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Doctoral Thesis

# Quantum Seiberg-Witten Curves and Universality in Argyres-Douglas Theories


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**Abstract**

The purpose of this thesis is to determine the quantum Seiberg-Witten (SW) periods of $\mathcal{N}=2$ supersymmetric gauge theories in the $\Omega$ background. In particular, we consider the quantum SW periods of the Argyres-Douglas (AD) theories in the Nekrasov-Shatashvili (NS) limit of the $\Omega$ background. We also study the universality for the Argyres-Douglas (AD) theories obtained from the $A_r$ type and $D_r$ type super Yang-Mills theories and supersummetric QCDs (SQCDs).

We first explain the $\mathcal{N}=2$ supersymmetry algebra and $\mathcal{N}=2$ supersymmetric Lagrangian in four-dimensional spacetime. We consider the low energy effective theories of the $\mathcal{N}=2$ supersymmetric theories, and introduce the prepotential that is calculated by the $\beta$-function. These moduli spaces of the low-energy effective theories are described by the Seiberg-Witten (SW) curves and the SW differentials. The prepotential is then calculated as the SW period that is given as the integral of the SW differential by the cycles.

Next, we explain the Argyres-Douglas (AD) theory, which is determined on the superconformal point of the moduli space. The superconformal points are realized at the intersection points of the mutually non-local massless BPS particles, which are at the strong coupling region of the moduli space. The SW curve and the SW differential around the superconformal point are given by the scaling limit of the SW curve and the SW differential.

We introduce the $\Omega$ background. This is four-dimensional spacetime which admits the two-torus action. In particular, we take the Nekrasov-Shatashvili (NS) limit of the Omega background. In this limit, the low-energy theory is described by the differential equation called the quantum SW curve. We use the method of quantization of the SW curves to obtain the quantum SW periods for $SU(N_c)$ SQCDs systematically. In particular, we construct the quantum SW curve in the AD theories for the $A_r$ type and $D_r$ type super Yang-Mills theories and the SQCDs, and calculate the quantum SW periods in the scaling limit. The quantum SW curve for the $A_r$ type super Yang-Mills is determined uniquely,

but there are two different quantizations of the SW curve in $D_r$ type super Yang-Mills theory. We also construct the differential operators which connect the classical periods and the quantum corrections.

To check the consistency of the quantum SW curves of the AD theories, we study the universality of AD theories at the quantum level by showing that the quantum SW periods are obtained from the different UV theories. We confirm some examples of the universality of AD theories associated with $SU(2)$ and $SU(3)$ SQCDs.

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# Contents

# Chapter 1

# Introduction

Quantum field theories are usually defined by quantizing fields based on a classical Lagrangian. The partition function or generating functional of correlation functions is an important object to calculate physical observables, which are given by perturbation in coupling constants. The perturbative expansion gives a good description of physics in the weak coupling region in general. However, in the strong coupling region, where the perturbation is not good and non-perturbative effects become important. Therefore, it is important to study the exact formula of the partition functions.

Recently, techniques of computing the partition functions have been developed for quantum field theories with supersymmetry. It is known as the localization method, where the infinite-dimensional path-integral reduces to the finite-dimensional integral or a sum [4]. In particular, for gauge theories with $\mathcal{N} = 2$ supersymmetry, by the localization method, the exact partition function has been calculated, which is called the Nekrasov partition function [63]. The Nekrasov partition function is defined in the weak coupling region but it contains non-perturbative instanton corrections.

$\mathcal{N} = 2$ super Yang-Mills theory plays an important role in studying quantum field theories and superstring theories. It is known that its low-energy effective theory is exactly determined by supersymmetry and duality [6], [7]. One can then study the low-energy physics of the theory, such as the condensation of a monopole. In string theory, $\mathcal{N} = 2$ theories are realized as a stack of D4-branes suspended between two NS5-branes in type IIA superstrings, which turns out to be a world-volume theory of M5-brane in M-theory at strong coupling [8]. In this thesis, we will study the Seiberg-Witten theory to investigate the strong coupling physics of the theory.

$\mathcal{N} = 2$ supersymmetric gauge theories are constructed from two types of supermultiplets. One is the vector multiplet which contains gauge fields and a complex scalar. The other is the hypermultiplet which contains the quark as component fields. The vacuum of the theory is characterized by the vacuum expectation values of the scalar and the scalar quarks, which corresponds to the Coulomb branch and the Higgs branch, respectively. In the Coulomb branch, the low-energy effective theory is described by the vector multiples associated with the Cartan subgroup of the gauge group. Due to $\mathcal{N} = 2$ supersymmetry, the low energy effective Lagrangian can be represented as prepotential, which is the function of the vector multiplets. The prepotential is expressed as the sum of the one-loop part and instanton corrections. The one-loop part is calculated by the $\beta$-function of the $\mathcal{N} = 2$ gauge theory. The second-order derivatives of the prepotential represent the complexified effective gauge couplings. In the Seiberg-Witten theory, the effective gauge couplings are identified as the ratio of the period integrals of certain meromorphic one-form over one-cycles on a certain Riemann surface. The Riemann surface is called the Seiberg-Witten(SW) curve and the meromorphic differential is called the SW differential [6], [7], [12]. Then, computing the periods as the functions of the vacuum moduli, one can determine the prepotential exactly. The SW curve and differential are determined by the monodromy of the period integrals (the SW periods) around a singularity, where certain monopoles or dyons become massless. The monodromy can be determined by electromagnetic duality. For a large class of gauge theories, we have the SW curve and the SW differential. We can then formally obtain prepotentials for any values of the moduli parameters. However, this approach is an indirect method and requires the assumptions for the singularities at the strong coupling region in the moduli space to obtain the prepotential. The localization approach is a direct approach to obtain the prepotential in the weak coupling without any knowledge in the strong coupling region.

$\mathcal{N} = 2$ supersymmetry contains a nilpotent supercharge, which determines the cohomological structure in the observables. In $\mathcal{N} = 2$ Super Yang-Mills theories, by the topological twist which identifies spinor indices and internal R-symmetry indices, one can define a nilpotent scalar supercharge and the BRST cohomology [64]. The action of the $\mathcal{N} = 2$ Super Yang-Mills theories is written as BRST-exact up to topological term.

One can perform the path-integral in the BRST symmetric manner. In particular, one can add the BRST-exact term with some deformation parameter $t \in \mathbb{R}$ such that the partition function is independent of $t$. Moreover, the contribution to the path-integral is localized on the solution to the instanton equations in the $t \to \infty$. These instanton solutions are obtained by the ADHM construction [9], which are the equations in matrix form and parametrized by the instanton moduli parameters. Then, the prepotentials can be given by the sum of instanton numbers and finite-dimensional integrals over the moduli space for instanton number. However, the integral over moduli spaces of each instanton is difficult to calculate due to the complicated structure of the ADHM construction.

To avoid this difficulty, Nekrasov introduced some regularization called $\Omega$-background [10], [63]. It is known that the four-dimensional $\mathcal{N} = 2$ Super Yang-Mills theories are given by dimensional reduction of the six-dimensional $\mathcal{N} = 1$ Super Yang-Mills [3]. We first compactify the 6-dimensional spacetime on the 2-dimensional torus. We then introduce the $\Omega$ background where the torus acts as rotations on four-dimensional spacetime parameterized by two real parameters [4]. This action also acts on the ADHM moduli space of instantons and the solutions are localized and labeled by the Young diagrams. Then, using the Localization method, we can obtain the partition function on the $\Omega$ background as the sum over the set of Young diagrams. Thus the $\Omega$ background is one of the regulators of the partition function of the four-dimensional $\mathcal{N} = 2$ Super Yang-Mills. We call this partition function as Nekrasov partition function. The $\Omega$ background was also applied to topological string [50], [51]. The relation to the conformal block of 2d CFT was studied in [52], [53], [54], [55].

So far, we have obtained the partition functions of $\mathcal{N} = 2$ supersymmetric gauge theories in the $\Omega$-background. The prepotential of the low-energy effective theory is found from the logarithm of the partition function by taking the $\Omega$-background parameters to be zero. The prepotential on the $\Omega$ background is defined in the weak coupling region. However, the prepotential on the $\Omega$ background calculated above cannot be applied in the strong coupling region, because the action on the $\Omega$ background is not valid in the strong coupling region. One might think that we can use the non-abelian version of duality between the strong coupling region and weak coupling region, which is not well-understood in the $\Omega$-background. It is interesting to explore the Nekrasov partition function in the strong coupling region, which is useful to understand strong coupling physics and the duality of field theories. However, the strong coupling region on the $\Omega$ background is meaningful to regularize the partition function for the $\mathcal{N} = 2$ Super Yang-Mills on the strong coupling region. Because of the dualities for the $\mathcal{N} = 2$ Super Yang-Mills, the partition function on the strong coupling and weak coupling region are related by dualities.

At strong coupling, the low-energy effective theory can be studied by using the Seiberg-Witten theory. It is not known how to define the low-energy action in general $\Omega$-background. However, at a special limit of the $\Omega$-background parameters, one can introduce the $\Omega$-deformation of the prepotential. This is called the Nekrasov-Shatashvili (NS) limit, which sends one of the $\Omega$-background parameters to zero. Then, we introduce the Nekrasov-Shatashvili limit of the $\Omega$-background which is a limit that fixes one parameter of the $\Omega$ background to zero [36]. On the NS limit of the $\Omega$ background, the SW periods (called the quantum SW periods) are deformed by the $\Omega$-background and can be calculated by using the differential equation which is called the quantum SW curve [33], [34]. For the $SU(2)$ super Yang-Mills theory, the differential equation is actually the differential equation which is known as Baxter's T-Q relation. The deformed SW periods in the weak coupling region are shown to agree with those obtained from the Nekrasov partition function up to some order in the deformation parameter. The quantum SW curves for $\mathcal{N} = 2$ gauge theories with higher rank gauge groups have been studied in [33], [34], [40], [41]. The systematic expansion in the deformation parameter is studied by using the WKB analysis. This Schrodinger equation is given by quantizing the SW curve and SW differential. In these papers, they obtained the prepotential or equivalently obtained the quantum SW periods as a WKB solution of some Schrodinger equation for some classical theory. The WKB method to calculating the quantum SW periods are studied in [37], [38], [39]. In the weak coupling region, the quantum SW curve obtained from the quantum SW curve is same as the quantum SW periods obtained from the Nekrasov partition function [33], [34], [40], [41]. Thus, we want to define the quantum SW periods in the strong coupling region by using the quantum SW curve.

We note that the quantum SW curve and the SW periods work both in the weak-coupling and strong coupling region. Then, in this thesis, we define the quantum SW periods on the $\Omega$ background in the strong coupling region by the quantum SW periods given by the WKB solutions of the quantum SW curves. We are interested in physics at the singularity locus in the moduli space, where some BPS particles become massless. Especially, on the massless monopole/dyon point (which is on the strong coupling region), the quantum SW periods are calculated by [48], [45], [49]. In this thesis and [46], [47], we consider the Argyres-Douglas theories which is defined in the strong coupling region [19], [20].

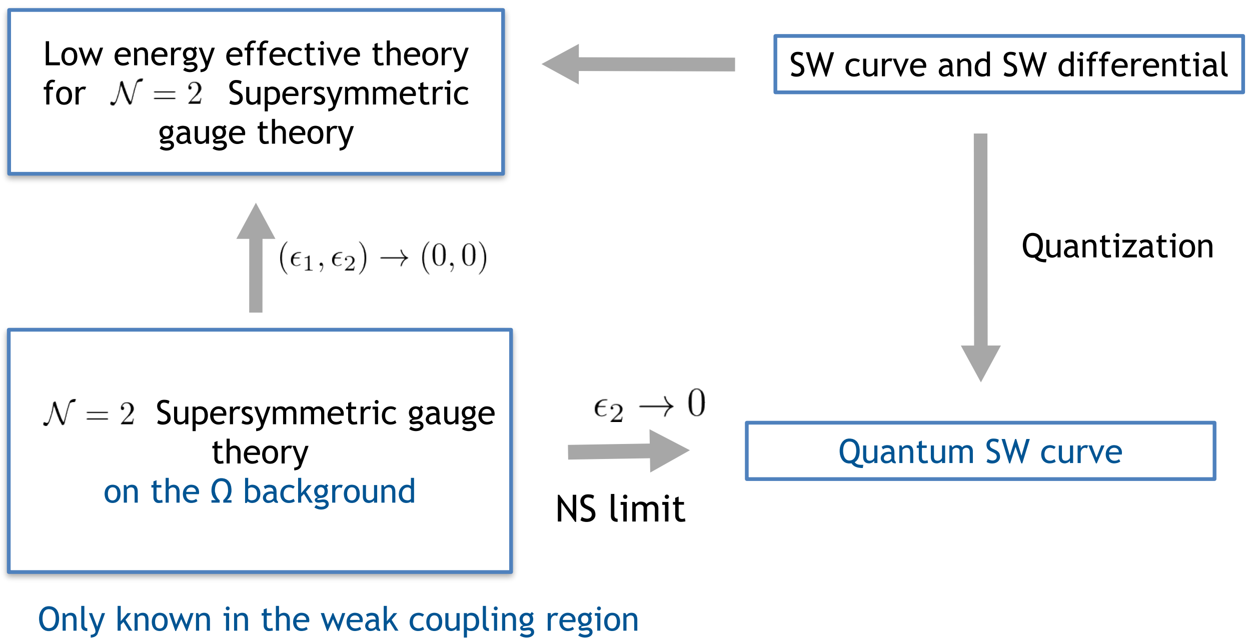


Figure 1.1: This is a conceptual diagram. The $\Omega$ background is parameterized by two real numbers $(\epsilon_1, \epsilon_2)$. If we take $(\epsilon_1, \epsilon_2) \to (0, 0)$, the theory becomes low energy effective theory of a $\mathcal{N} = 2$ supersymmetric gauge theory. If we take the NS limit, this theory can be represented by the quantum SW curve, which is given by quantizing the SW curve.

In the SW theory, there are massless singular points in the moduli space, where some massless BPS particles become massless. In general, two massless BPS particles are either mutually local or mutually non-local. In particular, in the $SU(3)$ Super Yang-Mills case, Argyres and Douglas [19] argued that if mutually non-local massless particles coexist on the moduli space. This point corresponds to a non-trivial new superconformal field theory, which is called the Argyres-Douglas (AD) theory. The Seiberg-Witten curve of the AD theory can be obtained by the scaling limit of the curve of $SU(3)$ theory, where two intersecting one-cycles on the genus two Riemann surfaces degenerate into the point. Near the superconformal point, we can consider the scaling limit, which is one of the deformations on the superconformal point [19]. By the scaling limit, the SW curve degenerates, and a pair $\alpha$-cycle and $\beta$-cycle which cross on the moduli space become trivial at the same time. [20] generalize the AD theory in the $SU(2)$ $N_f = 1, 2, 3$ cases, and in this paper, the CFT features of the AD theories are studied. This classification is generalized to $SU(N_c)$ $N_f$ SQCDs [23]. Superconformal field theories are particularly interesting examples of strongly coupled field theories. Since the theory is non-local, one cannot write down the Lagrangian explicitly by using fundamental fields. Therefore it is hard to study the theory based on the Lagrangian approach. Conformal field theories can be characterized by operators and their operator product expansions. In particular, relevant operators provide deformation around the superconformal point, which gives information about physics. In the AD theory, the relevant operators and the associated coupling constants are read off from the SW curve of the AD theories. One can then classify the AD theories by investigating the scaling limit of the gauge theories.

In this thesis and [46], [47], we study the quantum SW periods in the strong coupling region, especially around the superconformal point. We quantize the SW curve and the SW differential for $\mathcal{N} = 2$ supersymmetric QCD. Then, we will find the quantum SW periods as the WKB solution of the quantum SW curves. We can obtain the quantum SW periods in the AD theories by taking the scaling limit of these quantum SW periods in the UV theories. But we can also obtain the quantum SW periods in the scaling limit by using the quantum SW curve in the scaling limit : Fig:1.2. To confirm the consistency

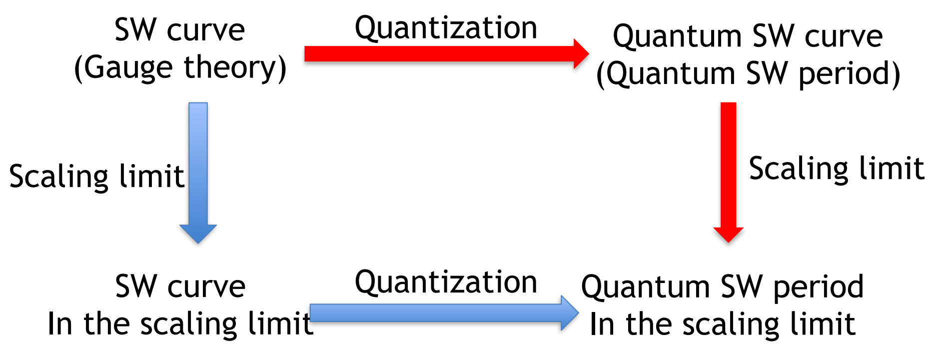


Figure 1.2: Quantization of the AD curve is consistent with the scaling limit.

of our method, we will check the equivalence of the two approaches. In particular, we focus on the universality of the AD theories. We say that two UV gauge theories belong to the same universality class if these two gauge theories become the same AD theory in the scaling limit. We confirm that this quantization is consistent with the scaling limit, by considering that the universality is preserved by the quantization of the SW curves.

This thesis is organized as follows: In chapter 2, we first explain supersymmetry algebras and supersymmetry multiplets. Then we construct supersymmetry Lagrangian of $\mathcal{N} = 2$ Super Yang-Mills and SQCDs. After that, we consider low energy effective action, which can be determined by prepotential. We argue that the one-loop part of the prepotential can be calculated by the $\beta$-function. We also consider the moduli space of vacua and explain the Coulomb branch and Higgs branch.

In chapter 3, we next explain the SW theory in the $\mathcal{N} = 2$ Super Yang-Mills case following [6] and [12]. We first explain that the fields of the effective theory on the moduli space of vacua can be interpreted as a base manifold of an $SL(2,\mathbb{Z})$ vector bundle. We also explain that the BPS mass can be written as a section on the vector bundle (which is only determined on the moduli points without a singular point), and calculate the monodromies around the singular points by using the fact that the BPS masses must be one-valued. Finally, we introduce the SW curve and the SW differential that is the Riemann surface and 1-form on the Riemann surface. Then, we explain that there exist homeomorphism between the torus (corresponding to the SW curve and the SW differential) and the moduli space which is written as a base manifold of an $SL(2,\mathbb{Z})$ bundle.

In chapter 4, we generalize the SW theory in $SU(N_c)$ $N_f$ SQCDs, $A_r$ type and $D_r$ type gauge theories [7]. In these cases, we determine the SW curve by using the symmetries on the moduli space. We also note that in these cases, there are massless singular points, not only massless dyon or massless monopole points but also massless quark points.

In chapter 5, we explain the AD theories. We first explain the AD theory for $SU(3)$ Super Yang-Mills case [19]. We introduce the AD theory at the superconformal points on the moduli parameter on which the SW curve degenerates. Then, we discuss the features of the AD theories as CFTs for $SU(2)$ $N_f = 1, 2, 3$ SQCDs following [20]. Then, we generalize these discussions and introduce AD theories obtained from $SU(N_c)$ $N_f$ SQCDs, $A_r$ and $D_r$ type gauge theories [23]. We also discuss the universality of AD theories [46], [47].

Finally, in chapter 6, we introduce the $\Omega$ background. We first explain the definition of the $\Omega$ background following [35]. Next, we explain the quantization of the SW curves [33], [34] and calculate the quantum SW periods as WKB solutions of the quantum SW curves. Then, we discuss the quantization of the SW curves for $A_r$ type and $D_r$ type gauge theories [46], and discuss the quantum SW curves in the scaling limit [46], [47], and calculate the quantum SW periods. For the consistency check, we confirm the universality of AD theories preserved by the quantization (introducing of the $\Omega$ background). To do this, we compare the quantum SW periods in the scaling limit by using the Picard-Fuchs equations [33], [34], [41], [48], [40], [50], [37], [56].

In Appendix, we will summarize the notations of the $\mathcal{N} = 1$ supersymmetry and some detailed calculations. In Appendix A, we first explain the notations of the $\mathcal{N} = 1$ supersymmetry algebra and multiplets. We also explain the $U(1)$ $\beta$-function which is necessary to describe the effective action around the dyon massless points. In Appendix B, we will examine some variable transformations in the SW curves. Finally, in Appendix C, we will calculate the Picard-Fuchs equations and higher-order corrections. We will also show some explicit examples of the theories with lower ranks.

# Chapter 2

# $\mathcal{N} = 2$ Supersymmetric Gauge Theory and the Prepotential

In this chapter, we will explain supersymmetry algebra and its representation. In particular, we will focus on $\mathcal{N} = 2$ supersymmetry (SUSY). The field contents of supersymmetric theories take the form of supersymmetry multiplets. For $\mathcal{N} = 2$ SUSY, they are vector and hyper-multiplets. We will explain examples of $\mathcal{N} = 2$ supersymmetric field theories by using supersymmetry multiplets. We finally explain that the low-energy effective action of these theories which are described by the prepotentials. The one-loop part of the prepotential is calculated by the $\beta$-functions.

## 2.1 $\mathcal{N} = 2$ Supersymmetry Algebra

Let us first explain the $\mathcal{N} = 2$ supersymmetry algebra following [12] and [6]. It is an extension of the Poincaré algebra by adding two sets of Weyl spinor charges. For $\mathcal{N} = 2$ algebra, it contains the central charge. The central charge provides the lower bound for the mass spectrum, which is called the BPS formula. We show the BPS formula gives a relationship between mass and central charge. The BPS formula will be also important to study the Seiberg-Witten theory, which defines the monodromy structure on the moduli space. We also introduce the $R$-symmetry which rotates the set of supercharges. We also define supersymmetry multiplets by using supersymmetry algebra. The $\mathcal{N} = 2$ supersymmetry algebra was first studied in [1].

### 2.1.1 $\mathcal{N}=2$ supersymmetry Algebra

Let us explain $\mathcal{N}=2$ supersymmetry algebra. Supersymmetry is fermionic symmetry that exchanges bosonic fields and fermionic fields. $\mathcal{N}$ extended supersymmetry is a symmetry that exchanges $\mathcal{N}$ sets of fermions with $\mathcal{N}$ sets of bosons. $\mathcal{N}=1$ supersymmetry algebra is reviewed in Appendix A. We will study $\mathcal{N}=2$ supersymmetry algebra in this chapter.

The supersymmetry generators $Q^I_\alpha, \overline{Q}_{\dot{\alpha}J}$ $(I,J=1,2)$ satisfy the following anti-commutation relations, where the $\alpha$ and $\dot{\alpha}$ are the spinor indices:

$$\{Q^I_\alpha, \overline{Q}_{\dot{\alpha}J}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J, \tag{2.1.1}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z^{IJ}, \tag{2.1.2}$$

$$\{\overline{Q}_{\dot{\alpha}I}, \overline{Q}_{\dot{\beta}J}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} Z_{IJ}. \tag{2.1.3}$$

Here $P_\mu$ is the translation operator, $\epsilon_{\alpha\beta}$ is an anti-symmetric invariant tensor of the spinor Lorentz transformation $SL(2,\mathbf{C})$. We can take $Z$ to be real by choosing the phase of the operators $Q$. $Z^{IJ} = \epsilon^{IJ} Z$ with $I,J=1,2$ are called the central charges. $Z$ commutes with all the generators of supersymmetry, therefore $Z$ is a fixed value for a representation of the algebra.

### 2.1.2 BPS Formula

Next, we construct the representation of $\mathcal{N}=2$ SUSY algebra. We then define the mass formula which is called the BPS formula. Let us define two sets of anti-commuting generators from supercharges as

$$a_\alpha := \frac{1}{2}\left(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right), \qquad b_\alpha := \frac{1}{2}\left(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right). \tag{2.1.4}$$

Then the algebra (2.1.1)-(2.1.3) for the massive state in the static frame $P^\mu = (M,0,0,0)$ reduces to

$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}\left(M + \sqrt{2}Z\right), \qquad \{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta}\left(M - \sqrt{2}Z\right). \tag{2.1.5}$$

These relations mean that $a$ and $b$ are anti-commuting harmonic oscillators. Thus from the unitarity of the representation, we obtain the following relation:

$$M \geq \sqrt{2}|Z| \tag{2.1.6}$$

This formula is called the BPS (Bogomolnyi-Prasad-Sommerfield) formula. Since this formula is defined by using supersymmetry algebra only, this formula does not change by the quantum corrections as far as $\mathcal{N}=2$ SUSY is preserved.

### 2.1.3 $R$-symmetry

In the above, we have introduced $\mathcal{N}=2$ supersymmetry algebra. The two supercharges are related by global $U(2)_R$ rotations (or rotations of the indices $I=1,2$). We can include this transformation in the supersymmetry algebra, which is called the $R$-symmetry. The $R$-symmetry can be written as $U(2)_R=(SU(2)_R\times U(1)_R)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ is the normal subgroup of $SU(2)_R\times U(1)_R$, and we identify $(1,e^{i\pi})\in SU(2)_R\times U(1)_R$ and $(-1,1)\in SU(2)_R\times U(1)_R$ by the $\mathbb{Z}_2$. Let us define the action of the $R$-symmetry on the supermultiplets. In $\mathcal{N}=2$ supersymmetric theories, there are two types of supermultiplets. We explicitly write the action of the $R$-symmetry on the following two types of supermultiplets.

- $\mathcal{N}=2$ vector multiplet ($\mathcal{N}=2$ chiral multiplet):

  This multiplet is consist with an $\mathcal{N}=1$ vector multiplet $V=(A_\mu,\lambda)$ and a chiral super field $\Phi=(\phi,\psi)$. The diagonal part of the $SU(2)_R$ is $U(1)_J\subset SU(2)_R$, which act on the superfield as follows:

$$U(1)_R:\Phi\to e^{2i\alpha}\Phi(e^{-i\alpha}\theta),\qquad V\to V(e^{-i\alpha}\theta) \tag{2.1.7}$$
$$U(1)_J:\Phi\to \Phi(e^{-i\alpha}\theta),\qquad V\to V(e^{-i\alpha}\theta). \tag{2.1.8}$$

  The other elements of $SU(2)_R$ mix these $V$ and $\Phi$. We also confirm that if $\alpha=\pi$, $U(1)_R$, and $U(1)_J$ act on superfields in the same way.

- $\mathcal{N}=2$ hypermultiplet:

  This multiplet contains two complex scalars $(q,\tilde{q}^\dagger)$ and two Weyl fermions $(\psi_q,\tilde{\psi}_q^\dagger)$. In $\mathcal{N}=1$ Language, this multiplet is decomposed by chiral multiplets $Q=(q,\psi_q)$ and $\tilde{Q}=(\tilde{q},\tilde{\psi}_q)$. These multiplets transform under the $U(1)_R$ and $U(1)_J$ as

$$U(1)_R:Q\to Q(e^{-i\alpha}\theta),\qquad \tilde{Q}\to\tilde{Q}(e^{-i\alpha}\theta), \tag{2.1.9}$$
$$U(1)_J:Q\to e^{i\alpha}Q(e^{-i\alpha}\theta),\qquad \tilde{Q}\to e^{i\alpha}\tilde{Q}(e^{-i\alpha}\theta). \tag{2.1.10}$$

For later convenience, we write the action of $U(1)_R$ and $U(1)_J$ on all component fields:

$$
\begin{aligned}
U(1)_R: \qquad & \phi & \to \quad & e^{2i\alpha}\phi, \\
& (\psi, \lambda) & \to \quad & e^{i\alpha}(\psi, \lambda), \\
& (\psi_q, \tilde{\psi}_q) & \to \quad & e^{-i\alpha}(\psi_q, \tilde{\psi}_q), \\
& (A_\mu, q, \tilde{q}) & \to \quad & (A_\mu, q, \tilde{q}), \\
U(1)_J: \qquad & (\lambda, q) & \to \quad & e^{i\alpha}(\lambda, q), \\
& (\psi, \tilde{q}^\dagger) & \to \quad & e^{-i\alpha}(\psi, \tilde{q}^\dagger), \\
& (\phi, A_\mu, \psi_q, \tilde{\psi}_q) & \to \quad & (\phi, A_\mu, \psi_q, \tilde{\psi}_q).
\end{aligned}
\tag{2.1.11}
$$

## 2.2 $\mathcal{N} = 2$ Supersymmetic Gauge Theories (Classical)

In this section, we construct some $\mathcal{N} = 2$ supersymmetric gauge theories by using the super multiplets. In the following, we consider $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theories and supersymmetric QCDs (SQCDs) following [12], [6], [7], [11]. In particular, we will consider the low energy effective theories of these theories, and specify the structures of the Coulomb branches of these vacuum moduli spaces. The $\mathcal{N} = 2$ supersymmetric Lagrangians were first found by [2]. These supersymmetric theories were also studied in the viewpoint of dimensional reduction [3].

### 2.2.1 $\mathcal{N} = 2$ $SU(N_c)$ Super-Yang-Mills Theory

Let us first introduce the $\mathcal{N} = 2$ Super Yang-Mills (SYM) theory with gauge group $SU(N_c)$. This theory is constructed by the $\mathcal{N} = 2$ vector multiplet, which is defined by $\mathcal{N} = 1$ vector multiplet $V$ and $\mathcal{N} = 1$ chiral superfield $\Phi$. Here, the $\mathcal{N} = 1$ vector superfield $V$ and the $\mathcal{N} = 1$ chiral superfield $\Phi$ belongs to the adjoint representation of the gauge group. The component field $A_\mu$ in the $\mathcal{N} = 1$ vector superfield $V$ is the gauge connection of the gauge group. In terms of $\mathcal{N} = 1$ superfields, the $\mathcal{N} = 2$ super

Yang-Mills theory has the following Lagrangian:

$$\mathcal{L} = \frac{1}{k}\mathrm{Imtr}\left\{\frac{1}{4\pi}\tau\int d^2\theta d^2\overline{\theta}\Phi^\dagger e^{2V}\Phi e^{-2V} + \frac{1}{8\pi}\int d^2\theta\tau W^\alpha W_\alpha\right\}, \qquad \tau = \frac{\Theta}{2\pi} + \frac{4\pi}{g^2}i, \tag{2.2.1}$$

where $\Theta$ is a real parameter called the theta angle which produces the theta term in the Lagrangian. $k$ is some normalization constant. $W_\alpha$ is the supersymmetric field strength of the vector superfield $V$. We can rewrite this Lagrangianin terms of component fields:

$$\begin{aligned}\mathcal{L} =& \frac{1}{kg^2}\mathrm{tr}\Big(-i\overline{\lambda}\overline{\sigma}^\mu D_\mu\lambda - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\Theta g^2}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}D^2 \\ &+ \tilde{F}F - i\psi\overline{\sigma}^\mu D_\mu\overline{\psi} - D_\mu\overline{\phi}D^\mu\phi + D[\phi,\overline{\phi}] + i\sqrt{2}\overline{\phi}[\lambda,\psi] - i\sqrt{2}[\overline{\psi},\overline{\lambda}]\phi\Big).\end{aligned} \tag{2.2.2}$$

Here, $D$ is an auxiliary field of vector multiplet $V$, and $F$ is an auxiliary field of chiral superfield $\Phi$. The potential term is

$$\frac{1}{g^2}\mathrm{tr}\left(\frac{1}{2}D^2 + D[\phi,\phi^\dagger] + \overline{F}F\right). \tag{2.2.3}$$

By using the equation of motion of $D$, and substitute it in this potential, we obtain the classical potential of the pure $\mathcal{N} = 2$ theory as

$$V(\phi) = \frac{1}{2g^2}\mathrm{Tr}[\phi,\phi^\dagger]^2 \geq 0. \tag{2.2.4}$$

Thus, the classical vacuum solutions are defined as solutions that satisfy $V(\phi) = 0$.

## The Coulomb Branch

Let us next consider the classical vacua of the theory. The classical vacuum of the $\mathcal{N} = 2$ $SU(N_c)$ SYM is given by the solution of $[\phi,\phi^\dagger] = 0$, where the $\phi$ belongs to $SU(N_c)$ gauge adjoint representation. $\phi$ can be written as a diagonalizable matrix:

$$U\phi U^{-1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & 0 \\ & & \cdots & \\ 0 & & & a_{N_c}\end{pmatrix}, \qquad a_i \in \mathbb{C}, \qquad \sum_{i=1}^{N_c} a_i = 0. \tag{2.2.5}$$

Here, $U$ is a unitary matrix. However, these vacuum solutions are independent up to permutation group $\mathcal{G}_{N_c} \subset SU(N_c)$ of $(a_1, \cdots, a_{N_c})$. These transformations are called Weyl reflections. Therefore, physically different vacua are

$$\{(a_1, \cdots, a_{N_c})\}/\mathcal{G}_{N_c}, \qquad a_i \in \mathbb{C}, \tag{2.2.6}$$

and parameterized by the Weyl invariants defined by

$$\det(\lambda - \phi) = \prod_i (\lambda - a_i) = \lambda^{N_c} + \lambda^{N_c - 2} \sum_{i<j} a_i a_j - \lambda^{N_c - 3} \sum_{i<j<k} a_i a_j a_k + \cdots (-1)^N \prod_{i=1}^{N_c} a_i = 0. \tag{2.2.7}$$

In this vacua, the gauge group $SU(N_c)$ broken to $U(1)^{N_c - 1}$, so these vacua are called the Coulomb branch. For example, in $SU(2)$ case, the Weyl invariant is $u = \frac{1}{2} < \mathrm{Tr}(\phi^2) >= a^2$, and the gauge group $SU(2)$ broken to $U(1)$ on the Coulomb branch. Then, because $u$ has $U(1)_R$ charge four, the $U(1)_R$ global symmetry also breaks to $\mathbb{Z}_4$. We also note that $-1 \in U(1)_R$ and $-1 \in U(1) \subset SU(2)$ gauge transformation must be identified. Thus, at $u \neq 0$, symmetries on this point is $(U(1) \times \mathbb{Z}_4)/\mathbb{Z}_2$. But at $u = 0$, the symmetry is $(SU(2) \times \mathbb{Z}_4)/\mathbb{Z}_2$.

## Higgs Mechanism

Let us explain in the $SU(2)$ SYM case. If Higgs field has non-zero vacuum expectation value $\langle \phi \rangle = a \neq 0$, then, substituting $\phi = a\sigma^3 + \tilde{\phi}_i \sigma^i$ to the Lagrangian (2.2.2), we obtain

$$\begin{aligned}
\mathcal{L} =& \frac{1}{g^2} \mathrm{tr} \left( -i\overline{\lambda}\overline{\sigma}^\mu D_\mu \lambda - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\Theta g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i\psi \overline{\sigma}^\mu D_\mu \overline{\psi} \right) \\
&- |a|^2 (A^1_\mu A^{\mu 1} + A^2_\mu A^{\mu 2}) + \frac{a^*}{g} (A^1_\mu \partial^\mu \tilde{\phi}_2 + A^2_\mu \partial^\mu \tilde{\phi}_1) + \frac{a}{g} (A^1_\mu \partial^\mu \tilde{\phi}^*_2 + A^2_\mu \partial^\mu \tilde{\phi}^*_1) \\
&- 8a^* (A^{\mu i} A^i_\mu \tilde{\phi}^3 - A^{\mu 3} A^i_\mu \tilde{\phi}^i) - 8a (A^{\mu i} A^i_\mu \tilde{\phi}^{3*} - A^{\mu 3} A^i_\mu \tilde{\phi}^{i*}) \\
&+ \frac{2}{g^2} (\partial_\mu \tilde{\phi}^*_i)(\partial^\mu \tilde{\phi}^i) + \frac{4i}{g} \epsilon^{ijk} \partial_\mu \tilde{\phi}^*_i A^\mu_j \tilde{\phi}_k + \frac{4i}{g} \epsilon^{ijk} \tilde{\phi}^*_k \partial^\mu \tilde{\phi}_i A^j_\mu \\
&- 8 (\tilde{\phi}^*_i \tilde{\phi}_i A^j_\mu A^{\mu j} - \tilde{\phi}^*_i \tilde{\phi}_j A^j_\mu A^{\mu i}) \\
&+ \frac{1}{g^2} a^* (\lambda^1 \psi^2 - \lambda^2 \psi^1) + \frac{2}{g^2} \epsilon^{ijk} \tilde{\phi}^*_i \lambda_j \psi_k \\
&+ \frac{1}{g^2} a (\psi^{1*} \lambda^{2*} - \psi^{2*} \lambda^{1*}) + \frac{2}{g^2} \epsilon^{ijk} \tilde{\phi}_i \lambda^*_j \psi^*_k \\
&+ 2 \left\{ \mathrm{Re}(\tilde{\phi}_2 \tilde{\phi}^*_3) + \mathrm{Im}(a \tilde{\phi}^*_2) \right\}^2 + 2 \left\{ \mathrm{Re}(\tilde{\phi}_1 \tilde{\phi}^*_3) + \mathrm{Im}(a \tilde{\phi}^*_1) \right\}^2 + 2\mathrm{Re}(\tilde{\phi}_1 \tilde{\phi}^*_2)^2.
\end{aligned} \tag{2.2.8}$$

Therefore, off-diagonal parts of each fields $A_\mu, \psi, \lambda, \tilde{\phi}$ obtain masses of order $a^2$. This is the Higgs mechanism. Only diagonal parts of $A_\mu, \psi, \lambda, \tilde{\phi}$ remain massless. Therefore, gauge symmetry $SU(2)$ is broken to $U(1)$. All massless fields $A^3_\mu, \psi_3, \lambda_3, \tilde{\phi}_3$ do not have $U(1)$ charge. Thus, the massless part of the theory is $U(1)$ gauge theory, and this $U(1)$ gauge theory is a free theory (its $\beta$ function is zero).

### 2.2.2 $\mathcal{N}=2$ SQCDs with Matters

Let us consider $\mathcal{N}=2$ $SU(N_c)$ SQCDs with $N_f$ matters which belong to the fundamental representation of the gauge group. These theories have the following fields:

- The matter hypermultiplet including two chiral multiplets $Q^i_a$ and $\tilde{Q}_{ia}$ ($i=1,\cdots,N_f$ denotes the flavor index, $a=1,\cdots,N_c$ the color index). These fields belong to fundamental representation and anti-fundamental representation of flavor symmetry.
- $\mathcal{N}=2$ vector multiplet including $\mathcal{N}=1$ vector multiplet $V$ and chiral multiplet $\Phi$.

The $\mathcal{N}=2$ $SU(N_c)$ with $N_f$ matters SQCD Lagrangian can be written as follows:

$$\mathcal{L} = \mathrm{Im}\frac{\tau}{4\pi}\left[\int d^2\theta d^2\overline{\theta}\left(\mathrm{tr}(\Phi^\dagger e^{2V}\Phi e^{-2V}) + Q^\dagger_i e^V Q^i + \tilde{Q}^{\dagger i} e^{-V}\tilde{Q}_i\right) + \int d^2\theta\left(\frac{1}{2}\mathrm{tr}W^2 + \mathcal{W}\right)\right]. \tag{2.2.9}$$

The superpotential $\mathcal{W}$ can be written as

$$\mathcal{W} = \sum_{i=1}^{N_f}\sqrt{2}(\tilde{Q}^a_i\Phi^b_a Q^i_b + m^j_i\tilde{Q}^a_i Q^j_a). \tag{2.2.10}$$

Here, the quark mass matrix $m^i_j$ must satisfy $[m, m^\dagger]=0$ to preserve $\mathcal{N}=2$ supersymmetry. Thus, this matrix $m$ can be diagonalized by a flavor symmetry as

$$m = \mathrm{diag}(m_1, \cdots, m_{N_f}). \tag{2.2.11}$$

### The classical vacua

Next, we consider the classical vacua. The scalar potential of the theory is

$$V = \frac{1}{2}\mathrm{Tr}(D^2) + F^{A\dagger}F_A, \tag{2.2.12}$$

where $F_A = \partial\mathcal{W}/\partial\varphi^A$ and $D^a = \varphi_A^\dagger (T^a)^A_B \varphi^B$, and $T^a$ are the generators of the gauge group $G = SU(N_c)$. The $\varphi^A = (\phi, Q, \tilde{Q})$ are the scalar field in the all $\mathcal{N} = 1$ chiral superfields. Thus, the classical vacua can be given as solutions of the $F$-term and $D$-term set to zero.

- The $D$-term equations:

$$0 = [\phi, \phi^\dagger], \qquad \nu\delta^b_a = Q^i_a (Q^\dagger)^b_i - (\tilde{Q}^\dagger)^i_a \tilde{Q}^b_i \tag{2.2.13}$$

- The $F$-term equations:

$$\rho\delta^b_a = Q^i_a \tilde{Q}^b_i, \qquad 0 = Q^j_a m^i_j + \Phi^b_a Q^i_b, \qquad 0 = m^j_i \tilde{Q}^a_j + \tilde{Q}^b_i \Phi^a_b \tag{2.2.14}$$

Here, $\nu \in \mathbb{R}$ and $\rho \in \mathbb{C}$ are arbitrary numbers.

Let us discuss the solutions to these vacua conditions. For simplicity, we only consider the $m^j_i = 0$ case. The solutions can be divided into the following three types, Coulomb branch, Higgs branch, and mixed-branch.

- Coulomb branch:

  The Coulomb phase is defined as the solution of $F$-term and $D$-term equations with $Q = \tilde{Q} = 0$. In this phase, the solution must satisfy $[\phi^\dagger, \phi] = 0$, so $\phi$ can be diagonalized by color rotation as

$$\phi = \mathrm{diag}(a_1, \cdots, a_{N_c}), \qquad \sum_i a_i = 0. \tag{2.2.15}$$

  Then, the gauge symmetry broken from $SU(N_c)$ to $U(1)^{N_c-1}$.

- Higgs branch:

  The Higgs phases are defined as a solution of $F$-term and $D$-term equations with $\langle\phi\rangle = 0$. Thus, the gauge symmetry is fully broken on the Higgs branch, and the $F$-term condition becomes the same as the following stationary condition:

$$\frac{\partial\mathcal{W}}{\partial Q_i} = \frac{\partial\mathcal{W}}{\partial\tilde{Q}_i} = 0. \tag{2.2.16}$$

- Mixed phases:

  Mixed phases are defined as solutions of $F$-term and $D$-term equations with expected values of $\Phi$ and $Q$ and $\tilde{Q}$ all are non-zero.

## 2.3 $\mathcal{N} = 2$ Supersymmetric Gauge Theories (Quantum)

Let us next consider symmetries in quantum moduli space in $\mathcal{N} = 2$ supersymmetric gauge theories.

### 2.3.1 Asymptotic Freedom of $SU(2)$ $N_f = 0, 1, 2, 3$ SQCDs

Let us first explain the asymptotic freedom of supersymmetric QCDs. The beta function of $SU(2)$ $N_f$ SQCD is given by

$$\beta(g) = -\frac{4 - N_f}{16\pi^2} g^3. \tag{2.3.1}$$

Here, $g$ is the effective coupling which is a function of the scale $t = \log E$ ($E$ is the energy scale). $g$ satisfy the renormalization group (RG) equation:

$$\frac{dg}{dt} = \beta(g). \tag{2.3.2}$$

Let us solve this equation. We can rewrite this equation as $dg/g^3 = -\frac{1}{4\pi^2} dt$, then we obtain

$$\int_{g_0}^{g} \frac{dg}{g^3} = -\frac{4 - N_f}{16\pi^2}(t - t_0), \qquad t = \log E,\ t_0 = \log \mu, \tag{2.3.3}$$

where $g_0 = g(\mu)$ and $g = g(E)$. Thus we obtain

$$\frac{1}{g(E)^2} = \frac{1}{g(\mu)^2} + \frac{4 - N_f}{8\pi^2} \log \frac{E}{\mu} = \frac{4 - N_f}{8\pi^2} \log \frac{E}{\Lambda}, \tag{2.3.4}$$

where $\Lambda = \mu \exp\left(-\frac{8\pi^2}{(4 - N_f) g(\mu)^2}\right)$ is the scale parameter. We can take the typical scale parameter $\mu$ as $\mu = a$. Therefore, $SU(2)$ $N_f = 0, 1, 2, 3$ SQCDs are asymptotic free, and thus in the low energy scale, effective coupling becomes large, then we cannot use the perturbative method in the low energy scale.

Then, we will use the low energy effective action for the quantum theory, which is defined by integral out all massive fields of (2.2.8). Global symmetries of the low energy effective theory are constrained of these of global symmetry of high energy theory ('t Hooft anomaly is the same in low energy effective action and high energy theory).

### 2.3.2 The Chiral Anomaly

Classically, these theories have $\mathcal{N} = 2$ supersymmetry, R-symmetry, gauge symmetry, and flavor symmetry. We seek the possibility of the breakdown of these symmetries by quantum effects. We see that $\mathcal{N} = 2$ supersymmetry and gauge symmetry is not broken by anomalies. The remaining possibility of symmetry breaking is the $R$-symmetry. The $R$-symmetry is written as $U(2)_R = (SU(2)_R \times U(1)_R)/\mathbb{Z}_2$. The $SU(2)_R$ cannot break because $\mathcal{N} = 2$ supersymmetry does not break. Let us show that the $U(1)_R$ symmetry is broken by the chiral anomaly.

The symmetry of the quantum theory is defined by the symmetry of the correlation functions and partition function. Thus, to find the symmetry breaking by the quantum correction, we should consider the non-zero valued correlation functions. The non-zero valued correlation function necessarily includes all fermion zero modes in the inserted operators. Therefore, we need to know the number of zero modes. By the Atiyah-Singer index theorem with $N_f$ fermions which belong to fundamental representation and two fermions in the adjoint representation of the gauge group $SU(N_c)$, the number of zero-modes $N$ in the one-instanton background is

$$N = \frac{2N_c + N_f + N_f}{8\pi^2} \int_{M_4} \mathrm{tr} F\tilde{F} = 2N_c + 2N_f. \tag{2.3.5}$$

Here, $M_4$ is four dimensional spacetime. Therefore, non-zero correlation functions for fermions in the one-instanton background is given by [12]:

$$G = \left\langle \lambda(x_1) \cdots \lambda(x_{2N_c}) \psi(y_1) \cdots \psi(y_{2N_c}) \psi_q(z_1) \cdots \psi_q(z_{N_f}) \tilde{\psi}_q(u_1) \cdots \tilde{\psi}_q(u_{N_f}) \right\rangle. \tag{2.3.6}$$

Notice that these fermion fields $\lambda, \psi, \psi_q, \tilde{\psi}_q$ has $U(1)_R$ charges $1, 1, -1, -1$. Therefore, under the $e^{i\alpha} \in U(1)_R$ transformation, correlation function $G$ transformed as

$$e^{i\alpha} : G \to e^{i\alpha(4N_c - 2N_f)} G. \tag{2.3.7}$$

Thus, the correlation function $G$ does not change only $\alpha = n/(4N_c - 2N_f)$ $(n \in \mathbb{Z})$ case. Therefore, $U(1)_R$ is broken to $\mathbb{Z}_{4N_c - 2N_f}$.

### 2.3.3 Symmetries for the $\mathcal{N} = 2$ Super Yang-Mills

Let us consider the symmetries on the Coulomb branch of the SYM. The gauge symmetry is broken to $SU(N_c) \to U(1)^{N_c - 1}$ because of the Higgs mechanism. We assume that $\mathcal{N} = 2$

supersymmetry does not break spontaneously. The $U(1)_R$ symmetry is broken to $\mathbb{Z}_{4N_c}$ by the chiral anomaly. This $\mathbb{Z}_{4N_c}$ symmetry is broken to the subgroup on the Coulomb branch by the vacuum expectation value of the Higgs fields. Let us summarize the $R$-symmetry breaking for $N_c$.

- Let us consider the $SU(2)$ case. Because $\phi$ has $U(1)_R$ charge 2, the composite field $\phi^2$ has $U(1)_R$ charge 4, and in the Coulomb branch, the vacuum is parameterized by the vacuum expectation value of the gauge invariant field $u = \frac{1}{2}\langle \mathrm{tr}\phi^2 \rangle$. The action of $\mathbb{Z}_8$ on $\phi^2$ is given by
$$e^{2\pi i\alpha} : \phi^2 \to e^{8\pi i\alpha}\phi^2, \qquad \alpha = \frac{1}{8}, \frac{2}{8}, \cdots, 1 \tag{2.3.8}$$
Thus, $\phi^2$ is invariant under only the subgroup $\mathbb{Z}_4 \subset \mathbb{Z}_8$ for non-zero value of $\langle \mathrm{tr}\phi^2 \rangle$. Thus the final $R$-symmetry group for $SU(2)$ gauge SYM is
$$(SU(2)_R \times \mathbb{Z}_4)/\mathbb{Z}_2. \tag{2.3.9}$$

- Let us next consider the $SU(3)$ case. $U(1)_R$ symmetry is broken to $\mathbb{Z}_{12}$ by the chiral anomaly. On the Coulomb branch, the vacuum is parameterized by the vacuum expectation value of $\langle \mathrm{tr}\phi^2 \rangle$ and $\langle \mathrm{tr}\phi^3 \rangle$. Because the $\phi$ has $U(1)_R$ charge 2, by $e^{2\pi in/12} \in \mathbb{Z}_{12}, (n \in \mathbb{Z})$, the moduli parameter $\langle \mathrm{tr}\phi^2 \rangle$ and $\langle \mathrm{tr}\phi^3 \rangle$ transform as $\langle \mathrm{tr}\phi^2 \rangle \to e^{2\pi in/3}\langle \mathrm{tr}\phi^2 \rangle$ and $\langle \mathrm{tr}\phi^3 \rangle \to e^{\pi in}\langle \mathrm{tr}\phi^3 \rangle$. For generic points on the Coulomb branch, only the subset $\mathbb{Z}_2 \subset \mathbb{Z}_{12}$ is invariant. Thus the final $R$-symmetry group for $SU(3)$ gauge SYM is
$$(SU(2)_R \times \mathbb{Z}_2)/\mathbb{Z}_2 = SU(2)_R \tag{2.3.10}$$

- In $SU(N_c)$ with $N_c \geq 4$, no subgroup of $\mathbb{Z}_{4N_c}$ survive for generic vacua.

### 2.3.4 Symmetries for the $SU(2)$ SQCDs

Let us next consider the symmetry of $SU(2)$ $N_f$ SQCD. We only consider the asymptotically free theories ($N_f = 0, 1, 2, 3$).

Let us first consider the massless case with $m_i = 0$. The classical theory has a global symmetry of $O(2N_f) \times SU(2)_R \times U(1)_R$. In the representation $O(2N_f)$, the squarks $Q^i_a$

and $\tilde{Q}_{ia}$ can be written as $Q^r$ with $r = 1, \cdots, 2N_f$ which is the vector representation of $O(2N_f)$. However, the actual symmetry is the quotient of the $O(2N_f) \times SU(2)_R \times U(1)_R$ by $\mathbb{Z}_2$. Here $\mathbb{Z}_2$ is the center of the $SU(2)_R$ and $O(2N_f)$. $\mathbb{Z}_2 \subset O(2N_F)$ symmetry is generated by "parity" transformation:

$$\rho : Q_1 \leftrightarrow \tilde{Q}_1, \tag{2.3.11}$$

and other squarks invariant. If we add mass terms which preserve the parity symmetry, the relevant term in the Lagrangian is

$$m_i^j \tilde{Q}_j^a Q_a^i = m_i^j \epsilon^{ab} Q_a^i Q_b^j \subset \mathcal{L}. \tag{2.3.12}$$

Then the mass changes as $m_1 \to -m_1$ under the parity transformation.

In quantum theory, the $U(1)_R$ symmetry and $\mathbb{Z}_2 \subset O(2N_f)$ are anomalous. For $N_f > 0$, anomaly free subgroup of $U(1)_R \times O(2N_f)$ is $\mathbb{Z}_{4(4-N_f)}$, which is generated by the following transformation.

$$\begin{aligned} W_\alpha \to& \omega W_\alpha(\omega^{-1}\theta) \\ \Phi \to& \omega^2 \Phi(\omega^{-1}\theta) \\ Q^1 \to& \overline{Q}_1(\omega^{-1}\theta) \\ \overline{Q}_1 \to& Q^1(\omega^{-1}\theta) \\ Q^i \to& Q^i(\omega^{-1}\theta), \qquad i \neq 1 \\ \overline{Q}_i \to& \overline{Q}_i(\omega^{-1}\theta), \qquad i \neq 1 \end{aligned} \tag{2.3.13}$$

where $\omega = \exp(\pi i/2(4 - N_f))$. Note that we identify $\mathbb{Z}_2 \subset \mathbb{Z}_{4(4-N_f)} \subset U(1)_R$ and $(-1)^F \in Spin(4)$. Combined with the symmetry (2.3.13) and $U(1)_J \subset SU(2)_R$, the anomaly free symmetry that commutes with the $\mathcal{N} = 1$ supersymmetry is the following.

$$\begin{aligned} \Phi \to& \omega^2 \Phi(\theta) \\ Q^1 \to& \omega^{-1} \overline{Q}_1(\theta) \\ \overline{Q}_1 \to& \omega^{-1} Q^1(\theta) \\ Q^i \to& \omega^{-1} Q_i(\theta), \qquad i \neq 1 \\ \overline{Q}_i \to& \omega^{-1} \overline{Q}_i(\theta), \qquad i \neq 1 \end{aligned} \tag{2.3.14}$$

Under this transformation, gauge invariant order parameter $u = \mathrm{Tr}\phi^2$ has charge 4 and transformed as $u \to \exp(2\pi i/(4-N_f))u$. This further breaks $\mathbb{Z}_{4(4-N_f)}$ down to $\mathbb{Z}_{4-N_f}$ (this $\mathbb{Z}_{4-N_f}$ symmetry does not change $u$).

### 2.3.5 Low energy Effective Action and the Prepotentials

In this subsection, we will introduce the low-energy effective action and express it by using the prepotential. The (Wilsonian) low energy effective action is defined by the path integral:

$$e^{-S_{\text{eff}}} := \int \prod_r \mathcal{D}\psi_r e^{-S}. \tag{2.3.15}$$

Here, $S$ is the action of $\mathcal{N} = 2$ supersymmetric gauge theory given at a point on the Coulomb branch (2.2.8), $\psi_r$ are fields that become massive by the Higgs mechanism. $S_{\text{eff}}$ is expressed in massless $U(1)^{N_c-1}$ fields. For example, in $SU(2)$ SYM case, these massless $U(1)$ fields are $A_3, \psi_3, \lambda_3, \tilde{\phi}_3$. However, it is difficult to perform this integral and write down $S_{\text{eff}}$ explicitly.

Let us consider the effective theory defined on the Coulomb branch. Note that the $\mathcal{N} = 2$ supersymmetry and maximal abelian subgroup in gauge symmetry survive at a generic point on the Coulomb branch. Then, on the Coulomb branch, the low energy effective Lagrangian is completely determined by the holomorphic function $\mathcal{F}(\mathcal{A}^i)$ called prepotential, which is a holomorphic function of $\mathcal{N} = 2$ massless vector multiplets $\mathcal{A}^i = (A^i, V^i), i = 1, \cdots, N_c - 1$ ( Here, $A^i = (a^i, \tilde{\psi}^i)$ is a $\mathcal{N} = 1$ chiral multiplet and $a^i$ is a scalar component, $V^i = (\tilde{\lambda}^i, \tilde{A}^i_\mu)$ is a $\mathcal{N} = 1$ vector multiplet) [5]. For example, in $SU(2)$ SYM case, we take $a(x) = \tilde{\phi}_3$, $\tilde{\psi}(x) = \psi_3(x)$, $\tilde{\lambda}(x) = \lambda_3(x)$ and $\tilde{A}_\mu = A^3_\mu$.. The general form of the Lagrangian can be written by the holomorphic function of $\mathcal{A}^i$ as

$$\begin{aligned}
\mathcal{L}_{\text{eff}} =& \frac{1}{4\pi}\mathrm{Im}\int d^2\theta_1 d^2\theta_2 \mathcal{F}(\mathcal{A}^i) \\
=& \frac{1}{4\pi}\mathrm{Im}\left[\int d^2\theta d^2\overline{\theta}\frac{\partial \mathcal{F}}{\partial A^i}\overline{A}^i + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A^i \partial A^j} W^{\alpha i} W^j_\alpha\right].
\end{aligned} \tag{2.3.16}$$

Here, the first line is the integral of the $\mathcal{N} = 2$ superfields, so the integral is defined on $\mathcal{N} = 2$ superspace. In the second line, we use the notation of the $\mathcal{N} = 1$ superfield [67].

The first term defines the Kahler potential, which induces the metric on the Coulomb branch:

$$ds^2 = \mathrm{Im}\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} da^i d\overline{a}^j. \tag{2.3.17}$$

Note that the second-order derivatives of the prepotential $\tau^{ij} = \partial^2 \mathcal{F}/\partial a_i \partial a_j$ represent the effective couplings.

This low energy effective action has $U(1)$ gauge symmetry, and all fields do not have $U(1)$ charge. Therefore, this $U(1)$ theory is free, and $\beta$-function is free.

Let us consider the prepotential of the $SU(2)$ SYM. In this case, we have a single abelian vector multiplet $\mathcal{A} = (\Phi, V)$. The low energy effective Lagrangian is given by

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A}\overline{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A \partial A} W^\alpha W_\alpha\right]. \tag{2.3.18}$$

Let us determine the prepotential $\mathcal{F}$ in the one-loop perturbative level. In this theory, the effective coupling is given by (2.3.4). Then the $k$-instanton correction of $\mathcal{F}$ can be written as $e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{a}\right)^{4k}$. Therefore, by using $\mathcal{F}'' = \tau = \frac{4\pi i}{g^2}$ we obtain the following result:

$$\mathcal{F} = \frac{i}{2\pi}\Phi^2 \log\frac{\Phi^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\Phi}\right)^{4k} \Phi^2. \tag{2.3.19}$$

Here, we include the instanton corrections to the prepotential [5].

In the same way, we obtain the prepotential for $SU(2)$ $N_f = 1, 2, 3$ SQCDs as follows:

$$\mathcal{F} = \frac{i(4-N_f)}{8\pi}\Phi^2 \log\frac{\Phi^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_{N_f,k} \left(\frac{\Lambda}{\Phi}\right)^{(4-N_f)k} \Phi^2. \tag{2.3.20}$$

### 2.3.6 Energy scale and Effective Coupling

Let us finally discuss the relation between effective coupling and energy scale. Because of the asymptotic freedom of SQCDs, the effective coupling of SQCDs becomes large when the energy scale of the theory is small. Thus, the perturbation method is not useful in low energy scale. Then, on a low energy scale, we use the low energy effective theory. The low energy effective theory is free $U(1)^{N_c-1}$ gauge theory, so the $\beta$-function is zero. The effective coupling is calculated as Fig:2.1. Therefore, $u = a^2 \to \infty$ region is weak coupling region, but finite $u = a^2$ region belongs to strong coupling region.

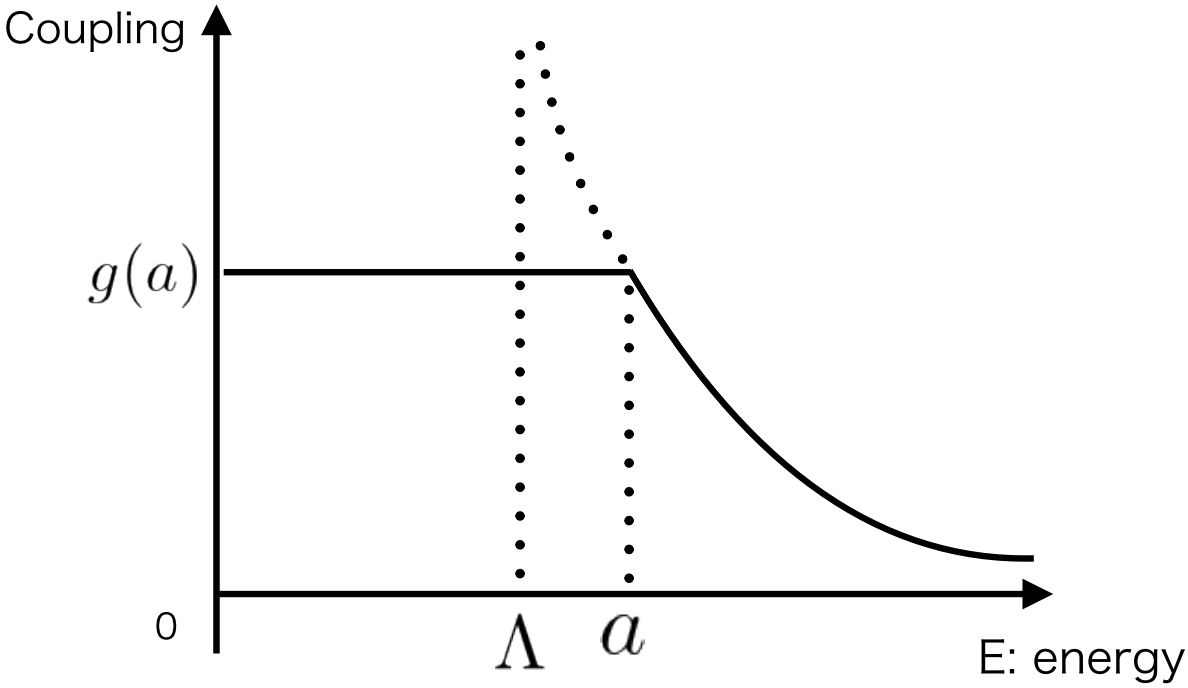


Figure 2.1: The horizontal direction of this figure represents the energy scale of the theory. If we take $|a|$ (masses of massive fields in SQCDs) large, the effective coupling becomes small. On the other hand, if we take $|a|$ finite, the effective coupling is large.

# Chapter 3

# The Seiberg-Witten Theory for $\mathcal{N} = 2$ $SU(2)$ Super Yang-Mills Theory

In this chapter, we will explain the Seiberg-Witten theory for $\mathcal{N} = 2$ $SU(2)$ super Yang-Mills Theory (SYM). Seiberg and Witten found the exact solution of the low-energy effective action of $SU(2)$ SYM [6] and $SU(2)$ SQCD [7].

First, we will argue the mathematical structure of the moduli space of vacua. In particular, we discuss $\mathcal{N} = 2$ $SU(2)$ super Yang-Mills theory and describe the moduli space of vacua as a base space of an $SL(2,\mathbb{Z})$-bundle over $u = \langle \mathrm{tr}\phi^2 \rangle$-plane, where an $SL(2,\mathbb{Z})$ transformation is regarded as the duality transformation. Next, we characterize the BPS masses as the section on the $u$-plane. We then will introduce the monodromies on the moduli space of vacua by requiring that the BPS masses does not change by the monodromy transformation. Finally, we explain the Seiberg-Witten (SW) solution for the $\mathcal{N} = 2$ $SU(2)$ SYM, which gives the effective coupling as a torus moduli parameter of a Riemann surface called the SW curve.

## 3.1 Mathematical Description of the Moduli Space

The Coulomb branch is parameterized by the complex parameter $u$. Complexified coupling constant $\tau$ is a function of $a$. However, if $\tau$ is defined globally, it cannot be positive definite (the harmonic function $\tau$ cannot have a minimum). Then $a$ is not defined globally on the $u$-plane. It is regarded as the section of a vector bundle on the $u$-plane.

To study the transformation properties of $a$, let us introduce the Lagrangian multiplier and write the effective Lagrangian in the dual description. Let us define $h(\Phi) := \partial\mathcal{F}/\partial\Phi$ and rewrite the effective Lagrangian (2.3.18) as

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\text{Im}\left[\int d^4\theta h(\Phi)\overline{\Phi} + \int d^2\theta \frac{1}{2}\tau(\Phi)W^\alpha W_\alpha\right]. \tag{3.1.1}$$

Note that the Bianchi identity is given by $\text{Im}\mathcal{D}W = 0$. Introducing a vector superfield $V_D$ as a Lagrangian multiplier and integrating out $W$, the effective Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\text{eff}} \to &\mathcal{L}_{\text{eff}} + \frac{1}{4\pi}\text{Im}\int d^4xd^4\theta V_D\mathcal{D}W \\ =&\mathcal{L}_{\text{eff}} - \frac{1}{4\pi}\text{Im}\int d^4xd^2\theta W_D W, \qquad W_D = \mathcal{D}V_D \\ =&\frac{1}{4\pi}\text{Im}\left[\int d^4\theta h(\Phi)\overline{\Phi} - \int d^2\theta\frac{1}{2}\frac{1}{\tau(\Phi)}W_D^\alpha W_{\alpha D}\right].\end{aligned} \tag{3.1.2}$$

This Lagrangian takes the same form as the (3.1.1), by introducing

$$\Phi_D := h(\Phi), \qquad h_D(\Phi_D) := -\Phi, \tag{3.1.3}$$

because note $h'(\Phi) = \tau(\Phi)$, we obtain $-\frac{1}{\tau(\Phi)} = -\frac{1}{h'(\Phi)} = h'_D(\Phi_D) = \tau_D(\Phi_D)$. Then, we can write (3.1.2) as

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\text{Im}\left[\int d^4\theta h_D(\Phi_D)\overline{\Phi}_D + \int d^2\theta\frac{1}{2}\tau_D(\Phi_D)W_D^\alpha W_{\alpha D}\right]. \tag{3.1.4}$$

This is the dual form of the effective Lagrangian (3.1.1). This dual transformation change the effective coupling as

$$\tau \to \tau_D = -\frac{1}{\tau}. \tag{3.1.5}$$

Notice also that the action (2.3.18) is invariant under $\tau \to \tau + 1$ (This is because we take imaginary part when we consider the effective action). These transformation of $\tau$ generates the $SL(2,\mathbb{Z})$ group

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{Z}. \tag{3.1.6}$$

Let us now define the $SL(2,\mathbb{Z})$ transformation of the $(a_D, a)$. Note that $\tau(a) = \frac{\partial a_D}{\partial a}$, we should define the $SL(2,\mathbb{Z})$ transformation of $(a_D, a)$ such that they do not contradict with the $SL(2,\mathbb{Z})$ transformation of the $\tau(a)$. Therefore an $SL(2,\mathbb{Z})$ transformation acts on $(a_D, a)$ as the following:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}. \tag{3.1.7}$$

This $SL(2,\mathbb{Z})$ transformation (3.1.7) is generated by the following transformation, which correspond to the transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.1.8}$$

These matrices generate the $SL(2,\mathbb{Z})$ group. $(a_D, a)$ construct an $SL(2,\mathbb{Z})$ bundle on the $u$-plane.

## 3.2 BPS Masses

In the previous section, we have defined the moduli space of vacua as a base space of a complex two-dimensional vector bundle on the $u$-plane. In this viewpoint, the $SL(2,\mathbb{Z})$ transformation can be regarded as the gauge transformation on the vector bundle. This $SL(2,\mathbb{Z})$ structure is given by the physical requirement that the dual description gives the same physical result. But, we also obtain another requirement that the physical observable does not change by the gauge transformation $SL(2,\mathbb{Z})$. The BPS mass represents a mass of a physical particle and therefore it can be regarded as a physical observable.

As we explained in section 2.1, the classical BPS bound $M \geq \sqrt{2}|Z|$ in $\mathcal{N} = 2$ supersymmetric gauge theories is given by

$$Z = a(n_e + \tau_{cl} n_m), \qquad \tau_{cl} = \frac{\Theta}{2\pi} + i\frac{4\pi}{\tau(a)^2}. \tag{3.2.1}$$

This inequality does not change by quantum correction, because this formula is defined by the supersymmetry algebra. However, the BPS masses may be modified by quantum corrections.

Let us consider theories containing matter fields represented as hypermultiplets. Let us write these hypermultiplets as $\mathcal{N} = 1$ chiral multiplets $M_i$ and $\tilde{M}_i$. These $\mathcal{N} = 1$ chiral multiplets $M_i$ and $\tilde{M}_i$ become massive if $a \neq 0$ because of (3.2.1). If a hypermultiplet which is consist of $M$ and $\tilde{M}$ carries charge $n_e$, then its coupling can be written as

$$\sqrt{2} n_e A M \tilde{M}. \tag{3.2.2}$$

From this coupling, in such a state, we find $Z = a n_e$. Similarly, we consider magnetic monopole which carries magnetic charge $n_m$. Then we find the BPS state has $Z = n_m a_D$. In general, $Z$ for dyon with pair of charges $(n_e, n_m)$ is given by

$$Z = a n_e + a_D n_m. \tag{3.2.3}$$

If the gauge group has rank $r$, we obtain

$$Z = a^i n_{e,i} + h_i(a) n_m^i = a \cdot n_e + a_D \cdot n_m. \tag{3.2.4}$$

The BPS mass can be seen as a section of the bundle on the $u$-plane defined by $(a_D(u), a(u))$.

## 3.3 Monodromies on the moduli space

We have seen that the BPS mass can be interpreted as a section of the $(a_D(u), a(u))$ bundle on the $u$-plane. Since the BPS mass is a physical observable, this should be a one-valued section on the $u$-plane. Especially, the BPS mass should be invariant under monodromy transformation on the $u$-plane.

### 3.3.1 The Physical Meaning of the Singular Points

First, we explain the physical meaning of the singular points in the moduli space. To obtain low-energy effective action, we have to integrate out all massive states. Then, the non-trivial metric on the moduli space can be got by integrating out the massive fields. However, one cannot define an effective coupling constant as the entire function of the moduli parameter. Then the effective coupling must have a singularity. The singularity implies the breakdown of the description of low-energy effective action and one needs additional massless degrees of freedom to describe the theory at the singularity point.

It is known that if a spin-1 multiplet become massless, the theory becomes inconsistent because we do not expect enhancement of gauge symmetry at a strong coupling region.

Therefore we assume spin $\leq 1/2$ hypermultiplets become massless. Then, the massless states are dyons that belong to $\mathcal{N}=2$ short multiplet, equivalently these massless states are BPS states. But, since we consider the strong coupling region, we consider the electro-magnetic duality transformation such that the dual photon is coupled to the dyon locally. For example, the monopole couple weakly to $W_D$ with the coupling $\tau_D = -1/\tau$, because of (2.3.4).

Next, we discuss the number of singular points in the moduli space. In principle, we cannot determine the number of singular points on the moduli space. It can be checked by the consistency of the solution. In the following, we assume a minimal model that has three singular points. We consider the monodromy around the singular point.

### 3.3.2 Monodromy at Large $u$

Let us first consider the monodromy at large $u$. At large $|a| \sim \sqrt{u}$, the theory is at weak coupling and asymptotically free, so we can use the classical formula for $u$, and we obtain $u = \frac{1}{2}a^2$. To obtain the monodromy around $u=\infty$, we have to obtain $a_D$ as a function of $u$. In $\mathcal{N}=2$ $SU(2)$ SYM, the prepotential in one-loop level is given as the first term in (2.3.19). From the prepotential, we obtain $a_D$ as

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi}\log\frac{a}{\Lambda} + \frac{ia}{\pi}. \tag{3.3.1}$$

If we take a closed large loop on the $u$-plane around $u=0$ as $u \to e^{2\pi i}u$, $(a_D(u), a(u))$ are changed as the following:

$$a_D \to -a_D + 2a, \qquad a \to -a. \tag{3.3.2}$$

Therefore the monodromy matrix act on $(a_D, a)$ is defined as follows:

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = PT^{-2}. \tag{3.3.3}$$

Here, $P := -1 \in SL(2,\mathbb{Z})$ and $T$ is a generator of $SL(2,\mathbb{Z})$ defined as before. As we explain before, the BPS mass should not change under the monodromy matrix. Therefore, the electric charge and magnetic charge $(n_e, n_m)$ in the BPS mass formula (3.2.3) should be changed by the monodromy matrix, not to change the BPS formula. Therefore $(n_e, n_m)$ changed by the monodromy matrix $M_\infty$ as the following:

$$n'_m = -n_m, \qquad n'_e = -n_e - 2n_m. \tag{3.3.4}$$

### 3.3.3 Monodromies at $u = \pm\Lambda^2$

Next, we consider the existence of monodromies around finite singular points on the $u$-plane. We will denote the monodromy matrices around the singular point $u$ on the $u$-plane as $M_u$. In the following, we explain that there exist at least two singular points on the $u$-plane.

To show this, we first show that at least one of these matrices $M_u$ must do not commute with $M_\infty$. Assume that all of these monodromy matrices commute with $M_\infty$. Then these $M_u$ cannot change $a^2$, and $u = a^2$ can be treated as global coordinate on the $u$-plane. However, if this global coordinate exists, the metric $\mathrm{Im}\tau(a)$ on the $SL(2,\mathbb{Z})$ bundle can be define globally. However, this global coordinate does not lead to a positive definite coupling constant on the $u$-plane. Thus, at least, one of the monodromy matrices must not commute with $M_\infty$. Therefore, there are at least 2 singular points at finite $u$.

We also note that this $u$-plane must have $\mathbb{Z}_2$ symmetry act by $u \to -u$. This $\mathbb{Z}_2$ symmetry is a unbroken $U(1)_R$ part, and we use the fact that this $e^{i\alpha} \in U(1)_R$ symmetry acts on $\phi$ as $\phi \to \phi' = e^{2i\alpha}\phi$. Therefore we can conclude that the minimal assumption is that there exist singular points at $u$ and $-u$. We can take these points at $u = \pm\Lambda^2$ without losing generality, where the $\Lambda$ is the scale parameter. Then, these monodromy matrices satisfy that

$$M_{\Lambda^2} M_{-\Lambda^2} = M_\infty. \tag{3.3.5}$$

We note that these massless hypermultiplets that appear at finite $u$ must have a magnetic charge. If these hypermultiplets have only electric charges, then the monodromy matrices could commute with $M_\infty$.

Let us calculate the monodromies around the singular points at $u = \pm\Lambda^2$. On these points, some massive dyons (having a non-zero magnetic charge) become massless. Note

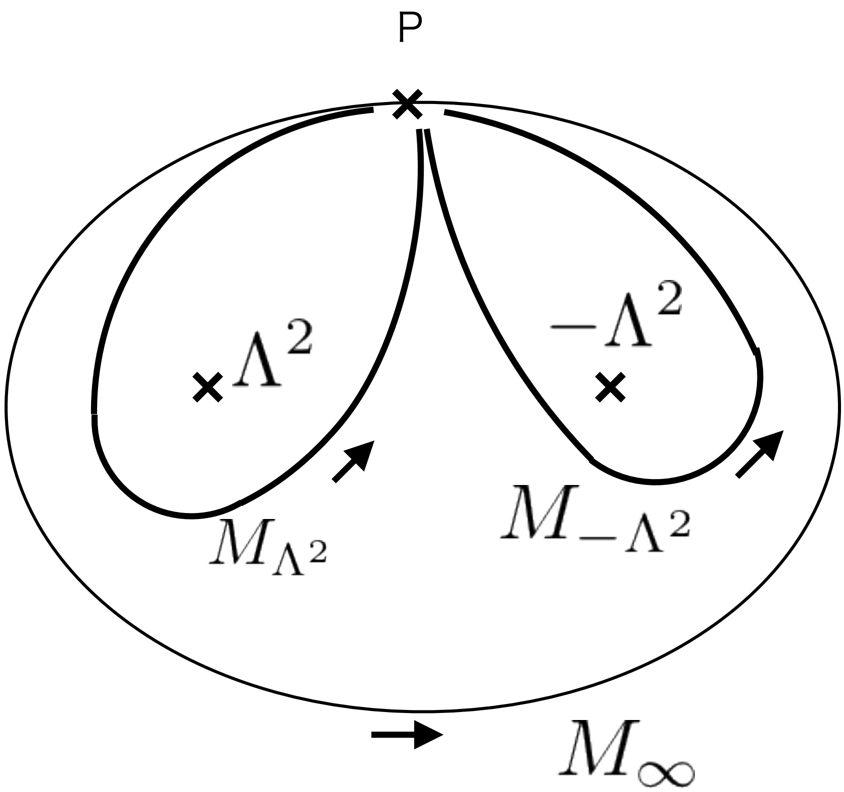


Figure 3.1: There are massless singular points on the $u$-plane. These singular points have monodromies.

that hypermultiplets which represent massive dyons with non-zero magnetic charge cannot couple to fundamental gauge fields. However, if we transform the description by the $SL(2,\mathbb{Z})$ duality transformation, some dual gauge fields couple to these dyons (that have non-trivial magnetic charges), and we can obtain the formula (3.2.3). Thus, to define monodromies around finite $u$, we should calculate the monodromy on the massless hypermultiplet points in the dual description given by some $SL(2,\mathbb{Z})$ dual transformations.

Let us calculate the monodromy around the point $u_q = \pm\Lambda^2$ by using this method. Let us take this massless hypermultiplet as $(a_D(u), a(u))$, and write the charge as $(n_m, n_e)$.

By using an $SL(2,\mathbb{Z})$ transformation, we transform as

$$\begin{pmatrix} \check{a}_D \\ \check{a} \end{pmatrix} = \begin{pmatrix} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{pmatrix}, \qquad \begin{pmatrix} \check{n}_m \\ \check{n}_e \end{pmatrix} = \begin{pmatrix} n_m\delta - n_e\gamma \\ -n_m\beta + n_e\alpha \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1. \tag{3.3.6}$$

Here, we denote the dual description of the section $(a_D(u), a(u))$ as $(\check{a}_D(u), \check{a}(u))$, and assume that this hypermultiplet in the dual description couple to some dual gauge field. Then, we can take $(\check{n}_m, \check{n}_e) = (0, 1)$ with taking some scaling, by using the Dirac quantization condition. On the singular point $u = u_q$, these hypermultiplets become massless, then we obtain $\check{a}(u_q) = 0$. Therefore, we can expand $\check{a}(u)$ near the singular point $u = u_q$ as follows:

$$\check{a}(u) \sim c_q(u - u_q). \tag{3.3.7}$$

By using the one-loop $U(1)$ beta function explains in the appendix, we obtain the effective coupling as

$$\tau(\check{a}(u)) = -\frac{i}{\pi}\log\frac{\check{a}(u)}{\Lambda}. \tag{3.3.8}$$

Thus we can obtain

$$\check{a}_D(u) = -\frac{i}{\pi}\check{a}(u)\log\frac{\check{a}(u)}{\Lambda} + \frac{i}{\pi}. \tag{3.3.9}$$

Then the monodromy matrix around the singular point $u = u_q$ is following:

$$\begin{pmatrix} \check{a}_D \\ \check{a} \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \check{a}_D \\ \check{a} \end{pmatrix}. \tag{3.3.10}$$

Let go back to the dual description. As we assume above, we take $(\check{n}_m, \check{n}_e) = (0, 1)$. So, the monodromy matrix is given as follows:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 + 2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1 - 2n_e n_m \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}. \tag{3.3.11}$$

We call this monodromy matrix as $M(n_m, n_e)$. We can confirm $M(n_m, n_e) \in SL(2,\mathbb{Z})$.

Finally, let us find possible charges $(n_m, n_e)$ of the hypermultiplet in the minimal model which has singular point at $u = \pm\Lambda^2$. Let us write the charges of the hypermultiplets that become massless at $u = \Lambda^2$ and $u = -\Lambda^2$ as $(m, n)$ and $(m', n')$. Then, the monodromy matrices must satisfy the following equation:

$$M_{\Lambda^2}(m,n) M_{-\Lambda^2}(m',n') = M_\infty. \tag{3.3.12}$$

The solution of these equations is the following:

$$\begin{array}{ccccc} (m,n): & (1,n), & (-1,n), & (-1,n), & (1,n), \\ (m',n'): & (1,n-1), & (1,-n-1), & (-1,n+1), & (-1,-n+1). \end{array} \tag{3.3.13}$$

The simple solution of (3.3.12) is $m = m' = 1, n = 0, n' = -1$. In this case, the monodromy matrices become

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{3.3.14}$$

We can conclude that the vector bundle on the $u$-plane defined by $(a_D(u), a(u))$ is the $SL(2,\mathbb{Z})$ bundle.

## 3.4 The Seiberg-Witten Theory

In the previous section, we have calculated the monodromies around the singular points of the $SL(2,\mathbb{Z})$ bundle on the $u$-plane. We can construct one to one map between the prepotential $\mathcal{F}$ as a function of $a$ and the $SL(2,\mathbb{Z})$ bundle over the $u$-plane. Note that the vector bundle is defined by fixing the local trivialization and the transformation function. We also note that the $SL(2,\mathbb{Z})$ group is a well-known modular transformation of the torus. Then if we treat these monodromy matrices as modular transformations of some torus, we can conversely define a prepotential $\mathcal{F}$. This is the main result of Seiberg-Witten theory explained below.

Let us consider the following curve parameterized by $u$:

$$E_u : \qquad y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u), \qquad u \in \mathbb{C}. \tag{3.4.1}$$

Here, $x, y \in \mathbb{C}$.

The curve defines a Riemann surface that is topologically equivalent to a torus Fig.3.2. In this viewpoint, we have to treat $x$-plane as a double cover of $\mathbb{C}$ with the point at infinity added to it. This double cover has 4 branch points at $x = -\Lambda^2, \Lambda^2, u, \infty$. We consider one branch cut from $-\Lambda^2$ to $\Lambda^2$, and another from $u$ to $\infty$. These two sheets are joint along with these cuts. On this double cover, $y$ is a single-valued function. This Riemann surface is a genus 1 Riemann surface. We can identify this Riemann surface as a torus $T^2$ topologically, we call this Riemann manifold as $E_u \simeq T^2$. We define the modular parameter $\tau_u$ of this torus $E_u$. We will obtain this moduli parameter $\tau_u$ by using the de-Rham cohomology of $E_u$.

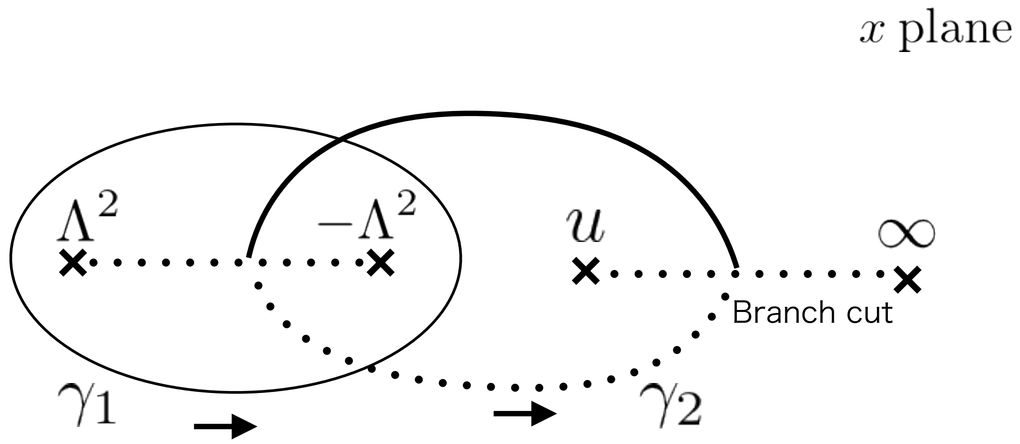


Figure 3.2: The SW curve is the Riemann surface. The zeroes on the $x$-plane are $x = \pm\Lambda^2, u$. We take branch cut that connect $-\Lambda^2$ to $\Lambda^2$, and $u$ and $\infty$.

Let us consider cycles on the torus $E_u$. We take the basis of the cycles as $\gamma_1, \gamma_2 \in V_u = H_1(E_u, \mathbb{C})$, which are normalized as the intersection number is 1:

$$\gamma_1 \cdot \gamma_2 = 1. \tag{3.4.2}$$

These cycles are linear transformations on 1-forms on $E_u$:

$$\gamma : \lambda \to \oint_\gamma \lambda. \tag{3.4.3}$$

Here $\lambda \in H^1(E_u, \mathbb{C})$ is meromorphic $(1,0)$-form on $E_u$. We can choose the following basis of meromorphic 1-forms at $E_u$:

$$\lambda_1 = \frac{dx}{y}, \qquad \lambda_2 = \frac{xdx}{y}. \tag{3.4.4}$$

Now we can construct modular parameter $\tau_u$ of the torus $E_u$. We define quantity

$$b_i = \oint_{\gamma_i} \lambda_1, \qquad i = 1, 2. \tag{3.4.5}$$

By methematical result, we can obtain

$$\tau_u = b_1/b_2, \qquad \mathrm{Im}(\tau_u) > 0. \tag{3.4.6}$$

For the purpose that we want to see this modular parameter $\tau_u$ of the torus $E_u$ as an effective coupling $\tau$ of the low energy theory, note that the effective coupling satisfies

$$\tau = \frac{da_D}{da} = \frac{da_D(u)/du}{da(u)/du}. \tag{3.4.7}$$

Generally, if we define some functions $f(u)$, $\tilde{a}(u)$ and $\tilde{a}_D(u)$, which satisfy

$$\frac{d\tilde{a}_D(u)}{du} = f(u)b_1, \qquad \frac{d\tilde{a}(u)}{du} = f(u)b_2, \qquad \tilde{\tau} := \frac{d\tilde{a}_D(u)/du}{d\tilde{a}(u)/du}, \tag{3.4.8}$$

we can obtain

$$\tilde{\tau} = \frac{b_1}{b_2} =: \tau_u. \tag{3.4.9}$$

Let us find this function $f(u)$, $\tilde{a}(u)$ and $\tilde{a}_D(u)$ which satisfy $\tau_u = \tau$. For this purpose, let us introduce arbitrary 1-form which can be written as

$$\lambda(u) = a_1(u)\lambda_1 + a_2(u)\lambda_2, \tag{3.4.10}$$

and assume that these $\tilde{a}(u)$ and $\tilde{a}_D(u)$ can be written as

$$\tilde{a}_D(u) = \oint_{\gamma_1} \lambda(u), \qquad \tilde{a}(u) = \oint_{\gamma_2} \lambda(u). \tag{3.4.11}$$

Then, by comparing with (3.4.8), these must satisfy

$$\frac{d\tilde{a}_D}{du} = \oint_{\gamma_1} \frac{d\lambda}{du} = f(u) \oint_{\gamma_1} \lambda_1, \qquad \frac{d\tilde{a}}{du} = \oint_{\gamma_2} \frac{d\lambda}{du} = f(u) \oint_{\gamma_2} \lambda_1. \tag{3.4.12}$$

Let us assume $f(u) = -\sqrt{2}/4\pi$. Then, $\lambda$ can be written as

$$\lambda = \frac{\sqrt{2}}{2\pi}\frac{dx\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} = \frac{\sqrt{2}}{2\pi}\frac{dxy}{x^2-\Lambda^4} = \frac{\sqrt{2}}{2\pi}\frac{dx}{y}(x-u). \tag{3.4.13}$$

We call this 1-form as the SW differential. To write $\tilde{a}_D$ and $\tilde{a}$ explicitly, we have to choose basis $\gamma_1, \gamma_2$. If we take $\gamma_2$ as an one-loop around $-1, 1$, then we obtain

$$\tilde{a}(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}. \tag{3.4.14}$$

In this choice, $\gamma_1$ is a cycle around $\Lambda^2$ and $u$, then we obtain

$$\tilde{a}_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u}\frac{dx\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}. \tag{3.4.15}$$

Finally, let us show that these $\tilde{a}_D(u)$ and $\tilde{a}(u)$ and $\tilde{\tau}$ with this choice of $f(u) = -\sqrt{2}/4\pi$ are same as $(a_D(u), a(u))$ and the effective coupling $\tau$. The asymptotic behavior of these $\tilde{a}(u)$ and $\tilde{a}_D(u)$ are the followings:

$$u \sim \infty : \quad \begin{cases} \tilde{a}(u) \simeq \sqrt{2u}, \\ \tilde{a}_D(u) \simeq i\frac{\sqrt{2u}}{\pi}\log u, \end{cases} \tag{3.4.16}$$

$$u = \Lambda^2 : \quad \begin{cases} \tilde{a}_D(u) \simeq c_0(u-\Lambda^2), \\ \tilde{a}(u) \simeq a_0 + \frac{i}{\pi}\tilde{a}_D\log\tilde{a}_D, \end{cases} \tag{3.4.17}$$

$$u = -\Lambda^2 : \quad \begin{cases} \tilde{a}(u) - \tilde{a}_D(u) \simeq c_0(u+\Lambda^2), \\ \tilde{a}(u) \simeq a_0 + \frac{i}{\pi}(\tilde{a}-\tilde{a}_D)\log(\tilde{a}-\tilde{a}_D). \end{cases} \tag{3.4.18}$$

For $u = -\Lambda^2$, we obtain similar behavior to $u = \Lambda^2$, but with $\tilde{a}_D$ replaced by $\tilde{a} - \tilde{a}_D$. The monodromy matrices at $\infty, \pm\Lambda^2$ is the same as the monodromy matrices (3.3.3) and (3.3.14) defined by prepotential. Thus, we can conclude these $\tilde{a}_D(u)$ and $\tilde{a}(u)$ and $\tilde{\tau}$ with this choice of $f(u) = -\sqrt{2}/4\pi$ are same as $(a_D(u), a(u))$ and the effective coupling $\tau$.

We have explained the exact solution to the periods for $SU(2)$ SYM. In the next chapter, we will explain its generalization to SQCD.

# Chapter 4

# The Seiberg-Witten Theory for $\mathcal{N} = 2$ $SU(N_c)$ SQCD with $N_f$ Matters

In the previous chapter, we explained the SW theory for $SU(2)$ SYM. In this chapter, we will explain the SW theories for the $\mathcal{N} = 2$ $SU(2)$ SQCDs following [7]. One of the important differences between these theories is that there are more possibilities of massless singular points on the moduli spaces. This feature is motivated us to consider the Argyres-Douglas theories as I will explain in the next chapter.

## 4.1 The Mathematical Structure of Moduli Space

We first discuss the mathematical structure of the quantum moduli space for $SU(2)$ SQCD with massless hypermultipltets. Coulomb branch is parameterized by $u = \langle \mathrm{tr}\phi^2 \rangle$. Then, we can define $(a_D(u), a(u))$ on the Coulomb branch. The prepotential $\mathcal{F}$ is also defined by $a_D = \frac{\partial \mathcal{F}}{\partial a}$. If we use the fact that $u = 2a^2$ at large $u$, and also use the prepotential at one-loop level, we obtain $(a_D, a)$ at large $u$ as

$$a \simeq \frac{1}{2}\sqrt{2u}, \qquad a_D \simeq i\frac{4-N_f}{2\pi}a(u)\log\frac{u}{\Lambda_{N_f}^2}. \tag{4.1.1}$$

When we include the instanton effects, we obtain the following expansions:

$$a = \frac{1}{2}\sqrt{2u}\left(1 + \sum_{n=1}^{\infty} a_n(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}\right), \tag{4.1.2}$$

$$a_D = i\frac{4-N_f}{2\pi}a(u)\log\frac{u}{\Lambda_{N_f}^2} + \sqrt{u}\sum_{n=0}^{\infty} a_{Dn}(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}. \tag{4.1.3}$$

## 4.2 Monodromies

When we turn on bare mass for the hypermultiplets, the low-energy theory is deformed. The BPS mass for $SU(N_c = 2)$ $N_f$ SQCD is given by $M = \sqrt{2}|Z|$ with

$$Z = n_e a + n_m a_D + \sum_{i=1}^{N_f} S_i \frac{m_i}{\sqrt{2}}, \tag{4.2.1}$$

where $S_i$'s are the electric $U(1)$ charges of the quarks.

Let us define the monodromies of the vector bundle $(a_D(u), a(u))$ around the singularities on the $u$-plane. In the case of the $\mathcal{N} = 2$ $SU(N_c = 2)$ SQCDs with $N_f \neq 0$, there exist other types of singular points on which there exist extra massless hypermultiplets with non-zero electric charges in addition to monopoles and dyons. These extra singular points correspond to the massless squark singularities and are at $c_i := m_i/\sqrt{2}$. Near the singular points, $(a_D(u), a(u))$ behave as

$$a \simeq c_i, \qquad a_D \simeq -\frac{i}{2\pi}(a - c_i)\log(a - c_i) + d_i, \qquad \forall i = 1, \cdots, N_f. \tag{4.2.2}$$

Therefore, the monodromies around these points are

$$a \to a, \qquad a_D \to a_D + a - \frac{m_i}{\sqrt{2}}. \tag{4.2.3}$$

Therefore the monodromy matrices around the singular points $c_i = m_i/\sqrt{2}$ can be written as

$$\begin{pmatrix} m_1/\sqrt{2} \\ \vdots \\ m_{N_f}/\sqrt{2} \\ a_D \\ a \end{pmatrix} \to \mathcal{M}_i \begin{pmatrix} m_1/\sqrt{2} \\ \vdots \\ m_{N_f}/\sqrt{2} \\ a_D \\ a \end{pmatrix}, \qquad \mathcal{M}_i = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0 \\ & & \mathbb{1}_{N_f} & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ \hline 0 & \cdots & -1 & \cdots & 0 & 1 & 1 \\ 0 & & & & & 0 & 1 \end{array}\right), \tag{4.2.4}$$

are the component $-1$ is the $(N_f + 1, i)$ component of the matrix $\mathcal{M}_i$. Note that the BPS masses do not change by the monodromies. Thus, if we write $\vec{v} := (S_1, \cdots, S_{N_f}, n_m, n_e)$, then this vector transformed by the monodromies as $\vec{v} \to \vec{v}\mathcal{M}_i^{-1}$.

## 4.3 SW curves for $SU(2)$ $N_f = 0, 1, 2, 3$ SQCDs

Let us next introduce the SW curves for $SU(2)$ $N_f = 1, 2, 3$ SQCDs. For the $SU(2)$ SYM case, the SW curve preserves the $\mathbb{Z}_2$ symmetry which acts on the moduli space. We will also use similar symmetries on the quantum moduli spaces to determine the form of the SW curves. We also use the fact that if we take the limit that the scaling parameter goes zero in flavor $N_f$ theory, the SW curve goes to the SW curve with $N_f - 1$ flavor. Therefore, we first discuss the $N_f = 0$ case, and then we explain the $N_f = 1$ case. After that, we will explain the $N_f = 2, 3$ cases.

### 4.3.1 SW curve for $N_f = 0$

Let us first rewrite the $N_f = 0$ SW curve. In $N_f = 0$ case, the SW curve and the SW differential are given as (3.4.1) and (3.4.13). This curve is the modular curve for the group $\Gamma(2)$ consist of the following matrices:

$$M = \begin{pmatrix} m & 2n \\ 2p & q \end{pmatrix}, \qquad \det M = 1. \tag{4.3.1}$$

Here, $m, n, p, q \in \mathbb{Z}$. In the $N_f > 0$ case, there are particles with half-integer valued electric charges in those units. Therefore, for $N_f > 0$, it is convenient to multiply $n_e$ by two and divide $a$ by two to preserve the BPS mass structure $Z = a_D n_m + a n_e$. Dividing $a$ by two and multiply $n_e$ by two in the $N_f = 0$ case, we replace the monodromy matrices (3.3.3) and (3.3.14) as $M \to M' = W^{-1} M W$ ($M = M_{\Lambda^2}, M_{-\Lambda^2}, M_\infty$):

$$M'_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad M'_{-\Lambda^2} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \qquad M'_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{4.3.2}$$

These matrices generate $\Gamma_0(4)$, and the corresponding to the following family of curves:

$$y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda_0^4 x. \tag{4.3.3}$$

Here, we write the scaling parameter of this theory as $\Lambda_0$. The discriminant of the curve is defined as following:

$$\Delta_0 = \frac{\Lambda_0^8}{16}(u - \Lambda_0^2)(u + \Lambda_0^2). \tag{4.3.4}$$

The monodromies define the topological structure of the Riemann surfaces. Thus, it is important to write discriminants of the SW curves. Here, generally, the discriminant of the curve $F(x) = \prod_i (x - e_i)$ is defined as follows:

$$\Delta := \prod_{i<j} (e_i - e_j)^2. \tag{4.3.5}$$

### 4.3.2 The SW curve for $N_f = 1$

Next, let us construct the SW curve for $SU(2)$ $N_f = 1$ SQCD. Let us first consider $N_f = 1$ with massless case. In this case, the $u$-plane has global $\mathbb{Z}_3$ symmetry. For $N_f = 1$, the $r$-th ($r \in \mathbb{Z}_{\geq 0}$) instanton correction is proportional to $\Lambda_1^{3r}$, where the $\Lambda_1$ is the scale parameter. However, due to the "parity" symmetry which acts on squarks as $Q_1 \leftrightarrow \tilde{Q}_1$, only even instanton numbers are possible. For $N_f = 1$, the $U(1)_R$ charge of $\Lambda_1$ is 2 (because the $U(1)_R$ charges of $a$ and $\Lambda_1$ is same, in the prepotential, these are paired to $(\Lambda_1/a)$). Then $u$ has $U(1)_R$ charge 4. For the cubic curve (4.3.3), we can assign the $U(1)_R$ charge of $x$ to 4, which implies that the $U(1)_R$ charge of $F(x, u)$ is 12. Then the only possible term in $F$ that depends on $\Lambda_1$ is a two instanton term. Thus the SW curve for massless $N_f = 1$ theory must be

$$y^2 = x^2(x - u) + t\Lambda_1^6, \tag{4.3.6}$$

where $t \in \mathbb{C}$ is some constant. We can verify $\mathbb{Z}_3 = \mathbb{Z}_{4-N_f}$ symmetry on the $u$-plane.

Next, we consider massive case $m \neq 0$, where the $m$ is the mass of a hypermultiplet. In this case, the parameter of this theory is $m$ and $\Lambda_1$. Therefore, the SW curve may include terms proportional to $m^k (k \in \mathbb{Z})$, and the SW curve can be written as the form

$$y^2 = x^2(x - u) + t\Lambda_1^6 + \sum_{k>0} m^k F_k(x, u, \Lambda_1). \tag{4.3.7}$$

We should determine the functions $F_k(x, u, \Lambda_1)$ by using the symmetries on the quantum moduli space, or natures of some limit. If we take the limit $\Lambda_1 \to 0$, the energy is enough small compare with $m$, so the theory must not depend on $m$. We also note that the mass $m$ transformed under the "parity" as $m \to -m$, as explained below of (2.3.11). Thus, the $m$ and $\Lambda_1$ must be included in the SW curve as the form $\Lambda_1^{k+1} m^l$ where $k, l \geq 0$ are integers and $k+l+1$ must be even. We also note that $m$ has $U(1)_R$ charge 2, then the most general possibility of the mass correction for the massless curve is

$$y^2 = x^2(x-u) + t\Lambda_1^6 + m\Lambda_1^3(ax+bu) + cm^3\Lambda_1^3. \tag{4.3.8}$$

Here, $a, b$ and $c$ are some constants.

Let us next determine these constants. Note that if we take large $m$, the quark can be integrated out, then the low energy theory become the pure $\mathcal{N}=2$ theory, and we obtain $\Lambda_0^4 = m\Lambda_1^3$. Then, we can obtain the SW curve for $SU(2)$ SYM, if we take limit $m \to \infty$ and $\Lambda_1 \to 0$ with $\Lambda_0^4 = m\Lambda_1^3$ fixed. Therefore we obtain $a = \frac{1}{4}$ and $b = c = 0$, and the SW curve for $N_f = 1$ case becomes

$$y^2 = x^2(x-u) + \frac{1}{4}m\Lambda_1^3 x + t\Lambda_1^6. \tag{4.3.9}$$

The degeneration structure of this SW curve can be characterized by the discriminant. The discriminant of this curve is given as

$$\Delta = -\frac{\Lambda_1^6}{16}(\Lambda_1^3 m^3 + 432\Lambda_1^6 t^2 + 72\Lambda_1^3 mtu - m^2u^2 - 64tu^3). \tag{4.3.10}$$

Now we determine the parameter $t$. Let us consider the large mass limit $\Lambda_1 \ll m$. For large $m$, there should be a singularity at $u \sim m^2$ because this point is the quark massless point classically. Then, $\Delta = 0$ is equivalent to

$$\Lambda_1^3 m^3 + 432\Lambda_1^6 t^2 + 72\Lambda_1^3 mtu = 0, \qquad -m^2u^2 - 64tu^3 = 0. \tag{4.3.11}$$

Then, we can find the following singularity:

$$u \simeq -\frac{m^2}{64t}, \qquad x \simeq -\frac{8t\Lambda_1^3}{m}. \tag{4.3.12}$$

Let us next fix the constant $t$. As we explained above, for large $m$, there should be a singularity at $u \sim m^2$ because this point is the quark massless point classically. Compare this singular point with (4.3.12), we obtain $t = -1/64$, and the curve becomes

$$y^2 = x^2(x-u) + \frac{1}{4}m\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6. \tag{4.3.13}$$

The discriminant of the curve is given as follows:

$$\Delta_1 = -\frac{\Lambda_1^6}{4096}(27\Lambda_1^6 + 16384\Lambda_1^3 m^3 - 1152\Lambda_1^3 mu - 4096m^2u^2 + 256u^3). \qquad (4.3.14)$$

### 4.3.3 The curve for $N_f = 2$

Let us next construct the SW curve for $N_f = 2$ case. The SW curve for the $SU(2)$ $N_f = 2$ theory is given as follows:

$$y^2 = \left(x^2 - \frac{1}{64}\Lambda_2^4\right)(x-u) + \frac{1}{4}m_1m_2\Lambda_2^2 x - \frac{1}{64}(m_1^2 + m_2^2)\Lambda_2^4. \qquad (4.3.15)$$

The discriminant of this curve is

$$\begin{aligned}\Delta =& \frac{\Lambda_2^{12}}{65536} - \frac{3\Lambda_2^{10}m_1m_2}{4096} + \frac{\Lambda_2^8}{4096}\{-27(m_1^4 + m_2^4) - 6m_1^2m_2^2 + 36u(m_1^2 + m_2^2) - 8u^2\} \\ &- \frac{\Lambda_2^6}{128}m_1m_2(8m_1^2m_2^2 - 9u(m_1^2 + m_2^2) + 10u^2) + \frac{\Lambda_2^4}{16}(m_1^2 - u)(m_2^2 - u)u^2. \qquad (4.3.16)\end{aligned}$$

Let us check that this curve preserves the symmetries on the moduli space and confirm some limits.

In this theory, the moduli space has the global symmetry $SO(2N_f)|_{N_f=2} = SU(2) \times SU(2)$, which exchange quarks $m_1$ and $m_2$. This theory also has "parity" symmetry (2.3.11), which act $m_i \to -m_i (i = 1, 2)$. This SW curve (4.3.15) does not change under these transformations.

Let us consider $\Lambda_2 \to 0$. In this limit, the bare masses $m_i >> \Lambda_2$, thus these bare masses do not contribute to the theory. Therefore, the SW curve includes $m_i$-dependence only the form $\Lambda_2^{j+1}m_1^k m_2^l (j, k, l \geq 0)$. We can confirm that (4.3.15) satisfy this condition. We also note that in the limit $m_2 \to 0$, this theory becomes $N_f = 1$ theory. The scale parameter satisfies $\Lambda_1^3 = m_2\Lambda_2^2$ [7], here the $\Lambda_1$ is the scale parameter in $N_f = 1$ theory. Therefore, if we take the limit $m_2 \to 0$, we can confirm that the SW curve (4.3.15) becomes the SW curve for $N_f = 1$ theory (4.3.13).

Finally, we require that the $N_f = 2$ theory has singular point at $u = m_1^2$ [7]. If we take $m_2 = 0$, we can check that the discriminant (4.3.16) of the curve (4.3.15) be zero at $u = m_1^2$.

### 4.3.4 The Curve for $N_f = 3$

Next, we construct the SW curve for $N_f = 3$ theory. The SW curve is given as

$$\begin{aligned} y^2 =& x^2(x-u) - \frac{1}{64}\Lambda_3^2(x-u)^2 - \frac{1}{64}(m_1^2+m_2^2+m_3^2)\Lambda_3^2(x-u) \\ &+ \frac{1}{4}m_1m_2m_3\Lambda_3 x - \frac{1}{64}(m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2)\Lambda_3^2. \end{aligned} \tag{4.3.17}$$

Let us check that this SW curve is consistent with some limits and symmetries on the moduli space.

Let us first consider symmetries. In the moduli space, there is $SO(2N_f)|_{N_f=3} = SO(6)$ global symmetry and "parity" symmetry (2.3.11) $m_i \to -m_i$. These symmetries preserve the form of the SW curve (4.3.17).

Next consider the limit $m_3 \to 0$. In this limit, the $N_f = 3$ theory becomes $N_f = 2$ theory. The scale parameter satisfies $\Lambda_2^2 = m_3\Lambda_3$ [7]. Therefore, in this limit the SW curve (4.3.17) becomes the SW curve for $N_f = 2$ theory (4.3.15).

We also note that the theory has singularity at $u \sim m$ and $u = m^2$ when $m_1 = m_2 = m_3 = m$ [7]. This curve is consistent with these singularities [7].

## 4.4 The SW Curves for $SU(N_c)$ $N_f$ SQCDs

Finally, we will write down the general SW curve for $SU(N_c)$ SQCD with $N_f$ flavors. In the previous sections, we can obtain the SW curve of $\mathcal{N} = 2$ $SU(2)$ SQCDs with $N_f = 0, 1, 2, 3$. We can write these SW curves in different forms, which have the same discriminant. In [13], the SW curve for $\mathcal{N} = 2$ $SU(N_c)$ SQCDs with flavor number $N_f \leq 2N_c$ has been presented systematically. Let us explain the results of the SW curve and SW differential in [13].

### 4.4.1 $N_f < N_c$

Let us first discuss the SW curves for $N_f < N_c$ cases. In these cases, the SW curve is given as follows [13]:

$$y^2 = C_{N_c}^2(x) - G(x), \qquad C_{N_c}(x) := x^{N_c} - \sum_{i=2}^{N_c} s_i x^{N_c-i}, \quad G(x) := \Lambda_{N_f}^{2N_c-N_f}\prod_{i=1}^{N_f}(x+m_i). \tag{4.4.1}$$

Here, the $m_i$'s are the mass parameters for the hypermultiplets, and $s_i$ are defined by Weyl invariant parameters made of $a_i$ defined in (2.2.7) as

$$s_k := (-1)^k \sum_{i_1<\cdots<i_k} a_{i_1}\cdots a_{i_k}, \qquad k=2,\cdots,N_c. \tag{4.4.2}$$

Let us check these SW curves have the same discriminants as that of the SW curves explained above. In the above sections, we have calculated them in $N_c=2$ and $N_f=0,1,2,3$ cases. In $N_c=2$ cases, the moduli parameter in the SW curve (4.4.1) is $s_2$ and $m_i(i=1,\cdots,N_f)$. In $N_c=2$ and $N_f=0$ case, the SW curve of (4.4.1) becomes

$$y^2=\left(x^2-\frac{u}{4}\right)^2-\left(\frac{\Lambda_0}{2}\right)^4. \tag{4.4.3}$$

Then, the discriminant of the curve is given as

$$\Delta_0=\frac{\Lambda_0^8}{16}(u-\Lambda_0^2)(u+\Lambda_0^2). \tag{4.4.4}$$

This is the same as the discriminant in (4.3.4).

In $N_c=2$ and $N_f=1$ case, the SW curve in (4.4.1) becomes

$$y^2=\left(x^2-\frac{u}{4}\right)^2-\frac{\Lambda_1^3}{8}(x+2m). \tag{4.4.5}$$

Then, the discriminant of the curve is

$$\Delta_1=-\frac{\Lambda_1^6}{4096}(27\Lambda_1^6+16384\Lambda_1^3m^3-1152\Lambda_1^3mu-4096m^2u^2+256u^3). \tag{4.4.6}$$

This discriminant matches the result of (4.3.14). The SW differential in this notation can be written as follows [13]:

$$\lambda_{SW}=xd\log\frac{C_{N_c}(x)-y}{C_{N_c}(x)+y}. \tag{4.4.7}$$

Then, the SW periods $\prod=(a_I,a_{DI}),(I=1,\cdots,N_c-1)$ are determined by the SW curve and the SW differential, and this is given as

$$a_I=\oint_{\alpha_I}\lambda_{SW}, \qquad a_{DI}=\oint_{\beta_I}\lambda_{SW}. \tag{4.4.8}$$

### 4.4.2 $N_f = N_c$

Let us next explain the $N_f = N_c$ cases. In these cases, the SW curves given in [13] are as follows:

$$y^2 = \mathcal{C}^2(x) - \mathcal{G}(x), \tag{4.4.9}$$

$$\begin{cases} \mathcal{C}(x) := C^2_{N_c}(x) + \frac{\Lambda^{N_c}_{N_f}}{4}, \quad \mathcal{G}(x) := \Lambda^{N_c}_{N_f} \prod_{i=1}^{N_f}(x+m_i), & N_c = N_f > 2, \\ \mathcal{C}(x) := x^2 - u + \frac{\Lambda_2^2}{8}, \quad \mathcal{G}(x) := \Lambda_2^4(x+m_1)(x+m_2) & N_c = N_f = 2. \end{cases} \tag{4.4.10}$$

Here, the $C_{N_c}(x)$'s are defined in (4.4.1). Let us confirm that the discriminant for the SW curve in (4.4.10) in the case of $N_c = N_f = 2$ is matches the discriminant determined in (4.3.16). In this case, the discriminant for (4.4.10) becomes

$$\begin{aligned} \Delta =& \frac{\Lambda_2^{12}}{16} - 3\Lambda_2^{10} m_1 m_2 + \Lambda_2^8 \{-27(m_1^4 + m_2^4) - 6m_1^2 m_2^2 + 36u(m_1^2 + m_2^2) - 8u^2\} \\ & - 32\Lambda_2^6 m_1 m_2 (8m_1^2 m_2^2 - 9u(m_1^2 + m_2^2) + 10u^2) + 256\Lambda_2^4 (m_1^2 - u)(m_2^2 - u)u^2, \end{aligned} \tag{4.4.11}$$

which is proportional for the discriminant determined in (4.3.16). The SW differentials are also given in [13]:

$$\lambda_{SW} = x d\log \frac{\mathcal{C}(x) - y}{\mathcal{C}(x) + y}. \tag{4.4.12}$$

Then, the SW periods $\prod = (a_I, a_{DI}), (I = 1, \cdots, N_c - 1)$ are given as

$$a_I = \oint_{\alpha_I} \lambda_{SW}, \qquad a_{DI} = \oint_{\beta_I} \lambda_{SW}. \tag{4.4.13}$$

Here $(\alpha_I, \beta_I)$ are the canonical pairs of 1-cycles on the SW curve.

### 4.4.3 $N_c < N_f < 2N_c$

Let us finally explain the $N_c < N_f < 2N_c$ cases. In $N_f = N_c + 1$ cases, the SW curves are given as follows:

$$y^2 = \left[ C_{N_c}(x) + \Lambda_{N_f}^{N_c - 1} \left( \frac{x}{4} + \frac{t_1(m)}{4} \right) \right]^2 - \Lambda_{N_f}^{N_c - 1} \prod_{i=1}^{N_f} (x + m_i). \tag{4.4.14}$$

Here, $C_{N_c}$ is defined in (4.4.1), and $t_k(m)$ are

$$t_k(m) := \sum_{i_1<\cdots<i_k} m_{i_1}\cdots m_{i_k}. \tag{4.4.15}$$

In general $N_c < N_f < N_c$ cases, the SW curves are given as

$$y^2 = \mathcal{C}^2(x) - \mathcal{G}(x), \tag{4.4.16}$$

$$\mathcal{C}(x) := C_{N_c}(x) + \Lambda_{N_f}^{2N_c-N_f} P, \quad \mathcal{G}(x) := \Lambda_{N_f}^{2N_c-N_f} \prod_{i=1}^{N_f}(x+m_i), \tag{4.4.17}$$

where $P(x, m_i, \Lambda_{N_f})$ is a polynomial of degree $N_f - N_c$ in $x, m_i$, and it is independent of $s_i$. The SW differentials are given as [13]:

$$\lambda_{SW} = x d\log \frac{\mathcal{C}(x) - y}{\mathcal{C}(x) + y}, \tag{4.4.18}$$

and the SW periods $\prod = (a_I, a_{DI}), (I = 1, \cdots, N_c - 1)$ are given as

$$a_I = \oint_{\alpha_I} \lambda_{SW}, \qquad a_{DI} = \oint_{\beta_I} \lambda_{SW}. \tag{4.4.19}$$

### 4.4.4 Summary (SQCDs for $0 \leq N_f < 2N_c$)

In the previous subsections, we obtain the SW curve in the SQCDs for $2N_c > N_f \geq 0$ cases. The SW curve in the SQCDs for $2N_c > N_f > N_c$ cases and $0 < N_f \leq N_c$ are slightly different. Therefore, let us rewrite these SW curves in a more useful form in the following way:

$$y^2 = C(x)^2 - G(x), \qquad \lambda_{SW} = x d\log \frac{C(x) - y}{C(x) + y}. \tag{4.4.20}$$

Here, we use

$$C(x) = x^{N_c} - \sum_{i=2}^{N_c} s_i x^{N_c - i}, \qquad G(x) = \Lambda_{N_f}^{2N_c-N_f} \prod_{a=1}^{N_f}(x+m_a), \tag{4.4.21}$$

by shifting or scaling the parameters $u_i, m_i, x$.

## 4.5 The SW curves for Other Gauge Theories

Let us finally show the results for the SW theories for other gauge theories. In the above, we showed the SW curves and the SW differentials for the SQCDs. But, there are other $\mathcal{N} = 2$ supersymmetric gauge theories. For example, let us show the SW curves and the SW differentials for $A_r$ type and $D_r$ type gauge theories (SYM). For other gauge groups the SW curves are studied in [59], [18], [16].

### 4.5.1 $A_r$ type Gauge Theories

Let us write the result of the SW curves for the $\mathcal{N} = 2$ SYM with $A_r$ type gauge groups. This curve is given by that of $SU(r+1)$ SQCD with $N_f = 0$. Here we write it in a different way, which is convenient to study the scaling limit to the AD theory in the next chapter. By setting $z = y + \mathcal{C}_{N_c}$ in (4.4.1), the SW curve and the SW differential for the $\mathcal{N} = 2$ SYM with $A_r$-type gauge group can be written as follows [14], [15], [16], [17]:

$$\frac{\Lambda^{r+1}}{2}\left(z + \frac{1}{z}\right) = W(x, u_i), \qquad \lambda_{SW} = x\frac{dz}{z}, \tag{4.5.1}$$
$$W(x, u_i) := x^{r+1} - u_2 x^{r-1} - \cdots - u_{r+1}.$$

Here, the $u_i$'s are the Coulomb moduli parameters, which corresponding to Casimir of Lie algebra. The $\Lambda$ is the QCD scale parameter. Then, the SW periods $\prod = (a_I, a_{DI}), (I = 1, \cdots, r)$ are given as

$$a_I = \oint_{\alpha_I} \lambda_{SW}, \qquad a_{DI} = \oint_{\beta_I} \lambda_{SW}. \tag{4.5.2}$$

### 4.5.2 $D_r$ type gauge theories

Finally, let us write the SW curve and the differential for the $\mathcal{N} = 2$ $SO(2r)$ SYM, which are called $D_r$ type gauge groups. In this case, the SW curve and the SW differential are as follows [16], [18]:

$$\frac{1}{2}\left(z + \frac{G(x)}{z}\right) = C(x), \qquad \lambda_{SW} = xd\mathrm{log}z - \frac{1}{2}xd\mathrm{log}G. \tag{4.5.3}$$

Here, we define

$$C(x) = x^{2r} + s_1 x^{2r-2} + \cdots + s_{r-1}x^2 + s_r, \qquad G(x) = \Lambda^{4r-4}x^4. \tag{4.5.4}$$

Where the $s_i$'s and $s_r^{1/2}$ are the Coulomb moduli parameters, which corresponding to Casimir of Lie algebra. The $\Lambda$ is the scaling parameter. The curve is regarded as the $SU(2r)$ SQCD with $N_f = 4$ massless hypermultiplets. Where $G(x)$ represents the contribution from the massless hypermultiplets. $C(x)$ from the coulomb moduli. Then, the SW periods $\prod = (a_I, a_{DI}), (I = 1, \cdots, r)$ are given as

$$a_I = \oint_{\alpha_I} \lambda_{SW}, \qquad a_{DI} = \oint_{\beta_I} \lambda_{SW}. \tag{4.5.5}$$

Introducing $w$ by $z = \sqrt{G}w$, we can rewrite the SW curve as

$$\frac{\Lambda^{2r-2}}{2}\left(w + \frac{1}{w}\right) = W(x, s_i) \tag{4.5.6}$$

where

$$W(x, s_i) = \frac{C(x)}{x^2}. \tag{4.5.7}$$

Then the SW differential becomes

$$\lambda_{SW} = xd\log w. \tag{4.5.8}$$

Then the SW curves take a similar form as $A_r$. But the $W$ has an inverse squared term. The two curves are equivalent. But if we quantize these curves, there arise differences, which will be discussed in Chapter 6.

The SW curves can be used to study the strong coupling physics of the gauge theories. In the next chapter, we study the superconformal field theories realized as the RG fixed point of the theory.

# Chapter 5

# Argyres-Douglas Theory

The SW theory determines the prepotentials for $\mathcal{N} = 2$ supersymmetric theories from the Riemann surface called the SW curve and the SW differential. The SW curves are characterized by the discriminants and the monodromies around each singular point. In the SQCDs, the singular points generate monodromies, which correspond to squark or monopole massless points. These singular points are separated for generic moduli, but if we choose a specific value of moduli, these points may coincide with each other.

In this case, we expect that there arise new phenomena happen. Argyres and Douglas showed that there arise new non-trivial theory if these two monodromy matrices do not commute each other in the pure $SU(3)$ SYM [19], and expected that the theory has superconformal symmetry. After that, [20] generalize this new phenomena to general $SU(2)$ $N_f$ SQCDs. These non-trivial CFT's are called Argyres-Douglas (AD) theories. By definition, these theories are in a strong coupling region on the moduli space. Therefore, AD theories cannot have a local Lagrangian description.

First, we review [19] and define the Argyres-Douglas theory in $SU(3)$ SYM case. Next, we generalize the AD theory of the $SU(3)$ SYM to AD theories for $SU(2)$ $N_f = 1, 2, 3$ SQCDs in another description [20]. After that, we will consider the AD theories for $SU(N_c)$ $N_f$ SQCDs [23]. Then, we discuss the universality of AD theories, which is an equivalence relation of two different UV gauge theories that have the same AD theory [44], [46], [47].

## 5.1 Mutual Locality and Dirac Quantization Condition

Let us first discuss the relations between two different monodromy matrices when two massless BPS particles co-exist. On the Coulomb branch, there exist singular points on which some massive dyons or monopoles become massless [6], [7]. We define the mutual locality of the two singular points on the Coulomb branch, as the monodromy matrices of these singularity points are commutative.

In SQCD, the singular point is either a squark massless point or monopole (dyon) massless point. As we have seen in (4.2.4), all singular points corresponding to the squark massless points commute with each other. But squark point and monopole (dyon) massless point, or pair of monopole (dyon) massless points may not commute each other [7]. Thus, these singular points are mutually non-local.

Let us show that the mutual non-locality of two monopoles (or dyon) massless points correspond to the non-trivial Dirac quantization condition in $N_c = 2$ cases. By using (3.3.11), the coummtator of the two monodromies around massless dyons with charges $(n_m, n_e)$ and $(n'_e, n'_m)$ are the following:

$$
\begin{aligned}
&\left[\begin{pmatrix} 1+2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1-2n_e n_m \end{pmatrix}, \begin{pmatrix} 1+2n'_e n'_m & 2{n'_e}^2 \\ -2{n'_m}^2 & 1-2n'_e n'_m \end{pmatrix}\right] \\
=&(n'_e n_m - n_e n'_m)\begin{pmatrix} 4(n_e n'_m + n'_e n_m) & 8n_e n'_e \\ -8n_m n'_m & -4(n_e n'_m + n'_e n_m) \end{pmatrix}.
\end{aligned}
\tag{5.1.1}
$$

Thus, the condition that two dyon massless points are mutually local whose charges are $(n_e, n_m)$ and $(n'_e, n'_m)$ is the same as the trivial case of the Dirac quantization condition:

$$
n_e n'_m - n'_e n_m = 0. \tag{5.1.2}
$$

The Dirac quantization condition or equivalently, the mutual locality is a concept which does not change by duality transformation.

Let us interpret the meaning of the mutual locality. If two dyons (hypermultiplets) $(n_e, n_m)$ and $(n'_e, n'_m)$ are mutually local, we can couple these hypermultiplets locally with the same $U(1)$ gauge field if we take some duality transformation. But if $(n_e, n_m)$ and $(n'_e, n'_m)$ are mutually non local, then these hypermultiplets cannot couple with the same gauge field. Therefore, we cannot construct Lagrangian that includes two different dyons that are mutually non-local [19].

## 5.2 Mutually Non-Local Solutions for $SU(3)$ SYM

Let us construct mutually non-local solutions for $SU(3)$ SYM case following [19]. In $SU(3)$ SYM, the SW curve can be written as

$$y^2 = P(x)^2 - \Lambda^6 := \left(x^3 - s_2 x - s_3\right)^2 - \Lambda^6 = Q_+(x)Q_-(x), \qquad Q_\pm(x) := x^3 - s_2 x - s_3 \pm \Lambda^3. \tag{5.2.1}$$

The discriminant of this curve is

$$\Delta = 64\Lambda^{18}\Delta(Q_+)\Delta(Q_-), \qquad \Delta(Q_\pm) = 4s_2^3 - 27(\Lambda^3 \mp s_3)^2. \tag{5.2.2}$$

Here, $\Delta(Q_\pm)$ is the discriminant of $Q_\pm$. The solutions of $\Delta = 0$ are made of two-dimensional space, which consists of two submanifolds $\Delta(Q_\pm) = 0$. They intersect at three points $(s_2, s_3) = (\omega\frac{3}{4^{1/3}}\Lambda^2, 0)$, where $\omega^3 = 1$. These points are called "$\mathbb{Z}_2$ vacua," because each point leaves a $\mathbb{Z}_2 \subset \mathbb{Z}_6$ symmetry, which acts as $s_3 \to -s_3$. Where the $\mathbb{Z}_6$ symmetry is symmetry on the moduli space defined in the first chapter, which acts as

$$s_2 \to e^{2\pi i/3}s_2, \qquad s_3 \to -s_3. \tag{5.2.3}$$

These vacua corresponding to two mutually local dyons are massless [15], [21].

There are other possibilities of singular points, which is the singular point of each curve $Q_\pm(x)$. For example, a point that satisfies

$$\frac{\partial\Delta(Q_+)}{\partial s_2} = \frac{\partial\Delta(Q_+)}{\partial s_3} = 0, \qquad \Delta(Q_+) = 0 \tag{5.2.4}$$

is one of the singular points on the $Q_+$ curve. The solution to this equation is

$$s_2 = 0, \qquad s_3 = \Lambda^3. \tag{5.2.5}$$

We can find such a singular point for $Q_-$ in the same way:

$$s_2 = 0, \qquad s_3 = -\Lambda^3. \tag{5.2.6}$$

We call these singular points as "$\mathbb{Z}_3$ vacua," because they have unbroken symmetry $\mathbb{Z}_3 \subset \mathbb{Z}_6$.

On this curve, there is two independent $Sp(4,\mathbb{Z})$ monodromies. We can verify that these monodromy matrices do not commute, and thus these dyons are mutually non-local. It is also shown in [22] that these three massless points are mutually non-local, and these massless dyons have the following charges:

$$\mathbf{n}^{(1)} = (1,0;\ 1,0), \qquad \mathbf{n}^{(2)} = (0,1;\ -1,1), \qquad \mathbf{n}^{(3)} = (1,1;\ 0,1). \tag{5.2.7}$$

Here, we use the notation $\mathbf{n}^{(i)} = (h_1^{(i)}, h_2^{(i)};\ q_1^{(i)}, q_2^{(i)})$, where $(h_1^{(i)}, q_1^{(i)})$ and $(h_2^{(i)}, q_2^{(i)})$ are two dyon charges of two $U(1)$ gauge groups ($U(1)^2$ is the Casimir of $SU(3)$).

## 5.3 The Scaling Limit

In the previous section, we have constructed mutually non-local dyon solutions. In this section, we will argue some deformation around the $\mathbb{Z}_3$ point on the moduli space, which go to the singular points on which mutually non-local singular points coincide.

Let us consider the moduli space near the $\mathbb{Z}_3$ point following [19]. The SW curve has genus two, then two pairs of conjugate cycles $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ are determined separetely. Near the $\mathbb{Z}_3$ point, we can determine two torus moduli parameters as the moduli parameters for the SW curve:

$$\tau^{ij} = \begin{pmatrix} \tau^{11} & 0 \\ 0 & \tau^{22} \end{pmatrix}. \tag{5.3.1}$$

Here, the torus parameter $\tau^{ij}$ is the complex coupling for the effective Lagrangian. $\tau^{11}$ is the moduli parameter for "small torus" and $\tau^{22}$ for "large" torus near the $\mathbb{Z}_3$ point (Fig.5.1).

Let us consider deformations around the $\mathbb{Z}_3$ point. The operators exist on the $\mathbb{Z}_3$ point is $s_2$ and $s_3$. Thus, if we consider some deformation, we obtain

$$s_2 = \delta s_2, \qquad s_3 = \pm\Lambda^3 + \delta s_3. \tag{5.3.2}$$

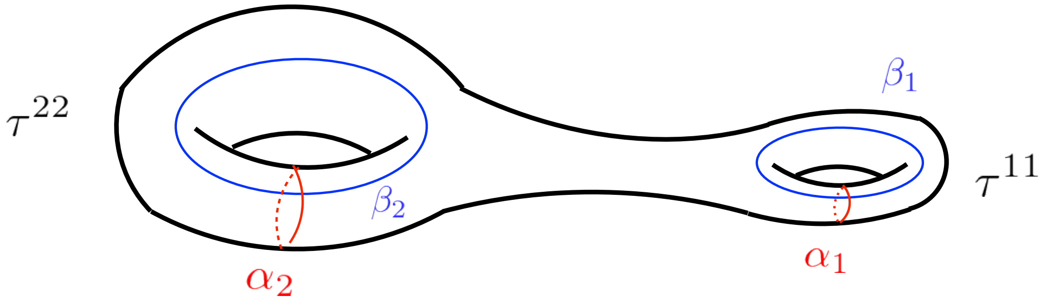


Figure 5.1: The SW curve in the scaling limit. At the $\mathbb{Z}_3$ point, the small torus becomes zero.

Here, we note that $s_2 = \mathrm{Tr}\phi^2$ and $s_3 = \mathrm{Tr}\phi^3$, the dimensionless parameters of the deformation are $\frac{\delta s_2}{\Lambda^2}$ and $\frac{\delta s_3}{\Lambda^3}$. In the following, we consider the $(s_2, s_3) = (0, \Lambda^3)$ point. On the $\mathbb{Z}_3$ point, the SW curve becomes

$$y^2 = x^3(x^3 - \Lambda^3), \tag{5.3.3}$$

and the small torus of the SW curve is represented as $y^2 = x^3$ [19]. Let us introduce parameters for the small torus as $\delta s_2 = 3\epsilon^2\Lambda^2\rho$ and $\delta s_3 = 2\epsilon^3\Lambda^3$. The $\rho$ has dimension zero. If we introduce $x = \epsilon\Lambda\tilde{x}$, the SW curve near $\mathbb{Z}_3$ point is

$$\begin{aligned} y^2 =& (x^3 - \delta s_2 x - (\Lambda^3 + \delta s_3))^2 - \Lambda^6 \\ =& \epsilon^6\Lambda^6\tilde{P}^2 - 2\epsilon^3\Lambda^6\tilde{P}, \qquad \tilde{P}(\tilde{x}) := \tilde{x}^3 - 3\rho\tilde{x} - 2. \end{aligned} \tag{5.3.4}$$

The second term is the SW curve for the small torus near the $\mathbb{Z}_3$ point:

$$\tilde{y}^2 = \tilde{x}^3 - 3\rho\tilde{x} - 2 = \tilde{P}(\tilde{x}). \tag{5.3.5}$$

The limit

$$s_2 = 3\epsilon^2\Lambda^2\rho, \qquad s_3 = \pm\Lambda^3 + 2\epsilon^3\Lambda^3, \qquad x = \epsilon\Lambda\tilde{x}, \qquad y = \epsilon^{3/2}\Lambda^{3/2}\tilde{y} \tag{5.3.6}$$

with $\epsilon \to 0$ is called the scaling limit. The moduli parameter of the small torus is given as

$$\tau^{11} = \tau(\rho) + \mathcal{O}\left(\frac{\delta s_2}{\Lambda^2}\right) + \mathcal{O}\left(\frac{\delta s_3}{\Lambda^3}\right). \tag{5.3.7}$$

Here, $\tau(\rho)$ is the period of the curve (5.3.5).

Let us next consider the SW differential near the $\mathbb{Z}_3$ point. If we substitute the scaling limit in the SW differential [13], we obtain

$$\lambda_{SW} = x\frac{P'(x)}{y}dx \sim \epsilon^{5/2}\Lambda\tilde{x}\frac{\tilde{P}'}{\tilde{P}^{1/2}}d\tilde{x} = 2\epsilon^{5/2}\Lambda\tilde{x}\tilde{y}'d\tilde{x} = -2\tilde{y}d\tilde{x} \tag{5.3.8}$$

on (5.3.5).

## 5.4 The Scaling limit as a Deformation

In the previous section, we define the AD theory for the $SU(3)$ pure YM theory. In this case, we first determine the $\mathbb{Z}_3$ point as the point that mutually non-local singular points coincide. In this consideration, we should calculate the commutation relations of the monodromy matrices or consider the Dirac quantization condition to consider that the monodromies are mutually local or not. We want to generalize this result in general $SU(N_c)$ $N_f$ SQCDs.

For this purpose, let us first note that AD theories are non-trivial CFTs. Especially, in the SW theories, the $U(1)_R$ charge $r$ and conformal dimension $D$ satisfy $2D = r$. Thus, we can obtain the scaling dimension by determining the $U(1)_R$ charges. We can determine the $U(1)_R$ charge by the form of the SW curve.

### 5.4.1 Super Conformal Algebra

To consider deformations around CFTs, we need to discuss the conformal multiplets and their dimensions. Therefore, we will first discuss superconformal symmetry and the representation of the algebra.

Let us first explain about 4-dimensional conformal symmetry. The conformal symmetry is defined as symmetry transformation act on a spacetime with some diffeomorphism which changes the metric up to the scale factor:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x). \tag{5.4.1}$$

These conformal symmetry vectors $\epsilon^\mu(x)$ are the solutions of the Conformal Killing equation defined as follows:

$$\nabla_\mu \epsilon_\nu(x) + \nabla_\nu \epsilon_\mu(x) = \frac{1}{4}(\nabla \cdot \epsilon(x)) g_{\mu\nu}(x). \tag{5.4.2}$$

There are 4 kinds of independent solutions to the equation, which are called dilatation, the Lorentz symmetry, translation, and special conformal symmetry. These generators are given by

- $\epsilon_\mu[d] = x_\mu$: dilatation
- $\epsilon_\mu[m^{\rho\sigma}] = x^\rho \delta^\sigma_\mu - x^\sigma \delta^\rho_\mu$: Lorentz symmetry
- $\epsilon_\mu[p^\nu] = \delta^\nu_\mu$: momentum generator
- $\epsilon_\mu[k^\nu] = 2x_\mu x^\nu - x^2 \delta^\nu_\mu$: special conformal symmetry

The superconformal generators are consist of the generators for the above transformations, a set of supercharges, superconformal charges, and the R-charge. In $\mathcal{N} = 2$ SCFT, there exists the operator-state correspondence, which means that states are labeled by fields that is a component field of some superconformal multiplet. Also, these superconformal multiplets are consist of one primary field and the corresponding descendants that are generated by applying the eight supercharges $Q^\alpha_i$, $\overline{Q}^{\dot\alpha}_i$ and the superconformal charges on the primary state.

Let $(j, \tilde{j})$ be spins of the Lorentz group $SO(4) = SU(2) \times SU(2)$. Let us explain the chiral supermultiplet whose primary satisfies $\tilde{j} = 0$ [20]. In this case, from the superconformal algebra and the unitarity condition, the primary satisfies

$$D = 2R + \frac{1}{2}r \geq 2j + 2R + 1. \tag{5.4.3}$$

Here, $j$ is the $SU(2)$ charge of the $SO(4)$, $R$ is the $SU(2)_R$ charge, $r$ is the $U(1)_R$ charge, and $D$ is the conformal dimension. A similar relation holds for anti-chiral fields with $j = 0$.

Next, we consider $\mathcal{N}=2$ vector multiplet $U$ [20]. For its primary field, which is the lowest component of $U$, $R=0$ and $j=0$ holds. Then from (5.4.3), we obtain

$$D = \frac{1}{2}r \geq 1, \tag{5.4.4}$$

where $r$ is the $U(1)_R$ charge of the scalar component. In the vector multiplet, the field strength $F^+_{\mu\nu} := F_{\mu\nu} + *F_{\mu\nu}$ is included as the second excited level, then the field strength has dimension $D(F^+) \geq 2$. If equality saturates, there exist an extra null state at the fourth level giving $dF^+ = 0$. In this case, the following constraint must be satisfied:

$$\overline{D}_{\dot{\alpha}i}U = 0. \tag{5.4.5}$$

If $r=2$, we obtain

$$D^{\alpha(i}D^{j)}_{\alpha}U = 0. \tag{5.4.6}$$

This is the well-known Maxwell equations, which is the equation of motion for free conformal theories.

### 5.4.2 Deformations around a Non-local Singular Point

In [19], it argued that the theories on the points where mutually non-local dyons become massless are CFTs. In the following, let us consider deformations around a superconformal point [20]. We will consider deformation on the Coulomb branch. Therefore, deformation should be determined by the $\mathcal{N}=2$ vector multiplets ($\mathcal{N}=2$ chiral multiplet) which satisfy (5.4.5).

If we deform around the fixed point by an operator $U$, the leading deformation is the following (we note that the superconformal point is non-trivial CFT, the chiral fields $U$ does not satisfy (5.4.6)):

$$\delta\mathcal{L} = \int d^2\theta_1 d^2\theta_2 mU, \qquad D(m) + D(u) = 2. \tag{5.4.7}$$

Here, we have used $\mathcal{N}=2$ superspace coordinates $\theta_{I,\alpha}$, $\overline{\theta}_{I,\dot{\alpha}}$, where $\alpha, \dot{\alpha}$ are spinor indices and $I=1,2$ $SU(2)_R$ indices. Here, for the deformation with $D(u) > 2$ ($D(u) < 2$), this deformation is called the irrelevant (relevant). When $D(u) = 2$, this deformation is called marginal.

## 5.5 Scaling Limits for $\mathcal{N}=2$ $SU(2)$ $N_f=1,2,3$ SQCDs

In section 5.2 and 5.3, we only have considered the scaling limit around the superconformal points for the $SU(3)$ SYM case. In this section, we will discuss a non-trivial fixed point in $\mathcal{N}=2$ $SU(2)$ $N_f$ SQCDs. Note that in pure $SU(2)$ SYM case, there is no fixed point with mutually non-local states [6]. Thus, in the following we consider $N_f=1,2,3$ cases following [20], [68]. In these cases, at the superconformal points, the three branch points of the SW curve coincide, and two intersecting cycles on the SW curve shrink to zero. Near the point, the SW curve becomes

$$y^2=\tilde{x}^3-f(u,m)\tilde{x}-g(u,m), \qquad f(0,0)=g(0,0)=0. \tag{5.5.1}$$

### 5.5.1 $N_f=1$

Let us first consider the superconformal point in the $N_f=1$ case. The SW curve for the massive $N_f=1$ SQCD is as follows:

$$y^2=x^2(x-u)+\frac{1}{4}M\Lambda_1^3x-\frac{1}{64}\Lambda_1^6. \tag{5.5.2}$$

Here, $M$ is the bare mass of the hypermultiplet, and $\Lambda_1$ is the scale parameter. We can take the scale parameter arbitrarily by rescaling the coordinates $x$ and $y$. We can take $\Lambda_1=2$. Then the curve becomes

$$y^2=x^2(x-u)+2Mx-1. \tag{5.5.3}$$

The discriminant of the curve is

$$\Delta_1=-4u^3+4u^2M^2+36uM-32M^3-27. \tag{5.5.4}$$

Let us first look for the superconformal point. On the superconformal point, the $\alpha$-cycle and $\beta$-cycle for one genus of the SW curve shrink to a point. Thus, on the superconformal point, the discriminant has more than two zeroes. We can check that the discriminant has double zero if we take $M=\frac{3}{2}\omega$ and $u=3\omega^{-1}$:

$$\Delta_1=-\frac{108}{\omega^3}(1-\omega)^2(1+\omega+\omega^2)^2. \tag{5.5.5}$$

On this point, the SW curve becomes

$$y^2 = (x - \omega^{-1})^3, \qquad \omega^3 = 1. \tag{5.5.6}$$

Let us consider the scaling limit at a superconformal point realized at $\omega = 1$. We should determine the scaling limit around the superconformal point, that the $\alpha$-cycle and $\beta$-cycle become zero at the same time. If we assume that the SW curve in the scaling limit is the form (5.5.1), we should take the scaling limit as

$$M = \frac{3}{2} + m, \qquad u = 3 + 2m + \tilde{u}, \qquad x = \frac{1}{3}u + \tilde{x}. \tag{5.5.7}$$

Substitute these shift of the parameters to (5.5.3), and taking the limit $m, \tilde{u} \to 0$ and ignore the higher terms, we obtain

$$y^2 = \tilde{x}^3 - 2(m + \tilde{u})\tilde{x} - (\tilde{u} + \frac{4}{3}m^2). \tag{5.5.8}$$

If we also require that the scaling dimensions as $D(\tilde{x}) : D(m) : D(u) = 1 : 2 : 3$, we can neglect $\tilde{u}\tilde{x}$ and $m^2$, and thus this curve becomes

$$y^2 = \tilde{x}^3 - 2m\tilde{x} - \tilde{u}. \tag{5.5.9}$$

This curve is the same form as (5.5.1).

### 5.5.2 $N_f = 2$

Next we consider $N_f = 2$ case. In this case, the SW curve is given as

$$y^2 = \left(x^2 - \frac{1}{64}\Lambda_2^4\right)(x - u) + \frac{1}{4}M_1 M_2 \Lambda_2^2 x - \frac{1}{64}(M_1^2 + M_2^2)\Lambda_2^4. \tag{5.5.10}$$

At the superconformal point, the theory has conformal symmetry, thus we can take $\Lambda_2^2 = 8$ to consider the scaling limit. Then the SW curve becomes

$$y^2 = (x^2 - 1)(x - u) + 2M_1 M_2 x - (M_1^2 + M_2^2). \tag{5.5.11}$$

The singular points of this curve are defined by the discriminant of the curve $\Delta_2$. To study the superconformal point, let us define

$$M := \frac{1}{2}\sum_i M_i, \qquad C_2 := \sum_i (M_i - M)^2. \tag{5.5.12}$$

If we take $M = \sqrt{2}$ and $C_2 = 0$, the SW curve becomes $y^2 = x^3$. Thus the superconformal point is defined at this point. If we consider a limit such that

$$M = \sqrt{2} + m, \qquad u = 3 + 2\sqrt{2}m - \frac{1}{3}m^2 + \tilde{u}, \qquad x = \frac{1}{3}\tilde{u} + \tilde{x}, \tag{5.5.13}$$

with the scaling dimensions are assumed to $D(m) : D(\tilde{u}) : D(C_2) = 1 : 2 : 3$, the SW curve becomes

$$y^2 = \tilde{x}^3 - 2\tilde{u}\tilde{x} - \frac{4\sqrt{2}}{3}m\tilde{u} + \frac{16\sqrt{2}}{27}m^3 - 2C_2. \tag{5.5.14}$$

This curve is the same form as (5.5.1). This curve is the SW curve in the scaling limit.

### 5.5.3 $N_f = 3$

Let us consider the scaling limit in the $N_f = 3$ theory. As in the same reason in $N_f = 1, 2$ theories, we can take the value of $\Lambda_3$ arbitrary, when we consider the scaling limit. If we take $\Lambda_3 = 8$ in SW curve (4.3.17), we obtain

$$\begin{aligned} y^2 =& x^2(x-u) - (x-u)^2 - (M_1^2 + M_2^2 + M_3^2)(x-u) + 2M_1M_2M_3x \\ &- (M_1^2M_2^2 + M_2^2M_3^2 + M_3^2M_1^2). \end{aligned} \tag{5.5.15}$$

Let us introduce the following parameters:

$$M := \frac{1}{3}\sum_i M_i, \qquad C_2 := \sum_i (M_i - M)^2, \qquad C_3 := \sum_i (M_i - M)^3. \tag{5.5.16}$$

The superconformal point is realized at $M_1 = M_2 = M_3 = 1$. The scaling limit around the superconformal point $M = 1$, $C_2 = C_3 = 0$ is defined as

$$M = 1 + m, \qquad u = 2 + 3m + m^2 - \frac{5}{6}m^3 + \tilde{u}, \qquad x = \frac{1}{3} + \frac{1}{3}\tilde{u} + \tilde{x}, \tag{5.5.17}$$

where the scaling dimensions of these deformation operators are $D(m) : D(u) : D(C_2) : D(C_3) = 1 : 3 : 4 : 6$. Thus the SW curve in the scaling limit is given as

$$y^2 = \tilde{x}^3 - 2(m\tilde{u} + C_2)\tilde{x} - \tilde{u}^2 - \frac{m^3\tilde{u}}{3} + \frac{m^6}{108} - \frac{2m^2C_2}{3} + \frac{8C_3}{3}. \tag{5.5.18}$$

This curve is the same form as (5.5.1).

### 5.5.4 Interpretation of the Scaling Limit as Deformations

Let us discuss the scaling limit as a deformation around the superconformal point. The scaling limit is a deformation around the superconformal point, which direction is on the Coulomb branch, this deformation must be realized by the $\mathcal{N}=2$ vector multiplets. Note that the scaling dimension and the $R$-charge of the primary field of the $\mathcal{N}=2$ vector multiplet are related by (5.4.4). We also note that a conformal field theory is characterized by the conformal quantum numbers. Therefore, if we interpret the scaling limit as a deformation around the superconformal point, we can characterize the scaling limit by the scaling dimensions of the deformation operators.

Let us summarize the dimensions of the operators around the superconformal point. To determine these dimensions, we should require that the dimension of the BPS mass must be one. This requirement is equivalent to the condition that the dimension of the $a \sim (\tilde{u}/\tilde{y})d\tilde{x}$ must be one. For the AD theories associated with $SU(2)$ SQCDs, the scaling dimensions are summarized in table 5.1. In these theories, the scaling limits are

| $N_f$ | $\tilde{u}$ | $m$ | $C_j$ |
|---|---|---|---|
| 1 | 6/5 | 4/5 | - |
| 2 | 4/3 | 2/3 | 1 |
| 3 | 3/2 | 1/2 | $D(C_2)=2,\ D(C_3)=3$ |

Table 5.1: scaling dimensions of AD theories associated to $SU(2)$ SQCDs

interpreted as relevant deformation.

## 5.6 Scaling Limits for $SU(N_c)$ $N_f$ SQCDs

In section 5.5, we have studied the scaling limits for $N_c=2$, $N_f=1,2,3$ theories. The idea of [20] to study conformal field theories can be generalized to $SU(N_c)$ $N_f$ SQCDs. In the following, let us discuss the scaling limits for $SU(N_c)$ $N_f$ SQCDs. This generalization is discussed by [23]. In these papers, the SW curve in the scaling limit is determined by the degeneration of the SW curve of the UV theory. In the following, we will discuss the scaling limit for the $SU(N_c)$ $N_f$ SQCDs following [23].

### 5.6.1 $N_f = 0$

Let us first explain the case of $N_f = 0$, which corresponds to the $SU(N_c)$ SYM. In this case, the SW curve and the SW differential are given as (4.4.20):

$$y^2 = C(x)^2 - \Lambda^{2N_c}, \qquad \lambda_{SW} = xd\log\frac{\{C(x)-y\}^2}{\Lambda^{2N_c}}, \tag{5.6.1}$$

where $C(x) := x^{N_c} - s_2 x^{N_c-2} - \cdots - s_{N_c}$. In this case, the superconformal point is given by $s_2 = \cdots = s_{N_c-1} = 0$ and $s_{N_c} = \pm\Lambda^{2N_c}$, and the scaling limit is defined as follows:

$$x = \epsilon\Lambda\tilde{x}, \qquad s_i = \pm\delta_{i,N_c}\Lambda^{N_c} + \epsilon^i\Lambda^i\tilde{s}_i \quad (i = 2,\cdots,N_c). \tag{5.6.2}$$

Then, the $C(x)$ becomes as $C(x) \to \epsilon^{N_c}\Lambda^{N_c}(\tilde{x}^{N_c} - \tilde{s}_2\tilde{x}^{N_c-2} - \cdots - \tilde{s}_{N_c}) \mp \Lambda^{N_c}$ in the scaling limit, and therefore the SW curve and the SW differential becomes

$$\begin{aligned} y^2 =& C(x)^2 - \Lambda^{2N_c} \\ &\to \mp 2\epsilon^{N_c}\Lambda^{2N_c}(\tilde{x}^{N_c} - \tilde{s}_2\tilde{x}^{N_c-2} - \cdots - \tilde{s}_{N_c}) + \mathcal{O}(\epsilon^{2N_c}) \\ &=: \mp 2\epsilon^{N_c}\Lambda^{2N_c}\tilde{y}^2 + \mathcal{O}(\epsilon^{2N_c}), \end{aligned} \tag{5.6.3}$$

$$\begin{aligned} \lambda_{SW} =& xd\log\frac{\{C(x)-y\}^2}{\Lambda^{2N_c}} \\ &\to \pm\,\epsilon^{\frac{N_c+3}{2}}\Lambda 2\sqrt{2}\tilde{y}d\tilde{x} + \cdots . \end{aligned} \tag{5.6.4}$$

Thus the SW curve and SW differential in the scaling limit are the follows:

$$\tilde{y}^2 = \tilde{x}^{N_c} - \sum_{i=2}^{N_c}\tilde{s}_i\tilde{x}^{N_c-i}, \qquad \tilde{\lambda}_{SW} = \tilde{y}d\tilde{x}. \tag{5.6.5}$$

The coefficients of the SW curve are the operators that are used to deform the theory. Note that the scaling dimension of the BPS spectrum must be one, we obtain $[\tilde{\lambda}_{SW}] = 1$. Then we find that the dimension of $x$ is given by $\frac{1}{N_c+2}$. Therefore, we obtain the scaling dimension of these coefficients of the SW curve in the scaling limit as follows:

$$[\tilde{s}_i] = 2i/(N_c+2), \qquad i = 2,\cdots,N_c. \tag{5.6.6}$$

For $N_c$ odd, the scaling dimensions take fractional value. But for even $N_c$, $[\tilde{s}_i] = 1$ for $i = \frac{N_c+2}{2}$, which indicates the theory has $U(1)$ global symmetry.

### 5.6.2 $N_f = 1$

Let us next discuss the $N_f = 1$ theories. The SW curve and the SW differential for these theories are given as (4.4.20) with $N_f = 1$. We can shift the $x$ by some constant to have the superconformal point at $x = 0$:

$$\begin{aligned} y^2 =& (x^{N_c} - s_1 x^{N_c-1} - \cdots - s_{N_c})^2 - \Lambda_1^{2N_c-1}(x + m') \\ =:& x^{2N_c} + t_1 x^{2N_c-1} + \cdots + t_{2N_c}. \end{aligned} \tag{5.6.7}$$

Here, we redefine the parameters $m$ and $s_i (i = 1, \cdots, N_c)$. These $t_1, \cdots, t_{2N_c}$ are the functions of $m', \Lambda, s_1, \cdots, s_{N_c}$:

$$\begin{aligned} t_{2N_c} =& s_{N_c}^2 - \Lambda^{2N_c-1} m', \\ t_{2N_c-1} =& 2 s_{N_c-1} s_{N_c} - \Lambda^{2N_c-1}, \\ t_{2N_c-2} =& s_{N_c-1}^2 + 2 s_{N_c-2} s_{N_c}, \\ t_{2N_c-k} =& \sum_{i+j=k} s_{N_c-i} s_{N_c-j}, \qquad k \geq 3. \end{aligned} \tag{5.6.8}$$

Let us now look for the superconformal point. The superconformal point is defined as $t_{2N_c} = \cdots = t_{N_c} = 0$. Let us write the superconformal point by parameters $s_1, \cdots, s_{N_c}$ and $m'$ by using (5.6.8). By introducing $m^*$ and $s_k^*$:

$$m' = m^* \Lambda, \qquad s_{N_c-k} = s_{N_c-k}^* \Lambda^{N_c-k}, \tag{5.6.9}$$

the superconformal point is given as follows (Note that the equations $t_{2N_c-k} = 0 (k = 0, \cdots, N_c$ are $N_c + 1$ equations):

$$s_{N_c}^* = \begin{cases} -\left( \frac{(\frac{1}{4})_n (\frac{3}{4})_n}{2 \times n! (\frac{3}{2})_n} \right)^{\frac{1}{4n+1}}, & N_c = 2n+1, \\ \left( \frac{(\frac{3}{4})_n (\frac{5}{4})_N}{8(n+1)! (\frac{3}{2})_n} \right)^{\frac{1}{4n+3}}, & N_c = 2n+2 \end{cases} \tag{5.6.10}$$

$$s_{N_c-k}^* = -\frac{(-1)^k (-\frac{1}{2})_k}{k!} \frac{1}{(s_{N_c}^*)^{2k-1}}, \qquad k \geq 1 \tag{5.6.11}$$

$$m^* = (s_{N_c}^*)^2. \tag{5.6.12}$$

Here, we use $(a)_n := a(a+1) \cdots (a+n-1)$, $(n \geq 1)$ and $(a)_0 := 1$. At the superconformal point, the SW curve becomes

$$y^2 = x^{N_c+1} \left( x^{N_c-1} + \cdots + \frac{2N_c - 1}{N_c + 1} \frac{\Lambda_1^{N_c-1}}{s_{N_c}^*} \right). \tag{5.6.13}$$

Let us check that this solutions satisfy the equations $t_{2N_c-k} = 0(k = 0, \cdots, N_c)$ in $N_c = 3$ case. In this case, the SW curve (5.6.7) becomes

$$y^2 = (x^3 - s_1x^2 - s_2x - s_3)^2 - \Lambda^5(x + m') = x^6 + t_1x^5 + \cdots + t_6. \tag{5.6.14}$$

Then, the $t_k = 0(k = 3, \cdots, 6)$ means that

$$\begin{aligned} t_6 =& s_3^2 - \Lambda^5 m' = \Lambda^6((s_3^*)^2 - m^*) = 0, & (5.6.15)\\ t_5 =& 2s_2s_3 - \Lambda^5 = \Lambda^5(2s_2^*s_3^* - 1) = 0, & (5.6.16)\\ t_4 =& s_2^2 + 2s_1s_3 = \Lambda^4((s_2^*)^2 + 2s_1^*s_3^*) = 0, & (5.6.17)\\ t_3 =& -s_3 + 2s_1s_2 = \Lambda^3(-2s_3^* + 2s_1^*s_2^*) = 0. & (5.6.18) \end{aligned}$$

Here we define $m' = m^*\Lambda$ and $s_{N_c-k} = s^*_{N_c-k}\Lambda^{N_c-k}$. We can solve the first 3 equations for $m^*$ and $s_1^*, s_2^*$ as

$$m^* = (s_3^*)^2, \qquad s_2^* = \frac{1}{2s_3^*}, \qquad s_1^* = -\frac{(s_2^*)^2}{2s_3^*} = -\frac{1}{8(s_3^*)^3}, \tag{5.6.19}$$

and finally if we substitute these solutions into the final equation, we obtain $(s_3^*)^5 = -\frac{1}{16}$.

We define the scaling limit as follows with $\epsilon \to 0$:

$$\begin{aligned} x =& \epsilon\Lambda_1\tilde{x}, \\ s_1 =& \Lambda_1 s_1^*, \\ s_i =& \Lambda_1^i \left( s_i^* + \sum_{k=1}^{i} \epsilon^k \tilde{s}_i^{(k)} \right), \qquad i = 2, \cdots, N_c - 1 \\ m =& \Lambda_1 \left( m^* + \sum_{a=1}^{N_c} \epsilon^a \tilde{m}_a \right), \\ s_{N_c} =& \Lambda_1^{N_c} \left( s_{N_c}^* + \sum_{k=1}^{N_c+1} \epsilon^i \tilde{s}_k \right). \end{aligned} \tag{5.6.20}$$

Explicit calculations for $N_c = 2, 3, \cdots$ suggest that the SW curve around the superconformal point takes the form:

$$y^2 = \Lambda^{2N_c}\epsilon^{N_c+1}(\tilde{u}_0\tilde{x}^{N_c+1} + \tilde{u}_2\tilde{x}^{N_c-1} + \cdots + \tilde{u}_{N_c+1}) + \mathcal{O}(\epsilon^{N_c+2}), \tag{5.6.21}$$

where $\tilde{u}_0 = t^*_{N_c-1}/\Lambda^{N_c-1} = \frac{2N_c-1}{N_c+1}\frac{1}{s^*_{N_c}}$. If we further rescaling $\tilde{u}_2, \cdots, \tilde{u}_{N_c+1}$, the SW curve around the superconformal point becomes the following form:

$$\tilde{y}^2 = \tilde{x}^{N_c+1} + \tilde{u}_2\tilde{x}^{N_c-1} + \cdots + \tilde{u}_{N_c+1}. \tag{5.6.22}$$

Then, the SW differential can be written as

$$\begin{aligned}
\lambda_{SW} =& x d\log\frac{C-y}{C+y} = x d\log\left(\frac{\sqrt{G+y^2}-y}{\sqrt{G+y^2}+y}\right) \\
=& x d\log\left(\frac{\sqrt{1+f^2}-f}{\sqrt{1+f^2}+f}\right), \qquad f := \frac{y}{\sqrt{G}} \\
=& x d\left(-2f + \frac{f^3}{3} + \cdots\right) \\
=& \epsilon^{\frac{N_c+3}{2}}\frac{2}{s^*_{N_c}}\Lambda\tilde{u}_0^{1/2}\tilde{y}d\tilde{x}.
\end{aligned} \tag{5.6.23}$$

Here, we have used

$$f = \frac{y}{\sqrt{G}} = \frac{\Lambda^{N_c}\epsilon^{\frac{N_c+1}{2}}\tilde{y} + \cdots}{\sqrt{\Lambda^{2N_c}m^* + \cdots}}, \qquad m^* = (s^*_{N_c})^2. \tag{5.6.24}$$

Therefore, we can obtain the SW curve and the SW differential as following:

$$\tilde{y}^2 = \tilde{x}^{N_c+1} + \tilde{u}_2\tilde{x}^{N_c-1} + \cdots + \tilde{u}_{N_c+1}, \qquad \tilde{\lambda}_{SW} = \tilde{y}d\tilde{x}. \tag{5.6.25}$$

Finally, let us calculate the scaling dimension of the operators in the scaling limit. Note that the BPS spectrum must have the scaling dimension one, we require that the dimension of the $\tilde{\lambda}_{SW}$ is one. Thus we obtain

$$[\tilde{u}_i] = \frac{2i}{N_c+3}, \qquad i = 2, \cdots, N_c+1, \qquad [\tilde{x}] = \frac{2}{N_c+3}. \tag{5.6.26}$$

We can see the spectrum is the same as the AD theory obtained from the scaling limit of $SU(N_c+1)$. This is an example of the universality of AD theories, which will be explained in section 5.9.

### 5.6.3 $N_f = 2n+1$ $(n \geq 1)$

Let us next consider the $N_f = 2n+1 (n \geq 1)$ case. In this case, the SW curve can be written as the following form when we shift $x$ some constant to obtain the scaling point at $x=0$:

$$\begin{aligned} y^2 =&(x^{N_c} - s_1 x^{N_c-1} - \cdots - s_{N_c})^2 - \Lambda_{N_f}^{2N_c-M_f} \prod_{a=1}^{N_f} (x+m_a) \\ =&C(x)^2 - G(x) \end{aligned} \tag{5.6.27}$$

To define the superconformal point, we first set $m_a = m$ and define $\hat{x} = x + m$. Then the SW curve becomes the following form:

$$y^2 = (\hat{x}^{N_c} + u_1 \hat{x}^{N_c-1} + \cdots + u_{N_c})^2 - \Lambda_{N_f}^{2N_c-N_f} \hat{x}^{N_f}. \tag{5.6.28}$$

Then, if we take

$$u_{N_c} = u_{N_c-1} = \cdots = u_{N_c-n-2} = 0, \tag{5.6.29}$$

the SW curve degenerates to

$$y^2 = \hat{x}^{2n+1} \left\{ -\Lambda_{N_f}^{2N_c-2n-1} + \hat{x}(\hat{x}^{N_c-1-1} + u_i \hat{x}^{N_c-n-2} + \cdots + u_{N_c-n-1})^2 \right\}. \tag{5.6.30}$$

Thus, the point is the superconformal point.

Let us next consider the scaling limit around the superconformal point. Let us consider a limit that is determined by

$$\begin{aligned} x =& - m - \epsilon \Lambda_{N_f} \tilde{m} + \epsilon^2 \Lambda_{N_f} \tilde{x}, \\ m_a =& m + \epsilon \Lambda_{N_f} \tilde{m} + \epsilon^2 \lambda_{N_f} \tilde{c}_a, \qquad \sum_{a=1}^{N_f} \tilde{c}_a = 0. \end{aligned} \tag{5.6.31}$$

Here, we take $\epsilon \to 0$. The SW curve in this limit is

$$\begin{aligned} y^2 =& \left\{ (-\epsilon \Lambda_{N_f} \tilde{m} + \epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c} + u_1 (-\epsilon \Lambda_{N_f} \tilde{m} + \epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c-1} + \cdots + u_{N_c} \right\}^2 - \Lambda_{N_f}^{2N_c} \epsilon^{2N_c} \prod_{a=1}^{N_f} (\tilde{x} + \tilde{c}_a) \\ =& \left\{ (\epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c} + \hat{u}_1 (\epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c-1} + \cdots + \hat{u}_{N_c} \right\}^2 - \Lambda_{N_f}^{2N_c} \epsilon^{2N_c} \prod_{a=1}^{N_f} (\tilde{x} + \tilde{c}_a) \end{aligned} \tag{5.6.32}$$

Here, we introduce

$$\begin{aligned} &(-\epsilon \Lambda_{N_f} \tilde{m} + \epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c} + u_1 (-\epsilon \Lambda_{N_f} \tilde{m} + \epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c-1} + \cdots + u_{N_c} \\ =:&(\epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c} + \hat{u}_1 (\epsilon^2 \Lambda_{N_f} \tilde{x})^{N_c-1} + \cdots + \hat{u}_{N_c}. \end{aligned} \tag{5.6.33}$$

Note the coefficients $\hat{u}_i$ do not depend on $\epsilon$. Therefore, we should introduce further scaling parameter to determine the scaling limit:

$$\hat{u}_{N_c-i} = \epsilon^{N_f-2i}\Lambda_{N_f}^{N_c-i}\tilde{u}_{n-i}, \qquad i = 0, 1, \cdots, n. \tag{5.6.34}$$

Here, $\hat{u}_i = O(1)$ $(i = 1, \cdots, N_c - n + 1)$. In these limits (5.6.31) and (5.6.34, the $C(x)$ and $G(x)$ in the SW curve becomes

$$C(x) = \epsilon^{N_f}\Lambda^{N_c}\tilde{C}(\tilde{x}) + \cdots, \qquad G(x) = \epsilon^{2N_f}\Lambda^{2N_c}\tilde{G}(\tilde{x}) + \cdots, \tag{5.6.35}$$

where

$$\tilde{C}(\tilde{x}) = \sum_{i=0}^{n}\tilde{u}_i\tilde{x}^{n-i}, \qquad \tilde{G}(\tilde{x}) = \prod_{a=1}^{2n+1}(\tilde{x}+\tilde{c}_a) = \sum_{a=0}^{N_f}\tilde{v}_a\tilde{x}^{N_f-a}, \qquad \sum_{a=1}^{2n+1}\tilde{c}_a = 0. \tag{5.6.36}$$

Therefore, we obtain

$$y^2 = \epsilon^{2N_f}\Lambda^{2N_c}(\tilde{C}(\tilde{x})^2 - \tilde{G}(\tilde{x})). \tag{5.6.37}$$

Thus, the SW curve in the scaling limit is the following:

$$\tilde{y}^2 = \tilde{C}(\tilde{x})^2 - \tilde{G}(\tilde{x}). \tag{5.6.38}$$

Next, we calculate the SW differential in the scaling limit. The SW differential in the scaling limit is

$$\lambda_{SW} = x d\log\frac{C-y}{C+y} = \epsilon^2\Lambda\tilde{x}d\log\frac{\tilde{C}-\tilde{y}}{\tilde{C}+\tilde{y}} + d(*). \tag{5.6.39}$$

Thus, the SW differential in the scaling limit is defined as follows:

$$\tilde{\lambda}_{SW} = \tilde{x}d\log\left(\frac{\tilde{C}-\tilde{y}}{\tilde{C}+\tilde{y}}\right). \tag{5.6.40}$$

Finally, let us calculate the scaling dimension of the parameters in the scaling limit. If we define the scaling dimension of the $\tilde{\lambda}_{SW}$ is one, we obtain

$$[\tilde{x}] = 1, \qquad [\tilde{u}_i] = \frac{2i+1}{2}, \qquad [\tilde{c}_a] = 1, \qquad a = 1, \cdots, N_f = 2n+1. \tag{5.6.41}$$

Here, $\tilde{u}_0$ and $\tilde{u}_1$ are relevant operator and $\tilde{u}_i$ for $i \geq 2$ are irrelevant operators. We can see that the theory has $SU(2n+1)$ global symmetry with $U(1)$ vector multiplet. This theory is referred as $D_2(SU(2n+1))$ theory in [60]. Its Schur index has been recently studied in [61].

### 5.6.4 $N_f = 2$

In $N_f = 2$ theories, the SW curve is given as follows:

$$y^2 = C(x)^2 - G(x), \tag{5.6.42}$$

where,

$$C(x) = x^{N_c} - \sum_{i=2}^{N_c} s_i x^{N_c - i}, \qquad G(x) = \Lambda_2^{2N_c-2}(x+m_1)(x+m_2). \tag{5.6.43}$$

In this case, the superconformal points are given by

$$s_i = \pm\delta_{i,N_c-1}\Lambda_2^{N_c-1}, \qquad m_1 = m_2 = 0. \tag{5.6.44}$$

On these points, the SW curve becomes

$$y^2 = x^{N_c+1}(x^{N_c-1} \mp 2\Lambda_2^{N_c-1}). \tag{5.6.45}$$

We define the scaling limit as following with $\epsilon \to 0$:

$$\begin{aligned}
x =&\epsilon^2\Lambda_2\tilde{x}, \\
s_i =&\epsilon^{2i}\Lambda_2^i\tilde{s}_i, \qquad (i = 2, \cdots, N_c - 2) \\
s_{N_c-1} =& \pm\Lambda_2^{N_c-1} + \epsilon^{2N_c-2}\Lambda_2^{N_c-1}\tilde{s}_{N_c-1}, \\
s_{N_c} =&\epsilon^2\tilde{m}\Lambda_2^{N_c} + \epsilon^{2N_c}\Lambda_2^{N_c}\tilde{s}_{N_c}, \\
m_a =&\epsilon^2\tilde{m}\Lambda_2 + \epsilon^{N_c+1}\Lambda_2\tilde{c}_a, \qquad a = 1, 2,
\end{aligned} \tag{5.6.46}$$

where $\tilde{c}_1 + \tilde{c}_2 = 0$. In this limit, $C$ and $G$ becomes

$$C(x) = -\epsilon^2\Lambda^{N_c}(\tilde{x}+\tilde{m}) + \epsilon^{2N_c}\Lambda^{N_c}\tilde{C}(\tilde{x}), \qquad G(x) = \Lambda^{2N_c}\left(\epsilon^4(\tilde{x}+\tilde{m})^2 + \epsilon^{2N_c+2}\tilde{c}_1\tilde{c}_2\right), \tag{5.6.47}$$

where,

$$\tilde{C}(\tilde{x}) = \tilde{x}^{N_c} - \sum_{i=2}^{N_c}\tilde{s}_i\tilde{x}^{N_c-i}. \tag{5.6.48}$$

Therefore, the SW curve becomes

$$y^2 = \mp 2\epsilon^{2N_c+2}\Lambda_2^{2N_c}\tilde{y}^2 + \cdots, \qquad \tilde{y}^2 = (\tilde{x}+\tilde{m})\tilde{C}(\tilde{x}) + \frac{1}{2}\tilde{c}_1\tilde{c}_2. \tag{5.6.49}$$

Next, we consider the SW differential in the scaling limit. If we substitute the scaling limit in the SW differential, we obtain

$$\begin{aligned}\lambda_{SW} =& x d\log\frac{C-y}{C+y} \\ =& \mp 2\sqrt{-2}\epsilon^{N_c}\Lambda\frac{\tilde{y}}{\tilde{x}+\tilde{m}}d\tilde{x}+\cdots \\ =& \mp 2\sqrt{-2}\epsilon^{N_c}\Lambda\tilde{y}d\log(\tilde{x}+\tilde{m}).\end{aligned} \tag{5.6.50}$$

Therefore, the SW differential in the scaling limit is the following:

$$\tilde{\lambda}_{SW} = \tilde{y}d\log(\tilde{x}+\tilde{m}). \tag{5.6.51}$$

Let us summarize the results. If we shift $\tilde{x} \to \tilde{x}-\tilde{m}$, we can obtain the SW curve and the SW differential in the scaling limit as follows:

$$\tilde{y}^2 = \tilde{x}^{N_c+1} + \tilde{u}_1\tilde{x}^{N_c} + \cdots + \tilde{u}_{N_c+1}, \qquad \tilde{\lambda}_{SW} = \tilde{y}d\log\tilde{x}. \tag{5.6.52}$$

The scaling dimension operators in the scaling limit are the following:

$$[\tilde{x}] = \frac{2}{N_c+1}, \qquad [\tilde{u}_i] = \frac{2i}{N_c+1}, \qquad i = 1, \cdots, N_c+1. \tag{5.6.53}$$

For $N_c$ odd, the theory has $U(1)$ global symmetry.

### 5.6.5 $N_f = 2n$ ($n \geq 2$)

Let us finally consider the $N_f = 2n$ case with $n \geq 2$. In this case, if we shift $x \to x - s_1/N_c$ in the SW curve (4.4.20), we can rewrite the SW curve as

$$y^2 = (x^{N_c} - s_1x^{N_c-1} - \cdots - s_{N_c})^2 - \Lambda_{N_f}^{2N_c-2n}\prod_{a=1}^{2n}(x+\hat{m}_a). \tag{5.6.54}$$

Here, $\hat{m}_a = m_a - \frac{s_1}{N_c}$, and we can choose the constant $s_1$ arbitrary. Let us choose $s_1$ to satisfy $\sum_{a=1}^{2n}\hat{m}_a = 0$. In other words, we define $s_1 := \frac{N_c}{2n}\sum_{a=1}^{2n} m_a$. Then, the superconformal point is given by

$$s_1 = s_2 = \cdots = s_{N_c-n-1} = \cdots = s_{N_c} = 0, \qquad s_{N_c-n} = \pm\Lambda^{N_c-n}, \qquad \hat{m}_a = 0. \tag{5.6.55}$$

At this point, the SW curve becomes

$$y^2 = x^{N_c+n}(x^{N_c-n} \mp 2\Lambda^{N_c-n}). \tag{5.6.56}$$

However, in this case, there are two possibilities of the scaling limit [24]. Let us discuss these two cases called type A and type B. Originally, in [23], the type B scaling was introduced. But it was shown in [24], there is another possibility of the scaling limit which is consistent with the $a$-theorem in 4-dimensional conformal field theories. We refer to this limit as type A.

## Type $B$ scaling

The type B scaling is defined as follows:

$$x = \epsilon_B^2 \Lambda \tilde{x}, \qquad \hat{m}_a = \epsilon_B^{N_c+2-n}\Lambda \tilde{c}_a, \qquad s_i = \pm\Lambda^{N_c-n}\delta_{i,N_c-n} + \epsilon_B^{2i}\Lambda^{2i}\tilde{s}_i, \tag{5.6.57}$$

where $\sum_{a=1}^{2n} \tilde{c}_a = 0$. In this limit, the $C(x)$ and $G(x)$ in the (5.6.54) become as

$$C(x) = \mp\, \Lambda^{N_c}\epsilon_B^{2n}\tilde{x}^n + \epsilon_B^{2N_c}\Lambda^{N_c}\tilde{C}(\tilde{x}), \tag{5.6.58}$$

$$G(x) = \Lambda^{2N_c}\left\{\epsilon_B^{4n}\tilde{x}^{2n} + \sum_{i=2}^{2n}\epsilon_B^{2(2n-i)+i(N_c+2-n)}\tilde{x}^{2n-i}\tilde{C}_i\right\}. \tag{5.6.59}$$

Here, $\tilde{C}_i(i = 2, \cdots, 2n)$ are elementary symmetric polynomial pf $\tilde{c}_a$, and

$$\tilde{C}(\tilde{x}) = \tilde{x}^{N_c} - \sum_{i=2}^{N_c}\tilde{s}_i\tilde{x}^{N_c-i}. \tag{5.6.60}$$

Thus, the SW curve becomes

$$\begin{aligned} y^2 =& \Lambda^{2N_c}\left(\epsilon_B^{4n}\tilde{x}^{2n} \mp 2\epsilon_B^{2n+2N_c}\tilde{x}^n\tilde{C}(\tilde{x}) - \epsilon_B^{4n}\tilde{x}^{2n} - \epsilon_B^{2N_c+2n}\tilde{x}^{2n-2}\tilde{C}_2 + \cdots\right) \\ =& \mp\, 2\Lambda^{2N_c}\epsilon_B^{2n+2N_c}\tilde{y}^2, \qquad \tilde{y}^2 = \tilde{x}^n\tilde{C}(\tilde{x}) \pm \frac{1}{2}\tilde{x}^{2n-2}\tilde{C}_2. \end{aligned} \tag{5.6.61}$$

The SW differential in the scaling limit is

$$\lambda_{SW} = xd\log\frac{C-y}{C+y} = 2\sqrt{-2}\epsilon_B^2\Lambda d\tilde{x}\frac{\tilde{y}}{(\tilde{x}+\tilde{m})^n} + \cdots. \tag{5.6.62}$$

Therefore, the SW differential in the scaling limit is the following:

$$\tilde{\lambda}_{SW} = \frac{\tilde{y}}{(\tilde{x}+\tilde{m})^n}d\tilde{x}. \tag{5.6.63}$$

Let us summarize the above SW curve and the SW differential in the scaling limit. It is convenient to shift $\tilde{x} \to \tilde{x} - \tilde{m}$. If we shift $\tilde{x} \to \tilde{x} - \tilde{m}$, we can write the SW curve and the SW differential defined in (5.6.61) and (5.6.63) as following form:

$$\tilde{y}^2 = \sum_{i=0}^{N_c} \tilde{u}_i \tilde{x}^{N_c+n-i}, \qquad \tilde{\lambda}_{SW} = \frac{\tilde{y}}{\tilde{x}^n} d\tilde{x}, , \qquad \tilde{u}_0 := 1. \tag{5.6.64}$$

Let us finally determine the scaling dimensions of the operators in the scaling limit. The scaling dimension of these operators is determined by requiring that the scaling dimension of the BPS masses is 1. Then, we obtain the dimensions of the operators as follows:

$$[\tilde{x}] = \frac{2}{N_c - n + 2}, \qquad [\tilde{u}_i] = \frac{2i}{N_c - n + 2}, \qquad i = 1, \cdots, N_c. \tag{5.6.65}$$

Thus, the operators $\tilde{u}_i (i = N_c - n + 3, \cdots, N_c)$ are irrelevant.

## Type $A$ scaling

The other scaling limit is defined as follows, which is called type A scaling:

$$\begin{aligned} x =& \epsilon_A^2 \Lambda \tilde{x}, & (5.6.66)\\ \hat{m}_a =& \epsilon_A^2 \Lambda \tilde{c}_a, \qquad \sum_{a=1}^{2n} \tilde{c}_a = 0, & (5.6.67)\\ s_i =& \pm \Lambda^{N_c - n} \delta_{i, N_c - n} + \epsilon_A^{-2N_c+2n+2i} \Lambda^i \tilde{u}_{i-N_c+n}, \qquad i = N_c - n, \cdots, N_c, & (5.6.68) \end{aligned}$$

where $s_1, \cdots s_{N_c-n+1}$ are $\mathcal{O}(1)$. $\epsilon_A$ and $\epsilon_B$ are related by $\epsilon_A^2 = \epsilon_B^{N_c-n+2}$. Then, the $C(x)$ and $G(x)$ in (5.6.54) can be written as

$$\begin{aligned} C(x) =& \Lambda^{N_c} \left\{ \epsilon_A^{2N_c} \tilde{x}^{N_c} - s_1 \epsilon_A^{2n_c-2} \tilde{x}^{N_c-1} - \cdots - \epsilon_A^{2n+2} s_{N_c-n} \tilde{x}^{n+1} \mp \epsilon_A^{2n} \tilde{x}^n - \epsilon_A^{2n} \tilde{u}_1 \tilde{x}^{n-1} - \cdots - \epsilon_A^{2n} \tilde{u}_n \right\} \\ =& - \epsilon_A^{2n} \Lambda^{N_c} (\pm \tilde{x}^n + \tilde{u}_1 \tilde{x}^{n-1} + \cdots + \tilde{u}_N) + \mathcal{O}(\epsilon_A^{2n}), \\ G(x) =& \epsilon_A^{4n} \Lambda^{2n} \prod_{a=1}^{2n} (\tilde{x} + \tilde{c}_a). \end{aligned} \tag{5.6.69}$$

Using this, we can obtain the SW curve in the scaling limit as

$$y^2 = \epsilon_A^{4n} \Lambda^{2N_c} \tilde{y}^2 + \cdots, \qquad \tilde{y}^2 = \tilde{C}(\tilde{x})^2 - \tilde{G}(\tilde{x}), \tag{5.6.70}$$

where we define

$$\tilde{C}(\tilde{x}) = \sum_{i=0}^{n} \tilde{u}_i \tilde{x}^{n-i}, \qquad \tilde{G}(\tilde{x}) = \prod_{a=1}^{2n} (\tilde{x} + \tilde{c}_a), \tag{5.6.71}$$

with $\tilde{u}_0 = \pm 1$. We also obtain the SW differential in the scaling limit as

$$\tilde{\lambda}_{SW} = \tilde{x} d\log\left(\frac{\tilde{C} - \tilde{y}}{\tilde{C} + \tilde{y}}\right). \tag{5.6.72}$$

Therefore, the SW curve and the SW differential in the scaling limit are given as

$$\tilde{y}^2 = \tilde{C}(\tilde{x})^2 - \tilde{G}(\tilde{x}), \qquad \tilde{\lambda}_{SW} = \tilde{x} d\log\left(\frac{\tilde{C} - \tilde{y}}{\tilde{C} + \tilde{y}}\right). \tag{5.6.73}$$

Let us finally discuss the dimension of the operator in the scaling limit. If we require that the dimension of the BPS masses must be one, the scaling dimensions of the operator in the scaling limit are the following:

$$[\tilde{x}] = 1, \qquad [\tilde{u}_i] = i, \qquad (i = 1, \cdots, n), \qquad [\tilde{c}_a] = 1, \qquad (a = 1, \cdots, N_f). \tag{5.6.74}$$

## 5.7 $A_r$ and $D_r$ type gauge theories

Finally, let us discuss the scaling limit for the $A_r = SU(r+1)$ and $D_r = SO(2r)$ type gauge theories.

### 5.7.1 $A_r$ type

Next, let us define the scaling limit for $A_r$ type gauge theories. This is the same as the $SU(r+1)$ $N_f = 0$ case. But we discuss the scaling limit again because this formalism is useful for the quantization of the SW curve. The SW curve for the $A_r$ type gauge theories are defined in (4.5.1). The superconformal point is defined as

$$u_2 = \cdots = u_r = 0, \qquad u_{r+1} = \pm \Lambda^{r+1}, \tag{5.7.1}$$

where the discriminant of this curve degenerates [17]. Let us consider the scaling limit for $u_{r+1} = -\Lambda^{r+1}$ case. In this case, the scaling limit is defined as follows [17], [25]:

$$
\begin{aligned}
z =& e^{\delta\xi}, && (5.7.2)\\
x =& \delta^{\frac{2}{r+1}}\Lambda\tilde{x}, && (5.7.3)\\
u_i =& \delta^{\frac{2i}{r+1}}\Lambda^i\tilde{u}_i, \qquad i = 2, \cdots, r && (5.7.4)\\
u_{r+1} =& -\Lambda^{r+1} + \delta^{\frac{2(r+1)}{r+1}}\Lambda^{r+1}\tilde{u}_{r+1}. && (5.7.5)
\end{aligned}
$$

Here, we take $\delta \to 0$. If we take the scaling limit, the function defined in (4.5.1) becomes

$$
W(x, u_i) \to \delta^2\Lambda^{r+1}(\tilde{x}^{r+1} - \tilde{u}_2\tilde{x}^{r-1} - \cdots - \tilde{u}_{r+1}) + \Lambda^{r+1} = \delta^2\Lambda^{r+1}W(\tilde{x}, \tilde{u}_i) + \Lambda^{r+1}, \tag{5.7.6}
$$

and the left-hand side of (4.5.1) becomes

$$
\frac{\Lambda^{r+1}}{2}\left(z + \frac{1}{z}\right) \to \Lambda^{r+1} + \frac{1}{2}\delta^2\xi^2\Lambda^{r+1} + \cdots . \tag{5.7.7}
$$

Thus, in this limit, the SW curve becomes

$$
\xi^2 = 2W(\tilde{x}, \tilde{u}_i). \tag{5.7.8}
$$

We can also consider the SW differential in the scaling limit:

$$
\lambda_{SW} = \delta^{\frac{r+3}{r+1}}\Lambda\tilde{x}d\xi. \tag{5.7.9}
$$

We define the SW differential in the scaling limit as the following:

$$
\tilde{\lambda}_{SW} = \tilde{x}d\xi. \tag{5.7.10}
$$

If we require that the dimension of the BPS mass must be one, we determine the dimension of $\tilde{\lambda}_{SW}$ equal one, so we obtain the dimension of the operator in the scaling limit as follows:

$$
[\tilde{u}_j] = \frac{2j}{r+3}, \qquad j = 2, \cdots, r+1. \tag{5.7.11}
$$

### 5.7.2 $D_r$ type

If we introduce $z = \sqrt{G}w$, we can rewrite the SW curve and the differential in (4.5.3) as follows:

$$
\frac{\Lambda^{2r-2}}{2}\left(w + \frac{1}{w}\right) = W(x, s_i), \qquad \lambda_{SW} = xd\text{log}w. \tag{5.7.12}
$$

Here we define

$$W(x, s_i) := \frac{C(x)}{x^2} = x^{2r-2} + s_1 x^{2r-4} + \cdots + s_{r-1} + \frac{s_r}{x^2}. \tag{5.7.13}$$

The superconformal point is defined as [23]:

$$s_1 = s_2 = \cdots = s_r = 0, \qquad s_{r-1} = \pm\Lambda^{2r-2}. \tag{5.7.14}$$

We can define the scaling limit around the point as

$$w = e^{\delta\xi}, \tag{5.7.15}$$

$$x = \delta^{\frac{2}{2r-2}}\Lambda\tilde{x}, \tag{5.7.16}$$

$$s_i = \delta^{\frac{4i}{2r-2}}\Lambda^{2i}\tilde{s}_i, \qquad i = 1, \cdots, r-2, \tag{5.7.17}$$

$$s_{r-1} = \Lambda^{2r-2} + \delta^2\Lambda^{2(r-1)}\tilde{s}_{r-1}, \tag{5.7.18}$$

$$s_r = \delta^{\frac{4r}{2r-2}}\Lambda^{2r}\tilde{s}_r, \tag{5.7.19}$$

with $\delta \to 0$ [17]. In this case, we can also obtain

$$W(x, s_i) \to \delta^2\Lambda^{2r-2}W(\tilde{x}, \tilde{s}_i) + \Lambda^{2r-2}, \tag{5.7.20}$$

and

$$\frac{\Lambda^{2r-2}}{2}\left(w + \frac{1}{w}\right) \to \Lambda^{2r-2} + \frac{1}{2}\delta^2\Lambda^{2r-2}\xi^2. \tag{5.7.21}$$

Then the SW curve and SW differential in the scaling limit is as follows:

$$\xi^2 = 2W(\tilde{x}, \tilde{s}_i), \qquad \tilde{\lambda}_{SW} = \tilde{x}d\xi. \tag{5.7.22}$$

This is the SW curve of the $(A_1, D_r)$-type AD theory. The parameters $\tilde{s}_i$ has the following scaling dimensions:

$$[\tilde{s}_i] = \frac{2j}{r}, \qquad j = 1, \cdots, r. \tag{5.7.23}$$

## 5.8 Summary

Let us summarize the results. As we explained in section 5.5, the scaling limits are one of the deformations around the superconformal points, which direction is on the Coulomb branch. Let us specify these scaling limits in the viewpoints of deformations.

We define the scaling limit not to break the $\mathcal{N}=2$ supersymmetries. We also note that the $U(2)_R$ must be broken by chiral anomaly as $U(2)_R \to SU(2)_R$, then the possible global symmetries in the scaling limit are only $SU(2)_R$ and flavor symmetry. But if these operators include $SU(2)_R$ generator in the SW curve in the scaling limit, the scaling limit cannot preserve the $\mathcal{N}=2$ supersymmetry. Thus, we can conclude that these operators that have integer dimensions correspond to flavor symmetry.

| Theory | Dimension of the Operators in AD | Flavor Symmetry |
|---|---|---|
| $SU(N_c=2n)\ N_f=0$ | $\frac{i}{n+1},\ i=2,\cdots,2n$ | U(1) |
| $SU(N_c=2n-1)\ N_f=0$ | $\frac{2i}{2n+1},\ i=2,\cdots,2n-1$ | - |
| $SU(N_c=2n)\ N_f=1$ | $\frac{2i}{2n+3},\ i=2,\cdots,2n+1$ | - |
| $SU(N_c=2n-1)\ N_f=1$ | $\frac{i}{n+1},\ i=2,\cdots,2n$ | U(1) |
| $SU(N_c)\ N_f=2n+1(n\geq 1)$ | $\frac{2i+1}{2},\ i=0,\cdots n$<br>and $N_f$ are with dimension one | - |
| $SU(N_c=2n)\ N_f=2$ | $\frac{2i}{2n+1},\ i=1,\cdots,2n+1$ | U(1) |
| $SU(N_c=2n-1)\ N_f=2$ | $\frac{i}{n},\ i=1,\cdots,2n$ | U(2) |
| $SU(N_c)\ N_f=2n(n\geq 2)$<br>Type B Scaling | $\frac{2i}{N_c-n+2},\ i=1,\cdots,N_c$ | - |
| $SU(N_c)\ N_f=2n(n\geq 2)$<br>Type A Scaling | $i,\ i=1,\cdots,N_c$<br>and $N_f$ are with dimension one | $SU(2n)$ |
| $A_r$ | $\frac{2j}{r+3},\ j=2,\cdots,r+1$ | U(1) or none |
| $D_r$ | $\frac{2j}{r},\ j=1,\cdots,r$ | U(1) or U(2) |

Table 5.2: List of the AD theorires associaed to $SU(N_c)$ SQCDs.

## 5.9 The Universality of the AD curves

In Table 5.2 of the AD theories, we observe that some different UV gauge theories lead to the same AD theory. This is called the universality of AD theories. In this section, we explain the universality of the AD theories by considering explicit examples [46], [47], [23]. We call two different UV gauge theories to belong to the same universality class, if and only if the AD theories of these two gauge theories are the same.

### 5.9.1 $SU(2)$ $N_f = 1$ and $A_2$ type

Let us first show the universality between $SU(2)$ $N_f = 1$ SQCD and $A_2$ type gauge theory. The SW curve and SW differential in the scaling limit of $SU(2)$ $N_f = 1$ SQCD are given as (5.6.25) with $N_c = 2$:

$$\tilde{y}^2 = \tilde{x}^3 + \tilde{u}_2\tilde{x} + \tilde{u}_3, \qquad \tilde{\lambda}_{SW} = \tilde{y}d\tilde{x}, \tag{5.9.1}$$

The dimensions of the operators are are given by $[\tilde{u}_2] = \frac{4}{5}$ and $[\tilde{u}_3] = \frac{6}{5}$. On the other hand, the SW curve and SW differential in the scaling limit of $A_2$ type gauge theory are

$$\frac{1}{2}\xi^2 = \tilde{x}^3 - \tilde{u}_2\tilde{x} - \tilde{u}_3, \qquad \tilde{\lambda}_{SW} = \tilde{x}d\xi, \tag{5.9.2}$$

and the dimensions of the operators are $[\tilde{u}_2] = \frac{4}{5}$ and $[\tilde{u}_3] = \frac{6}{5}$. We can confirm that these two AD theories have the same SW curve and the SW differential. Thus, these two gauge theories belong to the same universality class.

### 5.9.2 $SU(2)$ $N_f = 2$ and $A_3$

Next, let us examine the universality between $SU(2)$ $N_f = 2$ SQCD and $A_3$ type gauge theory. The SW curve and the SW differential in the scaling limit of $SU(2)$ $N_f = 2$ theory are given as (5.6.52) with $N_c = 2$:

$$\tilde{y}^2 = \tilde{x}^3 + \tilde{u}_1\tilde{x}^2 + \tilde{u}_2\tilde{x} + \tilde{u}_3, \qquad \tilde{\lambda}_{SW} = \tilde{y}d\log\tilde{x}. \tag{5.9.3}$$

Here, the dimensions of the operators are $[\tilde{u}_1] = \frac{2}{3}$, $[\tilde{u}_2] = \frac{4}{3}$ and $[\tilde{u}_3] = 2$. On the other hand, these of the scaling limit of the $A_3$ type gauge theories are given as (5.7.8) and

(5.7.10) with $r = 3$:

$$\frac{1}{2}\xi^2 = \tilde{x}^4 - \tilde{u}_2\tilde{x}^2 - \tilde{u}_3\tilde{x} - \tilde{u}_4, \qquad \tilde{\lambda}_{SW} = \tilde{x}d\xi. \tag{5.9.4}$$

Here, the dimensions of these operators are $[\tilde{u}_2] = \frac{2}{3}$, $[\tilde{u}_3] = 2$ and $[\tilde{u}_4] = \frac{4}{3}$. Thus, the operators of these two theories are the same, but these two SW curves and SW differential in the scaling limit looks like different. In the following let us show that these two SW curves and SW differential in the scaling limit are the same.

Let us change the variables in the SW curve and the SW differential in the scaling limit of $A_3$ type gauge theory. By the calculation of the appendix (B.1.8), the SW curve and the SW differential in the scaling limit of $A_3$ type gauge theory can be rewritten as

$$y^2 = x^3 - \tilde{u}_2x^2 + 4\tilde{u}_4x + (-4\tilde{u}_2\tilde{u}_4 + \tilde{u}_3^2). \tag{5.9.5}$$

Let us denote these $\tilde{u}_i (i = 2, 3, 4)$ in (5.9.5) as $\tilde{t}_i := \tilde{u}_i$. Then, if we identify

$$\tilde{u}_1 = -\tilde{t}_2, \qquad \tilde{u}_2 = 4\tilde{t}_4, \qquad \tilde{u}_3 = -4\tilde{t}_2\tilde{t}_4 + \tilde{t}_3^2, \tag{5.9.6}$$

these two curves (5.9.5) and (5.9.3) are the same. We can also obtain the SW differential becomes that of AD theory obtained from $SU(2)$ $N_f = 2$ SQCD. Therefore, we can conclude that these two gauge theories belong to the same universality class.

### 5.9.3 $SU(2)$ $N_f = 2$ and $D_3$

Let us next consider the universality between $SU(2)$ $N_f = 2$ and $D_3$ type gauge theory. The SW curve and the SW differential in the scaling limit of $D_3$ type gauge theory are given by substitute $r = 3$ in (5.7.22):

$$\frac{1}{2}\xi^2 = \tilde{x}^4 + \tilde{s}_1\tilde{x}^2 + \tilde{s}_2 + \frac{\tilde{s}_3}{\tilde{x}^2}, \qquad \tilde{\lambda}_{SW} = \tilde{x}d\xi. \tag{5.9.7}$$

Here, the dimeisnions of these operators are $[\tilde{s}_1] = \frac{2}{3}$, $[\tilde{s}_2] = \frac{4}{3}$ and $[\tilde{s}_3] = 2$. Thus this AD curve and differential look like different from that of $SU(2)$ $N_f = 2$ theory (5.9.3). But if we replace the variable in (5.9.7) as $\xi = \frac{\sqrt{2}y_1}{\tilde{x}}$, we obtain the SW curve and the SW differential in the scaling limit of $D_3$ type theory as

$$y_1^2 = \tilde{x}^6 + \tilde{s}_1\tilde{x}^4 + \tilde{s}_2\tilde{x}^2 + \tilde{s}_3, \qquad \tilde{\lambda}_{SW} = -\sqrt{2}y_1 d\log\tilde{x}. \tag{5.9.8}$$

Thus, this SW curve and the SW differential in the scaling limit is the same as that of $SU(2)$ $N_f = 2$ theory (5.9.3) if we identify

$$\tilde{u}_i = \tilde{s}_i, \qquad i = 1, 2, 3. \tag{5.9.9}$$

Thus, these two gauge theories belong to the same universality class.

### 5.9.4 $SU(2)$ $N_f = 3$ and $D_4$

Next, let us show the universality between $SU(2)$ $N_f = 3$ and $D_4$. The SW curve and SW differential in the scaling limit of $SU(2)$ $N_f = 3$ are given by substitute $n = 1$ in (5.6.38):

$$\tilde{y}^2 = \tilde{C}(\tilde{x})^2 - \tilde{G}(\tilde{x}), \qquad \tilde{\lambda}_{SW} = \tilde{x} d\log\frac{\tilde{C} - \tilde{y}}{\tilde{C} + \tilde{y}}. \tag{5.9.10}$$

Here,

$$\tilde{C} = \tilde{u}_0\tilde{x} + \tilde{u}_1, \qquad \tilde{G} = \tilde{x}^3 + \tilde{v}_2\tilde{x} + \tilde{v}_3, \tag{5.9.11}$$

and dimensions of these operators are $[\tilde{u}_0] = \frac{1}{2}$, $[\tilde{u}_1] = \frac{3}{2}$, $[\tilde{v}_2] = 2$ and $[\tilde{v}_3] = 3$.

On the other hand, the SW curve and SW differential in the scaling limit of $D_4$ type gauge theory are given by (5.7.22) with $r = 4$:

$$\frac{1}{2}\xi^2 = \tilde{x}^6 + \tilde{s}_1\tilde{x}^4 + \tilde{s}_2\tilde{x}^2 + \tilde{s}_3 + \frac{\tilde{s}_4}{\tilde{x}^2}, \qquad \tilde{\lambda}_{SW} = \tilde{x} d\xi. \tag{5.9.12}$$

Here, the dimensions of these opearators are $[\tilde{s}_1] = \frac{1}{2}$, $[\tilde{s}_2] = 1$, $[\tilde{s}_3] = \frac{3}{2}$ and $[\tilde{s}_4] = 2$. By the calculation of the appendix (B.2.4), we can rewrite this curve and the differential (5.9.12) as

$$\begin{aligned} y^2 =& x^3 + \frac{\tilde{s}_1^2}{4}x^2 + (\tilde{s}_1\tilde{s}_3 - 4\tilde{s}_4 - \frac{\tilde{s}_2^2}{3} + \frac{\tilde{s}_1^4}{48})x \\ &+ \tilde{s}_3^2 - \frac{\tilde{s}_1\tilde{s}_2\tilde{s}_3}{3} + \frac{\tilde{s}_1^3\tilde{s}_3}{12} - \frac{8\tilde{s}_2\tilde{s}_4}{3} + \frac{2\tilde{s}_1^2\tilde{s}_4}{3} + \frac{2\tilde{s}_2^3}{27} - \frac{\tilde{s}_1^2\tilde{s}_2^2}{36} + \frac{\tilde{s}_1^6}{1728}. \end{aligned} \tag{5.9.13}$$

We can identify these SW curves (5.9.13) and (5.9.10), if we identify these coefficients:

$$\begin{aligned} \tilde{u}_0^2 =& \frac{\tilde{s}_1^2}{4}, \\ 2\tilde{u}_0\tilde{u}_1 + \tilde{v}_2 =& \tilde{s}_1\tilde{s}_3 - 4\tilde{s}_4 - \frac{\tilde{s}_2^2}{3} + \frac{\tilde{s}_1^4}{48}, \\ \tilde{u}_1^2 + \tilde{v}_3 =& \tilde{s}_3^2 - \frac{\tilde{s}_1\tilde{s}_2\tilde{s}_3}{3} + \frac{\tilde{s}_1^3\tilde{s}_3}{12} - \frac{8\tilde{s}_2\tilde{s}_4}{3} + \frac{2\tilde{s}_1^2\tilde{s}_4}{3} + \frac{2\tilde{s}_2^3}{27} - \frac{\tilde{s}_1^2\tilde{s}_2^2}{36} + \frac{\tilde{s}_1^6}{1728}. \end{aligned} \tag{5.9.14}$$

This universality is confirmed by comparing the Picard-Fuchs equations of these two theories [46].

### 5.9.5 $SU(N_c+1)$ $N_f=0$ and $SU(N_c)$ $N_f=1$

The previous examples can be generalized as the following. The SW curve and the SW differential in the scaling limit of the $SU(N_c+1)$ $N_f=0$ theory are given by (5.6.5), and these of $SU(N_c)$ $N_f=1$ theory are given as (5.6.25). We can confirm that the SW curves and the SW differentials in the scaling limit for the $SU(N_c+1)$ $N_f=0$ and $SU(N_c)$ $N_f=1$ SQCDs are the same form. Therefore, these two theories belong to the same universality class.

### 5.9.6 $N_f=2n+1(n\geq 1)$ SQCDs

The SW curve and the SW differential in the scaling limit of the $SU(N_c)$ $N_f=2n+1(n\geq 1)$ SQCD are determined only $n$: (5.6.38). Thus, the SQCDs with $N_f=2n+1$ belong to the same universality class. For example, $SU(3)$ $N_f=3$ and $SU(2)$ $N_f=3$ belong to the same universality class. $SU(3)$ $N_f=5$ and arbitrary $SU(N_c)$ $N_f=5$ theories belong to the same universality class.

### 5.9.7 $SU(3)$ $N_f=1$ and $SU(2)$ $N_f=2$

The SW curve and the SW differential in the scaling limit of the $SU(3)$ $N_f=1$ theory is given as $N_c=3$ in (5.6.25):

$$\tilde{y}^2=\tilde{x}^4+\tilde{u}_2\tilde{x}^2+\tilde{u}_3\tilde{x}+\tilde{u}_4,\qquad \tilde{\lambda}_{SW}=\tilde{y}d\tilde{x}. \tag{5.9.15}$$

On the other hand, the SW curve and the SW differential in the scaling limit of the $SU(2)$ $N_f=2$ theory are given as (5.9.3). As we explained above, the $SU(2)$ $N_f=2$ theory belong to the same universality class with $A_3$. The SW curve and the SW differential in the scaling limit of the $A_3$ theory are given as (5.9.4). Let us denote the coefficients of (5.9.4) as $\tilde{t}_i=\tilde{u}_i$. Then we can rewrite the AD curve of $A_3$ theory (5.9.4) as

$$\frac{1}{2}\xi^2=\tilde{x}^4-\tilde{t}_2\tilde{x}^2-\tilde{t}_3\tilde{x}-\tilde{t}_4,\qquad \tilde{\lambda}_{SW}=\tilde{x}d\xi. \tag{5.9.16}$$

If we identify the coefficients of AD curves (5.9.16) and (5.9.15) as

$$\tilde{u}_i=-2\tilde{t}_i,\qquad i=2,3,4, \tag{5.9.17}$$

these two AD curves (5.9.16) and (5.9.15) are the same, and thus these two gauge theories belong to the same universality class.

### 5.9.8 $SU(3)$ $N_f = 2$ and $SU(2)$ $N_f = 3$

Let us next compare the $SU(3)$ $N_f = 2$ and the $D_4$ theory. The SW curve and the differential of the AD theory obtained from the $SU(3)$ $N_f = 2$ theory is given as (5.6.52) with $N_c = 3$:

$$\tilde{y}^2 = (\tilde{x} + \tilde{m})(\tilde{x}^3 - \tilde{s}_2\tilde{x} - \tilde{s}_3) + \frac{1}{2}\tilde{c}_1\tilde{c}_2 =: \tilde{x}^4 + \tilde{u}_1\tilde{x}^3 + \tilde{u}_2\tilde{x}^2 + \tilde{u}_3\tilde{x} + \tilde{u}_4, \qquad \tilde{\lambda}_{SW} = \tilde{y}d\mathrm{log}\tilde{x}. \tag{5.9.18}$$

On the other hand, the SW curve and SW differential in the scaling limit of $D_4$ theory are given by (5.9.12). If we change the variables as $Y = \tilde{y}\tilde{x}$ and $X = \tilde{x}^2$ in (5.9.12), we obtain

$$Y^2 = X^4 + \tilde{t}_1X^3 + \tilde{t}_2X^2 + \tilde{t}_3X + \tilde{t}_4, \qquad \tilde{\lambda}_{SW} = Yd\mathrm{log}X. \tag{5.9.19}$$

Here, we rewrite the coefficients of the curve (5.9.12) as $\tilde{t}_i := \tilde{s}_i (i = 1, 2, 3, 4)$. If we identify these coefficients of the AD curves (5.9.18) and (5.9.19) as

$$\tilde{t}_1 = \tilde{m} = \tilde{u}_1, \qquad \tilde{t}_2 = -\tilde{s}_2 = \tilde{u}_2, \qquad \tilde{t}_3 = -\tilde{s}_2\tilde{m} - \tilde{s}_3 = \tilde{u}_3, \qquad \tilde{t}_4 = -\tilde{s}_3\tilde{m} + \frac{1}{2}\tilde{c}_1\tilde{c}_2 = \tilde{u}_4, \tag{5.9.20}$$

these two gauge theories belong to the same universality class.

### 5.9.9 $SU(3)$ $N_f = 4$ (A) and $SU(2)$ $N_f = 4(A)$

The SW curve for the AD theories obtained from the $N_f = 2n (n \geq 2)$ SQCDs with type A scaling are only dependent on the flavor symmetry (or $n$): (5.6.73). Therefore, two UV theories $SU(3)$ $N_f = 4$ (A) and $SU(2)$ $N_f = 4(A)$ belong to the same universality class.

# Chapter 6

# $\Omega$ Background and Quantum SW curves

Next, let us introduce the $\Omega$ background which is four-dimensional spacetime admitting a torus action [4]. The $\Omega$ background is introduced as a regulator to calculate the partition function for the $\mathcal{N} = 2$ Super Yang-Mills and SQCDs. On the $\Omega$ background, the partition function called Nekrasov partition function can be calculated exactly and reproduce the prepotential at weak coupling region. But on the $\Omega$ background, it is not yet known S-dualities and we do not have an idea to define the partition function in the strong coupling region. The SW theory can describe in both weak and strong coupling region, but it is difficult to formulate the SW theory in the $\Omega$-background.

In this chapter, we introduce the Nekrasov-Shatashvili (NS) limit of the $\Omega$-background which is a limit that fixes one parameter of the $\Omega$ background to zero [36]. In the NS limit, the SW periods (called quantum SW periods) are calculated by using the quantum SW curve which is defined by quantizing the SW curve [33], [34] which is the ordinary differential equation of a complex variable. The quantum SW periods are analyzed by the WKB method [37], [38], [39]. In the weak coupling region, it has been shown that the quantum SW curve obtained from the quantum SW curve is same as the quantum SW periods obtained from the Nekrasov partition function [33], [34], [40], [41]. One of the main purposes of this thesis is to define the quantum SW periods in the strong coupling region by using the quantum SW curve. Especially, on the massless monopole/dyon point (which is on the strong coupling region), the quantum SW periods are calculated by [48], [45], [49]. The $\Omega$ background also studied by topological string [50], [51]. The relation to the CFT is studied by [52], [53], [54], [55]. In this thesis, we study the AD theories, which are typical field theories in the strong coupling region, and their quantum SW curves.

In this chapter, we first introduce the $\Omega$ background as a spacetime that admits the torus action. Next, we explain the quantization of the SW curve and obtain the quantum SW periods defined as the WKB solutions of the quantum SW curve. Finally, we calculate the quantum SW periods by taking the scaling limit of these quantum SW periods in UV, and check the consistency of the quantization with the scaling limit.

## 6.1 $\Omega$ background

Let us first explain the definition of the $\Omega$ background following [35]. $\mathcal{N}=2$ super Yang-Mills theories are given by dimensional reduction of $\mathcal{N}=1$ six-dimensional Super Yang-Mills theories. Now let us consider the six-dimensional spacetime compactified on the two-dimensional torus with the torus action on the four-dimensional spacetime $\mathbb{R}^4$ whose vector fields are given by

$$V_4^\mu = {\Omega_4}^\mu{}_\nu x^\nu, \qquad V_5^\mu = {\Omega_5}^\mu{}_\nu x^\nu. \tag{6.1.1}$$

Here, $\Omega_4, \Omega_5 \in SO(4)$. Since $\pi_1(T^2)$ is abelian, the Lie bracket of $V_4$ and $V_5$ should vanish, which is equivalent that the matrices $\Omega_4$ and $\Omega_5$ commute. Let us define the following metric:

$$\begin{aligned} ds^2 =& g_{\mu\nu}(dx^\mu + V_a^\mu dx^a)(dx^\nu + V_b^\nu dx^b) - (dx^4)^2 - (dx^5)^2 && (6.1.2)\\ =& G_{IJ} dx^I dx^J. && (6.1.3) \end{aligned}$$

Here $I, J = 0, \ldots, 5$ and the metric $G_{IJ}$ is given by

$$\begin{aligned} G_{\mu\nu} =& g_{\mu\nu}, \qquad G^{\mu\nu} = g^{\mu\nu} - V_a^\mu V_a^\nu, && (6.1.4)\\ G_{a\mu} =& V_{a\mu}, \qquad G^{a\mu} = V_a^\mu, && (6.1.5)\\ G_{ab} =& -\delta_{ab} + V_a^\mu V_{b\nu}, \qquad G^{ab} = -\delta^{ab}. && (6.1.6) \end{aligned}$$

This metric defines the six-dimensional spacetime on the $\Omega$ background. Let us define

$$\Omega := \frac{1}{\sqrt{2}}(\Omega_4 + i\Omega_5), \qquad \overline{\Omega} := \frac{1}{\sqrt{2}}(\Omega_4 - i\Omega_5). \tag{6.1.7}$$

We can write this matrix by choosing some basis as [35], [4]:

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -\epsilon_1 \\ 0 & 0 & -\epsilon_2 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ \epsilon_1 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon_1, \epsilon_2 \in \mathbb{C}. \tag{6.1.8}$$

Thus the $\Omega$ background on $\mathbb{R}^4$ is characterized by two real parameters $\epsilon_1, \epsilon_2 \in \mathbb{R}$. In the $\Omega$-background, the translational invariance is broken, and the $\mathcal{N} = 2$ supersymmetry is also broken. However, one can define scalar supercharge which acts as supersymmetry up to the torus action. This defines the BRST-cohomology in the Hilbert space of the theory. Using the localization with respect to this supercharge, Nekrasov has shown that the partition function is presented as the sum over several configurations of the ADHM moduli parameters labeled by the Young diagram [63]. The logarithm of the partition function which corresponds to the free energy reduces to the explicit formula for the prepotential of the Seiberg-Witten theory in the limit $\epsilon_1, \epsilon_2 \to 0$ [62], [10]. This implies that the assumptions in the argument of Seiberg and Witten to obtain the exact solution are justified by explicit calculations. Therefore, it is desirable to find a similar formula in the strong coupling region, which is not yet found at present. To search for the formula valid in the strong coupling region, we are going to find the effect of the $\Omega$-background to the SW curve. It turns out that we can see this effect in the limit where one of the epsilon parameters is set to be zero. In the next section, we will consider the SW theory in this limit.

## 6.2 Quantum SW curve for $SU(N_c)$ $N_f$ SQCDs

Next, we briefly explain the method of quantization of the SW curve. It is first found by A.Moronov and A.Morozov [33], [34]. Here, we briefly explain their results. Let us first consider the $\mathcal{N} = 2$ $SU(2)$ SYM theory. If we introduce a new variable $\phi$ by $e^{i\phi} = y + x^2$, the SW curve and the SW differential in the $SU(2)$ SYM take the form

$$x^2 - \frac{1}{2}(e^{i\phi} + \Lambda e^{-i\phi}) = u, \qquad \lambda_{SW} = 2i\phi dx. \tag{6.2.1}$$

In $\lambda_{SW}$, $\phi$ can be regarded as a canonical conjugate variable to $x$. Let us consider quantum mechanics described by the Schrödinger equation

$$\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\phi^2}+\Lambda^2\cos\phi\right)\Psi(\phi)=E\Psi(\phi). \tag{6.2.2}$$

Here, the parameter $E:=u$ is the energy eigenvalue of the Hamiltonian. This Schrödinger equation is given by replacing

$$x\to -i\hbar\frac{\partial}{\partial\phi}, \tag{6.2.3}$$

in the SW curve (6.2.1).

Let us next consider the Bohr-Sommerfeld (BS)periods of this system:

$$\Pi^{(0)}(C)=\oint_C\sqrt{2(E-\Lambda^2\cos\phi)}d\phi. \tag{6.2.4}$$

Here, $C$ is some contour that connects the turning points. There are two turning points $\phi=\pm\phi_0$, $E=\gamma\cos\phi_0$. Then, there are two bases of the contours $C=\alpha$ and $C=\beta$. The WKB solution of the Schrodinger equation is given as the following form:

$$\Psi(\phi)=\exp\left(\frac{i}{\hbar}\int^{\phi}P(\phi)d\phi\right). \tag{6.2.5}$$

The BS periods are now given by

$$\Pi(C)=\oint_C P(\phi)d\phi. \tag{6.2.6}$$

By using these BS periods, let us introduce the followings:

$$a:=\Pi(\alpha),\qquad \frac{\partial\mathcal{F}(a,\epsilon_1)}{\partial a}:=\Pi(\beta), \tag{6.2.7}$$

where we introduce some function $\mathcal{F}$ of $a$ and $\epsilon_1:=\hbar\in\mathbb{R}$. Identifying $\epsilon_1$ with $\hbar$, [33] checked that this function $\mathcal{F}(a,\epsilon_1)$ is same as the Nekrasov partition function $Z(a,\epsilon_1,\epsilon_2)$ in the limit of $\epsilon_2=0$ up some orders in the expansion of instanton number:

$$\mathcal{F}(a,\epsilon_1)=\mathrm{lim}_{\epsilon_2\to 0}\left\{\epsilon_1\epsilon_2\mathrm{log}Z(a,\epsilon_1,\epsilon_2)\right\} \tag{6.2.8}$$

It is also confirmed in [33] that the BS periods $\Pi^{(0)}(\alpha)$ and $\Pi^{(0)}(\beta)$ are the SW periods of the $\mathcal{N}=2$ $SU(2)$ SYM.

In the same way, [34], [40], [41] introduced the quantum SW curves and quantum SW periods in $\mathcal{N}=2$ $SU(N_c)$ $N_f$ SQCDs. Let us review these results very shortly. In $SU(N_c)$ $N_f$ SQCDs, the SW curve and the SW differential is given as (4.4.20). Introducing $z=y+C(x)$, we obtain the SW curve and differential as follows:

$$C(x)-\frac{1}{2}\left(z+\frac{G(x)}{z}\right)=0, \qquad \lambda_{SW}=x(d\mathrm{log}G(x)-2d\mathrm{log}z). \tag{6.2.9}$$

Differentiating the SW differential, we obtain holomorphic symplectic form $d\lambda_{SW}=-2dx\wedge d\mathrm{log}z$. Then, we replace $z\to\hat{z}=\exp\left(-i\hbar\frac{\partial}{\partial x}\right)$ in the SW curve, and we define the quantum SW curve as

$$\left[\frac{1}{2}\left(\exp\left(-i\hbar\frac{\partial}{\partial x}\right)+\exp\left(i\frac{\hbar}{2}\frac{\partial}{\partial x}\right)G(x)\exp\left(i\frac{\hbar}{2}\frac{\partial}{\partial x}\right)\right)-C(x)\right]\Psi(x)=0. \tag{6.2.10}$$

The BS periods are given by the WKB solution of the quantum SW curve as in the same way as (6.2.6). In [34], [40], [41], it was shown that if we introduce a function $\mathcal{F}$ as (6.2.7), this function $\mathcal{F}$ is the prepotential defined by the Nekrasov partition function in the limit $\epsilon_2=0$.

## Quantum SW periods for SQCDs

Let us calculate the quantum SW periods explicitly following [45], [47], [57]. Let us solve this differential equation (6.2.10) as a WKB solution:

$$\Psi(x)=\exp\left(\frac{i}{\hbar}S(x)\right), \qquad S(x)=\sum_{k=0}^{\infty}\hbar^k\int^x p_k(x)dx. \tag{6.2.11}$$

If we substitute this solution in (6.2.10), we obtain

$$\frac{1}{2}\left[\frac{\Psi(x-i\hbar)}{\Psi(x)}+G(x+i\hbar/2)\frac{\Psi(x+i\hbar)}{\Psi(x)}\right]=C(x). \tag{6.2.12}$$

Substituting this expansion in (6.2.12), and comparing the coefficient of order $\hbar^0$ in the left and right hand sides, we obtain

$$e^{p_0}+G(x)e^{-p_0}=C(x). \tag{6.2.13}$$

This equation is solved as $p_0=\log(C\pm\sqrt{C^2-G})$. We take the plus sign and obtain

$$p_0(x)dx=\log(C+y)dx, \qquad y:=\sqrt{C(x)^2-G(x)}. \tag{6.2.14}$$

Next, let us compare the coefficients for $\hbar^1$ in the left and right-hand sides of (6.2.12). We find that

$$e^{p_0}\left(-\frac{i}{2}p_0'+p_1\right)+-\frac{i}{2}e^{-p_0}(Gp_0'-2iGp_1-G')=0. \tag{6.2.15}$$

Substituting (6.2.14) into this equation, we obtain

$$p_1dx=\frac{i}{4}\frac{2CC'-G'}{y^2}=\frac{i}{2}\mathrm{dlog}y. \tag{6.2.16}$$

This is the total derivative, so we can ignore this term when we consider the quantum SW periods.

Next, let us consider the $p_2$. To calculate this, let us compare the coefficients of $\hbar^2$ in the left and right sides of (6.2.12), we obtain

$$\begin{aligned}\frac{1}{48}\Bigg\{&96p_2y-\frac{2CG'^2}{G^2}+\frac{2yG'^2}{G^2}-\frac{15C(G'-2CC')^2}{y^4}-\frac{2yG''}{G}\\&+\frac{G'^2+2C^2G''}{CG}-16C''+\frac{G'^2-16CC'G'+4C^2(13C'^2-3G''+6CC''')}{Cy^2}\Bigg\}=0.\end{aligned} \tag{6.2.17}$$

Then, we obtain

$$p_2dx=\left(\frac{C''}{8y}-\frac{Cy'^2}{8y^3}\right)dx+d\left\{\frac{1}{24}\frac{C'+y'}{C+y}-\frac{1}{4}\frac{Cy'}{y^2}\right\}. \tag{6.2.18}$$

Next, let us consider the $p_3$. To calculate this, let us compare the coefficients of $\hbar^3$ in the left and right sides of (6.2.12), we obtain

$$\begin{aligned}p_3=&-\frac{i}{8}\frac{C'^2y'}{y^3}+\frac{3i}{4}\frac{CC'y'^2}{y^4}-\frac{3i}{4}\frac{C^2y'^3}{y^5}+\frac{i}{8}\frac{y'^3}{y^3}+\frac{i}{16}\frac{C'C''}{y^2}-\frac{i}{4}\frac{CC''y'}{y^3}-\frac{3i}{8}\frac{CC'y''}{y^3}\\&+\frac{3i}{4}\frac{C^2y'y''}{y^4}-\frac{3i}{16}\frac{y'y''}{y^2}+\frac{i}{16}\frac{CC^{(3)}}{y^2}-\frac{iC^2y^{(3)}}{8y^3}+\frac{iy^{(3)}}{16y}\\=&\frac{d}{dx}\left\{\frac{i}{16}\frac{y''}{y}-\frac{i}{8}\frac{C^2y''}{y^3}+\frac{i}{16}\frac{CC''}{y^2}-\frac{i}{16}\frac{(y')^2}{y^2}-\frac{i}{8}\frac{CC'y'}{y^3}+\frac{3i}{16}\frac{C^2(y')^2}{y^4}\right\}.\end{aligned} \tag{6.2.19}$$

This is the total derivative, therefore we ignore the $p_3$ in the case of calculation of the quantum SW periods.

In the same way, we can obtain the following result:

$$
\begin{aligned}
p_4 = & \left[ \frac{(C')^2 C''}{768 y^3} - \frac{7}{1536} \frac{C (C'')^2}{y^3} - \frac{C (C')^2 y''}{256 y^4} + \frac{11}{512} \frac{C^2 C'' y''}{y^4} - \frac{11}{1536} \frac{C'' y''}{y^2} - \frac{5 C^3 (y'')^2}{256 y^5} \right. \\
& \left. + \frac{17 C (y'')^2}{1536 y^3} - \frac{C C' C^{(3)}}{384 y^3} + \frac{C^2 C' y^{(3)}}{128 y^4} - \frac{C' y^{(3)}}{384 y^2} - \frac{3 C^2 C^{(4)}}{1024 y^3} + \frac{11 C^{(4)}}{3072 y} + \frac{3 C^3 y^{(4)}}{1024 y^4} - \frac{5 C y^{(4)}}{1024 y^2} \right] \\
& + \frac{d}{dx} \left\{ \sum_{n=0}^{7} \frac{a_n}{9216} \frac{y^{3-n} C^n y'^3}{(C+y)^4} + \sum_{n=0}^{6} \frac{b_n}{9216} \frac{y^{3-n} C^n C'^3}{(C+y)^4} + \sum_{n=0}^{6} \frac{c_n}{3072} \frac{C^{6-n} y^{n-4} C' y''}{(C+y)^4} \right. \\
& + \sum_{n=0}^{7} \frac{d_n}{3072} \frac{C^{7-n} y^{n-5} y' C''}{(C+y)^4} + \sum_{n=0}^{5} \frac{e_n}{3072} \frac{C^{5-n} y^{n-3} C' y''}{(C+y)^4} + \sum_{n=0}^{6} \frac{f_n}{3072} \frac{C^{6-n} y^{n-4} y' C''}{(C+y)^4} \\
& + \frac{187}{1536} \frac{C C'^2 y'}{(C+y)^4} + \frac{33}{128} \frac{C^4 C'^2 y'}{y^3 (C+y)^4} + \frac{33}{128} \frac{C^2 C'^2 y'}{y (C+y)^4} + \frac{11}{192} \frac{y C'^2 y'}{(C+y)^4} + \frac{211}{384} \frac{C C' y'^2}{(C+y)^4} \\
& - \frac{1125}{512} \frac{C^4 C' y'^2}{y^3 (C+y)^6} + \frac{45}{128} \frac{C^2 C' y'^2}{y (C+y)^4} + \frac{277}{1536} \frac{y C' y'^2}{(C+y)^4} - \frac{99}{256} \frac{C^6 C' y'^2}{y^5 (C+y)^4} + \frac{99}{64} \frac{C^6 y'^3}{y^5 (C+y)^4} \\
& + \frac{33}{512} \frac{C^5 C'^2 y'}{y^4 (C+y)^4} + \frac{99}{256} \frac{C^3 C'^2 y'}{y^2 (C+y)^4} - \frac{99}{64} \frac{C^5 C' y'^2}{y^4 (C+y)^4} - \frac{135}{128} \frac{C^3 C' y'^2}{y^2 (C+y)^4} - \frac{803}{4608} \frac{C y'^3}{(C+y)^4} \\
& + \frac{297}{768} \frac{C^7 y'^3}{y^6 (C+y)^4} + \frac{3267}{1536} \frac{C^5 y'^3}{y^4 (C+y)^4} + \frac{297}{384} \frac{C^4 y'^3}{y^3 (C+y)^4} - \frac{297}{384} \frac{C^3 y'^3}{y^2 (C+y)^4} - \frac{297}{384} \frac{C^2 y'^3}{y (C+y)^4} \\
& + \frac{11}{576} \frac{y y'^3}{(C+y)^4} + \frac{25}{1152} \frac{C C'^3}{(C+y)^4} + \frac{1}{1536} \frac{C^4 C'^3}{y^3 (C+y)^4} + \frac{1}{384} \frac{C^3 C'^3}{y^2 (C+y)^4} + \frac{1}{256} \frac{C^2 C'^3}{y (C+y)^4} \\
& \left. + \frac{91}{4608} \frac{y C'^3}{(C+y)^4} \right\} . \qquad (6.2.20)
\end{aligned}
$$

Here, the coefficients $a_n, b_n, c_n, d_n, e_n, f_n$ are the follows:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $a_n$ | 88 | −171 | −1476 | −1973 | 456 | 2859 | 2196 | 549 |
| $b_n$ | 223 | 804 | 813 | −416 | −1431 | −1044 | −261 | − |
| $c_n$ | 579 | 2316 | 3275, | 1520 | −703 | −972 | −287 | − |
| $d_n$ | −1188 | −4752 | −6426 | −1944 | 3024 | 2720 | 526 | −88 |
| $e_n$ | −106 | −424 | −636 | −512 | −282 | −88 | − | − |
| $f_n$ | 411 | 1644 | 2287 | 928 | −751 | −892 | −267 | − |

(6.2.21)

Thus, we have obtained the quantum SW periods up to the fourth level. The odd-order corrections are total derivatives, then the non-trivial quantum SW periods are the only even-order corrections. In general, we can calculate higher-order $p_n$ in a similar way. But it is quite complicated. The quantum SW period is determined as

$$\begin{aligned}\Pi(\hbar) =&\left(\int_{\alpha_I} P(x)dx, \int_{\beta_I} P(x)dx\right)\\ =&\sum_{n=0}^{\infty} \hbar^{2n}\Pi^{(2n)}, \qquad \Pi^{(2n)} = \left(\int_{\alpha_I} p_{2n}(x)dx, \int_{\beta_I} p_{2n}(x)dx\right). \end{aligned} \tag{6.2.22}$$

## 6.3 Quantum SW curves of $A_r$ and $D_r$ type Theories

We examine the quantization of the SW curve for the $A_r$ type and $D_r$ type in the same way as [33], [34], [40], [41]. In this section, we first define the quantum SW curves by quantizing the SW curves. Then, by using the WKB method [37], [38], [39], we calculate the quantum SW periods [46], [57]. But we should check that this quantization is consistent with the result of the Nekrasov partition function or not. We check this consistency condition in the next sections.

### 6.3.1 $A_r$ type

Let us first explain the quantum SW curve for $A_r$ type SYM following [46]. For the SW curve and the SW differential (4.5.1), we obtain $d\lambda_{SW} = dx \wedge d\log z$. Thus, we quantize the curve by replacing $\log z \to -i\hbar\partial_x$ in the SW curve (4.5.1), and obtain the quantum SW as follows [25]:

$$\left[\frac{\Lambda^{r+1}}{2}(e^{-i\hbar\partial_x} + e^{i\hbar\partial_x}) - W(x, u_i)\right]\Psi(x) = 0. \tag{6.3.1}$$

This quantum SW curve is the same as that of SQCD (6.2.10) with $G(x) := 1$ and $C(x) := \frac{W(x,u_i)}{\Lambda^{r+1}} =: B(x, x_i)$. Thus, if we solve this quantum SW curve (6.3.1) by the WKB solution (6.2.11), we can calculate $p_n (n = 0, 1, 2, 3, 4)$ in the same way:

$$\begin{aligned}p_0(x) =&\log(B + \sqrt{B^2 - 1}),\\ p_1(x) =&\frac{iBB'}{2(B^2 - 1)} = \frac{i}{4}\frac{d}{dx}\log(B^2 - 1),\\ p_2(x) =& -\frac{\partial_x^2 B}{24(B^2 - 1)^{\frac{3}{2}}} + \frac{d}{dx}\left(-\frac{1}{12}\frac{B^2 B'}{(B^2 - 1)^{3/2}} - \frac{1}{8}\frac{B'}{(B^2 - 1)^{3/2}}\right),\end{aligned}$$

$$
\begin{aligned}
p_3(x) =& \frac{d}{dx}\left\{-\frac{i}{8}\frac{BB''}{(B^2-1)^2}+\frac{i}{4}\frac{B^2B'^2}{(B^2-1)^3}+\frac{i}{16}\frac{B'^2}{(B^2-1)^3}\right\}, \\
p_4(x) =& -\frac{B(23+12B^2)}{1920(B^2-1)^{\frac{7}{2}}}(\partial_x^2 B)^2+\frac{(7+8B^2)}{5760(B^2-1)^{\frac{5}{2}}}\partial_x^4 B \\
&+\frac{d}{dx}\left\{\frac{(-153+307B^2-43B^4-183B^6+32B^8+40B^{10})B^{(3)}}{5760(B^2-1)^{11/2}}\right. \\
&-\frac{B(657+408B^2+40B^4)B'B''}{1920(B^2-1)^{7/2}} \\
&\left.+\frac{(423+3654B^2+1368B^4+80B^6)B'^3}{5760(B^2-1)^{9/2}}\right\}. \qquad (6.3.2)
\end{aligned}
$$

$p_1, p_3$ are total derivatives. Here, we can ignore the total derivative terms, because we integrate these $p_n$ to compute the quantum SW periods.

### 6.3.2 $D_r$ type

Let us next consider the quantum SW periods for $D_r$ type gauge theories. In this case, we can write the SW curve and the SW differential in two different ways [46]. Therefore, there are two different quantum SW curves. Thus, we have to compare the quantum SW periods which are given by solving these two quantum SW curve.

Let us first quantize the SW curve of the form (5.7.12). Note that $d\lambda_{SW} = dx \wedge d\log w$, and replace $w \to -i\hbar\partial_x$ in the SW curve (5.7.12) to obtain the quantum SW curve. Then we obtain the quantum SW curve as

$$
\left[\frac{\Lambda^{2r-2}}{2}(e^{-i\hbar\partial_x}+e^{i\hbar\partial_x})-W(x,s_i)\right]\Psi(x)=0. \qquad (6.3.3)
$$

We write the WKB solution of the curve as $p_n^{(A_r)}$. This quantum SW curve is the same as the $A_r$ theory. Thus, we can obtain the quantum SW periods as follows [46], [57]:

$$
p_0^{(A_r)} = \log(B+\sqrt{B^2-1}), \qquad (6.3.4)
$$

$$
p_2^{(A_r)} = -\frac{\partial_x^2 B}{24(B^2-1)^{\frac{3}{2}}}, \qquad (6.3.5)
$$

$$
p_4^{(A_r)} = -\frac{B(23+12B^2)}{1920(B^2-1)^{\frac{7}{2}}}(\partial_x^2 B)^2+\frac{(7+8B^2)}{5760(B^2-1)^{\frac{5}{2}}}\partial_x^4 B, \qquad (6.3.6)
$$

where $B := W(x,s_i)/\Lambda^{2r-2}$.

We can also quantize the SW curve and the SW differential of the form (4.5.3). In this case, the differential of the SW differential is $d\lambda_{SW} = dx \wedge d\mathrm{log}z$. Thus, we quantize this SW curve by replacing $\mathrm{log}z \to -i\hbar\partial_x$. Then the quantum SW curve is as follows:

$$\frac{1}{2}\left[e^{-i\hbar\partial_x} + e^{i\hbar\partial_x/2}G(x)e^{i\hbar\partial_x/2}\right]\Psi(x) = C(x)\Psi(x). \tag{6.3.7}$$

Here, we choose the ordering of the operators as [40]. If we write the quantum SW periods obtained from this quantum SW curve as $p_n^{(SQCD)}$, we obtain the following results in the same way as SQCDs [46], [57]:

$$\begin{aligned}
p_0^{(SQCD)}(x) =&\log(C+y), \qquad y = \sqrt{C(x)^2 - G(x)},\\
p_2^{(SQCD)}(x) =& \left(-\frac{\partial_x^2 C}{8y} + \frac{C(\partial_x y)^2}{8y^3}\right),\\
p_4^{(SQCD)}(x) =& \left[\frac{(C')^2C''}{768y^3} - \frac{7}{1536}\frac{C(C'')^2}{y^3} - \frac{C(C')^2y''}{256y^4}\right.\\
&+\frac{11}{512}\frac{C^2C''y''}{y^4} - \frac{11}{1536}\frac{C''y''}{y^2} - \frac{5C^3(y'')^2}{256y^5} + \frac{17C(y'')^2}{1536y^3}\\
&-\frac{CC'C^{(3)}}{384y^3} + \frac{C^2C'y^{(3)}}{128y^4} - \frac{C'y^{(3)}}{384y^2}\\
&\left.-\frac{3C^2C^{(4)}}{1024y^3} + \frac{11C^{(4)}}{3072y} + \frac{3C^3y^{(4)}}{1024y^4} - \frac{5Cy^{(4)}}{1024y^2}\right].
\end{aligned} \tag{6.3.8}$$

We should check that these quantum SW periods $p_n^{(A_r)}$ and $p_n^{(SQCD)}$ are the same or not. For example, let us examine the difference in the first few quantum periods:

$$p_0^{(SQCD)} - p_0^{(A_r)} = \mathrm{log}G^{1/2}, \qquad p_2^{(SQCD)} - p_2^{(A_r)} = -\frac{\partial_x G}{G}\frac{\partial_x B}{16(B^2-1)^{3/2}}. \tag{6.3.9}$$

Here, we use

$$\partial_x y = \frac{1}{2}\frac{\partial_x(C^2-G)}{y} = \frac{2CC' - G'}{2y}. \tag{6.3.10}$$

Therefore, the two quantizations are different [46]. We should check whether these quantum SW periods are the same as that of using the Nekrasov partition function in the weak coupling. However, the Nekrasov partition function for $D_r$ type gauge group is known only for $\epsilon_1 = \pm\epsilon_2$ case [65]. For other gauge groups, for example, exceptional type and non-simply-laced type gauge groups, the SW curves are known. The quantization of the SW curves is left for future study.

## 6.4 Quantum SW curves of AD theories

Next, we consider the quantum SW curve for AD theories following [46] and [47]. We can obtain the quantum SW curve of AD theories by quantizing the SW curves for AD theories. Then, we can obtain the quantum SW periods for AD theories. But we also obtain quantum SW periods for AD theories by taking the scaling limit of the quantum SW periods for UV gauge theories. We confirm that these results are consistent with each other [46], [47].

### 6.4.1 $A_r$ type

Let us first obtain the quantum SW curve for AD theories obtained from $A_r$ theory. The AD curve of the $A_r$ theory is given as (5.7.8), and the SW differential in this AD theory is given as (5.7.10). Then, we obtain $d\lambda_{SW} = d\tilde{x} \wedge d\xi$, and thus we obtain the quantum SW curve for AD theories by replacing $\xi \to -i\hbar\partial_{\tilde{x}}$ in the SW curve (5.7.8) [46], [57]:

$$\left[-\hbar^2 \frac{\partial^2}{\partial^2 \tilde{x}} - 2W(\tilde{x}, \tilde{u}_i)\right] \tilde{\Psi}(\tilde{x}) = 0. \tag{6.4.1}$$

This is the Schrodinger type equation. Let us find the solution to the WKB expansion:

$$\tilde{\Psi}(\tilde{x}) = \exp\left(\frac{i}{\hbar}\int^{\tilde{x}} dx \tilde{P}(\tilde{x})\right), \qquad \tilde{P}(\tilde{x}) = \sum_{n=0}^{\infty} \hbar^n \tilde{p}_n(\tilde{x}). \tag{6.4.2}$$

The explicit calculation of $\tilde{p}_n$ is explained in appendix C.1. We obtain the follows [46], [57]:

$$\begin{aligned}
\tilde{p}_0 =& -\sqrt{2W}, \\
\tilde{p}_1 =& \frac{i}{4}\frac{d}{d\tilde{x}}\log W, \\
\tilde{p}_2 =& \frac{1}{48\sqrt{2}}\frac{\partial_{\tilde{x}}^2 W}{W^{3/2}} + \frac{d}{d\tilde{x}}\left\{\frac{5\sqrt{2}}{96}\frac{W'}{W^{3/2}}\right\}, \\
\tilde{p}_3 =& \frac{d}{d\tilde{x}}\left\{15i\frac{d^2}{d\tilde{x}^2}\frac{1}{W} - \frac{3i}{256}\frac{W''}{W^2}\right\} \\
\tilde{p}_4 =& \frac{7}{3072\sqrt{2}}\frac{(\partial_{\tilde{x}}^2 W)^2}{W^{7/2}} - \frac{1}{1536\sqrt{2}}\frac{\partial_{\tilde{x}}^4 W}{W^{5/2}} \\
&+ \frac{d}{d\tilde{x}}\left\{-\frac{23}{1536\sqrt{2}}\frac{W^{(3)}}{W^{5/2}} + \frac{221}{3072\sqrt{2}}\frac{W'W''}{W^{7/2}} - \frac{1105}{18432\sqrt{2}}\frac{W'^3}{W^{9/2}}\right\}.
\end{aligned} \tag{6.4.3}$$

By using these results, we obtain the quantum SW periods:

$$\tilde{\Pi}(\hbar) = \sum_{n=0}^{\infty} \hbar^{2n} \tilde{\Pi}^{(2n)}, \qquad \tilde{\Pi}^{(2n)} = \left( \int_{\alpha_I} \tilde{p}_{2n}(\tilde{x}) d\tilde{x}, \int_{\beta_I} \tilde{p}_{2n}(\tilde{x}) d\tilde{x} \right). \tag{6.4.4}$$

Here, the odd-order corrections are total derivatives, so we ignore contribution from $\tilde{p}_{2n+1}$.

Now let us compare this quantum SW period with that obtained by taking the scaling limit of the quantum SW periods for gauge theories. By using (5.7.6), we obtain $B = \delta^2 \tilde{W} + 1$, where $\tilde{W} := W(\tilde{x}, \tilde{u}_i)$. Therefore, if we take the scaling limit of $p_0$ obtained from the quantum SW curve in $A_r$ gauge theory (6.3.2), we obtain

$$p_0(x)dx = \log \left( \delta^2 \tilde{W} + 1 + \delta \sqrt{2\tilde{W}} \right) d(\delta^{\frac{2}{r+1}} \Lambda \tilde{x}) = -\delta^{\frac{r+3}{r+1}} \tilde{p}_0(\tilde{x}) d\tilde{x} + \cdots, \tag{6.4.5}$$

where $\tilde{p}_0$ on the right-hand side is the quantum period in the AD theory obtained in (C.1.5). In the same way, if we take the scaling limit for quantum SW periods for $A_r$ type gauge theories, we obtain

$$\begin{aligned}
p_2(x)dx &= -\frac{\delta^2 \cdot \left( \frac{1}{\delta^{\frac{2}{r+1}} \Lambda} \right)^2 \tilde{W}''}{24[(\delta^2 \tilde{W} + 1)^2 - 1]^{3/2}} d(\delta^{2/(r+1)} \Lambda \tilde{x}) = -\delta^{-\frac{r+3}{r+1}} \Lambda^{-1} \tilde{p}_2 d\tilde{x} + \cdots, \\
p_4(x)dx &= \left\{ -\frac{(\delta^2 \tilde{W} + 1)(23 + 12(\delta^2 \tilde{W} + 1))(\delta^2 \delta^{-\frac{4}{r+1}} \Lambda^{-2} \tilde{W}'')^2}{1920((\delta^2 \tilde{W} + 1)^2 - 1)^{7/2}} \right. \\
&\qquad \left. + \frac{\{7 + 8(\delta^2 \tilde{W} + 1)^2\} \delta^2 \delta^{-\frac{8}{r+1}} \Lambda^{-4} \tilde{W}^{(4)}}{5760((\delta^2 \tilde{W} + 1)^2 - 1)^{5/2}} \right\} d(\delta^{\frac{2}{r+1}} \Lambda \tilde{x}) \\
&= -\delta^{-3\frac{r+3}{r+1}} \Lambda^{-3} \tilde{p}_4 d\tilde{x} + \cdots.
\end{aligned} \tag{6.4.6}$$

Thus, we can conclude that the quantum SW periods obtained by the quantum SW curve of AD theories are the same as the quantum SW periods obtained by taking the scaling limit of the quantum SW periods of $A_r$ type gauge theories.

### 6.4.2 $D_r$ type

Let us next discuss the AD theories obtained from $D_r$ type gauge theories. The SW curve and the SW differential in this AD theory are given by (5.7.22). Then, we obtain $d\tilde{\lambda}_{SW} = d\tilde{x} \wedge d\xi$. Therefore, the quantum SW curve in this AD theory is given by replacing

$\xi \to -i\hbar\partial_{\tilde{x}}$ in the AD curve (5.7.22) [46], [57]:

$$\left[-\hbar^2\frac{\partial^2}{\partial^2\tilde{x}^2} - 2W(\tilde{x},\tilde{s}_i)\right]\tilde{\Psi}(\tilde{x}) = 0. \tag{6.4.7}$$

This quantum SW curve is the same as the quantum SW curve for AD theories obtained from $A_r$ type gauge theories, by changing $\tilde{u}_i \to \tilde{s}_i$. Thus, the quantum SW periods of this AD theory are the same as (6.4.3):

$$\begin{aligned}
\tilde{p}_0 =& -\sqrt{2W}, \\
\tilde{p}_2 =& \frac{1}{48\sqrt{2}}\frac{\partial_{\tilde{x}}^2 W}{W^{3/2}}, \\
\tilde{p}_4 =& \left[\frac{7}{3072\sqrt{2}}\frac{(\partial_{\tilde{x}}^2 W)^2}{W^{7/2}} - \frac{1}{1536\sqrt{2}}\frac{\partial_{\tilde{x}}^4 W}{W^{5/2}}\right].
\end{aligned} \tag{6.4.8}$$

These quantum SW periods (6.4.8) are also obtained by taking the scaling limit of $p_n^{(A_r)}$, which are the quantum periods for $D_r$ theory with $A_r$ type quantization. We can verify this fact by using (6.4.6).

Next, let us take the scaling limit of the $p_n^{(SQCD)}$, which are give by (6.3.8). Then, we find that the relation (6.4.6) does not hold due to the contribution from the pole at $x = 0$. To rescue this discrepancy, we will modify $W(x, s_i)$ in (6.3.3) as

$$W(\tilde{s}_i,\tilde{x}) \to W(\tilde{s}_i,\tilde{x}) + \hbar^2\frac{A}{\tilde{x}^2}, \qquad A \in \mathbb{R}. \tag{6.4.9}$$

Then, the WKB corrections are found to be [46]:

$$\begin{aligned}
\tilde{p}_0(A)d\tilde{x} =& \tilde{p}_0 d\tilde{x}, \\
\tilde{p}_2(A)d\tilde{x} =& \left[\frac{1}{48\sqrt{2}}\frac{\partial_{\tilde{x}}^2 W}{W^{3/2}} + \frac{A}{2\sqrt{2}}\frac{1}{\tilde{x}^2 W^{1/2}}\right]d\tilde{x}, \\
\tilde{p}_4(A)d\tilde{x} =& \left[-\frac{7}{3072\sqrt{2}}\frac{(\partial_{\tilde{x}}^2 W)^2}{W^{7/2}} + \frac{1}{1536\sqrt{2}}\frac{\partial_{\tilde{x}}^4 W}{W^{5/2}} - \frac{A}{64\sqrt{2}}\frac{\partial_{\tilde{x}}^2 W}{\tilde{x}^2 W^{5/2}} + \left(\frac{(1-A)A}{16\sqrt{2}}\right)\frac{1}{\tilde{x}^4 W^{3/2}}\right]d\tilde{x}
\end{aligned} \tag{6.4.10}$$

up to total derivatives. Explicit calculations are in Appendix C.1. Then we obtain [46]:

$$p_{2n}^{(SQCD)}dx = \delta^{\frac{(1-2n)r}{r-1}}\Lambda^{1-2n}\tilde{p}_{2n}(A = -1/2)d\tilde{x} + \cdots, \qquad n = 0, 2, 4. \tag{6.4.11}$$

### 6.4.3 Schrödinger Type Quantization of AD Curves

In the previous section, we have studied the quantum SW curves for AD theories obtained by the scaling limit of $SU(r+1)$ and $SO(2r)$ type super Yang-Mills theories. We now consider the quantum SW curves for AD theories obtained by scaling limit of $\mathcal{N}=2$ $SU(N_c)$ SQCD with $N_f$ hypermultiplets. The corresponding SW curves take different forms for each $N_f$. They are classified into the Schrödinger type and SQCD type.

For $N_f = 0, 1, 2$ and $N_f = 2n (n > 1)$ with type B scaling, the SW curve and the SW differential in the scaling limit are given by (5.6.5), (5.6.25), (5.6.52), (5.6.64). We will see that the quantization of these SW curves in the scaling limit are the same form, and the quantum SW curve in the scaling limit is the Schrödinger type equations. Let us discuss these curves following [47], [57]:

- $N_f = 0, 1$

  For $N_f = 0, 1$ case, the SW curve and the SW differential for AD theories are given by (5.6.5) and (5.6.25):

$$\tilde{y}^2 = -Q_0(\tilde{x}), \qquad Q_0(\tilde{x}) = -\tilde{x}^N + \sum_{i=2}^{N} \tilde{u}_i \tilde{x}^{N-i}, \qquad \tilde{\lambda}_{SW} = \tilde{y} d\tilde{x}. \tag{6.4.12}$$

  Here, $N = N_c$ for $N_f = 0$ and $N = N_c + 1$ for $N_f = 1$. We obtain the symplectic form $d\tilde{\lambda}_{SW} = d\tilde{y} \wedge d\tilde{x}$. The quantum SW curve is defined by replacing $\tilde{y} \to -i\hbar \frac{\partial}{\partial \tilde{x}}$ in the SW curve (6.4.12):

$$\left(-\hbar^2 \frac{\partial^2}{\partial \tilde{x}^2} + Q(\tilde{x})\right) \Psi(\tilde{x}) = 0. \tag{6.4.13}$$

  Thus, this quantum SW curve is the same form as that of AD theories of $A_r$ and $D_r$ type gauge theories: (6.4.1), (6.4.7). Then, the potential term $Q$ in this quantum SW curve (6.4.13) may have some quantum correction as the same reason of AD theory of $D_r$ theory:

$$Q(\tilde{x}) = Q_0(\tilde{x}) + \hbar^2 Q_2(\tilde{x}). \tag{6.4.14}$$

- $N_f = 2$

Let us next consider the $N_f = 2$ case. In this case, if we introduce $\xi = \log\tilde{x}$, the SW curve and the SW differential in the scaling limit is given as (5.6.52):

$$\tilde{y}^2 = -Q_0(\xi), \qquad \tilde{\lambda}_{SW} = \tilde{y}d\xi. \tag{6.4.15}$$

Here,

$$Q_0(\xi) = -\sum_{i=0}^{N_c+1} \tilde{u}_i e^{(N_c+1-i)\xi}, \qquad \tilde{u}_0 = 1 \tag{6.4.16}$$

We replace $y \to i\hbar\frac{\partial}{\partial\xi}$ to obtain the quantum SW curve:

$$\left(-\hbar^2\frac{\partial^2}{\partial\xi^2} + Q(\xi)\right)\Psi(\xi) = 0, \qquad Q(\xi) = Q_0(\xi) + \hbar^2 Q_2(\xi). \tag{6.4.17}$$

Here, note that the potential term $Q$ may have some quantum correction.

- $N_f = 2n(n > 1)$ with type B scaling

  The SW curve and the SW differential in the type $B$ scaling limit of the $N_f = 2n(n > 1)$ theories are given as (5.6.64). If we define $\xi = \tilde{x}^{1-n}$, the SW curve and the SW differential in these AD theories become the following:

$$\tilde{y}^2 = -Q_0(\xi), \qquad \tilde{\lambda}_{SW} = \frac{1}{1-n}\tilde{y}d\xi. \tag{6.4.18}$$

  Here,

$$Q_0(\xi) = -\sum_{i=0}^{N_c} \tilde{u}_i \xi^{\frac{N_c+n-i}{1-n}}. \tag{6.4.19}$$

  If we replace $\tilde{y} \to i\hbar\frac{\partial}{\partial\xi}$, we obtain the quantum SW curve with some quantum correction for the potential term $Q$:

$$\left(-\hbar^2\frac{\partial^2}{\partial\xi^2} + Q(\xi)\right)\Psi(\xi) = 0, \qquad Q(\xi) = Q_0(\xi) + \hbar^2 Q_2(\xi). \tag{6.4.20}$$

Let us now consider the quantum SW periods for these theories. We can obtain the quantum SW periods in the same way when we consider AD theories obtained from $D_r$

type gauge theories [47]:

$$\begin{aligned}
\tilde{p}_0(\tilde{x}) =& i\sqrt{Q_0}, \\
\tilde{p}_2(\tilde{x}) =& \frac{i}{2}\frac{Q_2}{\sqrt{Q_0}} + \frac{i}{48}\frac{Q_0''}{Q_0^{3/2}}, \\
\tilde{p}_4(\tilde{x}) =& -\frac{7i}{1536}\frac{Q_0''^2}{Q_0^{7/2}} + \frac{i}{768}\frac{Q_0^{(4)}}{Q_0^{5/2}} - \frac{i}{32}\frac{Q_2 Q_0''}{Q_0^{5/2}} + \frac{i}{48}\frac{Q_2''}{Q_0^{3/2}} - \frac{i}{8}\frac{Q_2^2}{Q_0^{3/2}}.
\end{aligned} \tag{6.4.21}$$

Explicit calculations are in Appendix C.1. In general, these $\tilde{p}_{2n}$ include the contribution from $Q_2$. But, in the following, we assume $Q_2 = 0$.

### 6.4.4 SQCD type quantization of AD Curves

Let us next consider the quantization of the AD theories obtained from $N_f = 2n+1 (n \geq 1)$ SQCDs and $N_f = 2n (n \geq 2)$ SQCDs with type A scaling. In this case, the SW curve and the SW differential in these AD theories have the same form as SQCDs:

$$\tilde{y}^2 = \tilde{C}(\tilde{x})^2 - \tilde{G}(\tilde{x}), \qquad \tilde{\lambda}_{SW} = \tilde{x} d\log\left(\frac{\tilde{C} - \tilde{y}}{\tilde{C} + \tilde{y}}\right). \tag{6.4.22}$$

with $\tilde{C}, \tilde{G}, \tilde{y}$ are defined in the previous section. In $N_f = 2n+1$ cases or $N_f = 2n (n \geq 2)$ SQCDs with type A scaling cases, $\tilde{C}$ and $\tilde{G}$ are given as

$$\tilde{C} = \sum_{i=0}^{n} \tilde{u}_i \tilde{x}^{n-i}, \qquad \tilde{G} = \prod_{a=1}^{2n+1} (\tilde{x} + \tilde{c}_a), \tag{6.4.23}$$

where $\sum_{a=1}^{2n+1} \tilde{c}_a = 0$. Thus, in these cases, the quantum SW curve or the quantum SW periods only depend on the flavor symmetry.

We can quantize these AD curves (6.4.22) in the same way as the quantizations of UV SQCDs: (6.2.10). Thus, we obtain the following quantum SW curve in the scaling limit [47], [57]:

$$\left[\frac{1}{2}\left(\exp\left(-i\hbar\frac{\partial}{\partial\tilde{x}}\right) + \exp\left(i\frac{\hbar}{2}\frac{\partial}{\partial\tilde{x}}\right)\tilde{G}(\tilde{x})\exp\left(i\frac{\hbar}{2}\frac{\partial}{\partial\tilde{x}}\right)\right) - \tilde{C}(\tilde{x})\right]\Psi(\tilde{x}) = 0. \tag{6.4.24}$$

This quantum SW curve is the same as (6.2.10). Thus, we can calculate the quantum SW periods of these AD theories in the same way as UV SQCDs. For odd $n$, $p_n$ becomes total derivative. For even $n$, the first few terms are the following:

$$\tilde{p}_0(\tilde{x}) = \log(C(x) + y), \tag{6.4.25}$$

$$\tilde{p}_2(\tilde{x}) = \frac{C''}{8y} - \frac{C(y')^2}{8y^3}, \tag{6.4.26}$$

$$\begin{aligned}\tilde{p}_4(\tilde{x}) =& \frac{(C')^2 C''}{768 y^3} - \frac{7}{1536}\frac{C(C'')^2}{y^3} - \frac{C(C')^2 y''}{256 y^4} \\ &+ \frac{11}{512}\frac{C^2 C'' y''}{y^4} - \frac{11}{1536}\frac{C'' y''}{y^2} - \frac{5C^3(y'')^2}{256 y^5} + \frac{17C(y'')^2}{1536 y^3} \\ &- \frac{CC'C^{(3)}}{384 y^3} + \frac{C^2 C' y^{(3)}}{128 y^4} - \frac{C' y^{(3)}}{384 y^2} \\ &- \frac{3C^2 C^{(4)}}{1024 y^3} + \frac{11 C^{(4)}}{3072 y} + \frac{3C^3 y^{(4)}}{1024 y^4} - \frac{5C y^{(4)}}{1024 y^2}.\end{aligned} \tag{6.4.27}$$

## 6.5 Summary

Let us summarize the quantum SW curves of the AD theories obtained from $SU(N_c)$ $N_f$ SQCDs and $A_r$ and $D_r$ theories.

| Gauge Theory | Quantum SW curve |
|---|---|
| $A_r$ | $[-\hbar^2\partial_{\tilde{x}}^2 - 2W(\tilde{x},\tilde{u}_i)]\Psi = 0,$<br>$W(\tilde{x},\tilde{u}_i) = \tilde{x}^{r+1} - \tilde{u}_2\tilde{x}^{r-1} - \cdots - \tilde{u}_{r+1}$ |
| $D_r$ | $\left[-\hbar^2\partial_{\tilde{x}}^2 - 2W(\tilde{x},\tilde{s}_i) + \hbar^2\frac{A}{x^2}\right]\Psi = 0,$<br>$W(\tilde{x},\tilde{s}_i) = \tilde{x}^{2r-2} + \tilde{s}_1\tilde{x}^{2r-4} + \cdots + \tilde{s}_{r-1} + \frac{\tilde{s}_r}{\tilde{x}^2}$ |
| $N_f = 0, 1$ SQCDs | $[-\hbar^2\partial_{\tilde{x}}^2 + Q_0(\tilde{x})]\Psi = 0,$<br>$Q_0(\tilde{x}) = -\tilde{x}^N + \sum_{i=2}^{N}\tilde{u}_i\tilde{x}^{N-i}$, $N := N_c + N_f$ |
| $N_f = 2$ SQCDs | $[-\hbar^2\partial_{\tilde{x}}^2 + Q_0(\tilde{x})]\Psi = 0,$<br>$Q_0(\tilde{x}) = -\sum_{i=0}^{N_c+1}\tilde{u}_i e^{(N_c+1-i)\tilde{x}}$, $\tilde{u}_0 = 1$ |
| $N_f = 2n(n > 1)$with type B scaling | $[-\hbar^2\partial_{\tilde{x}}^2 + Q_0(\tilde{x})]\Psi = 0,$<br>$Q_0 = -\sum_{i=0}^{N_c}\tilde{u}_i\tilde{x}^{\frac{N_c-n+1}{1-n}}$ |
| $N_f = 2n + 1(n \geq 1)$<br>$N_f = 2n(n \geq 2)$ with type A scaling | $\left[\frac{1}{2}(e^{-i\hbar\partial_{\tilde{x}}} + e^{i\hbar\partial_{\tilde{x}}/2}\tilde{G}(\tilde{x})e^{i\hbar\partial_{\tilde{x}}/2}) - C(\tilde{x})\right]\Psi = 0$<br>$\tilde{C} = \sum_{i=0}^{n}\tilde{u}_i\tilde{x}^{n-i}$, $\tilde{G} = \prod_{a=1}^{2n+1}(\tilde{x} + \tilde{c}_a)$<br>where $\sum_{a=1}^{2n+1}\tilde{c}_a = 0$. |

## 6.6 Quantum SW periods and the Picard-Fuchs Operators

So far we studied the quantum SW curves and their WKB solution up to the order of $\hbar^4$. In this section, we consider the quantum SW periods which are obtained by the integral over a 1-cycle on the SW curve. It is however difficult to evaluate the quantum corrections to the SW periods by performing integration directly. It is observed that the quantum corrections are shown to be related by the differential operators with respect to the moduli parameters [33], [34], [41], [48], [40], [50], [37], [56]:

$$\tilde{p}_{2n} = \tilde{\mathcal{O}}_{2n}\tilde{p}_0. \tag{6.6.1}$$

Here, the $\tilde{\mathcal{O}}_{2n}$ are differential operators with respect to moduli parameters $\tilde{u}_i$. We call this differential operator the Picard-Fuchs (PF) operator. By finding such relations, the quantum corrections can be evaluated from the classical SW periods. We can use this formula for the investigation of the quantum SW periods at strong coupling. In the following, we will find such operators for the AD theories. In this chapter, we construct the PF operators for $A_r$ type AD theories in detail. For other AD theories, the PF operators can be seen in Appendix C.2.

We can use the PF operator to investigate the universality of the AD theories. We recall that two different UV gauge theories to belong to the same universality class if these theories become the same AD theory if we take the scaling limit. We can check the universality of AD theories corresponding to different quantum SW curves by comparing the PF operators.

### 6.6.1 AD theory of $A_r$ theory

Let us consider the AD theories obtained from $A_r$ type gauge theories. In this AD theory, the quantum SW period $\tilde{p}_0$ in the scaling limit is given as (6.4.3):

$$\tilde{p}_0 = -\sqrt{2W}, \qquad W(\tilde{x}, \tilde{u}_i) = \tilde{x}^{r+1} - \tilde{u}_2\tilde{x}^{r-1} - \cdots - \tilde{u}_{r+1}. \tag{6.6.2}$$

To study the PF operators, we will explain some formulas for the derivatives of $W$ with respect to the moduli parameters. We first note that

$$\partial_{\tilde{u}_i} W = -x^{r+1-i}, \qquad \partial_{\tilde{u}_i} W^{1/2} = \frac{\partial_{\tilde{u}_i} W}{2W^{1/2}}. \tag{6.6.3}$$

This numerator of $\partial_{\tilde{u}_i} W^{1/2}$ does not depend on $\tilde{u}_i$'s. Thus, we obtain

$$\partial_{\tilde{u}_{j_1}} \cdots \partial_{\tilde{u}_{j_n}} W^{1/2} = \frac{(-1)^{n-1}(2n-3)!!}{2^n} \frac{\partial_{\tilde{u}_{j_1}} W \cdots \partial_{\tilde{u}_{j_n}} W}{W^{n-\frac{1}{2}}}. \tag{6.6.4}$$

Let us now calculate $\tilde{\mathcal{O}}_2$ by using (6.6.4). From the first equation of (6.6.3), we obtain

$$\begin{aligned}
\partial_{\tilde{x}}^2 W =& (r+1)r\tilde{x}^{r-1} - \sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j \tilde{x}^{r-1-j} \\
=& -(r+1)r\partial_{\tilde{u}_2} W + \sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j \partial_{\tilde{u}_{j+2}} W.
\end{aligned} \tag{6.6.5}$$

Then, by using (6.6.5), $\partial_{\tilde{u}_{r+1}} W = -1$ and $\partial_{\tilde{u}_i}\partial_{\tilde{u}_j} W^{1/2} = -\frac{\partial_{\tilde{u}_i} W \partial_{\tilde{u}_j} W}{4W^{3/2}}$, we obtain

$$\begin{aligned}
\tilde{p}_2 =& \frac{1}{48\sqrt{2}} \frac{\partial_{\tilde{x}}^2 W}{W^{3/2}} \\
=& \frac{1}{48\sqrt{2}} \frac{(r+1)r\partial_{\tilde{u}_2} W \partial_{\tilde{u}_{r+1}} W + \sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j \partial_{\tilde{u}_{j+2}} W \partial_{\tilde{u}_{r+1}} W}{W^{3/2}} \\
=& \frac{1}{12\sqrt{2}} \left( -(r+1)r\partial_{\tilde{u}_2}\partial_{\tilde{u}_{r+1}} + \sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j \partial_{\tilde{u}_{j+2}}\partial_{\tilde{u}_{r+1}} \right) W^{1/2} \\
=& -\frac{1}{24} \left( -(r+1)r\partial_{\tilde{u}_2}\partial_{\tilde{u}_{r+1}} + \sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j \partial_{\tilde{u}_{j+2}}\partial_{\tilde{u}_{r+1}} \right) \tilde{p}_0.
\end{aligned} \tag{6.6.6}$$

This is the PF operator $\mathcal{O}_2$ for $\tilde{p}_2$.

Next, let us consider the PF operator $\mathcal{O}_4$ for $\tilde{p}_4$. In this case, we have to rewrite

$(\partial_{\tilde{x}}^2 W)^2/W^{7/2}$ and $\partial_{\tilde{x}}^4 W/W^{5/2}$ by some derivatives of $\tilde{p}_0$ with respect to $\tilde{u}_i$. We find

$$\begin{aligned}\frac{(\partial_{\tilde{x}}^2\tilde{W})^2}{\tilde{W}^{7/2}} &= \frac{(r+1)^2r^2\partial_{\tilde{u}_2}\tilde{W}\partial_{\tilde{u}_2}\tilde{W}}{\tilde{W}^{7/2}} - 2(r+1)r\sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j\frac{\partial_{\tilde{u}_2}\tilde{W}\partial_{\tilde{u}_{j+2}}\tilde{W}}{\tilde{W}^{7/2}}\\
&\quad+\sum_{j=2}^{r-1}\sum_{k=2}^{r-1}(r+1-k)(r-k)(r+1-j)(r-j)\tilde{u}_j\tilde{u}_k\frac{\partial_{\tilde{u}_{j+2}}\tilde{W}\partial_{\tilde{u}_{k+2}}\tilde{W}}{\tilde{W}^{7/2}}\\
&= -\frac{16}{15}\left\{(r+1)^2r^2\partial_{\tilde{u}_2}^2 - 2(r+1)r\sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j\partial_{\tilde{u}_2}\partial_{\tilde{u}_{j+2}}\right.\\
&\quad\left.+\sum_{j=2}^{r-1}\sum_{k=2}^{r-1}(r+1-k)(r-k)(r+1-j)(r-j)\tilde{u}_j\tilde{u}_k\partial_{\tilde{u}_{j+2}}\partial_{\tilde{u}_{k+2}}\right\}\partial_{\tilde{u}_{r+1}}^2\tilde{W}^{1/2},\end{aligned}\tag{6.6.7}$$

$$\begin{aligned}\partial_{\tilde{x}}^4\tilde{W} &= (r+1)r(r-1)(r-2)\tilde{x}^{r-3} - \sum_{j=2}^{r-3}(r+1-j)(r-j)(r-1-j)(r-2-j)\tilde{u}_j\tilde{x}^{r-3-j}\\
&= -\frac{(r+1)!}{(r-3)!}\partial_{\tilde{u}_4}W + \sum_{j=2}^{r-3}\frac{(r+1-j)!}{(r-3-j)!}\tilde{u}_j\partial_{\tilde{u}_{j+4}}W,\end{aligned}\tag{6.6.8}$$

$$\frac{\partial_{\tilde{x}}^4\tilde{W}}{\tilde{W}^{5/2}} = \frac{8}{3}\left\{-\frac{(r+1)!}{(r-3)!}\partial_{\tilde{u}_4} + \sum_{j=2}^{r-3}\frac{(r+1-j)!}{(r-3-j)!}\tilde{u}_j\partial_{\tilde{u}_{j+4}}\right\}\partial_{\tilde{u}_{r+1}}^2\tilde{W}^{1/2}.\tag{6.6.9}$$

Combining (6.6.7) and (6.6.9), we obtain the PF operator for $\tilde{p}_4$:

$$\begin{aligned}\tilde{p}_4 &= \frac{7}{1536\cdot 2\sqrt{2}}\left(-\frac{16}{15}\right)\left\{(r+1)^2r^2\partial_{\tilde{u}_2}^2 - 2(r+1)r\sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j\partial_{\tilde{u}_2}\partial_{\tilde{u}_{j+2}}\right.\\
&\quad\left.+\sum_{j=2}^{r-1}\sum_{k=2}^{r-1}(r+1-k)(r-k)(r+1-j)(r-j)\tilde{u}_j\tilde{u}_k\partial_{\tilde{u}_{j+2}}\partial_{\tilde{u}_{k+2}}\right\}\partial_{\tilde{u}_{r+1}}^2\tilde{W}^{1/2}\\
&\quad-\frac{1}{768\cdot 2\sqrt{2}}\cdot\frac{8}{3}\left\{-\frac{(r+1)!}{(r-3)!}\partial_{\tilde{u}_4} + \sum_{j=2}^{r-3}\frac{(r+1-j)!}{(r-3-j)!}\tilde{u}_j\partial_{\tilde{u}_{j+4}}\right\}\partial_{\tilde{u}_{r+1}}^2\tilde{W}^{1/2}\\
&= \frac{7}{1536\cdot 4}\left(\frac{16}{15}\right)\left\{(r+1)^2r^2\partial_{\tilde{u}_2}^2 - 2(r+1)r\sum_{j=2}^{r-1}(r+1-j)(r-j)\tilde{u}_j\partial_{\tilde{u}_2}\partial_{\tilde{u}_{j+2}}\right.\\
&\quad\left.+\sum_{j=2}^{r-1}\sum_{k=2}^{r-1}(r+1-k)(r-k)(r+1-j)(r-j)\tilde{u}_j\tilde{u}_k\partial_{\tilde{u}_{j+2}}\partial_{\tilde{u}_{k+2}}\right\}\partial_{\tilde{u}_{r+1}}^2\tilde{p}_0\\
&\quad+\frac{1}{768\cdot 4}\cdot\frac{8}{3}\left\{-\frac{(r+1)!}{(r-3)!}\partial_{\tilde{u}_4} + \sum_{j=2}^{r-3}\frac{(r+1-j)!}{(r-3-j)!}\tilde{u}_j\partial_{\tilde{u}_{j+4}}\right\}\partial_{\tilde{u}_{r+1}}^2\tilde{p}_0.\end{aligned}\tag{6.6.10}$$

Let us write down the formulas for $r = 2, 3$. For $r = 2$, the PF operator $\mathcal{O}_2$ is given as

$$\mathcal{O}_2 = \frac{1}{4}\partial_{\tilde{u}_2}\partial_{\tilde{u}_3}. \tag{6.6.11}$$

For $r = 3$, we obtain

$$\mathcal{O}_2 = \frac{1}{2}\partial_{\tilde{u}_2}\partial_{\tilde{u}_4} - \frac{\tilde{u}_2}{12}\partial^2_{\tilde{u}_4}. \tag{6.6.12}$$

We can also obtain the PF operators $\mathcal{O}_2$ and $\mathcal{O}_4$ for $D_r$ type and general SQCDs. We explain these cases in Appendix C.2.

## 6.7 Universalities

Let us finally check that the quantization of the SW curve is consistent with the scaling limit or not. We have seen some examples of the universality of AD theories at the classical level in chapter 5. We will compare the quantum SW periods in the scaling limit which are in the same universality class in the classical level, by using the PF operators [46], [47]. We will see that the universality in the AD theories at the classical level is preserved by the quantum corrections.

### $SU(2)$ $N_f = 1$ and $A_2$

We first consider the universality of $SU(2)$ $N_f = 1$ and $A_2 = SU(3)$ theories. In these two UV theories, the SW curve and the SW differential in the scaling limit are the same forms. Thus, if we identify these coefficients of the SW curve and the SW differential in the scaling limit in the same way as the classical case of the universality, we can verify the universality also at the quantum level.

### $SU(2)$ $N_f = 2$ and $A_3$

Let us next consider the universality for $SU(2)$ $N_f = 2$ and $A_3$ type gauge theory. In this case, the quantum SW curves in the scaling limit are different. By using the result of (6.6.6), the PF operator for AD theory obtained from $A_3$ type theory is given as

$$\tilde{p}_2 = \left(\frac{1}{2}\frac{\partial}{\partial\tilde{t}_2}\frac{\partial}{\partial\tilde{t}_4} - \frac{\tilde{t}_2}{12}\frac{\partial^2}{\partial\tilde{t}_4^2}\right)\tilde{p}_0, \tag{6.7.1}$$

if we rewrite the coefficients in the AD theory obtained from the $A_3$ theory as $\tilde{t}_i := \tilde{u}_i$. On the other hand, the PF operator for the $SU(2)$ $N_f = 2$ case is given as (C.2.36):

$$\tilde{p}_2 = \left(\frac{3}{4}\partial_{\tilde{u}_1}\partial_{\tilde{u}_2} + \frac{1}{3}\tilde{u}_1\partial_{\tilde{u}_2}^2 + \frac{1}{12}\tilde{u}_2\partial_{\tilde{u}_3}\partial_{\tilde{u}_2}\right)\tilde{p}_0. \tag{6.7.2}$$

If we note $\tilde{p}_0 = \sqrt{\sum_{i=0}^{3}\tilde{u}_i e^{(3-i)\xi}}$, we obtain the following equation for AD theory obtained from $SU(2)$ $N_f = 2$ SQCD:

$$(3\partial_{\tilde{u}_1}\partial_{\tilde{u}_2} + 2\tilde{u}_1\partial_{\tilde{u}_2}^2 + \tilde{u}_2\partial_{\tilde{u}_2}\partial_{\tilde{u}_3})\tilde{p}_0 = 0. \tag{6.7.3}$$

Thus, we can write the PF operator for $SU(2)$ $N_f = 2$ SQCD (6.7.2) as following:

$$\tilde{p}_2 = \left(\frac{1}{2}\partial_{\tilde{u}_1}\partial_{\tilde{u}_2} + \frac{1}{6}\tilde{u}_1\partial_{\tilde{u}_2}^2\right)\tilde{p}_0. \tag{6.7.4}$$

Then, if we identify these coefficients as (5.9.6), these two PF operators (6.7.1) and (6.7.4) are the same. Thus, we can conclude that these two gauge theories belong to the same universality class.

### $SU(2)$ $N_f = 2$ and $D_3$

Next we consider $SU(2)$ $N_f = 2$ and $D_3$ type gauge theory. In this case, the quantum SW curves of these theories are different. By using (C.2.7), the PF operator for the $D_3$ theory is given as

$$\tilde{p}_2(A) = \frac{1}{24}(12\partial_{\tilde{s}_1}\partial_{\tilde{s}_2} + 2\tilde{s}_1\partial_{\tilde{s}_2}^2 + 6\tilde{s}_3\partial_{\tilde{s}_3}^2)\tilde{p}_0 - \frac{A}{2}\partial_{\tilde{s}_3}\tilde{p}_0. \tag{6.7.5}$$

Here, we take the quantum correction for the quantum SW curve in the scaling limit for $D_3$ type theory as $A \in \mathbb{C}$.

On the other hand, the PF operator for the $SU(2)$ $N_f = 2$ SQCD is given as (C.2.36). In the AD theory obtained from $SU(2)$ $N_f = 2$ SQCD, $\tilde{p}_0 = \sqrt{\sum_{i=0}^{3}\tilde{u}_i e^{(3-i)\xi)}}$, therefore we obtain the following equation:

$$\tilde{u}_2\partial_{\tilde{u}_2}\partial_{\tilde{u}_3}\tilde{p}_0 = \left\{(4k-3)\partial_{\tilde{u}_1}\partial_{\tilde{u}_2} + (2k-2)\tilde{u}_1\partial_{\tilde{u}_2}^2 - 2k\tilde{u}_3\partial_{\tilde{u}_3}^2 - k\partial_{\tilde{u}_3}\right\}\tilde{p}_0, \qquad k \in \mathbb{R}. \tag{6.7.6}$$

Then, if we substitute this equation in (C.2.36), the quantum SW period for AD theory obtained from the $SU(2)$ $N_f = 2$ becomes

$$\tilde{p}_2 = \frac{1}{6}\left\{(2k+3)\partial_{\tilde{u}_1}\partial_{\tilde{u}_2} + (k+1)\tilde{u}_1\partial_{\tilde{u}_2}^2 - k\tilde{u}_3\partial_{\tilde{u}_3}^2 - \frac{k}{2}\partial_{\tilde{u}_3}\right\}\tilde{p}_0. \tag{6.7.7}$$

If we take $k = -3/4$, the quantum SW period for AD theory obtained from the $SU(2)$ $N_f = 2$ can be written as

$$\tilde{p}_2 = \frac{1}{24}\left\{6\partial_{\tilde{u}_1}\partial_{\tilde{u}_2} + \tilde{u}_1\partial^2_{\tilde{u}_2} + 3\tilde{u}_3\partial^2_{\tilde{u}_3}\right\}\tilde{p}_0 + \frac{1}{16}\partial_{\tilde{u}_3}\tilde{p}_0. \tag{6.7.8}$$

Therefore, if we take $A = -1/8$, and identify the coefficients as (5.9.9), we can conclude that these two UV gauge theories belong to the same universality class.

### $SU(3)$ $N_f = 0$ and $SU(2)$ $N_f = 1$

In general, the SW curves and the SW differentials for the $SU(N_c+1)$ SYM and $SU(N_c)$ $N_f = 1$ SQCDs are equivalent. Thus, these two gauge theories belong to the same universality class. We can also obtain universality between $SU(N_c+1)$ $N_f = 0$ and $SU(N_c)$ $N_f = 1$.

### $SU(3)$ $N_f = 1$ and $SU(2)$ $N_f = 2$

The PF equations for $SU(3)$ $N_f = 1$ and $SU(2)$ $N_f = 2$ are given as (C.2.43) and (C.2.36). Thus, if we use the identification in (5.9.17), these two PF operators become the same. Thus, we can conclude that these two gauge theories belong to the same universality class.

### $SU(3)$ $N_f = 2$ and $D_4$

Let us next consider the universality between $SU(3)$ $N_f = 2$ and $D_4$ theory. The PF operator for the $SU(3)$ $N_f = 2$ theory is given as (C.2.45):

$$\tilde{p}_2 = \frac{1}{12}\left\{16\partial_{\tilde{u}_1}\partial_{\tilde{u}_3} + 9\tilde{u}_1\partial_{\tilde{u}_2}\partial_{\tilde{u}_3} + 4\tilde{u}_2\partial^2_{\tilde{u}_3} + \tilde{u}_3\partial_{\tilde{u}_4}\partial_{\tilde{u}_3}\right\}\tilde{p}_0. \tag{6.7.9}$$

The PF operator for the $D_4$ theory is given as

$$\tilde{p}_2 = \frac{1}{12}\left\{15\partial_{\tilde{s}_1}\partial_{\tilde{s}_3} + 6\tilde{s}_1\partial_{\tilde{s}_2}\partial_{\tilde{s}_3} + \tilde{s}_2\partial^2_{\tilde{s}_3} + 3\tilde{s}_4\partial^2_{\tilde{s}_4}\right\}\tilde{p}_0 - \frac{A}{2}\partial_{\tilde{s}_4}\tilde{p}_0, \tag{6.7.10}$$

where we take $r = 4$ in (C.2.7). Let us note that the $\tilde{p}_0$ for the AD theory of $SU(3)$ $N_f = 2$ theory is $\tilde{p}_0 = \sqrt{e^{4\xi} + \tilde{u}_1 e^{3\xi} + \tilde{u}_2 e^{2\xi} + \tilde{u}_3 e^{\xi} + \tilde{u}_4}$, $\tilde{p}_0$ satisfies the following equation:

$$\tilde{u}_3\partial_{\tilde{u}_4}\partial_{\tilde{u}_3}\tilde{p}_0 = (6k-4)\partial_{\tilde{u}_1}\partial_{\tilde{u}_3} + (4k-3)\tilde{u}_1\partial_{\tilde{u}_2}\partial_{\tilde{u}_3} + (2k-2)\tilde{u}_2\partial^2_{\tilde{u}_3} - 2k\tilde{u}_4\partial^2_{\tilde{u}_4} - k\partial_{\tilde{u}_4}. \tag{6.7.11}$$

If we take $k = -3/4$ and $A = -3$, we can confirm that the PF operators (6.7.9) and (6.7.10) are the same. Thus, these two gauge theories belong to the same universality class.

### $SU(3)$ $N_f = 3$ and $SU(2)$ $N_f = 3$

As I explained in the chapter on the quantum SW curve, the quantum SW curve for AD theories obtained from the $N_f = 2n + 1$ SQCDs is completely determined by the flavor symmetry of the gauge theory. Thus, the quantum SW curve for the $SU(3)$ $N_f = 3$ and $SU(2)$ $N_f = 3$ are the same. Therefore, we can verify that these two gauge theories belong to the same universality class. In the same way, $SU(3)$ $N_f = 5$ and $SU(N_c)$ $N_f = 5$ theories belong to the same universality class.

### $SU(3)$ $N_f = 4$ (A) and $SU(2)$ $N_f = 4$ (A)

The SW curve for the AD theories obtained from the $N_f = 2n(n \geq 2)$ SQCDs with type A scaling are only dependent on the flavor symmetry (or $n$). Therefore, these two quantum SW curve in the scaling limit are the same. Then, these two gauge theories belong to the same universality class.

To summarize, we obtain the quantum SW periods for the $SU(N_c)$ $N_f$ SQCDs and the $A_r$ type and $D_r$ type theories. We also introduce the PF operators to compare quantum SW periods for two different AD theories. We confirmed that quantization of the SW curve preserves the universality of AD theories. But we should also check the universality of AD theory for $SU(2)$ $N_f = 3$ and $D_4$ theory, which are in the same universality class in the classical level [66].

# Chapter 7

# Conclusion and Discussion

In this thesis, we studied the low energy effective theories of $\mathcal{N} = 2$ supersymmetric gauge theories. The low energy effective action can be determined by the SW theory. In the SW theory, there exists S-duality which relates the weak and strong coupling region. However, we can calculate the effective actions or equivalently prepotentials indirectly through the SW periods. On the other hand, we can calculate the prepotential directly from the Nekrasov partition function which is defined in the $\Omega$ background. But it is difficult to find the S-dualities on the $\Omega$ background, so we can only determine the Nekrasov partition function (which is the partition function on the $\Omega$ background) in the weak coupling region. The Nekrasov partition function has a mathematically rich structure. Then it is desirable to study physics in the $\Omega$-background at strong coupling.

Thus, the purpose of this thesis is to determine the quantum SW periods in the strong coupling region. For this purpose, we especially consider the quantum SW periods in the AD theories defined at the superconformal points. To calculate the $\Omega$-background deformation to the SW periods at the superconformal point, we use the method of quantum SW curves [33], [34]. In [33] and [34], the quantum SW curves for the pure SYM are introduced, and the quantum SW periods can be determined as WKB solutions of the quantum SW curves. This idea was generalized to $SU(2)$ SQCDs. It is confirmed that these quantum SW periods are equivalent to that of the quantum SW periods obtained from the Nekrasov partition function in the weak coupling region. Thus, in this thesis, we studied more general $SU(N_c)$ SQCD and we define the quantum SW periods by the method of the quantum SW curves [46], [47], [57].

We can define the quantum SW periods in the AD theories by taking the scaling limit of the quantum SW periods of the UV theories. But we can also obtain the quantum SW periods in the AD theories by using the quantum SW curve in the scaling limit. Therefore, we require that the quantization of the SW curves and the scaling limit is commutable to determine the quantum SW curve in the scaling limit. Then, we can see that in $A_r$ type gauge theories, the scaling limit and the quantization of the SW curve commute each other. But in the $D_r$ type gauge theories, the scaling limit and the quantization do not commute, therefore we should add the quantum correction for the quantum SW curve in the scaling limit. In the same way, we should add a quantum correction for the SW curve in the scaling limit in $SU(N_c)$ SQCDs.

These new quantum corrections for the SW curve in the scaling limit is important. This quantum correction can be interpreted as a mass shift in the QCD, therefore this means the mass term obtains the quantum correction in the scaling limit. Similar quantum correction is done in the calculations of the Nekrasov partition function [63]. This mass shift may change the flavor symmetry.

For the consistency check, we also consider the universality of the AD theories. We should check that the universality of the AD theories are preserved by the quantization of the SW curve, in the $A_r$ type, $D_r$ type gauge theories and the $SU(N_c)$ SQCDs. To do this, we use the PF equations satisfied by the classical periods, which are the differential operators of the moduli parameters act on $\tilde{p}_0$. Then, we compare these PF operators and we can confirm that the quantum SW periods in the scaling limit are the same if these two gauge theories belong to the same universality class in classical theories.

The AD theories are also studied in the viewpoint of Schur Index [61]. It is also obtained by compactification of the six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theory on a punctured Riemann surface [26], [27], [28], and specified by [29], [30], [31], [32].

Therefore, we can conclude that the quantization of the SW curve is consistent with the scaling limit. But we do not know that the quantum SW periods obtained from the quantum SW curves in the $A_r$ type and $D_r$ type gauge theories are equivalent to the quantum SW periods calculated by the Nekrasov partition function in the weak coupling region. Thus, we should look for the Nekrasov function at strong coupling and check this consistency condition.

To generalize the quantum SW curve in more general points on the strong coupling region, we need to consider this consistency check on general points on the strong coupling region. For example, at the massless dyon or massless monopole points, the quantum SW periods are calculated by [48], [45], [49]. But we should check in more general points.

In the above, we only consider the quantum SW periods in the NS limit of the $\Omega$ background. We can also consider other limits of the $\Omega$ background. If we take the two parameters of the $\Omega$ background in the same value $\epsilon_1 = \epsilon_2$, the quantum SW periods can be calculated by using the topological string [50], [51]. But we want to know more general quantum SW periods determined on the general $\Omega$ background. These relations would be clarified if we understand the deformed theories in general $\Omega$-background.

# Appendix A

# Supersymmetric Gauge Theories

In this Appendix, we explain the basics of supersymmetry following [58]. In this chapter, we explain $\mathcal{N}=1$ supersymmetric gauge theory.

First, I will write the definition of the $\mathcal{N}$-extended supersymmetric algebra. Supersymmetry is defined as symmetry transformation which exchanges bosonic fields and fermionic fields. So, if we can define such symmetry, the generator of supersymmetry must have Grassmannian statistics and have spinor index. Thus we can define supersymmetric generator $Q^I_\alpha, \overline{Q}_{\dot{\alpha}J}$ where $\alpha, \dot{\alpha} = 1, 2$ are spinor indices and $I, J = 1, \cdots, \mathcal{N})$ are used to label different supercharges. They satisfy the following anti-commutation relations:

$$\begin{aligned} \{Q^I_\alpha, \overline{Q}_{\dot{\alpha}J}\} =& 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J, \qquad \{Q^I_\alpha, Q^J_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z^{IJ}, \qquad \{\overline{Q}_{\dot{\alpha}I}, \overline{Q}_{\dot{\beta}J}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{IJ}, \\ \{(-1)^F, Q^I_\alpha\} =& \{(-1)^F, \overline{Q}_{\dot{\alpha}J}\} = 0. \end{aligned} \tag{A.0.1}$$

Here the $(-1)^F$ is the fermion number operator. $Z^{IJ} = -Z^{JI}$ is called the central charge, which commutes with other supersymmetry generators.

## A.1 $\mathcal{N}=1$ Supersymmetric Algebra and Lagrangian

In this section, we explain the $\mathcal{N}=1$ supersymmetry algebra and construct the $\mathcal{N}=1$ supersymmetric Lagrangian.

### A.1.1 $\mathcal{N}=1$ Super multiplets

Let us consider $\mathcal{N}=1$ case. In this case, there is no central charge. The supersymmetric algebra is the following:

$$[\xi Q, \overline{\xi} Q] = 2\xi\sigma^\mu\overline{\xi}P_\mu, \qquad [\xi Q, \xi Q] = [\overline{\xi Q}, \overline{\xi Q}] = 0, \qquad [P^\mu, \xi Q] = [P^\mu, \overline{\xi Q}] = 0, \quad \text{(A.1.1)}$$

where we write the supersymmetric generator as $Q_\alpha, \overline{Q}_{\dot{\alpha}}$, and the $\xi$ is a Grassmannian constant spinor, and use the following conventions:

$$\xi Q = \xi^\alpha Q_\alpha, \qquad \overline{\xi Q} = \overline{\xi}_{\dot{\alpha}}\overline{Q}^{\dot{\alpha}}. \quad \text{(A.1.2)}$$

Here, $P^\mu$ is the translation operator.

Let us construct $\mathcal{N}=1$ symmetric Lagrangians. We first construct supermultiplets, which are a set of fields such that the supersymmetric transformation can be a closed operator on these fields. The $\mathcal{N}=1$ supersymmetric Lagrangian is constructed from supersymmetric multiplets. The supersymmetric transformation for a field $\phi$ is defined as

$$\delta_\xi\phi := (\xi Q + \overline{\xi Q})\phi, \quad \text{(A.1.3)}$$

where $\xi$ is Grassmannian constant parameter with spinor index. If we write $Q_\alpha\phi = \lambda_\alpha$ and $\overline{Q}^{\dot{\alpha}}\phi = \overline{\lambda}^{\dot{\alpha}}$, we obtain

$$\delta_\xi\phi := \xi^\alpha\lambda_\alpha + \overline{\xi}_{\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}}. \quad \text{(A.1.4)}$$

These fields $\lambda_\alpha$ and $\overline{\lambda}^{\dot{\alpha}}$ satisfy different statistic from $\phi$. We can transform $\lambda$ and $\overline{\lambda}$ in the same way. Notice that $Q_\alpha$ and $\overline{Q}^{\dot{\alpha}}$ anti-commute themselves, supermultiplets constructed by these operations include at most $2^4$ fields. The fields are conveniently organized in the following way:

$$\begin{aligned} F(x,\theta,\overline{\theta}) =& f(x) + \theta\phi(x) + \overline{\theta}\overline{\chi}(x) \\ &+ \theta\theta m(x) + \overline{\theta\theta}n(x) + \theta\sigma^\mu\overline{\theta}v_\mu(x) \\ &+ \theta\theta\overline{\theta}\overline{\lambda}(x) + \overline{\theta\theta}\theta\psi(x) + \theta\theta\overline{\theta\theta}d(x), \end{aligned} \quad \text{(A.1.5)}$$

where $\theta$ and $\overline{\theta}$ are Grassmannian constants. We call the fields written as the form (A.1.5) are called superfields. Conversely, we can check that all superfields which are written as the form (A.1.5) give supermultiplets. Thus, supermultiplets are one to one correspondence between superfields.

### A.1.2 Super-Covariant Derivative

We constructed supermultiplets by superfields. In this subsection, we will define the supersymmetric transformations that act on the superfields. Supersymmetric transformations on the superfields are defined as follows:

$$\begin{aligned}\delta_\xi F(x,\theta,\overline{\theta}) =&(\xi Q+\overline{\xi Q})F\\ :=&\delta_\xi f(x)+\theta\delta_\xi\phi(x)+\overline{\theta}\delta_\xi\overline{\chi}(x)\\ &+\theta\theta\delta_\xi m(x)+\overline{\theta\theta}\delta_\xi n(x)+\theta\sigma^\mu\overline{\theta}\delta_\xi v_\mu(x)\\ &+\theta\theta\overline{\theta}\delta_\xi\overline{\lambda}(x)+\overline{\theta\theta}\theta\delta_\xi\psi(x)+\theta\theta\overline{\theta\theta}\delta_\xi d(x).\end{aligned}\tag{A.1.6}$$

Here, $\delta_\xi$ operation on each component field is a supersymmetric transformation on the supermultiplet. Therefore, the representation of $\xi Q+\overline{\xi Q}$ on superspace $(x^\mu,\theta,\overline{\theta})$ is the follows:

$$\xi Q+\overline{\xi Q}=\xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha}-i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu\right)+\overline{\xi}_{\dot{\alpha}}\left(\frac{\partial}{\partial\overline{\theta}}-i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu\right).\tag{A.1.7}$$

We also define the super-covariant derivatives that anti-commute with supersymmetric transformations $D$ and $\overline{D}$:

$$D_\alpha=\frac{\partial}{\partial\theta^\alpha}+i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu,\qquad \overline{D}_{\dot{\alpha}}=-\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}-i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.\tag{A.1.8}$$

These commutation relations are as

$$\{D_\alpha,\overline{D}_{\dot{\alpha}}\}=-2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu,\qquad \{D_\alpha,D_\beta\}=\{\overline{D}_{\dot{\alpha}},\overline{D}_{\dot{\beta}}\}=0,\tag{A.1.9}$$

and $D$ and $Q$ anti-commute:

$$\{D_\alpha,Q_\beta\}=\{D_\alpha,\overline{Q}_{\dot{\alpha}}\}=\{\overline{D}_{\dot{\alpha}},D_\alpha\}=\{\overline{D}_{\dot{\alpha}},\overline{Q}_{\dot{\beta}}\}=0.\tag{A.1.10}$$

### A.1.3 Chiral Multiplet

Chiral multiplet is a superfield that satisfies the condition $\overline{D}_{\dot{\alpha}}\Phi=0$. If we introduce $y^\mu=x^\mu+i\theta\sigma^\mu\overline{\theta}$, general chiral multiplets can be written as

$$\Phi=A(y)+\sqrt{2}\theta\psi(y)+\theta\theta F(y).\tag{A.1.11}$$

In the same way, the anti-chiral field $\Phi^\dagger$ is defined by $D_\alpha \Phi^\dagger = 0$, and which can be expanded as

$$\Phi^\dagger = A^*(y^\dagger) + \sqrt{2}\overline{\theta\psi}(y^\dagger) + \overline{\theta\theta} F^*(y^\dagger), \qquad y^{\dagger\mu} = x^\mu - i\theta\sigma^\mu\overline{\theta}. \tag{A.1.12}$$

Note that the general product of two chiral fields is a chiral superfield, the arbitrary function of chiral superfields is a chiral superfield:

$$\begin{aligned}\mathcal{W}(\Phi_i) =& \mathcal{W}(A_i + \sqrt{2}\theta\psi_i + \theta\theta F_i)\\ =& \mathcal{W}(A_i) + \frac{\partial \mathcal{W}}{\partial A_i}\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial \mathcal{W}}{\partial A_i}F_i - \frac{1}{2}\frac{\partial^2 \mathcal{W}}{\partial A_i \partial A_j}\psi_i\psi_j\right).\end{aligned} \tag{A.1.13}$$

$\mathcal{W}$ is called superpotential.

### A.1.4 Vector Superfield

Let us next to explain the vector superfield. The $\mathcal{N} = 1$ vector superfields are defined as superfields that satisfy the following condition:

$$V = V^\dagger. \tag{A.1.14}$$

We expand the vector superfields in the following form:

$$\begin{aligned}V(x,\theta,\overline{\theta}) =& C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x)\\ &+ \frac{i}{2}\theta\theta\left[M(x) + iN(x)\right] - \frac{i}{2}\overline{\theta\theta}\left[M(x) - iN(x)\right]\\ &- \theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\overline{\theta\theta}\left[\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi(x)\right]\\ &- i\overline{\theta\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)\right] + \frac{1}{2}\theta\theta\overline{\theta\theta}\left[D(x) + \frac{1}{2}\Box C(x)\right].\end{aligned} \tag{A.1.15}$$

All these component fields are real valued, and there are vector field $A_\mu(x)$ as a coefficient of $\theta\sigma^\mu\overline{\theta}$. Also the combination $\Phi + \Phi^+$ has $i\partial_\mu(A - A^*)$ as a coefficient of $\theta\sigma^\mu\overline{\theta}$. Therefore, we may motivate interpretation $A_\mu$ as a gauge field. In the following, we will interpret $A_\mu$ as a gauge field.

### A.1.5 $\mathcal{N}=1$ Supersymmetric Lagrangians without gauge symmetry

Let us construct $\mathcal{N}=1$ supersymmetric Lagrangian by using the chiral superfields and vector superfields. Let us first consider products of chiral and anti-chiral superfield $\Phi_i^\dagger\Phi_j$:

$$\Phi_i^\dagger\Phi_j\Big|_{\theta^2\overline{\theta}^2} = -\frac{1}{4}A_i^\dagger\Box A_j - \frac{1}{4}\Box A_i^\dagger A_j + F_i^\dagger F_j + \frac{1}{2}\partial_\mu A_i^\dagger\partial^\mu A_j - \frac{i}{2}\psi_j\sigma^\mu\partial_\mu\overline{\psi}_i + \frac{i}{2}\partial_\mu\psi_j\sigma^\mu\overline{\psi}_i. \tag{A.1.16}$$

Dropping some total derivatives and summing over $i=j$, we obtain a free Lagrangian:

$$\mathcal{L} = \Phi_i^\dagger\Phi_i\Big|_{\theta^2\overline{\theta}^2} = \partial_\mu A_i^\dagger\partial^\mu A_i + F_i^\dagger F_i - i\overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i. \tag{A.1.17}$$

In general, supersymmetric Lagrangian including the chiral superfields can be written as follows:

$$\mathcal{L} = \int d^4\theta K(\Phi,\Phi^\dagger) + \int d^2\mathcal{W}(\Phi) + \int d^2\overline{\theta}^2\overline{\mathcal{W}}(\Phi^\dagger). \tag{A.1.18}$$

Note that $\int d^2\theta\theta^2 = 1$, $\int d^2\overline{\theta\theta}^2 = 1$. In terms of non-holomorphic function $K(A,A^\dagger)$ called Kähler potential, the metric on the field space is given by

$$g^{ij} = \frac{\partial^2 K}{\partial A_i\partial A_j^\dagger}. \tag{A.1.19}$$

### A.1.6 $\mathcal{N}=1$ Supersymmetric Lagrangians with gauge symmetry

Let us write the generators of a Lie group $G$ as $T^a(T^a=(T^a)^\dagger)$. Here $a=1,\ldots,\mathrm{dim}G$. The algebra is specified by the structure contants $f^{abc}$. We define vector multiplet as $V=V^aT^a$. Let us define supersymmetric field strength associated with the gauge fields as

$$\begin{aligned} W_\alpha =& \frac{1}{8}\overline{D}^2e^{2V}D_\alpha e^{-2V} \\ =& T^a\left(-i\lambda_\alpha^a + \theta_\alpha D^a - \frac{i}{2}(\sigma^\mu\overline{\sigma}^\nu\theta)_\alpha F_{\mu\nu}^a + \theta^2\sigma^\mu D_\mu\overline{\lambda}^a\right), \end{aligned} \tag{A.1.20}$$

where $V=V_aT^a$ and $T^a$ are the generators of $G$, and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^bA_\nu^c, \qquad D_\mu\overline{\lambda}^a = \partial_\mu\overline{\lambda}^a + f^{abc}A_\mu^b\overline{\lambda}^c. \tag{A.1.21}$$

Using this, we obtain

$$\frac{1}{8\pi}\mathrm{Im}\left(\tau\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha\right) = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu} * F^{a\mu\nu} + \frac{1}{g^2}\left(\frac{1}{2}D^aD^a - i\lambda^a\sigma^\mu D_\mu\overline{\lambda}\right), \tag{A.1.22}$$

where $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ is some $\mathbb{C}$-valued constant.

Next, we introduce chiral superfields $\Phi_i$, which belongs to some representation $T^a_{ij}$ of $G$:

$$\Phi := \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \cdots \\ \Phi_N \end{pmatrix}, \qquad ((1 + T^a\epsilon_a)\Phi)^i = \Phi^i + \epsilon_a T^a_{ij}\Phi^j. \tag{A.1.23}$$

Note that the product of (anti-) chiral superfields are (anti-) chiral superfield, following gauge transformed (anti-) chiral superfields is also (anti-) chiral superfield:

$$\Phi' = e^{-i\Lambda}\Phi, \qquad \Lambda = \Lambda^a T^a. \tag{A.1.24}$$

Here, $\Lambda = \Lambda^a T^a$ is a (anti-) chiral superfield in the adjoint representation of $G$. We define gauge transformation on $V = V^a T^a$ as

$$e^{-2V} \to e^{-i\Lambda^\dagger}e^{-2V}e^{i\Lambda}. \tag{A.1.25}$$

Then the supersymmetric field strength transformed by gauge symmetry as

$$W_\alpha \to W'_\alpha = e^{-i\Lambda^\dagger}W_\alpha e^{i\Lambda}. \tag{A.1.26}$$

Then, $\Phi^\dagger e^{-2V}\Phi$ is invariant in gauge transformation. Thus, the general gauge $\mathcal{N} = 1$ supersymmetric Lagrangian is the following form:

$$\begin{aligned}\mathcal{L} =& \frac{1}{8\pi}\mathrm{Im}\left(\tau\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha\right) + \int d^2\theta d^2\overline{\theta}\Phi^\dagger e^{-2V}\Phi + \int d^2\theta\mathcal{W}(\Phi) + \int d^2\overline{\theta}\overline{\mathcal{W}}(\Phi^\dagger) \\ =& -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu} * F^{a\mu\nu} - \frac{i}{g^2}\lambda^a\sigma^\mu D_\mu\overline{\lambda}^a + \frac{1}{2g^2}D^aD^a \\ &+ (\partial_\mu A - iA^a_\mu T^a A)^\dagger(\partial^\mu A - iA^{a\mu}T^a A) - \overline{\psi}\overline{\sigma}^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi) \\ &- D^a A^\dagger T^a A - \sqrt{2}A^\dagger T^a\lambda^a\psi + i\sqrt{2}\overline{\psi}T^a A\overline{\lambda}^a + F^\dagger_i F_i \\ &+ \frac{\partial\mathcal{W}}{\partial A_i}F_i + \frac{\partial\overline{\mathcal{W}}}{\partial A^\dagger_i}F^\dagger_i - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i\partial A_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\overline{\mathcal{W}}}{\partial A^\dagger_i\partial A^\dagger_j}\overline{\psi}_i\overline{\psi}_j.\end{aligned} \tag{A.1.27}$$

The auxiliary fields $F$ and $D^a$ can be eliminated by the equation of motion. This Lagrangian have scalar potential as

$$V = \sum_i \left| \frac{\partial \mathcal{W}}{\partial A_i} \right|^2 - \frac{1}{2} g^2 (A^\dagger T^a A)^2. \tag{A.1.28}$$

## A.2 $U(1)$ $\beta$-function

One can develop a perturbation theory based on the Lagrangian described above. Let us calculate the effective coupling in the $U(1)$ gauge theories described by the $\mathcal{N} = 2$ hypermultiplets. We will obtain the effective coupling by solving the renormalization group equation. The beta functions for $U(1)$ gauge theories including Weyl fermions with $U(1)$ charge $Q_f$ and complex scalar fields with $U(1)$ charges $Q_s$ are given by

$$\beta(g) \equiv \frac{g^3}{16\pi^2} \left( \sum_f \frac{2}{3} Q_f^2 + \sum_s \frac{1}{3} Q_s^2 \right). \tag{A.2.1}$$

Then, the renormalization group equation for the $U(1)$ gauge theories are

$$\mu \frac{d}{d\mu} g = \beta(g). \tag{A.2.2}$$

Here the $\mu$ is the energy scale. In particular, let us consider $U(1)$ $\mathcal{N} = 2$ supersymmetric gauge theories with a $\mathcal{N} = 2$ hypermultiplet. In $\mathcal{N} = 1$ language, this $\mathcal{N} = 2$ hypermultiplet consists of two $\mathcal{N} = 1$ chiral multiplets $M, \tilde{M}$, and $M$ and $\tilde{M}$consist of two Weyl fermions and two complex scalars. Then, the theory includes two complex fermions with charge $Q$ and two complex scalars with charge $Q$, and the $\beta$-function (A.2.1) becomes

$$\beta(g) = \frac{g^3}{16\pi^2} Q^2 \left( 2 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} \right) = \frac{g^3}{8\pi^2} Q^2. \tag{A.2.3}$$

Thus, if we substitute this $\beta(g)$ in (A.2.2), we obtain

$$\mu \frac{d}{d\mu} g = \frac{g^3}{8\pi^2} Q^2. \tag{A.2.4}$$

Let us solve this equation. If we define $\alpha = g^2/4\pi$, this equation becomes

$$\mu \frac{d}{d\mu} \left( \frac{1}{\alpha} \right) = -8\pi b, \qquad b = \frac{Q^2}{8\pi^2}. \tag{A.2.5}$$

Then, we can solve this equation as follows:

$$\frac{1}{\alpha} = -8\pi b \log\frac{\mu}{\Lambda} = -\frac{Q^2}{\pi}\log\frac{\mu}{\Lambda}. \tag{A.2.6}$$

Here, $\Lambda$ is the scaling parameter. Therefore, if we take $Q = 1$, the effective coupling is given as follows:

$$\tau = -\frac{i}{\pi}\log\frac{a}{\Lambda}. \tag{A.2.7}$$

Here, $a$ is the natural scale, and we take $\mu = a$.

In particular, if we consider $U(1)$ gauge theory around the massless monopole or dyon point, the $U(1)$ charges of the fermions and scalars in the $\mathcal{N} = 2$ hypermultiplet is one. Therefore, we can obtain the effective coupling around the massless dyon points.

# Appendix B

# Changing the Formations of the SW curves

In this appendix, we explain the representations of the SW curves. If two SW curves have the same discriminants, these two SW curves are the same. Conversely, if two SW curves are the same up to change variables, these two SW curves have the same discriminants. In this appendix, we will see transformations of the variables in the SW curves, which cause the quartic SW curves to cubic SW curves. These results are necessary to consider the universality of the AD theories.

## B.1 Quartic Curve and Cubic Curve

In the following, we will consider the correspondence between quartic curves and cubic curves, which can be transformed into each other. Let us consider the following SW curve:

$$y^2 = \prod_{i=1}^{4} (\tilde{x} - e_i) = \tilde{x}^4 + a\tilde{x}^3 + b\tilde{x}^2 + c\tilde{x} + d. \tag{B.1.1}$$

If we introduce $y = -\frac{y_1}{x_1^2}$ and $\tilde{x} = \frac{1}{x_1} + e_4$, this curve becomes

$$y_1^2 = b_1 x_1^3 + b_2 x_1^2 + b_3 x_1 + b_4. \tag{B.1.2}$$

Here, we have defined

$$b_1 = -\frac{1}{E_1E_2E_3}, \tag{B.1.3}$$

$$b_2 = \frac{E_1+E_2+E_3}{E_1E_2E_3}, \tag{B.1.4}$$

$$b_3 = -\frac{E_1E_2+E_3E_1+E_2E_3}{E_1E_2E_3}, \tag{B.1.5}$$

$$b_4 = 1. \tag{B.1.6}$$

with $E_i = \frac{1}{e_i - e_4}$. If we rescale $y_1 = \frac{y_2}{b_1}$ and $x_1 = \frac{x_2}{b_1}$, this curve becomes

$$y_2^2 = x_2^3 + c_1x_2^2 + c_2x_2 + c_3. \tag{B.1.7}$$

Here, $c_1 = b_2$, $c_2 = b_1b_3$ and $c_3 = b_4b_1^2$. If we shift $x_2 = x_3 - e_4^2 + e_4(e_1+e_2+e_3)$, we can finally obtain

$$y_2^2 = x_3^3 + d_1x_3^2 + d_2x_3 + d_3, \tag{B.1.8}$$

where

$$d_1 = b, \qquad d_2 = ac - 4d, \qquad d_3 = a^2d - 4bd + c^2. \tag{B.1.9}$$

## B.2 $D_4$ SW curve

The SW curve and the SW differential of the AD theory obtained from the $D_4$ theory are given as follows:

$$\tilde{y}^2 = \tilde{x}^6 + \tilde{s}_1\tilde{x}^4 + \tilde{s}_2\tilde{x}^2 + \tilde{s}_3 + \frac{\tilde{s}_4}{\tilde{x}^2}, \qquad \tilde{\lambda}_{SW} = \tilde{y}d\tilde{x}. \tag{B.2.1}$$

If we change the variables as $Y = \tilde{y}\tilde{x}$ and $X = \tilde{x}^2$ we can rewrite the SW curve and the SW differential as follows:

$$Y^2 = X^4 + \tilde{s}_1X^3 + \tilde{s}_2X^2 + \tilde{s}_3X + \tilde{s}_4, \qquad \tilde{\lambda}_{SW} = \frac{Y}{2X}dX. \tag{B.2.2}$$

Following the procedure of Appendix B.1, we can write this curve as

$$y^2 = x^3 + \tilde{s}_2x^2 + (-4\tilde{s}_4 + \tilde{s}_1\tilde{s}_3)x + \tilde{s}_3^2 - 4\tilde{s}_2\tilde{s}_4 + \tilde{s}_1^2\tilde{s}_4. \tag{B.2.3}$$

If we shift the variable as $x \to x + \frac{1}{12}\tilde{s}_1^2$, we obtain

$$
\begin{aligned}
y^2 =& x^3 + \frac{\tilde{s}_1^2}{4}x^2 + (\tilde{s}_1\tilde{s}_3 - 4\tilde{s}_4 - \frac{\tilde{s}_2^2}{3} + \frac{\tilde{s}_1^4}{48})x \\
&+ \tilde{s}_3^2 - \frac{\tilde{s}_1\tilde{s}_2\tilde{s}_3}{3} + \frac{\tilde{s}_1^3\tilde{s}_3}{12} - \frac{8\tilde{s}_2\tilde{s}_4}{3} + \frac{2\tilde{s}_1^2\tilde{s}_4}{3} + \frac{2\tilde{s}_2^3}{27} - \frac{\tilde{s}_1^2\tilde{s}_2^2}{36} + \frac{\tilde{s}_1^6}{1728}.
\end{aligned} \tag{B.2.4}
$$

This form of $D_4$ SW curve is useful to consider the universality of AD theories for $D_4$ and $SU(2)$ $N_f = 3$.

# Appendix C

# Quantum SW Periods

In this appendix, we will first calculate the quantum SW periods for AD theories obtained from the $A_r$ and $D_r$ theories. Then, we will calculate the PF operators for the AD theories obtained from $D_r$ theories and SQCDs and will show the explicit form of the PF operators for $SU(2)$ SQCDs and $SU(3)$ SQCDs.

## C.1 Quantum SW periods for AD theories of $A_r$ and $D_r$ theories

In this section, we will calculate the quantum SW periods for AD theories obtained from $A_r$ and $D_r$ theories from the WKB solution of the quantum SW curves. We obtain the quantum SW curve for AD theories by replacing $\xi \to -i\hbar\partial_{\tilde{x}}$ in the SW curve (5.7.8) [46], [57], which is given by (6.4.1) or adding quantum correction (6.4.9). Let us consider the following quantum SW curve:

$$\left[-\hbar^2\frac{\partial^2}{\partial^2\tilde{x}^2} - \left\{Q_0(\tilde{x},\tilde{u}_i) + \hbar^2 Q_2(\tilde{x},\tilde{u}_i)\right\}\right]\tilde{\Psi}(\tilde{x}) = 0. \tag{C.1.1}$$

This is the Schrodinger type equation. Let us find the WKB solution:

$$\tilde{\Psi}(\tilde{x}) = \exp\left(\frac{i}{\hbar}\int^{\tilde{x}} dx\tilde{P}(\tilde{x})\right), \qquad \tilde{P}(\tilde{x}) = \sum_{n=0}^{\infty}\hbar^n\tilde{p}_n(\tilde{x}). \tag{C.1.2}$$

Substitute this solution in this quantum SW curve, we obtain

$$-i\hbar\tilde{P}'(\tilde{x}) + (\tilde{P}(\tilde{x}))^2 = Q_0(\tilde{x},\tilde{u}_i) + \hbar^2 Q_2(\tilde{x},\tilde{u}_i). \tag{C.1.3}$$

Let us compare the coefficients of $\hbar$ on the left and right sides. If we consider the coefficient of $\hbar^0$, we obtain

$$\tilde{p}_0^2 = Q_0(\tilde{x}, \tilde{u}_i). \tag{C.1.4}$$

Thus we obtain,

$$\tilde{p}_0 = -\sqrt{Q_0}. \tag{C.1.5}$$

The coefficient of $\hbar$ in (C.1.3) is as follows:

$$-i\tilde{p}_0' + 2\tilde{p}_0\tilde{p}_1 = 0. \tag{C.1.6}$$

Thus, if we substitute (C.1.5) in this equation, we obtain $\tilde{p}_1$:

$$\tilde{p}_1 = \frac{i}{4}\frac{Q_0'}{Q_0} = \frac{i}{4}\frac{d}{dx}\log Q_0. \tag{C.1.7}$$

This is total derivative. The coefficient of $\hbar^2$ in the equation (C.1.3) is

$$-i\tilde{p}_1' + 2\tilde{p}_0\tilde{p}_2 + \tilde{p}_1^2 = Q_2. \tag{C.1.8}$$

Substitute (C.1.5) and (C.1.7) in this equation, we obtain

$$\begin{aligned}\tilde{p}_2 =& \frac{-5{Q_0'}^2 + 4Q_0(-4Q_0Q_2 + Q_0'')}{32Q_0^{5/2}} \\ =& -\frac{1}{2}\frac{Q_2}{Q_0^{1/2}} + \frac{Q_0''}{48Q_0^{3/2}} + \frac{d}{dx}\left(\frac{5}{48}\frac{Q_0'}{Q_0^{3/2}}\right).\end{aligned} \tag{C.1.9}$$

In the same way, if we consider the coefficients of $\hbar^3$, we obtain

$$2\tilde{p}_1\tilde{p}_2 + 2\tilde{p}_0\tilde{p}_3 - i\tilde{p}_2' = 0, \tag{C.1.10}$$

and thus we obtain

$$\tilde{p}_3 = \frac{d}{dx}\left\{\frac{5i}{128}\frac{d^2}{dx^2}\frac{1}{Q_0} - \frac{3i}{128}\frac{Q_0''}{Q_0^2} + \frac{i}{4}\frac{Q_2}{Q_0}\right\}. \tag{C.1.11}$$

Thus we can verify that $\tilde{p}_3$ is total derivative. In the same way, if we consider the coefficients of $\hbar^4$, we can obtain the following equation:

$$\tilde{p}_2^2 + 2\tilde{p}_1\tilde{p}_3 + 2\tilde{p}_0\tilde{p}_4 - i\tilde{p}_3' = 0. \tag{C.1.12}$$

Thus we obtain

$$
\begin{aligned}
\tilde{p}_4 =& \frac{7Q_0''^2}{1536Q_0^{7/2}} - \frac{Q_0^{(4)}}{768Q_0^{5/2}} + \frac{Q_2^2}{8Q_0^{3/2}} - \frac{Q_2Q_0''}{32Q_0^{5/2}} + \frac{Q_2''}{48Q_0^{3/2}} \\
&+ \frac{d}{dx}\left\{ -\frac{23}{768}\frac{Q_0^{(3)}}{Q_0^{5/2}} + \frac{221}{1536}\frac{Q_0'Q_0''}{Q_0^{7/2}} - \frac{1105}{9216}\frac{Q_0'^3}{Q_0^{9/2}} - \frac{5}{32}\frac{Q_2Q_0'}{Q_0^{5/2}} + \frac{5}{48}\frac{Q_2'}{Q_0^{3/2}} \right\}. \qquad \text{(C.1.13)}
\end{aligned}
$$

If we take $Q_0 := 2W$ and $Q_2 := 0$, we obtain the quantum SW periods for AD theory obtained from $A_r$ theory: (6.4.3). If we take $Q_0 := 2W$ and $Q_2 = \frac{A}{x^2}$, we obtain the result of (6.4.10). If we replace $Q_0 \to -Q_0$ and $Q_2 \to -Q_2$, we obtain (6.4.21).

## C.2 PF Operators

In this section, we will calculate the PF equations for the AD theories of $D_r$ theories and SQCDs.

### C.2.1 AD theory of $D_r$ Theory

We consider the PF operator for AD theories obtained from $D_r$ type gauge theories. The quantum SW curve of this AD theory is given as (5.7.22), and the WKB solution is given by

$$
\tilde{\Psi}(\tilde{x}) = \exp\left(\frac{i}{\hbar}\int^{\tilde{x}} dx\tilde{P}(\tilde{x})\right), \qquad \tilde{P}(\tilde{x}) = \sum_{n=0}^{\infty} \hbar^n \tilde{p}_n(\tilde{x}). \qquad \text{(C.2.1)}
$$

Note that $\tilde{p}_0$ can be written as

$$
\tilde{p}_0 = -\sqrt{2W}, \qquad W(\tilde{x}, \tilde{s}_i) = \tilde{x}^{2r-2} + \tilde{s}_1\tilde{x}^{2r-4} + \cdots + \tilde{s}_{r-1} + \frac{\tilde{s}_r}{\tilde{x}^2}. \qquad \text{(C.2.2)}
$$

We should calculate $\partial_{\tilde{s}_{j_1}}\cdots\partial_{\tilde{s}_{jn}} W^{1/2}$ to obtain the PF operators. Note that

$$
\partial_{\tilde{s}_i} W = \tilde{x}^{2r-2i-2}, \qquad \partial_{\tilde{s}_{r-1}} W = 1, \qquad \partial_{\tilde{s}_r} W = \frac{1}{\tilde{x}^2} \qquad \text{(C.2.3)}
$$

do not depend on $\tilde{s}_i$'s. Then, we obtain

$$
\partial_{\tilde{s}_{j_1}}\cdots\partial_{\tilde{s}_{jn}} W^{1/2} = \frac{(-1)^{n-1}(2n-3)!!}{2^n}\frac{\partial_{\tilde{s}_{j_1}}W\cdots\partial_{\tilde{s}_{jn}}W}{W^{n-1/2}}. \qquad \text{(C.2.4)}
$$

By using these equations, let us first consider the PF operator for $\tilde{p}_2$ (6.4.8). Note that

$$\begin{aligned}\partial_{\tilde{x}}^2 W =&(2r-2)(2r-3)\tilde{x}^{2r-4} + \sum_{j=1}^{r}(2r-2-2j)(2r-3-2j)\tilde{s}_j\tilde{x}^{2r-4-2j}\\ =&(2r-2)(2r-3)\partial_{\tilde{s}_1}W\partial_{\tilde{s}_{r-1}}W + \sum_{i=1}^{r}(2r-2i-2)(2r-2i-3)\tilde{s}_i\partial_{\tilde{s}_{i+1}}W\partial_{\tilde{s}_{r-1}}W + 6\tilde{s}_r(\partial_{\tilde{s}_r}W)^2.\end{aligned} \tag{C.2.5}$$

Then, we obtain

$$\frac{\partial_{\tilde{x}}^2 W}{W^{3/2}} = -4\left((2r-2)(2r-3)\partial_{\tilde{s}_1}\partial_{\tilde{s}_{r-1}} + \sum_{i=1}^{r}(2r-2i-2)(2r-2i-3)\tilde{s}_i\partial_{\tilde{s}_{i+1}}\partial_{\tilde{s}_{r-1}} + 6\tilde{s}_r\partial_{\tilde{s}_r}^2\right)W^{1/2}, \tag{C.2.6}$$

and therefore, $\tilde{p}_2$ becomes

$$\begin{aligned}\tilde{p}_2 =& -\frac{4}{48\sqrt{2}}\left((2r-2)(2r-3)\partial_{\tilde{s}_1}\partial_{\tilde{s}_{r-1}} + \sum_{i=1}^{r}(2r-2i-2)(2r-2i-3)\tilde{s}_i\partial_{\tilde{s}_{i+1}}\partial_{\tilde{s}_{r-1}} + 6\tilde{s}_r\partial_{\tilde{s}_r}^2\right)W^{1/2}\\ &+\frac{2A}{2\sqrt{2}}\partial_{\tilde{s}_r}W^{1/2}\\ =&\frac{1}{24}\left((2r-2)(2r-3)\partial_{\tilde{s}_1}\partial_{\tilde{s}_{r-1}} + \sum_{i=1}^{r-1}(2r-2i-2)(2r-2i-3)\tilde{s}_i\partial_{\tilde{s}_{i+1}}\partial_{\tilde{s}_{r-1}} + 6\tilde{s}_r\partial_{\tilde{s}_r}^2\right)\tilde{p}_0 - \frac{A}{2}\partial_{\tilde{s}_r}\tilde{p}_0.\end{aligned} \tag{C.2.7}$$

Let us next consider the PF operator for the 4-th order $\tilde{p}_4$ (6.4.8). Note that

$$\begin{aligned}\frac{(\partial_{\tilde{x}}^2 W)^2}{W^{7/2}} =& -\frac{16}{15}\Big\{(2r-2)^2(2r-3)^2\tilde{\partial}_1^2\tilde{\partial}_{r-1}^2 + 12(2r-2)(2r-3)\tilde{s}_r\tilde{\partial}_1\tilde{\partial}_{r-1}\tilde{\partial}_r^2 + 36\tilde{s}_r^2\tilde{\partial}_r^4\\ &+2(2r-2)(2r-3)\sum_{i=1}^{r-1}(2r-2i-2)(2r-2i-3)\tilde{s}_i\tilde{\partial}_1\tilde{\partial}_{i+1}\tilde{\partial}_{r-1}^2\\ &+12\sum_{i=1}^{r-1}(2r-2i-2)(2r-2i-3)\tilde{s}_i\tilde{s}_r\tilde{\partial}_{i+1}\tilde{\partial}_{r-1}\tilde{\partial}_r^2\\ &+\sum_{i=1}^{r-1}\sum_{j=1}^{r-1}(2r-2i-2)(2r-2i-3)(2r-2j-2)(2r-2j-3)\tilde{s}_i\tilde{s}_j\tilde{\partial}_{i+1}\tilde{\partial}_{j+1}\tilde{\partial}_{r-1}^2\Big\}W^{1/2},\end{aligned} \tag{C.2.8}$$

$$
\begin{aligned}
\partial_{\tilde{x}}^4 W =&\frac{(2r-2)!}{(2r-6)!}\tilde{x}^{2r-6}+\sum_{i=1}^{r}\frac{(2r-2i-2)!}{(2r-2i-6)!}\tilde{s}_i\tilde{x}^{2r-2i-6} \\
=&\frac{(2r-2)!}{(2r-6)!}\tilde{\partial}_2 W+\sum_{i=1}^{r}\frac{(2r-2i-2)!}{(2r-2i-6)!}\tilde{s}_i\tilde{\partial}_{i+2}W+120\tilde{s}_r\tilde{\partial}_r^3 W,
\end{aligned}
\tag{C.2.9}
$$

$$
\frac{\partial_{\tilde{x}}^4 W}{W^{5/2}}=\frac{8}{3}\left\{\frac{(2r-2)!}{(2r-6)!}\tilde{\partial}_2\tilde{\partial}_{r-1}^2+\sum_{i=1}^{r}\frac{(2r-2i-2)!}{(2r-2i-6)!}\tilde{s}_i\tilde{\partial}_{i+2}\tilde{\partial}_{r-1}^2+120\tilde{s}_r\tilde{\partial}_r^3\right\}W^{1/2}. \tag{C.2.10}
$$

Then we obtain the following PF operator:

$$
\begin{aligned}
\tilde{p}_4 =& -\frac{7}{3072\cdot 2}\frac{16}{15}\Big\{(2r-2)^2(2r-3)^2\tilde{\partial}_1^2\tilde{\partial}_{r-1}^2+12(2r-2)(2r-3)\tilde{s}_r\tilde{\partial}_1\tilde{\partial}_{r-1}\tilde{\partial}_r^2+36\tilde{s}_r^2\tilde{\partial}_r^4 \\
&+2(2r-2)(2r-3)\sum_{i=1}^{r-1}(2r-2i-2)(2r-2i-3)\tilde{s}_i\tilde{\partial}_1\tilde{\partial}_{i+1}\tilde{\partial}_{r-1}^2 \\
&+12\sum_{i=1}^{r-1}(2r-2i-2)(2r-2i-3)\tilde{s}_i\tilde{s}_r\tilde{\partial}_{i+1}\tilde{\partial}_{r-1}\tilde{\partial}_r^2 \\
&+\sum_{i=1}^{r-1}\sum_{j=1}^{r-1}(2r-2i-2)(2r-2i-3)(2r-2j-2)(2r-2j-3)\tilde{s}_i\tilde{s}_j\tilde{\partial}_{i+1}\tilde{\partial}_{j+1}\tilde{\partial}_{r-1}^2\Big\}\,\tilde{p}_0 \\
&-\frac{1}{1536\cdot 2}\frac{8}{3}\left\{\frac{(2r-2)!}{(2r-6)!}\tilde{\partial}_2\tilde{\partial}_{r-1}^2+\sum_{i=1}^{r-3}\frac{(2r-2i-2)!}{(2r-2i-6)!}\tilde{s}_i\tilde{\partial}_{i+2}\tilde{\partial}_{r-1}^2+120\tilde{s}_r\tilde{\partial}_r^3\right\}\tilde{p}_0 \\
&+\frac{A}{64\cdot 2}\frac{8}{3}\left\{(2r-2)(2r-3)\tilde{\partial}_1\tilde{\partial}_{r-1}\tilde{\partial}_r+\sum_{i=1}^{r-1}(2r-2i-2)(2r-2i-3)\tilde{s}_i\tilde{\partial}_{i+1}\tilde{\partial}_{r-1}\tilde{\partial}_r+6\tilde{s}_r\tilde{\partial}_r^3\right\}\tilde{p}_0 \\
&+\left(\frac{(1-A)A}{16\cdot 2}\right)4\tilde{\partial}_r^2\tilde{p}_0.
\end{aligned}
\tag{C.2.11}
$$

We have obtained the PF operator in AD theory of $D_r$ gauge theory.

### C.2.2 AD theories of $SU(N_c)$ $N_f=0,1$ SQCDs

In these cases, the quantum SW periods are given in (6.4.21), and $Q_0$ is given as (6.4.12). Thus,

$$
\tilde{p}_0=i\sqrt{Q_0},\qquad Q_0=-\tilde{x}^N+\sum_{i=2}^{N}\tilde{u}_i\tilde{x}^{N-i}. \tag{C.2.12}
$$

Here we define $N=N_c$ for $N_f=0$ and $N=N_c+1$ for $N_f=1$. Then, we can obtain the PF operator in the same way as the AD theories obtained from $A_r$ type theories. Because the quantum SW curve is the same as the AD theories obtained from $A_r$ type theory. We can verify

$$\partial_{\tilde{u}_{j_1}} \cdots \partial_{\tilde{u}_{jn}} (-Q)^{1/2} = \left[\frac{1}{2}\right]_n \frac{\partial_{\tilde{u}_{j_1}}(-Q) \cdots \partial_{\tilde{u}_{jn}}(-Q)}{(-Q)^{n-1/2}}, \tag{C.2.13}$$

and

$$\partial_{\tilde{x}}^k Q = \sum_{i=0}^{N-k} [N-i]_k \tilde{u}_i \partial_{\tilde{u}_{i+k}} Q. \tag{C.2.14}$$

Here, we use $[a]_k := a(a-1)\cdots(a-k+1)$ and $[a]_0 := 1$. Thus, we obtain the following PF operators:

$$\tilde{p}_2 = -\frac{1}{24}\left(-[N]_2 \partial_{\tilde{u}_2}\partial_{\tilde{u}_N} + \sum_{j=2}^{N-2}[N-j]_2 \tilde{u}_j \partial_{\tilde{u}_{j+2}}\partial_{\tilde{u}_N}\right)\tilde{p}_0, \tag{C.2.15}$$

$$\begin{aligned}
\tilde{p}_4 = \Bigg(&\frac{7}{5760}\Bigg\{([N]_2)^2 \partial_{\tilde{u}_2}^2 - 2[N]_2 \sum_{j=2}^{N-2}[N-j]_2 \tilde{u}_j \partial_{\tilde{u}_2}\partial_{\tilde{u}_{j+2}} \\
&+ \sum_{j=2}^{N-3}\sum_{k=2}^{N-3}[N-k]_2[N-j]_2 \tilde{u}_j \tilde{u}_k \partial_{\tilde{u}_{j+2}}\partial_{\tilde{u}_{k+2}}\Bigg\}\partial_{\tilde{u}_N}^2 \\
&+ \frac{1}{1152}\left\{-[N]_4\partial_{\tilde{u}_4} + \sum_{j=2}^{N-4}[N-j]_4 \tilde{u}_j \partial_{\tilde{u}_{j+4}}\right\}\partial_{\tilde{u}_N}^2\Bigg)\tilde{p}_0.
\end{aligned} \tag{C.2.16}$$

### C.2.3 $N_f = 2$

In this case, the classical SW period can be written as $\tilde{p}_0 = iQ_0^{1/2}$, where

$$Q_0(\xi) = -\sum_{i=0}^{N_c+1} \tilde{u}_i e^{(N_c+1-i)\xi}. \tag{C.2.17}$$

Then, we obtain

$$\partial_{\tilde{u}_i} Q_0 = -e^{(N_c+1-i)\xi}, \qquad \partial_{\tilde{u}_i} Q_0^{1/2} = \frac{\partial_{\tilde{u}_i} Q_0}{2Q_0^{1/2}}, \tag{C.2.18}$$

and then we obtain

$$\partial_{\tilde{u}_{i_1}} \cdots \partial_{\tilde{u}_{i_k}} \tilde{p}_0 = -i\left[\frac{1}{2}\right]_k \frac{\exp(\sum_{j=1}^k (N_c+1-i_j))}{Q_0^{(2k-1)/2}}. \tag{C.2.19}$$

We also note

$$\partial_\xi^2 Q_0 = \sum_{i=0}^{N_c+1} \tilde{u}_i (N_c+1-i)^2 e^{(N_c+1-i)\xi}$$
$$= \sum_{i=1}^{N_c+1} \tilde{u}_i (N_c+1-i)^2 \partial_{\tilde{u}_{i+1}} Q_0 \partial_{\tilde{u}_{N_c}} Q_0. \tag{C.2.20}$$

Therefore we obtain the second PF operator if we set $Q_2 = 0$:

$$\tilde{p}_2 = \frac{i}{2}\frac{Q_2}{\sqrt{Q_0}} + \frac{i}{48}\frac{Q"_0}{Q_0^{3/2}}$$
$$= \frac{1}{12}\sum_{i=0}^{N_c} \tilde{u}_i (N_c+1-i)^2 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{N_c}} \tilde{p}_0. \tag{C.2.21}$$

Next, let us consider the 4-th PF operator in the $Q_2 = 0$ case. To do this, let us consider

$$Q_0^{(4)} = \sum_{i=0}^{N_c+1} \tilde{u}_i (N_c+1-i)^4 e^{(N_c+1-i)\xi}$$
$$= \sum_{i=0}^{N_c} \tilde{u}_i (N_c+1-i)^4 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{N_c+1}} \partial_{\tilde{u}_{N_c+1}} \tilde{p}_0. \tag{C.2.22}$$

By using this, we obtain

$$\tilde{p}_4 = -\frac{7i}{1536}\frac{Q_0''^2}{Q_0^{7/2}} + \frac{i}{768}\frac{Q_0^{(4)}}{Q_0^{5/2}}$$
$$= \frac{7}{1440}\sum_{i=0}^{N_c}\sum_{j=0}^{N_c} \tilde{u}_i \tilde{u}_j (N_c+1-i)^2 (N_c+1-j)^2 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{j+1}} \partial^2_{\tilde{u}_{N_c}} \tilde{p}_0$$
$$- \frac{1}{288}\sum_{i=0}^{N_c} \tilde{u}_i (N_c+1-i)^4 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{N_c}} \partial_{\tilde{u}_{N_c+1}} \tilde{p}_0. \tag{C.2.23}$$

### C.2.4 $N_f = 2n (n > 0)$ with type B scaling

Let us define

$$Q_2(\xi) = \frac{A}{\xi^2}, \qquad A = \frac{n}{4(1-n)}. \tag{C.2.24}$$

and consider the PF operators. The classical SW period is given by

$$\tilde{p}_0 = iQ_0^{1/2}, \qquad Q_0(\xi) = -\sum_{i=0}^{N_c} \tilde{u}_i \xi^{\frac{N_c-n+1}{1-n}}. \tag{C.2.25}$$

Then, we obtain

$$\partial_{\tilde{u}_i} Q_0 = \xi^{\frac{N_c-n+1}{1-n}}, \qquad \partial_{\tilde{u}_i} Q_0^{1/2} = \frac{\partial_{\tilde{u}_i} Q_0}{2Q_0^{1/2}}. \tag{C.2.26}$$

In the same way, we obtain

$$\partial_{\tilde{u}_{i_1}} \cdots \partial_{\tilde{u}_{i_k}} \tilde{p}_0 = -i \left[\frac{1}{2}\right]_k \frac{\prod_{j=1}^k \xi^{\frac{(N_c+1-i_j)}{1-n}}}{Q^{(2k-1)/2}}. \tag{C.2.27}$$

We also note

$$\begin{aligned}\partial_\xi^2 Q_0(\xi) =& -\sum_{i=0}^{N_c} \tilde{u}_i \frac{N_c-n+1}{1-n} \frac{N_c-2n+2}{1-n} \xi^{\frac{N_c-3n+3}{1-n}} \\ =& 4[\mu(i)]_2 \partial_{\tilde{u}_{i+1}} Q_0 \partial_{\tilde{u}_{N_c-n+1}} Q_0 + 4\tilde{u}_{N_c} \frac{n(2n-1)}{(1-n)^2} \partial_{\tilde{u}_{N_c}} Q_0 \partial_{\tilde{u}_{N_c-n+2}} Q_0,\end{aligned} \tag{C.2.28}$$

where

$$\mu(i) := \frac{N_c+n-i}{1-n}. \tag{C.2.29}$$

The $\tilde{p}_2$ of (6.4.21) can be written as

$$\tilde{p}_2 = \frac{1}{12} \sum_{i=0}^{N_c} \tilde{u}_i [\mu(i)]_2 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{N_c-n+1}} \tilde{p}_0 + \frac{1}{12} \tilde{u}_{N_c} \frac{n(2n-1)}{(1-n)^2} \partial_{\tilde{u}_{N_c}} \partial_{\tilde{u}_{N_c-n+2}} \tilde{p}_0 - A\partial_{\tilde{u}_{N_c-n+2}} \tilde{p}_0. \tag{C.2.30}$$

In the same way, we can obtain PF operator of the quantum SW period $\tilde{p}_4$ (6.4.21) [47]:

$$\begin{aligned}\tilde{p}_4 =& \frac{7}{1440} \left\{ \sum_{i=0}^{N_c-1} \sum_{j=0}^{N_c-1} \tilde{u}_i \tilde{u}_j [\mu(i)]_2 [\mu(j)]_2 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{j+1}} \partial^2_{\tilde{u}_{N_c-n+1}} \tilde{p}_0 \right. \\ & \left. + 2 \sum_{i=0}^{N_c-1} \tilde{u}_i \tilde{u}_{N_c} [\mu(i)]_2 [\mu(N_c)]_2 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{N_c}} \partial_{\tilde{u}_{N_c-n+1}} \partial_{\tilde{u}_{N_c-n+2}} \tilde{p}_0 + \tilde{u}_{N_c}^2 ([\mu(N_c)]_2)^2 \partial^2_{\tilde{u}_{N_c}} \partial^2_{\tilde{u}_{N_c-n+2}} \tilde{p}_0 \right\} \\ & + \frac{1}{288} \left\{ \sum_{i=0}^{N_c-1} \tilde{u}_i [\mu(i)]_4 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{N_c-n+2}} \partial_{\tilde{u}_{N_c-n+1}} \tilde{p}_0 + \tilde{u}_{N_c} [\mu(N_c)]_4 \partial_{\tilde{u}_{N_c}} \partial^2_{\tilde{u}_{N_c-n+2}} \tilde{p}_0 \right\} \\ & + \frac{A}{32} \left\{ \sum_{i=0}^{N_c-1} \tilde{u}_i \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{N_c-n+2}} \partial_{\tilde{u}_{N_c-n+1}} \tilde{p}_0 + \tilde{u}_{N_c} [\mu(N_c)]_2 \partial_{\tilde{u}_{N_c}} \partial^2_{\tilde{u}_{N_c-n+2}} \tilde{p}_0 \right\} \\ & - \frac{A(1-A)}{2} \partial^2_{\tilde{u}_{N_c-n+2}} \tilde{p}_0.\end{aligned} \tag{C.2.31}$$

### C.2.5 $N_f = 2n+1$ or $N_f = 2n (n > 0)$ with type A scaling

In this case, the quantum SW curve is of the SQCD type. We can obtain the PF operators in $N_f = 2n+1$ and $N_f = 2n (n > 0)$ with type A scaling as follows [47]:

$$\begin{aligned}\tilde{p}_2 =& \frac{1}{24}\left(\sum_{i=0}^{n}\sum_{j=0}^{n-2}\tilde{u}_i\tilde{u}_j[n-j]_2\partial_{\tilde{u}_i}\partial_{\tilde{u}_{j+2}} + \sum_{j=0}^{n-2}\tilde{u}_i[n-j]_2\partial_{\tilde{u}_{j+2}}\right)\tilde{p}_0 \\ & -\frac{1}{24}\sum_{i=0}^{n-1}\tilde{u}_i(n-i)\left(N_f\partial_{\tilde{u}_i}\partial_{\tilde{v}_2} + \sum_{a=2}^{N_f-1}\tilde{v}_a(N_f-a)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{v}_{a+1}}\right)\tilde{p}_0 \\ & +\frac{1}{24}\sum_{i=0}^{n}\sum_{a=0}^{N_f-2}\tilde{u}_i\tilde{v}_a[N_f-a]_2\partial_{\tilde{u}_i}\partial_{\tilde{v}_{a+2}}\tilde{p}_0,\end{aligned} \tag{C.2.32}$$

$$\begin{aligned}\tilde{p}_4 =& \left\{\frac{19}{5760}\mathcal{D}_{(0|4)} + \frac{41}{34560}\mathcal{D}_{(00|4)} + \frac{11}{34560}\mathcal{D}_{(001|3)} + \frac{19}{8640}\mathcal{D}_{(01|3)} + \frac{1}{360}\mathcal{D}_{(1|3)}\right. \\ & + \frac{7}{5760}\mathcal{D}_{(011|2)} + \frac{7}{1440}\mathcal{D}_{(11|2)} + \frac{7}{2880}\mathcal{D}_{(004|2)} + \frac{71}{5760}\mathcal{D}_{(04|2)} + \frac{19}{1920}\mathcal{D}_{(2|2)} \\ & + \frac{11}{34560}\mathcal{D}_{(00|13)} - \frac{7}{5760}\mathcal{D}_{(01|12)} + \frac{7}{5760}\mathcal{D}_{(0022|)} \\ & \left. +\frac{37}{5760}\mathcal{D}_{(022|)} + \frac{1}{180}\mathcal{D}_{(22|)} + \frac{1}{1152}\mathcal{D}_{(004|)} - \frac{17}{5760}\mathcal{D}_{(04|)} + \frac{7}{5760}\mathcal{D}_{(4|)}\right\}\tilde{p}_0.\end{aligned} \tag{C.2.33}$$

Here, we define

$$\begin{aligned}\mathcal{D}_{(n_1n_2\cdots|m_1m_2\cdots)} :=& \sum_{i_1,i_2,\cdots=0}^{N_c}\sum_{a_1,a_2,\cdots=0}^{N_f}\tilde{u}_{i_1}[N_c-i_1]_{n_1}\tilde{u}_{i_2}[N_c-i_2]_{n_2}\cdots \\ & \tilde{v}_{a_1}[N_f-a_1]_{m_1}\tilde{v}_{a_2}[N_f-a_2]_{m_2}\cdots \\ & \partial_{\tilde{u}_{i_1}+n_1}\partial_{\tilde{u}_{i_2}+n_2}\cdots\partial_{\tilde{v}_{a_1}+m_1}\partial_{\tilde{v}_{a_2}+m_2}\cdots .\end{aligned} \tag{C.2.34}$$

### C.2.6 $SU(2)$ cases

Let us write the PF operators for $SU(2)$ $N_f = 1, 2, 3$ SQCDs.

- $N_f = 1$:

$$\tilde{p}_2 = \frac{1}{4}\partial_{\tilde{u}_2}\partial_{\tilde{u}_3}\tilde{p}_0, \qquad \tilde{p}_4 = \frac{7}{160}\partial_{\tilde{u}_2}^2\partial_{\tilde{u}_3}^2\tilde{p}_0. \tag{C.2.35}$$

- $N_f = 2$:

$$\tilde{p}_2 = \frac{1}{12}\sum_{i=0}^{2} \tilde{u}_i (3-i)^2 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_2} \tilde{p}_0, \tag{C.2.36}$$

$$\tilde{p}_4 = \frac{7}{1440}\sum_{i=0}^{2}\sum_{j=0}^{2} \tilde{u}_i \tilde{u}_j (3-i)^2 (3-j)^2 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_{j+1}} \partial^2_{\tilde{u}_2} \tilde{p}_0 - \frac{1}{288}\sum_{i=0}^{2} \tilde{u}_i (3-i)^4 \partial_{\tilde{u}_{i+1}} \partial_{\tilde{u}_2} \partial_{\tilde{u}_3} \tilde{p}_0. \tag{C.2.37}$$

- $N_f = 3$:

$$\tilde{p}_2 = -\frac{1}{24}\tilde{u}_0 (3\partial_{\tilde{u}_0}\partial_{\tilde{v}_2} + \tilde{v}_2 \partial_{\tilde{u}_1}\partial_{\tilde{v}_3})\tilde{p}_0 + \frac{1}{24}6(\tilde{u}_0\partial_{\tilde{u}_0}\partial_{\tilde{v}_2} + \tilde{u}_1\partial_{\tilde{u}_1}\partial_{\tilde{v}_2})\tilde{p}_0, \tag{C.2.38}$$

$$\begin{aligned}\tilde{p}_4 = &\left\{ \frac{11}{34560}\mathcal{D}_{(001|3)} + \frac{19}{8640}\mathcal{D}_{(01|3)} + \frac{1}{360}\mathcal{D}_{(1|3)} + \frac{7}{5760}\mathcal{D}_{(011|2)} + \frac{7}{1440}\mathcal{D}_{(11|2)} \right.\\ &+ \frac{19}{1920}\mathcal{D}_{(2|2)} + \frac{11}{34560}\mathcal{D}_{(00|13)} - \frac{7}{5760}\mathcal{D}_{(01|12)} + \frac{7}{5760}\mathcal{D}_{(0022|)}\\ &\left. + \frac{37}{5760}\mathcal{D}_{(022|)} + \frac{1}{180}\mathcal{D}_{(22|)} \right\} \tilde{p}_0. \end{aligned}\tag{C.2.39}$$

Here, we use $\partial^3_{\tilde{x}}\tilde{C} = \partial^4_{\tilde{x}}\tilde{C} = 0$ and thus we obtain

$$\mathcal{D}_{(0|4)}\tilde{p}_0 = \mathcal{D}_{(00|4)}\tilde{p}_0 = \mathcal{D}_{(004|2)}\tilde{p}_0 = \mathcal{D}_{(04|2)}\tilde{p}_0 = \mathcal{D}_{(004|)}\tilde{p}_0 = \mathcal{D}_{(04|)}\tilde{p}_0 = \mathcal{D}_{(4|)}\tilde{p}_0 = 0. \tag{C.2.40}$$

### C.2.7 $SU(3)$ cases

Finally, let us consider the universality with the $SU(3)$ SQCDs. We can write the PF operators for the $SU(3)$ SQCDs as follows:

- $N_f = 0$:

$$\tilde{p}_2 = \frac{1}{4}\partial_{\tilde{u}_2}\partial_{\tilde{u}_3}\tilde{p}_0, \tag{C.2.41}$$

$$\tilde{p}_4 = \frac{7}{160}\partial^2_{\tilde{u}_2}\tilde{p}_0. \tag{C.2.42}$$

- $N_f = 1$:

$$\tilde{p}_2 = \left(\frac{1}{2}\partial_{\tilde{u}_2}\partial_{\tilde{u}_4} - \frac{1}{6}\tilde{u}_2\partial^2_{\tilde{u}_4}\right)\tilde{p}_0, \tag{C.2.43}$$

$$\tilde{p}_4 = \left(\frac{7}{40}\partial^2_{\tilde{u}_2}\partial^2_{\tilde{u}_4} - \frac{1}{48}\partial^3_{\tilde{u}_4}\right)\tilde{p}_0. \tag{C.2.44}$$

- $N_f = 2$:

$$\tilde{p}_2 = \frac{1}{12}\sum_{i=0}^{3}\tilde{u}_i(4-i)^2\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_3}\tilde{p}_0, \tag{C.2.45}$$

$$\tilde{p}_4 = \frac{7}{1440}\sum_{i=0}^{3}\sum_{j=0}^{3}\tilde{u}_i\tilde{u}_j(4-i)^2(4-j)^2\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_{j+1}}\partial^2_{\tilde{u}_3}\tilde{p}_0 - \frac{1}{288}\sum_{i=0}^{3}\tilde{u}_i(4-i)^4\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_3}\partial_{\tilde{u}_4}\tilde{p}_0. \tag{C.2.46}$$

- $N_f = 3$:

$$\tilde{p}_2 = -\frac{1}{24}\tilde{u}_0(3\partial_{\tilde{u}_0}\partial_{\tilde{v}_2} + \tilde{v}_2\partial_{\tilde{u}_1}\partial_{\tilde{v}_3})\tilde{p}_0 + \frac{1}{24}6(\tilde{u}_0\partial_{\tilde{u}_0}\partial_{\tilde{v}_2} + \tilde{u}_1\partial_{\tilde{u}_1}\partial_{\tilde{v}_2})\tilde{p}_0. \tag{C.2.47}$$

Next we consider $\tilde{p}_4$. In this case, $\partial^3_{\tilde{x}}\tilde{C} = 0$ and $\partial^4_{\tilde{x}}\tilde{G} = 0$. Then, we obtain

$$\mathcal{D}_{(0|4)}\tilde{p}_0 = \mathcal{D}_{(00|4)}\tilde{p}_0 = \mathcal{D}_{(004|2)}\tilde{p}_0 = \mathcal{D}_{(04|2)}\tilde{p}_0 = \mathcal{D}_{(004|)}\tilde{p}_0 = \mathcal{D}_{(04|)}\tilde{p}_0 = \mathcal{D}_{(4|)}\tilde{p}_0. \tag{C.2.48}$$

Then we obtain

$$\begin{aligned}\tilde{p}_4 = \Bigg\{ &\frac{11}{34560}\mathcal{D}_{(001|3)} + \frac{19}{8640}\mathcal{D}_{(01|3)} + \frac{1}{360}\mathcal{D}_{(1|3)} + \frac{7}{5760}\mathcal{D}_{(011|2)} + \frac{7}{1440}\mathcal{D}_{(11|2)} \\ &+ \frac{19}{1920}\mathcal{D}_{(2|2)} + \frac{11}{34560}\mathcal{D}_{(00|13)} - \frac{7}{5760}\mathcal{D}_{(01|12)} + \frac{7}{5760}\mathcal{D}_{(0022|)} \\ &+ \frac{37}{5760}\mathcal{D}_{(022|)} + \frac{1}{180}\mathcal{D}_{(22|)}\Bigg\}\tilde{p}_0. \end{aligned} \tag{C.2.49}$$

- $N_f = 4$ type A scaling:

$$\begin{aligned}\tilde{p}_2 = &\frac{1}{24}\left(\sum_{i=0}^{4}\sum_{j=0}^{2}\tilde{u}_i\tilde{u}_j(4-j)(3-j)\partial_{\tilde{u}_i}\partial_{\tilde{u}_{j+2}} + \sum_{j=0}^{2}\tilde{u}_j(4-j)(3-j)\partial_{\tilde{u}_{j+2}}\right)\tilde{p}_0 \\ &- \frac{1}{24}\sum_{i=0}^{3}\tilde{u}_i(4-i)\left(4\partial_{\tilde{u}_i}\partial_{\tilde{v}_2} + \sum_{a=2}^{3}\tilde{v}_a(4-a)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{v}_{a+1}}\right)\tilde{p}_0 \\ &+ \frac{1}{24}\sum_{i=0}^{4}\sum_{a=0}^{2}\tilde{u}_i\tilde{v}_a(4-a)(3-a)\partial_{\tilde{u}_j}\partial_{\tilde{v}_{a+2}}\tilde{p}_0, \end{aligned} \tag{C.2.50}$$

$$\begin{aligned}\tilde{p}_4 = \Bigg\{ &\frac{19}{5760}\mathcal{D}_{(0|4)} + \frac{41}{34560}\mathcal{D}_{(00|4)} + \frac{11}{34560}\mathcal{D}_{(001|3)} + \frac{19}{8640}\mathcal{D}_{(01|3)} + \frac{1}{360}\mathcal{D}_{(1|3)} \\ &+ \frac{7}{5760}\mathcal{D}_{(011|2)} + \frac{7}{1440}\mathcal{D}_{(11|2)} + \frac{19}{1920}\mathcal{D}_{(2|2)} + \frac{11}{34560}\mathcal{D}_{(00|13)} - \frac{7}{5760}\mathcal{D}_{(01|12)} \\ &+ \frac{7}{5760}\mathcal{D}_{(0022|)} + \frac{37}{5760}\mathcal{D}_{(022|)} + \frac{1}{180}\mathcal{D}_{(22|)}\Bigg\}\tilde{p}_0. \end{aligned} \tag{C.2.51}$$

Here, we use $\partial_{\tilde{x}}^4 \tilde{C} = 0$ and thus

$$\mathcal{D}_{(004|2)}\tilde{p}_0 = \mathcal{D}_{(04|2)}\tilde{p}_0 = \mathcal{D}_{(004|)}\tilde{p}_0 = \mathcal{D}_{(04|)}\tilde{p}_0 = \mathcal{D}_{(4|)}\tilde{p}_0 = 0. \tag{C.2.52}$$

- $N_f = 4$ type B scaling:

$$\tilde{p}_2 = \frac{1}{12}\sum_{i=0}^{3} \tilde{u}_i(i-5)(i-6)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_2}\tilde{p}_0 + \frac{1}{12}\tilde{u}_3 6\partial_{\tilde{u}_3}^2\tilde{p}_0 + \frac{1}{2}\partial_{\tilde{u}_3}\tilde{p}_0. \tag{C.2.53}$$

$$\begin{aligned}\tilde{p}_4 =& \frac{7}{1440}\left\{\sum_{i=0}^{2}\sum_{j=0}^{2}\tilde{u}_i\tilde{u}_j(i-5)(i-6)(j-5)(j-6)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_{j+1}}\partial_{\tilde{u}_2}^2\tilde{p}_0\right.\\ &\left.+ 12\sum_{i=0}^{2}\tilde{u}_i\tilde{u}_2(i-5)(i-6)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_3}^2\partial_{\tilde{u}_2}\tilde{p}_0 + 36\tilde{u}_3^2\partial_{\tilde{u}_3}^4\tilde{p}_0\right\}\\ &+ \frac{1}{288}\left\{\sum_{i=0}^{2}\tilde{u}_i(i-5)(i-6)(i-7)(i-8)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_3}\partial_{\tilde{u}_2}\tilde{p}_0 + 120\tilde{u}_3\partial_{\tilde{u}_3}\partial_{\tilde{u}_3}^2\tilde{p}_0\right\}\\ &- \frac{1}{64}\left\{\sum_{i=0}^{2}\tilde{u}_i(i-5)(i-6)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{u}_3}\partial_{\tilde{u}_2}\tilde{p}_0 + 6\tilde{u}_3\partial_{\tilde{u}_3}\partial_{\tilde{u}_3}^2\tilde{p}_0\right\}.\end{aligned} \tag{C.2.54}$$

- $N_f = 5$:

$$\begin{aligned}\tilde{p}_2 =& \frac{1}{24}\left(\sum_{i=0}^{3}\sum_{j=0}^{1}\tilde{u}_i\tilde{u}_j(3-j)(2-j)\partial_{\tilde{u}_i}\partial_{\tilde{u}_{j+2}} + \sum_{j=0}^{1}\tilde{u}_j(3-j)(2-j)\partial_{\tilde{u}_{j+2}}\right)\tilde{p}_0\\ &- \frac{1}{24}\sum_{i=0}^{2}\tilde{u}_i(3-i)\left(N_f\partial_{\tilde{u}_i}\partial_{\tilde{v}_2} + \sum_{a=2}^{4}\tilde{v}_a(5-a)\partial_{\tilde{u}_{i+1}}\partial_{\tilde{v}_{a+1}}\right)\tilde{p}_0\\ &+ \frac{1}{24}\sum_{i=0}^{3}\sum_{a=0}^{3}\tilde{u}_i\tilde{v}_a(5-a)(4-a)\partial_{\tilde{u}_i}\partial_{\tilde{v}_{a+2}}\tilde{p}_0,\end{aligned} \tag{C.2.55}$$

$$\begin{aligned}\tilde{p}_4 =& \left\{\frac{19}{5760}\mathcal{D}_{(0|4)} + \frac{41}{34560}\mathcal{D}_{(00|4)} + \frac{11}{34560}\mathcal{D}_{(001|3)} + \frac{19}{8640}\mathcal{D}_{(01|3)} + \frac{1}{360}\mathcal{D}_{(1|3)}\right.\\ &+ \frac{7}{5760}\mathcal{D}_{(011|2)} + \frac{7}{1440}\mathcal{D}_{(11|2)} + \frac{19}{1920}\mathcal{D}_{(2|2)} + \frac{11}{34560}\mathcal{D}_{(00|13)}\\ &\left.- \frac{7}{5760}\mathcal{D}_{(01|12)} + \frac{7}{5760}\mathcal{D}_{(0022|)} + \frac{37}{5760}\mathcal{D}_{(022|)} + \frac{1}{180}\mathcal{D}_{(22|)}\right\}\tilde{p}_0.\end{aligned} \tag{C.2.56}$$

Here, we use $\partial_{\tilde{x}}^4 \tilde{C} = 0$ and thus

$$\mathcal{D}_{(004|2)}\tilde{p}_0 = \mathcal{D}_{(04|2)}\tilde{p}_0 = \mathcal{D}_{(004|)}\tilde{p}_0 = \mathcal{D}_{(04|)}\tilde{p}_0 = \mathcal{D}_{(4|)}\tilde{p}_0 = 0. \tag{C.2.57}$$

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