\theta; \alpha, m)$ is decreasing for $\theta_1 \leq \theta \leq \theta_2$ and is increasing for $\theta_2 \leq \theta \leq \theta_1 + 2\pi$.*

Proof. Let $0 \le \tilde{\theta}_1 < \tilde{\theta}_2 < 2\pi$ denote the zeros of $\lambda'_r(\theta;\alpha,m)$. If $\lambda_r(\tilde{\theta}_1;\alpha,m) < \lambda_r(\tilde{\theta}_2;\alpha,m)$, then we have

$$\lambda'_r(\tilde{\theta};\alpha,m) \begin{cases} < 0 & \text{for } \tilde{\theta}_1 < \theta < \tilde{\theta}_2, \\ > 0 & \text{for } \tilde{\theta}_2 < \theta < \tilde{\theta}_1 + 2\pi \end{cases}$$

since there exist no zeros except for $\tilde{\theta}_1$ and $\tilde{\theta}_2$ by Lemma B.1.1. Then the result follows by taking $\theta_j = \tilde{\theta}_j$. If $\lambda_r(\tilde{\theta}_1;\alpha,m) > \lambda_r(\tilde{\theta}_2;\alpha,m)$, we take $\theta_1 = \tilde{\theta}_2$ and $\theta_2 = \tilde{\theta}_1 + 2\pi$. Then we obtain the desired result similarly. □

Put $\lambda_r(\theta;\alpha) = \lambda_r(\theta;\alpha,0)$. Then it is elementary to see that the upper bound

$$\lambda_r^{(n)}(\theta;\alpha) \ll n!\, r \tag{B.2}$$

holds uniformly for all $n \ge 0$ and $\theta \in \mathbb{R}$. Indeed, relation (B.1) yields

$$\lambda_r^{(n)}(\theta;\alpha) \ll \sum_{k=1}^{\infty} k^n r^k =: S_n(r).$$

Remark that we have

$$(1-r)S_n(r) = r + \sum_{k=1}^{\infty} \{(k+1)^n - k^n\}\, r^{k+1} = r\left(1 + \sum_{j=0}^{n-1} \binom{n}{j} S_j(r)\right)$$

for $n \ge 1$. Then we derive (B.2) by a simple induction on n. We further prove the following lower bound.

Lemma B.1.3. *Let α be a real number. Denote by θ_1 and θ_2 the real numbers of Proposition B.1.2. If $0 < r < 0.278$, then we have uniformly*

$$\lambda''_r(\theta_1;\alpha) \gg r.$$

Furthermore, there exists an absolute constant $d > 0$ such that $|\theta_2 - \theta_1| \ge d$.

Proof. Since $\lambda'_r(\theta_1;\alpha) = 0$, the equality

$$r\sin(\theta_1 - \alpha) = -\sum_{k=2}^{\infty} r^k \sin(k\theta_1 - \alpha)$$

holds by definition. Therefore we obtain

$$|\sin(\theta_1 - \alpha)| \le \sum_{k=2}^{\infty} r^{k-1} = \frac{r}{1-r}.$$

For $0 < r < 1/2$, it gives the inequality

$$|\cos(\theta_1 - \alpha)| > \frac{\sqrt{1-2r}}{1-r}. \tag{B.3}$$

Furthermore, we note that the inequality

$$\left|\lambda''_r(\theta_1;\alpha) + r\cos(\theta_1 - \alpha)\right| \le \sum_{k=2}^{\infty} k r^k = \frac{r(2r - r^2)}{(1-r)^2} \tag{B.4}$$

holds by the definition of $\lambda_r(\theta;\alpha)$. From (B.3) and (B.4), we deduce

$$\lambda_r''(\theta_1;\alpha) < -\left(\frac{\sqrt{1-2r}}{1-r} - \frac{2r-r^2}{(1-r)^2}\right) r = -\frac{(1-r)\sqrt{1-2r}-(2r-r^2)}{(1-r)^2} r.$$

We see that $(1-r)\sqrt{1-2r}-(2r-r^2) > 0$ for $0 < r < 0.2787\ldots$. Hence there exists an absolute constant $c > 0$ such that $|\lambda_r''(\theta_1;\alpha)| = -\lambda_r''(\theta_1;\alpha) > cr$ for $0 < r < 0.278$. To see the second assertion, we remark that there exists a positive absolute constant c' satisfying

$$\cos(\theta_1-\alpha) > c' \quad \text{and} \quad \cos(\theta_2-\alpha) < -c'$$

if $0 < r < 0.278$. Hence the inequality $|\theta_2-\theta_1| \geq |\cos^{-1}(-c') - \cos^{-1}(c')| > 0$ follows as desired. □

Remark that $r = p^{-\sigma}$ satisfies the condition $0 < r < 0.278$ if $p \geq 13$ and $\sigma \geq 1/2$ since we have $1/\sqrt{13} = 0.2773\ldots$.

B.2. The gamma function

We hereby collect several results on the gamma function

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}. \tag{B.1}$$

All proofs are seen, for example, in Whittaker–Watson [186]. First, we recall that the function Γ satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s). \tag{B.2}$$

Hence we have $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Although the right-hand side of (B.1) converges only for $\operatorname{Re} s > 0$, we extend the function Γ as a meromorphic function on $\mathbb{C}$ by applying formula (B.2). Note that it has no zeros and has poles only at $s \in \mathbb{Z}_{\leq 0}$. All poles are simple. We have also the relations

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad \Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\pi^{1/2}\Gamma(2s).$$

Stirling's formula is useful to study the asymptotic behavior of the gamma function.

THEOREM B.2.1 (Stirling's formula). *Let $\delta > 0$. We have*

$$\Gamma(s) = \exp\left(\left(s-\frac{1}{2}\right)\operatorname{Log} s - s + \frac{1}{2}\log(2\pi) + O\left(|s|^{-1}\right)\right)$$

if $|s|$ is sufficiently large and $-\pi+\delta < \operatorname{Arg} s < \pi-\delta$ is satisfied.

B.3. Bessel functions

We further collect several results on Bessel functions, whose proofs are seen, for example, in Watson [185]. Let $\nu \in \mathbb{Z}$. The Bessel function of the first kind of order ν is defined as

$$J_\nu(z) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(i(z\sin\theta - \nu\theta)\right)\, d\theta,$$

and the modified Bessel function of the first kind of order ν is defined as

$$I_\nu(z) = i^{-\nu} J_\nu(iz).$$

In particular, we see that $I_0(z)$ is equal to the moment-generating function

$$I_0(z) = \mathbb{E}[e^{z \operatorname{Re} X}] = \mathbb{E}[e^{z \operatorname{Im} X}],$$

where X is a random variable uniformly distributed on the unit circle. The Bessel functions $J_\nu(z)$ and $I_\nu(z)$ are entire functions. Moreover, their zeros are lie on the coordinate axes as follows.

Theorem B.3.1. *Let $\nu \in \mathbb{Z}_{\geq 0}$. Then $J_\nu(z)$ has infinitely many zeros, and all zeros lie on the real axis. Equivalently, $I_\nu(z)$ has infinitely many zeros, and all zeros lie on the imaginary axis. The order of the zero at $z = 0$ is ν, and the other zeros are simple.*

For $\nu, m \in \mathbb{Z}_{\geq 0}$, we define

$$(\nu, m) = \frac{\Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu - m + \frac{1}{2})}.$$

Then we have the following asymptotic formula.

Theorem B.3.2. *Let $\nu \in \mathbb{Z}_{\geq 0}$ and $\delta > 0$. For any $N \geq 1$, we obtain*

$$\begin{aligned} I_\nu(z) = {} & \frac{e^z}{\sqrt{2\pi z}}\left(\sum_{m=0}^{N} \frac{(-1)^m(\nu, m)}{(2z)^m} + O\left(|z|^{-N}\right)\right) \\ & + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}}\left(\sum_{m=0}^{N} \frac{(\nu, m)}{(2z)^m} + O\left(|z|^{-N}\right)\right) \end{aligned}$$

if $|z|$ is sufficiently large and $-\pi/2 + \delta < \operatorname{Arg} z < 3\pi/2 - \delta$ is satisfied.

By definition, we see that $I_0(x) \in \mathbb{R}_{>0}$ for any $x \in \mathbb{R}$. Then we define $g(x) = \log I_0(x)$. Moreover, the function g is continued to a holomorphic function in the half plane $\operatorname{Re} z > 0$ by Theorem B.3.1.

Lemma B.3.3. *Let $u > 0$ be a real number. Then we have*

$$g(u) \ll \begin{cases} u^2 & \text{if } 0 \leq u \leq 1, \\ u & \text{if } u \geq 1, \end{cases} \qquad g'(u) \ll \begin{cases} u & \text{if } 0 \leq u \leq 1, \\ 1 & \text{if } u \geq 1, \end{cases}$$

and for $n \geq 2$,

$$g^{(n)}(u) \ll \begin{cases} n! & \text{if } 0 \leq u \leq 1, \\ 2^n n! u^{1-n} & \text{if } u \geq 1. \end{cases}$$

PROOF. The first and second upper bounds follow from [**93**, Lemma 7.4]. We also know that $g(z) \ll |z|$ holds by Theorem B.3.2. Hence the third upper bound follows by Cauchy's integral formula. $\square$

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