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Pochette surgery on 4-manifolds and the d-invariants of Brieskorn homology 3-spheres

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Overview

This doctoral thesis consists of the following two research contents:
Part 1 (Chapter 1): The boundary sum of the product of a circle with a 3-ball and the product of a disk with a 2-sphere is called a pochette. The pochette surgery, which was discovered by Iwase and Matsumoto, is a generalization of the Gluck surgery. For a pochette P embedded in a 4-manifold M, a pochette surgery on M is the operation of removing the interior of P and gluing P by a diffeomorphism of the boundary of P.

In this paper, we first explain that non-trivial cords may exist if embedding the pochette into a 4-manifold. Then, we explain a presentation of the fundamental group of any pochette surgery. Furthermore, we define a mod 2 framing and a linking number for pochette surgery. By Iwase and Matsumoto, the diffeomorphism type of the manifold M' obtained by a pochette surgery along P embedded in M is determined by the embedding $e : P \to M$, an element p/q of $\mathbb{Q} \cup \{\infty\}$ called the slope, and an element ε of $\{0, 1\}$ called the mod 2 framing because the isotopy class of the gluing diffeomorphism of ∂P is characterized by p/q and ε. We give a construction of handle diagrams for pochette surgeries based on the continued fraction expansions of slopes. In particular, when the slope is $1/p$ or $p/(p+1)$, we specifically introduce how to draw handle diagrams for those. We also calculate the homology of any pochette surgery on any homology 4-sphere and the homology of any pochette surgery for a embedding on any simply connected closed 4-manifold. Furthermore, under the above assumptions, we present a necessary and sufficient condition for the homeomorphism type to be the same before and after Pochette surgery.

With the above preparations, we prove that if the cord is trivial in a pochette surgery on the 4-sphere, it can be reduced to the problem of Gluck surgery. We also show that if the core sphere is trivial, any homology 4-sphere obtained by a pochette surgery on the 4-sphere is diffeomorphic to the 4-sphere. Furthermore, we show that a non-trivial cord can be obtained if the core sphere is any non-trivial ribbon 2-knot. Then, we show that if a non-trivial ribbon 2-knot is a ribbon 2-knot of 1-fusion, there exists always a non-trivial cord such that a pochette surgery on the 4-sphere is diffeomorphic to 4-sphere. Furthermore, for any integer $n \geq 2$, we show that there exists a ribbon 2-knot of n-fusion that has a non-trivial cord such that a pochette surgery on the 4-sphere is diffeomorphic to the 4-sphere. In particular, we present a countably infinite number of sufficient conditions for the existence of a non-trivial cord for any n by using a finite representation of the Andrews-Curtis trivial group obtained from the handle decomposition of pochette surgery on the 4-sphere.

Furthermore, we show that for a finite representation group R that is a finite representation of any knot group plus one arbitrary relation, there exists a pochette surgery on a ribbon 2-knot whose fundamental group is isomorphic to R. In particular, the fundamental group of any Dehn surgery on the 3-sphere with coefficient $1/q$ along any knot is isomorphic to that of a homology 4-sphere consisting of pochette surgery on the 4-sphere along a ribbon 2-knot.

Part 2 (Chapter 2): It is unclear when a homology 3-sphere becomes a smooth boundary of a homology 4-sphere. This problem is unsolved even in the Brieskorn homology 3-spheres, which is one of the most basic homology 3-spheres. There is a rational homology spin^c cobordism invariant called a d-invariant, which was introduced by Ozsváth and Szabó in 2003. This invariant is useful in showing that a homology 3-sphere is not a smooth boundary of any rational homology 4-ball. There is the formula of the d-invariant for any Brieskorn homology 3-sphere by Can and Karakurt by taking the maximum value of rational numbers in a set, which depend on p, q and r. In 2020, Karakurt and Şavk investigated the d-invariant of any Brieskorn homology 3-sphere $\Sigma(p, q, r)$ satisfying with $pq + pr - qr = 1$. They calculated the case of p is even, and derived the formula for the case of p is odd.

In this paper, when $pq + pr - qr = 1$ and p is odd, we develop a more precise formula based on the formula for the d-invariant of $\Sigma(p, q, r)$ proposed by Karakurt and Şavk. Using this refined formula, we can calculate the d-invariant of $\Sigma(p, q, r)$ with $q - p = 2$ or $p \leq 23$ and $q - p \geq n$. Furthermore, if $pq + pr - qr = 1$, we derive a natural inequality relation for the d-invariant of $\Sigma(p, q, r)$, which does not depend on the evenness of p. Based on these results, we present a countable infinite number of new Brieskorn homology 3-spheres in which two Brieskorn homology 3-spheres are not homology cobordant. Also, by generalizing the reciprocity law of d-invariants of two lens spaces by Ozsváth and Szabó, we can change the order of the set in the formula for the d-invariant of $\Sigma(p, q, r)$ to a number pq that does not depend on r.

Parts of this thesis are based on [S1], [S2] and [ST].

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Contents

Chapter 1

Pochette surgery on 4-manifolds

1.1 Introduction for this chapter

1.1.1 Pochette surgery

Let D^n be an n-dimensional disk and S^n an n-dimensional sphere. Let P denote the boundary-sum $S^1 \times D^3 \natural D^2 \times S^2$. It is called a *pochette*. Throughout this paper, all manifolds are assumed smooth, and connected, and all maps are smooth. For a manifold M, the open tubular neighborhood for a submanifold A of M is denoted by $N(A)$. Let $E(X)$ denote the exterior $M - N(X)$ of a submanifold X in M.

Here we define pochette surgery, which was initially defined by Iwase and Matsumoto in [IM]. Let e be an embedding $P \hookrightarrow M$ in a 4-manifold M. Let Q_e denote the image $e(Q)$ of a submanifold Q in P.

Definition 1.1.1. Let g be a diffeomorphism $g : \partial P \to \partial E(P_e)$. Gluing $E(P_e)$ and P via g, we construct a manifold $M(e, g) := E(P_e) \cup_g P$. We call this operation a *pochette surgery*. We say that the diffeomorphism g is a *gluing map* for the pochette surgery.

We call the curves $l := S^1 \times \{*\}$ and $m := \partial D^2 \times \{*\}$ on ∂P a *longitude* and a *meridian* of P, respectively. The diffeomorphism type of $M(e, g)$ is uniquely determined by the following data.

Theorem 1.1.2 (Iwase-Matsumoto [IM])**.** *The diffeomorphism type of $M(e, g)$ is determined by*
(1) *an embedding $e : P \to M$,*
(2) *a slope p/q, and*
(3) *a mod 2 framing around $g(m)$.*

The mod 2 framing will be defined in Subsection 1.2.3. The induced map g_* maps the primitive element $[m]$ in $H_1(\partial P)$ to $p[m_e]+q[l_e]$ in $H_1(\partial E(P_e))$, where p, q are relatively prime integers. Then, we call the element $p/q \in \mathbb{Q} \cup \{\infty\}$ a *slope* of the pochette surgery. Any slope p/q gives an unoriented image $g(m)$ of m. Hence, for some embedding e, the pochette surgery with the slope p/q and the mod 2 framing ε is called $(p/q,\varepsilon)$-*pochette surgery* of M and this denotes $M(e,p/q,\varepsilon)$. We call the 2-sphere $S := \{*\}\times S^2$ of P a *core sphere* of P and the meridian 2-sphere $B := \{*\}\times \partial D^3$ of P a *belt sphere* of P.

Consider P as $D^2\times S^2\cup h^1$, where h^1 is a 1-handle. In order to embed P into a 4-manifold M, we have only to determine an embedding of $D^2\times S^2$ and the 1-handle h^1. First we take an embedding $e : D^2\times S^2 \hookrightarrow M$.

Definition 1.1.3 (cord)**.** The 1-handle gives a properly embedded, simple arc in $E(S^2_e)$ by taking the core of h^1. We call this arc a *cord* here. If a cord is boundary parallel, then the cord is said to be *trivial*.

1.1.2 Gluck surgery and circle surgery

Let S' be an embedded sphere with a product neighborhood in a 4-manifold M. *Gluck surgery* along S' is an operation $\mathrm{Gl}(S') := E(S')\cup_\varphi (D^2\times S^2)$, where φ is a diffeomorphism $\partial D^2\times S^2 \to \partial N(S') \cong S^1\times S^2$ which is not homotopy equivalent to the identity. From the construction of pochette surgery, for an embedding $e : P \hookrightarrow M$, any $(\infty,0)$-pochette surgery is the trivial surgery and any $(\infty,1)$-pochette surgery yields $\mathrm{Gl}(S_e)$. In the case of $(0,\varepsilon)$-pochette surgery, it is an operation $E(l_e)\cup(D^2\times S^2)$ along the curve l_e of M. This surgery means that the result is one side of the manifold obtained by attaching 5-dimensional 2-handle on $M\times I$ along l_e. We call the result an S^1*-surgery* (*circle surgery*). Thus, any pochette surgery with the slope p/q can be regarded as an intermediate between a Gluck surgery and an S^1-surgery.

Pochette surgery is a generalization of Gluck surgery as mentioned above. Gluck surgery gave exotic nonorientable 4-manifolds in [A1]. It is natural to think pochette surgery may give interesting orientable 4-manifolds, possibly exotic 4-spheres and so on. In this article, we focus on pochette surgeries yielding homotopy 4-spheres.

1.1.3 Torus surgery

We define $ST := S^1\times D^2$. Let $e_0 : S^1\times ST \to M$ be an embedding and $g_0 : \partial(S^1\times ST) \to \partial E((S^1\times ST)_{e_0})$ a diffeomorphism. The diffeomorphism type of the manifold $E((S^1\times ST)_{e_0})\cup_{g_0}(S^1\times ST)$ obtained by a torus surgery on M is determined by e_0 and $(g_0)_*([\{*\}\times S^1\times\{*\}])$ in $H_1(S^1\times\partial ST)$. Let H be a 0-framed 2-handle attached to $S^1\times ST$ along $S^1\times\{*\}\times\{*\}$. Fix an identification between $S^1\times ST\cup H$ and P. Then $\{*\}\times S^1\times\{*\}$ is the meridian m of P, $\{*\}\times\{*\}\times\partial D^2$ is the longitude l of P. We define $s := S^1\times\{*\}\times\{*\}$. Then $\{[m],[l],[s]\}$ are a basis of $H_1(S^1\times\partial ST)$. If $e_0 = e|_{S^1\times ST}$, then we have the manifold $M(e,p/q,\varepsilon)$ is the torus surgery with e_0 and $(g_0)_*([m]) = p[m]+q[l]+\varepsilon p[s]$ (see [IM, Section 3]). This means that any pochette surgery on M is nothing but a torus surgery on M.

1.1.4 Other results

Since the definition of pochette surgery was done, some people have studied pochette surgery. Murase [Mu] studied pochette surgeries of the double of P. Let $D(P)$ be the double of P which is nothing but $P\cup_{\mathrm{id}}(-P)$. In fact, $D(P)$ is diffeomorphic to $S^1\times S^3\# S^2\times S^2$. Let i_P be the inclusion map $i_P : P\to D(P)$.

He shows the resulting manifold $D(P)(i_P, p/q, \varepsilon)$ is diffeomorphic to a rational homology 4-sphere with type L, which is defined in [P].

In the next subsection, we will share Okawa's result with readers. He investigates pochette surgeries yielding homotopy 4-spheres with the core sphere ribbon and with the cord trivial. We generalize this in Theorem 1.1.5.

Suzuki [S1] computed the homology of some types of pochette surgeries. These results are generalized in this paper (Proposition 1.2.9).

Pochette surgery can easily extend to a surgery along $\natural^a S^1 \times D^3 \natural^b D^2 \times S^2$ for some positive integers a, b. This is called *outer surgery* defined in [N]. In the future, we expect to find many exotic 4-manifolds by pochette surgery or outer surgery. See Section 1.6 for questions for pochette surgery or outer surgery.

1.1.5 Pochette surgery with trivial cord or trivial core sphere

After the definition of pochette surgery by Iwase and Matsumoto, pochette surgeries for embedding of P with trivial cord or trivial core sphere in S^4 have been considered to construct a new type of homotopy 4-spheres.

The case of trivial cord.

In this paper, we clarify diffeomorphism types of pochette surgeries of closed 4-manifolds with the trivial cord. Okawa proved the following.

Theorem 1.1.4 (Okawa [O, Theorem 1.2])**.** *Let e be an embedding of P into S^4 with the cord trivial. If the core sphere S_e is a ribbon 2-knot, then any pochette surgery $S^4(e, 1/q, \varepsilon)$ is diffeomorphic to S^4 for any integer q.*

Here we state the first main theorem.

Theorem 1.1.5. *Let e be an embedding of P into a closed 4-manifold M with the trivial cord. Then for any integer q, the following holds:*

$$M(e, 1/q, \varepsilon) \cong \begin{cases} M & (\varepsilon = 0), \\ \mathrm{Gl}(S_e) & (\varepsilon = 1). \end{cases}$$

The Gluck surgery along any ribbon 2-knot is diffeomorphic to the standard 4-sphere; see for example [GS]. Hence, Theorem 1.1.5 implies Theorem 1.1.4. It is also known that Gluck surgeries of some nonribbon 2-knots give the standard S^4; see for example, [G], [Me], [NS]. Pochette surgeries for such examples give the standard S^4.

Theorem 1.1.5 determines diffeomorphism types of $(1/q, \varepsilon)$-pochette surgeries with the trivial cord. As a corollary, we clarify the diffeomorphism type of any pochette surgery on a homology 4-sphere with the complement of the core sphere homotopically trivial.

Gluck surgery can produce nonorientable exotic 4-manifolds due to Akbulut [A1]. Hence, Theorem 1.1.5 implies that pochette surgery also produces nonorientable exotic 4-manifolds. As in the case of Gluck surgery, it remains uncertain whether pochette surgery has the potential to produce orientable exotic 4-manifolds (Question 1.6.6).

The case of trivial core sphere

Suzuki [S1] proved that several examples of infinitely many homotopy 4-spheres with the trivial core sphere are all diffeomorphic to the standard 4-sphere.

Theorem 1.1.5 immediately leads to the following theorem. This is a generalization of Suzuki's result.

Theorem 1.1.6. *Let M be a homology 4-sphere. Let e be an embedding $P \hookrightarrow M$ with $\pi_1(E(S_e)) = \mathbb{Z}$. If a pochette surgery produces a homology 4-sphere, then the result is diffeomorphic to M or $\mathrm{Gl}(S_e)$. In particular suppose M is S^4 and $e : P \hookrightarrow S^4$ is an embedding that the core sphere S_e is the unknot. Then if a pochette surgery by e yields a homology 4-sphere M', then M' is diffeomorphic to S^4.*

1.1.6 Pochette surgeries with nontrivial core sphere and cord

Next, we consider several examples of pochette surgeries with nontrivial core sphere and cord.

First, we prove the existence of such an example.

Theorem 1.1.7. *There exists a pochette embedding $e : P \hookrightarrow S^4$ with a nontrivial core sphere and a nontrivial cord such that the pochette surgery $S^4(e, g)$ is diffeomorphic to S^4.*

Further, the following theorem gives a sufficient condition for the existence of nontrivial cords whose surgery yielding homotopy 4-sphere is trivialized.

Theorem 1.1.8. *Let a subset S of S^4 be any ribbon 2-knot of 1-fusion with $\pi_1(E(S)) \not\cong \mathbb{Z}$. Then there exists a nontrivial cord c in $E(S)$ and an embedding*

$$e : P \to P_e = N(S) \cup N(c)$$

such that the pochette surgery $S^4(e, p/(p + 1), \varepsilon)$ is diffeomorphic to S^4.

Actually, as proven in Theorem 1.1.8, the core sphere of e is any nontrivial ribbon 2-knot of 1-fusion. Furthermore, there exist infinitely many cords for such a ribbon 2-knot such that the results all obtain the standard S^4.

Theorem 1.1.9. *Let a subset S of S^4 be any ribbon 2-knot with $\pi_1(E(S)) \not\cong \mathbb{Z}$. Then there exists a nontrivial cord C in $E(S)$ satisfying the following conditions:*

1. *The embedding $e : P \hookrightarrow S^4$ has the core sphere S and the cord C.*
2. *If for a gluing map g, $S^4(e, g)$ is a homology 4-sphere then it is diffeomorphic to the double of a homology 4-ball H without 3-handles.*

For a general ribbon 2-knot, it is uncertain whether the homology 4-ball H is contractible or not. In Theorem 1.1.9 we show that for any nontrivial ribbon 2-knot there exists a nontrivial cord such that any pochette surgery yielding a homology 4-sphere gives the double of a homology 4-ball without 3-handles.

Furthermore, when $S^4(e, g)$ is a homotopy 4-sphere, for $S^4(e, g)$ to be the standard S^4, we have only to assume the AC-triviality of the presentation of π_1. As a result, we obtain the following theorem.

Theorem 1.1.10. *If the homology 4-ball H obtained in Theorem 1.1.9 is contractible and the presentation of $\pi_1(H)$ for a handle decomposition of H without 3-handles is AC-trivial, then $S^4(e,g)$ is standard S^4.*

In Lemma 1.4.5, we actually give infinitely many presentations for $\pi_1(H)$ satisfying this condition. This means that such a type of ribbon 2-knots has a nontrivial cord satisfying $S^4(e,g) = S^4$.

It is unknown whether a pochette surgery with nontrivial S_e gives an exotic manifold or not. In general, even if S_e is trivial in a 4-manifold M, then it is unclear whether the pochette surgery is trivial or not. We expect that some pochette surgery creates a new exotic 4-manifold.

In Section 1.5, we investigate that finite representations of groups are isomorphic to the fundamental groups of pochette surgeries on the 4-sphere.

1.1.7 Aims of this chapter

The first aim of this paper is to investigate pochette surgeries $M(e,g)$ yielding homotopy 4-spheres and to determine the diffeomorphism types. What occurs in the case of nontrivial core sphere? The second aim is what even in this case, we clarify the existence of nontrivial cords that pochette surgeries give the standard S^4.

1.1.8 Organization of this paper

In Section 1.2, we give a review for pochette surgery. We define several definitions and lemmas. To carry out the second aim above, we compute the homology of $M(e,g)$ for any homology 4-sphere M. In order to compute the homology, we need to introduce the notion of a linking number for an embedding of a pochette as well as the slope which was defined by Iwase and Matsumoto [IM]. The linking number of an embedded pochette is the usual linking number of the embedded core sphere S_e and the longitude l_e in M. It depends on the choice of a meridian m, a longitude l and an embedding $e : P \hookrightarrow M$. Actually, we show that the homology of any pochette surgery on the 4-sphere is uniquely determined by the slope and the linking number (Proposition 1.2.9).

In Section 1.3, first, we prove Theorem 1.1.5 and clarify that pochette surgeries $M(e,g)$ of the case where the cord is trivial is diffeomorphic to M or some Gluck surgery. Second, we prove Theorem 1.1.6, by using this result, and we give a sufficient condition that any pochette surgery of M for some core sphere gives the same manifold M or the Gluck surgery. As a particular condition, any $(1/q, \varepsilon)$-pochette surgery of 4-sphere whose core sphere is the unknot is diffeomorphic to S^4.

In Section 1.4, we investigate cases where the core sphere S_e is a nontrivial 2-knot and the cord is a nontrivial (Theorem 1.1.7). These surgeries give the standard 4-sphere. Actually, we use a ribbon 2-knot of 1-fusion as S_e. The proof is essentially proven in Theorem 1.1.8. We generalize this situation to some cases where the core spheres are any general nontrivial ribbon 2-knots S with $\pi_1(E(S)) \not\cong \mathbb{Z}$ (Theorem 1.1.9). However, we did not see whether the resulting manifold is a homotopy 4-sphere or not. In Theorem 1.1.10, we give a sufficient condition of ribbon 2-knots for the existence of a nontrivial cord such that any surgery yielding homotopy 4-sphere gives the standard S^4.

In Section 1.5, we discuss a relationship between pochette surgery and finite representation groups.

1.2 Preliminaries for this chapter

1.2.1 Embedding of P

To consider an embedding of P in a 4-manifold M, as mentioned in the previous section, we embed a 2-sphere S in M with product neighborhood and embed a cord in the exterior $E(S)$. In 4-dimension, the isotopy class of any 1-manifold coincides with the homotopy class. Thus, the isotopy class of any embedding of P is determined by a 2-knot with product neighborhood and the homotopy class of a cord as a proper embedding in $E(S)$.

Let S be a 2-knot in a homology 4-sphere M. Here we clarify the isotopy classes of embedding e of P with $S_e = S$. We put $G(S) = \pi_1(E(S))$. $G(S)$ includes a subgroup $\langle m\rangle$ that is isomorphic to $\mathbb{Z}$. In this section, m is regarded as the class represented by the meridian circle. Here we call $\langle m\rangle$ a *boundary-subgroup*.

In fact, the abelianization map induces the surjection $G(S) \to H_1(E(S)) \cong \mathbb{Z}$ and the meridian is mapped to a generator in an infinite cyclic subgroup of $H_1(E(S))$. Thus m is nontorsion in $G(S)$. We define the set of isotopy classes of cords in $E(S)$ to be

$$\Pi_1(E(S), \partial E(S)) := [(I, \partial I), (E(S), \partial E(S))],$$

and the double coset space $G(S)//\langle m\rangle := \langle m\rangle\backslash G(S)/\langle m\rangle$. Let

$$\varphi : \pi_1(E(S), \partial E(S)) \to \Pi_1(E(S), \partial E(S))$$

be the natural map.

Lemma 1.2.1. *Let S be a 2-knot in a homology 4-sphere M. The set of properly embedded cords up to isotopy with the end points included in $\partial E(S)$ has a bijection to the double coset space $G(S)//\langle m\rangle$.*

Proof. By the short exact sequence:

$$1 \to \pi_1(\partial E(S)) \to \pi_1(E(S)) = G(S) \to \pi_1(E(S), \partial E(S)) \to 1$$

induced from the homotopy long exact sequence of the pair $(E(S), \partial E(S))$, we have the bijection

$$\pi_1(E(S), \partial E(S)) \cong \langle m\rangle\backslash G(S).$$

Here $\pi_1(E(S), \partial E(S))$ is the relative homotopy set.

Any element in $\Pi_1(E(S), \partial E(S))$ can be realized as one in $\pi_1(E(S), \partial E(S))$ by homotoping a starting point of the path to the base point x_0 of $\pi_1(E(S), \partial E(S))$. If $\varphi(\gamma_0) = \varphi(\gamma_1)$ for some $\gamma_0, \gamma_1 \in \pi_1(E(S), \partial E(S))$, then

$$\gamma_0(0) = \gamma_1(0) = x_0, \gamma_0(1), \gamma_1(1) \in \partial E(S).$$

There is a homotopy $H : I \times I \to E(S)$ such that $H(i, \cdot) = \gamma_i$ and $H(t, i) \in \partial E(S)$ $(i = 0, 1)$. Then $c(t) := H(t, 0)$ is a loop in $\partial E(S)$ with a base point x_0, we have $\gamma_0 = \gamma_1 \cdot c \in \pi_1(E(S), \partial E(S))$. Therefore, φ is surjective. If

$$\gamma_0 = \gamma_1 \cdot c \in \pi_1(E(S), \partial E(S))$$

for some $c \in \pi_1(\partial E(S))$, then $\gamma_0 = \gamma_1$ in $\Pi_1(E(S), \partial E(S))$. Thus

$$\pi_1(E(S), \partial E(S))/\langle m\rangle \to \Pi_1(E(S), \partial E(S))$$

is bijective.

Then we obtain the bijection

$$\Pi_1(E(S), \partial E(S)) \to \pi_1(E(S), \partial E(S))/\langle m\rangle \to G(S)//\langle m\rangle.$$

□

Let [[id]] be the element in $G(S)//\langle m\rangle$ represented by the trivial cord. Here the class in the double coset is represented by $[[\cdot]]$ and id stands for the identity element in $G(S)$. Hence, if the boundary-subgroup $\langle m\rangle$ is a proper subgroup in $G(S)$, then $G(S)//\langle m\rangle \neq \{[[\mathrm{id}]]\}$. If S is the trivial 2-knot in the 4-sphere, then $G(S) = \langle m\rangle$ and it has a unique isotopy class of a cord. If $G(S)$ is not isomorphic to $\mathbb{Z}$, then there exists a nontrivial cord.

1.2.2 Fundamental group of pochette surgery

In general, to find a homotopy 4-sphere obtained by applying pochette surgery, we need to compute the fundamental group. Let M be a 4-manifold and e an embedding $e : P \hookrightarrow M$. According to [IM], we see that a free isotopy class of an unoriented curve with slope p/q is uniquely determined as an image of m. We call the class a *natural lift.* Let $c_{p,q}$ be the natural lift of $p[m_e] + q[l_e]$ to $\pi_1(\partial E(P_e))$, which is defined in [IM]. Let l', and m' be the images on $\pi_1(\partial E(P_e))$ of the based, oriented, longitude and meridian in ∂P via e respectively. Let $c'_{p,q}$ be an element in $\pi_1(\partial E(P_e))$ presenting $c_{p,q}$. Concretely, the element is given by

$$c'_{p,q} = l'^{\lfloor q/p\rfloor} m' l'^{\lfloor 2q/p\rfloor - \lfloor q/p\rfloor} m' l'^{\lfloor 3q/p\rfloor - \lfloor 2q/p\rfloor} \cdots m' l'^{\lfloor pq/p\rfloor - \lfloor((p-1)q)/p\rfloor} m'.$$

See Theorem 6 in [IM].

We assume that the group presentation of $\pi_1(E(S))$ is $\pi_1(E(S)) = \langle \mathcal{S}|\mathcal{R}\rangle$, where $\mathcal{S}$ is a set of generators and $\mathcal{R}$ is a set of relators. For the inclusion maps $i : \partial P_e \to E(P_e)$ and $j : \partial P \to P$, the following maps are induced:

$$i_\# : \pi_1(\partial P_e) \to \pi_1(E(P_e)), j_\# : \pi_1(\partial P) \to \pi_1(P).$$

From the Seifert-Van Kampen theorem, we have

$$\pi_1(M(e, p/q, \varepsilon)) = \langle \mathcal{S}|\mathcal{R}, c'_{p,q}\rangle. \tag{1.1}$$

1.2.3 Mod 2 framing

In pochette surgery on a 4-manifold, after attaching $D^2 \times S^2$ to P along $g(m)$, the method of attaching $S^1 \times D^3$ is unique. Therefore, when gluing P, it is sufficient to consider an identification between neighborhoods of m and $g(m)$ via g.

Fix an identification between ∂P and $S^1 \times \partial D^3 \# \partial D^2 \times S^2 = S^1 \times S^2 \# S^1 \times S^2$. The meridian m of P has the natural product framing. By embedding e, we get identification $\iota : \partial E(P_e) \to S^1 \times S^2 \# S^1 \times S^2$. Then, $S^1 \times S^2 \# S^1 \times S^2$

can be expressed as the 2-component unlink which consists of 2 0-framed knots. Therefore, g maps the natural framing on m of ∂P to a framing on $g(m)$. This framing on $g(m)$ is represented by some integer determined by ι. The pochette can be regarded as $S^1 \times D^3$ attaching a 2-handle with the cocore m. Let $g_1, g_2 : \partial P \to \partial E(P_e)$ be two gluing maps. If $g_1(m)$ and $g_2(m)$ are the same and a difference between the framing on $g_1(m)$ and that of $g_2(m)$ is even, the map $g_1^{-1}g_2|_{N(m)}$ can be extended to the inside of the 2-handle. Here, $N(A)$ is the open tubular neighborhood for a submanifold A of P. Therefore, when considering the diffeomorphism type of the pochette surgery, we should consider an integer modulo 2 as the framing on $g(m)$. This framing on $g(m)$ is called a *mod 2 framing* and write it as ε. The mod 2 framing of $g(m)$ for the gluing map $g : \partial P \to \partial E(P_e)$ was first introduced in [IM, First paragraph in p.162].

1.2.4 Linking number

Let l and S be the longitude and the core sphere of a pochette P respectively. Let M be an oriented homology 4-sphere and $e : P \hookrightarrow M$ an embedding. The images l_e, and S_e in M give submanifolds of M. Then they can give the linking number

$$\ell = L(S_e, l_e)$$

according to [B, Section 15 of Chapter II]. In fact, we extend an embedding $e|_S : S \to M$ to a map $\mathcal{B}^3 \to M$, where $\mathcal{B}^3$ is a homology 3-ball. The orientation of $\mathcal{B}^3$ is induced by the one of S_e. We count the intersection points between the image of $\mathcal{B}^3$ and l_e with sign. Here we deform l_e in $E(S_e)$ so that l_e can meet with $\mathcal{B}^3$ transversely. For each intersection point if the concatenation of orientations on $\mathcal{B}^3$ and l_e at the point coincides with the orientation of M, then the sign is $+1$, otherwise -1. We call the sign a *local intersection number* at the intersection point. In the end, we sum up the local intersection numbers through all the intersection points. In the same way, we can compute $L(l_e, S_e)$ by changing the order of l_e and S_e.

In the general theory of linking number, the absolute values of $L(S_e, l_e)$ and $L(l_e, S_e)$ are the same. Actually, by the careful consideration of orientation we can easily obtain $L(S_e, l_e) = -L(l_e, S_e) =: \ell$. We call this number ℓ *linking number* of the embedding e. We must notice that the linking number is *not* an invariant of the embedding of P. If we fix the coordinate m and l, then the linking number can be determined. This is due to what the 3-disk separating $S^1 \times D^3$ and $D^2 \times S^2$ is not unique.

Here let us reinterpret the linking number $L(S_e, l_e)$ in terms of the homology. We use the intersection pairing:

$$\langle \cdot, \cdot \rangle_3^4 : H_3(E(S_e), \partial(E(S_e))) \times H_1(E(S_e)) \to \mathbb{Z}.$$

Let $\mathcal{M}^3$ be a Seifert hypersurface of S_e in $E(S_e)$, namely $\mathcal{M}^3$ is a properly embedded 3-manifold in $E(S_e)$ satisfying $\partial \mathcal{M}^3 = S_e$. $H_3(E(S_e), \partial(E(S_e)))$ is isomorphic to $\mathbb{Z}[\mathcal{M}^3]$. Here $\mathcal{M}^3 \cap E(S_e)$ and $\mathcal{M}^3$ are identified.
$H_3(E(P_e), \partial E(P_e))$ is isomorphic to $\mathbb{Z}[\mathcal{M}^3]$.

The intersection point between $\mathcal{M}^3$ and m_e is one point. Here we give an orientation on $\mathcal{M}^3$ satisfying $\langle [\mathcal{M}^3], [m_e] \rangle_3^4 = +1$.

By the definition of linking number, it follows that $\langle [\mathcal{M}^3], [l_e] \rangle_3^4 = \ell$. Since $H_1(E(S_e))$ is also isomorphic to $\mathbb{Z}$ generated by $[m_e]$, we have $[l_e] = \ell[m_e]$.

In the similar way we consider the next intersection pairing:

$$\langle \cdot, \cdot \rangle_2^4 : H_2(E(l_e), \partial E(l_e)) \times H_2(E(l_e)) \to \mathbb{Z}.$$

Here we take a proper embedded surface Σ satisfying $\partial\Sigma = l_e$ in $E(l_e)$. We take the usual orientation of the meridian B_e of l_e and the orientation on Σ by using $\langle [\Sigma], [B_e] \rangle_2^4 = +1$. From the computation $L(l_e, S_e) = -\ell$ of the linking number, we obtain $\langle [\Sigma], [S_e] \rangle_2^4 = -\ell$. Since $H_2(E(l_e))$ is isomorphic to $\mathbb{Z}$ generated by the belt sphere $[B_e]$, $[S_e] = -\ell[B_e]$ holds.

1.2.5 Handle diagram for pochette surgery

In this subsection we give a construction of handle diagrams for pochette surgeries under special conditions. Let M be a 4-manifold and $e : P \to M$ an embedding from a pochette P into M. Let p, q be coprime integers and ε an element of $\{0, 1\}$. In Figure 1.1 we describe the subsets m, l of $\partial P = \#^2 S^2 \times S^1$. Suppose that the diagram depicted in Figure 1.2 is a part of a handle diagram for M, where all the curves partially drawn in Figure 1.2 are framed knots, and any framed knot entwined with the dotted circle in Figure 1.2 has a 0-framed meridian. The pochette P_e consists of the 0-handle, the 1-handle presented by the leftmost dotted circle, and the 2-handle presented by the rightmost 0-framed unknot in Figure 1.2.

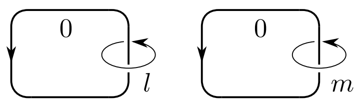

Figure 1.1: The orientation of m and l in $\partial P = \#^2 S^2 \times S^1$.

Proposition 1.2.2. *A handle diagram of $M(e, p/q, \varepsilon)$ is depicted in Figure 1.3, where p' is 1 if $p = 0$ and p otherwise.*

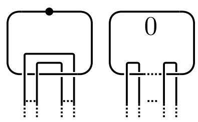

Figure 1.2: A part of the handle diagram of M.

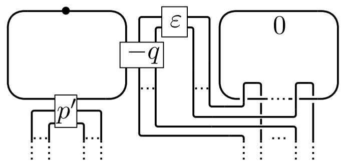

Figure 1.3: A part of the handle diagram of $M(e, p/q, \varepsilon)$.

Proof. Here we will consider the case where only a framed knot is entwined with the 0-framed knot on the right side exactly once. The case where framed knots are entwined with the 0-framed knot on the right side can be proved in the same way. If $|p|$ and $|q|$ are coprime positive integers, then there exist a positive integer n, a nonnegative integer a_0 and positive integers $a_1, \ldots, a_n$ such that

$$\frac{|p|}{|q|} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

We define the diffeomorphism E_0, E_1, E_2, E_3, E_4 and $E_5 : \partial P \to \partial P$ as the 1-Rolfsen twist for the leftmost $\langle 0 \rangle$-framed knot, the handle slide in Figure 1.4, 1.5, 1.6 and 1.7, the operation changing the direction of the meridian m, respectively.

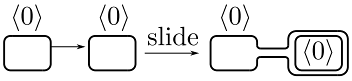

Figure 1.4: E_1.

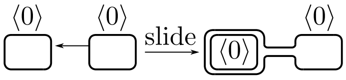

Figure 1.5: E_2.

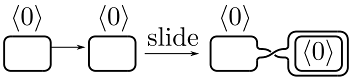

Figure 1.6: E_3.

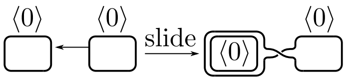

Figure 1.7: E_4.

Then we have

$$E_{i*}([m]) = \begin{cases} [m] & (i = 0, 2, 4), \\ [m] + [l] & (i = 1), \\ [m] - [l] & (i = 3), \\ -[m] & (i = 5), \end{cases} \qquad E_{i*}([l]) = \begin{cases} [l] & (i = 0, 1, 3, 5), \\ [m] + [l] & (i = 2), \\ -[m] + [l] & (i = 4). \end{cases}$$

We define $E_{p/q,\varepsilon}$ to be

$$\begin{cases} E_0^\varepsilon & (p = 1, q = 0) \\ E_4 E_1 E_0^\varepsilon & (p = 0, q = 1) \\ E_5 E_0^\varepsilon & (p = -1, q = 0) \\ E_4 E_1 E_5 E_0^\varepsilon & (p = 0, q = -1) \\ E_2^{a_0} E_1^{a_1} E_2^{a_2} \cdots E_1^{a_{n-1}} E_2^{a_n - 1} E_1 E_0^\varepsilon & (p > 0, q > 0, n \in 2\mathbb{N}), \\ E_2^{a_0} E_1^{a_1} E_2^{a_2} \cdots E_2^{a_{n-1}} E_1^{a_n} E_0^\varepsilon & (p > 0, q > 0, n \in 2\mathbb{N} - 1), \\ E_4^{a_0} E_3^{a_1} E_4^{a_2} \cdots E_3^{a_{n-1}} E_4^{a_n - 1} E_3 E_0^\varepsilon & (p > 0, q < 0, n \in 2\mathbb{N}), \\ E_4^{a_0} E_3^{a_1} E_4^{a_2} \cdots E_4^{a_{n-1}} E_3^{a_n} E_0^\varepsilon & (p > 0, q < 0, n \in 2\mathbb{N} - 1), \\ E_4^{a_0} E_3^{a_1} E_4^{a_2} \cdots E_3^{a_{n-1}} E_4^{a_n - 1} E_3 E_5 E_0^\varepsilon & (p < 0, q > 0, n \in 2\mathbb{N}), \\ E_4^{a_0} E_3^{a_1} E_4^{a_2} \cdots E_4^{a_{n-1}} E_3^{a_n} E_5 E_0^\varepsilon & (p < 0, q > 0, n \in 2\mathbb{N} - 1), \\ E_2^{a_0} E_1^{a_1} E_2^{a_2} \cdots E_1^{a_{n-1}} E_2^{a_n - 1} E_1 E_5 E_0^\varepsilon & (p < 0, q < 0, n \in 2\mathbb{N}), \\ E_2^{a_0} E_1^{a_1} E_2^{a_2} \cdots E_2^{a_{n-1}} E_1^{a_n} E_5 E_0^\varepsilon & (p < 0, q < 0, n \in 2\mathbb{N} - 1) \end{cases}$$

and $g_{p/q,\varepsilon} = eE_{p/q,\varepsilon}$. Then we have $E_{p/q,\varepsilon*}([m]) = p[m] + q[l]$ for any $p/q \in \mathbb{Q} \cup \{\infty\}$ and $\varepsilon \in \{0, 1\}$. By Theorem 1.1.2, the pochette surgery $M(e, p/q, \varepsilon)$ is diffeomorphic to $M(e, g_{p/q,\varepsilon})$ for any $p/q \in \mathbb{Q} \cup \{\infty\}$ and $\varepsilon \in \{0, 1\}$. A part of a handle diagram of M is depicted in Figure 1.8. By several handle slides on the 0-framed meridians of the framed knots entwined with the dotted circle, we obtain a part of a handle diagram of $M(e, p/q, \varepsilon)$ depicted in Figure 1.9. Concretely, the homotopy class of $g_{p/q,\varepsilon}(m)$ is the natural lift defined in [IM]:

$$\begin{cases} m'^p l'^q & (pq = 0), \\ \prod_{k=1}^{|p|} l'^{q/|q|(\lfloor k|q|/|p| \rfloor - \lfloor (k-1)|q|/|p| \rfloor)} m'^{p/|p|} & (pq \neq 0). \end{cases}$$

Here, l', m' are the images on $\pi_1(\partial E(P_e))$ of based, oriented longitude and meridian in ∂P via e.

If $q = 0$, then we reach the desired result.

If $q \neq 0$, we obtain Figure 1.10 by creating a 2-handle/3-handle pair in Figure 1.9. By the handle slide in Figure 1.10 and several handle slides on the 0-framed meridians of the framed knots entwined with the dotted circle, we obtain the handle diagram depicted in Figure 1.11. By the handle slides between the leftmost 0-framed knot and the rightmost 0-framed knot in Figure 1.11:

$$\begin{cases} E_0^{\varepsilon} E_{p/q,\varepsilon}^{-1} & (p > 0 \text{ or } (p,q) = (0,1)), \\ E_5 E_0^{\varepsilon} E_{p/q,\varepsilon}^{-1} & (p < 0 \text{ or } (p,q) = (0,-1)), \end{cases}$$

we obtain the handle diagram depicted in Figure 1.12. Changing the self-intersection of the framed knot in Figure 1.12 by several handle slides on the 0-framed meridian and canceling the 2-handle/3-handle pair, we obtain the handle diagram depicted in Figure 1.13. Therefore, we also obtain the conclusion in the case of $q \neq 0$. □

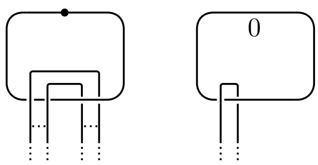

Figure 1.8: The first handle diagram in the proof of Proposition 1.2.2.

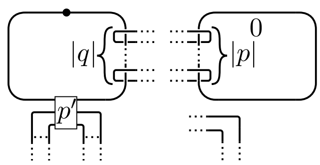

Figure 1.9: The second handle diagram in the proof of Proposition 1.2.2.

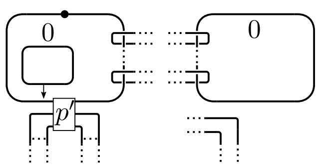

Figure 1.10: The third handle diagram in the proof of Proposition 1.2.2.

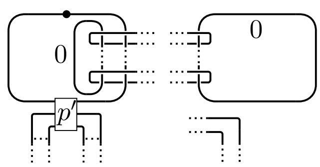

Figure 1.11: The fourth handle diagram in the proof of Proposition 1.2.2.

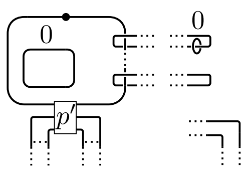

Figure 1.12: The fifth handle diagram in the proof of Proposition 1.2.2.

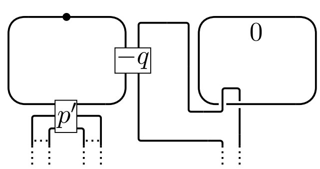

Figure 1.13: The sixth handle diagram in the proof of Proposition 1.2.2.

Remark 1.2.3. For any 4-manifold M, an embedding $e : P \to M$, coprime integers p, q and an element ε of $\{0, 1\}$, we give the explicit diffeomorphism $eE_{p/q,\varepsilon} : \partial P \to \partial E(P_e)$ for constructing the pochette surgery $M(e, p/q, \varepsilon)$.

Remark 1.2.4. If $p = 1$, Proposition 1.2.2 holds even without the 0-framed meridians, so any pochette surgery $M(e, 1/q, \varepsilon)$ is given by Figure 1.3.

Remark 1.2.5. Proposition 1.2.2 is a generalization of [K, Theorem 1, 2] and [Mu, Theorem 1.1, 1.2]. The curve $g_{p/q,\varepsilon}(m)$ in ∂P is depicted in Figure 1.14 in the case of $|p| = 1$, $q = 0$, in Figure 1.15 in the case of $p = 0$, $|q| = 1$, in Figure 1.16 in the case of $|q| > |p| > 0$, $pq > 0$, in Figure 1.17 in the case of $|q| > |p| > 0$, $pq < 0$, in Figure 1.18 in the case of $|p| > |q| > 0$, $pq > 0$, in Figure 1.19 in the case of $|p| > |q| > 0$, $pq < 0$.

We perform a Rolfsen twist just before or just after performing each handle slide E_1, E_2, E_3, E_4 by using either of two $\langle 0 \rangle$-framed knots in ∂P_e. Then we can obtain the curves depicted in Figure 1.16–1.19. The curve depicted in Figure 1.17 was first discovered by Murase [Mu].

Figure 1.14: $|p| = 1$, $q = 0$.

Figure 1.15: $p = 0$, $|q| = 1$.

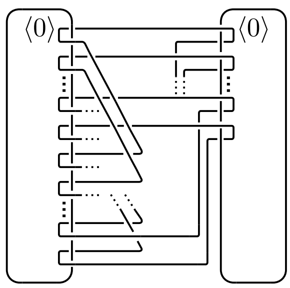

Figure 1.16: $|q| > |p| > 0$, $pq > 0$.

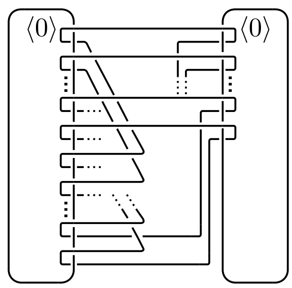

Figure 1.17: $|q| > |p| > 0$, $pq < 0$.

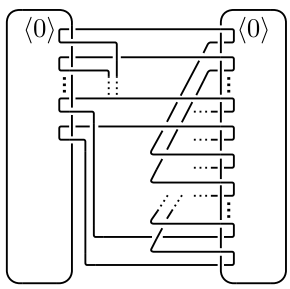

Figure 1.18: $|p| > |q| > 0$, $pq > 0$.

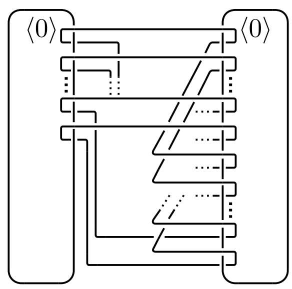

Figure 1.19: $|p| > |q| > 0$, $pq < 0$.

1.2.6 Images of the meridian by diffeomorphism

In this subsection we describe images of m via some gluing maps $g : \partial P \to \partial E(P_e)$ with slope $1/p$ and $p/(p+1)$. In the first diagram in Figure 1.20 we describe the subsets m, l of $\partial P = \#^2 S^2 \times S^1$. By sliding along the dashed arrow in the first picture, m is moved to a curve represented by $[m] + [l]$ in the second picture. Furthermore, sliding the diagram along the dashed arrow, we obtain the third picture. Then $[m] + [l]$ is moved to a curve by represented by $[m]+2[l]$. By the same diffeomorphism, $[m]+2[l]$ is moved to a curve represented by $[m] + 3[l]$ in the fourth picture.

Thus, by the diffeomorphism $h : \#^2 S^2 \times S^1 \to \#^2 S^2 \times S^1$ with slope $1/p$, meridian m is moved to a curve represented in $[m]+p[l]$ as in the bottom picture in Figure 1.20. This position will be used when we describe the handle diagram of $M(e, 1/p, \varepsilon)$.

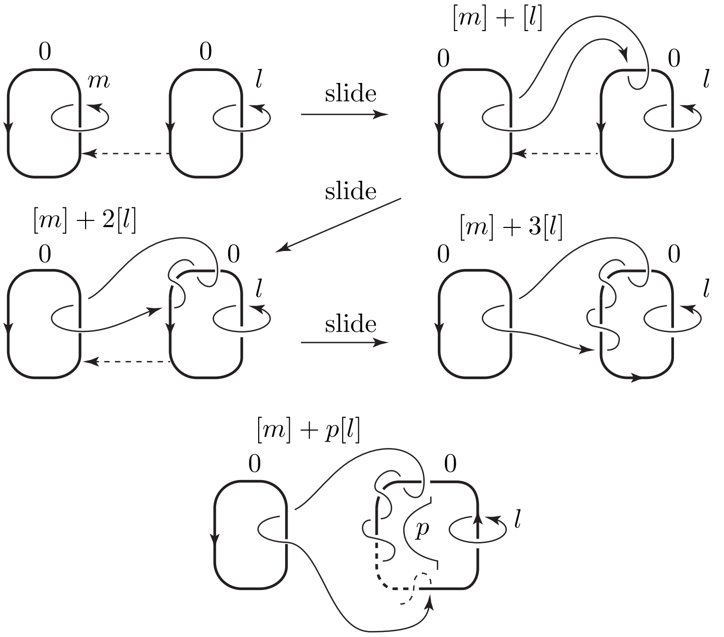

Figure 1.20: Images of m and l via a gluing map $\#^2 S^2 \times S^1 \to \#^2 S^2 \times S^1$.

Furthermore, exchanging m and l in the last picture in Figure 1.20 and doing an isotopy, we obtain a curve represented by $p[m] + [l]$ as in the first picture in Figure 1.21 and 1.22. We call these cases Case (I) and Case (II) respectively. Sliding a 0-framed 2-handle, we obtain the second picture. The thin curves in the figures are represented by $p[m] + (p+1)[l]$. By an isotopy we obtain the last pictures in Figure 1.21 and 1.22.

1.2.7 The homology of a pochette surgery

The case of any homology 4-sphere.

Let M be a homology 4-sphere. Here we compute the homology of the result by pochette surgery. Let $g : \partial P \to \partial E(P_e)$ be a gluing map with the slope p/q and the mod 2 framing ε. Let i be the inclusion map $\partial E(P_e) \to E(P_e)$.

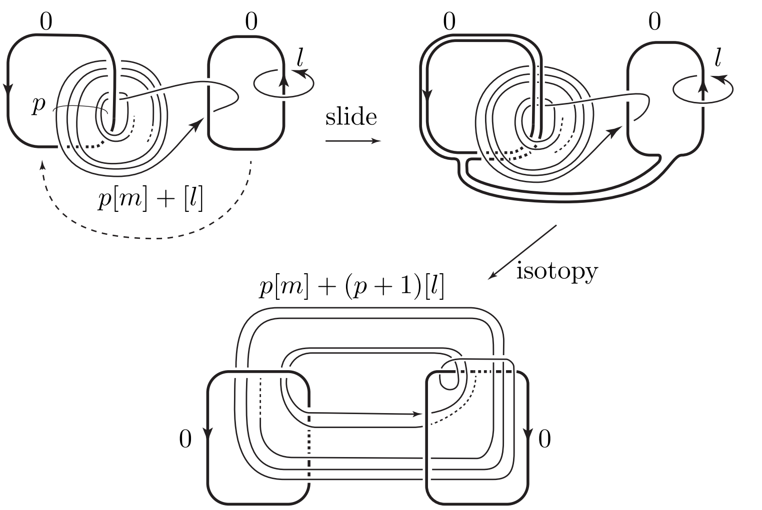

Figure 1.21: Case (I).

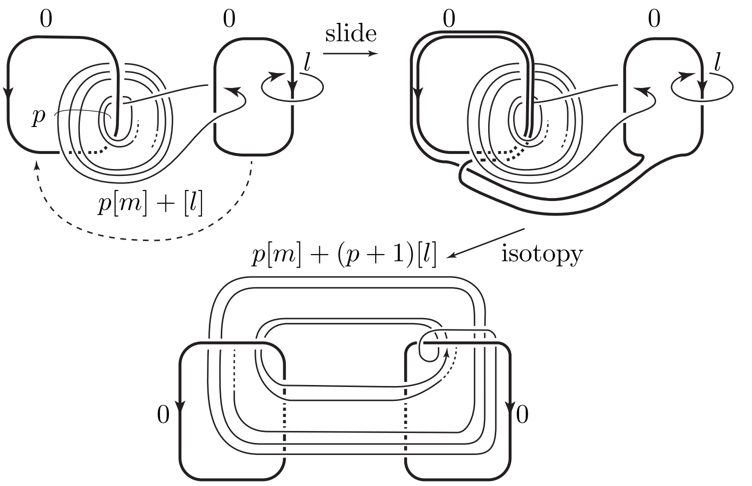

Figure 1.22: Case (II).

To compute the homology group of any pochette surgery of a homology 4-sphere, we prove lemmas needed later. First, we compute the homology of $E(P_e)$ here. Since $E(P_e)$ is connected, we have $H_0(E(P_e)) \cong \mathbb{Z}$.

Lemma 1.2.6. *$E(P_e)$ has the following homology groups:*

$$H_n(E(P_e)) = \begin{cases} \mathbb{Z}[m_e] & (n = 1), \\ \mathbb{Z}[B_e] & (n = 2), \\ 0 & (n \geq 3). \end{cases}$$

Proof. Let h^3 be a 4-dimensional 3-handle. Attaching h^3 on the belt sphere of P_e, we obtain $E(P_e) \cup h^3 = E(S_e)$ and $E(P_e) \cap h^3 = \partial D^3 \times D^1 = S^2 \times D^1$. The homology of $E(S_e)$ is the same as the homology of S^1 and the first homology group is generated by the meridian m_e. Since H_1 is independent of attaching any 3-handle, we have $H_1(E(P_e)) = H_1(E(P_e) \cup h^3) = H_1(E(S_e)) = \mathbb{Z}[m_e] \cong \mathbb{Z}$. Then we obtain the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H_n(S^2 \times D^1) \longrightarrow H_n(E(P_e)) \oplus H_n(h^3) \longrightarrow H_n(E(S_e)) \longrightarrow \cdots .$$

Thus, we can easily check

$$H_n(E(P_e)) = \begin{cases} \mathbb{Z} & (n = 2), \\ 0 & (n = 3, 4). \end{cases}$$

The generator of H_1 clearly corresponds to the meridian m_e of $E(S_e)$ and the one of H_2 corresponds to the generator, the belt sphere B_e which is the image of $H_2(S^2 \times D^1)$. □

From this lemma, we obtain natural isomorphisms $H_1(E(P_e)) \cong H_1(E(S_e))$ and $H_2(E(P_e)) \cong H_2(E(l_e))$. The isomorphisms are induced by the inclusions and connect the corresponding elements $[m_e]$ and $[B_e]$.

Let g be a gluing map from ∂P to $\partial E(P_e)$. Suppose that $g_*([m]) = p[m_e] + q[l_e]$ is satisfied on the first homology group.

Lemma 1.2.7. *If $g_*([m]) = p[m_e] + q[l_e]$, then we have $g_*([B]) = p[B_e] - q[S_e]$.*

Proof. We put $g_*([l]) = r[m_e] + s[l_e]$, $g_*([B]) = x[B_e] + y[S_e]$. Then, we can define the nondegenerate bilinear form $\langle \cdot, \cdot \rangle_3 : H_1(\partial P) \times H_2(\partial P) \to \mathbb{Z}$ from the cup product $H^2(\partial P) \times H^1(\partial P) \to H^3(\partial P)$.

By defining

$$\langle [m], [B] \rangle_3 = 0, \langle [l], [B] \rangle_3 = 1, \langle [m], [S] \rangle_3 = 1, \text{ and } \langle [l], [S] \rangle_3 = 0,$$

we determine the orientations on m and B. These orientations coincide with the ones determined Subsection 1.2.4 via the map $H_n(\partial P_e) \to H_n(E(P_e))$. Since $g : \partial P \to \partial E(P_e)$ is a diffeomorphism, we can define the nondegenerate bilinear form $\langle \cdot, \cdot \rangle_3^e : H_1(\partial E(P_e)) \times H_2(\partial E(P_e)) \to \mathbb{Z}$ from the nondegenerate bilinear form $\langle \cdot, \cdot \rangle_3 : H_1(\partial P) \times H_2(\partial P) \to \mathbb{Z}$. Since $g : \partial P \to \partial E(P_e)$ is an orientation preserving diffeomorphism, the determinant of the matrix given by

$$\begin{pmatrix} g_*([m]) & g_*([l]) \end{pmatrix} = \begin{pmatrix} [m_e] & [l_e] \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

is 1. Hence we obtain $ps - qr = 1$. Thus the inverse is as

$$\begin{pmatrix} g_*^{-1}([m_e]) & g_*^{-1}([l_e]) \end{pmatrix} = \begin{pmatrix} [m] & [l] \end{pmatrix} \begin{pmatrix} s & -r \\ -q & p \end{pmatrix}.$$

Since

$$\langle g_*(\alpha), g_*(\beta)\rangle_3^e = \langle \alpha, \beta\rangle_3 \text{ for all } \alpha \in H_1(\partial P),\ \beta \in H_2(\partial P),$$

we have

$$\begin{aligned} x = \langle [l_e], x[B_e] + y[S_e]\rangle_3^e = \langle [l_e], g_*([B])\rangle_3^e &= \langle g_*^{-1}([l_e]), [B]\rangle_3 \\ &= \langle -r[m] + p[l], [B]\rangle_3 = p \end{aligned}$$

and

$$\begin{aligned} y = \langle [m_e], x[B_e] + y[S_e]\rangle_3^e = \langle [m_e], g_*([B])\rangle_3^e &= \langle g_*^{-1}([m_e]), [B]\rangle_3 \\ &= \langle s[m] - q[l], [B]\rangle_3 = -q. \end{aligned}$$

Therefore, we obtain the desired result above. □

Lemma 1.2.8. *Let e be an embedding $P \hookrightarrow M$ with linking number ℓ. Let i be an inclusion $i : \partial E(P_e) \to E(P_e)$. Then $i_*([l_e]) = \ell[m_e]$ and $i_*([S_e]) = -\ell[B_e]$ are satisfied.*

Proof. The image of $[l_e] \in H_1(\partial E(P_e))$ by i_* is also $[l_e]$ in $H_1(E(P_e))$. Since $H_1(E(P_e))$ and $H_1(E(S_e))$ are identified with each other by the natural isomorphism by the inclusion, the elements $[m_e]$ having in these homology groups are mapped. Hence, from Subsection 1.2.4, $[l_e] = \ell[m_e]$ also holds in $H_1(E(P_e))$. In the same way, we have $i_*([S_e]) = -\ell[B_e]$. □

Here, we compute the homology groups of the pochette surgery $M(e, p/q, \varepsilon)$. Since M is connected and oriented, $H_0(M(e, p/q, \varepsilon)) \cong H_4(M(e, p/q, \varepsilon)) \cong \mathbb{Z}$ is satisfied. We compute H_n of M for $n = 1, 2, 3$.

Proposition 1.2.9. *Let M be a homology 4-sphere. Let e be an embedding with linking number ℓ. Then, $M(e, p/q, \varepsilon)$ has the following homology groups:*

(i) If $p + q\ell \neq 0$, then

$$H_n(M(e, p/q, \varepsilon)) \cong \begin{cases} \mathbb{Z}/(p + q\ell)\mathbb{Z} & (n = 1, 2), \\ 0 & (n = 3). \end{cases}$$

(ii) If $p + q\ell = 0$, then

$$H_n(M(e, p/q, \varepsilon)) \cong \begin{cases} \mathbb{Z} & (n = 1, 3) \\ \mathbb{Z}^2 & (n = 2). \end{cases}$$

Note that the case of $p + q\ell = 0$ is nothing but $(p, q) = (\ell, -1), (-\ell, 1)$ because p, q are relatively prime.

Proof. The embedding map $e : P \hookrightarrow M$ induces the map

$$H_n(\partial P) \xrightarrow{g_*} H_n(\partial E(P_e)) \xrightarrow{i_*} H_n(E(P_e)).$$

Then we have $H_1(\partial E(P_e)) = \mathbb{Z}[m_e] \oplus \mathbb{Z}[l_e]$, $H_2(\partial E(P_e)) = \mathbb{Z}[B_e] \oplus \mathbb{Z}[S_e]$ and obtain $g_*([m]) = p[m_e] + q[l_e]$, $i_*([m_e]) = [m_e]$ and $i_*([B_e]) = [B_e]$. By Lemma 1.2.7, we obtain $g_*([B]) = p[B_e] - q[S_e]$. By Lemma 1.2.8, we have $i_*([l_e]) = \ell[m_e]$ and $i_*([S_e]) = -\ell[B_e]$. By Lemma 1.2.11 and the Mayer-Vietoris sequence

$$\cdots \to H_*(E(P_e)) \oplus H_*(P) \to H_*(M(e, p/q, \varepsilon)) \to H_{*-1}(\partial P) \to \cdots,$$

we obtain the following:

$$\begin{array}{ccccccc}
\longrightarrow & 0 & \longrightarrow & H_3(M(e,p/q,\varepsilon)) & \xrightarrow{\partial_3} & \mathbb{Z}[B] \oplus \mathbb{Z}[S] \\
\xrightarrow{j_{21}\oplus j_{22}} & \mathbb{Z}[B_e] \oplus \mathbb{Z}[S] & \xrightarrow{i_2} & H_2(M(e,p/q,\varepsilon)) & \xrightarrow{\partial_2} & \mathbb{Z}[m] \oplus \mathbb{Z}[l] \\
\xrightarrow{j_{11}\oplus j_{12}} & \mathbb{Z}[m_e] \oplus \mathbb{Z}[l] & \xrightarrow{i_1} & H_1(M(e,p/q,\varepsilon)) & \xrightarrow{\partial_1 = 0} & H_0(\partial P).
\end{array}$$

We put $j_n = j_{n1} \oplus j_{n2}$ for any $n \in \mathbb{Z}$. Since we have $\partial_1 = 0$, i_1 is a surjection. Since we have $j_1([m]) = (p + q\ell)[m_e]$ and $j_1([l]) = (r + s\ell)[m_e] + [l]$, we obtain

$$\begin{aligned}
H_1(M(e,p/q,\varepsilon)) &= \operatorname{Im} i_1 \\
&\cong \mathbb{Z}[m_e] \oplus \mathbb{Z}[l]/\langle (p+q\ell)[m_e], (r+s\ell)[m_e] + [l] \rangle \\
&\cong \mathbb{Z}[m_e]/\langle (p+q\ell)[m_e] \rangle \cong \mathbb{Z}/(p+q\ell)\mathbb{Z}.
\end{aligned}$$

Here r, s are the same coefficients as the ones used in the proof of Lemma 1.2.7.

Next, we compute H_2 and H_3 of the result of the pochette surgery.

If $p + q\ell \neq 0$, then j_1 is an injection. Since i_2 is a surjection, we obtain the following isomorphism:

$$\begin{aligned}
H_2(M(e,p/q,\varepsilon)) &= \operatorname{Im} i_2 \\
&\cong \mathbb{Z}[B_e] \oplus \mathbb{Z}[S]/\langle (p+q\ell)[B_e], (r'+s'\ell)[B_e] + [S] \rangle \\
&\cong \mathbb{Z}[B_e]/\langle (p+q\ell)[B_e] \rangle \cong \mathbb{Z}/(p+q\ell)\mathbb{Z}.
\end{aligned}$$

Here, r', s' are some integers satisfying $ps' + qr' = 1$. In this case, $\operatorname{Im} \partial_3 = \operatorname{Ker} j_2 = 0$. Thus we have

$$H_3(M(e,p/q,\varepsilon)) = \operatorname{Ker} \partial_3 = 0.$$

If $p + q\ell = 0$, then $\operatorname{Im} \partial_2 = \operatorname{Ker} j_1 = \mathbb{Z}[m]$. Thus we have

$$H_2(M(e,p/q,\varepsilon)) \cong \operatorname{Im} i_2 \oplus \mathbb{Z}[m] \cong \mathbb{Z}[B_e] \oplus \mathbb{Z}[m].$$

In this case, $\operatorname{Im} \partial_3 = \operatorname{Ker} j_2 = \mathbb{Z}[B]$. Thus we have

$$H_3(M(e,p/q,\varepsilon)) \cong \mathbb{Z}[B].$$

Therefore, we obtain the desired result above. □

The theorems by Whitehead [W], Freedman [F] and Proposition 1.2.9 imply the next corollary.

Corollary 1.2.10. *Let M be a homology 4-sphere. $M(e, p/q, \varepsilon)$ is homeomorphic to S^4 if and only if $M(e, p/q, \varepsilon)$ is a simply connected 4-manifold and $|p+q\ell|$ is equal to 1.*

Proof. By Freedman's theorem, $M(e, p/q, \varepsilon)$ is homeomorphic to S^4 if and only if $M(e, p/q, \varepsilon)$ is homotopy equivalent to S^4. We will only show that $M(e, p/q, \varepsilon)$ is homotopy equivalent to S^4 if and only if $M(e, p/q, \varepsilon)$ is a simply connected 4-manifold and $|p + q\ell| = 1$. By the Whitehead theorem, the necessary and sufficient condition for a manifold to be homotopy equivalent to S^4 is $\pi_1 = \{\mathrm{id}\}$ and $H_n = 0$ for $n = 1, 2, 3$. From Proposition 1.2.9, we can easily check this corollary follows. □

The case of any simply connected closed 4-manifold.

Let p, q be coprime integers and ε an element of $\{0, 1\}$. Let M be a simply connected closed 4-manifold and $e : P \to M$ an embedding from a pochette P into M. Here, we prove lemmas needed later.

Lemma 1.2.11. *If the homomorphism $t_2 : H_2(M) \to H_2(M, E(P_e))$ induced by the inclusion map $(M, \varnothing) \to (M, E(P_e))$ is a zero map, the homology groups of the exterior $E(P_e)$ of P_e are calculated as follows:*

$$H_n(E(P_e)) = \begin{cases} \mathbb{Z}[x_e] & (n = 0), \\ \mathbb{Z}[m_e] & (n = 1), \\ \mathbb{Z}[B_e] \oplus H_2(M) & (n = 2), \\ 0 & (\text{otherwise}). \end{cases}$$

Here x is a point in ∂P.

Proof. By the long exact sequence of the pair $(P, \partial P)$:

$$\cdots \xrightarrow{\partial_{n+1}} H_n(\partial P) \xrightarrow{s_n} H_n(P) \xrightarrow{t_n} H_n(P, \partial P) \xrightarrow{\partial_n} \cdots,$$

we have

$$H_n(P, \partial P) = \begin{cases} \mathbb{Z}[D^2 \times \{*\}] & (n = 2), \\ \mathbb{Z}[\{*\} \times D^3] & (n = 3), \\ \mathbb{Z}[P] & (n = 4), \\ 0 & (\text{otherwise}). \end{cases}$$

By the Excision Theorem, we obtain

$$H_n(M, E(P_e)) \cong H_n(P, \partial P) \text{ for any } n \in \mathbb{Z}.$$

By the long exact sequence of pair $(M, E(P_e))$:

$$\cdots \xrightarrow{\partial_{n+1}} H_n(E(P_e)) \xrightarrow{s_n} H_n(M) \xrightarrow{t_n} H_n(M, E(P_e)) \xrightarrow{\partial_n} \cdots,$$

we have the homology groups above. □

Lemma 1.2.12. *Let $t_n : H_n(M) \to H_n(M, E(P_e)), i_{n1} : H_n(\partial P) \to H_n(E(P_e))$, $i_{n2} : H_n(\partial P) \to H_n(P)$ be homomorphisms induced by inclusion maps. If $t_2 = 0$, $i_{11}([l]) = 0$ and $i_{21}(H_2(\partial P))$ is included in $\mathbb{Z}[B_e] \oplus \mathbb{Z}[S]$, then we have $i_{21}([B]) = \pm p[B_e]$ and $i_{21}([S]) = 0$.*

Proof. By the definitions of E_0, E_1, E_2, E_3, E_4 and E_5 in the proof of Proposition 1.2.2, we obtain

$$E_{i*}([m]) = \begin{cases} [m] & (i = 0, 2, 4), \\ [m] + [l] & (i = 1), \\ [m] - [l] & (i = 3), \\ -[m] & (i = 5), \end{cases} \quad E_{i*}([l]) = \begin{cases} [l] & (i = 0, 1, 3, 5), \\ [m] + [l] & (i = 2), \\ -[m] + [l] & (i = 4), \end{cases}$$

$$E_{i*}([B]) = \begin{cases} [B] & (i = 0, 2, 4), \\ [B] \mp [S] & (i = 1), \\ [B] \pm [S] & (i = 3), \\ \pm[B] & (i = 5), \end{cases} \quad E_{i*}([S]) = \begin{cases} [S] & (i = 0, 1, 3), \\ \mp[B] + [S] & (i = 2), \\ \pm[B] + [S] & (i = 4), \\ \mp[S] & (i = 5) \end{cases}$$

(double-sign corresponds). Then, there exist some integers r, s such that $g_{p/q,\varepsilon *}([m]) = p[m_e] + q[l_e]$, $g_{p/q,\varepsilon *}([l]) = r[m_e] + s[l_e]$, $g_{p/q,\varepsilon *}([B]) = \pm p[B_e] \pm q[S_e]$ and $g_{p/q,\varepsilon *}([S]) = \pm r[B_e] \pm s[S_e]$. Let $i_{\partial E(P_e)} : \partial E(P_e) \to E(P_e)$ be the inclusion map. Then we have $i_{n1} = (i_{\partial E(P_e)} g_{p/q,\varepsilon})_{n*}$ for any $n \in \mathbb{Z}$. Here $f_{n*} : H_n(A) \to H_n(B)$ is the n-th induced homomorphism on homology of a continuous map $f : A \to B$. If $t_2 = 0$, by Lemma 1.2.11, $H_1(E(P_e)) = \mathbb{Z}[m_e]$ and $H_2(E(P_e)) = \mathbb{Z}[B_e] \oplus H_2(M)$. If $i_{11}([l]) = 0$ and $i_{21}(H_2(\partial P))$ is included in $\mathbb{Z}[B_e] \oplus \mathbb{Z}[S]$, then we have $i_{21}([S]) = 0$. By $i_{11}([m]) = p[m_e]$, we also have $i_{21}([B]) = \pm p[B_e]$. Therefore, we obtain the desired result above. □

For a simply connected closed 4-manifold M, we give a necessary condition for a pochette surgery of M to have the same homology as M.

Proposition 1.2.13. *Let $t_n : H_n(M) \to H_n(M, E(P_e)), i_{n1} : H_n(\partial P) \to H_n(E(P_e))$,
$i_{n2} : H_n(\partial P) \to H_n(P)$ be homomorphisms induced by inclusion maps. If $t_2 = 0$, $i_{11}([l]) = 0$ and $i_{21}(H_2(\partial P))$ is included in $\mathbb{Z}[B_e] \oplus \mathbb{Z}[S]$, then the homology groups of the pochette surgery $M(e, p/q, \varepsilon)$ are calculated as follows:*

$$H_n(M(e, p/q, \varepsilon)) = \begin{cases} \mathbb{Z} & (n = 0, 4), \\ \mathbb{Z}/p\mathbb{Z} & (n = 1). \end{cases}$$

Moreover, if $|p|$ is equal to 1, then $M(e, p/q, \varepsilon)$ has the same homology groups as M.

Proof. Since M is connected and oriented, $H_n(M(e, p/q, \varepsilon)) \cong \mathbb{Z}$ for any $n = 0, 4$. We compute $H_1(M(e, p/q, \varepsilon))$ here. By Lemma 1.2.11 and the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\partial_{n+1}} H_n(\partial P) \xrightarrow{i_{n1} \oplus i_{n2}} H_n(E(P_e)) \oplus H_n(P) \xrightarrow{j_n} H_n(M(e, p/q, \varepsilon)) \xrightarrow{\partial_n} \cdots,$$

we obtain the following:

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & H_3(M(e, p/q, \varepsilon)) & \xrightarrow{\partial_3} & \mathbb{Z}[B] \oplus \mathbb{Z}[S] \\ \xrightarrow{i_{21} \oplus i_{22}} & (\mathbb{Z}[B_e] \oplus H_2(M)) \oplus \mathbb{Z}[S] & \xrightarrow{j_2} & H_2(M(e, p/q, \varepsilon)) & \xrightarrow{\partial_2} & \mathbb{Z}[m] \oplus \mathbb{Z}[l] \\ \xrightarrow{i_{11} \oplus i_{12}} & \mathbb{Z}[m_e] \oplus \mathbb{Z}[l] & \xrightarrow{j_1} & H_1(M(e, p/q, \varepsilon)) & \xrightarrow{\partial_1} & H_0(\partial P). \end{array}$$

Then, we have $i_{11}([m]) = p[m_e], i_{12}([m]) = 0, i_{12}([l]) = [l]$ and $i_{22}([B]) = 0, i_{22}([S]) = [S]$. If $i_{11}([l]) = 0$, then we have

$$\mathrm{Ker}\,\partial_1 = \mathrm{Im}\,j_1 \cong \mathbb{Z}[m_e] \oplus \mathbb{Z}[l]/\mathrm{Im}\,(i_{11} \oplus i_{12}) \cong \mathbb{Z}[m_e]/p\mathbb{Z}[m_e] \cong \mathbb{Z}/p\mathbb{Z}$$

and $\mathrm{Im}\,\partial_1 = 0$. Thus $H_1(M(e, p/q, \varepsilon)) \cong \mathbb{Z}/p\mathbb{Z}$. If $|p| = 1$, then we have $H_n(M(e, 1/q, \varepsilon)) \cong H_n(M)$ for any $n = 0, 1, 4$. By Lemma 1.2.12, we have $i_{21}([B]) = \pm p[B_e], i_{21}([S]) = 0$. Therefore, we have $\mathrm{Ker}\,\partial_3 = 0, \mathrm{Im}\,\partial_3 = \mathrm{Ker}\,(i_{21} \oplus i_{22}) = 0$, and

$$\mathrm{Ker}\,\partial_2 = \mathrm{Im}\,j_2 \cong (\mathbb{Z}[B_e] \oplus H_2(M)) \oplus \mathbb{Z}[S]/\mathrm{Im}\,(i_{21} \oplus i_{22}) = H_2(M),$$

$\mathrm{Im}\,\partial_2 = \mathrm{Ker}\,(i_{11} \oplus i_{12}) = 0$. Then we have $H_n(M(e, 1/q, \varepsilon)) \cong H_n(M)$ for any $n = 2, 3$. □

Remark 1.2.14. Proposition 1.2.13 is a generalization of [O, Theorem 1.1].

The next corollary follows from Proposition 1.2.13 and the Freedman theorem [F], [FQ].

Corollary 1.2.15. *If $t_2 = 0$, $i_{11}([l]) = 0$ and $i_{21}(H_2(\partial P))$ is included in $\mathbb{Z}[B_e] \oplus \mathbb{Z}[S]$, then $M(e, p/q, \varepsilon)$ is homeomorphic to M if and only if $M(e, p/q, \varepsilon)$ is a simply connected 4-manifold and $|p|$ is equal to 1.*

Proof. If $M(e, p/q, \varepsilon)$ is homeomorphic to M, then $M(e, p/q, \varepsilon)$ has the same homology groups as M. By Proposition 1.2.13, $M(e, p/q, \varepsilon)$ is a simply connected 4-manifold and $|p| = 1$. Conversely, if $M(e, p/q, \varepsilon)$ is a simply connected 4-manifold and $|p| = 1$, we obtain a natural isomorphism $H_2(M(e, p/q, \varepsilon)) \cong H_2(M)$ by the proof of Proposition 1.2.13. Hence, $Q_{M(e,p/q,\varepsilon)} \cong Q_M$. Here, Q_Y is the intersection form of the 4-manifold Y. Since $M(e, p/q, \varepsilon)$ and M are simply connected 4-dimensional closed manifolds with differential structures, $M(e, p/q, \varepsilon) \times \mathbb{R}$ and $M \times \mathbb{R}$ have differential structures. Therefore, we obtain $ks(M(e, p/q, \varepsilon)) = 0 = ks(M)$. Here, $ks(Y)$ is the Kirby-Siebenmann invariant of Y. By the Freedman theorem, $M(e, p/q, \varepsilon)$ is homeomorphic to M. Therefore, we obtain the desired result above. □

1.3 Proof of main theorems of this chapter

In this section we prove Theorem 1.1.5.

Proof of Theorem 1.1.5. Let e be an embedding $P \hookrightarrow M$ with a trivial cord. The exterior $E(P_e)$ is obtained by attaching a 0-framed 2-handle on $E(S_e)$ in a separated position from the diagram of $E(S_e)$ as in the left picture of Figure 1.23. The circle m_e in the figure is the image of meridian of P. For example, when we describe $E(S_e)$ along the motion picture as in [GS, Section 6.2], it is a meridian of a 1-handle corresponding to a 0-handle of the embedded sphere. Hence, the pochette surgery on M can be obtained by attaching an ε-framed 2-handle on $E(P_e)$ plus a 3-handle and a 4-handle. The position of the ε-framed 2-handle is understood from the argument in Subsection 1.2.6. The right picture in Figure 1.23 is the local picture of the handle diagram of $M(e, 1/q, \varepsilon)$.

Here, we prove that the rightmost 0-framed knot in Figure 1.23 is isotopic to the unknot in $\partial(E(S_e) \cup h^2(\varepsilon)) = S^3$, where $h^2(\varepsilon)$ is the ε-framed 2-handle. We remove the previous 3- and 4-handle in $M(e, 1/q, \varepsilon)$. Since the boundary of obtained manifold is diffeomorphic to the ε-Dehn surgery of $\partial E(S_e)$. By several handle moves, we obtain the Hopf link surgery that the framing coefficients of the two components are $\langle 0 \rangle$ and $\langle \varepsilon \rangle$. Then we get the second picture in Figure 1.24. From this point, doing slides by q-times, we obtain the fifth picture. Canceling the Hopf link component, we obtain 0-framed knot as in the last picture in Figure 1.24. Hence, this 0-framed unknot is isotopic to the unknot.

Since we can move the 0-framed unknot in the last picture in Figure 1.23 to the unlink position in the same picture, we cancel this component with a 3-handle. The remaining diagram is obtained by attaching an ε-framed 2-handle and a 4-handle on $E(S_e)$. Therefore, the resulting manifold is the trivial surgery or the Gluck surgery along S_e depending on $\varepsilon = 0$ or 1 respectively. □

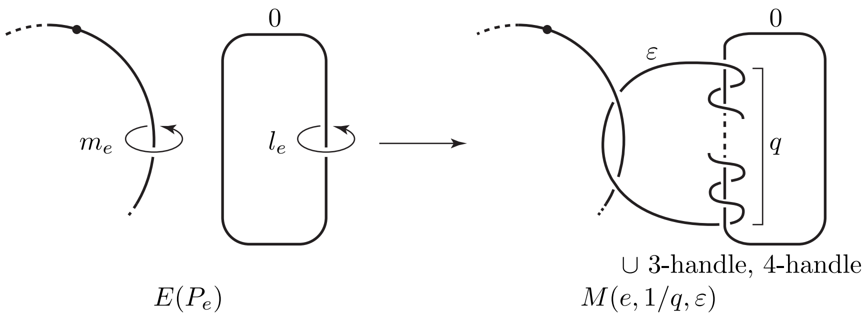

Figure 1.23: Attaching P on $E(P_e)$ with the trivial cord.

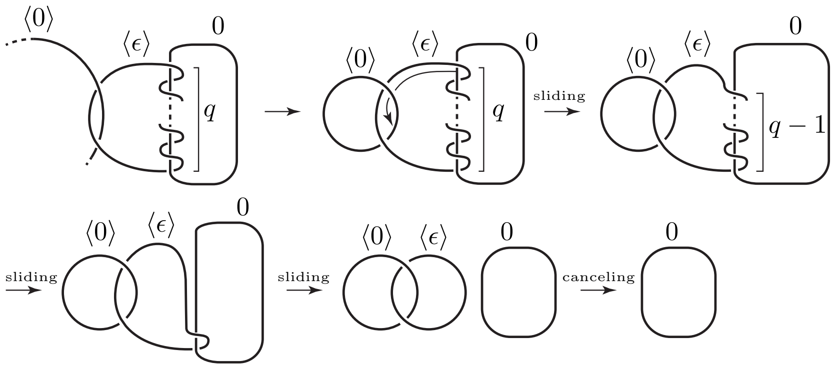

Figure 1.24: The isotopy type of the rightmost component.

Using this theorem, we can prove Theorem 1.1.6.

Proof of Theorem 1.1.6. Let e be an embedding $P \hookrightarrow M$. If $G(S_e) \cong \mathbb{Z}$ holds, then $\pi_1(E(S_e), \partial E(S))$ consists of one element. This means that any cord in $E(S_e)$ is isotopic to the trivial cord. Moving the embedded 1-handle in P around the meridian $\partial D^2 \times \{*\}$ as an isotopy of e, we can make the linking number zero. Hence, if the pochette surgery produces a homology 4-sphere, then the slope is $1/q$ for some meridian and longitude in P. From Theorem 1.1.5, the result is M (when $\varepsilon = 0$) or $\mathrm{Gl}(S_e)$ (when $\varepsilon = 1$).

If M is diffeomorphic to S^4 and S_e is the unknot, then any cord is isotopic to the trivial one. In the same way as above, any pochette surgery yielding a homology 4-sphere gives S^4. □

1.4 Examples

1.4.1 Pochette surgeries along ribbon 2-knots of 1-fusion

In this subsection, we consider diffeomorphism types of pochette surgeries on the 4-sphere with nontrivial core spheres and nontrivial cords.

Now we define *ribbon 2-knot* and *fusion*.

Definition 1.4.1 (ribbon 2-knot)**.** Let $\{D_1^3, \cdots, D_m^3\}$ be m pairwise disjoint 3-disks in S^4. We take $m-1$ pairwise disjoint embeddings $f_1, \dots, f_{m-1} : D^2 \times [0,1] \to S^4$. We assume that the embeddings satisfy the following conditions:

- $f_k(D^2 \times [0,1]) \cap \bigcup_{u=1}^m \partial D_u^3 = f_k(D^2 \times \{0,1\})$ for any $1 \leq k \leq m-1$.
- $\bigcup_{k=1}^{m-1} f_k(D^2 \times [0,1]) \cup \bigcup_{u=1}^m \partial D_u^3$ is connected.

Then the boundary of union of these m 3-disks and $m-1$ $D^2 \times [0,1]$

$$\bigcup_{u=1}^{m} \partial D_u^3 \cup \bigcup_{k=1}^{m-1} f_k(\partial D^2 \times [0,1])$$

is a 2-knot and called a *ribbon 2-knot* of $(m-1)$*-fusion*.

We take any ribbon 2-knot of 1-fusion as core spheres. Let S denote a ribbon 2-knot of 1-fusion in the 4-sphere. The sphere S is the double of a disk obtained by attaching one band over two 2-disks as presented by the left picture in Figure 1.25. The right diagram is the handle diagram of the complement of S. Let a subset m' of $\partial E(S_e)$ be the oriented meridian of a dotted 1-handle indicated in Figure 1.25 with a base point p. Let l' be an oriented meridian of the other dotted 1-handle passing p. Pushing the complement (the dashed line in the right picture in Figure 1.25) of the neighborhood of l' in the interior of $E(S_e)$, we obtain a cord c. Then the following holds.

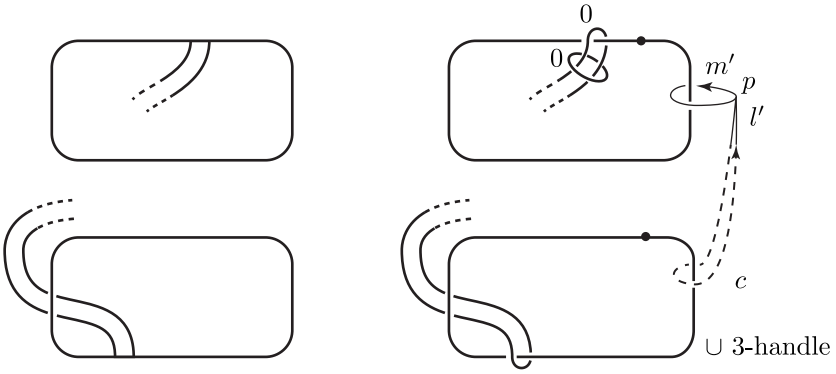

Figure 1.25: A ribbon 2-knot of 1-fusion (left) and the diagram of the complement of the 2-knot (right).

Lemma 1.4.2. *If $G(S_e)$ is not isomorphic to $\mathbb{Z}$, then this cord c is nontrivial.*

Recall the triviality of a cord was defined in Definition 1.1.3.

Proof. The fundamental group $G(S_e)$ is presented by

$$\langle x, y | wxw^{-1}y^{\pm 1} \rangle,$$

where x and y are the elements presented by the meridian m' and the longitude l' respectively, and w is a word obtained by reading x, y along the 2-handle corresponding to the band. Here the boundary-subgroup in $G(S_e)$ is $\langle x \rangle$.

Let $p : G(S_e) \to G(S_e)//\langle x \rangle$ be the projection for the double coset. Let [[id]] be the trivial coset in $G(S_e)//\langle x \rangle$, which is the coset including the identity element id $\in G(S_e)$. The inverse image $p^{-1}([[\text{id}]])$ is equal to $\langle x \rangle$. In fact $\langle x \rangle$ is a subset of $p^{-1}([[\text{id}]])$ is clear. For any $z \in p^{-1}([[\text{id}]])$, there exist some integers r, s such that $x^r z x^s = \text{id}$ is satisfied. Then $z = x^{-r-s} \in \langle x \rangle$.

The homotopy class of the cord c corresponds to $[[y]] \in G(S_e)//\mathbb{Z}$. If the cord c is trivial, then $y \in p^{-1}([[\text{id}]]) = \langle x \rangle$ holds. Hence we have $y = x^n$ for some integer n. This means $G(S_e)$ is an abelian group. Since the abelianization of $G(S_e)$ is $\mathbb{Z}$, we have $G(S_e) \cong \mathbb{Z}$. □

In general, it is well-known that $G(S_e) \not\cong \mathbb{Z}$ is satisfied for many nontrivial 2-knot S_e. Then the cord c is nontrivial.

By using this cord c, we obtain an embedding $e : P \hookrightarrow S^4$ whose core sphere is S. Then the handle diagram of the complement $E(P_e)$ of P is Figure 1.26. The meridian m_e is isotopic to l_1 or l_2 in $E(S_e)$. Here we assume that m_e is isotopic to l_i. Then, we put the orientation of the longitude as $[l_e] = -[l_i]$ in $E(P_e)$. Then $[m_e] = -[l_e]$ in $H_1(E(S_e))$ is satisfied. In this situation, the linking number of P_e is -1. Consider the $(p/(p+1), \varepsilon)$-pochette surgery by using the embedding e and these oriented meridian and longitude in P. The element $y \in \pi_1(E(P_e))$ is a lift of $-[l_1]$ and y^{-1} is a lift of $-[l_2]$, and hence $y^{\pm 1}$ is a lift of the longitude l_e.

According to the last pictures in Figure 1.21 and 1.22, the cases (I) and (II) in Figure 1.27 are obtained as results of attaching P along $p[m_e] + (p+1)[l_e]$ with the mod 2 framing ε. The case (I) is the one which m_e is isotopic to l_1 (as an oriented loop), while (II) is the case where m_e is isotopic to l_2 in the same way.

To prove Theorem 1.1.8, we first prove the following:

Proposition 1.4.3. *$S^4(e, p/(p+1), \varepsilon)$ is diffeomorphic to the double of a contractible 4-manifold without no 3-handles.*

Proof. Here we will consider the case where m_e is isotopic to l_2. The case where m_e is isotopic to l_1 can be proved in the same way.

We deform the handle diagram of (II) as in Figure 1.28. Continuously, we deform the handle diagram according to Figure 1.29. We show that the last picture presents that $S^4(e, p/(p+1), \varepsilon)$ is diffeomorphic to the double of a contractible 4-manifold C. The fundamental group $\pi_1(C)$ of C has the following presentation

$$\langle x, y | wxw^{-1}y^{\pm 1}, y^{\pm 1}(xy^{\pm 1})^p \rangle, \tag{1.2}$$

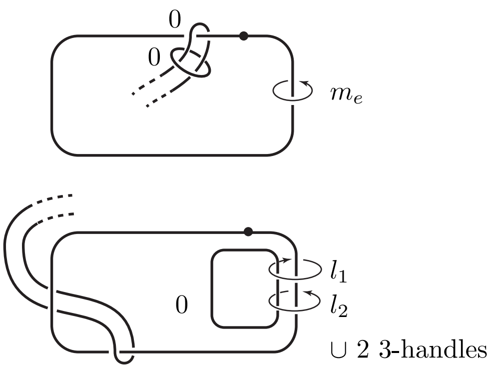

Figure 1.26: The pochette complement whose core sphere is a ribbon 2-knot of 1-fusion.

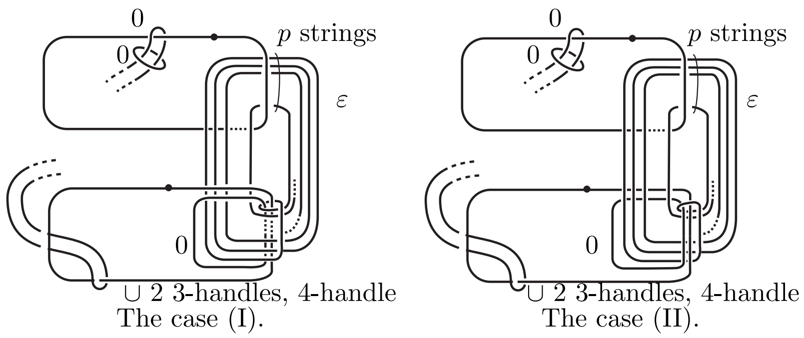

Figure 1.27: (I): m_e is isotopic to l_1. (II): m_e is isotopic to l_2.

according to the last picture in Figure 1.29. The proof of the triviality of this group is postponed in Lemma 1.4.4. The homology group of C is easily found out to be trivial from the handle decomposition. $\square$

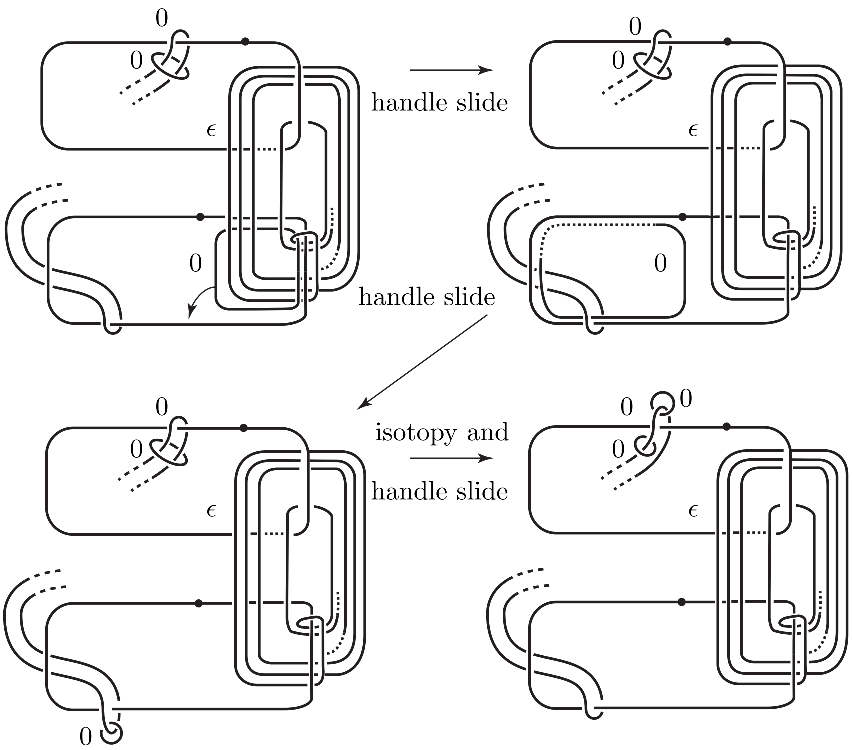

Figure 1.28: Handle moves.

As mentioned in [A2, the second paragraph in p. 36], the following result holds. Let $\mathcal{C}$ be a contractible 4-manifold with n 1-handles, n 2-handles and no 3-handles. If the presentation $\pi_1(\mathcal{C})$ with respect to the handle decomposition is AC-trivial, which is defined in the next subsection, then the double satisfies $D(\mathcal{C}) := \mathcal{C} \cup_{\mathrm{id}} (-\mathcal{C}) = \partial(\mathcal{C} \times I)$. Since the handle decomposition of $\mathcal{C} \times I$ depends only on the homotopy classes of the 2-handles, $\mathcal{C} \times I$ is diffeomorphic to the standard D^5. In the next subsection, we give a brief review of Andrews-Curtis moves and Andrews-Curtis trivial.

1.4.2 Andrews-Curtis triviality

Let $F = F(X)$ be a free group of rank $n \geq 2$ with a basis $X = \{x_1, \ldots, x_n\}$ and $W = (w_1, \ldots, w_n)$ an n-tuple of words of X. Consider the following three types of transformations of W:

(AC1) Replace w_i by $w_i w_j$ if $j \neq i$.

(AC2) Replace w_i by w_i^{-1}.

(AC3) Replace w_i by $v w_i v^{-1}$ for some $v \in F$, and leave w_k fixed for all $k \neq i$.

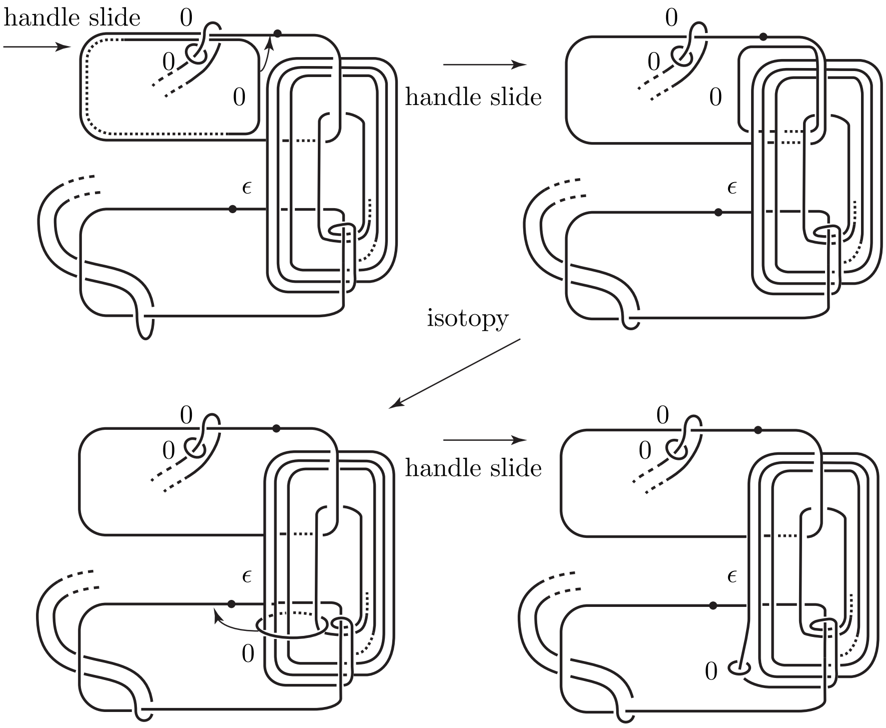

Figure 1.29: Handle moves.

Let $R = \langle x_1, \ldots, x_n | w_1, \ldots, w_n \rangle$ be a presentation of the trivial group. We call base transformations (inversion and permutation of generators and relators) of X, the transformations (AC1)-(AC3) for relators $w_1, \ldots, w_n$, and adding or deleting a generator g and a relator g as the same element *Andrews-Curtis moves* (or *AC-moves*). If R can be reduced to the empty presentation $\langle \varnothing | \varnothing \rangle$ by a finite sequence of AC-moves for the basis and relators, then R is called an *Andrews-Curtis trivial* (or *AC-trivial*) presentation.

Lemma 1.4.4. *The presentation* (1.2) *is an AC-trivial presentation of the trivial group.*

Proof. We give the following sequence of AC-moves:

$$\begin{aligned} & \langle x, y | wxw^{-1}y^{\pm 1}, y^{\pm 1}(xy^{\pm 1})^p \rangle \\ = \ & \langle x, xy^{\pm 1} | wxw^{-1}x^{-1}(xy^{\pm 1}), x^{-1}(xy^{\pm 1})^{p+1} \rangle \\ = \ & \langle x, z | wxw^{-1}x^{-1}z, x^{-1}z^{p+1} \rangle \\ = \ & \langle x^{-1}z^{p+1}, z | wxw^{-1}x^{-1}z, x^{-1}z^{p+1} \rangle \\ = \ & \langle u, z | w(z^{p+1}u^{-1})w^{-1}uz^{-p}, u \rangle = \langle z | z^m \rangle. \end{aligned}$$

Here since this group is trivial, $m = \pm 1$. Thus the presentation is AC-trivial. □

We left the proof of the triviality of $\pi_1(C)$ in Proposition 1.4.3. Lemma 1.4.4 implies the proof of Proposition 1.4.3 completes.

Proof of Theorem 1.1.8. Let $e : S^2 \hookrightarrow S^4$ be a ribbon 2-knot of 1-fusion. We take the same cord c as the one chosen in Subsection 1.4.1, which is used in

Figure 1.25. By using Proposition 1.4.3, the pochette surgery $S^4(e, p/(p+1), \varepsilon)$ is diffeomorphic to the double of a contractible 4-manifold C. The C has an AC-trivial presentation of π_1 coming from a handle decomposition of C with no 3-handles. By applying the method in [A2], $S^4(e, p/(p+1), \varepsilon) = D(C)$ is diffeomorphic to the standard 4-sphere. □

Proof of Theorem 1.1.7. Let S and c be the ribbon 2-knot and the cord that we dealt with in Theorem 1.1.8. Then S is nontrivial and c is nontrivial. The pochette surgery gives the standard S^4. □

1.4.3 A case of spun trefoil knot

As an example, we give a concrete diagram for the spun trefoil knot as a ribbon 2-knot of 1-fusion. Figure 1.30 is the handle diagram of the complement.

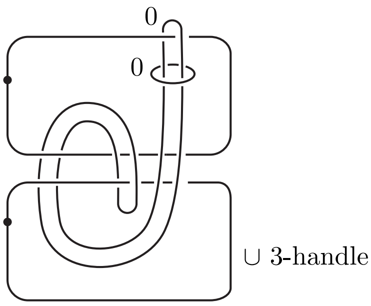

Figure 1.30: A handle diagram of a ribbon 2-knot exterior.

We choose m_e and l_e as in Figure 1.31 (left), then the embedding $i : \partial P_e \hookrightarrow E(P_e)$ gives $i_*([l_e]) = -[m_e]$. Namely the linking number is $\ell = -1$. Let x, y be lifts in $\pi_1\left(S^4\left(e, \frac{1}{2}, \varepsilon\right)\right)$ of generators m_e and l_e respectively. Then the presentation of $\pi_1\left(S^4\left(e, \frac{1}{2}, \varepsilon\right)\right)$ is the following:

$$\langle x, y | yx^{-1}yxy^{-1}x, y^2x\rangle \cong \{\mathrm{id}\}.$$

The diagram of this homotopy 4-sphere becomes the right picture in Figure 1.31. In this case, we can deform this diagram into the double of a contractible 4-manifold with no 3-handles as in Figure 1.32.

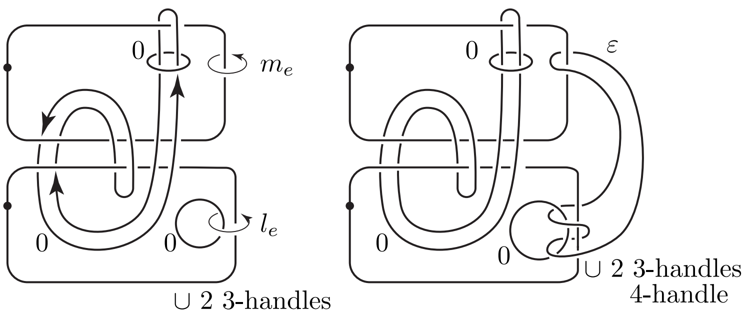

Figure 1.31: A pochette surgery $S^4\left(e, \frac{1}{2}, \varepsilon\right)$ with a nontrivial 2-knot S_e and a nontrivial cord.

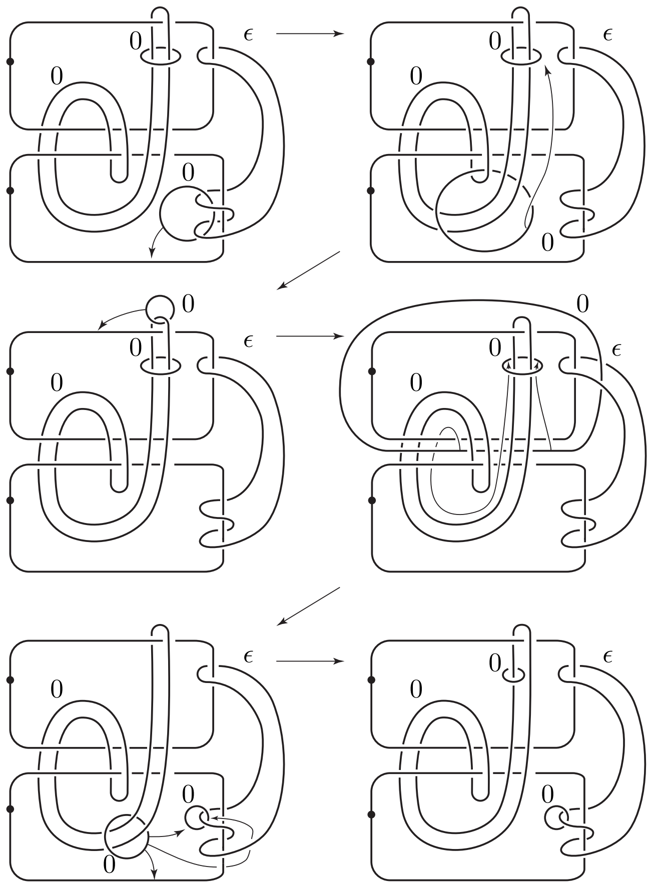

Figure 1.32: A diffeomorphism to the double of a contractible 4-manifold.

1.4.4 Pochette surgeries along ribbon 2-knots of n-fusion

The method to prove Theorem 1.1.8 can be easily extended to the case of the surgery that the core sphere is any ribbon 2-knot of n-fusion. Here we prove Theorem 1.1.9.

Proof of Theorem 1.1.9. Let S be any ribbon 2-knot of n-fusion. We fix the handle decomposition of $E(S)$ corresponding to the fusion. That is, the decomposition has one 0-handle, $n+1$ dotted 1-handles, n 2-handles and n dual 2-handles and $n+1$ 3-handles and one 4-handle. See [GS, Section 6.2] for the description of ribbon 2-knot complement. We take two based meridians m' and l' of the dotted 1-handles with a base point $p_0 \in \partial E(S)$. We suppose that m' lies in $\partial E(S)$ and is a meridian of $\partial E(S)$. Let x, y be elements in $\pi_1(E(S))$ corresponding to m' and l' respectively. Here we can assume that $y^{\pm 1}$ is conjugate to x but $y^{\pm 1} \notin \langle x \rangle$. Actually, if any based meridian of each dotted 1-handle of $E(S)$ is in an element in $\langle x \rangle$, then $\pi_1(E(S))$ is a quotient of $\mathbb{Z}$, because the set of the meridians of the dotted 1-handles is a generator of $\pi_1(E(S))$. Actually using the abelianization map $\pi_1(E(S)) \overset{ab}{\to} H_1(E(S)) = \mathbb{Z}$, we conclude that $\pi_1(E(S))$ is isomorphic to $\mathbb{Z}$. Now this case is ruled out. Thus, there exists a based meridian l' of $E(S)$ such that $y := [l']$ is conjugate to x but $y \notin \langle x \rangle$.

In the same way as the proof of Theorem 1.1.8, from l' we produce a cord in $E(S)$. Thus, by taking such a cord, we obtain a pochette embedding $e : P \hookrightarrow S^4$. By moving the 0-framed 2-handle by the process in Figure 1.28 and 1.29, we can take the 0-framed 2-handle in the position of the meridian of the ε-framed 2-handle.

If the graph for the n-fusion is as in Figure 1.33. This is just a schematic picture for the fusion, and the edges stand for connecting 0-framed 2-handles coming from the bands of the ribbon disk. Actually, in the true picture, the edges should be drawn as some bands and might be linking to several dotted 1-handles. For our proof, we may omit these data because sliding the 0-framed 2-handle to dual 2-handles, we can ignore the linking.

We take the two based oriented meridians m' and l' in the positions in the figure. We suppose that the below 0-framed 2-handle in the first picture in Figure 1.28 is attached in the dashed circle in Figure 1.33 in our situation. From the 1-handle k linking to l' to the 1-handle k' linking to m', the 0-framed 2-handle can be moved by doing several handle slides and some isotopy. See Figure 1.34 for the handle moves. This also generalizes the moves from the first picture in Figure 1.28 to the second picture in Figure 1.29. Hence, we can freely move the 0-framed 2-handle from a dotted 1-handle to another dotted 1-handle.

By these handle slides, all 0-framed 2-handles corresponding to the dual bands can be moved in the meridians of all 2-handles. This means that $S^4(e, p/(p+1), \varepsilon)$ is the double of a homology 4-ball H without 3-handles. □

As mentioned in Section 1.1 as well, it is unclear whether any homology 4-sphere obtained by this pochette surgery is simply connected or not.

If the 2-knot is n-fusion ribbon knot, the fundamental group of $S^4(e, p/(p+1), \varepsilon)$ has the form

$$\langle x_1, \ldots, x_{n+1} | w_1 x_{i_1} w_1^{-1} x_{j_1}^{-1}, \ldots, w_n x_{i_n} w_n^{-1} x_{j_n}^{-1}, x_s^{-1}(x_r x_s^{-1})^p \rangle, \tag{1.3}$$

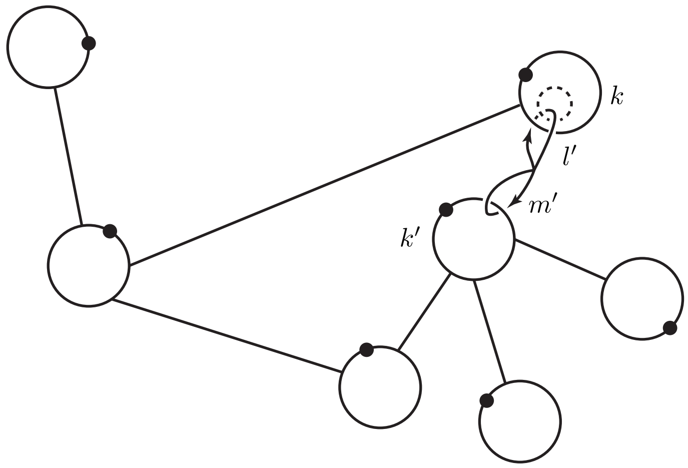

Figure 1.33: A graph for the fusion of a ribbon 2-knot.

where for $k = 1, 2, \ldots, n$, w_k is a word in $x_1, \ldots, x_{n+1}$, the set

$$\{\{i_k, j_k\} | k = 1, \cdots, n\}$$

is the set of edges of the graph, and r, s are some integers in $\{1, \ldots, n\}$. Even if H in the proof of Theorem 1.1.9 is contractible, that is, the fundamental group is trivial then it is unclear whether $S^4(e, p/(p+1), \varepsilon)$ is diffeomorphic to S^4 or not.

Proof of Theorem 1.1.10. If the homology 4-ball H in the proof above is contractible, then $S^4(e, g) = H \cup (-H)$ is a homotopy 4-sphere. Furthermore, if the presentation of π_1 coming from the handle decomposition is AC-trivial, then from the method mentioned right after the proof of Theorem 1.1.8, therefore, $S^4(e, g)$ is diffeomorphic to the standard S^4. □

We give a sufficient condition that the presentation (1.3) is AC-trivial. Let $\{x_1, \ldots, x_{n+1}\}$ be a generator of the free group F_{n+1}. For any word w of $x_1, \ldots, x_{n+1}$, we put $r_{2i-1} = w x_{2i} w^{-1} x_{2i+1}^{-1}$ for $2i-1 < n$, $r_{2i} = w x_{2i+2} w^{-1} x_{2i+1}^{-1}$ for $2i < n$ and

$$r_n = \begin{cases} w x_{n+1} w^{-1} x_1^{-1} & (n \text{ is odd}), \\ w x_1 w^{-1} x_{n+1}^{-1} & (n \text{ is even}). \end{cases}$$

Then we consider the presentation

$$\langle x_1, \ldots, x_{n+1} | r_1, \ldots, r_n \rangle.$$

This presentation gives the fundamental group of the complement of a ribbon 2-knot of n-fusion.

Lemma 1.4.5. *Let n be a positive integer. For any word w of $x_1, \ldots, x_{n+1}$, the relators $r_1, \ldots, r_n$ are the same as above. For $r_{n+1} = x_1^{-1}(x_2 x_1^{-1})^p$, the presentation*

$$\langle x_1, \ldots, x_{n+1} | r_1, \ldots, r_n, r_{n+1} \rangle \tag{1.4}$$

is the trivial group presentation with AC-trivial.

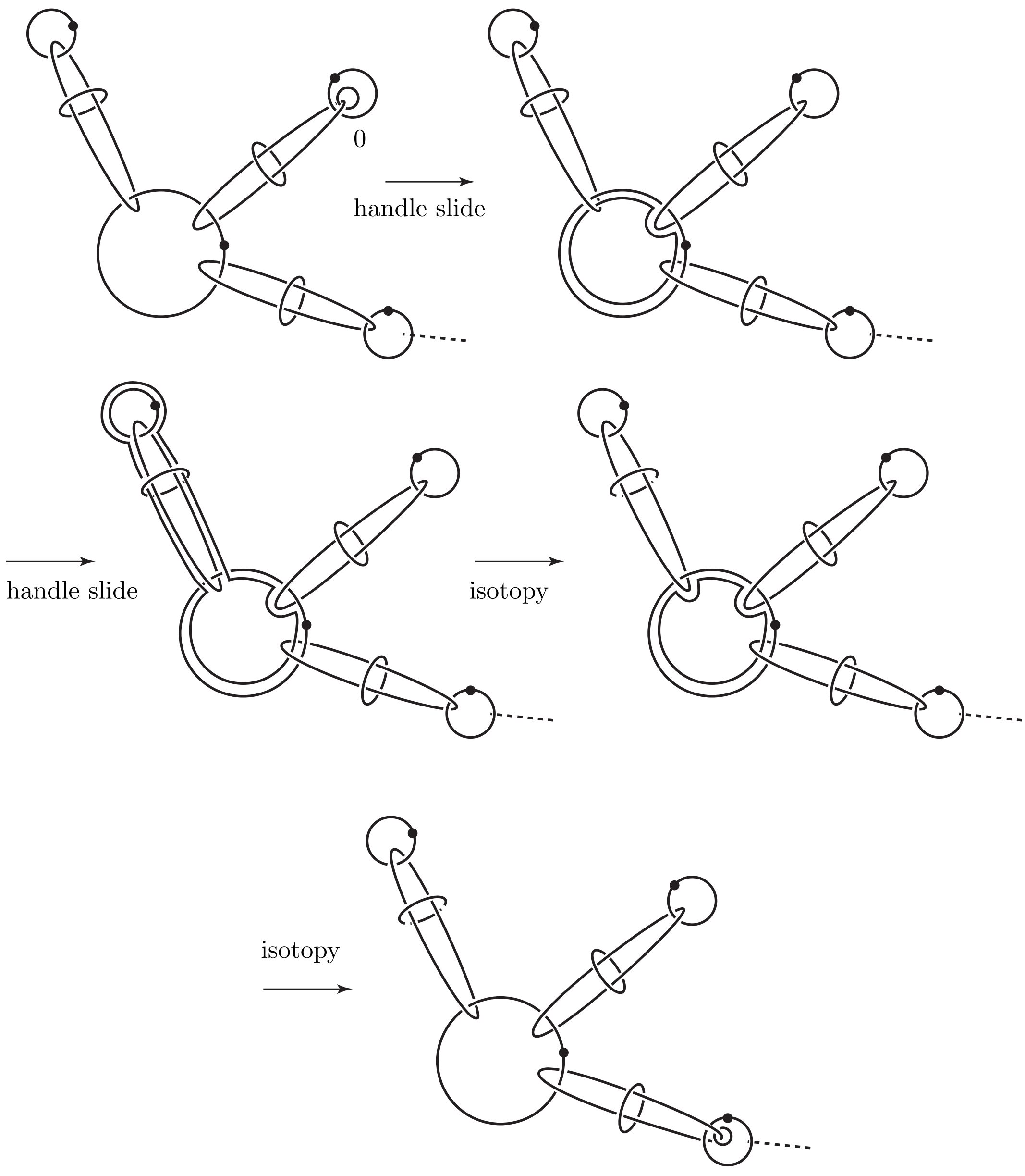

Figure 1.34: Deformations to move the 0-framed meridian in the position.

Proof. We obtain

$$r_{2i-1}r_{2i}^{-1} = wx_{2i}x_{2i+2}^{-1}w^{-1} \sim x_{2i}x_{2i+2}^{-1} \text{ and } r_{2i+1}^{-1}r_{2i} = x_{2i+3}^{-1}x_{2i+2},$$

$r_{n-1}^{-1}r_n = x_n x_1^{-1}$ if n is odd, $r_{n-1}r_n^{-1} = wx_n x_1^{-1}w^{-1} \sim x_n x_1^{-1}$ if n is even, where $\sim$ presents the relation between conjugate elements. Then we have

$$\begin{aligned}
& \langle x_1, \ldots, x_{n+1} | r_1, r_2, r_3, \ldots, r_n, r_{n+1} \rangle \\
\cong\ & \langle x_1, \ldots, x_{n+1} | r_1 r_2^{-1}, r_2, r_3, \ldots, r_n, r_{n+1} \rangle \\
\cong\ & \langle x_1, \ldots, x_{n+1} | r_1 r_2^{-1}, r_2^{-1} r_3, r_3, \ldots, r_n, r_{n+1} \rangle \\
\cong\ & \langle x_1, \ldots, x_{n+1} | r_1 r_2^{-1}, r_2^{-1} r_3, r_3 r_4^{-1}, \ldots, r_n, r_{n+1} \rangle \\
\cong\ & \begin{cases} \langle x_1, \ldots, x_{n+1} | r_1 r_2^{-1}, r_2^{-1} r_3, r_3 r_4^{-1}, \ldots, r_{n-1}^{-1} r_n, r_n, r_{n+1} \rangle & (n \text{ is odd}), \\ \langle x_1, \ldots, x_{n+1} | r_1 r_2^{-1}, r_2^{-1} r_3, r_3 r_4^{-1}, \ldots, r_{n-1} r_n^{-1}, r_n, r_{n+1} \rangle & (n \text{ is even}) \end{cases} \\
\cong\ & \langle x_1, \ldots, x_{n+1} | x_2 x_4^{-1}, x_3 x_5^{-1}, x_4 x_6^{-1}, \ldots, x_{n-1} x_{n+1}^{-1}, x_n x_1^{-1}, r_n, r_{n+1} \rangle.
\end{aligned}$$

Replacing $x_i x_{i+2}^{-1}$ with x_i' for $i = 2, \ldots, n-1$ and $x_n x_1^{-1}$ with x_n', we give

$$x_2 = \begin{cases} x_2' x_4' x_6' \cdots x_{n-1}' x_{n+1} & (n \text{ is odd}), \\ x_2' x_4' x_6' \cdots x_n' x_1 & (n \text{ is even}), \end{cases}$$

$$r_n = \begin{cases} w' x_{n+1} (w')^{-1} x_1^{-1} & (n \text{ is odd}), \\ w' x_1 (w')^{-1} x_{n+1}^{-1} & (n \text{ is even}), \end{cases}$$

where w' is a word of x_1, x_i' and x_{n+1} and we have

$$\begin{aligned}
& \langle x_1, \ldots, x_{n+1} | r_1, r_2, r_3, r_4, \ldots, r_n, r_{n+1} \rangle \\
\cong\ & \langle x_1, x_2', \ldots, x_n', x_{n+1} | x_2', x_3', x_4', \ldots, x_{n-1}', x_n', r_n, r_{n+1} \rangle \\
\cong\ & \begin{cases} \langle x_1, x_{n+1} | w' x_{n+1} (w')^{-1} x_1^{-1}, x_1^{-1} (x_{n+1} x_1^{-1})^p \rangle & (n \text{ is odd}), \\ \langle x_1, x_{n+1} | w' x_1 (w')^{-1} x_{n+1}^{-1}, x_1^{-1} \rangle & (n \text{ is even}). \end{cases}
\end{aligned}$$

By applying Lemma 1.4.4, we see that this presentation is AC-trivial. Therefore, we obtain the desired result above. $\square$

1.5 Pochette surgeries and finite presentation of groups

In this section, we mention that a relationship of a pochette surgery on a ribbon 2-knot and a finite representation group of any knot group plus one arbitrary relation.

Theorem 1.5.1. *Let K be any knot in the 3-sphere. Then, there exists an embedding $e : P \hookrightarrow S^4$ and a diffeomorphism $g : \partial P \to \partial E(P_e)$ that satisfy the followings:*

(1) The core sphere S_e is a ribbon 2-knot,

(2) There exists a finite representation of a knot group of K

$$R(K) = \langle x_1, \dots, x_k | r_1, \dots, r_{k-1} \rangle$$

such that for any word r of $x_1, \dots, x_k$, we have

$$\pi_1(S^4(e, 1, \varepsilon)) \cong \langle x_1, \dots, x_k | r_1, \dots, r_{k-1}, r \rangle.$$

Proof. By [GS, Section 6.2], there exists an embedding $e : P \hookrightarrow S^4$ such that $E(S_e)$ is the complement of a ribbon 2-knot S_e in S^4 and $\pi_1(E(S_e)) \cong \pi_1(E(K))$. Then we have $\pi_1(E(P_e)) = \pi_1(E(S_e))$. Furthermore, by using the dotted circles in a handle diagram of $E(S_e)$ as the generator, a finite representation of the knot group of K $R(K) = \langle x_1, \dots, x_k | r_1, \dots, r_{k-1} \rangle$ be obtained from the way the framed link intertwines with each dotted circle. Thus we have

$$\pi_1(E(P_e)) \cong \langle x_1, \dots, x_k | r_1, \dots, r_{k-1} \rangle.$$

Thus, any word r of $x_1, \dots, x_k$, we can obtain a cord C that satisfies $m' = [m] = x_i$, $l' = [l] = x_i^{-1} r$. Then, there exists a diffeomorphism $g : \partial P \to \partial E(P_e)$ such that

$$\pi_1(S^4(e, 1, \varepsilon)) \cong \langle x_1, \dots, x_k | r_1, \dots, r_{k-1}, r \rangle.$$

Therefore, we obtain the desired result above. □

Let $S^3_{p/q}(K)$ be the Dehn surgery on S^3 with the coefficient p/q in $\mathbb{Q} \cup \{\infty\}$.

Corollary 1.5.2. *, The fundamental group of any Dehn surgery on the 3-sphere along any knot is isomorphic to that of a pochette surgery on the 4-sphere along a ribbon 2-knot.*

Proof. By the proof of Theorem 1.5.1, there exists an embedding $e : P \hookrightarrow S^4$ such that $E(S_e)$ is the complement of a ribbon 2-knot $e(S)$ in S^4. By Theorem 1.5.1, for any Dehn surgery on the 3-sphere, there exists a word r of $x_1, \dots, x_k$ and a diffeomorphism $g : \partial P \to \partial E(P_e)$ such that the fundamental group of the Dehn surgery is isomorphic to

$$\langle x_1, \dots, x_k | r_1, \dots, r_{k-1}, r \rangle \cong \pi_1(S^4(e, 1, \varepsilon))$$

for any $\varepsilon \in \{0, 1\}$. □

Corollary 1.5.3. *The fundamental group of any Dehn surgery on the 3-sphere with coefficient $1/q$ along any knot is isomorphic to that of a homology 4-sphere which is diffeomorphic to a pochette surgery on the 4-sphere along a ribbon 2-knot.*

Proof. By Theorem 1.5.1 and Corollary 1.5.2, we can take the embedding $e : P \hookrightarrow S^4$ such that we have

$$\pi_1(S^4(e, 1, \varepsilon)) \cong \pi_1(S^3_{1/q}(K)).$$

Thus we have $H_1(S^4(e, 1, \varepsilon)) \cong H_1(S^3_{1/q}(K)) = 0$. By Proposition 1.2.9, we have $H_2(S^4(e, 1, \varepsilon)) = 0$ and $H_3(S^4(e, 1, \varepsilon)) = 0$. Also, since $H_k(S^4(e, 1, \varepsilon)) \cong \mathbb{Z}$ $(k = 0, 4)$, the pochette surgery $S^4(e, 1, \varepsilon)$ is a homology 4-sphere for any $\varepsilon \in \{0, 1\}$. □

1.6 Questions in pochette surgery and outer surgery

In this section we raise several questions. We leave the following problem about Theorem 1.1.9.

Question 1.6.1. *Let S be any ribbon 2-knot with $G(S) \ncong \mathbb{Z}$. Does there exist a nontrivial cord c in $E(S)$ such that any nontrivial surgery with respect to the embedding $e : P \hookrightarrow S^4$ with the cord c and the core sphere S yielding a homology 4-sphere gives the standard 4-sphere?*

Since pochette surgery is a generalization of Gluck surgery, the triviality of Gluck surgery on any ribbon 2-knot might also hold in the pochette surgery situation.

Question 1.6.2. *Let S be any ribbon 2-knot with $G(S) \ncong \mathbb{Z}$. Suppose that $e : P \hookrightarrow S^4$ is any embedding with $S_e = S$. Does any pochette surgery $S^4(e, g)$ yielding a homology 4-sphere for some gluing map g give the 4-sphere?*

It might be possible that we answer the following question affirmatively.

Question 1.6.3. *Let S be any ribbon 2-knot in S^4 with $G(S) \ncong \mathbb{Z}$. If a pochette surgery with the core sphere S yields a homology 4-sphere, is the pochette surgery the standard 4-sphere?*

Can the diffeomorphisms in the previous section be generalized to cases of any nontrivial core sphere?

Question 1.6.4. *Let S be any 2-knot with $G(S) \ncong \mathbb{Z}$. Then, does there exist a nontrivial cord in $E(S)$ such that any pochette surgery for a pochette embedding $e : P \hookrightarrow S^4$ with the core sphere S is S^4 or $\mathrm{Gl}(S)$?*

Can we construct a homotopy 4-sphere other than $\mathrm{Gl}(S)$ by pochette surgery? Furthermore, we raise two questions in more generalized settings.

Question 1.6.5. *Can a pochette surgery of S^4 construct an exotic S^4?*

More generally, we ask the following question.

Question 1.6.6. *Can a pochette surgery of an oriented 4-manifold M construct an exotic structure on M?*

Pochette surgery can be generalized to a surgery on a *generalized pochette* $P_{a,b} = \natural^a S^1 \times D^3 \natural^b D^2 \times S^2$. Such a surgery is called an *outer surgery* and it is studied by Nakamura in [N]. Would studying outer surgery lead to the construction of interesting 4-manifolds? Investigating outer surgery is a potential avenue for future research about exotic 4-manifolds.

Chapter 2

The d-invariants of Brieskorn homology 3-spheres

2.1 The d-invariant and the Brieskorn homology 3-spheres

One of the most important problems in 3-manifold topology is which integral homology 3-spheres smoothly bound integral homology 4-balls. In 2003, Ozsváth and Szabó [OS] introduced a d-invariant. The d-invariant $d(Y, \mathfrak{s}(Y))$ is a rational homology spin^c cobordism invariant that assigns a rational number to any rational homology 3-sphere Y and any spin^c structure $\mathfrak{s}(Y)$ over Y. Since there exists only one spin^c structure on any homology 3-sphere, we will not represent these spin^c structures. In particular, if Y is a homology sphere, d-invariant is an even value, and is a homology cobordism invariant.

We assume that p, q and r are pairwise relatively prime, positive integers. We define a homology sphere $\Sigma(p, q, r)$ as

$$\{(x, y, z) \in \mathbb{C}^3 \cap S^5_\varepsilon | x^p + y^q + z^r = 0\},$$

which is called a *Brieskorn homology* 3*-sphere*. Here, S^5_ε is a 5-sphere whose radius is a sufficiently small positive real number ε.

In Section 2.2 we examine the value of the d-invariant of $\Sigma(p, q, r)$ only when $pq + pr - qr = 1$. In Section 2.3 we attempt to construct a formula to obtain the value of the d-invariant of any Brieskorn homology 3-sphere.

2.2 The case of $pq + pr - qr = 1$

2.2.1 Introduction for this section

In this section, we assume that $1 < p < q < r$ and

$$pq + pr - qr = 1.$$

Then a surgery diagram of the 3-manifold $\Sigma(p,q,r)$ is the plumbed 3-manifold corresponding to the graph shown in Figure 2.1.

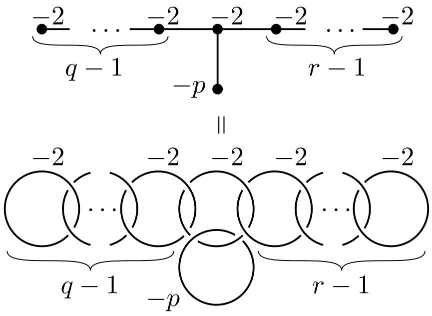

Figure 2.1: A surgery diagram of $\Sigma(p,q,r)$ with $pq+pr-qr=1$.

We call such graphs almost simple linear graphs. The most well-known example of Brieskorn homology 3-spheres is the Poincaré homology sphere $\Sigma(2,3,5)$ and its almost simple linear graph is the E_8 graph.

In 2020, Karakurt and Şavk [KŞ] studied the Ozsváth-Szabó d-invariant of $\Sigma(p,q,r)$. They computed the case of p is even.

Proposition 2.2.1 (Karakurt-Şavk [KŞ, Proposition 4.5])**.** *If p is even, then we have*

$$d(\Sigma(p,q,r)) = \frac{q+r}{4} = \frac{r^2-1}{4(r-p)} = \frac{q^2-1}{4(q-p)}.$$

They also introduced a formula of the case of p is odd. We compute more explicitly when p is odd. Let t and α be the quotient and the remainder when we divide n by $q-p$, respectively. We define the two-variable quadratic function

$$f(x,y) := -(q+r)x^2 + 4qxy - 4(q-p)y^2 - 4y, \text{ and } F(x,y) := \frac{f(x,y)+q+r}{4}.$$

Let $\mathfrak{M}' := \{(a,m)\in\mathbb{Z}^2 | a \text{ is odd, } 3\le a<m\le n, F(a,m)\ge F(1,1)\}$ and

$$\mathfrak{M} := \{(1,t+1)\}\sqcup\mathfrak{M}'.$$

Theorem 2.2.2. *If p is odd, then we have*

$$d(\Sigma(p,q,r)) = \max_{(a,m)\in\mathfrak{M}} F(a,m) \ge (t+1)(n+\alpha).$$

Remark 2.2.3. If t is an odd integer, then $t+1$ is even. If t is an even integer, then $n+\alpha = t(q-p)+2\alpha$ is even. Thus, we have $(t+1)(n+\alpha)$ is even for any integer t.

Remark 2.2.4. In all the cases of [KŞ, Theorem 1.3], the equality of the inequality in Theorem 2.2.2 holds. In other words, if q is $p+1$, $(5p+1)/4$, $(5p+3)/4$, $(3p-1)/2$, $(3p+1)/2$ or $2p-1$, then we have

$$d(\Sigma(p,q,r)) = (t+1)(n+\alpha).$$

In Subsection 2.2.2 we review results of [KŞ] for the case where p is odd. In Subsection 2.2.3 we prove the main result. In Subsection 2.2.4 we present inequality relationship between d-invariants and conject the Ozsváth-Szabó d-invariant of any Brieskorn homology 3-sphere $\Sigma(p,q,r)$ with p odd and $pq+pr-qr=1$. In Subsection 2.2.5 we present some methods for computation of $d(\Sigma(p,q,r))$ with p odd and $pq+pr-qr=1$.

2.2.2 Preliminaries for this section

We consider the Brieskorn homology 3-sphere $\Sigma(p,q,r)$ with $pq+pr-qr=1$. If p is odd, then we put $p=2n+1$ for some positive integer n. We define the sets

$$\mathfrak{L} := \{\pm 1, \pm 3, \dots, \pm p\} \times \{0, 1, \dots, (p-1)/2\}$$

and

$$R := \{(x,y) \in \mathbb{R}^2 | -p \le x \le p, 0 \le y \le (p-1)/2, F(x,y) \ge F(1,1)\}.$$

Theorem 2.2.5 (Karakurt-Şavk [KŞ, Theorem 1.1]). *If p is odd, then we have*

$$d(\Sigma(p,q,r)) = \max_{(a,m)\in \mathfrak{L}\cap R} F(a,m).$$

We also define

$$\Delta(y) := 4(2y-(q+r))^2 - 16(q+r)(p-1),\ \mathfrak{d}(m) := \min_{x\in 2\mathbb{Z}+1} |\mathfrak{c}(m)-x|,$$

$$\mathfrak{c}(m) := \frac{2qm}{q+r} \text{ and } \mathfrak{t}(m) := \frac{\sqrt{\Delta(m)}}{2(q+r)}.$$

There is a necessary and sufficient condition for a pair of integers (a,m) to be an element of R.

Proposition 2.2.6 (Karakurt-Şavk [KŞ, Proposition 4.8]). *A pair of integers (a,m) in R if and only if (a,m) is equal to $(1,1)$ or all of the following conditions are satisfied:*
(1) $m \ge 2$.
(2) $\Delta(m) \ge 0$.
(3) $\mathfrak{d}(m) \le \mathfrak{t}(m)$.
(4) $|\mathfrak{c}(m)-a| = \mathfrak{d}(m)$.

2.2.3 Proofs of main theorems for this section

To evaluate the d-invariant of any Brieskorn homology 3-sphere $\Sigma(p,q,r)$ with p odd and $pq+pr-qr=1$, we prove lemmas needed later.

Lemma 2.2.7.

$$p+1 \le q \le 2p-1.$$

Proof. By p odd and $pq+pr-qr=1$, we obtain

$$p \ge 3 \text{ and } q < r = \frac{pq-1}{q-p}.$$

Thus we have $q^2-2pq+1<0$. Therefore, we have

$$p-\sqrt{p^2-1} < q < p+\sqrt{p^2-1}.$$

By $q>p>p-\sqrt{p^2-1}$ and $2p-1<p+\sqrt{p^2-1}<2p$, we obtain the inequality above. □

Lemma 2.2.8.
$$F(a,m) = -\frac{1}{4(q-p)}\left((2(q-p)m - aq + 1)^2 - (q-a)^2\right).$$

Proof. This can be proven by simple calculations:

$$\begin{aligned}
& F(a,m) \\
= & \frac{1}{4}(-(q+r)a^2 + 4aqm - 4(q-p)m^2 - 4m + q + r) \\
= & \frac{1}{4}(-4(q-p)m^2 + 4(aq-1)m - (q+r)(a^2-1)) \\
= & -(q-p)m^2 + (aq-1)m - \frac{(a^2-1)(q^2-1)}{4(q-p)} \\
= & -(q-p)\left(m - \frac{aq-1}{2(q-p)}\right)^2 + (q-p)\left(\frac{aq-1}{2(q-p)}\right)^2 - \frac{(a^2-1)(q^2-1)}{4(q-p)} \\
= & -(q-p)\left(m - \frac{aq-1}{2(q-p)}\right)^2 + \frac{(aq-1)^2 - (a^2-1)(q^2-1)}{4(q-p)} \\
= & -\frac{(2(q-p)m - aq + 1)^2}{4(q-p)} + \frac{(q-a)^2}{4(q-p)} \\
= & -\frac{1}{4(q-p)}\left((2(q-p)m - aq + 1)^2 - (q-a)^2\right).
\end{aligned}$$

□

We divide n by $q-p$ as $n = t(q-p) + \alpha$, where t is the quotient and α is the remainder. We define the set
$$\mathfrak{L}' := \{1, 3, \ldots, p\} \times \{1, \ldots, (p-1)/2\}.$$

Lemma 2.2.9.
$$\mathfrak{L} \cap R = \mathfrak{L}' \cap R.$$

Proof. By Proposition 2.2.6, $(a, 0) \notin R$ for any $a \in \{\pm 1, \pm 3, \ldots, \pm p\}$. From
$$\mathfrak{c}(m) = \frac{2qm}{(q+r)} \geq 0,$$
we have
$$|\mathfrak{c}(m) - a| > |\mathfrak{c}(m) - 1| \geq \mathfrak{d}(m)$$
for any $a \in \{-1, -3, \ldots, -p\}$. By the condition (4) in Proposition 2.2.6, we have $(a, m) \notin R$ for any $a \in \{-1, -3, \ldots, -p\}$.

Therefore, we obtain the desired result above. □

Lemma 2.2.10. *For any (a, m) in $\mathfrak{L}' \cap R$, if $m \leq a$, then we have $F(a, m) \leq F(1, 1)$.*

Proof. If $m \leq a$, then we have
$$\begin{aligned}
aq - 2(q-p)m - 1 &\geq aq - 2(q-p)a - 1 \\
&= -aq + 2ap - 1 \\
&= a(2p - q) - 1 \\
&\geq 2p - q - 1 \\
&\geq 2p - (2p-1) - 1 = 0.
\end{aligned}$$

By Lemma 2.2.8, we have

$$\begin{aligned}
& 4(q-p)(F(1,1)-F(a,m)) \\
= \;& 4(q-p)\Big(-\frac{1}{4(q-p)}\left((2(q-p)-q+1)^2-(q-1)^2\right) \\
+ \;& \frac{1}{4(q-p)}\left((2(q-p)m-aq+1)^2-(q-a)^2\right)\Big) \\
= \;& 4(q-p)\left(p-1+\frac{1}{4(q-p)}\left((2(q-p)m-aq+1)^2-(q-a)^2\right)\right) \\
= \;& (aq-2(q-p)m-1)^2-(q-a)^2+4(p-1)(q-p) \\
\geq \;& (aq-2(q-p)a-1)^2-(q-a)^2+4(p-1)(q-p) \\
= \;& 4(p-1)(q-p)+(-aq+2ap-1)^2-(a-q)^2 \\
= \;& (a^2-1)q^2+4(1-a^2)pq+4(a^2-1)p^2+4(1-a)p+4(a-1)q-a^2+1 \\
= \;& (a^2-1)(q^2-4pq+4p^2)+(a-1)(q-p)-(a^2-1) \\
= \;& (a-1)((a+1)((2p-q)^2-1)+(q-p)) \geq 0.
\end{aligned}$$

The last inequality follows from the fact that Lemma 2.2.7 and $a \geq 1$.

Therefore, we obtain the desired inequality above. □

Proposition 2.2.11.

$$\max_{1\leq m\leq n} F(1,m) = F(1, t+\min\{\alpha,1\}) = (t+1)(n+\alpha).$$

Proof. By

$$\frac{q-1}{2(q-p)} = \frac{p-1+q-p}{2(q-p)} = \frac{n}{q-p}+\frac{1}{2} = t+\frac{\alpha}{q-p}+\frac{1}{2},$$

we have

$$\begin{aligned}
& \max_{1\leq m\leq n} F(1,m) \\
= \;& \max_{1\leq m\leq n}(-(q-p)m^2+(q-1)m) \\
= \;& \max_{1\leq m\leq n}\left(-(q-p)\left(m-\frac{q-1}{2(q-p)}\right)^2+(q-p)\left(\frac{q-1}{2(q-p)}\right)^2\right) \\
= \;& \max_{1\leq m\leq n}\left(-(q-p)\left(m-\left(t+\frac{\alpha}{q-p}+\frac{1}{2}\right)\right)^2+(q-p)\left(t+\frac{\alpha}{q-p}+\frac{1}{2}\right)^2\right).
\end{aligned}$$

If $\alpha = 0$, then we obtain

$$\min_{m\in\mathbb{Z}}\left|m-\left(t+\frac{\alpha}{q-p}+\frac{1}{2}\right)\right| = \min_{1\leq m\leq n}\left|m-\left(t+\frac{1}{2}\right)\right| = \left|t-\left(t+\frac{1}{2}\right)\right| = \frac{1}{2}.$$

By $0 < t \leq n$ and

$$\begin{aligned}
& f(1,1)-f(1,t) \\
= \;& (-(q+r)+4q-4(q-p)-4)-(-(q+r)+4qt-4(q-p)t^2-4t) \\
= \;& 4((q-p)t^2-(q-1)t+p-1) = 4((t+2)n-(q-1)t) \\
= \;& 4((t+2)n-((q-p)+2n)t) = 4(nt+2n-(q-p)t-2nt) \\
= \;& 4(-nt+n) = -4n(t-1) \leq 0,
\end{aligned}$$

we have $(1,t) \in \mathfrak{L} \cap R$. Thus, we obtain

$$\begin{aligned}
\max_{1 \le m \le n} F(1,m) &= -(q-p)\left(t - \left(t + \frac{1}{2}\right)\right)^2 + (q-p)\left(t+\frac{1}{2}\right)^2 \\
&= -\frac{(q-p)}{4} + (q-p)\left(t+\frac{1}{2}\right)^2 \\
&= (q-p)(t^2+t) = t(q-p)(t+1) = n(t+1) \\
&= (t+1)(n+\alpha).
\end{aligned}$$

If $\alpha \neq 0$, then we have

$$\min_{m\in\mathbb{Z}} \left| m - \left(t + \frac{\alpha}{q-p} + \frac{1}{2}\right)\right| = \left| t+1 - \left(t + \frac{\alpha}{q-p} + \frac{1}{2}\right)\right| = \left|\frac{1}{2} - \frac{\alpha}{q-p}\right|$$

by

$$0 < \frac{\alpha}{q-p} < 1.$$

Furthermore, we obtain $t+1 \ge 0$ and $n-(t+1) = ((q-p)t+\alpha)-(t+1) \ge 0$. If $t=0$, then we obtain $f(1,1) = f(1,t+1)$. If $t>0$, then we obtain

$$\begin{aligned}
& f(1,1) - f(1,t+1) \\
= \; & (-(q+r) + 4q - 4(q-p) - 4) \\
- \; & (-(q+r) + 4q(t+1) - 4(q-p)(t+1)^2 - 4(t+1)) \\
= \; & 4((q-p)(t+1)^2 - (q-1)(t+1) + p - 1) \\
= \; & 4((q-p)t^2 + (q-2p+1)t) \\
= \; & 4t((q-p)t + q - 2p + 1) \\
= \; & 4t((n-\alpha) + (q-p) - p + 1) \\
\le \; & 4t(2(n-\alpha) - (2n+1) + 1) \\
= \; & -8t\alpha \le 0.
\end{aligned}$$

Thus, we have $(1,t+1) \in \mathfrak{L} \cap R$ and we obtain

$$\begin{aligned}
\max_{1\le m\le n} F(1,m) &= -(q-p)(t+1)^2 + (q-1)(t+1) \\
&= (t+1)(-(q-p)(t+1) + q - 1) \\
&= (t+1)(-(q-p)t + p - 1) \\
&= (t+1)(-(n-\alpha) + 2n) \\
&= (t+1)(n+\alpha).
\end{aligned}$$

Therefore, we obtain the desired result above. □

Remark 2.2.12. *If $\alpha = 0$, then we have*

$$\begin{aligned}
& F(1,t+1) - F(1,t) \\
= \; & -(q-p)(t+1)^2 + (q-1)(t+1) - (-(q-p)t^2 + (q-1)t) \\
= \; & -(q-p)(2t+1) + (q-1) \\
= \; & -2(q-p)t - (q-p) + (q-1) \\
= \; & -(p-1) + (p-1) = 0.
\end{aligned}$$

Thus we have

$$\max_{1\leq m\leq n} F(1,m) = F(1,t+1) = (t+1)(n+\alpha).$$

To prove Theorem 2.2.2, we define

$$\mathfrak{N}' := \{(a,m)\in\mathbb{Z}^2 | a \text{ is odd}, 1\leq a<m\leq n, F(x,y)\geq F(1,1)\}.$$

Proof of Theorem 2.2.2. By Lemma 2.2.9 and 2.2.10, we have

$$d(\Sigma(p,q,r)) = \max_{(a,m)\in\mathfrak{L}\cap R} F(a,m) = \max_{(a,m)\in\mathfrak{N}'} F(a,m).$$

Thus, from Proposition 2.2.11 and Remark 2.2.12, we obtain

$$d(\Sigma(p,q,r)) = \max_{(a,m)\in\mathfrak{M}} F(a,m) \geq (t+1)(n+\alpha).$$

□

2.2.4 Inequality between d-invariants

We define

$$D(p,q,r) := \begin{cases} (t+1)(n+\alpha) & (p \text{ is odd}), \\ d(\Sigma(p,q,r)) & (p \text{ is even}). \end{cases}$$

Let p_i, q_i and r_i be pairwise relatively prime, ordered, positive integers satisfying $p_iq_i+p_ir_i-q_ir_i=1$ $(i=1,2)$.

Proposition 2.2.13. *If p_1 and p_2 are equal to p and $q_1\geq q_2$, then we have*

$$2\left\lfloor\frac{p}{2}\right\rfloor \leq D(p_1,q_1,r_1)\leq D(p_2,q_2,r_2)\leq \left\lfloor\frac{p}{2}\right\rfloor^2+\left\lfloor\frac{p}{2}\right\rfloor.$$

Proof. If p is odd, then we have

$$n=\frac{p-1}{2}=(q_i-p)t_i+\alpha_i \text{ and } 0\leq\alpha_i<q_i-p \ (i=1,\,2).$$

Hence, if $q_1\geq q_2$, then we have $t_1\leq t_2$.

<u>The case of $t_1=t_2$.</u>

Since $\alpha_1=\alpha_2$, we have

$$\begin{aligned} & (t_2+1)(n+\alpha_2)-(t_1+1)(n+\alpha_1) \\ = \; & (t_1+1)(n+\alpha_1)-(t_1+1)(n+\alpha_1)=0. \end{aligned}$$

<u>The case of $t_1<t_2$.</u>

Since $t_1+1\leq t_2$, we have

$$\begin{aligned} & (t_2+1)(n+\alpha_2)-(t_1+1)(n+\alpha_1) \\ \geq \; & (t_1+2)(n+\alpha_2)-(t_1+1)(n+\alpha_1) \\ \geq \; & (t_1+1)(\alpha_2-\alpha_1)+(n+\alpha_2) \\ = \; & (t_1+1)(\alpha_2-\alpha_1)+(q_1-p)t_1+\alpha_1+\alpha_2 \\ = \; & t_1(q_1-p+\alpha_2-\alpha_1)+2\alpha_2 \\ \geq \; & 2\alpha_2\geq 0. \end{aligned}$$

If $q = 2p - 1$, the quotient when dividing n by $q - p = p - 1$ is 0, and the remainder is $(p-1)/2$. Thus, we obtain

$$D(p, 2p-1, 2p+1) = (0+1)\left(\frac{p-1}{2} + \frac{p-1}{2}\right) = p - 1.$$

If $q = p + 1$, the quotient when dividing n by $q - p = 1$ is $(p-1)/2$, and the remainder is 0. Thus, we obtain

$$D(p, p+1, p^2+p-1) = \left(\frac{p-1}{2} + 1\right)\left(\frac{p-1}{2} + 0\right) = \frac{p^2-1}{4}.$$

If p is even and $q_1 \geq q_2$, then we have

$$\begin{aligned}
& 4(q_1 - p_1)(q_2 - p_2)(D(p_2, q_2, r_2) - D(p_1, q_1, r_1)) \\
= \;& (q_1 - p)(q_2^2 - 1) - (q_2 - p)(q_1^2 - 1) \\
= \;& q_1(q_2^2 - 1) - p(q_2^2 - 1) - q_2(q_1^2 - 1) + p(q_1^2 - 1) \\
= \;& q_1 q_2^2 - q_1 - p q_2^2 + p - q_2 q_1^2 + q_2 + p q_1^2 - p \\
= \;& q_1 q_2^2 - q_1 - p q_2^2 - q_2 q_1^2 + q_2 + p q_1^2 \\
= \;& q_1 q_2^2 - q_2 q_1^2 - p q_2^2 + p q_1^2 - q_1 + q_2 \\
= \;& -q_1 q_2 (q_1 - q_2) + p(q_1 + q_2)(q_1 - q_2) - (q_1 - q_2) \\
= \;& (-q_1 q_2 + p(q_1 + q_2) - 1)(q_1 - q_2) \\
\geq \;& (-q_1 q_2 + 2p q_2 - 1)(q_1 - q_2) \\
\geq \;& (-(2p-1) q_2 + 2p q_2 - 1)(q_1 - q_2) \\
\geq \;& (q_2 - 1)(q_1 - q_2) \geq 0
\end{aligned}$$

by using Lemma 2.2.7. Furthermore, we obtain

$$D(p, 2p-1, 2p+1) = (2p - 1 + 2p + 1)/4 = p$$

and

$$D(p, p+1, p^2+p-1) = (p + 1 + p^2 + p - 1)/4 = (p^2 + 2p)/4.$$

Therefore, by Lemma 2.2.7 we obtain the desired inequality above. □

Proposition 2.2.14. *If p is odd, then $D(p, q, r) = p - 1$ if and only if $q - p \geq n$.*

Proof. The case of $q - p > n$.

The quotient when dividing n by $q - p$ is 0, and the remainder is n. Thus, we obtain

$$D(p, q, r) = (0+1)(n+n) = 2n = p - 1.$$

The case of $q - p = n$.

The quotient when dividing n by $q - p$ is 1, and the remainder is 0. Thus, we obtain

$$D(p, q, r) = (1+1)(n+0) = 2n = p - 1.$$

The case of $1 \leq q - p < n$.

Let t and α be the quotient and the remainder when we divide n by $q - p$, respectively. Then we have $t \geq 2$ or $t = 1$ and $\alpha \geq 1$. If $t \geq 2$, then we have

$$D(p, q, r) \geq (2 + 1)(n + \alpha) = 3(n + \alpha) \geq 3n > 2n = p - 1.$$

If $t = 1$ and $\alpha > 0$, then we have

$$D(p, q, r) \geq (1 + 1)(n + \alpha) = 2(n + \alpha) \geq 2(n + 1) > 2n = p - 1.$$

Therefore, we have the desired result above. □

We end this subsection by raising a conjecture for the d-invariant of any Brieskorn homology 3-sphere $\Sigma(p, q, r)$ with p odd and $pq + pr - qr = 1$.

Conjecture 2.2.15. If p is odd and $pq + pr - qr$ is equal to 1, then we have

$$d(\Sigma(p, q, r)) = (t + 1)(n + \alpha).$$

2.2.5 Examples

If $F(a, m) \leq (t + 1)(n + \alpha)$ for any $(a, m) \in \mathfrak{N}$, then we obtain $d(\Sigma(p, q, r)) = (t + 1)(n + \alpha)$ by Theorem 2.2.2. We present some examples.

Corollary 2.2.16. *If $q - p = 2$, then we have*

$$d(\Sigma(p, q, r)) = (t + 1)(n + \alpha).$$

Proof. If $q - p = 2$ and $a \geq 3$, then we have

$$\begin{aligned} aq - 2(q - p)m - 1 &= ap + 2a - 4m - 1 \\ &\geq ap + 2a - 2p + 1 \\ &= (a - 2)p + 2a + 1 > 0 \end{aligned}$$

by $m \leq n = (p - 1)/2$. Thus, we obtain

$$\begin{aligned} & 4(q - p)(F(1, 1) - F(a, m)) \\ = \ & (aq - 2(q - p)m - 1)^2 - (q - a)^2 + 4(p - 1)(q - p) \\ = \ & (ap + 2a - 4m - 1)^2 - (p - a + 2)^2 + 8(p - 1) \\ \geq \ & ((a - 2)p + 2a + 1)^2 - (p - a + 2)^2 + 8(p - 1) \\ = \ & (a(p + 1) - p + 3)(a(p + 3) - 3p - 1) + 8(p - 1) \\ \geq \ & (3(p + 1) - p + 3)(3(p + 3) - 3p - 1) + 8(p - 1) \\ = \ & 24p + 40 > 0 \end{aligned}$$

by

$$a(p + 1) - p + 3 \geq 3(p + 1) - p + 3 = 2p + 6 > 0$$

and

$$a(p + 3) - 3p - 1 \geq 3(p + 3) - 3p - 1 = 8 > 0.$$

Therefore, we have $F(a, m) < F(1, 1)$ for any $(a, m) \in \mathfrak{N}$. Thus, we have the desired result above. □

Corollary 2.2.17. *If $p \leq 23$ and $q-p \geq n$, then we have $d(\Sigma(p,q,r)) = p-1$.*

Proof. By Lemma 2.2.7, we have

$$(p+a-2)-(q-p) = (2p-q)+(a-2) \geq 0.$$

By $a \leq n$, we have

$$\frac{3p-1}{2} - a \geq p > 0.$$

If $q-p \geq n$, then we have

$$\begin{aligned}
& 4(q-p)(F(1,1)-F(a,m)) \\
= \ & (aq-2(q-p)m-1)^2-(q-a)^2+4(p-1)(q-p) \\
\geq \ & -(q-a)^2+4(p-1)(q-p) \\
= \ & -((p+a-2)-(q-p))^2+(p+a-2)^2-(p-a)^2 \\
\geq \ & -\left((p+a-2)-\frac{p-1}{2}\right)^2+(p+a-2)^2-(p-a)^2 \\
= \ & -a^2+(3p-1)a-\frac{p^2+10p-7}{4} \geq -\frac{p^2-26p+41}{4}.
\end{aligned}$$

Since $p^2-26p+41 \leq 0$ and $3 \leq p \leq 23$ are equivalent, thus we have the desired result above. □

Remark 2.2.18. We define the sets

$$A_0 := \{(p,q,r) \in \mathbb{Z}^3 | \gcd(p,q) = \gcd(q,r) = \gcd(r,p) = 1, \ pq+pr-qr = 1\},$$

$$A_1 := \{(p,q,r) \in A_0 | p \in \{(3p-1)/2, (3p+1)/2, 2p-1\} \text{ or } p \leq 23 \text{ and } q-p \geq n\}$$

and $A_2 := \{(p,q,r) \in A_0 | q-p < n\}$. For any (p_i,q_i,r_i) in $A_i \quad (i=1,2)$, we have

$$d(\Sigma(p_1,q_1,r_1)) = p-1 < D(p_2,q_2,r_2) \leq d(\Sigma(p_2,q_2,r_2))$$

by Theorem 2.2.2, Remark 2.2.4, Proposition 2.2.14 and Corollary 2.2.17. Then we know that $\Sigma(p_1,q_1,r_1)$ and $\Sigma(p_2,q_2,r_2)$ are not homology cobordant.

The following example is the most complex example described in [KŞ], and is the only example without an explicit calculation result.

Example 2.2.19. Let F_k be the k-th Fibonacci number. F_k is even if and only if $k \in 3\mathbb{Z}$. If $(p,q,r) = (F_{2n+1}, F_{2n+2}, F_{2n+3})$, then we have

$$\begin{aligned}
pq+pr-qr &= F_{2n+1}F_{2n+2}+F_{2n+1}F_{2n+3}-F_{2n+2}F_{2n+3} \\
&= F_{2n+1}F_{2n+3}+(F_{2n+1}-F_{2n+3})F_{2n+2} \\
&= F_{2n+1}F_{2n+3}-F_{2n+2}^2 = (-1)^{2n+2} = 1.
\end{aligned}$$

If $2n+1 \in 3\mathbb{Z}$, then we have

$$d(\Sigma(F_{2n+1}, F_{2n+2}, F_{2n+3})) = \frac{F_{2n+2}+F_{2n+3}}{4} = \frac{F_{2n+4}}{4}$$

by Theorem 2.2.2.

If $2n+1 \notin 3\mathbb{Z}$, then we have $q-p = F_{2n+2} - F_{2n+1} = F_{2n}$ and

$$\begin{aligned} F_{2n} - \frac{F_{2n+1}-1}{2} &= \frac{2F_{2n} - F_{2n+1} + 1}{2} \\ = \quad \frac{2F_{2n} - F_{2n+1} + 1}{2} &= \frac{F_{2n} - F_{2n-1} + 1}{2} > 0. \end{aligned}$$

Thus, we have the quotient when dividing $(F_{2n+1}-1)/2$ by F_{2n} is 0, and the remainder is $(F_{2n+1}-1)/2$. Therefore, we obtain

$$D(\Sigma(F_{2n+1}, F_{2n+2}, F_{2n+3})) = (0+1)\left(\frac{F_{2n+1}-1}{2} + \frac{F_{2n+1}-1}{2}\right) = F_{2n+1} - 1$$

and we conject that $d(\Sigma(F_{2n+1}, F_{2n+2}, F_{2n+3}))$ is equal to $F_{2n+1}-1$.

2.3 The case of any Brieskorn homology 3-sphere

2.3.1 Introduction for this section

Let $L(p,q)$ be the lens space of type (p,q). We note that

$$L(p, q+pk) = L(p,q)$$

for any integer k. By $\mathrm{Spin}^c(L(p,q))$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, any spin^c structure of the lens space $L(p,q)$ is corresponding to an integer. We note that

$$d(L(p,q), n+p) = d(L(p,q), n)$$

for any integer n. The reciprocity law of d-invariants of lens spaces is proved in Ozsváth-Szabó [OS].

Proposition 2.3.1 (Ozsváth-Szabó [OS, Proposition 4.8])**.** *If* $\gcd(p,q) = 1$, $p > q$ *and* $0 \le n < p+q$, *then we have*

$$d(L(p,q), n) = -d(L(q,p), n) - \frac{pq - (2n+1-p-q)^2}{4pq}.$$

Note that the orientation of the lens space $L(p,q)$ is opposite to that of [OS].

We define $\varepsilon(p,q;n) := \#\{(a,b) \in \mathbb{N}^2 | ap + bq \le n\}$ for any integer n. Note that $\varepsilon(p,q;n) = 0$ for any $n < p+q$. We extend Proposition 2.3.1 to any case of p and q are pairwise relatively prime, positive integers.

Theorem 2.3.2. *If* $\gcd(p,q) = 1$, *then we have*

$$d(L(p,q), n) = -d(L(q,p), n) - \frac{pq - (2n+1-p-q)^2}{4pq} - 2\varepsilon(p,q;n)$$

for any nonnegative integer n.

Since $\Sigma(p,q,r)$ is a homology 3-sphere, there exist unique integers e_0, $0 < p' < p$, $0 < q' < q$ and $0 < r' < r$ satisfied with the Diophantine equation:

$$e_0 pqr + p'qr + pq'r + pqr' = -1. \tag{2.1}$$

We define

$$\Delta(p,q,r;n) := 1 - e_0 n - \left\lceil \frac{p'n}{p} \right\rceil - \left\lceil \frac{q'n}{q} \right\rceil - \left\lceil \frac{r'n}{r} \right\rceil \quad (n \geq 0)$$

and

$$\tau(p,q,r;n) := \begin{cases} 0 & (n=0), \\ \sum_{i=0}^{n-1} \Delta(p,q,r;i) & (n \in \mathbb{N}). \end{cases}$$

Let $\lambda(p,q,r)$ be the Casson invariant of $\Sigma(p,q,r)$. We also define $\kappa(p,q,r)$ to be the number of lattice points in

$$\left\{ (x,y,z) \in \mathbb{N}^3 \middle| \frac{x}{p} + \frac{y}{q} + \frac{z}{r} < 1 \right\}.$$

Let $\mathcal{T}(p,q,r;n)$ be $2(\tau(p,q,r;n+1) + \lambda(p,q,r) + \kappa(p,q,r))$. There is a formula that can help derive the specific value of the d-invariant of any Brieskorn homology 3-sphere.

Proposition 2.3.3 (Can-Karakurt [CK])**.**

$$d(\Sigma(p,q,r)) = -\min_{n \geq -1} \mathcal{T}(p,q,r;n).$$

We define $N_0(p,q,r) := pqr - pq - qr - rp$ and

$$R(p,q,r) := \left\{ n \in \mathbb{Z} \middle| \max\left\{ \frac{N_0(p,q,r)}{2} - pq, -1 \right\} \leq n \leq \frac{N_0(p,q,r)-1}{2} \right\}.$$

We restricted the candidates for the minimum value in Proposition 2.3.3 to pq cases.

Theorem 2.3.4.

$$d(\Sigma(p,q,r)) = -\min_{n \in R(p,q,r)} \mathcal{T}(p,q,r;n).$$

In Subsection 2.3.2 we review a precise definition and known properties for proving the main results. In Subsection 2.3.3 we generalize the reciprocity law of d-invariants of lens spaces in [OS, Proposition 4.8]. In Subsection 2.3.4 we compute the value of $\mathcal{T}(p,q,r;n+pq) - \mathcal{T}(p,q,r;n)$ to prove the main result. In Subsection 2.3.5 we prove the main result for the d-invariant of any Brieskorn homology 3-sphere.

2.3.2 Preliminaries for this section

We define $\{x\} := x - \lfloor x \rfloor$ and the sawtooth function $((\cdot)) : \mathbb{R} \to (-1/2, 1/2)$ as

$$((x)) = \begin{cases} \{x\} - 1/2 & (x \in \mathbb{R} - \mathbb{Z}), \\ 0 & (x \in \mathbb{Z}). \end{cases}$$

Note that $((-x)) = -((x))$ for any real number x. We also define the (classical) Dedekind sum

$$s(b,c) = \sum_{n=1}^{c-1} \left(\left(\frac{n}{c}\right)\right) \left(\left(\frac{bn}{c}\right)\right)$$

for any relatively prime nonzero integers b and c. Note that $s(-b, c) = -s(b, c)$. There exists the well-known reciprocity law of the Dedekind sums:

$$s(a, b) + s(b, a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right)$$

for a and b are coprime positive integers.

We also define the following δ-function for a positive integer s:

$$\delta_s(n) = \begin{cases} 1 & (n \equiv 0 \pmod{s}), \\ 0 & (\text{otherwise}). \end{cases}$$

In this subsection, we prove lemmas need later. We rewrite p, q, and r as p_1, p_2, and p_3, respectively.

Lemma 2.3.5. *We have*

$$\Delta(p, q, r; n) = -\frac{1}{2} + \frac{n}{pqr} - \sum_{i=1}^{3}\left(\left(\left(-\frac{p_i' n}{p_i}\right)\right) - \frac{1}{2}\delta_{p_i}(n)\right)$$

for any nonnegative integer n.

Proof. We have

$$\begin{aligned}
\Delta(p, q, r; n) &= 1 - e_0 n - \sum_{i=1}^{3}\left\lceil \frac{p_i' n}{p_1} \right\rceil \\
&= 1 - e_0 n + \sum_{i=1}^{3}\left\lfloor -\frac{p_i' n}{p_i} \right\rfloor \\
&= 1 - e_0 n + \sum_{i=1}^{3}\left(-\frac{p_i' n}{p_i} - \left\{-\frac{p_i' n}{p_i}\right\}\right) \\
&= 1 - e_0 n + \sum_{i=1}^{3}\left(-\frac{p_i' n}{p_i} - \left(\left(-\frac{p_i' n}{p_i}\right)\right) - \frac{1}{2} + \frac{1}{2}\delta_{p_i}(n)\right) \\
&= 1 - e_0 n - \sum_{i=1}^{3}\frac{p_i' n}{p_i} - \sum_{i=1}^{3}\left(\left(\left(-\frac{p_i' n}{p_i}\right)\right) - \frac{1}{2}\delta_{p_i}(n)\right) - \frac{3}{2} \\
&= -\frac{1}{2} + \frac{n}{p_1 p_2 p_3} - \sum_{i=1}^{3}\left(\left(\left(-\frac{p_i' n}{p_i}\right)\right) - \frac{1}{2}\delta_{p_i}(n)\right)
\end{aligned}$$

for any $n \geq 0$. ∎

In the last equation, we use the Diophantine equation (2.1).

We define $N(p, q, r)$ to be the number of lattice points in the tetrahedron:

$$\left\{(x, y, z) \in \mathbb{Z}^3 \,\middle|\, x, y, z \geq 0, 0 < \frac{x}{p} + \frac{y}{q} + \frac{z}{r} < 1\right\}.$$

The value $N(p, q, r)$ is, as computed in [R, Theorem 1.1],

$$\frac{pqr}{6} + \frac{pq + qr + rp}{4} + \frac{p + q + r}{4} + \frac{(pq)^2 + (qr)^2 + (rp)^2 + 1}{12pqr} - 2$$
$$-s(qr, p) - s(pr, q) - s(pq, r).$$

Lemma 2.3.6.

$$\begin{aligned}\kappa(p,q,r) &= \frac{pqr}{6} - \frac{pq+qr+rp}{4} + \frac{p+q+r}{4} + \frac{(pq)^2+(qr)^2+(rp)^2+1}{12pqr} - \frac{1}{2} \\ &\quad -(s(qr,p)+s(pr,q)+s(pq,r)).\end{aligned}$$

Proof. We have

$$\begin{aligned}\kappa(p,q,r) &= N(p,q,r)+1 \\ &\quad -\#\left\{(x,y,z)\in\mathbb{Z}^3 \middle| x,y,z\geq 0, xyz=0, \frac{x}{p}+\frac{y}{q}+\frac{z}{r}<1\right\} \\ &= N(p,q,r)+1 \\ &\quad -\left(q+\sum_{x=1}^{p-1}\left(\left\lfloor q\left(1-\frac{x}{p}\right)\right\rfloor+1\right)+r+\sum_{y=1}^{q-1}\left(\left\lfloor r\left(1-\frac{y}{q}\right)\right\rfloor+1\right)\right. \\ &\quad \left.+p+\sum_{z=1}^{r-1}\left(\left\lfloor p\left(1-\frac{z}{r}\right)\right\rfloor+1\right)\right)+(p+q+r)-1 \\ &= N(p,q,r)-\left(\frac{pq+p+q-1}{2}+\frac{qr+q+r-1}{2}+\frac{rq+r+q-1}{2}\right) \\ &\quad +(p+q+r) \\ &= N(p,q,r)-\frac{pq+qr+rp}{2}+\frac{3}{2} \\ &= \frac{pqr}{6} - \frac{pq+qr+rp}{4} + \frac{p+q+r}{4} + \frac{(pq)^2+(qr)^2+(rp)^2+1}{12pqr} - \frac{1}{2} \\ &\quad -(s(qr,p)+s(pr,q)+s(pq,r)).\end{aligned}$$

□

Next, we compute the Casson invariant $\lambda(p,q,r)$ of $\Sigma(p,q,r)$.

Lemma 2.3.7.

$$\lambda(p,q,r) = -\frac{pqr}{24}+\frac{(pq)^2+(qr)^2+(rp)^2+1}{24pqr}-\frac{1}{8}-\frac{1}{2}(s(qr,p)+s(pr,q)+s(pq,r)).$$

Proof. By [FS, Theorem 2.10], we have

$$\lambda(p,q,r) = \frac{1}{8}\sigma(B(p,q,r)),$$

where

$$B(p,q,r) := \{(z_1,z_2,z_3)\in\mathbb{C}^3 | z_1^p+z_2^q+z_3^r=\varepsilon\}$$

and $\sigma(X)$ is the signature of a manifold X. Furthermore, by [FS, p. 116–117] and Lemma 2.3.6, we have

$$\begin{aligned}
&-\sigma(B(p,q,r)) \\
=\ & (p-1)(q-1)(r-1) - 4\#\left\{(k,l,m)\in\mathbb{N}^3 \,\middle|\, \frac{k}{p}+\frac{l}{q}+\frac{m}{r}<1\right\} \\
=\ & (p-1)(q-1)(r-1) - 4\kappa(p,q,r) \\
=\ & pqr - (pq+qr+rp) + (p+q+r) - 1 \\
& -\frac{2pqr}{3} + (pq+qr+rp) - (p+q+r) - \frac{(pq)^2+(qr)^2+(rp)^2+1}{3pqr} + 2 \\
& +4(s(qr,p)+s(pr,q)+s(pq,r)) \\
=\ & \frac{pqr}{3} - \frac{(pq)^2+(qr)^2+(rp)^2+1}{3pqr} + 1 + 4(s(qr,p)+s(pr,q)+s(pq,r)).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\lambda(p,q,r) = \frac{1}{8}\sigma(B(p,q,r)) \\
=\ & -\frac{pqr}{24} + \frac{(pq)^2+(qr)^2+(rp)^2+1}{24pqr} - \frac{1}{8} - \frac{1}{2}(s(qr,p)+s(pr,q)+s(pq,r)).
\end{aligned}$$

□

We abbreviate $d(L(p,q),n)$ as $d(p,q;n)$. Let $[n]_p$ be the remainder of n divided by p.

Lemma 2.3.8.

$$d(p,q;n) = 3s(q,p) + \frac{p-1-2[n]_p}{2p} - 2\sum_{k=0}^{[n]_p}\left(\left(-\frac{q^- k}{p}\right)\right)$$

for any integer n. Here q^- is uniquely determined the integer by

$$q^- q \equiv 1 \pmod p \text{ and } 1 \le q^- \le p-1.$$

Proof. By [T, Theorem 4], the d-invariant $d(p,q;n)$ is computed as

$$d(p,q;n) = \begin{cases} 3s(q,p) + \dfrac{p-1}{2p} & (n\equiv 0 \pmod p), \\ 3s(q,p) + \dfrac{p-1}{2p} + 2\displaystyle\sum_{k=1}^{[n]_p}\left(\left(\frac{2q^- k - 1}{2p}\right)\right) & (n \not\equiv 0 \pmod p). \end{cases}$$

for any $n\in\mathbb{Z}$. Note that the orientation of the lens space $L(p,q)$ is opposite to that of [T].

The case of $n\equiv 0 \pmod p$.

In this case, we have

$$\begin{aligned}
d(p,q;n) &= 3s(q,p) + \frac{p-1}{2p} \\
&= 3s(q,p) + \frac{p-1-2[0]_p}{2p} - 2\left(\left(-\frac{q^-[0]_p}{p}\right)\right).
\end{aligned}$$

The case of $n \not\equiv 0 \pmod p$.

In this case, we have

$$\begin{aligned}
& d(p,q;n) \\
= \ & 3s(q,p) + \frac{p-1}{2p} + 2\sum_{k=1}^{[n]_p} \left(\left(\frac{2q^-k-1}{2p}\right)\right) \\
= \ & 3s(q,p) + \frac{p-1}{2p} + 2\sum_{k=1}^{[n]_p} \left(\left(\frac{q^-k}{p} - \frac{1}{2p}\right)\right) \\
= \ & 3s(q,p) + \frac{p-1}{2p} - 2\sum_{k=1}^{[n]_p} \left(\left(-\frac{q^-k}{p} + \frac{1}{2p}\right)\right) \\
= \ & 3s(q,p) + \frac{p-1-2[n]_p}{2p} - 2\sum_{k=1}^{[n]_p} \left(\left(\left(-\frac{q^-k}{p} + \frac{1}{2p}\right)\right) - \frac{1}{2p}\right).
\end{aligned}$$

By q^-, $k \in \mathbb{Z} - p\mathbb{Z}$, we have $2q^-k \in 2\mathbb{Z} - 2p\mathbb{Z}$. Hence, $2 \leq [2q^-k]_{2p} \leq 2p-2$. Thus, we obtain

$$0 < \frac{2p - [2q^-k]_{2p}}{2p} < \frac{2p+1-[2q^-k]_{2p}}{2p} < 1.$$

Therefore, we have

$$\begin{aligned}
\left(\left(-\frac{q^-k}{p} + \frac{1}{2p}\right)\right) - \frac{1}{2p} &= \left(\left(\frac{1-2q^-k}{2p}\right)\right) - \frac{1}{2p} \\
&= \left\{\frac{1-2q^-k}{2p}\right\} - \frac{1}{2} - \frac{1}{2p} \\
&= \frac{2p+1-[2q^-k]_{2p}}{2p} - \frac{1}{2} - \frac{1}{2p} \\
&= \frac{2p-[2q^-k]_{2p}}{2p} - \frac{1}{2} = \left\{\frac{2p-2q^-k}{2p}\right\} - \frac{1}{2} \\
&= \left\{-\frac{q^-k}{p}\right\} - \frac{1}{2} = \left(\left(-\frac{q^-k}{p}\right)\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
& d(p,q;n) \\
= \ & 3s(q,p) + \frac{p-1-2[n]_p}{2p} - 2\sum_{k=1}^{[n]_p} \left(\left(\left(-\frac{q^-k}{p} + \frac{1}{2p}\right)\right) - \frac{1}{2p}\right) \\
= \ & 3s(q,p) + \frac{p-1-2[n]_p}{2p} - 2\sum_{k=1}^{[n]_p} \left(\left(-\frac{q^-k}{p}\right)\right) \\
= \ & 3s(q,p) + \frac{p-1-2[n]_p}{2p} - 2\sum_{k=0}^{[n]_p} \left(\left(-\frac{q^-k}{p}\right)\right).
\end{aligned}$$

Therefore, we have

$$d(p,q;n) = 3s(q,p) + \frac{p-1-2[n]_p}{2p} - 2\sum_{k=0}^{[n]_p}\left(\left(-\frac{q^- k}{p}\right)\right)$$

for any $n \in \mathbb{Z}$. □

Lemma 2.3.9.

$$(p')^- \equiv -qr \pmod p,\ (q')^- \equiv -rp \pmod q,\ (r')^- \equiv -pq \pmod r.$$

Proof. By the Diophantine equation (2.1), we have

$$p'qr \equiv -1 \pmod p.$$

Hence, we have

$$p'(-qr) \equiv 1 \pmod p.$$

Therefore, we obtain

$$(p')^- \equiv -qr \pmod p.$$

The remaining two proofs are similar. □

Lemma 2.3.10.

$$\begin{aligned}
&\tau(p,q,r;n+1) \\
= \ &\frac{n^2}{2pqr} - \frac{pqr-pq-qr-rp-1}{2pqr}n + \frac{1}{4} + \frac{1}{4}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) \\
&+\frac{1}{2}(d(p,-qr;n) + d(q,-rp;n) + d(r,-pq;n)) \\
&+\frac{3}{2}(s(qr,p) + s(rp,q) + s(pq,r))
\end{aligned}$$

for any nonnegative integer n.

Proof. Since we have

$$\sum_{j=k}^{k+p_i-1}\left(\left(-\frac{p_i'j}{p_i}\right)\right) = \sum_{j=0}^{p_i-1}\left(\left(-\frac{p_i'j}{p_i}\right)\right) = \sum_{j=1}^{p_i-1}\left(\left(-\frac{p_i'j}{p_i}\right)\right) = 0,$$

then we have

$$\sum_{j=0}^{n}\left(\left(-\frac{p_i'j}{p_i}\right)\right) = \sum_{j=1}^{[n]_{p_i}}\left(\left(-\frac{p_i'j}{p_i}\right)\right)$$

for any $i = 1, 2, 3$ and integer n. From Lemma 2.3.5 and this, we have

$$\tau(p, q, r; n+1) = \sum_{j=0}^{n} \Delta(p, q, r; j)$$

$$= \sum_{j=0}^{n} \left(-\frac{1}{2} + \frac{j}{pqr} - \sum_{i=1}^{3} \left(\left(\left(-\frac{p_i' j}{p_i} \right) \right) - \frac{1}{2} \delta_{p_i}(j) \right) \right)$$

$$= -\frac{n+1}{2} + \frac{n(n+1)}{2pqr} - \sum_{j=0}^{n} \left(\sum_{i=1}^{3} \left(\left(\left(-\frac{p_i' j}{p_i} \right) \right) - \frac{1}{2} \delta_{p_i}(j) \right) \right)$$

$$= -\frac{n+1}{2} + \frac{n(n+1)}{2pqr} - \sum_{i=1}^{3} \left(\sum_{j=1}^{[n]_{p_i}} \left(\left(-\frac{p_i' j}{p_i} \right) \right) - \frac{1}{2} \left(\left\lfloor \frac{n}{p_i} \right\rfloor + 1 \right) \right)$$

$$= 1 - \frac{n}{2} + \frac{n(n+1)}{2pqr} - \sum_{i=1}^{3} \left(\sum_{j=0}^{[n]_{p_i}} \left(\left(-\frac{p_i' j}{p_i} \right) \right) - \frac{1}{2} \left\lfloor \frac{n}{p_i} \right\rfloor \right)$$

for any $n \geq 0$.

By Lemma 2.3.8, we have

$$d(p_i, (p_i')^-, n) = 3s((p_i')^-, p_i) + \frac{p_i - 1 - 2[n]_{p_i}}{2p_i} - 2 \sum_{j=0}^{[n]_{p_i}} \left(\left(-\frac{p_i' j}{p_i} \right) \right),$$

for any $i = 1, 2, 3$ and Lemma 2.3.9, we have

$$
\begin{aligned}
&\tau(p,q,r;n+1)\\
= \quad & 1 - \frac{n}{2} + \frac{n(n+1)}{2pqr} + \frac{1}{2}\sum_{i=1}^{3}\left(-2\sum_{j=0}^{[n]_{p_i}}\left(\left(-\frac{p_i'j}{p_i}\right)\right) + \left\lfloor \frac{n}{p_i}\right\rfloor\right)\\
= \quad & 1 - \frac{n}{2} + \frac{n(n+1)}{2pqr}\\
& + \frac{1}{2}\sum_{i=1}^{3}\left(d(p_i,(p_i')^-,n) - 3s((p_i')^-,p_i) - \frac{p_i-1}{2p_i} + \frac{n}{p_i}\right)\\
= \quad & \frac{1}{4} - \frac{n}{2} + \frac{n(n+1)}{2pqr} + \left(\frac{1}{4} + \frac{n}{2}\right)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\\
& + \frac{1}{2}(d(p,-qr;n) + d(q,-rp;n) + d(r,-pq;n))\\
& - \frac{3}{2}(s(-qr,p) + s(-rp,q) + s(-pq,r))\\
= \quad & \frac{1}{4} - \frac{n}{2} + \frac{n(n+1)}{2pqr} + \left(\frac{1}{4} + \frac{n}{2}\right)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\\
& + \frac{1}{2}(d(p,-qr;n) + d(q,-rp;n) + d(r,-pq;n))\\
& + \frac{3}{2}(s(qr,p) + s(rp,q) + s(pq,r))\\
= \quad & \frac{n^2}{2pqr} - \frac{pqr - pq - qr - rp - 1}{2pqr}n + \frac{1}{4} + \frac{1}{4}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\\
& + \frac{1}{2}(d(p,-qr;n) + d(q,-rp;n) + d(r,-pq;n))\\
& + \frac{3}{2}(s(qr,p) + s(rp,q) + s(pq,r))
\end{aligned}
$$

for any $n \geq 0$. □

Proposition 2.3.11.

$$
\begin{aligned}
&\mathcal{T}(p,q,r;n)\\
= \quad & \frac{1}{pqr}\left(n - \frac{N_0(p,q,r)-1}{2}\right)^2 - \frac{1}{4} + d(p,-qr;n) + d(q,-pr;n) + d(r,-pq;n)
\end{aligned}
$$

for any $n \geq -1$.

Proof. By Lemma 2.3.6 and 2.3.7, we obtain

$$
\begin{aligned}
&\lambda(p,q,r) + \kappa(p,q,r)\\
= \quad & \frac{pqr}{8} - \frac{pq+qr+rp}{4} + \frac{p+q+r}{4} + \frac{(pq)^2 + (qr)^2 + (rp)^2 + 1}{8pqr}\\
& - \frac{5}{8} - \frac{3}{2}(s(qr,p) + s(rp,q) + s(pq,r)).
\end{aligned}
$$

From this and Lemma 2.3.10, we have

$$\begin{aligned}
&\tau(p,q,r;n+1)+\lambda(p,q,r)+\kappa(p,q,r)\\
=\;&\frac{n^2}{2pqr}-\frac{pqr-pq-qr-rp-1}{2pqr}n+\frac{pqr}{8}-\frac{pq+qr+rp}{4}+\frac{p+q+r}{4}\\
&+\frac{(pq)^2+(qr)^2+(rp)^2}{8pqr}-\frac{3}{8}+\frac{1}{4}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)+\frac{1}{8pqr}\\
&+\frac{1}{2}(d(p,-qr;n)+d(q,-rp;n)+d(r,-pq;n)).
\end{aligned}$$

for any $n \geq 0$. By

$$\begin{aligned}
&\frac{1}{2pqr}\left(n-\frac{N_0(p,q,r)-1}{2}\right)^2-\frac{1}{8}\\
=\;&\frac{1}{2pqr}\left(n^2-(N_0(p,q,r)-1)n-\frac{(N_0(p,q,r)-1)^2}{4}\right)-\frac{1}{8}\\
=\;&\frac{n^2}{2pqr}-\frac{pqr-pq-qr-rp-1}{2pqr}n+\frac{(pqr-pq-qr-rp-1)^2}{8pqr}-\frac{1}{8}\\
=\;&\frac{n^2}{2pqr}-\frac{pqr-pq-qr-rp-1}{2pqr}n+\frac{pqr}{8}\\
&-\frac{pq+qr+rp+1}{4}+\frac{(pq+qr+rp+1)^2}{8pqr}-\frac{1}{8}\\
=\;&\frac{n^2}{2pqr}-\frac{pqr-pq-qr-rp-1}{2pqr}n+\frac{pqr}{8}\\
&-\frac{pq+qr+rp}{4}+\frac{(pq+qr+rp+1)^2}{8pqr}-\frac{3}{8}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(pq+qr+rp+1)^2}{8pqr}\\
=\;&\frac{(pq+qr+rp)^2+2(pq+qr+rp)+1}{8pqr}\\
=\;&\frac{(pq)^2+(qr)^2+(rp)^2+2pqr(p+q+r)}{8pqr}+\frac{1}{4}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)+\frac{1}{8pqr}\\
=\;&\frac{(pq)^2+(qr)^2+(rp)^2}{8pqr}+\frac{p+q+r}{4}+\frac{1}{4}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)+\frac{1}{8pqr}
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{2pqr}\left(n-\frac{N_0(p,q,r)-1}{2}\right)^2-\frac{1}{8}\\
=\;&\frac{n^2}{2pqr}-\frac{pqr-pq-qr-rp-1}{2pqr}n+\frac{pqr}{8}-\frac{pq+qr+rp}{4}+\frac{p+q+r}{4}\\
&+\frac{(pq)^2+(qr)^2+(rp)^2}{8pqr}-\frac{3}{8}+\frac{1}{4}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)+\frac{1}{8pqr}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\tau(p,q,r;n+1)+\lambda(p,q,r)+\kappa(p,q,r)\\
= \ &\frac{1}{2pqr}\left(n-\frac{N_0(p,q,r)-1}{2}\right)^2-\frac{1}{8}+\frac{1}{2}(d(p,-qr;n)+d(q,-pr;n)+d(r,-pq;n))
\end{aligned}$$

for any $n \geq 0$.

Next, we will prove the case of $n=-1$. By [CK, Theorem 1.3], we obtain

$$\Delta(p,q,r;n) = -\Delta(p,q,r;N_0(p,q,r)-n)$$

for any $0 \leq n \leq N_0(p,q,r)$.

If $N_0(p,q,r) \equiv 0 \pmod 2$, then we have $\Delta(p,q,r;N_0(p,q,r)/2)$ $= -\Delta(p,q,r;N_0(p,q,r)-N_0(p,q,r)/2) = -\Delta(p,q,r;N_0(p,q,r)/2)$. Hence, we obtain $\Delta(p,q,r;N_0(p,q,r)/2)=0$. Therefore, we have

$$\begin{aligned}
&\tau(p,q,r;N_0(p,q,r)+1) = \sum_{i=0}^{N_0(p,q,r)} \Delta(p,q,r;i)\\
= \ &\sum_{i=0}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i) + \sum_{i=N_0(p,q,r)/2+1}^{N_0(p,q,r)} \Delta(p,q,r;i)\\
= \ &\sum_{i=0}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i) - \sum_{i=N_0(p,q,r)/2+1}^{N_0(p,q,r)} \Delta(p,q,r;N_0(p,q,r)-i)\\
= \ &\sum_{i=0}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i) - \sum_{i=0}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i)\\
= \ &0 = \tau(p,q,r;0).
\end{aligned}$$

If $N_0(p,q,r) \equiv 1 \pmod 2$, then we have

$$\begin{aligned}
&\tau(p,q,r;N_0(p,q,r)+1) = \sum_{i=0}^{N_0(p,q,r)} \Delta(p,q,r;i)\\
= \ &\sum_{i=0}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i) + \sum_{i=(N_0(p,q,r)+1)/2}^{N_0(p,q,r)} \Delta(p,q,r;i)\\
= \ &\sum_{i=0}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i) - \sum_{i=(N_0(p,q,r)+1)/2}^{N_0(p,q,r)} \Delta(p,q,r;N_0(p,q,r)-i)\\
= \ &\sum_{i=0}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i) - \sum_{i=0}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i)\\
= \ &0 = \tau(p,q,r;0).
\end{aligned}$$

Hence, we have

$$\tau(p,q,r;N_0(p,q,r)+1) = \tau(p,q,r;0).$$

Thus, we obtain

$$\begin{aligned}
& \tau(p,q,r;0)+\lambda(p,q,r)+\kappa(p,q,r) \\
& \tau(p,q,r;N_0(p,q,r)+1)+\lambda(p,q,r)+\kappa(p,q,r) \\
= \;& \frac{1}{2pqr}\left(N_0(p,q,r)-\frac{N_0(p,q,r)-1}{2}\right)^2-\frac{1}{8} \\
& +\frac{1}{2}(d(p,-qr;N_0(p,q,r))+d(q,-pr;N_0(p,q,r))+d(r,-pq;N_0(p,q,r))) \\
= \;& \frac{1}{2pqr}\left(-1-\frac{N_0(p,q,r)-1}{2}\right)^2-\frac{1}{8} \\
& +\frac{1}{2}(d(p,-qr;-qr)+d(q,-rp;-rp)+d(r,-pq;-pq)).
\end{aligned}$$

By [T, Lemma 1] and Lemma 2.3.8, we have

$$\begin{aligned}
& d(p,-qr;-qr)+d(q,-rp;-rp)+d(r,-pq;-pq) \\
= \;& d(p,p-qr;-qr)+d(q,q-rp;-rp)+d(r,r-pq;-pq) \\
= \;& -d(p,qr;0)-d(q,rp;0)-d(r,pq;0) \\
= \;& -3(s(qr,p)+s(rp,q)+s(pq,r))+\frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)-\frac{3}{2} \\
= \;& 3(s(-qr,p)+s(-rp,q)+s(-pq,r))+\frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)-\frac{3}{2} \\
= \;& \left(3s(-qr,p)+\frac{p-1-2[p-1]_p}{2p}-2\sum_{k=0}^{[p-1]_p}\left(\left(-\frac{(-qr)^{-}k}{p}\right)\right)\right) \\
& +\left(3s(-rp,q)+\frac{q-1-2[q-1]_q}{2q}-2\sum_{k=0}^{[q-1]_q}\left(\left(-\frac{(-rp)^{-}k}{q}\right)\right)\right) \\
& +\left(3s(-pq,r)+\frac{r-1-2[r-1]_r}{2r}-2\sum_{k=0}^{[r-1]_r}\left(\left(-\frac{(-pq)^{-}k}{r}\right)\right)\right) \\
= \;& d(p,-qr;p-1)+d(q,-rp;q-1)+d(r,-pq;r-1) \\
= \;& d(p,-qr;-1)+d(q,-rp;-1)+d(r,-pq;-1).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \tau(p,q,r;0)+\lambda(p,q,r)+\kappa(p,q,r) \\
= \;& \frac{1}{2pqr}\left(-1-\frac{N_0(p,q,r)-1}{2}\right)^2-\frac{1}{8} \\
& +\frac{1}{2}(d(p,-qr;-1)+d(q,-rp;-1)+d(r,-pq;-1)).
\end{aligned}$$

Therefore, we obtain the desired result. □

2.3.3 A generalized reciprocity law

The reciprocity law of d-invariants of lens spaces in Proposition 2.3.1 is restricted to $0 \leq n < p+q$. Here we prove the reciprocity law of the general version.

Proof of Theorem 2.3.2. We define

$$E(p,q;n) := -\frac{1}{2}\left(d(p,q;n) + d(q,p;n) + \frac{pq-(2n+1-p-q)^2}{4pq}\right)$$

for any $n \geq 0$. We will show that $E(p,q;n) = \varepsilon(p,q;n)$.

By Lemma 2.3.8, we have

$$\begin{aligned} d(p,q;n) &= 3s(q,p) + \frac{p-1-2[n]_p}{2p} - 2\sum_{k=0}^{[n]_p}\left(\left(-\frac{q^- k}{p}\right)\right) \\ &= 3s(q,p) + \frac{p-1}{2p} - \frac{[n]_p}{p} - 2\sum_{k=0}^{[n]_p}\left(\left(-\frac{q^- k}{p}\right)\right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} d(p,q;n) - d(p,q;n-1) &= -\frac{[n]_p}{p} + \frac{[n-1]_p}{p} - 2\left(\left(-\frac{q^-[n]_p}{p}\right)\right) \\ &= 1 - \frac{1}{p} - \frac{2[-q^- n]_p}{p}. \end{aligned}$$

By

$$\begin{aligned} & \frac{pq-(2n+1-p-q)^2}{4pq} - \frac{pq-(2n-1-p-q)^2}{4pq} \\ = & \frac{(2n-1-p-q)^2-(2n+1-p-q)^2}{4pq} \\ = & -\frac{2n-p-q}{pq} = -\frac{2n}{pq} + \frac{1}{p} + \frac{1}{q}, \end{aligned}$$

we have

$$\begin{aligned} & E(p,q;n) - E(p,q;n-1) \\ = & -\frac{1}{2}\Big((d(p,q;n)-d(p,q;n-1)) + (d(q,p;n)-d(q,p;n-1)) \\ & + \left(\frac{pq-(2n+1-p-q)^2}{4pq} - \frac{pq-(2n-1-p-q)^2}{4pq}\right)\Big) \\ = & -\frac{1}{2}\Big(\left(1-\frac{1}{p}-\frac{2[-q^- n]_p}{p}\right) + \left(1-\frac{1}{q}-\frac{2[-p^- n]_q}{q}\right) \\ & + \left(-\frac{2n}{pq}+\frac{1}{p}+\frac{1}{q}\right)\Big) \\ = & -\frac{1}{2}\left(-\frac{2n}{pq} + 2 - \frac{2[-q^- n]_p}{p} - \frac{2[-p^- n]_q}{q}\right) \\ = & \frac{n}{pq} - 1 + \frac{[-q^- n]_p}{p} + \frac{[-p^- n]_q}{q}. \end{aligned}$$

Let s_1 be $\lfloor n/p \rfloor$ and s_2 $\lfloor n/q \rfloor$.

We consider the solutions of the equation $pk + ql = n$ for integers k, l.

The case of $n \equiv 0 \pmod{pq}$.

In this case $s_1 = n/p$ holds. Then, the equation $pk + ql = n$ is equivalent to $p(s_1 - k) = ql$. Thus, the solutions of $pk + ql = n$ are $(k, l) = (s_1 - qr, pr)$ for $1 \le r \le s_1/q - 1$. Hence, we have

$$\#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\} = \frac{s_1}{q} - 1 = \frac{n}{pq} - 1.$$

Therefore, we have

$$E(p,q;n) - E(p,q;n-1) = \frac{n}{pq} - 1 = \#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\}.$$

The case of $n \equiv 0 \pmod{p}$ and $n \not\equiv 0 \pmod{q}$.

In this case $s_1 = n/p$ holds. Then, the equation $pk + ql = n$ is equivalent to $p(s_1 - k) = ql$. Thus, the solutions of $pk + ql = n$ are $(k, l) = (s_1 - qr, pr)$ for $1 \le r \le \lfloor s_1/q \rfloor$. Hence, we have

$$\#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\} = \left\lfloor \frac{s_1}{q} \right\rfloor = \left\lfloor \frac{n}{pq} \right\rfloor.$$

Therefore, we have

$$\begin{aligned}
& E(p,q;n) - E(p,q;n-1) \\
= \ & \frac{n}{pq} - 1 + \frac{[-p^- n]_q}{q} = \frac{n}{pq} - 1 + \frac{[-s_1]_q}{q} = \frac{n}{pq} - \frac{q - [-s_1]_q}{q} \\
= \ & \frac{n}{pq} - \frac{[s_1]_q}{q} = \frac{n}{pq} - \frac{[ps_1]_{pq}}{pq} = \frac{n}{pq} - \frac{[n]_{pq}}{pq} = \left\lfloor \frac{n}{pq} \right\rfloor \\
= \ & \#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\}.
\end{aligned}$$

The case of $n \not\equiv 0 \pmod{p}$ and $n \equiv 0 \pmod{q}$.

In this case $s_2 = n/q$ holds. Then, the equation $pk + ql = n$ is equivalent to $q(s_2 - l) = pk$. Thus, the solutions of $pk + ql = n$ are $(k, l) = (qr, s_2 - pr)$ for $1 \le r \le \lfloor s_2/p \rfloor$. Hence, we have

$$\#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\} = \left\lfloor \frac{s_2}{p} \right\rfloor = \left\lfloor \frac{n}{pq} \right\rfloor.$$

Therefore, we have

$$\begin{aligned}
& E(p,q;n) - E(p,q;n-1) \\
= \ & \frac{n}{pq} - 1 + \frac{[-q^- n]_p}{p} = \frac{n}{pq} - 1 + \frac{[-s_2]_p}{p} = \frac{n}{pq} - \frac{p - [-s_2]_p}{p} \\
= \ & \frac{n}{pq} - \frac{[s_2]_p}{p} = \frac{n}{pq} - \frac{[qs_2]_{pq}}{pq} = \frac{n}{pq} - \frac{[n]_{pq}}{pq} = \left\lfloor \frac{n}{pq} \right\rfloor \\
= \ & \#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\}.
\end{aligned}$$

The case of $n \not\equiv 0 \pmod{p}$ and $n \not\equiv 0 \pmod{q}$.

We suppose that $p < q$. The solutions of the equation $pk + ql = n$ for integers k, l are presented by $(k, l) = (nx + qs, ny - ps)$ for some integer s. Here (x, y) is a solution of the equation $px + qy = 1$. Hence, we obtain

$$E_0(p,q;n) := \{(k,l) \in \mathbb{N}^2 | pk + ql = n\} = \{(nx + qs, ny - ps) \in \mathbb{N}^2 | s \in \mathbb{Z}\}.$$

The set $E_0(p,q;n)$ is bijective to

$$S(p,q;n) := \left\{ nx+qs \in \mathbb{N} \middle| nx+qs < \frac{n}{p}, s \in \mathbb{Z} \right\}.$$

If $S(p,q;n) \neq \varnothing$, then the minimal number in $S(p,q;n)$ is $k_0 := [p^- n]_q$. Because k_0 is positive and less than q and $pk_0 \equiv n \pmod q$, therefore there exists a positive integer l_0 such that $pk_0 + ql_0 = n$. The maximal number in $S(p,q;n)$ is $(n - q[q^- n]_p)/p$, because this number is the k-coordinate of a point with the minimal l-coordinate in the points in $E_0(p,q;n)$. Indeed the minimum coordinate is $[q^- n]_p$. Then we have $(n-q[q^- n]_p)/p \leq s$ by $p<q$. The solution of $pk+ql=n$ is described in Figure 2.2. The circled dots are the solutions of $pk+ql=n$ and the square dots are $\{[s]_q + nq | n \in \mathbb{Z}\}$.

In this case, we have

$$\begin{aligned} E(p,q;n) - E(p,q;n-1) &= \frac{n}{pq} - 1 + \frac{[-q^- n]_p}{p} + \frac{[-p^- n]_q}{q} \\ &= \frac{n}{pq} + 1 - \frac{[q^- n]_p}{p} - \frac{[p^- n]_q}{q} \\ &= \frac{n}{pq} + 1 - \frac{p[p^- n]_q + q[q^- n]_p}{pq}. \end{aligned}$$

<u>The case A: $(0 < [p^- n]_q \leq [s]_q)$.</u>

In this case, the set $S(p,q;n)$ is bijective to

$$\{[s]_q, [s]_q + q, \cdots, s-2q, s-q, s\}.$$

Therefore, we obtain

$$\begin{aligned} \#\{(k,l) \in \mathbb{N}^2 | pk+ql=n\} &= \#S(p,q;n) = \frac{s-[s]_q}{q} + 1 = \left\lfloor \frac{s}{q} \right\rfloor + 1 \\ &= \left\lfloor \frac{1}{q} \left\lfloor \frac{n}{p} \right\rfloor \right\rfloor + 1 \\ &= \frac{1}{q} \left\lfloor \frac{n}{p} \right\rfloor - \left\{ \frac{1}{q} \left\lfloor \frac{n}{p} \right\rfloor \right\} + 1 \\ &= \frac{n}{pq} + 1 - \frac{[n]_p}{pq} - \frac{[s]_q}{q} \\ &= \frac{n}{pq} + 1 - \frac{[n]_p + p[s]_q}{pq}. \end{aligned}$$

We note that

$$q[q^- n]_p + p([p^- n]_q + s - [s]_q) = n.$$

Hence, we have

$$\begin{aligned} q[q^- n]_p - [n]_p &= n - p([p^- n]_q + s - [s]_q) - [n]_p \\ &= (n - [n]_p) - ps + p([s]_q - [p^- n]_q) \\ &= p[s]_q - p[p^- n]_q. \end{aligned}$$

Thus, we obtain

$$p[p^- n]_q + q[q^- n]_p = [n]_p + p[s]_q.$$

Therefore, we have

$$E(p,q;n) - E(p,q;n-1) = \#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\}.$$

The case B: ($[s]_q < [p^- n]_q$).

In this case, the set $S(p,q;n)$ is bijective to

$$\{[s]_q + q, [s]_q + 2q, \ldots, s - 2q, s - q, s\}.$$

Therefore, we obtain

$$\begin{aligned}
\#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\} &= \#S(p,q;n) = \frac{s - ([s]_q + q)}{q} + 1 = \left\lfloor \frac{s}{q} \right\rfloor \\
&= \frac{n}{pq} - \frac{[n]_p + p[s]_q}{pq} \\
&= \frac{n}{pq} + 1 - \frac{[n]_p + p[s]_q + pq}{pq}.
\end{aligned}$$

We note that

$$q[q^- n]_p + p([p^- n]_q + s - ([s]_q + q)) = n.$$

Hence, we have

$$\begin{aligned}
q[q^- n]_p - [n]_p &= n - p([p^- n]_q + s - ([s]_q + q)) - [n]_p \\
&= (n - [n]_p) - ps + p([s]_q - [p^- n]_q) + pq \\
&= p[s]_q - p[p^- n]_q + pq.
\end{aligned}$$

Thus, we obtain

$$p[p^- n]_q + q[q^- n]_p = [n]_p + p[s]_q + pq.$$

Therefore, we have

$$E(p,q;n) - E(p,q;n-1) = \#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\}.$$

These mean

$$\#\{(k,l) \in \mathbb{N}^2 | pk + ql = n\} = E(p,q;n) - E(p,q;n-1).$$

for any $n \in \mathbb{N}$. Using $\varepsilon(p,q;0) = \#\{(k,l) \in \mathbb{N}^2 | pk + ql = 0\} = 0$ and

$$\begin{aligned}
E(p,q;0) &= -\frac{1}{2}\left(d(p,q;0) + d(q,p;0) + \frac{pq-(1-p-q)^2}{4pq}\right)\\
&= -\frac{1}{2}\left(3(s(q,p)+s(p,q)) + \frac{p-1}{2p} + \frac{q-1}{2q}\right.\\
&\quad\left. - \frac{p^2+pq+q^2-2p-2q+1}{4pq}\right)\\
&= -\frac{1}{2}\left(3\left(\frac{1}{12}\left(\frac{q}{p}+\frac{1}{pq}+\frac{p}{q}\right)-\frac{1}{4}\right) + \frac{2pq-p-q}{2pq}\right.\\
&\quad\left. - \frac{p^2+pq+q^2-2p-2q+1}{4pq}\right)\\
&= -\frac{1}{2}\left(\frac{p^2-3pq+q^2+1}{4pq} + \frac{4pq-2p-2q}{4pq}\right.\\
&\quad\left. - \frac{p^2+pq+q^2-2p-2q+1}{4pq}\right)\\
&= 0,
\end{aligned}$$

we have $\varepsilon(p,q;0) = E(p,q;0)$ and

$$\begin{aligned}
\varepsilon(p,q;n) &= \varepsilon(p,q;n) - \varepsilon(p,q;0)\\
&= \sum_{t=1}^{n}(\varepsilon(p,q;t) - \varepsilon(p,q,t-1))\\
&= \sum_{t=1}^{n}\#\{(k,l) \in \mathbb{N}^2 | pk+ql = t\}\\
&= \sum_{t=1}^{n}(E(p,q;t) - E(p,q,t-1))\\
&= E(p,q;n) - E(p,q;0)\\
&= E(p,q;n)
\end{aligned}$$

for any $n \in \mathbb{N}$.

By changing the roles of p and q, we can prove the case $p > q$ using the same argument above.

Therefore, we have the desired result above. □

Remark 2.3.12. *For any integer n, by Theorem 2.3.2, then we have*

$$\begin{aligned}
&d(p,q;n) = d(p,q;[n]_{pq})\\
= \quad &-d(q,p;[n]_{pq}) - \frac{pq-(2[n]_{pq}+1-p-q)^2}{4pq} - 2\varepsilon(p,q;[n]_{pq}).
\end{aligned}$$

We define

$$D(p,q,r;n) := d(p,-qr;n) + d(q,-rp;n) + d(pq,r;n)$$

and

$$F(p,q,r;n) := -\frac{pq-p-q-1}{pq}\left(n - \frac{N_0(p,q,r)+r-pq-2}{4}\right)$$

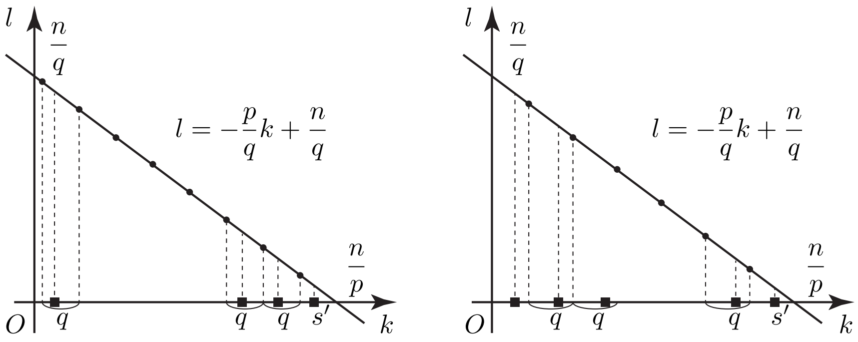

Figure 2.2: The case A: $(0 < [p^- n]_q \leq [s]_q)$ and the case B: $([s]_q < [p^- n]_q)$.

for any integer n.

Corollary 2.3.13.

$$\mathcal{T}(p,q,r;n) = D(p,q,r;n) + F(p,q,r;n) + 2\varepsilon(r,pq;n+pq)$$

for any $n \geq -1$.

Proof. By [T, Lemma 1] and Theorem 2.3.2, we have

$$\begin{aligned} & d(r,-pq;n) = d(r,r-pq;n) = -d(r,pq;n+pq) \\ = \quad & d(pq,r;n+pq) + \frac{pqr - (2(n+pq)+1-r-pq)^2}{4pqr} + 2\varepsilon(r,pq;n+pq) \end{aligned}$$

for any $n \geq -1$. Then, by Proposition 2.3.11 and

$$\begin{aligned} & \frac{1}{pqr}\left(n - \frac{N_0(p,q,r)-1}{2}\right)^2 - \frac{(2n+1+pq-r)^2}{4pqr} \\ = \quad & \frac{1}{pqr}\left(n - \frac{N_0(p,q,r)-1}{2}\right)^2 - \frac{1}{pqr}\left(n - \frac{r-pq-1}{2}\right)^2 \\ = \quad & \frac{1}{pqr}\left(2n - \frac{N_0(p,q,r)+r-pq-2}{2}\right)\frac{-N_0(p,q,r)+r-pq}{2} \\ = \quad & -\frac{pq-p-q-1}{pq}\left(n - \frac{N_0(p,q,r)+r-pq-2}{4}\right) \\ = \quad & F(p,q,r;n), \end{aligned}$$

we have

$$
\begin{aligned}
& \mathcal{T}(p,q,r;n) \\
= \ & 2(\tau(p,q,r;n+1)+\lambda(p,q,r)+\kappa(p,q,r)) \\
= \ & \frac{1}{pqr}\left(n-\frac{N_0(p,q,r)-1}{2}\right)^2 + d(p,-qr;,n)+d(q,-rp;n) \\
& +d(r,-pq;n)-\frac{1}{4} \\
= \ & \frac{1}{pqr}\left(n-\frac{N_0(p,q,r)-1}{2}\right)^2 + D(p,q,r;n) \\
& +\frac{pqr-(2(n+pq)+1-r-pq)^2}{4pqr}+2\varepsilon(r,pq;n+pq)-\frac{1}{4} \\
= \ & D(p,q,r;n)+\frac{1}{pqr}\left(n-\frac{N_0(p,q,r)-1}{2}\right)^2-\frac{(2n+1+pq-r)^2}{4pqr} \\
& +2\varepsilon(r,pq;n+pq) \\
= \ & D(p,q,r;n)+F(p,q,r;n)+2\varepsilon(r,pq;n+pq).
\end{aligned}
$$

for any $n \geq -1$. □

2.3.4 The value of $\mathcal{T}(p,q,r;n+pq)-\mathcal{T}(p,q,r;n)$

In this subsection, we first introduce a remark and prove lemmas need later.

Remark 2.3.14. *The function $D(p,q,r;n)$ is a function in n with period pq. Therefore, we have*

$$D(p,q,r;n+pq)=D(p,q,r;n)$$

for any integer n.

We define

$$\sigma(p,q) := \frac{pq-p-q-1}{2}.$$

Lemma 2.3.15.

$$F(p,q,r;n+pq)-F(p,q,r;n)=-2\sigma(p,q)$$

for any integer n.

Proof. By the definition of F, we have

$$
\begin{aligned}
& F(p,q,r;n+pq)-F(p,q,r;n) \\
= \ & -\frac{pq-p-q-1}{pq}\left(n+pq-\frac{N_0(p,q,r)+r-pq-2}{4}\right) \\
& +\frac{pq-p-q-1}{pq}\left(n-\frac{N_0(p,q,r)+r-pq-2}{4}\right) \\
= \ & -(pq-p-q-1) = -2\sigma(p,q).
\end{aligned}
$$

□

Lemma 2.3.16.

$$\varepsilon(p,q;n+q)-\varepsilon(p,q;n)=\left\lfloor\frac{n}{p}\right\rfloor$$

for any nonnegative integer n.

Proof.

$$\begin{aligned}
\varepsilon(p,q;n+q)-\varepsilon(p,q;n) &= \sum_{k=1}^{\lfloor n/p\rfloor}\left\lfloor\frac{n+q-pk}{q}\right\rfloor-\sum_{k=1}^{\lfloor (n-q)/p\rfloor}\left\lfloor\frac{n-pk}{q}\right\rfloor\\
&= \sum_{k=1}^{\lfloor n/p\rfloor}\left(\left\lfloor\frac{n-pk}{q}\right\rfloor+1\right)-\sum_{k=1}^{\lfloor (n-q)/p\rfloor}\left\lfloor\frac{n-pk}{q}\right\rfloor\\
&= \left\lfloor\frac{n}{p}\right\rfloor+\sum_{k=\lfloor (n-q)/p\rfloor+1}^{\lfloor n/p\rfloor}\left\lfloor\frac{n-pk}{q}\right\rfloor.
\end{aligned}$$

By

$$\frac{n-pk}{q}\le\frac{1}{q}\left(n-p\left(\left\lfloor\frac{n-q}{p}\right\rfloor+1\right)\right)<\frac{1}{q}\left(n-p\left(\frac{n-q}{p}\right)\right)=1$$

and

$$\frac{n-pk}{q}\ge\frac{1}{q}\left(n-p\left\lfloor\frac{n}{p}\right\rfloor\right)\ge 0,$$

we have

$$0\le\frac{n-pk}{q}<1$$

for any $\lfloor (n-q)/p\rfloor+1\le k\le\lfloor n/p\rfloor$.

Therefore, we obtain the desired equation. □

Proposition 2.3.17.

$$\mathcal{T}(p,q,r;n)-\mathcal{T}(p,q,r;n-pq)=-2\left(\sigma(p,q)-\left\lfloor\frac{n}{r}\right\rfloor\right)$$

for any $n\ge pq-1$.

Proof. Using Remark 2.3.14, Lemma 2.3.15, and Lemma 2.3.16, we have

$$\begin{aligned}
&\mathcal{T}(p,q,r;n)-\mathcal{T}(p,q,r;n-pq)\\
=\ &F(p,q,r;n)-F(p,q,r;n-pq)+2(\varepsilon(r,pq;n+pq)-\varepsilon(r,pq;n))\\
=\ &-2\sigma(p,q)+2\left\lfloor\frac{n}{r}\right\rfloor.
\end{aligned}$$

Therefore, we have the desired result above. □

2.3.5 Proof of main theorem for this section

To prove Theorem 2.3.4, we show the following:

Lemma 2.3.18.

$$d(\Sigma(p,q,r))=-\min_{-1\le n\le (N_0(p,q,r)-1)/2}\mathcal{T}(p,q,r;n).$$

Proof. By [CK, Theorem 1.3], we obtain $\Delta(p,q,r;n) \geq 0$ for any $n > N_0(p,q,r)$. Then we have

$$\min_{n\geq 0} \tau(p,q,r;n) = \min_{0\leq n\leq N_0(p,q,r)+1} \tau(p,q,r;n).$$

By [CK, Theorem 1.3], we obtain

$$\Delta(p,q,r;n) = -\Delta(p,q,r;N_0(p,q,r)-n)$$

for any $0 \leq n \leq N_0(p,q,r)$.

If $N_0(p,q,r) \equiv 0 \pmod 2$, then we have $\Delta(p,q,r;N_0(p,q,r)/2) = 0$. Therefore, we have

$$\begin{aligned}
&\tau(p,q,r;N_0(p,q,r)+1-n) = \sum_{i=0}^{N_0(p,q,r)-n} \Delta(p,q,r;i) \\
=& \sum_{i=0}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i) + \sum_{i=N_0(p,q,r)/2+1}^{N_0(p,q,r)-n} \Delta(p,q,r;i) \\
=& \sum_{i=0}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i) - \sum_{i=N_0(p,q,r)/2+1}^{N_0(p,q,r)-n} \Delta(p,q,r;N_0(p,q,r)-i) \\
=& \sum_{i=0}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i) - \sum_{i=n}^{N_0(p,q,r)/2-1} \Delta(p,q,r;i) \\
=& \tau(p,q,r;n).
\end{aligned}$$

for any $0 \leq n \leq N_0(p,q,r)/2$. Therefore, we have

$$\min_{-1\leq n\leq N_0(p,q,r)} \tau(p,q,r;n+1) = \min_{-1\leq n\leq N_0(p,q,r)/2-1} \tau(p,q,r;n+1).$$

If $N_0(p,q,r) \equiv 1 \pmod 2$, then we have

$$\begin{aligned}
&\tau(p,q,r;N_0(p,q,r)+1-n) = \sum_{i=0}^{N_0(p,q,r)-n} \Delta(p,q,r;i) \\
=& \sum_{i=0}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i) + \sum_{i=(N_0(p,q,r)+1)/2}^{N_0(p,q,r)-n} \Delta(p,q,r;i) \\
=& \sum_{i=0}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i) - \sum_{i=(N_0(p,q,r)+1)/2}^{N_0(p,q,r)-n} \Delta(p,q,r;N_0(p,q,r)-i) \\
=& \sum_{i=0}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i) - \sum_{i=n}^{(N_0(p,q,r)-1)/2} \Delta(p,q,r;i) \\
=& \tau(p,q,r;n).
\end{aligned}$$

for any $0 \leq n \leq (N_0(p,q,r)+1)/2$. Hence, we have

$$\min_{-1\leq n\leq N_0(p,q,r)} \tau(p,q,r;n+1) = \min_{-1\leq n\leq (N_0(p,q,r)-1)/2} \tau(p,q,r;n+1).$$

Thus we have

$$\min_{-1\leq n\leq N_0(p,q,r)} \tau(p,q,r;n+1) = \min_{-1\leq n\leq \lfloor (N_0(p,q,r)-1)/2\rfloor} \tau(p,q,r;n+1).$$

Therefore, we have

$$d(\Sigma(p,q,r)) = -\min_{-1\leq n\leq \lfloor (N_0(p,q,r)-1)/2\rfloor} \mathcal{T}(p,q,r;n).$$

Thus, we obtain the desired result. □

Here we prove Theorem 2.3.4.

Proof of Theorem 2.3.4. By Proposition 2.3.17, we have

$$\mathcal{T}(p,q,r;n) \leq \mathcal{T}(p,q,r;n-pq)$$

if and only if

$$\left\lfloor \frac{n}{r} \right\rfloor \leq \sigma(p,q)$$

for any $n \geq pq-1$. If $pq-1 \leq n \leq \lfloor (N_0(p,q,r)-1)/2 \rfloor$, then we have

$$\begin{aligned} & \left\lfloor \frac{n}{r} \right\rfloor \leq \left\lfloor \frac{N_0(p,q,r)-1}{2r} \right\rfloor = \left\lfloor \frac{pqr-pq-qr-rp-1}{2r} \right\rfloor \\ = & \left\lfloor \frac{(pq-p-q-1)r+r-pq-1}{2r} \right\rfloor = \sigma(p,q) + \left\lfloor \frac{r-pq-1}{2r} \right\rfloor \leq \sigma(p,q). \end{aligned}$$

From this and Lemma 2.3.18, we have

$$\begin{aligned} & d(\Sigma(p,q,r)) \\ = & -\min_{-1\leq n\leq \lfloor (N_0(p,q,r)-1)/2\rfloor} \mathcal{T}(p,q,r;n) \\ = & -\min_{\max\{\lfloor (N_0(p,q,r)-1)/2\rfloor - pq+1,-1\}\leq n\leq \lfloor (N_0(p,q,r)-1)/2\rfloor} \mathcal{T}(p,q,r;n). \end{aligned}$$

Therefore, we obtain the desired result. □

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