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Combinatorics of the quantum alcove model —Thesis outline—

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This article describes the outline of the Ph. D. thesis, entitled "Combinatorics of the quantum alcove model" ([2]).

The quantum alcove model and Schubert calculus The quantum alcove model, introduced in [4] (and generalized to arbitrary weights in [6]), is a combinatorial tool, which appears in many branches of mathematics; for example, the quantum alcove model is used to describe explicitly the $t = 0$ specialization of symmetric (or nonsymmetric) Macdonald polynomials ([8] and [7]; see also [9]).

As another example, we focus on Schubert calculus. Let G be a connected, simply-connected simple algebraic group over $\mathbb{C}$, $H \subset G$ a maximal torus, W the Weyl group, P the weight lattice, and $Q^\vee$ the coroot lattice of G; also, we denote by $W_{\mathrm{af}} = W \ltimes Q^\vee$ the affine Weyl group associated to G, and set $W_{\mathrm{af}}^{\geq 0} := W \ltimes Q^{\vee,+}$, where $Q^{\vee,+} := \{\alpha^\vee \in Q^\vee \mid \alpha^\vee \geq 0\}$. Let $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ denote the $(H \times \mathbb{C}^*)$-equivariant K-group of the semi-infinite flag manifold $\mathbf{Q}_G$ associated to G; in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$, we have the line bundle classes $[\mathcal{O}(\lambda)]$ for $\lambda \in P$, and the semi-infinite Schubert classes $[\mathcal{O}_x]$ for $x \in W_{\mathrm{af}}^{\geq 0}$. In the Schubert calculus, one considers the expansion in $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ of the following form:

$$[\mathcal{O}(\lambda)] \cdot [\mathcal{O}_x] = \sum_{y \in W_{\mathrm{af}}} c_{x,\lambda}^{y} [\mathcal{O}_y],$$

where $c_{x,\lambda}^{y} \in \mathbb{Z}[P](\!(q^{-1})\!)$. By the *Chevalley formula* in [6], the coefficients $c_{x,\lambda}^{y}$ are explicitly described in terms of the quantum alcove model.

In the quantum alcove model, one considers the collection $\mathcal{A}(w, \Gamma)$ of *admissible subsets* associated to a certain sequence of roots Γ corresponding to $\lambda \in P$, called a λ-*chain*.

Quantum Lakshmibai-Seshadri paths *Quantum Lakshmibai-Seshadri (QLS) paths*, introduced in [7], are combinatorial objects related to the representation theory of quantum affine algebras; for example, QLS paths appear in the study of Kirillov-Reshetikhin modules and level-zero extremal weight modules.

For a dominant weight λ, let us take a certain λ-chain (called the *lex λ-chain*). It is known in [7] that there exists a bijection from $\mathcal{A}(e, \Gamma)$ to the set $\mathrm{QLS}(\lambda)$ of QLS paths of shape λ which preserves crystal structures; here $e \in W$ is the identity element of W.

Interpolated QLS paths —Main result 1— We consider the following problem: "Is there a relationship between the quantum alcove model and the set of QLS paths for an arbitrary weight $\lambda \in P$?" To give an answer to this problem, we define the set $\mathrm{IQLS}(\lambda)$ of *interpolated QLS paths* of shape λ, which can be thought of as a generalization of QLS paths. We then construct an explicit injection $\mathcal{A}(w, \Gamma) \to \mathrm{IQLS}(\lambda) \times W$ which preserves some important statistics including weights. In the thesis, we give some applications of this injection to the equivariant K-group of ordinary flag manifolds. Remark that the extended abstract of this part of the thesis is contained in [1].

Quantum Yang-Baxter moves —Main result 2— In general, there exists two or more λ-chains for a fixed $\lambda \in P$. Hence we need to consider the relationship between two collections $\mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}(w, \Gamma_2)$ for λ-chains Γ_1 and Γ_2, where $\lambda \in P$ and $w \in W$. If λ is dominant and Γ_2 is obtained from Γ_1 by a certain deformation procedure called the *Yang-Baxter transformation*, then it is known in [5] that there exists a bijection from $\mathcal{A}(e, \Gamma_1)$ to $\mathcal{A}(e, \Gamma_2)$ which preserves some important statistics including weights and heights. This bijection is given in terms of *quantum Yang-Baxter moves*. As our second main result, we provide a generalization of quantum Yang-Baxter moves. Let $\lambda \in P$ be an arbitrary weight, and $w \in W$. Take λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by the Yang-Baxter transformation. We prove that there exist certain (explicit) subsets $\mathcal{A}_0(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}_0(w, \Gamma_2) \subset \mathcal{A}(w, \Gamma_2)$ such that

- there exists a "sign-preserving" bijection $Y : \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}_0(w, \Gamma_2)$ which preserves some important statistics including weights and heights,
- there exists a "sign-reversing" involution I_1 on $\mathcal{A}(w, \Gamma_1) \setminus \mathcal{A}_0(w, \Gamma_2)$ (resp., I_2 on $\mathcal{A}(w, \Gamma_2) \setminus \mathcal{A}_0(w, \Gamma_2)$) which preserves some important statistics including weights and heights.

In the thesis, we give an application of this generalization of quantum Yang-Baxter moves to the representation theory of quantum affine algebras. Remark that this part of the thesis is based on [3].

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