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On the Exactly Solvable Conditions of Quadratically Constrained Quadratic Program with Sparsity Structures

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ABSTRACT

Quadratically constrained quadratic program (QCQP) is a nonconvex optimization problem which minimizes a quadratic function over quadratic inequality constraints. The QCQP has a wide range of applications such as sensor network localization problems, and the max-cut problems; however, its solution is an NP-hard problem in general. One of attractive method to approximately solve QCQPs is by semidefinite programming (SDP) relaxation, whose optimal value gives a lower bound for the optimal value of the corresponding QCQP. The SDP relaxation is called exact when their optimal values coincide. In this case, we can recover the optimal solution of the QCQP from its SDP relaxation, and hence these conditions for QCQPs are extensively studied. It is known that the SDP relaxation is exact if and only if it has a rank-1 solution. Recently, based on the rank of solutions, exactness conditions for QCQPs whose data matrices are all diagonal were proposed.

This thesis studies exactness conditions from the aspect of sparsity structures of QCQPs. Contributions of this thesis are roughly divided into three. The first contribution is to provide two exactness conditions for QCQPs with two types of the aggregated sparsity patterns: connected forest and connected bipartite. This result is obtained mainly from the rank estimation in the dual problem of the SDP relaxation by using the rank-bound conditions for sparse matrices. Second, the exactness conditions in the first contribution are generalized by employing a perturbation technique to a QCQP. The connectivity condition of the aggregated sparsity pattern is removed. Hence, the exactness conditions can be applied to QCQPs with disconnected forest or disconnected bipartite sparsity patterns. Third, this thesis discusses special classes of QCQPs such as QCQPs with only one equality constraint, QCQPs with all nonnegative off-diagonal elements, and the generalized trust-region subproblem. In particular, alternative proofs for the exactness of the latter two classes are provided by developing two distinct conversion methods for QCQPs.

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NOTATION

Sets

$\mathbb{R}$	Real numbers
$\mathbb{R}^n$	Real n-dimensional vectors
$\mathbb{R}^{m\times n}$	Real $m \times n$ matrices
$\mathbb{R}_+$	Nonnegative real numbers
$\mathbb{R}^n_+$	Nonnegative orthant of dimension n
$\mathbb{S}^n$	Symmetric $n \times n$ matrices
$\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$)	Symmetric positive semidefinite (definite) $n \times n$ matrices
$\mathbb{N}$	Natural integers $\{1, 2, \dots, n\}$
$[m]$	Natural integers from 1 to m, i.e., $\{1, \dots, m\}$
$[n, m]$	Natural integers from n to m, i.e., $\{n, \dots, m\}$
$A \subseteq B$	Set A is a subset of or equal to set B
$A \subsetneq B$	Set A is a proper subset of set B
K^*	Dual cone of a cone K (see (2.1))

Operators

$\mathcal{A}^*$	Adjoint operator ($W \to V$) of linear map $\mathcal{A} : V \to W$

Assignment and inequalities

$A := B$	A is defined by B
$A \succeq B$	$A - B$ is positive semidefinite
$A \succ B$	$A - B$ is positive definite

Vectors

v_i	ith element of vector $\boldsymbol{v}$
$\boldsymbol{v}^{\mathrm{T}}$	Transpose of vector $\boldsymbol{v}$
$\mathbf{0}_n, \mathbf{0}$	n-dimensional zero vector (n is often omitted for simplicity)
$\mathbf{1}_n, \mathbf{1}$	n-dimensional one vector (n is often omitted for simplicity)
$\boldsymbol{e}_i$	ith unit vector
$\langle \boldsymbol{u}, \boldsymbol{v} \rangle$	Inner product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, i.e., $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}$
$\Vert\boldsymbol{v}\Vert_p$	p-norm of vector $\boldsymbol{v} \in \mathbb{R}^n$, i.e., $\Vert\boldsymbol{v}\Vert_p = \left(\sum_{i=1}^{n} \lvert v_i \rvert^p \right)^{1/p}$
$\Vert\boldsymbol{v}\Vert_\infty$	Maximum norm of vector $\boldsymbol{v} \in \mathbb{R}^n$, i.e., $\Vert\boldsymbol{v}\Vert_\infty = \max_{i=1,\ldots,n} \lvert v_i \rvert$

Matrices

M_{ij}	(i, j)th element of matrix M
M_{IJ}	Submatrix of matrix M, generated by removing rows and columns indexed in I and J, respectively
M_I	Principal submatrix of matrix M, or equivalently, M_{II}
M^{T}	Transpose of matrix M
$\operatorname{tr} M$	Trace of matrix M
$\operatorname{rank} M$	Rank of matrix M
I, I_n	$n \times n$ identity matrix
O, O_n	$n \times n$ zero matrix
$A \bullet B$	Frobenius inner product of matrices A and B, i.e., $A \bullet B = \operatorname{tr}\left(A^{\mathrm{T}}B\right)$
$\Vert M\Vert_{\max}$	Maximum norm of matrix $M \in \mathbb{S}^n$, i.e., $\Vert M\Vert_{\max} = \max_{i,j=1,\ldots,n} \lvert M_{ij} \rvert$
$\operatorname{svec}(M)$	Vectorization of matrix $M \in \mathbb{S}^n$ (see (2.6))

CHAPTER 1

INTRODUCTION

Convex optimization is an important class of mathematical optimization problems including linear programming (LP), semidefinite programming (SDP), and conic linear programming problems. Since many kinds of optimization problems in this class can be solved in polynomial time by a variety of solving methods [16, 23, 31, 42, 59, 57], convex optimization problems arise in many applications such as data fitting [39], portfolio optimization [40], and machine learning. However, many practical problems are often formulated as nonconvex optimization problems, and the analysis of their problem structure is still an important task.

Quadratically constrained quadratic programs (QCQPs) are nonconvex optimization problems which seek a minimization of a quadratic function over quadratic inequality constraints, i.e.,

$$\begin{aligned} v^* \;=\; \min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \le b_p, \quad p = 1, \dots, m, \end{aligned} \tag{1.1}$$

where $Q^0, \dots, Q^m$ is a family of $n \times n$ symmetric matrices, and $\boldsymbol{b} \in \mathbb{R}^m$. Even though it is well-known that solving a general QCQP is NP-hard, the application range of QCQPs include optimal power flow problems [36, 62], pooling problems [33], sensor network localization problems [6, 32, 49], quadratic assignment problems [45, 61], and the max-cut problem [18]. Indeed, the QCQP can be expressive as the polynomial optimization problems since any polynomial optimization problem can be rewritten as a QCQP by introducing auxiliary variables.

Semidefinite programming (SDP) relaxation, one of convex relaxations, is an attractive method to approximately solve QCQPs due to its tractability. The optimal values of a QCQP and its SDP relaxation do not coincide in general since the SDP relaxation is constructed by replacing the matrix $\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}$ with X and dropping the rank-1 constraint in the QCQP. However, it is known that under certain conditions their optimal values coincide. When all the matrices $Q^0, \dots, Q^m$ are positive semidefinite, the QCQP can be regarded as a convex optimization problem [15], which implies that these optimal values are equal. In this thesis, the SDP relaxation is called exact if it and the original QCQP has the same optimal value. Examining such conditions helps to identify the gap between the QCQP and the convex optimization. Hence, the main issue in this thesis is to reveal the following question.

Question: What conditions of QCQPs guarantee the exact SDP relaxation?

Recently, this question has drawn the attention of researchers, and many exactness conditions have been identified [2, 3, 4, 10, 11, 24, 28, 37, 50, 53, 54]. Yakubovich's S-lemma [44, 58] (also known as S-procedure) can be regarded as one of the most important results. The S-lemma proves that the trust-region subproblem, a special class of QCQPs with only one constraint (i.e., (1.1) with $m = 1$), always admits an exact SDP relaxation. The exactness for general classes of the trust-region subproblems has been also studied. Wang and Xia [56] discussed a class of QCQPs satisfying $m = 2$ and $Q^1 = -Q^2$ under some mild assumptions. In addition, Jeyakumar [24] considered the extended trust-region subproblem whose constraints consist of one ellipsoid and linear inequalities, and proposed the exactness condition that the algebraic multiplicity of the minimum eigenvalue of Q^0 is strictly greater than the dimension of the space spanned by the coefficient vectors of the linear inequalities. This condition has been generalized in [22, 38].

Some researches focus on structures of QCQPs and signs of its elements. Burer and Ye [10] proposed an LP-based detecting method of the exactness for QCQPs where all the matrices are diagonal, and demonstrated that it can detect the exactness for some general QCQPs via a certain conversion. The sign-definite QCQP is defined as a QCQP where all the off-diagonal elements are either all nonnegative or all nonpositive element-wisely. QCQPs with nonpositive off-diagonal data matrices, a special class of sign-definite QCQPs, were shown to have an exact SDP and SOCP relaxation by Kim and Kojima [32]. Sojoudi and Lavaei generalized this result to whole sign-definite QCQPs, and they provide a sufficient condition which uses problem structures characterized by the aggregated sparsity pattern.

Stronger conditions also have been studied. Wang and Kılınç-Karzan [54] proposed sufficient conditions for the convex hull exactness, i.e., the coincidence of the convex hull of the epigraph of a QCQP and the projected epigraph of its SDP relaxation. In this context, the polyhedral property of the feasible region of the dual SDP relaxation was assumed; however, in [53], they eliminated this assumption and improved their conditions. Argue et al. [2] employed the rank-one generated (ROG) cone property for the convex hull exactness. They proposed sufficient conditions for the feasible set of the SDP relaxation to be a ROG cone, and connected the ROG cone with both the objective value and the exactness of the convex hull.

This thesis focuses on the sparsity structures of QCQPs, and we propose new exactness conditions for QCQPs. In addition, we characterize subclasses of QCQPs such that the SDP relaxation is exact.

Outline and Contributions

We here provide an outline of this thesis. Chapters 3 to 5 correspond to our contributions.

Chapter 2: Strong duality of the SDP relaxation and the aggregated sparsity pattern for QCQP play important role in this thesis. Chapter 2 first presents an overview of the conic linear programming, a general class of the semidefinite programming (SDP), focusing on the duality theory. Then, we introduce quadratically constrained quadratic programming (QCQP) and its homogeneous form (1.1). Throughout this thesis, we are interested in the relationship between a QCQP and its semidefinite programming relaxation which has the following inequality between the two objective values:

$$v^* \;\geq\; \min \left\{ Q^0 \bullet X \;\middle|\; \begin{array}{l} Q^p \bullet X \leq b_p, \quad p = 1, \dots, m \\ X \succeq O \end{array} \right\}. \tag{1.2}$$

Subsequently, this chapter introduces the aggregated sparsity pattern of QCQP after the preparation of basic terms in graph theory. This pattern allows us to represent the sparse structure in a QCQP as a undirected graph. Finally, we summarize existing exactness conditions [10, 24, 28, 50, 53, 54], i.e., sufficient conditions in which an equality holds in (1.2).

Chapter 3: This chapter provides two exactness conditions for QCQPs with two types of the aggregated sparsity pattern: connected forest and connected bipartite. We first present three conditions for the rank-bound of sparse matrices, and prove the positive semidefiniteness of the matrix which is obtained by replacing some off-diagonal elements of a sparse positive semidefinite matrix with zeros. The existence of a rank-1 solution, i.e., the exactness, of the SDP relaxation can be ensured by the rank estimation of optimal solutions for its dual problem under the strong duality. Since the sparsity pattern of dual feasible points is the same as the aggregated sparsity pattern, the rank-bound conditions for sparse matrices can be used for this rank estimation in the dual side. As a result, we obtain two exactness conditions which require checking the infeasibility of systems with

(a) a positive semidefinite constraint, m linear inequality constraints, and one additional linear equality for the forest aggregated sparsity pattern, or

(b) a positive semidefinite constraint, m linear inequality constraints, and one additional linear inequality for the bipartite aggregated sparsity pattern.

Chapter 4: In this chapter, the exactness conditions in Chapter 3 are generalized by exploiting perturbed problems of a QCQP. We first introduce ε-perturbed QCQP, a main tool for the generalization, of the form:

$$v^* \;\geq\; \min\left\{ \boldsymbol{x}^{\mathrm{T}}(Q^0 + \varepsilon P)\boldsymbol{x} \;\middle|\; \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p, \quad p = 1, \dots, m \right\}, \tag{1.3}$$

where $\varepsilon P \in \mathbb{S}^n$ is a perturbation by a negative semidefinite matrix. We then prove that if this problem has the exact SDP relaxation for any $\varepsilon > 0$, the original QCQP also admits it under each assumption of the exactness conditions in Chapter 3. When these conditions are partially satisfied, the existence of such a matrix P is always guaranteed in QCQPs with disconnected aggregated sparsity pattern. Therefore, the connectivity condition of the aggregated sparsity pattern is dropped from the exactness conditions in Chapter 3.

Chapter 5: This chapter discusses special classes of QCQPs such as QCQPs with only one equality constraint, QCQPs with all nonnegative off-diagonal elements, and the generalized trust-region subproblems. For the first class, this chapter provides a simple exactness condition which requires checking the positive semidefiniteness of matrices instead of the infeasibility of systems. To examine the exactness for QCQPs without bipartite or forest sparsity structures, we develop conversion methods into QCQPs with tractable aggregated sparsity pattern. One of them decomposes matrices in a general QCQP to that with a bipartite sparsity pattern by using a new variable $z := -x$ (see Section 5.3.2). These conversions enable us to prove that our exactness conditions cover several existing results [28, 50] based on the sparsity structures and the element-wise signs of QCQPs. The other is based on the simultaneous tridiagonalization [48]. Using it with a proper parameter γ, we present an alternative proof for the exactness of the generalized trust-region subproblem since any problem can be transformed into another QCQP satisfying our exactness conditions.

The work in this thesis are based on the following publications.

- G. Azuma, M. Fukuda, S. Kim, and M. Yamashita. Exact SDP relaxations of quadratically constrained quadratic programs with forest structures. Journal of Global Optimization, 82(2):243–262, 2022.
  DOI: `10.1007/s10898-021-01071-6`

- G. Azuma, M. Fukuda, S. Kim, and M. Yamashita. Exact SDP relaxations for quadratic programs with bipartite graph structures. Journal of Global Optimization, 2022.
  DOI: `10.1007/s10898-022-01268-3`

The first publication discusses QCQPs whose aggregated sparsity pattern is a forest, and its main results are described in Sections 3.3 and 4.3 of this thesis. Additionally, the second publication generalizes these results to QCQPs whose aggregated sparsity pattern is a bipartite, and obtains the results in Sections 3.4 and 4.4.

CHAPTER 2

PRELIMINARIES

2.1 Convex Cone

Let $\mathbb{E}$ be a finite dimensional vector space in which the inner products $\langle\cdot,\cdot\rangle$ are well-defined. This section summarizes definitions of a convex set and a cone in $\mathbb{E}$.

Convex Set

A set $C \subseteq \mathbb{E}$ is called convex if, for any two point $\boldsymbol{x}, \boldsymbol{y} \in C$, and any scalar $\alpha \in [0,1]$, the following inclusion holds:

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in C.$$

Roughly speaking, since the set $\left\{ \alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \,|\, \alpha \in [0,1] \right\}$ is a line segment between fixed points $\boldsymbol{x}, \boldsymbol{y} \in C$, the line segment between any two points in a convex set C lies in C. The empty set $\emptyset$, a singleton $\{\boldsymbol{x}\}$, a line segment, and the whole space $\mathbb{E}$ are convex sets.

Cone

A set $K \subseteq \mathbb{E}$ is cone if the inclusion $\alpha\boldsymbol{x} \in K$ holds for any $\boldsymbol{x} \in K$ and any scalar $\alpha \geq 0$. In addition, a cone K is called a convex cone if K is also convex. The set of symmetric positive semidefinite $n \times n$ matrices, called the positive semidefinite cone, is an important convex cone on the space $\mathbb{S}^n$.

Dual cone of a cone K is the set

$$K^* = \left\{ \boldsymbol{y} \in \mathbb{E} \,|\, \langle \boldsymbol{x}, \boldsymbol{y} \rangle \geq 0 \quad \forall \boldsymbol{x} \in K \right\}. \tag{2.1}$$

Even when a given set K is not convex, the dual cone K^* must be a convex cone. An interesting property of the dual cone is that $K \subseteq (K^*)^*$ holds. In particular, if K is closed convex, then $K = (K^*)^*$.

2.2 Conic Linear Programming

Conic linear programming is well-known as an extension of linear programming where the variable is chosen from a closed convex cone instead of the vector space $\mathbb{R}^n$. It includes many noteworthy convex optimization problems such as semidefinite programming problems, second-order cone programming problems. In this section, the formulation of the conic linear programming and the several optimization problem are presented.

Let $\mathbb{E}_p$ and $\mathbb{E}_d$ be finite dimensional vector spaces in which the inner products $\langle \cdot, \cdot \rangle$ are well-defined. Let $\mathbb{J}_p$ and $\mathbb{J}_d$ be closed convex cones of $\mathbb{E}_p$ and $\mathbb{E}_d$, respectively. For a given $\boldsymbol{b} \in \mathbb{E}_d$, $\boldsymbol{c} \in \mathbb{E}_p$, and a linear map $\mathcal{A} : \mathbb{E}_p \to \mathbb{E}_d$, the general form of the conic linear programming and its dual are as follows:

$$\begin{aligned} \theta_p &= \inf \quad \langle \boldsymbol{c}, \boldsymbol{u} \rangle \\ &\quad\ \text{s.t.} \quad \boldsymbol{u} \in \mathbb{J}_p,\ \mathcal{A}\boldsymbol{u} - \boldsymbol{b} \in \mathbb{J}_d, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \theta_d &= \sup \quad \langle \boldsymbol{b}, \boldsymbol{v} \rangle \\ &\quad\ \text{s.t.} \quad \boldsymbol{v} \in \mathbb{J}_d^*,\ \boldsymbol{c} - \mathcal{A}^*\boldsymbol{v} \in \mathbb{J}_p^*, \end{aligned} \tag{2.3}$$

where $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, and $\mathbb{J}_p^*$ and $\mathbb{J}_d^*$ are the dual cones of $\mathbb{J}_p$ and $\mathbb{J}_d$, respectively. Note that we use here infimum/supremum instead of minimize/maximize since the optimal value of problems may not be attained. Throughout this thesis, we will use minimize/maximize if we implicitly assume that the optimal value can be attained.

2.2.1 Semidefinite Programming

Semidefinite programming (SDP) is an important class of convex optimization problems where the objective function and constraints are linear and the domain of variables is a positive semidefinite cone. More precisely, the inequality form of SDP is to find a symmetric matrix X of the following optimization problem:

$$\begin{aligned} \min \quad & Q^0 \bullet X \\ \text{s.t.} \quad & X \in \mathbb{S}^n_+,\ Q^p \bullet X - b_p \leq 0, \quad p = 1, \dots, m \end{aligned} \tag{2.4}$$

where $Q^0, \dots, Q^p \in \mathbb{S}^n$ and $b_p \in \mathbb{R}$ for $p = 1, \dots, m$ are problem data. Since the equality constraint $Q^p \bullet X - b_p = 0$ can be divided into two inequality constraints

$$Q^p \bullet X - b_p \leq 0 \quad \text{and} \quad -Q^p \bullet X + b_p \leq 0 \tag{2.5}$$

in a similar way to the linear programming, the form (2.4) covers the equality form of SDPs.

The SDP (2.4) is a special case of the conic linear programming. To confirm this, we take $\mathbb{J}_p = \mathbb{S}^n_+$ and $\mathbb{J}_d = -\mathbb{R}^+$. Then, by the definition of the dual cone, $\mathbb{J}_p$ and $\mathbb{J}_d$ are self-dual, i.e., $\mathbb{J}^*_p = \mathbb{S}^n_+$ and $\mathbb{J}^*_d = -\mathbb{R}^+$. For a symmetric matrix $X \in \mathbb{S}^n$, we define the vectorization of X as follows:

$$\operatorname{svec}(X) := \begin{bmatrix} X_{11} \\ \sqrt{2}X_{21} \\ \vdots \\ \sqrt{2}X_{n1} \\ X_{22} \\ \sqrt{2}X_{32} \\ \vdots \\ \sqrt{2}X_{n2} \\ \vdots \\ X_{nn} \end{bmatrix} \in \mathbb{R}^{n(n+1)/2}. \tag{2.6}$$

Using $\boldsymbol{u} := \operatorname{svec}(X)$ and $\boldsymbol{q}^p := \operatorname{svec}(Q^p)$, we have

$$Q^p \bullet X = \operatorname{svec}(Q^p)^{\mathrm{T}} \operatorname{svec}(X) = \langle \boldsymbol{q}^p, \boldsymbol{u} \rangle \quad \text{for all } p = 0, \dots, m.$$

Taking

$$\boldsymbol{c} = \boldsymbol{q}^0, \quad \mathcal{A}\boldsymbol{u} = \begin{bmatrix} \langle \boldsymbol{q}^1, \boldsymbol{u} \rangle \\ \vdots \\ \langle \boldsymbol{q}^m, \boldsymbol{u} \rangle \end{bmatrix},$$

a SDP (2.4) can be formulated by a conic linear programming (2.2).

2.2.2 Strong Duality

On convex optimization problems, the duality of the primal and the dual problems plays an important role to analyze their behaviour. In this subsection, we briefly explain the weak and strong duality of the conic linear programming, and we introduce the recent results which propose the sufficient conditions of the strong duality for the conic linear programming.

Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be feasible regions of (2.2) and (2.3), respectively. The weak duality holds between (2.2) and (2.3).

Proposition 2.1. *Suppose that both $\mathcal{F}_p$ and $\mathcal{F}_d$ are nonempty. Then,*

$$\langle \boldsymbol{c}, \boldsymbol{u} \rangle \geq \langle \boldsymbol{b}, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{u} \in \mathcal{F}_p \text{ and } \boldsymbol{v} \in \mathcal{F}_d.$$

Proof. Let $\boldsymbol{s} \in \mathbb{J}_p^*$ such that $\boldsymbol{s} = \boldsymbol{c} - \mathcal{A}^* \boldsymbol{v}$. Then,

$$\begin{aligned} \langle \boldsymbol{c}, \boldsymbol{u} \rangle - \langle \boldsymbol{b}, \boldsymbol{v} \rangle &= \langle \boldsymbol{s}, \boldsymbol{u} \rangle + \langle \mathcal{A}^* \boldsymbol{v}, \boldsymbol{u} \rangle - \langle \boldsymbol{b}, \boldsymbol{v} \rangle \\ &= \langle \boldsymbol{s}, \boldsymbol{u} \rangle + \langle \mathcal{A} \boldsymbol{u} - \boldsymbol{b}, \boldsymbol{v} \rangle \qquad \geq 0. \end{aligned}$$

The last inequality follows from $\boldsymbol{s} \in \mathbb{J}_p^*$, $\boldsymbol{u} \in \mathbb{J}_p$, $\mathcal{A}\boldsymbol{u} - \boldsymbol{b} \in \mathbb{J}_d$, and $\boldsymbol{v} \in \mathbb{J}_d$. □

In [30], the authors have proposed new conditions for the strong duality of the primal-dual pair of (2.2) and (2.3).

Proposition 2.2. *(Corollary 4.3 [30]) If the set of optimal solutions (or feasible region) of primal cone linear programming* (2.2) *or its dual form* (2.3) *is nonempty and bounded. then $-\infty < \theta_p = \theta_d < \infty$.*

2.3 Quadratically Constrained Quadratic Programs

Quadratically constrained quadratic program (QCQP) is a nonlinear optimization problem where the objective function and all the constraints are quadratic. More precisely, it is a minimization problem of the form

$$\begin{aligned} \min \quad & \boldsymbol{x}^T Q^0 \boldsymbol{x} + 2(\boldsymbol{q}^0)^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^T Q^p \boldsymbol{x} + 2(\boldsymbol{q}^p)^T \boldsymbol{x} \leq b_p, \quad p = 1, \dots, m. \end{aligned} \tag{2.7}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is a variable vector, and $(Q^p, \boldsymbol{q}^p) \in \mathbb{S}^n \times \mathbb{R}^n$ for $p = 0, \dots, m$ and $\boldsymbol{b} \in \mathbb{R}^m$ are problem data. In this formulation, the constant $b_0 \in \mathbb{R}$ in the objective function is omitted because it does not affect the optimal solutions. When all the matrices $Q^0, \dots, Q^m$ are positive semidefinite, (2.7) can be transformed to a variety of convex optimization problems, e.g., a SDP, which implies that it can be solved in polynomial time. However, solving a general QCQP is known to be a NP-hard problem in general, and many relaxation techniques for the QCQP has been studied. In Section 2.3.1, the semidefinite programming relaxation, one of relaxation technique, will be introduced.

The formulation (2.7) has linear terms $2(\boldsymbol{q}^0)^{\mathrm{T}}\boldsymbol{x}$ and $2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x}$; however, such a formulation can be transformed into the homogeneous form with no linear terms:

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \le b_p, \quad p = 1, \dots, m. \end{aligned} \tag{2.8}$$

Indeed, since for any $p = 0, \dots, m$,

$$\boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} + 2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x} = \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & (\boldsymbol{q}^p)^{\mathrm{T}} \\ \boldsymbol{q}^p & Q^p \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix},$$

the problem (2.7) is equivalent to the following QCQPs

$$\begin{aligned} \min \quad & \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & (\boldsymbol{q}^0)^{\mathrm{T}} \\ \boldsymbol{q}^0 & Q^0 \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & (\boldsymbol{q}^p)^{\mathrm{T}} \\ \boldsymbol{q}^p & Q^p \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix} \le b_p, \quad p = 1, \dots, m, \end{aligned} \tag{2.9}$$

and linear terms are vanished. The variable of (2.9) is $\bar{x} := [x_0; x]$ with the linear constraint $x_0 = 1$. To avoid this linear constraint, two quadratic constraints $x_0^2 \le 1$ and $-x_0^2 \le -1$ are usually employed, as follows:

$$\begin{aligned} \min \quad & \bar{\boldsymbol{x}}^{\mathrm{T}} \begin{bmatrix} 0 & (\boldsymbol{q}^0)^{\mathrm{T}} \\ \boldsymbol{q}^0 & Q^0 \end{bmatrix} \bar{\boldsymbol{x}} \\ \text{s.t.} \quad & \bar{\boldsymbol{x}}^{\mathrm{T}} \begin{bmatrix} 0 & (\boldsymbol{q}^p)^{\mathrm{T}} \\ \boldsymbol{q}^p & Q^p \end{bmatrix} \bar{\boldsymbol{x}} \le b_p, \quad p = 1, \dots, m, \\ & \bar{\boldsymbol{x}}^{\mathrm{T}} E_{11} \bar{\boldsymbol{x}} \le 1, \quad -\bar{\boldsymbol{x}}^{\mathrm{T}} E_{11} \bar{\boldsymbol{x}} \le -1. \end{aligned} \tag{2.10}$$

An optimal solution of (2.10) may not seem to satisfy the constraints of (2.9), since $x_0^2 \leq 1$ and $-x_0^2 \leq -1$ allow that $x_0 = -1$. However, we can recover a solution of (2.9) as $x_0^{-1}\bar{x}$ even if $x_0 = -1$. Putting appropriate $(n+1) \times (n+1)$ matrices, (2.10) can be regarded as the homogeneous QCQP (2.8). Since the homogeneous QCQP is a simpler form than the original (2.7), we often use the homogeneous form in this thesis.

2.3.1 Semidefinite Programming Relaxation

Semidefinite programming relaxation (SDP relaxation) [18] is an attractive technique for solving QCQPs approximately, generating an SDP that provides a lower bound on the optimal value of the original QCQP. As described in Section 2.2, since the conic linear programming including the SDP can be solved in polynomial time by the primal-dual interior-point method, the SDP relaxation allows us to obtain lower bounds in polynomial time, In this subsection, we first explain a method to relax the QCQP to the corresponding SDP.

All the quadratic terms $\boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x}$ in (2.7) can be expressed as $Q^p \bullet \left(\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\right)$ where $\bullet$ denotes the Frobenius inner product. Introducing a matrix variable $X = \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}$, (2.7) is equivalent to

$$
\begin{aligned}
\min \quad & Q^0 \bullet X + 2(\boldsymbol{q}^0)^{\mathrm{T}} \boldsymbol{x} \\
\text{s.t.} \quad & Q^p \bullet X + 2(\boldsymbol{q}^p)^{\mathrm{T}} \boldsymbol{x} \leq b_p, \quad p = 1, \dots, m, \\
& X = \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}.
\end{aligned}
$$

Then, relaxing the equality constraint $X = \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}$ to positive semidefinite constraint $X \succeq \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}$, we obtain the SDP relaxation of (2.7):

$$
\begin{aligned}
\min \quad & Q^0 \bullet X + 2(\boldsymbol{q}^0)^{\mathrm{T}} \boldsymbol{x} \\
\text{s.t.} \quad & Q^p \bullet X + 2(\boldsymbol{q}^p)^{\mathrm{T}} \boldsymbol{x} \leq b_p, \quad p = 1, \dots, m, \\
& X \succeq \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}.
\end{aligned} \tag{2.11}
$$

To show that (2.11) is a SDP, we continue the transformation. By the Schur complement [60], the last constraint in (2.11) is represented by

$$
\begin{bmatrix} 1 & \boldsymbol{x}^{\mathrm{T}} \\ \boldsymbol{x} & X \end{bmatrix} \succeq O.
$$

Since for any p,

$$Q^p \bullet X + 2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x} = \begin{bmatrix} 0 & \left(\boldsymbol{q}^p\right)^{\mathrm{T}} \\ \boldsymbol{q}^p & Q^p \end{bmatrix} \bullet Y, \quad Y := \begin{bmatrix} 1 & \boldsymbol{x}^{\mathrm{T}} \\ \boldsymbol{x} & X \end{bmatrix},$$

the following formulation is obtained:

$$\begin{aligned} \min \quad & \begin{bmatrix} 0 & \left(\boldsymbol{q}^0\right)^{\mathrm{T}} \\ \boldsymbol{q}^0 & Q^0 \end{bmatrix} \bullet Y \\ \text{s.t.} \quad & Y \succeq O, \\ & \begin{bmatrix} 0 & \left(\boldsymbol{q}^p\right)^{\mathrm{T}} \\ \boldsymbol{q}^p & Q^p \end{bmatrix} \bullet Y - b_p \leq 0, \quad p = 1, \dots, m, \\ & \left(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}\right) \bullet Y - 1 \leq 0, \quad -\left(\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}\right) \bullet Y + 1 \leq 0. \end{aligned}$$

The last two inequalities are introduced to fix the $(1, 1)$-st element of Y as 1. Comparing with (2.4), we conclude that the above formulation is a SDP with $m + 2$ linear constraints and an $(n + 1) \times (n + 1)$ matrix variable Y.

Let v^*_{QCQP} denote the optimal value of (2.7), v^*_{SDP} that of (2.11). For an optimal solution $\boldsymbol{x}^*$ of (2.7) attaining v^*_{QCQP}, since

$$Q^p \bullet \left\{\boldsymbol{x}^*(\boldsymbol{x}^*)^{\mathrm{T}}\right\} + 2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x}^* = (\boldsymbol{x}^*)^{\mathrm{T}}Q^p\boldsymbol{x}^* + 2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x}^* \leq b_p, \quad p = 1, \dots, m,$$

the $n \times n$ matrix $\boldsymbol{x}^*(\boldsymbol{x}^*)^{\mathrm{T}}$ is a feasible point of (2.11). An observation on the objective function

$$v^*_{\mathrm{QCQP}} = (\boldsymbol{x}^*)^{\mathrm{T}}Q^0\boldsymbol{x}^* + 2(\boldsymbol{q}^0)^{\mathrm{T}}\boldsymbol{x}^* = Q^0 \bullet \left\{\boldsymbol{x}^*(\boldsymbol{x}^*)^{\mathrm{T}}\right\} + 2(\boldsymbol{q}^0)^{\mathrm{T}}\boldsymbol{x}^*,$$

implies that (2.11) attains v^*_{QCQP}. Therefore, $v^*_{\mathrm{SDP}} \leq v^*_{\mathrm{QCQP}}$ follows. In general, the equality between them does not hold. The difference of them ($v^*_{\mathrm{QCQP}} - v^*_{\mathrm{SDP}}$) is often called the relaxation gap.

2.4 Aggregated Sparsity Pattern

Aggregated sparsity pattern (or aggregate sparsity pattern) is originally used to describe the sparsity structure of SDPs. Using this sparsity pattern, it is possible to structurally express whether the (i, j)th element of the matrix variable X appears in a SDP (2.4). Exploiting the aggregated sparsity pattern to decompose a SDP to smaller SDPs has been a popular subject [16, 31]. Recently, several researches incorporate the aggregated sparsity pattern in other optimization problems, e.g., the second-order cone programming [47] and the polynomial optimization problem [29].

In this section, to describe the aggregated sparsity pattern, we first introduce basic terminologies and concepts in the graph theory. Then, we will define the aggregated sparsity pattern for the SDP, and illustrate an aggregated sparsity pattern of a simple instance of SDP.

2.4.1 Basic of Graph Theory

A graph G is a pair consisting of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, denoted by $G(\mathcal{V}, \mathcal{E})$ [46]. Each edge $e \in E$ is represented by a pair (u, v) of two distinct vertices $u \in \mathcal{V}$ and $v \in \mathcal{V}$. A given graph is called an undirected graph if all edges do not have a direction. On the undirected graph, the edges (u, v) and (v, u) coincide, and $(u, v) \in \mathcal{E}$ if and only if $(v, u) \in \mathcal{E}$. In this thesis, we only think undirected graphs, and hence we simply refer to an undirected graph as a graph.

An walk W is a nonempty sequence $W = (v_1, e_1, v_2, \dots, e_{f-1}, v_f)$ where vertices and edges in G are alternately listed and for any $i = 1, \dots, f-1$, the edge (v_i, v_{i+1}) exists in $\mathcal{E}$. Since the edge e_i can be uniquely determined from vertices, the walk is often represented by the vertex sequence $W = (v_1, v_2, \dots, v_f)$. The start and end vertices of W denote v_1 and v_f, respectively.

A path is an walk in which the each vertex appears in W only once except the end vertex, that is, the path W does not pass each vertices twice. In particular, when the start and end vertices coincide ($v_1 = v_f$), it is called a cycle in G. The path and cycle are important concepts in the graph theory. We next introduce graph properties defined by them.

A connected graph is a graph G where there exists a path between any pair of two vertices in G. Otherwise, the graph G is called a disconnected graph. A maximal connected subgraph $G'(\mathcal{V}', \mathcal{E}')$ where $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$ is called a connected component of a graph $G(\mathcal{V}, \mathcal{E})$. The number of connected components is uniquely determined for a given graph G. The connected graphs have only one connected component, and the disconnected graph must have two or more connected components.

A tree is a connected graph which has no cycles. For example, a path which has different start and end vertices is always a tree. When a tree T has n vertices, it must have $n-1$ edges. A path between any pair of two vertices in a tree T is unique, a well property. The union of one or more trees are called a forest. Since a tree is always a forest, the forest is a wider class than the tree.

A bipartite graph is a graph which has no cycles of odd length. In other words, a bipartite graph can contain only even length cycles. Since the forest cannot contain cycles of any length, the bipartite graph is more general than forest.

A cycle basis of a graph G is a minimal set of cycles in G where every cycle in G can be expressed as a symmetric difference of one or more cycles in the cycle basis.

2.4.2 Sparsity Pattern of Matrix

The sparsity pattern of a given symmetric matrix $M \in \mathbb{S}^n$ can be represented by its nonzero elements as follows:

$$\mathcal{E}_M := \left\{ (i,j) \in [n] \times [n] \,\middle|\, M_{ij} \neq 0 \right\}. \tag{2.12}$$

By definition, if $(i,j) \notin \mathcal{E}_M$, then the (i,j)th element of M must be zero. Since M is symmetric, $(i,j) \in \mathcal{E}_M$ if and only if $(j,i) \in \mathcal{E}_M$. A small number $|\mathcal{E}_M|$ implies that M has many zero elements, thus M is close to the zero matrix.

To visualize the sparsity pattern $\mathcal{E}_M$, the graph $G\left([n], \overline{\mathcal{E}}_M\right)$ is often used where

$$\overline{\mathcal{E}}_M := \mathcal{E}_M \setminus \{(u,u) \mid u \in [n]\}.$$

We call symmetric matrices as forest-structured matrices if the graphs of their sparsity patterns are forests.

2.4.3 Aggregated Sparsity Pattern of SDPs and QCQPs

For the SDP of the form (2.4), its aggregated sparsity pattern is the graph $G(\mathcal{V}, \mathcal{E})$ where the vertex set $\mathcal{V}$ is $[n]$, and the edge set $\mathcal{E}$ contains all the off-diagonal indices at which Q^p takes nonzero for some $p \in [0, m]$. More precisely,

$$\mathcal{E} := \left\{ (i, j) \in \mathcal{V} \times \mathcal{V} \,\middle|\, i \neq j \text{ and } Q^p_{ij} \neq 0 \text{ for some } p \in [0, m] \right\}. \tag{2.13}$$

Similarly, the aggregated sparsity pattern of the QCQP (2.8) is defined as that of its SDP relaxation. The graph G is also called an aggregated sparsity pattern graph. If $\mathcal{E}$ corresponds to an adjacency matrix $\mathbb{Q}$ of n vertices, then $\mathbb{Q}$ is called the aggregated sparsity pattern matrix.

The aggregated sparsity pattern can also be defined by the sparsity patterns of $Q^0, \dots, Q^m$. By definition, $\mathcal{E}$ is a union of their sparsity patterns:

$$\mathcal{E} = \overline{\mathcal{E}}_{Q^0} \cup \cdots \cup \overline{\mathcal{E}}_{Q^m}.$$

We next discuss the following example of a SDP.

Example 2.3.

$$\begin{array}{rl} \min & Q^0 \bullet X \\ \text{s.t.} & Q^1 \bullet X \leq 10, \quad Q^2 \bullet X \leq 10, \quad Q^3 \bullet X \leq 5, \end{array} \tag{2.14}$$

where

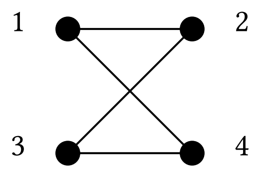
$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \quad Q^3 = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

The aggregated sparsity pattern of this example is

$$\mathcal{E} = \{(1, 2), (1, 4), (2, 3), (3, 4)\}.$$

Since $(1, 3)$th and $(2, 4)$th elements are zeros in all of Q^0, Q^1, Q^2, Q^3, the indices $(1, 3)$ and $(2, 4)$ are not contained in $\mathcal{E}$. On the other hand, for any other index (i, j) with $i \neq j$, at least one of $Q^0_{ij}, Q^1_{ij}, Q^2_{ij}, Q^3_{ij}$ is nonzero, i.e., there exist the other edges. For example, for the



$$\mathbb{Q} = \begin{bmatrix} \star & \star & 0 & \star \\ \star & \star & \star & 0 \\ 0 & \star & \star & \star \\ \star & 0 & \star & \star \end{bmatrix}.$$

Figure 2.1: The aggregated sparsity pattern graph and matrix of Example 2.3. The sign $\star$ denotes an arbitrary nonzero value.

index $(i, j) = (1, 4)$, we have $(Q^0_{14}, Q^1_{14}, Q^2_{14}, Q^3_{14}) = (2, 1, 0, 0)$, and hence the first two nonzero elements cause the edge between 1 and 4. Figure 2.1 illustrates the aggregated sparsity pattern graph $G(\{1, 2, 3, 4\}, \mathcal{E})$, and the corresponding aggregated sparsity pattern matrix $\mathbb{Q}$. As shown in the figure, G has only one cycle with 4 vertices. It is the simplest example of connected bipartite graphs with cycles.

2.5 Exactness of SDP Relaxation of QCQPs

We give the precise definition of the exactness of the SDP relaxation for QCQPs.

Definition 2.4. *For the QCQP* (2.7)*, its SDP relaxation is called exact if the optimal values of the QCQP* (2.7) *and the SDP relaxation* (2.11) *are equal and attained. In addition, when the SDP relaxation is exact, we say that QCQP has the exact SDP relaxation.*

The rank of an optimal solution of the SDP relaxation plays an important role when the exactness is discussed. Indeed, the following proposition is well-known to be a sufficient condition of the exactness:

Proposition 2.5. *If there exists an optimal solution $(\boldsymbol{x}^*, X^*)$ of SDP relaxation* (2.11) *such that the matrix*

$$\begin{bmatrix} 1 & (\boldsymbol{x}^*)^\top \\ \boldsymbol{x}^* & X^* \end{bmatrix}$$

is of rank-1, the original QCQP (2.7) *has the exact SDP relaxation.*

Proof. Let $(\boldsymbol{x}^*, X^*)$ be an optimal solution of SDP relaxation (2.11). Since the above matrix is of rank-1, the equality $X^* = \boldsymbol{x}^*(\boldsymbol{x}^*)^{\mathrm{T}}$ holds. Hence, we have

$$Q^p \bullet X^* + 2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x}^* = (\boldsymbol{x}^*)^{\mathrm{T}}Q^p\boldsymbol{x}^* + 2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x}^* \quad \text{for any } p \in [0, m],$$

which implies $\boldsymbol{x}^*$ is a feasible point for (2.7). It is concluded that the optimal value of (2.7) is less than or equal to that of (2.11). □

Thus, from a practical point of view, the exactness of the SDP relaxation can be determined by solving it by an SDP solver and checking the rank of an obtained solution.

A different approach on the exactness of the SDP relaxation for QCQPs is to study the convex hull exactness, i.e., the coincidence of the convex hull of the epigraph of a QCQP and the projected epigraph of its SDP relaxation [2, 34, 52, 53, 54]. By definition, the convex hull exactness is stronger than the exactness in Definition 2.4, and hence the exact SDP relaxation exists whenever the convex hull exactness holds. In this thesis, when we simply say the exactness, it is based on Definition 2.4, not the convex hull exactness.

The rest of this section consists of (i) the summary of known classes in which the SDP relaxation is exact, and (ii) the introduction of known exactness conditions for QCQPs.

2.5.1 Generalized Trust-region Subproblem

The trust-region subproblem is an auxiliary problem in the trust-region method [12], and to minimize a quadratic function over the full-dimensional ellipsoid:

$$\begin{array}{rl} \min & \boldsymbol{x}^{\mathrm{T}}Q^0\boldsymbol{x} + 2(\boldsymbol{q}^0)^{\mathrm{T}}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x}^{\mathrm{T}}Q^p\boldsymbol{x} + 2(\boldsymbol{q}^p)^{\mathrm{T}}\boldsymbol{x} \le b_1, \end{array} \tag{2.15}$$

where $Q^0, Q^1 \in \mathbb{S}^n$, $\boldsymbol{q}^0, \boldsymbol{q}^1 \in \mathbb{R}^n$, and $b_1 \in \mathbb{R}$. Since $\boldsymbol{x}$ is within the full-dimensional ellipsoid, we can assume without loss of generality that $Q^1 \succ O$ and $b_1 > 0$. It is known that the SDP relaxation of the trust-region subproblem always has an optimal solution of rank-1 [44]. Hence, it is exact by Proposition 2.5. Since the SDP is solvable in polynomial time as mentioned in Section 2.2.1, so is the trust-region subproblem regardless of the positive semidefiniteness of Q^0. Recently, finding an approximate solution of the trust-region subproblem in linear time has been a popular subject [20, 21, 55].

Generalized trust-region subproblem (GTRS) is a generalized class of the trust-region subproblem, where the constraint is not limited to the full-dimensional ellipsoid. Thus, $Q^1 \nsucc O$ and $b_1 \ngtr 0$ are allowed in (2.15). As the same as the trust-region subproblem, Yakubovich's S-lemma [44, 58] (also known as S-procedure) proves that the GTRS also always admits an exact SDP relaxation [51]. In addition, a linear-time algorithm for the GTRS was already proposed [25].

2.5.2 Conditions using Diagonal QCQPs

In this subsection, we briefly present a method in Burer and Ye [10] to determine the upper bound of the minimum rank in all the optimal solutions of the SDP relaxation for a class of QCQPs. This method is based on the n systems of linear inequalities constructed by the data of a given QCQP, and by counting the number of feasible systems among them, the desired maximum rank can be estimated. By Proposition 2.5, the special case of this estimation derives the exactness conditions for SDP relaxation of the QCQP.

The following assumptions of QCQPs (2.7) are used in [10].

Assumption 2.6.

(a) There exists a feasible point for (2.7)*.*

(b) There exists $\bar{\boldsymbol{y}} \geq \boldsymbol{0}$ *satisfying* $\sum_{p=1}^{m} \bar{y}_p Q^p \succ O$.

(c) There exists an interior feasible points for (2.11)*.*

In addition, data matrices $Q^0, \ldots, Q^m$ are assumed to be diagonal. QCQPs satisfying this diagonal assumption are called diagonal QCQPs.

Let r^* be the smallest rank in all the optimal solutions

$$\begin{bmatrix} 1 & (\boldsymbol{x}^*)^{\mathrm{T}} \\ \boldsymbol{x}^* & X^* \end{bmatrix}$$

of the SDP relaxation (2.11) of a QCQP. The value of r^* can measure the tractability of the SDP relaxation. For example, $r^* = 1$ implies that at least one optimal solution of the SDP relaxation (2.11) is of rank-1, and so (2.11) is exact by Proposition 2.5. In addition, the low-rank factorization method [9] may help to solve the SDP relaxation of a QCQP when r^* is small. Several upper bounds of r^* have been reported [5, 13, 35, 43].

We define the feasibility number f of a diagonal QCQP. The number f is nonnegative, and counts the number of $\ell \in [n]$ such that the following system has no solutions:

$$\left.\begin{aligned} &\boldsymbol{y} \geq \boldsymbol{0}, \quad Q^0 + \sum_{p=1}^m y_p Q^p \succeq O, \\ &\left[Q^0 + \sum_{p=1}^m y_p Q^p\right]_{\ell\ell} = 0, \quad \left[\boldsymbol{q}^0 + \sum_{p=1}^m y_p \boldsymbol{q}^p\right]_{\ell} = 0 \end{aligned}\right\} \tag{2.16}$$

Since all the matrices $Q^0, \dots, Q^m$ are diagonal, the second inequality of (2.16) can be rewritten as linear inequalities

$$\left[Q^0 + \sum_{p=1}^m y_p Q^p\right]_{jj} \geq 0 \quad \text{for all } j \in [n].$$

The last two equalities are also linear, thus (2.16) is a system of linear inequalities. Therefore, by Farkas' lemma [8, Section 5.8.3], (2.16) has no solutions if and only if the following linear system has a feasible solution:

$$\begin{aligned} Q^0 \bullet X + q_\ell^0 x_\ell &= -1, \\ Q^p \bullet X + q_\ell^p x_\ell &\leq 0, \quad p = 1, \dots, m, \\ X_{jj} &\geq 0, \quad j \in [n] \setminus \{\ell\}, \\ X \in \mathbb{S}^n, &\quad x_\ell \in \mathbb{R}. \end{aligned} \tag{2.17}$$

The variable X is a matrix; however, only its diagonal elements appears in (2.17) due to the diagonal QCQP. Hence, the number of variables in (2.17) is just $n + 1$. Using the transformed system, the feasibility number f also can be expressed as

$$f = \left|\{\ell \in [n] \mid (2.17) \text{ has a solution}\}\right|.$$

Recently, Burer and Ye [10] proposed an upper bound of r^* using f.

Theorem 2.7. *[10, Theorem 1] Assume that Assumption 2.6 holds. Let (x^*, X^*) be a pair of an optimal solution of the SDP relaxation (2.11) of a given QCQP (2.7). Then,*

$$1 \leq \operatorname{rank}\left(\begin{bmatrix} 1 & (\boldsymbol{x}^*)^\mathrm{T} \\ \boldsymbol{x}^* & X^* \end{bmatrix}\right) \leq n - f + 1.$$

In addition, $r^* \leq n - f + 1$.

In particular, if $n - f + 1 = 1$ (or equivalently, $f = n$), then the matrix

$$\begin{bmatrix} 1 & (\boldsymbol{x}^*)^{\mathrm{T}} \\ \boldsymbol{x}^* & X^* \end{bmatrix}$$

in the above inequality must be of rank-1, since the left-hand side and the right-hand side bound its rank by 1. By Proposition 2.5, we can conclude that the SDP relaxation is always exact as below.

Corollary 2.8. *Assume that Assumption 2.6 holds. If the system* (2.17) *has a feasible solution for any* $\ell \in [n]$*, then the SDP relaxation* (2.11) *of a given QCQP* (2.7) *is exact.*

2.5.3 Conditions under Sign-definite Property

A sign-definite set $S \subseteq \mathbb{R}$ is a finite set of real numbers whose members are either all nonnegative or all nonpositive values, i.e., $ab \geq 0$ for any $a, b \in S$. For example, the set $\{0, 100, 0, 2\}$ is sign-definite while sets $\{0, 100, 0, -2\}$ and $\{-1, 1\}$ are not sign-definite.

For the discussion on QCQPs with sign-definiteness, we employ the following notation from [50]. Let $\mathcal{E}$ be the aggregated sparsity pattern of a QCQP. For each edge $(i, j) \in \mathcal{E}$, the edge sign σ_{ij} is defined as

$$\sigma_{ij} = \begin{cases} +1 & \text{if } Q_{ij}^0, \dots, Q_{ij}^m \geq 0, \\ -1 & \text{if } Q_{ij}^0, \dots, Q_{ij}^m \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Note that since the sign σ_{ij} corresponds to an edge $(i, j) \in \mathcal{E}$, at most one of cases holds in the above definition. Obviously, $\sigma_{ij} \in \{-1, +1\}$ if and only if $\{Q_{ij}^0, \dots, Q_{ij}^m\}$ is sign-definite. A QCQP is called a sign-definite QCQP if $\sigma_{ij} \neq 0$ for all $(i, j) \in \mathcal{E}$.

Sojoudi and Lavaei [50] proposed the following condition for sign-definite QCQPs.

Theorem 2.9 ([50, Theorem 2])**.** *The SDP relaxation of* (2.8) *are exact if both of the following* *hold:*

$$\sigma_{ij} \neq 0, \qquad \forall (i,j) \in \mathcal{E}, \tag{2.19}$$

$$\prod_{(i,j)\in\mathcal{C}_r} \sigma_{ij} = (-1)^{|\mathcal{C}_r|}, \qquad \forall r \in \{1,\dots,\kappa\}, \tag{2.20}$$

where the set of cycles $\mathcal{C}_1, \dots, \mathcal{C}_\kappa \subseteq \mathcal{E}$ denotes a cycle basis for G.

As mentioned above, (2.19) implies that a given QCQP (2.8) is a sign-definite QCQP. For several graph families of the aggregated sparsity pattern graph G, they also presented the following corollary:

Corollary 2.10 ([50, Corollary 1])**.** *The SDP relaxation and the SOCP relaxation of* (2.8) *are exact if one of the following holds:*

(a) G is forest with $\sigma_{ij} \in \{-1, 1\}$ for all $(i,j) \in \mathcal{E}$,

(b) G is bipartite with $\sigma_{ij} = 1$ for all $(i,j) \in \mathcal{E}$,

(c) G is arbitrary with $\sigma_{ij} = -1$ for all $(i,j) \in \mathcal{E}$.

CHAPTER 3

EXACTNESS CONDITION UNDER CONNECTED CASES

3.1 Introduction

Throughout this chapter, we consider the homogeneous version of QCQPs:

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p, \quad p = 1, \ldots, m, \end{aligned} \tag{3.1}$$

and we assume that the aggregated sparsity pattern graph $G := G(\mathcal{V}, \mathcal{E})$ of a given homogeneous QCQP is connected. The disconnected graph G will be dealt with in the next chapter. For the homogeneous QCQP, the SDP relaxation and its dual problem can be expressed as

$$\begin{aligned} \min \quad & Q^0 \bullet X \\ \text{s.t.} \quad & Q^p \bullet X \leq b_p, \quad p = 1, \ldots, m, \\ & X \succeq O, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \max \quad & -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} \\ \text{s.t.} \quad & Q^0 + \sum_{p=1}^{m} y_p Q^p \succeq O, \quad \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned} \tag{3.3}$$

To make the discussion easier, we define a matrix-valued function $S(\boldsymbol{y})$ as

$$S(\boldsymbol{y}) := Q^0 + \sum_{p=1}^{m} y_p Q^p,$$

then the feasible region of (3.3) is represented by $S(\boldsymbol{y}) \succeq O$ and $\boldsymbol{y} \geq \boldsymbol{0}$.

In this chapter, we present sufficient conditions for the SDP relaxation to be exact under two certain graph classes of G. The first result is for a forest graph G and it is provided as Theorem 3.7 in Section 3.3. As well as Theorem 2.7, this result is based on the upper bound of the maximum rank of SDP solutions. To detect the exactness of a given QCQP, we only need to check the infeasibility of n systems. We next expand a class of QCQPs with forests to that with bipartite at Theorem 3.10 in Section 3.4. Even though these theorems require the connectivity of G, they are extended to the ones for the disconnected aggregated sparsity in Chapter 4.

At the end of this chapter, Section 3.5 and Section 3.6, we compare Theorem 3.7 for the forest graphs, Theorem 3.10 for the bipartite graphs, and existing results shown in Theorem 2.9 and Theorem 2.7, using two examples. For these examples, we will check the feasibility systems in analytical and computational aspects.

3.2 Properties of Matrix Classes

3.2.1 Tridiagonal Matrix

A tridiagonal matrix is a matrix $M \in \mathbb{R}^{n\times n}$ where the element M_{ij} is zero for any $i, j \in [n]$ such that $|i-j| \geq 2$, i.e.,

$$\begin{bmatrix} * & * & 0 & \cdots & & 0 \\ * & * & * & 0 & & \vdots \\ 0 & * & * & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & * & 0 \\ & & & * & * & * \\ 0 & & \cdots & 0 & * & * \end{bmatrix}$$

is a tridiagonal matrix even if elements represented by $*$ take nonzero values. In this subsection, we introduce two properties for tridiagonal matrices: (i) a sufficient condition for a tridiagonal $M \in \mathbb{S}^n$ to be of rank at least $(n-1)$; (ii) the positive semidefiniteness of a matrix dropping some off-diagonal elements from a symmetric matrix $M \in \mathbb{S}^n_+$.

We now consider a tridiagonal and symmetric matrix $M \in \mathbb{S}^n$. Let $\mathcal{E}$ be the sparsity pattern of M defined in (2.12). Since the number of super-diagonal elements of M is exactly $n-1$, the cardinality of $\mathcal{E}$ is at most $2(n-1)$ due to the symmetricity of M. Any super-diagonal element on $(i, i+1) \in \mathcal{E}$ corresponds to $(i+1, i)$ on sub-diagonal. Thus, the graph $G([n], \mathcal{E})$

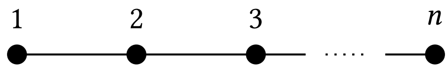


Figure 3.1: Sparsity pattern graph of tridiagonal matrix

of the sparsity pattern can be expressed by strings of vertices as displayed in Figure 3.1.

We discuss a method to estimate a lower bound on the rank of tridiagonal matrices. For general matrices, this estimation is hard. In the case of a diagonal matrix, we know that its rank equals the number of nonzero elements on its main diagonal. For a tridiagonal matrix, we can show that its rank can be bounded from below by the number of off-diagonal nonzero elements. The following lemma is immediately obtained from the result of [26].

Lemma 3.1. *Let $M \in \mathbb{S}^n$ be a tridiagonal matrix. If all the superdiagonal elements of M are nonzeros, then* $\operatorname{rank} M \geq n-1$.

Proof. Let λ be an arbitrary eigenvalue of M. We consider an $(n-1)\times(n-1)$ matrix L constructed by removing the first row and the nth column from $M-\lambda I$, i.e.,

$$L := (M-\lambda I)_{\{2,\ldots,n\}\{1,\ldots,n-1\}} = \begin{bmatrix} M_{12} & M_{22}-\lambda & M_{23} & & \\ & M_{23} & M_{33}-\lambda & \ddots & \\ & & M_{34} & \ddots & M_{n-2,n-1} \\ & & & \ddots & M_{n-1,n-1}-\lambda \\ & & & & M_{n-1,n} \end{bmatrix}.$$

Then L is a nonsingular matrix as all elements on the main diagonal are nonzeros. Since L is a submatrix of $M-\lambda I$, the rank of $M-\lambda I$ must be greater than or equal to $n-1$, and hence both the geometric and the algebraic multiplicity of λ are 1. It implies that $\operatorname{rank}(M) = n-1$ if the eigenvalues of M contain 0; otherwise $\operatorname{rank}(M) = n$. □

For symmetric positive semidefinite matrices $\mathbb{S}^n \ni M \succeq O$, it is difficult to determine whether the positive semidefiniteness is maintained after replacing some of off-diagonal elements with zeros. However, in the case of positive semidefinite tridiagonal matrices, we show in the following lemma that they remain to be positive semidefinite even if some of off-diagonal elements are replaced by zeros.

Lemma 3.2. *Let $M \in \mathbb{S}^n$ be a positive semidefinite and tridiagonal matrix, and $\mathcal{E}$ be the sparsity pattern of M. For a subset $E \subseteq \mathcal{E}$, let $L \in \mathbb{S}^n$ be the tridiagonal matrix constructed by replacing the matrix elements of M indexed by E with zeros, i.e.,*

$$L_{ij} := \begin{cases} 0 & \text{if } (i,j) \in E \text{ or } (j,i) \in E, \\ M_{ij} & \text{otherwise.} \end{cases}$$

Then, $L \succeq O$.

Proof. To use induction on the size of the set E, let us first consider the case $|E| = 1$. Then, E has one element $(i, i+1) \in E$ for some i. For any $I \subseteq [n]$, we examine the principal submatrix L_I which is generated by removing rows and columns not indexed in I. We have the following two cases:

(a) If $i \notin I$ or $i + 1 \notin I$, the principal L_I does not have $(i, i+1)$th and $(i+1, i)$th elements of M, therefore

$$\det(L_I) = \det(M_I) \geq 0.$$

(b) If $i \in I$ and $i + 1 \in I$, the principal L_I has $(i, i+1)$th and $(i+1, i)$th elements of M replaced by zeros. Since the submatrix L_I is a block diagonal matrix with two blocks,

$$\begin{aligned} \det(L_I) &= \det\left(L_{I\cap\{1,\ldots,i\}}\right) \det\left(L_{I\cap\{i+1,\ldots,n\}}\right) \\ &= \det\left(M_{I\cap\{1,\ldots,i\}}\right) \det\left(M_{I\cap\{i+1,\ldots,n\}}\right) \quad \geq 0, \end{aligned}$$

where the last inequality follows from the fact that all the principal minors of M are nonnegative.

Since all the principal minors of L are nonnegative, $L \succeq O$ follows.

Suppose the result is true for $|E| = k - 1$, and consider the case $|E| = k$. The set E can be divided into two sets: $F := \{(j, j+1)\}$ and $E \setminus F$. Let $N \in \mathbb{S}^n$ be a tridiagonal matrix constructed by replacing the $(j, j+1)$ and $(j+1, j)$ elements of M with zeros. Then, from the case mentioned above, $N \succeq O$ holds. Since $|E \setminus F| = k - 1$, by the induction hypothesis, we have $L \succeq O$. □

3.2.2 Forest-structured Matrix

We call a matrix $M \in \mathbb{S}^n$ a forest-structured matrix if its sparsity pattern graph is forest: for example, an arrow-type matrix of form

$$V = \begin{bmatrix} v_1 & & & & w_1 \\ & v_2 & & & w_2 \\ & & \ddots & & \vdots \\ & & & v_{n-1} & w_{n-1} \\ w_1 & w_2 & \cdots & w_{n-1} & v_n \end{bmatrix}. \tag{3.4}$$

In Section 3.2.1, we considered the tridiagonal matrix whose sparsity pattern can be represented by a string of vertices. Since its sparsity pattern graph is obviously a forest (see Figure 3.1), a tridiagonal matrix is always a forest-structured matrix. In this subsection, we introduce generalized results of Lemmas 3.1 and 3.2.

In [27], Johnson et al. have proposed the lower bound conditions for a positive semidefinite and forest-structured matrix.

Lemma 3.3. *[27, Corollary 3.9] Let $M \in \mathbb{S}^n$ be a positive semidefinite and forest-structured matrix with the sparsity pattern $\mathcal{E}$. If the sparsity pattern graph $G([n], \mathcal{E})$ of M is connected and the off-diagonal element M_{ij} is nonzero for all $(i, j) \in \mathcal{E}$, then* $\operatorname{rank} M \geq n - 1$.

To illustrate this lemma, we now consider an arrow-type matrix represented by (3.4). We can easily check that if $v_1, \ldots, v_{n-1}$ are nonzeros, then the rank of V is at least $n - 1$. Simultaneously, by Lemma 3.3, the same result holds if $V \succeq O$ and $w_1, \ldots, w_{n-1}$ are nonzeros.

The following lemma is an extension of Lemma 3.2 to forest-structured matrices.

Lemma 3.4. *Let $M \in \mathbb{S}^n$ be a positive semidefinite and forest-structured matrix, and $\mathcal{E}$ be the sparsity pattern of M. For a subset $E \subseteq \mathcal{E}$, let $L = [\ell_{ij}] \in \mathbb{S}^n$ be the forest-structured matrix constructed by replacing the matrix elements of M indexed by E with zero, i.e.,*

$$L_{ij} := \begin{cases} 0 & \text{if } (i, j) \in E \text{ or } (j, i) \in E, \\ M_{ij} & \text{otherwise.} \end{cases}$$

Then, $L \succeq O$.

Proof. It suffices to consider the case $|E| = 1$ since similar inductive arguments to the proof of Lemma 3.2 can be applied. In this case, there exists only one element (i, j) in E. By removing the edge (i, j) from $G(\mathcal{V}, \mathcal{E})$ of the forest-structured matrix M, a tree in $G(\mathcal{V}, \mathcal{E})$ is divided into two trees: one with the node i, and the other with the node j. The set $\mathcal{V}$ is also separated into two sets: $W_1 \subseteq \mathcal{V}$, the set of nodes in the component including the node i in the graph $G(\mathcal{V}, \mathcal{E}\setminus\{(i, j), (j, i)\})$, and $W_2 \subseteq \mathcal{V}$, the set of the other nodes. Without loss of generality, we may assume that the indices in W_1 and W_2 are consecutive integers, i.e., there exists a positive number ℓ such that $W_1 = [\ell]$ and $W_2 = [n] \setminus [\ell]$. For any $I \subseteq [n]$, we have the following two cases:

(a) If $i \notin I$ or $j \notin I$, the principal L_I does not include (i, j)th and (j, i)th elements of M, then
$$\det(L_I) = \det(M_I) \geq 0.$$

(b) If $i \in I$ and $j \in I$, the principal L_I includes (i, j)th and (j, i)th elements of M, and their values are zeros. Since the submatrix L_I is a block diagonal matrix with two blocks, we have that
$$\det(L_I) = \det\left(L_{I\cap W_1}\right)\det\left(L_{I\cap W_2}\right) = \det\left(M_{I\cap W_1}\right)\det\left(M_{I\cap W_2}\right) \geq 0,$$
where the last inequality follows from the fact that all the principal minors of M are nonnegative.

Since all the principal minors of L are nonnegative, we have $L \succeq O$. □

3.2.3 Matrix with Bipartite Structure

The following lemma is an immediate consequence of Proposition 1 of [19]. It shows that the rank of a nonnegative positive semidefinite matrix can be bounded below by $n - 1$ under some sparsity conditions if the sum of every row of the matrix is positive.

Lemma 3.5 ([19, Proposition 1])**.** *Let $M \in \mathbb{R}^{n\times n}$ be a nonnegative and positive semidefinite matrix with $M\mathbf{1} > \mathbf{0}$. If the sparsity pattern graph of M is bipartite and connected, then* $\operatorname{rank}(M) \geq n-1$.

3.3 Forest Sparsity Structure Cases

In this section, we assume that the aggregated sparsity pattern graph G is forest. A class of QCQPs whose G is forest is called a forest-structured QCQP. We also assume the following for a given forest-structured QCQP and its SDP relaxation:

Assumption 3.6.

(a) *There exists a feasible point for* (3.1).

(b) *There exists* $\bar{\boldsymbol{y}} \geq 0$ *satisfying* $\sum_{p=1}^{m} \bar{y}_p Q^p \succ O$.

(c) *There exists an interior feasible points for* (3.2).

We note that these assumptions were also used for the diagonal QCQPs to establish the results on the exact SDP relaxations in [10]. Assumption 3.6(b) can be also represented as $\bar{\boldsymbol{y}} \leq 0$ and $\sum_{p=1}^{m} \bar{y}_p Q^p \prec O$.

By Assumption 3.6(a) and (b), we see that the feasible regions of (3.1) and (3.2) are bounded, and a solution to (3.1) exists. In fact, multiplying the constraint $\boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \leq b_p$ by $\bar{y}_p$, and adding these for p, we have

$$\boldsymbol{x}^{\mathrm{T}} \left(\sum_{p=1}^{m} \bar{y}_p Q^p \right) \boldsymbol{x} \leq \boldsymbol{b}^{\mathrm{T}} \bar{\boldsymbol{y}}.$$

Thus, since $\sum_{p=1}^{m} \bar{y}_p Q^p \succ O$, all the feasible points of (3.1) are in the ellipsoid given by the above inequality. It also implies that all the feasible points of (3.2) is bounded as:

$$\left(\sum_{p=1}^{m} \bar{y}_p Q^p \right) \bullet X \leq \boldsymbol{b}^{\mathrm{T}} \bar{\boldsymbol{y}}.$$

By Assumption 3.6*(b)* and *(c)*, the strong duality holds for the primal SDP (3.2).

We are ready to present our main results on the exactness of SDP relaxations for forest-structured QCQPs.

Theorem 3.7. *Suppose that Assumption 3.6 holds and the aggregated sparsity pattern graph* $G(\mathcal{V}, \mathcal{E})$ *of* (3.1) *is a forest and connected graph, i.e., a tree. Then, its SDP relaxation* (3.2) *is exact if for all* $(k, \ell) \in \mathcal{E}$, *the following system has no solutions:*

$$\boldsymbol{y} \geq 0,\ S(\boldsymbol{y}) \succeq O,\ [S(\boldsymbol{y})]_{k\ell} = 0, \tag{3.5}$$

Proof. Let X^* be any optimal solution for (3.2). By Assumption 3.6, there exists an optimal solution $\boldsymbol{y}^*$ for (3.3). Since $\boldsymbol{y}^* \geq 0$ and $S(\boldsymbol{y}^*) \succeq O$, we have $S(\boldsymbol{y}^*)_{k\ell} \neq 0$ for every $(k, \ell) \in \mathcal{E}$ by the assumption. This implies that all the off-diagonal elements on $\mathcal{E}$ of the forest-structured matrix $S(\boldsymbol{y}^*)$ are nonzeros, thus $\operatorname{rank} S(\boldsymbol{y}^*) \geq n-1$ by Lemma 3.1. Since Assumption 3.6(b) and (c) hold, it follows $X^* S(\boldsymbol{y}^*) = O$ by the strong duality. From the Sylvester's rank inequality [1], $\operatorname{rank} X^* + \operatorname{rank} S(\boldsymbol{y}^*) \leq n + \operatorname{rank} X^* S(\boldsymbol{y}^*)$ holds for X^* and $S(\boldsymbol{y}^*)$. Therefore, $\operatorname{rank} X^* \leq n + \operatorname{rank} O - \operatorname{rank} S(\boldsymbol{y}^*) \leq n + 0 - (n-1) = 1$. By Proposition 2.5, we obtain the desired result. □

We note that, although we can construct n^2 systems (3.5), only $n-1$ systems are used in Theorem 3.7 because a tree has just $n-1$ edges.

3.4 Bipartite Sparsity Structure Cases

Throughout this section, we assume that the aggregated sparsity pattern graph G is bipartite. The forest-structured QCQP discussed in Section 3.3 does not have a cycle in the aggregated sparsity pattern graph. On the other hand, since a bipartite allows even-length cycles, QCQPs in this section is more general than QCQPs in Section 3.3.

In this section, we introduce Assumption 3.8 instead of Assumption 3.6. Assumption 3.6 has been introduced to use the strong duality in the proof of Theorem 3.7 in Section 3.3. As shown in Remark 3.9 below, Assumption 3.8 is a weaker condition than Assumption 3.6; however, the strong duality holds under this weaker condition. Under the new assumption, we provide a new exactness conditions that can be applied to wider problems than that in Section 3.3

Assumption 3.8. *The following two conditions hold:*

(i) the sets of optimal solutions for (3.2) *and* (3.3) *are nonempty; and*

(ii) at least one of the following two conditions holds:

  (a) the feasible region of (3.2) *is bounded; or*

  (b) the set of optimal solutions for (3.3) *is bounded.*

Remark 3.9. *Assumption 3.8 is weaker than Assumption 3.6. To compare these assumptions, we suppose that there exists* $\bar{\boldsymbol{y}} \geq 0$ *such that* $\sum_p \bar{y}_p Q^p \succ O$*. Then, there obviously exists sufficiently large* $\lambda > 0$ *such that*

$$\lambda \bar{\boldsymbol{y}} \geq \mathbf{0} \quad \textit{and} \quad Q^0 + \sum_p \lambda \bar{y}_p Q^p \succ O,$$

which implies (3.3) *has an interior feasible point. It follows that the set of optimal solutions of* (3.2) *is bounded. Similarly, since* (3.2) *has an interior point by Assumption 3.6, the set of optimal solutions of* (3.3) *is also bounded. This indicates Assumption Assumption 3.8 (i) and (ii)(b).*

In addition, as mentioned right after Assumption 3.6, the feasible region of (3.2) *is bounded. Thus, Assumption 3.8(ii)(a) is also satisfied under Assumption 3.6.*

We utilize Lemma 3.5 to estimate the rank of solutions of the dual SDP relaxation, and establish conditions for the exact SDP relaxation in this section.

Theorem 3.10. *Suppose that Assumption 3.8 holds and the aggregated sparsity pattern* $G(\mathcal{V}, \mathcal{E})$ *of* (3.1) *is a bipartite and connected graph. Then, its SDP relaxation* (3.2) *is exact if, for all* $(k, \ell) \in \mathcal{E}$*, the following system has no solutions:*

$$\boldsymbol{y} \geq \mathbf{0}, \; S(\boldsymbol{y}) \succeq O, \; S(\boldsymbol{y})_{k\ell} \leq 0. \tag{3.6}$$

Proof. Let X^* be any optimal solution for (3.2) which exists by Assumption 3.8. By Proposition 2.2, the optimal values of (3.2) and (3.3) are finite and equal. Thus, there exists an optimal solution $\boldsymbol{y}^*$ for (3.3) such that the complementary slackness holds, i.e.,

$$X^* S(\boldsymbol{y}^*) = O.$$

Since $\boldsymbol{y}^* \geq \mathbf{0}$ and $S(\boldsymbol{y}^*) \succeq O$, by the infeasibility of (3.6), we obtain $S(\boldsymbol{y}^*)_{k\ell} > 0$ for every $(k, \ell) \in \mathcal{E}$. Furthermore, for each $i \in \mathcal{V}$, the ith element of $S(\boldsymbol{y}^*)\mathbf{1}$ is

$$\begin{aligned}[S(\boldsymbol{y}^*)\mathbf{1}]_i &= \sum_{j=1}^n S(\boldsymbol{y}^*)_{ij} \\ &= S(\boldsymbol{y}^*)_{ii} + \sum_{(i,j)\in\mathcal{E}} S(\boldsymbol{y}^*)_{ij} > 0.\end{aligned}$$

From Lemma 3.5, it follows $\operatorname{rank}\left\{S(\boldsymbol{y}^*)\right\} \geq n-1$. By the Sylvester's rank inequality [1],

$$\begin{aligned}\operatorname{rank}(X^*) &\leq n - \operatorname{rank}\left\{S(\boldsymbol{y}^*)\right\} + \operatorname{rank}\left\{X^*S(\boldsymbol{y}^*)\right\} \\ &\leq n-(n-1) \\ &= 1.\end{aligned}$$

Therefore, by Proposition 2.5, the SDP relaxation is exact. □

The exactness of a given QCQP can be determined by checking the infeasibility of $|\mathcal{E}|$ systems. Each system (3.6) should be infeasible if a SDP with objective function 0 and constraints (3.6) has no solutions, or if the optimal value of an SDP,

$$\begin{aligned}\mu^* = \min \quad & S(\boldsymbol{y})_{k\ell} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \boldsymbol{0},\ S(\boldsymbol{y}) \succeq O,\end{aligned}$$

is positive. Thus, we can check the infeasibility of them by using SDP solvers. In Section 3.6, we used the latter SDP to determine the exactness of an instance QCQP under the default parameters of the solver MOSEK [41] in Julia. Moreover, as seen in the following subsections, we can also analytically check the infeasibility if the SDP has nice properties. For instance, some of such properties were discussed in [50]. As a result, Theorem 3.10 can be considered as a generalization of the previous result.

3.5 Comparison between Our Results

We now compare Theorem 3.7 and Theorem 3.10 in this section.

Compared with Theorem 3.7, Theorem 3.10 can determine the exactness of a wider class of QCQPs in terms of the required assumption and sparsity. As mentioned in Remark 3.9, the assumptions in Theorem 3.10 are weaker than those in Theorem 3.7, and the aggregated sparsity pattern of G is extended from forest graphs to bipartite graphs.

When the assumption of Theorem 3.7 or Theorem 3.10 is satisfied, the optimal values of a given QCQP and its SDP relaxation are equal. Simultaneously, in their proofs, it is also guaranteed that all feasible points of the SDP relaxations are of rank-1, which implies that any optimal solution of the SDP relaxation can recover the corresponding optimal solution of the original QCQP.

3.6 Numerical Experiment

We investigate analytical and computational aspects of the conditions in Theorems 3.7 and 3.10 with two QCQP instances below. The first QCQP consists of 2×2 data matrices. We show the exactness of its SDP relaxation by checking the feasibility systems in Theorems 3.7 and 3.10 without SDP solvers. Next, the QCQP whose SDP relaxation is Example 2.3 is considered as the second QCQP. As the size n of the second QCQP is 4, it is difficult to handle the positive semidefinite constraint $S(\boldsymbol{y}) \succeq O$ without numerical computation. We present a numerical method for testing the exactness of the SDP relaxation with a computational solver.

We also detail the difference between our results (Theorems 3.7 and 3.10) and the existing result (Theorems 2.7 and 2.9). For the two QCQP instances in this section, we cannot apply Theorem 2.7 because their aggregated sparsity patterns $\mathcal{E}$ are a forest and a bipartite. In addition, even though Theorem 2.9 can be applied to any pattern $\mathcal{E}$, it cannot detect their exactness because the sign edge $\sigma_{ij} = 0$ for some $(i, j) \in \mathcal{E}$, which implies that the instances do not satisfy the condition (2.19). In Section 5.3, we will prove that Theorem 3.10 is more general than Theorem 2.9.

For numerical experiments, JuMP [14] was used with the solver MOSEK [41] and SDPs were solved with tolerance 1.0×10^{-8}. All numerical results are shown with four significant digits.

3.6.1 A QCQP instance with $n = 2$

Example 3.11. *Consider the QCQP* (3.1) *with*

$$n = 2, \quad m = 1, \quad \boldsymbol{b} = \begin{bmatrix} 1 \end{bmatrix},$$
$$Q^0 = \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix}, \quad Q^1 = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}.$$

We first verify whether the problem satisfies the assumption of Theorem 3.10. The aggregated sparsity pattern graph G is bipartite and connected as it has only two vertices and $Q^0_{12} \neq 0$. Since Q^1 is positive definite, the problem satisfies Assumption 3.6. By the discussion in Remark 3.9, it also satisfies Assumption 3.8. It only remains to show that the system

$$y_1 \geq 0, \quad \hat{S}(y_1) := \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix} + y_1 \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \succeq O, \quad -1 + 4y_1 \leq 0$$

has no solutions. By definition, $\hat{S}(y_1) \succeq O$ holds if and only if all the principal minors of $\hat{S}(y_1)$ are nonnegative, or equivalently, $-3 + 3y_1 \geq 0$, $-2 + 6y_1 \geq 0$, and $2y_1^2 - 16y_1 + 5 \geq 0$. Hence, if $y_1 \geq 4 + 3\sqrt{6}/2 \simeq 7.674$, then the first two inequalities of the system are satisfied. Since $-1 + 4y_1 \geq -1 + 4(4 + 3\sqrt{6}/2) = 15 + 6\sqrt{6} > 0$, the last inequality does not hold for such y_1. The problem therefore admits an exact SDP relaxation.

Actually, we numerically obtained an optimal solution of the above QCQP in Example 3.11 and its SDP relaxation as

$$\boldsymbol{x}^* \simeq \begin{bmatrix} 1.731 \\ -1.167 \end{bmatrix} \quad \text{and} \quad X^* \simeq \begin{bmatrix} 2.997 & -2.021 \\ -2.021 & 1.362 \end{bmatrix},$$

respectively. From $(\boldsymbol{x}^*)^{\mathrm{T}} Q^0 \boldsymbol{x}^* - Q^0 \bullet X^* \simeq 5.379 \times 10^{-10}$, we observe numerically that the SDP relaxation provided the exact optimal value.

Since G is clearly a forest (no cycles), we can also apply Theorem 3.7. From the discussion above, the system (3.5) has no solutions for $(k, \ell) = (1, 2)$ and Assumption 3.6*(a)* is satisfied. By taking

$$\hat{X} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \succ O,$$

we know $Q^1 \bullet \hat{X} = 0.9 \leq 1 = b_1$. Hence, the exactness of the SDP relaxation can be proved by Theorem 3.7. We mention that this result cannot be obtained by Theorem 2.9 proposed in [50]. Since $Q^0_{12} = -1$ and $Q^1_{12} = 4$, the edge sign σ_{12} of the edge $(1, 2)$ must be zero by definition, contradicting (2.19).

3.6.2 A QCQP instance with $n = 3$

We next consider the following QCQP.

Example 3.12.

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^2 \boldsymbol{x} \leq 10, \quad \boldsymbol{x}^{\mathrm{T}} Q^3 \boldsymbol{x} \leq 5, \end{aligned}$$

where

$$Q^0 = \begin{bmatrix} 0 & -2 & 0 & 2 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 \\ 2 & 0 & 1 & -4 \end{bmatrix}, \ Q^1 = \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix},$$
$$Q^2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \ Q^3 = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

Since the SDP relaxation of Example 3.12 is (2.14) obviously, the aggregated sparsity pattern of Example 3.12 is a cycle graph with four vertices (Figure 2.1), as well as (2.14).

We computed an optimal solution of Example 3.12 and that of its SDP relaxation as

$$\boldsymbol{x}^* \simeq \begin{bmatrix} 3.412 \\ -4.536 \\ 1.513 \\ -4.642 \end{bmatrix} \text{ and } X^* \simeq \begin{bmatrix} 11.65 & -15.48 & 5.163 & -15.84 \\ -15.48 & 20.57 & -6.862 & 21.06 \\ 5.163 & -6.862 & 2.289 & -7.024 \\ -15.84 & 21.06 & -7.024 & 21.55 \end{bmatrix} \in \mathbb{S}^4_+,$$

respectively. From $(\boldsymbol{x}^*)^{\mathrm{T}} Q^0 \boldsymbol{x}^* - Q^0 \bullet X^* \simeq 2.450 \times 10^{-8}$, we observe again numerically that the SDP relaxation resulted in the exact optimal value.

We first see whether Example 3.12 satisfies the assumption of Theorem 3.10. We compute $3Q^1 + 4Q^2 + 0Q^3$ as

$$3 \begin{bmatrix} 5 & 2 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & -1 & 4 \end{bmatrix} + 4 \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} + 0 \cdot Q^3 = \begin{bmatrix} 11 & 10 & 0 & 3 \\ 10 & 13 & 5 & 0 \\ 0 & 5 & 33 & 1 \\ 3 & 0 & 1 & 4 \end{bmatrix},$$

Table 3.1: Optimal values of (3.7) for each (k, ℓ)

(k,ℓ)	$(1,2)$	$(2,3)$	$(1,4)$	$(3,4)$
μ^*	7.828	7.301	5.653	0.3215

and its minimum eigenvalue is approximately 0.1577. Thus, there exists $\bar{\boldsymbol{y}} \geq 0$ such that $\bar{y}_1 Q^1 + \bar{y}_2 Q^2 + \bar{y}_3 Q^3 \succ O$, e.g., $\bar{\boldsymbol{y}} = [3; 4; 0]$. As mentioned in Remark 3.9, it follows that the second problem satisfies Assumption 3.8. To show the exactness of the SDP relaxation for the problem, it only remains to show that the system (3.6) has no solutions for all $(k, \ell) \in \mathcal{E}$. Using an SDP solver, we observe that there is no solution for the system. Indeed, for every $(k, \ell) \in \mathcal{E}$, the SDP

$$\begin{aligned} \mu^* = \min \quad & S(\boldsymbol{y})_{k\ell} \\ \text{s.t.} \quad & \boldsymbol{y} \geq \mathbf{0},\ S(\boldsymbol{y}) \succeq O, \end{aligned} \tag{3.7}$$

returns the optimal values shown in Table 3.1, which implies that no solution exists for (3.6) since $S(\boldsymbol{y})_{k\ell}$ cannot attain a nonpositive value. Therefore, the SDP relaxation of Example 3.12 is exact by Theorem 3.10.

On the contrary, Theorem 2.9 in [50] is not sufficient to show the exactness of the SDP relaxation. The edge sign σ_{12} for $(1, 2)$th element is 0 by definition. Since the cycle basis of $\mathcal{G}$ is only $\mathcal{C}_1 = \mathcal{G}$, the left-hand side of (2.20) is $\sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41} = 0$. However, its right-hand side only takes -1 or $+1$. This implies that Theorem 2.9 cannot be applied to Example 3.12.

CHAPTER 4

EXACTNESS CONDITION UNDER DISCONNECTED CASES

4.1 Introduction

The connectivity of the aggregated sparsity pattern graph G has played an important role for our main theorems in Chapter 3. For QCQPs with sparse data matrices, the connectivity assumption might be a difficult condition to be satisfied. In this chapter, we relax this condition, and we extend our conditions to QCQPs whose aggregated sparsity pattern graphs are disconnected. In other words, we assume that the aggregated sparsity pattern graph G of a given QCQP is a disconnected graph.

Continuing from the previous chapter, we use the homogeneous version of QCQPs (3.1). In this chapter, we introduce an ε-perturbed QCQP to relax the condition.

4.2 ε-perturbed QCQP

We perturb the objective function of a given QCQP to remove the connectivity of G from results in Chapter 3. Let $P \in \mathbb{S}^n$ be an $n \times n$ nonzero matrix, and let $\varepsilon > 0$ denote the magnitude of the perturbation. An ε-perturbed QCQP is described as follows:

$$\begin{array}{ll} \min & \boldsymbol{x}^{\mathrm{T}} \left(Q^0 + \varepsilon P \right) \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} \le b_p, \quad p = 1, \ldots, m. \end{array} \tag{4.1}$$

Note that for a given QCQP (3.1), the corresponding ε-perturbed QCQP is given by adding a term $\varepsilon \boldsymbol{x}^{\mathrm{T}} P \boldsymbol{x}$ to its objective function. Hence, the feasible region of (4.1) is the same as the original QCQP (3.1). To generalize $S(\boldsymbol{y})$ in the dual problem for the ε-perturbed QCQP, we define

$$S(\boldsymbol{y};\, \varepsilon) := Q^0 + \varepsilon P + \sum_{p=1}^{m} y_p Q^p.$$

Then, it follows from the definition that $S(\boldsymbol{y};\, \varepsilon) = S(\boldsymbol{y}) + \varepsilon P$.

For an ε-perturbed QCQP, its primal and dual SDP relaxation can be written as

$$\begin{array}{ll} \min & \left(Q^0 + \varepsilon P\right) \bullet X \\ \text{s.t.} & Q^p \bullet X \leq b_p, \quad p = 1, \ldots, m, \\ & X \succeq O, \end{array} \tag{4.2}$$

$$\begin{array}{ll} \max & -\boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} \\ \text{s.t.} & S(\boldsymbol{y};\, \varepsilon) \succeq O, \quad \boldsymbol{y} \geq \boldsymbol{0}. \end{array} \tag{4.3}$$

As same as (4.1), the primal SDP relaxations of a given QCQP and the corresponding ε-perturbed QCQP, i.e, (3.2) and (4.2), have the same feasible region. On the other hand, since the perturbation term εP appears in the constraints of (4.3), the similar relationship does not hold in the dual side.

In the following sections, for a disconnected version of QCQPs, we will choice a suitable matrix P such that the ε-perturbed QCQP has a connected aggregated sparsity pattern graph. Then, we can apply our results in Chapter 3 to the obtained ε-perturbed QCQP instead of the original disconnected QCQP. The equivalence between the exactness of the original QCQP and that of the ε-perturbed QCQP will be discussed in each section, since this equivalence depends on the difference of assumptions (Assumptions 3.6 and 3.8).

4.3 Forest-structured QCQPs

In this section, we continue considering the forest-structured QCQP from Section 3.3 under the same assumption (Assumption 3.6). Since the feasible regions of QCQP (3.1) and its SDP relaxation are bounded by Assumption 3.6, the feasible regions for the perturbed problems (4.1) and (4.2) are also bounded. The following lemma states that the exactness of the SDP relaxation for the original problem (3.1) can be determined by that of perturbed problems (4.1) and (4.2).

Lemma 4.1. *Let $P \neq O$ be an $n\times n$ nonzero matrix, and $\{\varepsilon_t\}_{t=1}^{\infty}$ be a sequence such that $\lim_{t\to\infty} \varepsilon_t = 0$. If the SDP relaxation of the ε_t-perturbed problem* (4.1) *is exact for all $t = 1, 2, \ldots$, then the original problem* (3.1) *also has an exact SDP relaxation.*

We mention that Lemma 4.1 was first proved by Zhou et al. [62, Lemma 1] in the context of optimal power flow problems. Their proof is based on the closedness of the set of all the rank-1 matrices in their problems. However, since this set is not closed in general QCQPs, the proof needs to be modified to use in our context. In the following proof, we employ the closed set A instead of the set of all rank-1 matrices.

Proof. Let A and B be the feasible regions of (3.1) and (3.2), respectively:

$$
\begin{aligned}
A &:= \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, Q^p \bullet (\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}) \leq b_p,\ p = 1, \ldots, m \right\}, \\
B &:= \left\{ X \in \mathbb{S}^n \,\middle|\, X \succeq O,\ Q^p \bullet X \leq b_p,\ p = 1, \ldots, m \right\}.
\end{aligned}
$$

Note that A is a closed set and both A and B are bounded by Assumption 3.6. Thus, A is a compact set in $\mathbb{R}^n$. For any $t \geq 1$, let $\boldsymbol{x}_t$ and X_t be optimal solutions of (4.1) and (4.2) satisfying $\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}} = X_t$, which follows from the assumption on the exactness of the relaxation of (4.1). As a result, a sequence $\{\boldsymbol{x}_t\}_{t=1}^{\infty}$ can be defined.

Since the feasible regions of (3.1) and (4.1) are identical, we have $\boldsymbol{x}_t \in A$. From the compactness of A, there exists $\boldsymbol{x}_{\mathrm{lim}} := \lim_{t\to\infty} \boldsymbol{x}_t$ in A. As $X_{\mathrm{lim}} := \boldsymbol{x}_{\mathrm{lim}}\boldsymbol{x}_{\mathrm{lim}}^{\mathrm{T}} \in B$ by the relationship between A and B, the rank-1 matrix X_{lim} is also feasible for (3.2).

To show that X_{lim} is an optimal solution of (3.2), we assume that there exists another feasible $X_{\mathrm{opt}} \neq X_{\mathrm{lim}}$ such that $\nu := Q^0 \bullet X_{\mathrm{lim}} - Q^0 \bullet X_{\mathrm{opt}} > 0$. Since B is bounded, there exists μ such that $\|X\|_{\max} < \mu$ for any $X \in B$, which implies $\|\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\|_{\max} < \mu$ for any $\boldsymbol{x} \in A$. For a sufficiently large t such that

$$
\varepsilon_t < \frac{\nu}{4n^2\mu\|P\|_{\max}} \quad \text{and} \quad \|\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}} - X_{\mathrm{lim}}\|_{\max} < \frac{\nu}{2n^2\|Q^0\|_{\max}},
$$

we have

$$\begin{aligned} Q^0 \bullet \left(\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}} - X_{\mathrm{lim}}\right) &> -\frac{\nu}{2}, \\ \varepsilon_t P \bullet (\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}}) &> -\frac{\nu}{4}, \\ Q^0 \bullet X_{\mathrm{lim}} &= Q^0 \bullet X_{\mathrm{opt}} + \nu, \\ \frac{\nu}{4} &> \varepsilon_t P \bullet X_{\mathrm{opt}}. \end{aligned}$$

Consequently, adding these inequalities and the equality,

$$(Q^0 + \varepsilon_t P) \bullet (\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}}) > (Q^0 + \varepsilon_t P) \bullet X_{\mathrm{opt}},$$

which contradicts the optimality of $\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}}$ in (4.2). The desired result follows. □

Using the above lemma, we can generalize the exactness conditions for the forest-structured QCQPs with a connected aggregated sparsity pattern graph as follows.

Theorem 4.2. *Suppose that Assumption 3.6 holds and the aggregated sparsity pattern graph* $G(\mathcal{V}, \mathcal{E})$ *of* (3.1) *is a forest. Then, its SDP relaxation* (3.2) *is exact if for all* $(k, \ell) \in \mathcal{E}$*, the system* (3.5) *has no solutions.*

Proof. Let $G(\mathcal{V}, \mathcal{E})$ be the aggregate sparsity graph of the SDP relaxation defined in (2.13) for the forest-structured QCQP, and let κ denote the number of connected components of $G(\mathcal{V}, \mathcal{E})$. Since $G(\mathcal{V}, \mathcal{E})$ consists of one or more trees, we can construct a set $\mathcal{D}$ with $\kappa - 1$ edges such that $G(\mathcal{V}, \mathcal{E} \cup \mathcal{D})$ is a tree (i.e., a connected graph with no cycles). Let $P := \sum_{(i,j)\in\mathcal{D}} \boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}$ be a perturbation matrix defined by the $n \times n$ matrices $\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}$. With $\varepsilon > 0$, consider the ε-perturbed QCQP (4.1) with this P. Obviously, the aggregate sparsity pattern graph of the ε-perturbed QCQP (4.1) is $G(\mathcal{V}, \mathcal{E} \cup \mathcal{D})$. The system (3.5) that corresponds to the ε-perturbed QCQP (4.1) can be written as:

$$\boldsymbol{y} \geq 0, \quad Q^0 + \sum_{(i,j)\in\mathcal{D}} \varepsilon \left(\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}\right) + \sum_{p=1}^{m} y_p Q^p \succeq O, \tag{4.4a}$$

$$\left[Q^0 + \sum_{(i,j)\in\mathcal{D}} \varepsilon \left(\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}\right)\right]_{k\ell} + \sum_{p=1}^{m} y_p [Q^p]_{k\ell} = 0. \tag{4.4b}$$

For the exact SDP relaxation of (4.1), we need to show that (4.4) has no feasible solutions for all $(k, \ell) \in \mathscr{E} \cup \mathscr{D}$. First, suppose $(k, \ell) \in \mathscr{D}$. Since $[Q^p]_{k\ell} = 0$ ($\forall p = 0, 1, \dots, m$), the left hand side of (4.4b) becomes

$$\varepsilon \sum_{(i,j)\in\mathscr{D}} \left[\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}\right]_{k\ell} = \varepsilon > 0.$$

We have shown that (4.4b) does not hold for any $\boldsymbol{y} \geq 0$.

Next, we suppose $(k, \ell) \in \mathscr{E}$. Assume that (4.4) has a solution $\hat{\boldsymbol{y}}$. Then,

$$\hat{\boldsymbol{y}} \geq 0, \quad Q^0 + \sum_{(i,j)\in\mathscr{D}} \varepsilon\left(\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}\right) + \sum_{p=1}^{m} \hat{y}_p Q^p \succeq O, \quad [Q^0]_{k\ell} + \sum_{p=1}^{m} \hat{y}_p [Q^p]_{k\ell} = 0. \tag{4.5}$$

We now define a forest-structured matrix $\overline{S} \in \mathbb{S}^n$ as:

$$[\overline{S}]_{qr} = \begin{cases} \left[Q^0 + \sum\limits_{(i,j)\in\mathscr{D}} \varepsilon\left(\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}\right) + \sum\limits_{p=1}^{m} \hat{y}_p Q^p\right]_{qr} & (q,r) \notin \mathscr{D}, \\ 0 & (q,r) \in \mathscr{D}. \end{cases} \tag{4.6}$$

If $(q,r) \notin \mathscr{D}$, by definition, we have $\left[\boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}\right]_{qr} = 0$ for any $(i,j) \in \mathscr{D}$. If $(q,r) \in \mathscr{D}$, we have $\left[Q^0 + \sum_{p=1}^{m} \hat{y}_p Q^p\right]_{qr} = 0$ since $[Q^p]_{qr} = 0$ for any $p \in \{0\} \cup [m]$. Thus, we obtain that $\overline{S} = Q^0 + \sum\limits_{p=1}^{m} \hat{y}_p Q^p$. By Lemma 3.4, it follows that $\overline{S} \succeq O$, which implies that $\hat{\boldsymbol{y}}$ solves the following system:

$$\boldsymbol{y} \geq 0, \quad Q^0 + \sum_{p=1}^{m} y_p Q^p \succeq O, \quad [Q^0]_{k\ell} + \sum_{p=1}^{m} y_p [Q^p]_{k\ell} = 0. \tag{4.7}$$

We know that the system (4.7), which is equivalent to (3.5) for (3.1), has no feasible points by the assumption. This is a contradiction. Thus, (4.4) has no feasible points. By Theorem 3.7, the SDP relaxation for (4.1) with considering P is exact for all $\varepsilon > 0$.

We now take a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ which converges to zero so that the SDP relaxation for ε_t-perturbed QCQP (4.1) is exact for all $t = 1, 2, \dots$. By Lemma 4.1, we conclude that the SDP relaxation for (3.1) is exact. □

We note that the system (3.5) must be tested for feasibility for all $(k, \ell) \in \mathscr{E}$ in Theorem 4.2. In Theorem 3.7, we need to additionally examine whether the aggregate sparsity pattern graph is connected. Thus, the sufficient condition presented in Theorem 4.2 can be applied to more general QCQPs than the one in Theorem 3.7.

4.4 Bipartite Sparsity Structure Cases

In this section, we replace the assumption for connected graphs by a slightly different assumption (Assumption 4.3), and present a new condition for the exact SDP relaxation.

The following assumption is slightly stronger than Assumption 3.8 in the sense that it requires the existence of a feasible interior point of the dual SDP relaxation (3.3). However, it can be satisfied in practice without much difficulty.

Assumption 4.3. *The following two conditions hold:*

(i) the sets of optimal solutions for (3.2) *and* (3.3) *are nonempty; and*

(ii) at least one of the following two conditions holds:

(a) the feasible region of (3.2) *is bounded; or*

(b) for (3.3)*, the set of optimal solutions is bounded, and the interior of the feasible region is nonempty.*

Under the condition that the feasible region of a QCQP is bounded, Lemma 4.1 proved that the SDP relaxation is exact if a sequence of perturbed QCQPs that satisfy the exactness condition converges to the original one. In Theorem 4.2, this result was used to eliminate the requirement that the aggregated sparsity pattern graph is connected from Theorem 3.7. The following lemmas are extensions of the results in Section 4.3 under a weaker assumption.

Lemma 4.4. *Suppose that Assumption 4.3(i) and (ii)(a) hold. Let P be an $n \times n$ nonzero matrix, and $\{\varepsilon_t\}_{t=1}^{\infty}$ be a monotonically decreasing sequence such that $\lim_{t\to\infty} \varepsilon_t = 0$. If the SDP relaxation of the ε_t-perturbed problem* (4.1) *is exact for all $t = 1, 2, \dots$, then the SDP relaxation of the original problem* (3.1) *is also exact.*

Proof. Let A and B be the feasible regions of (3.1) and (3.2), respectively:

$$A := \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, Q^p \bullet (\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}) \le b_p, \quad p = 1, \dots, m \right\},$$
$$B := \left\{ X \in \mathbb{S}^n_+ \,\middle|\, Q^p \bullet X \le b_p, \quad p = 1, \dots, m \right\}.$$

Note that B is a compact set by the assumption. The intersection of B and the set of at most rank-1 matrices

$$\begin{aligned} B_1 &:= B \cap \{X \in \mathbb{S}^n \mid \operatorname{rank}(X) \leq 1\} \\ &= \left\{ X \succeq O \,\middle|\, \operatorname{rank}(X) \leq 1,\ Q^p \bullet X \leq b_p,\ p = 1, \ldots, m \right\} \end{aligned}$$

is also a compact set since $\{X \in \mathbb{S}^n \mid \operatorname{rank}(X) \leq 1\}$ is closed. There exists a bijection $f : A \to B_1$ given by $f(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}$, thus A is also a compact set. Then, we can apply an argument similar to the proof of Lemma 4.1.

For any $t \geq 1$, let $\boldsymbol{x}_t$ and X_t be optimal solutions of (4.1) and (4.2) satisfying $\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}} = X_t$, whose existence follows from the assumption on the exactness of the relaxation of (4.1). As a result, a sequence as $\{\boldsymbol{x}_t\}_{t=1}^{\infty}$ can be defined.

Since the feasible regions of (3.1) and (4.1) are identical, we have $\boldsymbol{x}_t \in A$. From the compactness of A, there exists $\boldsymbol{x}_{\lim} := \lim_{t\to\infty} \boldsymbol{x}_t \in A$. As $X_{\lim} := \boldsymbol{x}_{\lim}\boldsymbol{x}_{\lim}^{\mathrm{T}} \in B$ by the relationship between A and B, the rank-1 matrix $X_{\lim}$ is also feasible for (3.2).

To show that $X_{\lim}$ is an optimal solution of (3.2), we assume that there exists another feasible $X_{\mathrm{opt}} \neq X_{\lim}$ such that $\nu := Q^0 \bullet X_{\lim} - Q^0 \bullet X_{\mathrm{opt}} > 0$. Since B is bounded, there exists μ such that $\|X\|_{\max} < \mu$ for any $X \in B$, which implies $\|\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\|_{\max} < \mu$ for any $\boldsymbol{x} \in A$. For a sufficiently large t satisfying

$$\varepsilon_t < \frac{\nu}{4n^2\mu\|P\|_{\max}} \quad \text{and} \quad \|\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}} - X_{\lim}\|_{\max} < \frac{\nu}{2n^2\|Q^0\|_{\max}},$$

we have

$$\begin{aligned} Q^0 \bullet \left(\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}} - X_{\lim}\right) &> -\frac{\nu}{2}, \\ \varepsilon_t P \bullet (\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}}) &> -\frac{\nu}{4}, \\ Q^0 \bullet X_{\lim} &= Q^0 \bullet X_{\mathrm{opt}} + \nu, \\ \frac{\nu}{4} &> \varepsilon_t P \bullet X_{\mathrm{opt}}. \end{aligned}$$

Consequently, adding these inequalities and the equality, we obtain

$$(Q^0 + \varepsilon_t P) \bullet (\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}}) > (Q^0 + \varepsilon_t P) \bullet X_{\mathrm{opt}},$$

which contradicts the optimality of $\boldsymbol{x}_t\boldsymbol{x}_t^{\mathrm{T}}$ in (4.2). The desired result follows. □

Lemma 4.5. *Suppose that Assumption 4.3(i) and (ii)(a) hold. Let P be an $n\times n$ negative semidefinite nonzero matrix, and $\{\varepsilon_t\}_{t=1}^{\infty}$ be a monotonically decreasing sequence such that $\lim_{t\to\infty}\varepsilon_t = 0$. If the SDP relaxation of the ε_t-perturbed problem (4.1) is exact for all $t = 1, 2, \ldots$, then the SDP relaxation of the original problem (3.1) is also exact.*

Proof. Let $\Gamma := \left\{ \boldsymbol{y} \geq \boldsymbol{0} \,|\, S(\boldsymbol{y}) \succeq O \right\}$ be the feasible region of (3.3). Define

$$\Gamma(\varepsilon) := \left\{ \boldsymbol{y} \geq \boldsymbol{0} \,|\, S(\boldsymbol{y};\, \varepsilon) \succeq O \right\}$$

as the feasible region of the dual SDP relaxation (4.3) with the perturbation. Since P is negative semidefinite, we have $S(\boldsymbol{y};\, \varepsilon_1) \preceq S(\boldsymbol{y};\, \varepsilon_2)$ for any $\boldsymbol{y} \geq \boldsymbol{0}$ and $\varepsilon_1 > \varepsilon_2 > 0$, which indicates a monotonic structure of the sequence $\{\Gamma(\varepsilon_t)\}_{t=1}^{\infty}$:

$$\Gamma = \Gamma(0) \supseteq \cdots \supseteq \Gamma(\varepsilon_{t+1}) \supseteq \Gamma(\varepsilon_t) \supseteq \cdots.$$

From Assumption 4.3*(ii)(b)*, there exists a point $\bar{\boldsymbol{y}} \in \Gamma$ such that $S(\bar{\boldsymbol{y}}) \succ O$. Since each $\Gamma(\varepsilon_t)$ is a closed set and $\lim_{t\to\infty}\varepsilon_t = 0$, there exists an integer T such that $S(\bar{\boldsymbol{y}}; \varepsilon_T) \succ O$. In addition, it holds that $S(\bar{\boldsymbol{y}}; \varepsilon_t) \succeq S(\bar{\boldsymbol{y}}; \varepsilon_T)$ for $t \geq T$.

Let v_t^* and $B^*(\varepsilon_t)$ be the optimal value and the set of the corresponding optimal solutions of the SDP relaxation of (4.1) with the perturbation ε_t, respectively. From the assumptions that (3.1) has a feasible point and P is negative semidefinite, there is an upper bound $\bar{v}$ such that $v_t^* \leq \bar{v}$ for any t. Therefore, it holds that, for any $t \geq T$,

$$\begin{aligned}
B^*(\varepsilon_t) &= \left\{ X \in \mathbb{S}^n \,\middle|\, X \succeq O,\ (Q^0 + \varepsilon_t P) \bullet X = v_t^*,\ Q^p \bullet X \leq b_p \text{ for all } p \in [m] \right\} \\
&\subseteq \left\{ X \in \mathbb{S}^n \,\middle|\, X \succeq O,\ \left(Q^0 + \varepsilon_t P + \sum_{p=1}^{m} \bar{y}_p Q^p \right) \bullet X \leq v_t^* + \bar{\boldsymbol{y}}^{\mathrm{T}} \boldsymbol{b} \right\} \\
&= \left\{ X \in \mathbb{S}^n \,\middle|\, X \succeq O,\ S(\bar{\boldsymbol{y}};\, \varepsilon_t) \bullet X \leq v_t^* + \bar{\boldsymbol{y}}^{\mathrm{T}} \boldsymbol{b} \right\}, \\
&\subseteq \left\{ X \in \mathbb{S}^n \,\middle|\, X \succeq O,\ S(\bar{\boldsymbol{y}};\, \varepsilon_T) \bullet X \leq \bar{v} + \bar{\boldsymbol{y}}^{\mathrm{T}} \boldsymbol{b} \right\},
\end{aligned}$$

which implies $\bigcup_{t=T}^{\infty} B^*(\varepsilon_t)$ is bounded since $S(\bar{\boldsymbol{y}};\, \varepsilon_T) \succ O$. With the exact SDP relaxation of the perturbed problems and strong duality, we can consider $X^t \in B^*(\varepsilon_t)$, a rank-1 solution of the primal SDP relaxation, and $\boldsymbol{y}^t \in \Gamma(\varepsilon_t)$, an optimal solution of (4.3) with the perturbation ε_t satisfying $X^t S(\boldsymbol{y}^t;\, \varepsilon_t) = O$. We define a closed set as

$$U := \left(\bigcup_{t=T}^{\infty} B^*(\varepsilon_t) \right)$$

so that the sequence $\{X^t\}_{t=T}^{\infty} \subseteq U$. Since $\bigcup_{t=T}^{\infty} B^*(\varepsilon_t)$ is bounded, the set U is a compact set. As the sequence has an accumulation point, we let $X^{\lim} := \lim_{t\to\infty} X^t \in U$ by taking an appropriate subsequence from $\{X^t \mid t \geq T\}$. Moreover, since $\bigcup_{t=T}^{\infty} B^*(\varepsilon_t)$ is included in the feasible region of (3.2), its closure U is also in the same set, which implies that $X^{\lim}$ is an at most rank-1 feasible point of (3.2).

Finally, we show the optimality of $X^{\lim}$ for (3.2). Let v^* be the optimal value of the SDP relaxation of (3.1). For any t, we have

$$Q^0 \bullet X^t + \varepsilon \left(P \bullet X^t\right) = v_t^* \leq v^* \leq Q^0 \bullet X^t.$$

The first inequality comes from the negative semidefiniteness of P. By taking the limit of both sides, $Q^0 \bullet X^{\lim} = v^*$ follows. □

We note that the negative semidefiniteness of P assumed in Lemma 4.5 is not included in Lemma 4.4. In the subsequent discussion, we remove the assumption on the connectivity of G from Theorem 3.10 using Lemmas 4.4 and 4.5. We present an improved version of Theorem 3.10 for QCQPs with disconnected aggregated sparsity pattern graphs G.

Theorem 4.6. *Suppose that Assumption 4.3 holds and the aggregated sparsity pattern graph $G(\mathcal{V}, \mathcal{E})$ of (3.1) is a bipartite. Then, its SDP relaxation (3.2) is exact if, for all $(k, \ell) \in \mathcal{E}$, the system (3.6) has no solutions.*

Proof. Let L denote the number of connected components of G, and choose an arbitrary vertex u_i from the connected components indexed by $i \in [L]$. Then, we define the edge set

$$\mathcal{F} = \{(u_i, u_{i+1}) \mid i \in [L-1]\}.$$

Since $\mathcal{F}$ connects the ith and $(i+1)$th component, the graph $\tilde{G}(\mathcal{V}, \tilde{\mathcal{E}} := \mathcal{E} \cup \mathcal{F})$ is a connected and bipartite graph. Let $P \in \mathbb{S}^n$ be the negative of the Laplacian matrix of a subgraph $\hat{G}(\mathcal{V}, \mathcal{F})$ of $\tilde{G}$ induced by $\mathcal{F}$, i.e.,

$$P_{ij} = \begin{cases} -\deg(i) & \text{if } i = j, \\ 1 & \text{if } (i, j) \in \mathcal{F} \text{ or } (j, i) \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg(i)$ denotes the degree of the vertex i in the subgraph $\hat{G}(\mathcal{V}, \mathcal{F})$. Since the Laplacian matrix is positive semidefinite, P is negative semidefinite. By adding a perturbation εP with any $\varepsilon > 0$ into (3.1), we obtain an ε-perturbed QCQP (4.1) whose aggregated sparsity pattern graph is $\tilde{G}(\mathcal{V}, \tilde{\mathcal{E}})$.

To check the exactness of the SDP relaxation for (4.1) by Theorem 3.10, it suffices to show that the following system

$$\boldsymbol{y} \geq \mathbf{0}, \; S(\boldsymbol{y};\, \varepsilon) \succeq O, \; S(\boldsymbol{y};\, \varepsilon)_{k\ell} \leq 0$$

has no solutions for all $(k, \ell) \in \tilde{\mathcal{E}}$, where $S(\boldsymbol{y};\, \varepsilon) := (Q^0 + \varepsilon P) + \sum_{p \in [m]} y_p Q^p$. Let $\hat{\boldsymbol{y}}$ be an arbitrary vector satisfying the first two constraints, i.e., $\hat{\boldsymbol{y}} \geq \mathbf{0}$ and $S(\hat{\boldsymbol{y}};\, \varepsilon) \succeq O$.

(i) If $(k, \ell) \in \mathcal{F}$, then $P_{k\ell} = 1$ and $Q^p_{k\ell} = 0$ for any $p \in [0, m]$ by definition. Thus, we have

$$S(\hat{\boldsymbol{y}};\, \varepsilon)_{k\ell} = \varepsilon P_{k\ell} > 0.$$

(ii) If $(k, \ell) \in \tilde{\mathcal{E}} \setminus \mathcal{F} = \mathcal{E}$, the system (3.6) with (k, ℓ) has no solutions, which implies $S(\hat{\boldsymbol{y}})_{k\ell} > 0$. Since $(k, \ell) \notin \mathcal{F}$, we have $P_{k\ell} = 0$. Hence, it follows

$$S(\hat{\boldsymbol{y}};\, \varepsilon)_{k\ell} = S(\hat{\boldsymbol{y}})_{k\ell} > 0.$$

Therefore, all the systems have no solutions, and the SDP relaxation of (4.1) is exact.

Let $\{\varepsilon_t\}_{t=1}^{\infty} \subseteq \mathbb{R}_+$ be a monotonically decreasing sequence converging to zero, then the SDP relaxation of the ε_t-perturbed QCQP is exact as discussed above. By Lemmas 4.4 or 4.5, the desired result follows. □

CHAPTER 5

EXACT SDP RELAXATION OF SPECIAL CLASSES OF QCQPS

5.1 Introduction

In Chapters 3 and 4, we have proposed four exactness conditions for sparse QCQPs. In this chapter, by using these conditions, we discuss the exactness for special classes of QCQPs such as QCQPs with one equality constraint (Section 5.2), sign-definite QCQPs (Section 5.3), and the generalized trust-region subproblem (Section 5.4). In particular, for the second and the third classes, we generalize our exactness conditions to dense QCQPs by developing conversion methods which transform dense QCQPs into sparse ones.

5.2 QCQPs with One Equality Constraint

In this section, we consider the following QCQP:

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}} Q^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} = b_1. \end{aligned} \tag{5.1}$$

To write (5.1) in the form of (3.1), the equality constraint is converted into two inequality constraints $\boldsymbol{x}^{\mathrm{T}} Q^1 \boldsymbol{x} \leq b_1$ and $\boldsymbol{x}^{\mathrm{T}}(-Q^1)\boldsymbol{x} \leq -b_1$. It is clear that problem (5.1) is a special case of QCQP (3.1) with two constraints. Since both matrices Q^1 and $-Q^1$ appear in the inequality constraints of (5.1), the set $\{Q^0_{ij}, Q^1_{ij}, -Q^1_{ij}\}$ is not sign-definite unless $Q^1_{ij} = 0$. As a result, Corollary 5.2 cannot be used to determine the exactness of the SDP relaxation of (3.1). For (5.1), we propose the following exactness condition.

Corollary 5.1. *Let $\mathcal{E}$ be the aggregated sparsity pattern of* (5.1)*. Suppose that Assumption 3.6 holds, and* (5.1) *is a forest-structured QCQP. If*

$$Q^0 - \frac{Q^0_{ij}}{Q^1_{ij}} Q^1 \not\succeq O$$

for all $(i, j) \in \mathcal{E}$ such that $Q^1_{ij} \neq 0$, then the SDP relaxation for (5.1) *is exact.*

Proof. Let $G(\mathcal{V}, \mathcal{E})$ be the aggregate sparsity pattern graph for Assumption 3.6. If (3.5) is infeasible for all $(k, \ell) \in \mathcal{E}$, then we know from Theorem 4.2 that the SDP relaxation is exact. Thus, we analyze the feasibility of the system (3.5) for each $(k, \ell) \in \mathcal{E}$. We compute $S(\boldsymbol{y})$ in the system as:

$$\begin{aligned} S(\boldsymbol{y}) &= Q^0 + y_1 Q^1 + y_2(-Q^1), \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \mathbf{0}, \\ &= Q^0 + zQ^1, \quad z := y_1 - y_2 \in \mathbb{R}. \end{aligned}$$

Then, the system (3.5) is equivalent to the following system:

$$Q^0 + zQ^1 \succeq O, \quad Q^0_{k\ell} + zQ^1_{k\ell} = 0. \tag{5.2}$$

(a) If $[Q^1]_{k\ell} = 0$, then $[Q^0]_{k\ell} \neq 0$ by $(k, \ell) \in \mathcal{E}$. Since any $z \in \mathbb{R}$ does not satisfy the second equation in (5.2), the system (5.2) is infeasible.

(b) If $[Q^1]_{k\ell} \neq 0$, we assume that a solution z^* to (5.2) exists. By solving the second equation in (5.2) for z^* and substituting it into the first equation, we have

$$Q^0 - \frac{[Q^0]_{ij}}{[Q^1]_{ij}} Q^1 \succeq O,$$

which is a contradiction. Thus, the desired result follows. □

Burer and Ye [10] proposed several methods to extend their result of diagonal QCQPs to a general class of QCQPs with exact SDP relaxations. However, for some QCQPs, it is difficult to show the exactness of relaxations by their conditions; for example, it is hard to prove the existence of an exact relaxation for the generalized trust-region subproblem (see Section 5.4).

5.3 Sign-definite QCQPs

In this section, we focus on sign-definite QCQPs whose data matrices are element-wise sign-definite, that is, the set $\{Q^0_{ij}, \dots, Q^m_{ij}\}$ is sign-definite for all $(i, j) \in \mathcal{E}$. For subclasses of the sign-definite QCQP, several researches [7, 28, 36] proved the exactness of its SDP relaxation (see Section 2.5.3). Using the feasibility of the system (3.5), we provide an alternative proofs for them.

We first provide the exactness of the SDP relaxation for a forest-structured and sign-definite QCQP, which can be derived from Theorem 4.2.

Corollary 5.2. *Let $G(\mathcal{V}, \mathcal{E})$ be the aggregated sparsity pattern graph of* (3.1)*. Suppose that Assumption 3.6 holds, and that G is a forest. If the set $\{Q^0_{k\ell}, \dots, Q^m_{k\ell}\}$ is sign-definite for all $(k, \ell) \in \mathcal{E}$, then the SDP relaxation* (3.2) *is exact.*

Proof. Define $P \in \mathbb{S}^n$ given by

$$P_{ij} = \begin{cases} +1 & \text{if } Q^0_{ij} = 0 \text{ and } \sum_{p=1}^m Q^p_{ij} \geq 0, \\ -1 & \text{if } Q^0_{ij} = 0 \text{ and } \sum_{p=1}^m Q^p_{ij} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\varepsilon_t\}_{t=0}^\infty \subseteq \mathbb{R}$ be a sequence converging to zero. Consider the ε_t-perturbed problem (4.1) with a perturbation εP defined in Section 4.2. The corresponding system (3.5) is

$$\boldsymbol{y} \geq 0, \quad Q^0 + \varepsilon_t P + \sum_{p=1}^m y_p Q^p \succeq O, \quad Q^0_{k\ell} + \varepsilon_t P_{k\ell} + \sum_{p=1}^m y_p Q^p_{k\ell} = 0. \tag{5.3}$$

Now we analyze the feasibility of the system for each $(k, \ell) \in \mathcal{E}$.

(a) If $Q^0_{k\ell} \neq 0$, then for any $\boldsymbol{y} \geq \boldsymbol{0}$, the right-hand side of the third equation in (5.3) is

$$Q^0_{k\ell} + \sum_{p=1}^m y_p Q^p_{k\ell} = \begin{cases} > 0 & \text{if } Q^0_{k\ell} > 0, \\ < 0 & \text{if } Q^0_{k\ell} < 0, \end{cases}$$

by the sign-definite assumption on the set $\{Q^0_{k\ell}, \dots, Q^m_{k\ell}\}$. Therefore the system (5.3) has no solution.

(b) If $Q^0_{k\ell} = 0$, then for any $\boldsymbol{y} \geq \boldsymbol{0}$,

$$Q^0_{k\ell} + \sum_{p=1}^{m} y_p Q^p_{k\ell} = \begin{cases} \geq 0 & \text{if } \sum_{p=1}^{m} Q^p_{k\ell} \geq 0, \\ \leq 0 & \text{otherwise.} \end{cases}$$

From

$$\varepsilon_t P_{k\ell} = \begin{cases} \varepsilon > 0 & \text{if } \sum_{p=1}^{m} Q^p_{k\ell} \geq 0, \\ -\varepsilon < 0 & \text{otherwise,} \end{cases}$$

the right-hand side of the third equation in (5.3) is nonzero, which implies that the system has no solution.

As (a) and (b) cover all possible cases, the SDP relaxation of the ε_t-perturbed problem (4.1) is exact for any ε_t by Theorem 4.2. Therefore, the original QCQP (3.1) is also exact by Lemma 4.1. □

We will use Corollary 5.2 to prove the exact SDP relaxation for another class of QCQPs in Section 5.4.2.

As shown in the above corollary, for the forest-structured QCQP, it is easy to show that the element-wise sign-definiteness is a sufficient conditions for the exact SDP relaxation. However, for a wider class of QCQPs, we need to prepare (i) the exactness for nonnegative off-diagonal QCQPs under bipartite sparsity structure, and (ii) a conversion method from QCQPs to the corresponding QCQPs with bipartite sparsity structure. The following subsections discuss them, and then we generalize Corollary 5.2.

5.3.1 Nonnegative Off-diagonal QCQPs

By Theorem 4.6, we can also prove a known result, i.e., the exactness of the SDP relaxation for QCQPs with nonnegative off-diagonal data matrices $Q^0, \ldots, Q^m$, which was referred as Corollary 2.10(b). In this section, we provide an alternative proof for a specialized case, in which a QCQP has bipartite sparsity pattern graph.

Corollary 5.3. *Suppose that Assumption 3.8 holds, and that the aggregated sparsity pattern graph $G(\mathcal{V}, \mathcal{E})$ of (3.1) is bipartite and connected. If $Q^0_{ij} > 0$ for all $(i, j) \in \mathcal{E}$, and $Q^p_{ij} \geq 0$ for all $(i, j) \in \mathcal{E}$ and all $p \in [m]$, then the SDP relaxation is exact.*

Proof. Let $\hat{\boldsymbol{y}} \geq \mathbf{0}$ be any nonnegative vector satisfying $S(\hat{\boldsymbol{y}}) \succeq O$. By the assumption, for any $(i, j) \in \mathcal{E}$,

$$S(\hat{\boldsymbol{y}})_{ij} = Q^0_{ij} + \sum_{p \in [m]} \hat{y}_p Q^p_{ij} \geq Q^0_{ij} > 0.$$

Hence, the system (3.6) for every $(i, j) \in \mathcal{E}$ has no solutions. Therefore, by Theorem 3.10, the SDP relaxation is exact. □

Corollary 5.4. *Suppose that Assumption 4.3 holds, and that the aggregated sparsity pattern graph $G(\mathcal{V}, \mathcal{E})$ of* (3.1) *is bipartite. If $Q^p_{ij} \geq 0$ for all $(i, j) \in \mathcal{E}$ and for all $p \in [0, m]$, then the SDP relaxation is exact.*

Proof. Let $P \in \mathbb{S}^n$ be the negative of the Laplacian matrix of $G(\mathcal{V}, \mathcal{E})$, i.e.,

$$P_{ij} = \begin{cases} -\deg(i) & \text{if } i = j, \\ 1 & \text{if } (i, j) \in \mathcal{E} \text{ or } (j, i) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the Laplacian matrix is positive semidefinite, P is negative semidefinite. By adding a perturbation εP with any $\varepsilon > 0$, we obtain an ε-perturbed QCQP (4.1) whose aggregated sparsity pattern graph remains the same as the graph $G(\mathcal{V}, \mathcal{E})$.

To determine whether the SDP relaxation is exact for this ε-perturbed QCQP (4.1), it suffices to check the infeasibility of the system, according to Theorem 4.6:

$$\boldsymbol{y} \geq \mathbf{0}, \;\; S(\boldsymbol{y};\, \varepsilon) \succeq O, \;\; S(\boldsymbol{y};\, \varepsilon)_{k\ell} \leq 0.$$

Let $\hat{\boldsymbol{y}} \geq \mathbf{0}$ be an arbitrary vector satisfying the first two constraints, i.e., $\hat{\boldsymbol{y}} \geq \mathbf{0}$ and $S(\hat{\boldsymbol{y}};\, \varepsilon) \succeq O$. For every $(k, \ell) \in \mathcal{E}$, since $S(\hat{\boldsymbol{y}})_{k\ell} \geq 0$ and $P_{k\ell} > 0$, we have

$$S(\hat{\boldsymbol{y}};\, \varepsilon)_{k\ell} \geq \varepsilon P_{k\ell} > 0,$$

which implies that the system above has no solutions. Hence, by Theorem 4.6, the SDP relaxation of the ε-perturbed QCQP (4.1) is exact.

Let $\{\varepsilon_t\}_{t=1}^{\infty} \subseteq \mathbb{R}_+$ be a monotonically decreasing sequence converging to zero, then the SDP relaxation of the ε-perturbed QCQP is exact as discussed above. By Lemmas 4.4 and 4.5, the SDP relaxation of a QCQP with nonnegative off-diagonal elements and bipartite structures is also exact. □

Comparing Corollaries 5.3 and 5.4, the assumption of Corollary 5.3 contains the connectivity of the aggregated sparsity pattern graph. In this sense, Corollary 5.4 can be applied to more general QCQPs. However, Assumption 4.3 in Corollary 5.4 is slightly weaker than Assumption 3.8 in Corollary 5.3, and hence the use of these corollaries should be addressed for each sign-definite QCQP.

In Section 5.3.3, we will use Corollaries 5.3 and 5.4 to construct more generalized conditions.

5.3.2 Conversion

To transform a QCQP into an equivalent QCQP with bipartite aggregated sparsity pattern graph, we define a diagonal matrix $D^p \in \mathbb{S}^n$ with a positive number from the diagonal of Q^p for every p. In addition, off-diagonal elements of Q^p are divided into two nonnegative symmetric matrices $2N_+^p,\ 2N_-^p \in \mathbb{S}^n$ according to their signs such that $Q^p = D^p + 2N_+^p - 2N_-^p$. More precisely, for an arbitrary positive number $\delta > 0$,

$$D_{ii}^p = Q_{ii}^p + 2\delta,$$

$$2[N_+^p]_{ij} = \begin{cases} +Q_{ij}^p & \text{if } i \neq j \text{ and } Q_{ij}^p > 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$2[N_-^p]_{ij} = \begin{cases} -Q_{ij}^p & \text{if } i \neq j \text{ and } Q_{ij}^p < 0, \\ 2\delta & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}$$

We introduce a new variable $\boldsymbol{z}$ such that $\boldsymbol{z} := -\boldsymbol{x}$. Then,

$$\boldsymbol{x}^{\mathrm{T}} Q^p \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} D^p + 2N_+^p & N_-^p \\ N_-^p & O \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix}, \tag{5.4}$$

The constraint $\boldsymbol{z} = -\boldsymbol{x}$ can be expressed as $\|\boldsymbol{x} + \boldsymbol{z}\|^2 \leq 0$, which can be written as

$$(\boldsymbol{x} + \boldsymbol{z})^{\mathrm{T}}(\boldsymbol{x} + \boldsymbol{z}) = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & I \\ I & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} \leq 0.$$

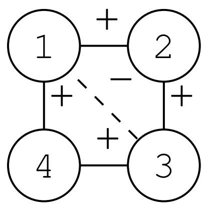


Figure 5.1: An aggregated sparsity pattern graph with edge signs. The solid and dashed lines show that the corresponding σ_{ij} are $+1$ and -1, respectively. Both lines indicate the existence of nonzero elements in some Q^p.

Thus, we have an equivalent QCQP:

$$\begin{aligned} \min \quad & \begin{bmatrix} x \\ z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} D^0 + 2N_+^0 & N_-^0 \\ N_-^0 & O \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} x \\ z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} D^p + 2N_+^p & N_-^p \\ N_-^p & O \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq b_p, \quad p \in [m], \\ & \begin{bmatrix} x \\ z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & I \\ I & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq 0. \end{aligned} \tag{5.5}$$

Note that (5.5) includes $m + 1$ constraints and all off-diagonal elements of data matrices are nonnegative since N_+^p and N_-^p are nonnegative. Let $\bar{G}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ denote the aggregated sparsity pattern graph of (5.5). The number of vertices in $\bar{G}$ is twice as many as that in G due to the additional variable z. If $\bar{G}$ is bipartite and $Q_{ij}^0 \neq 0$ for all $(i, j) \in \mathcal{E}$, the SDP relaxation of (5.5) is exact since the assumptions of Corollary 5.3 are satisfied.

The rest of this subsection illustrates this conversion using an easy-to-understand example.

Example 5.5. *Now, consider an instance of QCQP (3.1) with $n = 4$, $Q_{24}^p = 0\,(p \in [0, m])$ and the edge signs*

$$\sigma_{12} = +1, \quad \sigma_{13} = -1, \quad \sigma_{14} = +1, \quad \sigma_{23} = +1, \quad \sigma_{34} = +1.$$

Figure 5.1 illustrates the above signs. We also suppose that $Q_{ij}^0 \neq 0$ for all $(i, j) \in \mathcal{E}$. Then, for any distinct $i, j \in [n]$, the set $\left\{ Q_{ij}^0, \ldots, Q_{ij}^m \right\}$ is sign-definite by definition. Since there exist odd cycles, e.g., $\{(1,2), (2,3), (3,1)\}$, the aggregated sparsity pattern graph of the QCQP with the above edge *signs is not bipartite. Next, we transform the QCQP instance into an equivalent QCQP with bipartite aggregated sparsity pattern graph. Since* $n = 4$*, we see* $\bar{\mathcal{V}} = [8]$*. Figure 5.2a displays* $\bar{G}$ *from*

$$\begin{bmatrix} D^p + 2N_+^p & N_-^p \\ N_-^p & O \end{bmatrix} = \left[\begin{array}{cccc|cccc} Q_{11}^p & Q_{12}^p & 0 & Q_{14}^p & 0 & 0 & -\frac{1}{2}Q_{13}^p & 0 \\ Q_{21}^p & Q_{22}^p & Q_{23}^p & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{32}^p & Q_{33}^p & Q_{34}^p & -\frac{1}{2}Q_{31}^p & 0 & 0 & 0 \\ Q_{41}^p & 0 & Q_{43}^p & Q_{44}^p & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2}Q_{13}^p & 0 & & & & \\ 0 & 0 & 0 & 0 & & O & & \\ -\frac{1}{2}Q_{31}^p & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \end{array}\right] + \delta \left[\begin{array}{c|c} 2I & I \\ \hline I & O \end{array}\right]$$

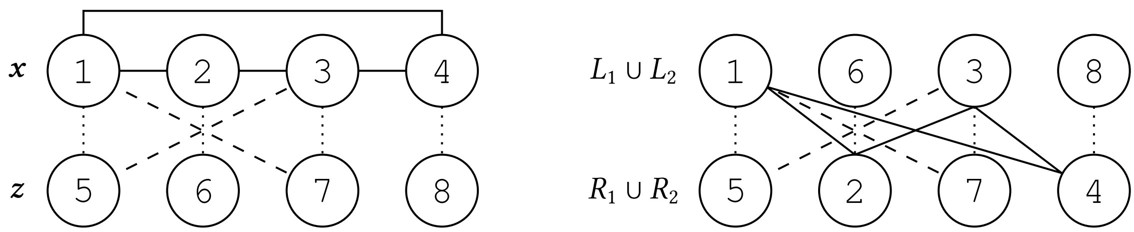
and $[I\ I; I\ I]$*. There exist three types of edges:*

$$\begin{cases} (i) & (1,2),(2,3),(3,4),(1,4); \\ (ii) & (1,7),(3,5); \\ (iii) & (1,5),(2,6),(3,7),(4,8). \end{cases}$$

The edges in (i) and (ii) are derived from four elements of N_+^p *on the upper-left of the data matrices, and two elements of* N_-^p *on the upper-right and the lower-left of the data matrices, respectively. The edges for (iii) represent off-diagonal elements in* $[I\ I; I\ I]$ *in the new constraint. In Figure 5.2a, the cycle in the solid lines is bipartite with the vertices* $\{1,2,3,4\}$*, and hence its vertices can be divided into two distinct sets* $L_1 = \{1,3\}$ *and* $R_1 = \{2,4\}$*. If we let* $L_2 := \{6,8\}$ *and* $R_2 := \{5,7\}$*, there are no edges between any distinct* i, j *in* $L_1 \cup L_2$*, and the same is true for* $R_1 \cup R_2$*. The graph* $\bar{G}$ *is thus bipartite (Figure 5.2b). We can conclude that the SDP relaxation of* (5.5) *is exact by Corollary 5.3.*

5.3.3 Graph-based Condition of Sign-definite QCQP

Similar to the last part of Example 5.5, the SDP relaxation of any QCQP that satisfies Theorem 2.9 can be shown to be exact by the above conversion. Therefore, Theorem 3.10 includes a wider classes of QCQPs than Theorem 2.9. We prove this assertion in the following.



(a) Vertices are divided into two groups: the upper vertices correspond to $\boldsymbol{x}$ while the lower ones correspond to $\boldsymbol{z}$.

(b) Vertices are reorganized to show the bipartite structure of the graph.

Figure 5.2: Aggregated sparsity pattern graph of the transformed example. The solid lines and the dashed lines come from N_+^p and N_-^p, respectively. The dotted lines are for the new constraint $\|\boldsymbol{x}+\boldsymbol{z}\|^2 \leq 0$.

Proposition 5.6. *Suppose that Assumption 3.8 holds, the aggregated sparsity pattern graph $G(\mathcal{V},\mathcal{E})$ of (3.1) is connected, and that $Q^0_{ij} \neq 0$ for all $(i,j) \in \mathcal{E}$. If (3.1) satisfies the assumption of Theorem 2.9, then the SDP relaxation of the equivalent QCQP (5.5) is exact by Corollary 5.3. In addition, the SDP relaxation of (3.1) is also exact.*

Proof. Let $\bar{G}(\bar{\mathcal{V}},\bar{\mathcal{E}})$ be the aggregated sparsity pattern graph of (5.5). Since the number of variables is $2n$, $\bar{\mathcal{V}} = [2n]$ holds. The edges in $\bar{G}$ are:

$$\begin{cases} \text{(i)} \quad (i,j) & \text{for } i,j \in \mathcal{V} \text{ such that } \sigma_{ij} = +1, \\ \text{(ii)} \quad (i,j+n),(j,i+n) & \text{for } i,j \in \mathcal{V} \text{ such that } \sigma_{ij} = -1, \\ \text{(iii)} \quad (i,i+n) & \text{for } i \in \mathcal{V}. \end{cases}$$

Note that no edges exist among the vertices in $\{n+1,\ldots,2n\}$. By the definition of (5.5), an edge (i,j) with $\sigma_{ij} = -1$ in G is decomposed into two paths with positive signs in $\bar{G}$: (a) the edges $(j,i+n)$ and $(i+n,i)$; (b) the edges $(i,j+n)$ and $(j+n,j)$, as shown in Figure 5.3. Since G is connected, so is the graph $\bar{G}$. Recall that all off-diagonal elements of the data matrices in (5.5) are nonnegative, since both N_+^p and N_-^p are nonnegative matrices. In particular, for each $(i,j) \in \bar{\mathcal{E}}$, the (i,j)th element of the matrix in the objective function is not only nonnegative but also positive by assumption. Thus, to apply Corollary 5.3, it remains to show that $\bar{G}$ is

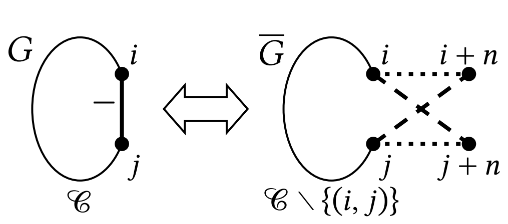


Figure 5.3: An edge with the negative sign. If the cycle $\mathscr{C}$ has the edge (i, j) with $\sigma_{ij} = -1$, then (i, j) is decomposed into two paths: (a) $(j, i + n)$ and $(i + n, i)$ via the vertex $i + n$; (b) $(i, j + n)$ and $(j + n, j)$ via the vertex $j + n$.

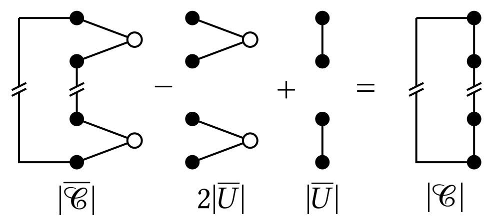


Figure 5.4: Removing and adding edges, and calculating of the number of edges if $\bar{\mathscr{U}} = 2$. The black circles are the vertices in $[n]$ while the white circles represent those in $[n + 1, 2n]$.

bipartite.

Assume, to the contrary, that there exists an odd cycle $\bar{\mathscr{C}}$ in $\bar{G}$. Let $\bar{\mathscr{U}} \subseteq [n+1, 2n]$ denote the set of vertices on $[n + 1, 2n]$ in $\bar{\mathscr{C}}$. As illustrated in Figure 5.3, any vertex $v := i + n \in \bar{\mathscr{U}}$ connects with i and $j \in \mathscr{V}$ in $\bar{\mathscr{C}}$. Hence for every vertex $v \in \bar{\mathscr{U}}$, by removing the edges (i, v) and (v, j) from $\bar{\mathscr{C}}$ and adding the edge (i, j) with the negative sign to $\bar{\mathscr{C}}$, we obtain a new cycle $\mathscr{C}$ in G. Since $2|\bar{\mathscr{U}}|$ edges are removed and $|\bar{\mathscr{U}}|$ edges are added in this procedure, it follows $|\mathscr{C}| = |\bar{\mathscr{C}}| - 2|\bar{\mathscr{U}}| + |\bar{\mathscr{U}}| = |\bar{\mathscr{C}}| - |\bar{\mathscr{U}}|$. Figure 5.4 displays a case for $|\bar{\mathscr{U}}| = 2$. Thus, if $|\bar{\mathscr{U}}|$ is even (odd), $|\mathscr{C}|$ is odd (resp., even), hence, by (2.20) in Theorem 2.9, the number of negative edges in $\mathscr{C}$ must be odd (resp., even). However, the number of negative edges in $\mathscr{C}$ is equal to $|\bar{\mathscr{U}}|$ since $\bar{\mathscr{C}}$ has no negative edges and all the additional edges in the conversion from $\bar{\mathscr{C}}$ to $\mathscr{C}$ are negative. This is a contradiction. Therefore, there are no odd cycles in $\bar{G}$, which implies $\bar{G}$ is bipartite.

By Corollary 5.3, the SDP relaxation of (5.5) is exact. Since there exists a bijection $\boldsymbol{x} \mapsto [\boldsymbol{x}; -\boldsymbol{x}]$ between optimal solutions of (3.1) and those of (5.5), the SDP relaxation of (3.1) is also exact. □

Proposition 5.6 is proved under the assumptions that (i) G is connected; (ii) $Q^0_{ij} \neq 0$ for all $(i, j) \in \mathscr{E}$. Although these assumptions may seem strong, we next show that they can be removed by using Corollary 5.4 instead of Corollary 5.3.

Proposition 5.7. *Suppose that Assumption 4.3 holds and no conditions on sparsity are considered.*

If (3.1) *satisfies the assumption of Theorem 2.9, then* (5.5) *also satisfies that of Corollary 5.4. In addition, the exactness of its SDP relaxation can be proved by Theorem 4.6.*

5.3.4 Nonpositive Off-diagonal QCQPs

At the end of this section, we apply Proposition 5.6 to a class of QCQPs where all the off-diagonal elements of every matrix $Q^0, \ldots, Q^m$ are nonpositive. We call QCQPs in this class nonpositive off-diagonal QCQPs. It is well-known that their SDP relaxations are exact [28]. By applying the same transformation above, we obtain (5.5) with $N_+^p = O$ for every p since no positive off-diagonal elements exist. The diagonal elements of D^p do not generate edges in the aggregated sparsity pattern graph, thus, the data matrices in (5.5) induce a bipartite sparsity pattern graph. Therefore, the SDP relaxation is exact. This can be regarded as an alternative proof for [28] and Corollary 2.10(c).

Corollary 5.8. *Under Assumption 3.8, the SDP relaxation of a nonpositive off-diagonal QCQP is exact if the aggregate sparsity pattern graph $G(\mathcal{V}, \mathcal{E})$ of* (3.1) *is connected and $Q^0_{ij} < 0$ for all $(i, j) \in \mathcal{E}$.*

Corollary 5.9. *Under Assumption 4.3, the SDP relaxation of a nonpositive off-diagonal QCQP is exact.*

5.4 Trust-region Subproblems via Tridiagonalization

To apply the results in Chapters 3 and 4 to a wider class of QCQPs, we consider QCQPs (3.1) which is not forest-structured. In this section, we assume that all the matrices of a QCQP are simultaneously tridiagonalizable. In Section 5.4.1, simultaneous tridiagonalization is discussed in detail when $m = 1$ (only one quadratic constraint). Then, a method to determine the exactness of the SDP relaxation for these QCQPs is described. In addition, we provide an alternative proof for the exactness of generalized trust-region subproblems (GTRSs).

The matrices $Q^0, Q^1, \ldots, Q^m \in \mathbb{S}^n$ are called simultaneous tridiagonalizable if there exist a nonsingular matrix U such that $U^\top Q^0 U, U^\top Q^1 U, \ldots, U^\top Q^m U$ are tridiagonal matrices. The simultaneous tridiagonalization is a generalization of the simultaneous diagonalization used in [10]. By replacing $U^{-1}\boldsymbol{x}$ with $\hat{\boldsymbol{x}}$, we obtain the QCQP with tridiagonal data matrices which is equivalent to (2.7):

$$\begin{array}{ll} \min & \hat{\boldsymbol{x}}^{\mathrm{T}}\left(U^{\mathrm{T}}Q^{0}U\right)\hat{\boldsymbol{x}} + 2\left(U^{\mathrm{T}}\boldsymbol{q}^{0}\right)^{\mathrm{T}}\hat{\boldsymbol{x}} \\ \text{s.t.} & \hat{\boldsymbol{x}}^{\mathrm{T}}\left(U^{\mathrm{T}}Q^{p}U\right)\hat{\boldsymbol{x}} + 2\left(U^{\mathrm{T}}\boldsymbol{q}^{p}\right)^{\mathrm{T}}\hat{\boldsymbol{x}} \leq b_p, \quad p = 1,\dots,m. \end{array} \tag{5.6}$$

The standard SDP relaxation of (5.6) is

$$\begin{array}{ll} \min & \left(U^{\mathrm{T}}Q^{0}U\right)\bullet\widehat{X} + 2\left(U^{\mathrm{T}}\boldsymbol{q}^{0}\right)^{\mathrm{T}}\hat{\boldsymbol{x}} \\ \text{s.t.} & \left(U^{\mathrm{T}}Q^{p}U\right)\bullet\widehat{X} + 2\left(U^{\mathrm{T}}\boldsymbol{q}^{p}\right)^{\mathrm{T}}\hat{\boldsymbol{x}} \leq b_p, \quad p = 1,\dots,m, \\ & \widehat{X} \succeq \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^{\mathrm{T}}. \end{array} \tag{5.7}$$

Obviously, if $\hat{\boldsymbol{x}}$ is an optimal solution of (5.6), then $U\hat{\boldsymbol{x}}$ is an optimal solution of the original problem (2.7). Therefore, the SDP relaxation (2.11) of (2.7) is at least as strong as the corresponding SDP relaxation (5.7) for (5.6). As a result, if (5.6) has an exact relaxation, then (2.7) also has an exact relaxation. When (5.6) becomes a forest-structured QCQP in the homogeneous form, the exactness conditions in Chapters 3 and 4 can be applied to (5.6). We note that the exactness of the SDP relaxation for QCQPs (2.7) can be determined if their data matrices are simultaneous tridiagonalizable even when they are not forest-structured.

5.4.1 Simultaneous Tridiagonalization

Simultaneous tridiagonalization of multiple matrices is an extension of simultaneous diagonalization, and it can be achieved by finding a nonsingular matrix that transforms all matrices to tridiagonal matrices. Recently, Sidje [48] (See also Garvey et al. [17]) introduced conditions under which two matrices are simultaneous tridiagonalizable.

Proposition 5.10. *(Section 2 [48]) Let $K, M \in \mathbb{S}^n$ and $0 \neq \gamma \in \mathbb{R}$. Suppose that the matrix pencil $K - \gamma M$ is nonsingular. Then, K and M are simultaneously tridiagonalizable.*

If the data matrices are simultaneously tridiagonalizable with the nonsingular matrix U, we also need a method to compute U. Sidje proposed a method to compute such a nonsingular matrix U on the basis of Householder reflections. We briefly describe his method for the simultaneous tridiagonalization, and then analyze this method to find new properties which will be used in the proof of the GTRS.

In the beginning of the Sidje's recursive procedure [48], the matrices are initialized as $K^n := K, M^n := M \in \mathbb{S}^n$. Then, an appropriate nonsingular matrix $U^k = \left[1, \mathbf{0}_{k-1}^{\mathrm{T}}; \boldsymbol{u}_k, \widetilde{U}^k\right] \in \mathbb{R}^{k\times k}$ with nonsingular $\widetilde{U}^k \in \mathbb{R}^{(k-1)\times(k-1)}$ and $\boldsymbol{u}_k \in \mathbb{R}^{k-1}$ is chosen such that

$$(U^k)^{\mathrm{T}} K^k U^k = \left[\begin{array}{c|cc} \xi_k & \tau_k & \mathbf{0}_{k-2}^{\mathrm{T}} \\ \hline \tau_k & & \\ & \multicolumn{2}{c}{K^{k-1}} \\ \mathbf{0}_{k-2} & & \end{array}\right], \quad (U^k)^{\mathrm{T}} M^k U^k = \left[\begin{array}{c|cc} \nu_k & \sigma_k & \mathbf{0}_{k-2}^{\mathrm{T}} \\ \hline \sigma_k & & \\ & \multicolumn{2}{c}{M^{k-1}} \\ \mathbf{0}_{k-2} & & \end{array}\right] \tag{5.8}$$

at each step $k = n, \ldots, 2$. Here $\xi_k, \tau_k, \nu_k, \sigma_k \in \mathbb{R}$, and $K^{k-1}, M^{k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$. This procedure generates two tridiagonal matrices:

$$U^{\mathrm{T}} K^n U = \left[\begin{array}{cccc|c} \xi_n & \tau_n & & & \\ \tau_n & \xi_{n-1} & \ddots & & \\ & \ddots & \ddots & \tau_3 & \\ & & \tau_3 & \xi_2 & \tau_2 \\ \hline & & & \tau_2 & K^1 \end{array}\right] \quad \text{and} \quad U^{\mathrm{T}} M^n U = \left[\begin{array}{cccc|c} \nu_n & \sigma_n & & & \\ \sigma_n & \nu_{n-1} & \ddots & & \\ & \ddots & \ddots & \sigma_3 & \\ & & \sigma_3 & \nu_2 & \sigma_2 \\ \hline & & & \sigma_2 & M^1 \end{array}\right],$$

where

$$U := U^n \left[\begin{array}{c|c} I_1 & \\ \hline & U^{n-1} \end{array}\right] \cdots \left[\begin{array}{c|c} I_{n-2} & \\ \hline & U^2 \end{array}\right].$$

Now, consider the kth step. To have nonzero elements only on the diagonals, superdiagonals, and subdiagonals by the operation shown in (5.8), $\widetilde{U}^k$ should satisfy the following equations:

$$\begin{aligned} \left(\widetilde{U}^k\right)^{\mathrm{T}} \left(K^k_{\{1\},\{2,\ldots,k\}} + K^k_{\{2,\ldots,k\}} u_k\right) &= \tau_k \boldsymbol{e}_1, \\ \left(\widetilde{U}^k\right)^{\mathrm{T}} \left(M^k_{\{1\},\{2,\ldots,k\}} + M^k_{\{2,\ldots,k\}} u_k\right) &= \sigma_k \boldsymbol{e}_1, \end{aligned} \tag{5.9}$$

where $\boldsymbol{e}_1 \in \mathbb{R}^{k-1}$ is the one vector and $K^k_{\{2,\ldots,k\}}$ means the submatrix of K^k obtained by removing the first row and column from K^k. In his procedure, $\widetilde{U}^k$ is chosen to be a Householder reflection, and therefore, nonsingular, and the existence of $\boldsymbol{u}_k$ follows imposing

$$\left(\widetilde{U}^k\right)^{\mathrm{T}} \left(K^k_{\{1\},\{2,\ldots,k\}} + K^k_{\{2,\ldots,k\}} u_k\right) = \gamma \left(\widetilde{U}^k\right)^{\mathrm{T}} \left(M^k_{\{1\},\{2,\ldots,k\}} + M^k_{\{2,\ldots,k\}} u_k\right) \tag{5.10}$$

for $0 \neq \gamma \in \mathbb{R}$ since $K - \gamma M$ is nonsingular. By restricting γ to be positive, we extend Proposition 5.10.

Lemma 5.11. *Let $K, M \in \mathbb{S}^n$ and $\mathbb{R} \ni \gamma > 0$. Suppose that the matrix pencil $K - \gamma M$ is nonsingular. Then, there exists a nonsingular matrix $U \in \mathbb{S}^n$ that simultaneously tridiagonalizes K and M. Moreover, for any $i \in [n-1]$, the $(i, i+1)$th elements of $U^\mathrm{T} K U$ and $U^\mathrm{T} M U$ are sign-definite.*

Proof. Substituting (5.9) into (5.10), we obtain $\tau_k = \gamma \sigma_k$. Since $\gamma > 0$, the set $\{\tau_k, \sigma_k\}$ is sign-definite for any $k = n, \ldots, 2$. The variables τ_k and σ_k appear on the $(1,2), \ldots, (n-1, n)$th elements of $U^\mathrm{T} K^n U$ and $U^\mathrm{T} M^n U$, respectively. □

If the elements on superdiagonal of all matrices can be transformed to sign-definite elements, then we can apply Corollary 5.2 to show the exactness of the SDP relaxation, which is discussed next in Section 5.4.2.

5.4.2 Generalized Trust-region Subproblem

Consider the Generalized trust-region subproblem (GTRS):

$$\begin{aligned} \min \quad & \boldsymbol{x}^\mathrm{T} Q^0 \boldsymbol{x} + 2(\boldsymbol{q}^0)^\mathrm{T} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^\mathrm{T} Q^1 \boldsymbol{x} + 2(\boldsymbol{q}^1)^\mathrm{T} \boldsymbol{x} \le b_1. \end{aligned} \tag{5.11}$$

The GTRS can be considered a generalization of the classical TRS that minimizes a quadratic objective function over an Euclidean ball, i.e., the GTRS with $Q^1 \succ O$. This fact can be seen by substituting $\sqrt{Q^1}\boldsymbol{x}$ as a new variable $\tilde{\boldsymbol{x}}$, where $\sqrt{Q^1}$ denotes the Cholesky factor of Q^1, i.e., $Q^1 = \sqrt{Q^1}^\mathrm{T} \sqrt{Q^1}$. In the TRS, Q^0 is not necessarily positive definite. Although the TRS is nonlinear and nonconvex, its SDP relaxation is well-known to be exact [44]. The GTRS shares nice properties with the TRS. For example, by using S-lemma, it is proved that the SDP relaxation of the GTRS is always exact under the Slater's condition.

Since the GTRS is a QCQP (2.7) with only one constraint ($m = 1$), we can also prove that the GTRS admits an exact SDP relaxation with the exactness conditions presented in Chapters 3 and 4. Although the result is not new, our proof shows a procedure on how to apply the exactness conditions for tridiagonal QCQPs to wider classes of QCQPs. In fact, the proof demonstrates how to determine the exactness of a given QCQP in practice, and it can be used to analyze the exactness conditions for broader classes of QCQPs.

Any QCQP can be formulated in the equivalent homogeneous QCQP as in (3.1), thus, it is sufficient to consider the following QCQP with an additional variable to discuss the exactness for the GTRS (5.11):

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}} \bar{Q}^0 \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}} \bar{Q}^1 \boldsymbol{x} \leq 0, \\ & \boldsymbol{x}^{\mathrm{T}} E_{11} \boldsymbol{x} = 1. \end{aligned} \tag{5.12}$$

where

$$\bar{Q}^p := \begin{bmatrix} -b_p & \boldsymbol{q}^{p\mathrm{T}} \\ \boldsymbol{q}^p & Q^p \end{bmatrix} \quad (p = 0, 1) \quad \text{and} \quad b_0 = 0.$$

For simplicity, we assume that the number of variables in (5.12) is n. As (5.12) has the additional equality constraint, the problem (5.12) is a QCQP with three inequality constraints, and (5.12) is no longer a GTRS. In the subsequent discussion, we describe the exactness for (5.12) using the simultaneous tridiagonalization. In particular, we show that the SDP relaxation of GTRS (5.11) is exact as follows.

Theorem 5.12. *Suppose that GTRS* (5.12) *satisfies Assumption 3.6. Then, the SDP relaxation of* (5.12) *is exact.*

Proof. Let us first consider the case when $\bar{Q}^0 - \gamma \bar{Q}^1$ is nonsingular for some $\gamma > 0$. By Lemma 5.11, we obtain a nonsingular matrix $U \in \mathbb{S}^n$ that simultaneously tridiagonalizes Q^0 and Q^1, and the $(i, i+1)$th elements of $U^{\mathrm{T}} \bar{Q}^0 U$ and $U^{\mathrm{T}} \bar{Q}^1 U$ become sign-definite for any $i \in [n-1]$. Let these tridiagonal matrices

$$R^p = [r_{ij}^p] := U^{\mathrm{T}} \bar{Q}^p U, \quad p = 0, 1.$$

For any $i \in [n-1]$, the set $\{r_{i,i+1}^0, r_{i,i+1}^1\}$ is sign-definite. By letting $\boldsymbol{y} = U^{-1}\boldsymbol{x}$, the homogeneous TRS (5.12) can be transformed to an equivalent tridiagonal QCQP:

$$\begin{aligned} \min \quad & \boldsymbol{y}^{\mathrm{T}} R^0 \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{y}^{\mathrm{T}} R^1 \boldsymbol{y} \leq 0, \\ & \boldsymbol{y}^{\mathrm{T}} E_{11} \boldsymbol{y} = 1. \end{aligned} \tag{5.13}$$

Notice that the first row of U is $[1 \ \boldsymbol{0}_{n-1}{}^{\mathrm{T}}]$ by the construction of section 5.4.1. By Corollary 5.2, the SDP relaxation of (5.13) is exact. As U is nonsingular, the SDP relaxation of the original problem (5.12) is also exact.

Now consider the other case, i.e., there is no $\gamma > 0$ such that $\bar{Q}^0 - \gamma\bar{Q}^1$ is nonsingular. We will show that, for a fixed $\gamma > 0$ and any $\varepsilon > 0$, the SDP relaxation of the following ε-perturbed problem is exact:

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}}\left(\bar{Q}^0 + \varepsilon I_n\right)\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}^{\mathrm{T}}\bar{Q}^1\boldsymbol{x} \leq 0, \\ & \boldsymbol{x}^{\mathrm{T}}E_{11}\boldsymbol{x} = 1. \end{aligned} \tag{5.14}$$

Then, by Lemma 4.1, the SDP relaxation of the original problem (5.12) is also exact. Since $\det\left(\bar{Q}^0 - \gamma\bar{Q}^1\right) = 0$, $\bar{Q}^0 - \gamma\bar{Q}^1$ can be diagonalized, as below:

$$\begin{bmatrix} \Lambda & \\ & O_{n-\mathrm{rk}} \end{bmatrix} = P^{\mathrm{T}}\left(\bar{Q}^0 - \gamma\bar{Q}^1\right)P, \tag{5.15}$$

where $\mathrm{rk} := \mathrm{rank}\left(\bar{Q}^0 - \gamma\bar{Q}^1\right)$, $P \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, and $\Lambda \in \mathbb{S}^{\mathrm{rk}}$ is a diagonal matrix. By adding perturbation with sufficiently small $\varepsilon > 0$ to both sides of (5.15),

$$\begin{bmatrix} \Lambda & \\ & O_{n-\mathrm{rk}} \end{bmatrix} + \varepsilon I_n = P^{\mathrm{T}}\left(\bar{Q}^0 - \gamma\bar{Q}^1\right)P + \varepsilon P^{\mathrm{T}}I_nP = P^{\mathrm{T}}\left(\bar{Q}^0 + \varepsilon I_n - \gamma\bar{Q}^1\right)P,$$

$\left(\bar{Q}^0 + \varepsilon I_n\right) - \gamma\bar{Q}^1$ becomes nonsingular. From the first case of this proof, the SDP relaxation of (5.14) must be exact. □

CHAPTER 6

CONCLUSION AND OUTLOOK

Many practical optimization problems have sparsity structures on their data. In particular, on QCQPs, such sparsity structures can be exploited to characterize their SDP relaxations. This thesis has focused on developing sufficient conditions of QCQPs, under which their optimal solution can be recovered from the SDP relaxation, by dividing sparsity to four classes of graphs: connected forest (i.e., tree) structures, connected bipartite structures, disconnected forest structures, and disconnected bipartite structures. In this chapter, we conclude this thesis by summarizing the main results and discussing several research directions.

6.1 Summary

The sparsity pattern of a matrix plays an important role for the determination of the rank of matrices as discussed in Section 3.2. In the context of the SDP relaxation of the QCQP, when an optimal solution of the relaxation is rank-1, the original QCQP with n variables can be exactly solved (c.f., Proposition 2.5). Under the strong duality, this rank condition derives another condition in the dual SDP, i.e., the existence of an optimal solution of at least $n-1$. The main idea of this thesis is to estimate the rank of solutions of dual SDP using the aggregated sparsity pattern of the QCQP.

Connected Aggregated Sparsity QCQPs

Chapter 3 focused on connected structures of the aggregated sparsity pattern for the QCQP with n variables. We first presented conditions for matrices under which their ranks are at least $n-1$ for three sparsity structures: tridiagonal matrices (strings of vertices), forest structures, and bipartite structures. In particular, the condition for the forest structures is more general than that for tridiagonal matrices and thus we proposed an exactness condition for QCQPs with forest-structured matrices. It requires the infeasibility of $n-1$ systems with a linear matrix inequality and $n+1$ ordinary inequalities. We then considered the other structure, and proposed another condition for QCQPs whose aggregated sparsity pattern is bipartite. The main difference between these two conditions are that the one of linear inequalities in the system is changed to an equality. The two instances of QCQPs were provided to show our results in analytical and computational aspects.

Disconnected Aggregated Sparsity QCQPs

In Chapter 4, we extended both exactness conditions of Chapter 3 to QCQPs which do not satisfy the connectivity conditions, that is, where the aggregated sparsity pattern is disconnected. Section 4.2 provided the definition of the ε-perturbed QCQP which is constructed by adding a perturbation to the objective function of the original QCQP. We first proved that if the SDP relaxation of the ε-perturbed QCQP for each epsilon that converges to zero is exact, then that of the original QCQP is also exact under each assumption for forest-structured and bipartite-structured QCQPs in Sections 4.3 and 4.4. The remaining task is to find a perturbation of ε-perturbed QCQP which admits the exact SDP relaxation. Proofs of exactness conditions for disconnected version theorems (Theorems 4.2 and 4.6) constructed a perturbation matrix P for each of forest-structured QCQPs and bipartite-structured QCQPs.

Special Classes of QCQPs

In Chapter 5, we applied exactness conditions of Chapters 3 and 4 to the three subclasses of QCQPs: QCQPs with only one equality constraints, the generalized trust-region subproblems, and QCQPs whose data matrices are element-wise either all nonnegative or all nonpositive. For the first subclasses, Section 5.2 presented a tractable exactness conditions which require checking the positive semidefiniteness of only $n-1$ matrices instead of solving the systems.

We next considered sign-definite QCQPs in Section 5.3. As an immediate consequence of the exactness condition for bipartite aggregated sparsity patterns, it was proved that QCQPs whose off-diagonal elements are all nonnegative admit the exact SDP relaxation. We next developed a conversion method from a sign-definite QCQP to the corresponding QCQP with bipartite aggregated sparsity pattern in order to handle all the sign-definite QCQPs. Employing this conversion method, we proved that our conditions cover all the sparsity conditions in the existing results [7, 28, 36] for the sign-definite QCQP (Proposition 5.6).

Section 5.4 considered the simultaneous tridiagonalization of matrices to transform the generalized version of trust-region subproblems (GTRS) into tractable QCQPs on the exactness condition for tridiagonal QCQPs, and revealed that all the off-diagonal elements of the obtained QCQP are nonnegative for a positive parameter γ. Using this property, we proved that any GTRS can be solved by the exact SDP relaxation.

6.2 Future Directions

For our future work, sufficient conditions for the exactness of a wider class of QCQPs than those with bipartite aggregated sparsity pattern graph will be investigated. We will also study whether there may be simpler conditions based on the optimal solutions of (3.2) and (3.3) than Assumption 4.3 in Theorem 4.6. Furthermore, examining our conditions to analyze the exact SDP relaxation of QCQPs transformed from polynomial optimization would be an interesting subject.

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