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DOCTORAL DISSERTATION

Theoretical Study on Compositeness in Hadron Physics

by

Zanpeng Yin

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in partial fulfillment of the requirements for the degree of

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Abstract

In this thesis, we study the consistency and evaluation of compositeness under a theoretical approach. Compositeness is a quantity first proposed by Steven Weinberg in order to account for the problem of whether a particle is elementary or not. Compositeness is defined as a proportion of the origin of a bound state that can be attributed to a scattering state, and elementariness is the proportion attributed to an elementary state in a similar manner. However, the calculation of the deuteron compositeness generates unrealistic quantity, and we attempt to solve this problem.

We first attempt to perform numerical calculation of the deuteron using a modern interpretation of compositeness with different models, in the hope of finding a type of interaction that can recover the deuteron properties and yield physical compositeness at the same time. We attempt low energy constants (LEC) models, one pion exchange potential (OPEP), Yamaguchi potential, and square well potential. However, none of the interactions can succeed in both deuteron property and physical compositeness. This raises the suspicion that the unphysical compositeness is caused by a deeper physical reason rather than the absence of a good interaction model.

We postulate that the unphysical compositeness is a consequence of attributing any energy dependency into other states, which we call the surjective interpretation. To test this postulation, we formalize a modified definition of compositeness with explicit energy dependency in interaction. Utilizing this method, we are able to perform a perturbative calculation of compositeness and confirm our postulation. We conclude that an attractive energy-dependent perturbation from an energy-independent theory is highly likely to enhance the compositeness from unity, which is the case of the deuteron. Our outcome indicates that, rather than the lack of a realistic interaction potential, it is the formalism of compositeness that requires modification.

The first proposed solution is to explicitly separate different contributions from the interaction potential. In the deuteron, the pions play an important role, and it may be counterintuitive to attribute such interaction to elementariness due to the lack of a one-body state. We propose a new quantity interactioness to account for this contribution and combine interactionesss into compositeness considering its dynamical origin. However, this proposed solution makes compositeness no longer model-independent, diminishing its meaning in a phenomenology setting.

We then propose a second solution, which tries to be model-independent as well as always generates physical quantity. We are motivated by S. Weinberg's notion when compositeness was originally defined and the extreme case that the hydrogen atom is always considered a composite particle. We observe that the energy scale is of great importance in this process. As a result, we define energy decomposition as compositeness. For the delta function in the 1-dimensional scenario, it yields the same outcome as the original definition.

Contents

CHAPTER 1

Introduction

Hadrons are particles governed by Quantum Chromodynamics (QCD) and are considered bound states of quarks and gluons. However, the full picture is much more complex than composing the particle like putting blocks together. Even though QCD can be calculated perturbatively, and quarks are asymptotically free in the high-energy regime, non-perturbative effects are of great importance in the low-energy regime where the hadron self-energy resides. It is caused by the increasing coupling constant of QCD at low energy, which also causes quark confinement. As a result, the hadron structure is much more complex than a bound state caused by other forces, like a hydrogen atom when viewed as a bound state of proton and electron mediated by electromagnetic force, if the inner structure of the proton is ignored.

Due to the complexity of low-energy QCD, the study of hadron physics usually utilizes effective models. For example, hadrons can be seen as constructed by constituent quarks in the quark model, where the mass of a hadron is just the addition of the mass of constituent quarks and the interaction among them (different from the quarks in QCD, which is usually called current quark to distinguish between them). In the constituent quark model, we can categorize most of the hadrons into two categories: baryons that have three quarks and mesons that have two quarks, and they are the free states in the effective theory for hadrons. However, there are some hadrons, called exotic hadrons, that cannot be categorized as such. Some of the most famous exotic hadrons include $\Lambda(1405)$[1], $X(3872)$[2], and $Y(4260)$[3].

Compositeness is a quantity proposed by S. Weinberg in Refs.[4, 5] that attempts to distinguish if a particle is composed of other particles when such particle manifests itself in a scattering process. This should be rather straightforward in a classical scenario but can face some difficulty in Quantum Mechanics (QM) when several states exist at a similar energy scale, making these states compete with each other in a Quantum Mechanical way. This is exactly the case in hadron physics, where quark pair creation, meson creation, and normal excitation can all be on a similar energy scale. As a result, instead of a binary answer of yes or no, compositeness is defined as a proportion of the origin of a bound state that can be attributed to a scattering state. On the other hand, elementariness is defined as the proportion contributed by elementary states.

Currently, the main research interest of compositeness in hadron physics lies in the exotic hadrons when trying to understand their inner structure. It has been proposed that they may be an elementary state bound by QCD directly and cannot be expressed by hadron states, or a hadron scattering state which is also called a composite state. The bound state can be a mixture of those states.

Compositeness has attracted much attention during the 20th century, some works were performed on compositeness in a QFT scheme in Refs.[6, 7], and a modern QFT

formalism was discussed in Ref.[8]. On the other hand, along with the QM formalism of compositeness as it was proposed, T. Sekihara et. al. developed a modern formalism in Refs.[9–12]. This approach brings about a pseudo-model-independent calculation of compositeness and elementariness, which expands its usability in interpreting experimental observations. The weak-binding limit of compositeness is also researched vastly in Refs.[13–16], as particles of interest usually stay close to the scattering threshold. Further, some other developments on the calculation method have also been developed in Refs.[17–19]. Finally, the interpretation of complex compositeness in resonance instead of bound state has been studied in Ref.[12], and some interpretations of the theory of compositeness itself have been discussed in Refs.[20, 21].

On the phenomenological side, compositeness has been utilized on many particles, especially exotic hadrons. Since a weak-binding limit calculation is relatively straightforward, we will only mention the model calculations. Those calculations have been performed in Refs.[9, 22–24]. It is worth mentioning that some of the calculations utilized Lattice QCD as in Refs.[25, 26].

Even though it is suggested in Ref.[5] that the deuteron is a composite particle of proton and neutron, which aligns with our understanding of nuclear physics, it yields an unrealistic negative quantity for elementariness when considering its probabilistic origin. This unrealistic quantity of elementariness (and compositeness) in the deuteron can be seen in many calculations, yet we believe there was not a convincing solution or explanation to this problem. In this thesis, we will perform a theoretical study on the consistency of compositeness.

In chapter 2, we will discuss hadron physics as the background of this thesis. We will discuss the position of hadron physics, the quark model, and exotic hadrons. This serves as a motivation of our research. In chapter 3, we outline the method to generate bound states in the scattering process using the Lippmann-Schwinger equation. Feshbach partitioning that reduces the model space will also be explored. In chapter 4, we will build the formalism of compositeness mostly based on Ref.[9] but with energy-dependent interaction. This formalism will be the theoretical basis of this whole thesis. We also discussed one way to evaluate compositeness from experiments based on the surjective interpretation, the assumption that energy-dependent interactions always originate from other states. In chapter 5, we will perform numerical calculations of the deuteron compositeness based on the surjective interpretation. However, all of these models will give unphysical compositeness, which is larger than unity. In chapter 6, we give a formal statement of the problem, and conclude that the surjective interpretation is the cause of this problem. In chapter 7, a relatively straightforward extension that explicitly accounts for the energy dependency is discussed, and we revisit the deuteron. In chapter 8, we give another solution to the problem, which is defining compositeness differently. After discussing the motivations, we give a new definition that interprets energy decomposition as compositeness. An analytical calculation that has a connection with the original definition is discussed. In chapter 9, we will summarize this thesis, and give some final discussions.

Chapter 4 to Chapter 7 are related to the author's paper: *A Possible Solution to the Difficulty in the Interpretation of Deuteron Compositeness*[27].

CHAPTER 2

Hadron Physics

In this chapter, we will review hadron physics focusing on the important backgrounds to be considered in this thesis. As a result, we will start the discussion from the standard model, and introduce hadrons as bound states of the strong interaction.

1. The Standard Model

The standard model is the best-tested theory that includes both the building blocks of this universe and the interactions between them. This model uses Quantum Field theory (QFT) as the framework and describes three out of the four fundamental forces, the electromagnetic force, the weak force, and the strong force, leaving out only the gravitational force. The gravity is extremely small and should be neglected in hadron physics.

Hadrons are a type of subatomic particle built of quarks, and quarks are elementary particles that have fractional charges. As a result, hadrons are composite particles, and they are held by the strong interaction, i.e. by gluons. Most of the hadrons can be divided into two categories: baryons and mesons. Baryons are hadrons that consist of three quarks, and mesons are composed of a quark and an antiquark. There are some hadrons outside of this model, which are called exotic hadrons. They are believed to be 4 or 5 quark states, or hadron scattering states, i.e. composed by other hadrons.

The strong force has a strength of around 1. Compared to it, the electromagnetic force has a strength at the order of 10^{-3}, and the strength of the weak force is at the order of 10^{-8}.[28] As a result, the strong force plays a very important role in the properties of hadrons. It is true that hadrons (as well as quarks) are subject to all of the forces, and many observations are related to the electromagnetic force and the weak force. However, this thesis is most related to the strong interaction due to the energy scale that is around the scattering threshold of hadrons and also the relatively shorter time scale we are interested in.

2. Hadrons in QCD

The strong interaction is described by Quantum Chromodynamics (QCD), which is mediated by gluons. Similar to the case in the electromagnetic force, which only particles holding electric charge are sensitive to, only the particles holding the color charge are sensitive to the strong interaction. As a result, among all the elementary particles, only quarks are sensitive to this force.

To be clear, the quarks mentioned are treated as free states in QCD, which are called current quarks or bare quarks. As a theory based on QFT, these current quarks can be dressed, and the dressed quarks are called the constituent quarks (or valence quarks), which will be discussed later.

One of the important properties of the QCD is color confinement. Intuitively, color confinement states that colored objects are always confined to color singlet states, and no objects with non-zero color charge can propagate as free particles.[28] It is believed that the interaction potential increases approximately linearly with the distance between any two particles due to gluon-gluon interaction. As a result, when attempting to separate two quarks, it will require infinite energy, so that color-singlet hadrons are produced in this process, causing the jets in hadron colliders.[29]

When viewed from another direction, the coupling constant α_S increases with a decrease of the energy. Even though quarks have asymptotic freedom at high energy, it can be observed from Figure 1 that α_S increases drastically at low energy, making the situation in the hadron physics complicated.

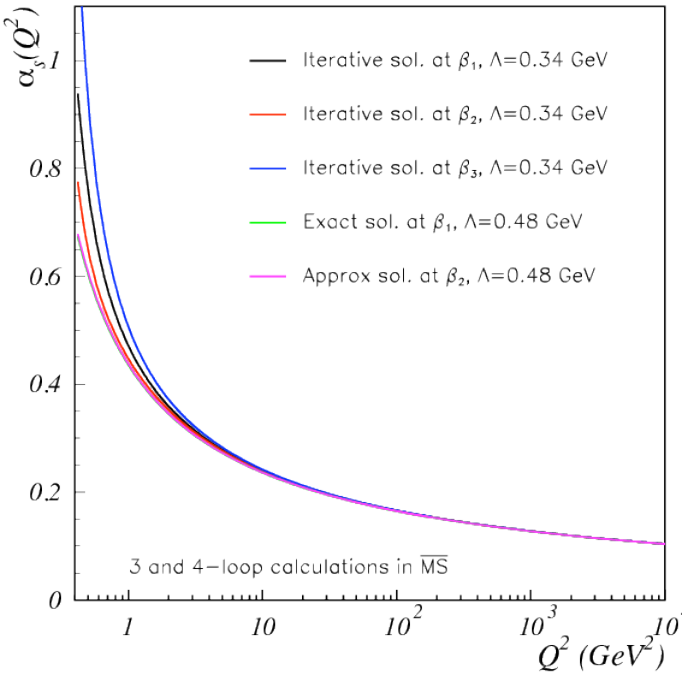

FIGURE 1. The computed perturbative coupling $\alpha_S(Q^2)$ shown in Ref.[30].

From experiments, we know that protons and neutrons have mass of around $1GeV$, while the u and d current quarks in the standard model have only a mass of several MeV. Instead of attributing the mass of hadrons to the current quarks, most of the mass of the hadrons is actually contributed by the interaction between the current quarks. This limits the usage of perturbative QCD in hadron physics, and we have to use effective models or numerical methods to gain physical understanding.

3. The Eightfold Way

The eightfold way is a method that focuses on organizing the hadron states based on their spin and isospin. It can be viewed as the periodic table in hadron physics. There are in total 6 quarks in the standard model, while the ones that are most related to the discussion here are three of them with the lowest energies. Consider a basic $SU(3)$ triplet of quarks, and the properties of the quarks can be found in

Quark	I_3	S	Q
u	1/2	0	+2/3
d	-1/2	0	-1/3
s	0	-1	-1/3

TABLE 1. Properties of quarks. I_3 is the isospin, S is strangeness, and Q is electric charge.[31]

Table 1,

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \bar{q} = \begin{pmatrix} \bar{u} & \bar{d} & \bar{s} \end{pmatrix}, \tag{1}$$

With this representation, we can consider meson as a quark-antiquark state, and we are able to find from dimension analysis,[31]

$$q_b \bar{q}^a = \hat{T}^a_b + \frac{1}{3}\delta^a_b T, \tag{2}$$

where $\hat{T}^a_b$ is traceless, so that $dim\,\hat{T} = 8$, and $\mathbf{3} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}$ as its group structure. They are separated into pseudoscalar mesons and vector mesons due to the spin-orbit coupling.

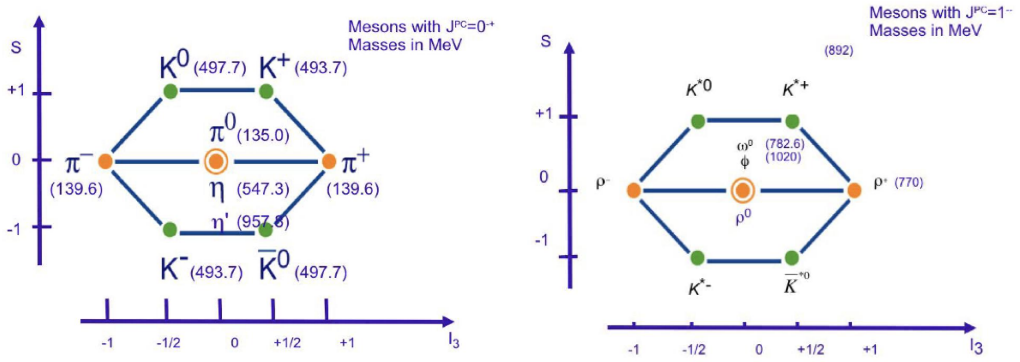

FIGURE 2. Eightfold way for pseudoscalar (left) and vector (right) mesons.[31]

The same can be performed for baryons, as three quark states. The group structure can be written as $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\bar{\mathbf{3}} \oplus \mathbf{6}) = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$. Among them, $\mathbf{10}$ represents completely symmetric states of particle exchange under $SU(3)$, related to the baryon decimet of $J^P = 3/2^+$ states. One of the $\mathbf{8}$ represents mixed-antisymmetric states under $SU(3)$, and the other represents the mixed-symmetric states. Both of them require at least one of the quarks to be different. Finally, $\mathbf{1}$ represents the antisymmetric state of three different quarks. Combining the discussion above, we have,[32]

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1}^A \oplus \mathbf{8}^{MA} \oplus \mathbf{8}^{MS} \oplus \mathbf{10}^S, \tag{3}$$

where MA is shorthand for mixed-antisymmetric, and MS is mixed-symmetric. The spin of the particles can be written in a similar fashion. Since the spin follows the $SU(2)$ group, we have,

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2}^{MA} \oplus \mathbf{2}^{MS} \oplus \mathbf{4}^S. \tag{4}$$

Since baryons are fermions, the total wave function of them has to be antisymmetric. The color charge carried by the states shall finally be antisymmetric, and the ground state should be the S wave, i.e. symmetric in the orbital angular momentum. As a result, the observed states are required to have,

$$\psi_{spin}\psi_{SU(3)} = symmetric. \tag{5}$$

The baryon decimet with $J^P = 3/2^+$ are the $\mathbf{10}^S_{SU(3)}$ combined with the symmetric spin states,

$$\text{decimet: } \psi_{spin}\psi_{SU(3)} = \psi^S_{spin}\psi^S_{SU(3)}, \tag{6}$$

and the $\mathbf{8}_{MS}$ has to combine with symmetric spin states, while $\mathbf{8}_{MA}$ has to combine with antisymmetric spin states, resulting in the $J = 1/2^+$ baryon octet,

$$\text{octet: } \psi_{spin}\psi_{SU(3)} = \frac{1}{\sqrt{2}}(\psi^{MS}_{SU(3)}\psi^{MS}_{spin} + \psi^{MA}_{SU(3)}\psi^{MA}_{spin}). \tag{7}$$

The singlet $\mathbf{1}$ is ruled out due to the fact that there is no fully antisymmetric state for the spin, and all the observed baryon states composed only by u, d, and s quarks can be expressed.

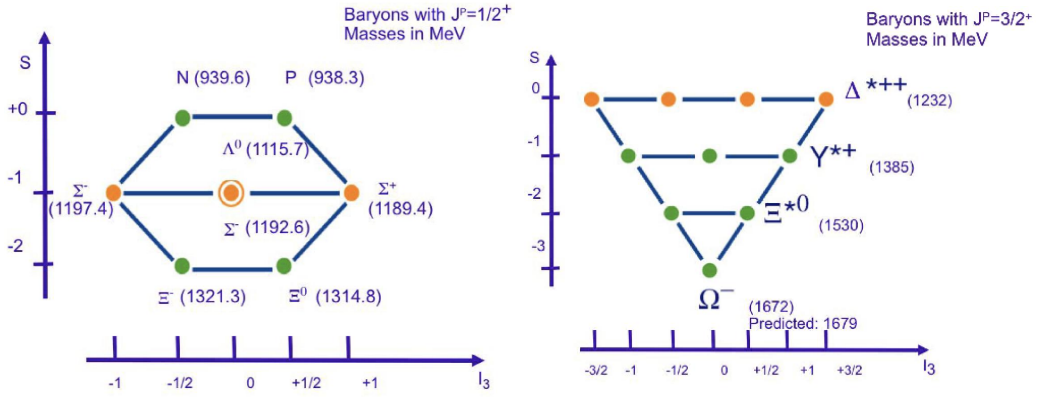

FIGURE 3. Eightfold way for spin-1/2 (left) and spin-3/2 (right) baryons.[31]

4. Consituent Quark Model

Different from the quark model as a part of the standard model, the constituent quark model in hadron physics has a different meaning. It is a low energy effective model where current quarks are dressed with gluons and quark-antiquark pairs. The hadron mass is expressed as the sum of its component constituent quark mass and the interaction among them. The interaction is expressed as,

$$H_{ij} = 2\kappa_{ij}(\boldsymbol{s_i} \cdot \boldsymbol{s_j}), \tag{8}$$

where $\boldsymbol{s_i}$ and $\boldsymbol{s_j}$ are spin operators, and,

$$\kappa_{ij} = -\lambda(R) \times \frac{g_s^2}{3m_i m_j}|\psi(0)|^2. \tag{9}$$

When only considering the u, d, and s quarks, κ can be viewed as scaling with $1/m_i m_j$, and we have a way to calculate hadron mass with only a few parameters, which satisfies both meson and baryon mass composed by u, d, and s quarks relatively well with $\sim 30MeV$ accuracy. [31]

Different from the u and d current quarks, the u and d constituent quarks have mass of around $330MeV$. The exact value of the constituent quark mass is model-dependent.

5. Exotic Hadrons

With the constituent quark model, we have a relatively clear view of the construction of mesons as $q\bar{q}$ states and baryons as qqq states. However, there exist some hadronic states that cannot be explained by this model, and they are generally called exotic hadrons.

5.1. Observations of Exotic Hadrons. First, we will discuss some candidates of the exotic hadrons, which have already been observed.

The first candidates are the hadrons that decay into one J/ψ particle in the final state. It is because those particles are likely to have a $c\bar{c}$ or $b\bar{b}$ pair. This type of exotic hadrons include $X(3872)$[2] and $Y(4260)$[3]. [31]

The second candidate is hidden charm or bottom states with electric charges like $Z(4430)$. [31, 33]

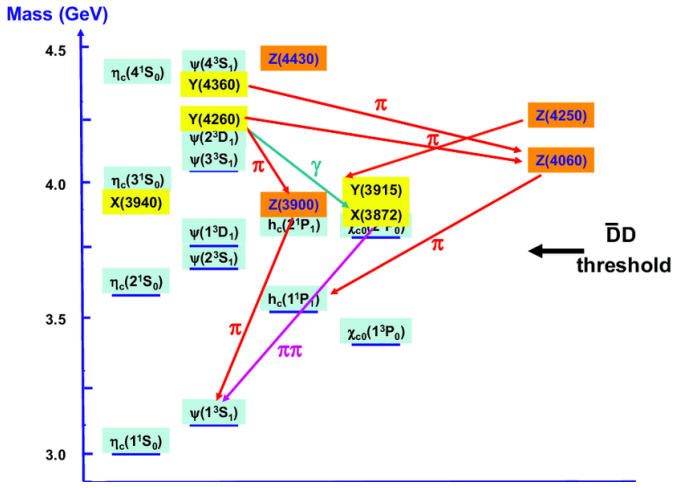

FIGURE 4. The Charmonium spectrum.[34]

5.2. Theoretical Explanations of Exotic Hadrons. One of the ways to explain the exotic hadron states is the multiquark state constructed by more than three constituent quarks. This explanation includes tetraquarks (4 quark states), pentaquarks (5 quark states), etc. We may replace an antiquark with a diquark, or a pair of two qurks from, for example, an antibaryon. This is possible because the quarks in the diquark can merge into a $\bar{\mathbf{3}}$ structure in the $SU(3)$ color group, which is identical to the antiquark. As an example, the $X(3872)$ may be interpreted as a $\bar{c}c\bar{q}q$ state,[35] where q represents either a u or a d quark. This type of explanation is shown in Figure 5(c).

On the other hand, the exotic hadrons may be a hadron molecule state, where two hadrons are bound by nuclear force, like the one-pion exchange interaction. This type of explanation can be seen as an extension of the deuteron, where the pn pair is bound by nuclear forces. The $X(3872)$ exotic hadron can also be interpreted in a similar manner. Since it has an energy close to the $D^*\bar{D}$ threshold, it can be

expressed in the molecule model.[36] This type of explanation is shown in the first figure in Figure 5(d).

Other than the two possibilities mentioned above, there are also purely QCD-origin states like glueball and hybrid meson. The glueballs are states that contain no constituent quarks but only gluons, and hybrid mesons consist of both constituent quarks and gluons.

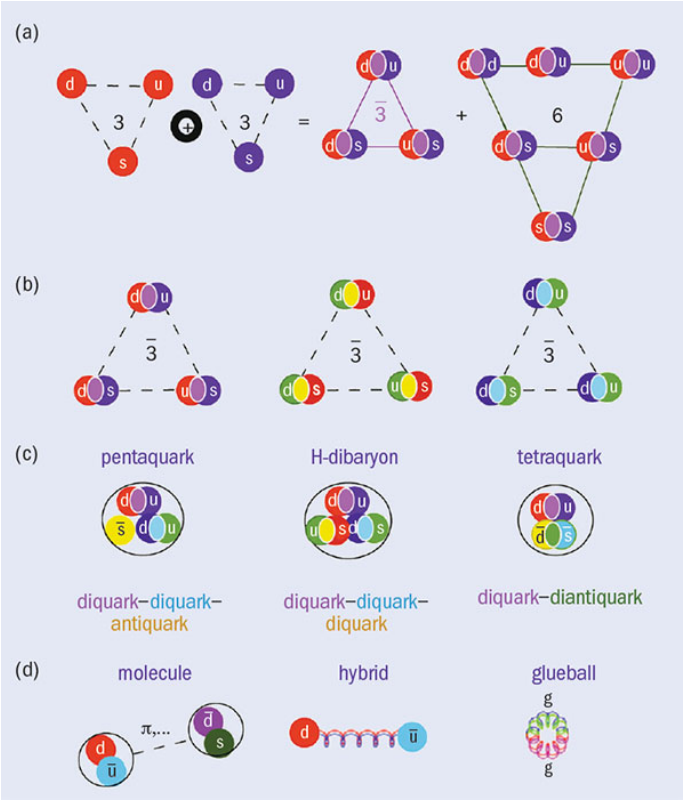

FIGURE 5. The schematic graph of possible hadrons allowed by QCD:(a) Combining a red and blue quark triplet produces a magenta (anti-green) antitriplet and sextet. (b) The three anticolored diquark antitriplets. (c) Some of the multiquark, color-singlet states that can be formed from quarks, antiquarks, diquarks and diantiquarks. (d) Other possible multiquark/gluon systems.[37]

5.3. The Role of Compositeness. Even though one may expect the nature to choose one of the explanations, it is possible that these schemes may compete with each other in a quantum mechanical way. It is because the dynamics are governed by quantum mechanics (or QFT), and radial excitation, quark pair creation, and meson creation all fall around the same energy scale of $\sim 500MeV$. Consequently, as all the schemes are possible, it is natural to attempt to calculate which schemes may be taking the lead. Compositeness is used to solve this exact problem. We may write the exotic hadron state as,

$$|\Psi\rangle = c_{ml}\,|molecule\rangle + c_{mq}\,|multiquark\rangle + c_{gb}\,|glueball\rangle\,, \tag{10}$$

and compositeness is the attempt to calculate the $|c|^2$ without knowing all the physical details.

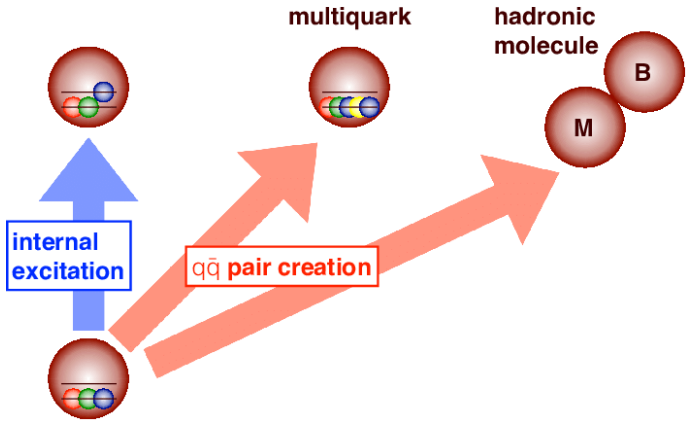

FIGURE 6. The schematic graph of excitation modes of hadrons.[38]

At first, compositeness was used to distinguish if the deuteron is a composite particle or not. In Ref.[5], S. Weinberg gave a clue that the deuteron is a composite particle of proton and neutron. With the current understanding of the deuteron state, we know that the deuteron is a pn state bound by the nuclear force, and hadron molecule states shall share the same situation. As a result, we may expect compositeness $X = |c_{ml}|^2$. On the other hand, the multiquark states and glueballs both have a QCD origin and shall be interpreted as elementariness, the counterpart of compositeness, $Z = |c_{mq}|^2 + |c_{gb}|^2$.

However, the calculation in Ref.[5] gives compositeness $X = 1.68$ with the scattering observations of the deuteron. It is true that as long as $X > 1$, we shall be able to say the particle is composite, but it breaks the probability understanding that should have been the virtue of compositeness. Further, it shows a problem with the calculation procedure, as there are possibilities that the calculation can produce an outcome that cannot be interpreted normally.

Regardless of this problem with the deuteron, compositeness has found its use in many exotic hadrons. This problem on the deuteron will be examined closely in chapter 6.

CHAPTER 3

Scattering Theory

Scattering experiments are one of the most utilized research methods in modern physics, and the observations of exotic hadrons usually come from such experiments. In this chapter, we will review the old-fashioned perturbation theory from which the Lippmann-Schwinger equation is derived, the effective range expansion which describes the low energy behavior of scattering experiments, the Weinberg program which is important for the discussion of the deuteron, and Feshbach partitioning which can be utilized to transform the model space.

1. Old Fashioned Perturbation Theory

Different from calculating the Feynman diagrams from the field theory, the old-fashioned perturbation theory (OFPT) starts from the Schrodinger equation. As a result, the states in OFPT are all on the mass shell, but the energy is no longer conserved at vertices. In this section, we will outline the derivation of the Lippmann-Schwinger equation, which will become useful in the formalism of compositeness.

Consider the Hamiltonian $\boldsymbol{H}$ which can be divided into the free theory $\boldsymbol{H}_0$ and the interaction $\boldsymbol{V}$,

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{V}, \tag{11}$$

where $\boldsymbol{H}_0$ have a set of eigenstates $|\phi\rangle$,

$$\boldsymbol{H}_0 |\phi\rangle = E |\phi\rangle . \tag{12}$$

Since we are considering the scattering process where the energy is continuous, we will be able to find eigenstate of $\boldsymbol{H}$, $|\psi\rangle$, which satisfies,

$$\boldsymbol{H} |\psi\rangle = E |\psi\rangle . \tag{13}$$

Combining Eqs.(12)(13), one may write formally,

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - \boldsymbol{H}_0 + i\epsilon} \boldsymbol{V} |\psi\rangle , \tag{14}$$

where the complex part in the denominator is added to avoid making $(E - \boldsymbol{H}_0)^{-1}$ singular and should finally be removed. The validity of Eq.(14) can be verified through multiplying both sides with $E - \boldsymbol{H}_0 + i\epsilon$. Eq.(14) is the original form of the Lippmann-Schwinger equation, but we would want to remove the $|\psi\rangle$ state from the formalism for practical usage. To do this, we define the transfer matrix, or T-matrix, $\boldsymbol{T}$,

$$\boldsymbol{V} |\psi\rangle = \boldsymbol{T} |\phi\rangle . \tag{15}$$

By adding $\boldsymbol{V}$ to both sides of Eq.(14) and ignoring the imaginary part, we will have,

$$\boldsymbol{T} |\phi\rangle = \boldsymbol{V} |\phi\rangle + \boldsymbol{V} \frac{1}{E - \boldsymbol{H}_0} \boldsymbol{T} |\phi\rangle . \tag{16}$$

By removing the state $|\phi\rangle$, we have the final form of the Lippmann-Schwinger equation,

$$\boldsymbol{T} = \boldsymbol{V} + \boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{T}, \tag{17}$$

where $\boldsymbol{\Pi}$ is the Lippmann-Schwinger kernel.[39]

With Eq.(17), one may calculate $\boldsymbol{T}$ perturbatively as,

$$\boldsymbol{T} = \boldsymbol{V} + \boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V} + \dots . \tag{18}$$

To better understand the physical significance of the Lippmann-Schwinger equation, one may take matrix elements of the matrices by inserting a complete set of $|\phi\rangle$, resulting in,

$$T_{ij} = V_{ij} + \sum_k V_{ik}\Pi_k V_{kj} + \sum_{k,l} V_{ik}\Pi_k V_{kl}\Pi_l V_{lj} + \dots , \tag{19}$$

where $T_{ij} = \langle\phi_i|\,\boldsymbol{T}\,|\phi_j\rangle$, $V_{ij} = \langle\phi_i|\,\boldsymbol{V}\,|\phi_j\rangle$, and $\Pi_k = \frac{1}{E-E_k}$. From it, we may observe that the first term is the first Born approximation, and the following terms follow the higher order Born approximation in the same manner. [39, 40]

We would like to note that $\boldsymbol{V}$ is assumed to be small to give the expansion in Eq.(18) and also the interpretation of the Born approximation. As a result, when the interaction potential is strong enough to create a bound state, such approximation shall not be considered available.

One may have noticed that this formalism is constructed with dependency on E. This E can be traced back as representing the total energy of the system. As a result, we will have three variables for energy, E, E_i, and E_j. However, this problem can be solved by taking $E = E_i = E_j$, which is the only scenario of our interest.[39]

2. Generating the Bound State

To approach the problem with the bound state, especially the deuteron bound state, the methodology is to put the bound state in by the construction of the effective theory.

Instead of taking the Lippmann-Schwinger equation in a perturbative way, we may interpret the series without cutting off as an effective theory, so that Eq.(18) can be written in the following form,

$$\boldsymbol{T} = \boldsymbol{V} + \boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V} + \dots = \frac{1}{\frac{1}{\boldsymbol{V}} - \boldsymbol{\Pi}}. \tag{20}$$

For example, in the single channel scenario, we may find that $\boldsymbol{T}$ matrix, which has only one element, can yield a pole at,

$$1 - V\Pi = 0, \tag{21}$$

where V and Π are the only elements of $\boldsymbol{V}$ and $\boldsymbol{\Pi}$. This pole is generated dynamically, instead of existing in either V or Π alone. In this way, instead of using the Lippmann-Schwinger equation perturbatively, we utilize the entire series to generate the bound state that is required for this effective theory.

3. Separable Interaction

We may focus on the scattering channels so that $|\phi_i\rangle$ is a state with fixed quantum numbers and momentum. As a result, we can notate V_{ij} more precisely as,

$$V_{ij} = \langle q_i | \boldsymbol{V} | q_j' \rangle \,, \tag{22}$$

where i and j notate only the discrete quantum numbers, and q is the momentum, which is different from the notation above where each i specifies both discrete quantum numbers and momentum.

It is common that the interaction is assumed to take separable form,[1]

$$V_{ij} = f_i(q^2) v_{ij}(E) f_j(q'^2), \tag{23}$$

Eq.(19) can be reordered into,

$$\begin{aligned} T_{ij} = f_i(q^2) v_{ij}(E) f_j(q'^2) + \sum_k f_i(q^2) v_{ik}(E) G_k(E) v_{kj}(E) f_j(q') \\ + \sum_{k,l} f_i(q^2) v_{ik}(E) G_k(E) v_{kl}(E) G_l(E) v_{lj}(E) f_j(q') + \dots \,, \end{aligned} \tag{24}$$

where

$$G_k(E) := \int dq [f_k(q^2)]^2 \Pi_k(E, E_q). \tag{25}$$

We may further define,

$$T_{ij} := f_i(q^2) t_{ij}(E) f_j(q'^2), \tag{26}$$

giving us

$$t_{ij} = v_{ij} + \sum_k v_{ik} G_k v_{kj} + \sum_{kl} v_{ik} G_k v_{kl} G_l v_{lj} + \dots \,. \tag{27}$$

We can furthur writing it into matrix form,

$$\boldsymbol{t} = \boldsymbol{v} + \boldsymbol{vGv} + \boldsymbol{vGvGv} + \dots = \frac{1}{\frac{1}{\boldsymbol{v}} - \boldsymbol{G}}. \tag{28}$$

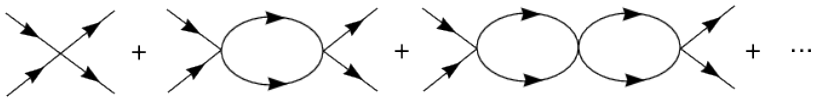

FIGURE 1. The bubble chain generating the bound state.[41]

We can interpret Eq.(27) as a series of diagrams added together. This process is shown in Figure 1, which is called the bubble chain.

[1] Conventionally, the separable form requires there is no explicit energy dependency in v, i.e. $v(E) = const.$ However, the discussion around energy dependency is at the core of this thesis, and we will use this extended separable form.

To show it, we may calculate the single loop integral in a phenomenological scenario where the form factor is unknown and thus set to one,[41]

$$\begin{aligned} I_0 &= -i(\frac{\mu}{2})^{4-n}\int\frac{d^nk}{(2\pi)^n}(\frac{i}{E+k_0-k^2/2M+i\epsilon})(\frac{i}{-k_0-k^2/2M+i\epsilon}) \\ &= (\frac{\mu}{2})^{4-n}\int\frac{d^{n-1}k}{(2\pi)^{n-1}}(\frac{1}{E-k^2/M+i\epsilon}) \\ &\equiv G(E), \end{aligned} \tag{29}$$

where M is nucleon mass, k_0 is the energy of one of the nucleons, and k is the three momentum of it. Under the $\overline{MS}$ scheme, we will take the $n \to 4$ limit, so that the single-loop integral will take the same form as $G(E)$.[41]

4. Effective Range Expansion

The effective range expansion is a way to study the low energy scattering properties in nuclear physics. When considering only the low energy section of NN scattering, we can expect the S wave to dominate. As a result, the scattering matrix can be written as,

$$S = e^{2i\delta} = 1 + \frac{2iq}{qcot\delta - iq} = 1 - i\frac{qM}{2\pi}T, \tag{30}$$

where δ is the phase shift. Besides, s and t are scattering matrix $\boldsymbol{S}$ and transfer matrix $\boldsymbol{T}$ projected into a single channel so that they can be seen as a function of momentum q, or energy since the observed particles are on the mass shell. We can further expand the unknown $qcot\delta$ part to give it a form only related with q,[41]

$$qcot\delta = -\frac{1}{a} + \frac{1}{2}r_0q^2 + r_1q^4 + \dots. \tag{31}$$

By combining Eq.(30) and Eq.(31), we can find that the low energy scattering matrix does not rely on the dynamical details, but only on the effective range expansion parameters. Among these parameters, a is called scattering length, and r_0 is called effective range.

We need to note that, different from section 2, the study of scattering amplitude does not have a predefined form factor in the interaction. As a result, s and t are related with S and T only by taking matrix elements, and the form factor is taken as one.

When combined with Eq.(20), Eq.(30) can be written into,

$$s = 1 - \frac{qM}{2\pi}\frac{1}{\frac{1}{v} - G}, \tag{32}$$

so that the form of the interaction v at low energy can in principle be derived from the effective range parameters. This type of theory is called the effective range theory.

5. Weinberg Program

Attempts were made to calculate the NN with a more realistic and physical model other than the effective range theory. The pion plays an important role in nuclear physics, and the perturbative pion exchange was formulated. This formulation utilizes the KSW-counting, which indicates the energy range where pion can be treated perturbatively. However, the perturbative pion causes the contact

interaction to dominate over pion exchange at low energy,[42] regardless of our expectation that the pion shall play an important role. Further, it is observed that the calculation based on this formalism deviates significantly from data at momentum $q \gtrsim m_\pi/2$.[41]

One solution to this problem is to follow the Weinberg Program,[43] which uses the chiral perturbation theory to calculate NN interaction potential. This method utilizes a different power counting, which does not manifest in the amplitudes for each individual process. On the other hand, the power-counting works as an a priori and shall be checked later.[41]

For the nucleon-nucleon interaction at leading order, which is important in this thesis, the interaction has only two pieces,[41]

$$V(\boldsymbol{q}_\pi) = -\frac{g_A^2}{4f_\pi^2}\frac{\sigma_1 \cdot \boldsymbol{q}_\pi \sigma_2 \cdot \boldsymbol{q}_\pi}{\boldsymbol{q}_\pi^2 + m_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2) + C_S + C_T \sigma_1 \cdot \sigma_2, \tag{33}$$

where g_A is the axial coupling constant, f_π is the pion decay constant, and C_S and C_T are coefficient for contact interactions. If we focus on only the ${}^3S_1$ channel, it can be further reduced into a constant term and a Yukawa potential term.

6. Feshbach Partitioning

Feshbach partitioning, also called Feschbach-Fano partitioning, is a method originally proposed to give a generalized calculation method for nuclear interaction.[44, 45] This method separates the Hilbert space into the open channel subspace and closed channel subspace, and represents the effective Hamiltonian of the open channel subspace, essentially allowing us to disregard the details of the closed subspace.

Consider the Hamiltonian $\boldsymbol{H}$ with eigenstates $|\psi\rangle$,

$$\boldsymbol{H}\,|\psi\rangle = E\,|\psi\rangle\,. \tag{34}$$

We can define projection operators $\boldsymbol{P}$ and $\boldsymbol{Q}$ which represent the subspace of the open channel and closed channel respectively. By definition, they satisfy $\boldsymbol{P}+\boldsymbol{Q} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity operator in the full Hilbert space.

By acting $\boldsymbol{P}$ and $\boldsymbol{Q}$ on Eq.(34), we have,

$$(E - \boldsymbol{H}_{PP})\boldsymbol{P}\,|\psi\rangle = \boldsymbol{H}_{PQ}\boldsymbol{Q}\,|\psi\rangle\,, \tag{35}$$

$$(E - \boldsymbol{H}_{QQ})\boldsymbol{Q}\,|\psi\rangle = \boldsymbol{H}_{QP}\boldsymbol{P}\,|\psi\rangle\,, \tag{36}$$

where,

$$\boldsymbol{H}_{PP} = \boldsymbol{P}\boldsymbol{H}\boldsymbol{P}, \tag{37}$$

$$\boldsymbol{H}_{PQ} = \boldsymbol{P}\boldsymbol{H}\boldsymbol{Q}, \tag{38}$$

$$\boldsymbol{H}_{QP} = \boldsymbol{Q}\boldsymbol{H}\boldsymbol{P}, \tag{39}$$

$$\boldsymbol{H}_{QQ} = \boldsymbol{Q}\boldsymbol{H}\boldsymbol{Q}. \tag{40}$$

Eq.(36) has a formal solution,

$$\boldsymbol{Q}\,|\psi\rangle = \frac{1}{E - \boldsymbol{H}_{QQ}}\boldsymbol{H}_{QP}\boldsymbol{P}\,|\psi\rangle\,. \tag{41}$$

Substituting Eq.(41) back into Eq.(35), we are left with the solution of $\boldsymbol{P}\,|\psi\rangle$,

$$(E - H^{\text{eff}})\boldsymbol{P}\,|\psi\rangle = 0, \tag{42}$$

where the effective Hamiltonian is expressed as,

$$H^{\text{eff}} = \boldsymbol{H}_{PP} + \boldsymbol{H}_{PQ}\frac{1}{E - \boldsymbol{H}_{QQ}}\boldsymbol{H}_{QP}. \tag{43}$$

As a result, we are able to reduce the closed channels from the Hilbert space. It gives us a great tool to interchange between interaction and states.

CHAPTER 4

Compositeness with Energy-dependent Interaction

In this chapter, we will give an energy-dependent formalism of compositeness based on Refs[9, 27], which will become the theoretical basis of this thesis.

1. Physical Model and the Definition of Compositeness

To properly define compositeness, we need to have a physical system where the bound state can take contribution from both the scattering states $|q_j\rangle$ and the elementary states $|\psi_a\rangle$. Thus, both of them shall be taken as a part of the free Hamiltonian $\boldsymbol{H}_0$. [1] Formally,

$$\boldsymbol{H}_0 |q_j\rangle = E_{q_j} |q_j\rangle , \tag{44}$$
$$\boldsymbol{H}_0 |\psi_a\rangle = E_{\psi_a} |\psi_a\rangle , \tag{45}$$

where E_{q_j} is the eigenenergy of scattering state $|q_j\rangle$ in $\boldsymbol{H}_0$, and E_{ψ_a} is the eigenenergy of elementary state $|\psi_a\rangle$ in $\boldsymbol{H}_0$. $|q_j\rangle$ represents a scattering state of two particles, one with mass m_j and the other with mass M_j, and q_j is the momentum in the center of mass (CM) frame of these states. As a convention, we will use subscript j, k, etc. as the notation for scattering channels, and subscript a, b, etc. for elementary states. In quantum mechanics, states are always on the mass shell, and we will take a non-relativistic form of the eigenenergy,

$$E_{q_j} = M_{th,j} + \frac{q_j^2}{2\mu_j}, \tag{46}$$
$$M_{th,j} = m_j + M_j, \tag{47}$$
$$\mu_j = \frac{m_j M_j}{m_j + M_j}. \tag{48}$$

$M_{th,j}$ is the scattering threshold, and μ_j is the reduced mass of channel j. These states have normalization,

$$\langle q_j' | q_k \rangle = (2\pi)^3 \delta_{jk} \delta(q - q'), \tag{49}$$
$$\langle \psi_a | \psi_b \rangle = \delta_{ab}, \tag{50}$$
$$\langle \psi_a | q_k \rangle = 0. \tag{51}$$

They form an orthognal basis of the Hilbert space of $\boldsymbol{H}_0$,

$$1 = \sum_a |\psi_a\rangle \langle\psi_a| + \sum_j \int \frac{d^3 q_j}{(2\pi)^3} |q_j\rangle \langle q_j| . \tag{52}$$

[1] We would like to note that in this chapter, the operators (or matrices) are notated in bold text while scalars are not.

Up to now, we have built the free states to be used in this formalism. However, they are yet to form any bound state, so we need to introduce the interaction between them as $\boldsymbol{V}$, and the full Hamiltonian $\boldsymbol{H}$ can be expressed by,

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{V}. \tag{53}$$

With interaction $\boldsymbol{V}$, the free states generate a bound state $|\Psi\rangle$ as an eigenstate of $\boldsymbol{H}$,

$$\boldsymbol{H}\,|\Psi\rangle = M_\Psi\,|\Psi\rangle\,, \tag{54}$$

which is normalized as,

$$\langle\Psi|\Psi\rangle = 1. \tag{55}$$

As discussed above, compositeness is related to the contribution from the scattering states to the bound state, and elementariness is related to the contribution from the elementary state to the bound state. Thus, we may formalize them as,

$$X_j = \int \frac{d^3q_j}{(2\pi)^3}\,\langle\Psi|q_j\rangle\,\langle q_j|\Psi\rangle\,, \tag{56}$$

$$Z_a = \langle\Psi|\psi_a\rangle\,\langle\psi_a|\Psi\rangle\,, \tag{57}$$

where X_j is the compositeness of channel j, i.e. contribution of channel j towards the bound state, and Z_a is the elementariness of elementary state $|\psi_a\rangle$. We may also write compositeness in the form of wave functions,

$$X_j = \int \frac{d^3q_j}{(2\pi)^3}\tilde{\Psi}_j^*(q)\tilde{\Psi}_j(q), \tag{58}$$

where $\tilde{\Psi}_j(q) = \langle q_j|\Psi\rangle$.

Since we have assumed the completeness of the states in the Hilbert space (52), we may have the normalization of compositeness and elementariness,

$$1 = \sum_j X_j + \sum_a Z_a. \tag{59}$$

2. Compositeness in Single Channel

In this section, we will calculate compositeness in a single channel scenario with the theoretical model defined above. Since there is only one scattering channel and one elementary state, we will ignore the subscripts and call them $|q\rangle$ and $|\psi\rangle$ accordingly.

There are two possible ways to interpret the energy-dependent interaction. The first one is an explicit dependency on the external energy, which can be written as $\boldsymbol{V}(q, q', E)$, and the other is that the interaction $\boldsymbol{V}$ relates to only velocity, which can be written as $\boldsymbol{V}(q, q', E(q^2))$. Regardless, we will show that they will give the same compositeness and elementariness under an ansatz, which will be validated by recovering the bound state condition.

The explicitly energy-dependent way is the method we will be mainly using. The velocity-dependent way is included mainly for the completeness of our discussion.[2]

[2]These two subsections will have some naming collision. We would like to note that the namings in subsection 2.2 are only effective in that subsection, while the ones in subsection 2.1 are effective throughout the thesis.

2.1. Explicitly Energy-dependent Interaction. In this part (and as well as most of this thesis), we will interpret the interaction $\boldsymbol{V}$ as explicitly energy-dependent. Formally, we can write,

$$\langle q_j | \boldsymbol{V} | q'_k \rangle = v_{jk}(E) f_j(q^2) f_k(q'^2), \tag{60}$$

$$\langle \psi_a | \boldsymbol{V} | q'_k \rangle = g_{(0),ak} f_k(q'^2), \tag{61}$$

$$\langle \psi_a | \boldsymbol{V} | \psi_b \rangle = 0. \tag{62}$$

In it, $f(q)$ is the form factor, and $g_{(0),ak}$ is assumed to be constant. Further, by choosing an appropriate phase, both $v_{jk}(E)$ and $g_{(0),ak}$ can be real. In this formalism, we assume $v_{jk}(E)$ can be explicitly related to an external energy E, which is different from Ref.[9].

The first step of this calculation is to write out the coupled channel Schrodinger equation. By taking equation elements of $\boldsymbol{H}$, inserting a complete set through Eq.(52), and using the matrix elements of $\boldsymbol{V}$ as described in Eqs.(60)-(62), we have

$$\langle q | \boldsymbol{H} | \Psi \rangle = E_q \tilde{\Psi}(q) + g_{(0)} f(q^2) \langle \psi | \Psi \rangle + f(q^2) \int \frac{d^3q'}{(2\pi)^3} v(E) f(q'^2) \tilde{\Psi}(q') = M_\Psi \tilde{\Psi}(q), \tag{63}$$

$$\langle \psi | \boldsymbol{H} | \Psi \rangle = E_\psi \langle \psi | \Psi \rangle + g_{(0)} \int \frac{d^3q'}{(2\pi)^3} f(q'^2) \tilde{\Psi}(q') = M_\Psi \langle \psi | \Psi \rangle . \tag{64}$$

To reduce the formalism to a more understandable form, we can utilize Feshbach partitioning [44], and reduce the influence of the elementary state into the integrated interaction v^{int} in the scattering channel as, [3]

$$E_q \tilde{\Psi}(q) + f(q^2) \int \frac{d^3q'}{(2\pi)^3} v^{\text{int}}(E) f(q'^2) \tilde{\Psi}(q') = M_\Psi \tilde{\Psi}(q), \tag{65}$$

where the integrated potential is given as,

$$v^{\text{int}}(E) = v(E) + \frac{(g_{(0)})^2}{E - E_\psi}. \tag{66}$$

We would like to note that the influence of the elementary state resembles the form of a propagator, with the numerator fixed as positive, whose physical significance will be discusssed in chapter 6.

This Schrodinger equation has a formal solution,

$$\tilde{\Psi}(q) = -\frac{c f(q^2)}{E_q - M_\Psi}, \tag{67}$$

$$c = v^{\text{int}}(M_\Psi) \int \frac{d^3q'}{(2\pi)^3} f(q'^2) \tilde{\Psi}(q'). \tag{68}$$

It is worth mentioning that this is merely a formal solution and is still yet to be solved.

We can substitute the $\tilde{\Psi}(q')$ in Eq.(68) with Eq.(67). We require the equation to have a non-trivial solution so that c can not be zero. We have,

$$1 - v^{\text{int}}(M_\Psi) G(M_\Psi) = 0, \tag{69}$$

[3]The integrated potential was originally called effective potential in Ref.[9]. However, we would like to distinguish its origin from the effective potential that originates from phenomenology. Similarly, this quantity was named as integrated potential in Ref.[27]

where G is the Green's function,

$$G(E) = \int \frac{d^3q}{(2\pi)^3} \frac{[f(q^2)]^2}{E - E_q}. \tag{70}$$

We would like to note that the bound state condition (69) is the same one as the one in the Lippmann-Schwinger equation, which will be discussed later.

By utilizing Eqs.(67) and (70) to the definition of compositeness (58), we have,

$$X = \int \frac{d^3q}{(2\pi)^3} \tilde{\Psi}^*(q)\tilde{\Psi}(q) \tag{71}$$

$$= |c|^2 \int \frac{d^3q}{(2\pi)^3} \frac{[f(q^2)]^2}{[E_q - M_\Psi]^2} \tag{72}$$

$$= -|c|^2 [\frac{dG}{dE}]_{E=M_\Psi}. \tag{73}$$

The elementariness Z can be calculated in a similar way. By utilizing Eq.(64) and (70),

$$Z = \langle \Psi | \psi \rangle \langle \psi | \Psi \rangle \tag{74}$$

$$= |c|^2 G(M_\Psi) \frac{(g_{(0)})^2}{(M_\Psi - E_\psi)^2} G(M_\Psi) \tag{75}$$

$$= -|c|^2 [G \frac{d(v^{\rm int} - v)}{dE} G]_{E=M_\Psi}. \tag{76}$$

By the normalization of $|\Psi\rangle$ and completeness of the Hilbert space, we have the normalization of compositeness X and elementariness Z,

$$1 = X + Z = -|c|^2 [\frac{dG}{dE} + G \frac{d(v^{\rm int} - v)}{dE} G]_{E=M_\Psi}. \tag{77}$$

As a result, we are able to interpret them as probability as the definition implies.

2.2. Velocity Dependent Interaction. In this subsection, we would like to discuss a different way to interpret energy dependency for the completeness of the discussion. Its shorthand was written as $\boldsymbol{V}(q, q', E(q^2))$, and we will define this interaction formally as,

$$\langle q_j | \boldsymbol{V} | q'_k \rangle = v_{jk}(E_{q_j}) f_j(q^2) f_k(q'^2), \tag{78}$$

$$\langle \psi_a | \boldsymbol{V} | q'_k \rangle = g_{(0),ak} f_k(q'^2), \tag{79}$$

$$\langle \psi_a | \boldsymbol{V} | \psi_b \rangle = 0, \tag{80}$$

where v_{jk} now relies on on-shell energy of state $|q_j\rangle$. This is because we are working under the old-fashioned perturbation theory (OFPT) framework so that the particles are always on the mass shell. Besides, this form can represent any $\boldsymbol{V}$ as long as the q_j and q_k dependency can be separated like $\langle q_j | \boldsymbol{V} | q'_k \rangle = f_j(q^2) v(q_j) f_k(q'^2)$ due to the freedom in choosing the form factor $f(q^2)$.

The coupled-channel Schrodinger equation under the one scattering channel one elementary state scenario can be written in a similar form as in subsection 2.1,

$$\langle q | \boldsymbol{H} | \Psi \rangle = E_q \tilde{\Psi}(q) + g_{(0)} f(q^2) \langle \psi | \Psi \rangle + f(q^2) \int \frac{d^3q'}{(2\pi)^3} v(E_q) f(q'^2) \tilde{\Psi}(q') = M_\Psi \tilde{\Psi}(q), \tag{81}$$

$$\langle\psi|\,\boldsymbol{H}\,|\Psi\rangle = E_\psi\,\langle\psi|\Psi\rangle + g_{(0)}\int\frac{d^3q'}{(2\pi)^3}f(q'^2)\tilde{\Psi}(q') = M_\Psi\,\langle\psi|\Psi\rangle\,. \tag{82}$$

Different from subsection 2.1, we can precisely reduce the influence of the elementary state in the same way as Ref.[9], resulting in

$$E_q\tilde{\Psi}(q) + f(q^2)v^{\rm int}(E_q, M_\Psi)\int\frac{d^3q'}{(2\pi)^3}f(q'^2)\tilde{\Psi}(q') = M_\Psi\tilde{\Psi}(q), \tag{83}$$

where the integrated potential is given by,

$$v^{\rm int}(E_1, E_2) = v_\alpha(E_1, E_2) + v_\beta(E_2), \tag{84}$$

$$v_\alpha(E_1, E_2) = v(E_1) - v(E_2), \tag{85}$$

$$v_\beta(E_2) = v(E_2) + \frac{(g_{(0)})^2}{E_2 - E_\psi}. \tag{86}$$

With this definition, we can have the formal solution of Eq.(83) in a similar manner,

$$\tilde{\Psi}(q) = -\frac{[c_\alpha(E_q) + c_\beta]f(q^2)}{E_q - M_\Psi}, \tag{87}$$

with a non-constant part of c_α. The renormalization factors then can be written as

$$c_\alpha(E_q) = v_\alpha(E_q, M_\Psi)\int\frac{d^3q'}{(2\pi)^3}f(q'^2)\tilde{\Psi}(q'), \tag{88}$$

$$c_\beta = v_\beta(M_\Psi)\int\frac{d^3q'}{(2\pi)^3}f(q'^2)\tilde{\Psi}(q'). \tag{89}$$

If we follow the same route in subsection 2.1, we will encounter an integral that cannot be easily transformed into the Green's function, which we define as h,

$$h := \int\frac{d^3q'}{(2\pi)^3}f(q'^2)\tilde{\Psi}(q') \tag{90}$$

$$= \int\frac{d^3q'}{(2\pi)^3}\frac{[c_\alpha(E_q) + c_\beta][f(q'^2)]^2}{M_\Psi - E_q}. \tag{91}$$

We may observe that $c_\alpha(M_\Psi) = 0$ due to its definition of v_α in Eq.(85), so that h shall be dominated by c_β instead of c_α. Thus, we may set an ansatz to ignore c_α from the integral,

$$h \simeq \int\frac{d^3q'}{(2\pi)^3}\frac{c_\beta[f(q'^2)]^2}{M_\Psi - E_{q'}}. \tag{92}$$

From it, we are able to define the same form of Green's function as subsection 2.1,

$$G(E) = \int\frac{d^3q'}{(2\pi)^3}\frac{[f(q'^2)]^2}{E - E_{q'}}. \tag{93}$$

Further, we can utilize this ansatz to Eq.(89), resulting in,

$$1 - v_\beta(M_\Psi)\int\frac{d^3q'}{(2\pi)^3}\frac{[f(q'^2)]^2}{E - E_{q'}} = 0. \tag{94}$$

With the definition of v_α in Eq.(85) and definition of G in Eq.(93), we are able to recover the bound state condition in the same form as subsection 2.2 and also in

accordance with the Lippmann-Schwinger equation,

$$1 - v^{\rm int}(M_\Psi, M_\Psi) \int \frac{d^3q'}{(2\pi)^3} \frac{[f(q'^2)]^2}{E - E_{q'}} = 0, \tag{95}$$

so that the ansatz is justified.

From here, we can also calculate compositeness and elementariness under this ansatz. Inserting Eqs.(87) and (93) into the definition of compositeness (58),

$$\begin{aligned} X &= \int \frac{d^3q}{(2\pi)^3} |\tilde{\Psi}(q)|^2 && (96)\\ &\simeq \int \frac{d^3q}{(2\pi)^3} \frac{|c_\beta|^2 [f(q^2)]^2}{(E_q - M_\Psi)} && (97)\\ &= -|c_\beta|^2 [\frac{dG}{dE}]_{E=M_\Psi} && (98)\\ &= -[|c|^2]_{E_q=M_\Psi} [\frac{dG}{dE}]_{E=M_\Psi}. && (99) \end{aligned}$$

Similarly, the elementariness can be expressed by utilizing Eqs.(82) and (57),

$$\begin{aligned} Z &= \langle \Psi | \psi \rangle \langle \psi | \Psi \rangle && (100)\\ &\simeq |c_\beta|^2 G(M_\Psi) \frac{(g_{(0)})^2}{(M_\Psi - E_\psi)^2} G(M_\Psi) && (101)\\ &= -|c_\beta|^2 [G \frac{d(v^{\rm int} - v)}{dE} G]_{E=M_\Psi} && (102)\\ &= -[|c|^2]_{E_q=M_\Psi} [G \frac{d(v^{\rm int} - v)}{dE} G]_{E=M_\Psi}. && (103) \end{aligned}$$

For clarity, we would like to give some explanations to this outcome. Firstly, we have to enforce $E_q = M_\Psi$ on $|c|^2$ so that this formalism can be mathematically rigorous. However, we argue that this notation does not give a significant difference, because $|c|^2$ has the meaning of renormalization factor and is also related to the residue of the pole so that it is only meaningful at the pole position. Secondly, the notation in Z may be confusing because of the derivative part $\frac{d(v^{\rm int} - v)}{dE}$. We write it in this way to give it a direct correlation to the formalism in subsection 2.1, and also it is technically correct. We may write it in a more detailed form like $\frac{d(v^{\rm int}(E_q, E) - v(E_q))}{dE}$, while it is not meaningful to do so, because $v^{\rm int}(E_q, E) - v(E_q)$ does not have any correlation to E_q.

3. Compositeness in Multiple Channels

In this section, we would like to discuss the formalism of compositeness when there are multiple channels, which will be useful in later discussions. Similar to the single channel scenario, we are able to write the Schrodinger equation, but with multiple scattering states and elementary states. We can use Feshbach partitioning and give the coupled channel Schrodinger equation in a similar way. For each scattering channel, we can write,

$$E_{q_j} \tilde{\Psi}_j(q) + \sum_k f_j(q^2) \int \frac{d^3q'}{(2\pi)^3} v^{\rm int}_{jk}(E) f_k(q'^2) \tilde{\Psi}_k(q') = M_\Psi \tilde{\Psi}_j(q), \tag{104}$$

with integrated potential $v_{jk}^{\rm int}(E)$ defined as,

$$v_{jk}^{\rm int}(E) = v_{jk}(E) + \sum_a \frac{g_{(0),aj} g_{(0),ak}}{E - E_{\psi_a}}. \tag{105}$$

Following the same procedure as in single channel case, we can have access to the formal solution,

$$\tilde{\Psi}_j(q) = -\frac{c_j f_j(q^2)}{E_{q_j} - M_\Psi}, \tag{106}$$

$$c = \sum_k v_{jk}^{\rm int}(M_\Psi) \int \frac{d^3q'}{(2\pi)^3} f_k(q'^2) \tilde{\Psi}_k(q'). \tag{107}$$

For non-trivial solution, we have to have non-zero c_j. As a result, we have

$$det[1 - v^{\rm int}(M_\psi) G(M_\Psi)] = 0, \tag{108}$$

$$G_j(E) = \int \frac{d^3q}{(2\pi)^3} \frac{[f_j(q^2)]^2}{E - E_{q_j}}, \tag{109}$$

in which G_j is the Green's function of each scattering channel. If we consider each channel as a matrix element, we are able to consider the determinants of this matrix, so that such a form can be possible. It also ensures the existence of a bound state at $E = M_\Psi$.

Combining Eqs.(106) and (109), we are able to calculate compositeness in the same manner,

$$X_j = -|c_j|^2 [\frac{dG_j}{dE}]_{E=M_\Psi}. \tag{110}$$

We did not explicitly give the Schrodinger equation that governs the elementary states, but similar to the single channel case, we are able to write out the overlap between the bound state and the elementary state as,

$$\langle \psi_a | \Psi \rangle = \frac{1}{M_\Psi - E_{\psi_a}} \sum_j c_j g_{(0),aj} G_j(M_\Psi), \tag{111}$$

which finally leads to,

$$Z_a = \sum_{j,k} c_j c_k^* G_j(M_\Psi) \frac{g_{(0),aj} g_{(0),ak}}{M_\Psi - E_{\psi_a}} G_j(M_\Psi). \tag{112}$$

We may find that this form is very different from the one we have in the single channel scenario. However, the sum of elementariness will finally generate a similar form,

$$\begin{aligned} Z := \sum_a Z_a &= \sum_{j,k} c_j c_k^* G_j(M_\Psi) G_j(M_\Psi) \sum_a \frac{g_{(0),aj} g_{(0),ak}}{M_\Psi - E_{\psi_a}} \\ &= -\sum_{j,k} c_j c_k [G_j \frac{v_{jk}^{\rm int}}{dE} G_k]_{E=M_\Psi}. \end{aligned} \tag{113}$$

Please note that we are only considering the bound state scenario, so that we may have c_j and c_k both being real by an appropriate choice of phase.

The compositeness and elementariness will have normalization as determined by the normalization of the bound state and the completeness of Hilbert space,

$$1 = \sum_j X_j + \sum_a Z_a = -\sum_{j,k} c_j c_k [\delta_{jk} \frac{dG_j}{dE} + G_j \frac{v^{\text{int}}_{jk}}{dE} G_k]_{E=M_\Psi}. \tag{114}$$

Up till now, we have only discussed how to include explicit channels. It is also worth discussing a scenario where another channel is implicit. For example, we may consider a high energy scattering channel N that is not explicitly considered, and its influence on the other channels can be written according to the Feshbach partitioning as,[9, 46]

$$w_{jk}(E) = v^{\text{int}}_{jk}(E) + v^{\text{int}}_{jN}(E) \frac{G_N(E)}{1 - v^{\text{int}}_{NN} G_N(E)} v^{\text{int}}_{Nk}(E). \tag{115}$$

We may consider two ways of understanding this state. The first is to view it as a scattering state, and this state will have a compositeness X_N,

$$X_N = -c_N^2 [\frac{dG_N}{dE}]_{E=M_\Psi}. \tag{116}$$

The second is to reduce this state from the Hilbert space so that its influence shall appear in the elementariness instead of compositeness. Formally, we can have,

$$Z^w = Z + X_N, \tag{117}$$

where Z is determined by Eq.(113). The justification will be omitted due to its length and lower relevance, interested readers may consult appendix A1 in [9]. Even though we assumed $v(E)$ can be energy-dependent, the content in that appendix can be used, because both the v^{int} in this thesis and the v^{eff} in Ref.[9] are energy-dependent, so the mathematical implications are the same.

4. Evaluation of Compositeness with the Surjective Interpretation

To this point, we have discussed how compositeness is formalized theoretically without touching any of its calculations. When trying to perform a phenomenological calculation, one may find that this calculation is model-dependent, and as a result, may be begging the question. From a phenomenological viewpoint, one can only know v^{int} and G as a phenomenological model. However, one can always enforce $v = v^{\text{int}}$ so that any particle will have $X = 1$ and thus is a compositeness particle. This actually can be an option only when the interaction is well-understood, but it will be discussed in a later chapter.

To make a phenomenological calculation possible, we have to add some assumptions to the formalism. In this thesis, we will call the assumption made to allow a phenomenological calculation as an "interpretation". Compositeness and elementariness calculated through different interpretations are separated by different diacritic marks added above X and Z, including breve $\breve{X}$ and acute $\acute{X}$.

In this section, we will discuss the calculation method presented in Ref.[9], which gives the same outcome as the original paper by Weinberg in Ref.[5]. This way of calculation is based on an assumption that energy dependency always comes from other states, so that $v(E)$ is always constant, and no intrinsic energy dependency exists. We call this interpretation the surjective interpretation, whose name is borrowed from the mathematical idea of the surjective function.

We may consider the scattering theory in a single channel case, where the bound state is generated through a scattering process of state $|q\rangle$ which follows the Lippmann-Schwinger equation as discussed in chapter 3,[39]

$$\boldsymbol{t} = \boldsymbol{v} + \boldsymbol{vGt}, \tag{118}$$

in which $\boldsymbol{G}$ does not have a specific form, but practically it should be related to the single loop integral.

By using the same method discussed in chapter 3, we may take the matrix elements of the T matrix, and write it as a function,

$$t(E) = \frac{1}{1 - v^{\text{int}}(E)G(E)} v^{\text{int}}(E), \tag{119}$$

in which we follow our convention that bold texts are matrix, so that $v^{\text{int}}(E)$ is matrix element of $\boldsymbol{V}^{\text{int}}$, and $G(E)$ is matrix element of $\boldsymbol{G}(E)$. Phenomenologically, we should have already seen a pole to do this analysis, so there should be a pole in $t(E)$ at $E = M_\Psi$, giving us the bound state condition which is also seen above,

$$1 - v^{\text{int}}(M_\Psi)G(M_\Psi) = 0. \tag{120}$$

Further, the residue of this pole can be written as,

$$g^2 = \lim_{E \to M_\Psi} (E - M_\Psi)t(E) = -\frac{1}{[\frac{dG}{dE} + \frac{1}{(v^{\text{int}})^2}\frac{dv^{\text{int}}}{dE}]_{E=M_\Psi}}. \tag{121}$$

Utilizing Eq.(120), we can find the it will yield a normalization with a similar form as Eq.(77),

$$1 = -|g|^2[\frac{dG}{dE} + G\frac{dv^{\text{int}}}{dE}G]_{E=M_\Psi}, \tag{122}$$

where we added the absolute value to make it the same form as Eq.(77), which will not cause any difference because the residue of the bound state shall be positive. As a result, we can connect c and g like,

$$|g|^2 = \frac{|c|^2}{1 + |c|^2[\frac{dG}{dE} + G\frac{dv}{dE}G]_{E=M_\Psi}}. \tag{123}$$

By taking the surjective interpretation, i.e. constant v, so that $\frac{dv}{dG}$ is zero, and $c = g$, we can have a calculation method of compositeness and elementariness as,

$$\breve{X} = -|g|^2[\frac{dG}{dE}]_{E=M_\Psi}, \tag{124}$$

$$\breve{Z} = -|g|^2[G\frac{dv^{\text{int}}}{dE}G]_{E=M_\Psi}. \tag{125}$$

We only considered single-channel scenarios but the same outcome can also be arrived in a multi-channel scenario. A detailed discussion of the multi-channel scenario under the surjective interpretation can be found in Ref.[9]. However, we omit this discussion because its relevance to the whole discussion in this thesis is low.

5. Discussion

In this chapter, we have built an energy-dependent formalism of compositeness, which sets the theoretical basis of this thesis. From this formalism, assuming the surjective interpretation grants us the ability to perform a phenomenological calculation of compositeness and elementariness on a real particle.

The great power of the surjective interpretation lies in the fact that it allows a model-independent calculation. For a real bound state, we have already had the knowledge of the information of the incident particles, so we have access to the loop function $G(E)$ by performing single loop integral and assume form factor as one. Further, by performing a scattering experiment, we can also gain information about the residue of the pole $|g|^2$. As a result, we are able to calculate compositeness, and in turn elementariness, with just those information.

Evaluations based on the surjective interpretation, especially in the weak-binding limit, have been widely performed. Ref.[9] performed the calculations on $\Lambda(1405)$ and $f_0(980)$, and concluded that these states are dominated by $\bar{K}N$ scattering and $\bar{K}K$ scattering respectively. However, the calculation based on the surjective interpretation fails on the deuteron, by resulting in a negative elementariness.

We would like to note that this formalism is formulated in a quantum mechanical way instead of QFT. As a result, it is based on Old-fashioned Perturbation Theory (OFTP)[39]. The main difference between a QFT formalism and an OFTP one is that every state, including intermediate states, is on the mass shell, but energy is not conserved at vertices. Despite the difference in formulation, they provide equivalent results on the observations.

CHAPTER 5

Interlude: Numerical Calculation on the Deuteron

In this chapter, we will show the numerical calculation of the deuteron compositeness that we perform with the surjective interpretation. We will show these models cannot recover the deuteron properties and give a physical elementariness at the same time. This works as a justification showing the surjective interpretation results in a negative evaluation of elementariness, and also a motivation for a more theoretical discussion to be performed in later chapters.

1. Calculation Settings

As a preparation for further calculations in this chapter, we will list out a set of parameters used in this calculation, and discuss the Green's function we utilized.

We consider an isospin symmetric model, where the nucleon mass is set as 938.9185 MeV. The binding energy is chosen to be 2.2 MeV when we need to fit towards it. We would like to note that even though the difference between the significant figures of the nucleon mass and binding energy is large, it actually does not influence our discussion. We are only trying to show that these calculations can not give a physical elementariness value, and the exact value is not important.

The loop function we choose is the relativistic dimensional regularized loop function, which can be written as,

$$G = -i(4M^2)\int \frac{d^4k}{(2\pi)^4}\frac{1}{(P-k)^2 - M^2 + i\epsilon}\frac{1}{k^2 - M^2 + i\epsilon} \tag{126}$$

$$= \frac{4M^2}{16\pi^2}\{a_\Lambda + \frac{q}{E}(2ln(E^2+2qE) - 2ln(-E^2+2qE))\}, \tag{127}$$

where k and P are 4-momenta, M is nucleon mass, and q is three momentum in the center of mass frame. The subtraction constant a_Λ is determined by mapping to a 3-momentum cutoff scheme,

$$a_\Lambda = -\frac{M}{2\pi^2}\Lambda, \tag{128}$$

where Λ is the cutoff, which is chosen to be 150 MeV. In the energy regime that we are interested in, which is close to the scattering threshold, this Green's function is practically equivalent to the non-relativistic Green's function in Ref.[41],

$$G = -\frac{iqM}{4\pi}. \tag{129}$$

Regardless of the different forms of Green's function, we expect no difference will be found in the calculation caused by it because of the mathematical form of the Lippmann-Schwinger equation.

2. Low Energy Constants Model

The first model we consider is a low energy constants (LEC) model, where the interaction is given by,

$$v^{\text{int}} = c_0 + c_1 q^2 + ... \quad , \tag{130}$$

where c_n are the low energy constants that determine the low energy behaviors of the interaction. These constants can be mapped to the effective expansion of the proton-neutron scattering. Considering the effective range expansion discussed in chapter 3, the effective range expansion coefficients of proton-neutron scattering are $a = 5.424fm$, $r_0 = 1.759fm$, $r_1 = 0.040fm^3$.[47] As a result, the LECs each can be expressed by the effective range expansion parameters.

First, we may look at the 0th-order LEC theory, where only the constant term exists. Considering the bound state condition, we know that as long as the sign of a, and in turn c_0, is chosen correctly, there will always be one pole. If we take the surjective interpretation, elementariness will always be evaluated as zero, as indicated in Eq.(125). When taking c_0 in accordance with scattering property in a, we can find that the binding energy is 1.477 MeV, which indicates insufficient attraction when compared with the 2.2 MeV binding energy from the experiment.

By including the 1st order of the LECs, the binding energy is calculated to be 2.232 MeV, which is close to the experiment. However, the compositeness increases from one, resulting in $\breve{X} = 1.479$.

3. One Pion Exchange Potential

The next interaction we consider is the One Pion Exchange Potential (OPEP), which is very important in nuclear physics. In the position space, it is normally written as,

$$V_\pi(r) = \frac{m_\pi^2}{12\pi}(\frac{g_A}{\sqrt{2}f_\pi})^2(\tau_1\cdot\tau_2)[S_{12}(1+\frac{3}{m_\pi r}+\frac{3}{(m_\pi r)^2})+\sigma_1\cdot\sigma_2]\frac{e^{-m_\pi r}}{r}-\frac{4\pi}{3}\sigma_1\cdot\sigma_2\delta^3(r), \tag{131}$$

where m_π is the mass of pion, g_A and f_π are constants about this interaction, S_{12} is the tensor operator, σ is the pauli matrix of spin, and τ is the pauli matrix of isospin. When considering this exact problem of the deuteron, we know that the deuteron is dominated by the component with $S = 1$ and $T = 1$, so that $\sigma_1 \cdot \sigma_2 = 1$, $\tau_1 \cdot \tau_2 = -3$, $S_{12} = 0$. Further, since we will finally be performing the calculation in the momentum space, and the delta function part will finally turn out as a constant in the interaction in momentum space under S-wave, which dominates the deuteron. Considering we will be using a constant term with this interaction, we have the freedom to ignore the final term here.

As a result, we may simplify the OPEP potential of the deuteron under S-wave as,

$$V_\pi(r) = c_{opep}\frac{e^{-m_\pi r}}{r}, \tag{132}$$

where $c_{opep} = -\frac{m_\pi^2}{4\pi}(\frac{g_A}{\sqrt{2}f_\pi})^2$. Since the calculation is performed in the momentum space, we need to perform the Fourier transformation. We have,

$$V_\pi(q_\pi) = c_{opep}\frac{4\pi}{m_\pi}\frac{1}{q_\pi^2+m_\pi^2}, \tag{133}$$

where q_π is the momentum of the pion being exchanged. As discussed above, we shall project it into the S wave component since the deuteron is dominated by S wave. We have,

$$V_\pi(q) = c_{opep}\frac{\pi}{2q^2} ln\frac{4q^2+m_\pi^2}{m_\pi^2}. \tag{134}$$

It is worth noting that, even though there is q^2 in the denominator, this interaction does not diverge at $q \to 0$. Since c_{opep} is negative, the OPEP interaction will always be attractive. By combining it with a constant term, the compositeness is calculated as $\breve{X} = 1.870$.

We are working in the Weinberg Program as described in [41], so that the interaction is an effective interaction that originates from the Chiral Perturbation Theory (ChPT), where OPEP interaction is at the leading order. As a result, this calculation is in essence a calculation based on ChPT. However, as discussed above, the dynamics of the Lippmann-Schwinger equation are governed by quantum mechanics, and these interactions are merely effective potentials. The pion exchange diagram, which is a part of ChPT, should not be mixed in the same Feynmann diagram with the bubble chain of the Lippmann-Schwinger equation. As a result, it is not true that the calculation will introduce pion loops.[1]

4. Yamaguchi Potential

Yamaguchi potential is a form of separable potential, which is defined by the form factor,

$$f(q^2) = \frac{\beta^2}{q^2+\beta^2}, \tag{135}$$

where we follow the same convention that q is the momentum in the center of mass frame, and β is an energy scale that characterizes the interaction.

First, we combine this form factor with the Low energy constant(LEC) model up to the 1st order $V(q) = c_0 + c_1 q^2$. We found that the compositeness increases with increasing binding energy, which is the observation we had in LEC models. We are able to recover $\breve{X} = 1$ as $\beta = 336MeV$ with LECs fixed to the effective range expansion. However, the binding energy is $B = 2.01MeV$ so that $\breve{X}$ will be larger than unity if we reproduce the deuteron properties. We then included the 2nd order term in the LEC model $V(q) = c_0 + c_1 q^2 + c_2 q^4$. The result shows that to reproduce the bound state properties, the compositeness turns out to be $\breve{X} = 1.21$.

As a result, we can conclude that Yamaguchi potential cannot help us to have a reasonable compositeness for the deuteron. Physically, Yamaguchi potential can be interpreted as a finite range interaction, so that finite range interaction may not be the reason for the problem in the deuteron compositeness.

5. Square Well Potential

As a more simplistic model, we attempted the square well potential, which is converted into the momentum space through Fourier transformation. In the position

[1]The pion loop is introduced in the interaction at higher orders of the Weinberg Program and is absent at leading order.

space, it is defined as,

$$V(r) = \begin{cases} A & r < R \\ 0 & r > R \end{cases}, \tag{136}$$

which can be transformed into,

$$V(q) = \frac{4\pi A}{q}(-\frac{1}{q}Rcos(Rq) + \frac{1}{q^2}sin(Rq)). \tag{137}$$

In it, we assume $R = 1fm$. As a result, we are left with one free parameter, and we try to fix the scattering length to it. It results in binding energy $B = 1.472MeV$, and compositeness is calculated as $\breve{X} = 1.046$.

We further include a repulsive core in addition to this attractive well. In this case, we have to take $r_{\text{core}} = 1fm$, then $r_{\text{attraction}}$ can take a larger range so that we can fix the scattering properties from the experiment. We have a free parameter, but we failed to reproduce the deuteron property and reasonable $\breve{X}$ at the same time.

6. Coupled Channels

We are able to generalize the LEC model into two channels. As next-to-next-to-leading order (NNLO), the interaction can be written as,[41]

$$V(q^2) = \begin{pmatrix} c_0 + c_1q^2 + c_2q^4 & c_{SD,1}q^2 + c_{SD,2}q^4 \\ c_{SD,1}q^2 + c_{SD,2}q^4 & c_{DD}q^4 \end{pmatrix}. \tag{138}$$

where each c is a free parameter we can fix the physical observations with. To avoid generating elementariness in crossing channels, we introduce a transformation for the Lippmann-Schwinger equation,

$$\overset{*}{\boldsymbol{T}} = \boldsymbol{P}^{-1}\boldsymbol{T}\boldsymbol{P}^{-1}, \tag{139}$$

$$\overset{*}{\boldsymbol{V}} = \boldsymbol{P}^{-1}\boldsymbol{V}\boldsymbol{P}^{-1}, \tag{140}$$

$$\overset{*}{\boldsymbol{G}} = \boldsymbol{P}\boldsymbol{G}\boldsymbol{P}, \tag{141}$$

where $\boldsymbol{P}$ is defined as

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}. \tag{142}$$

In this model, the number of free parameters is larger than the number of observables, so we have much freedom in the choice of freedom. We attempted to search for a set of parameters that can reproduce the deuteron property and have a reasonable compositeness $\breve{X}$, but we could not find such parameters. We, on the other hand, observed that to make the pole go deeper, compositeness usually increases.

7. Discussion

In this chapter, we have discussed several calculations we performed on the deuteron compositeness based on the surjective interpretation. We can gather the outcomes in Table 1.

Model	$\breve{X}$	Binding Energy/MeV
c_0	1.000	1.477
$c_0 + c_1 q^2$	1.479	2.232
$c_0 + c_1 q^2 + c_2 q^4$	1.727	2.423
$c_0 + OPEP$	1.870	2.200(fixed)
$c_0 + c_1 q^2 + OPEP$	1.427	2.200(fixed)
D channel with c_0	1.133	2.200(fixed)

TABLE 1. The deuteron compositeness calculated under the surjective interpretation.

In this table, we only included a part of the numerical outcomes. Some of the models are overcomplete, where we have more free parameters than needed, and only an example is displayed.

In these attempts, none of them can reproduce the deuteron property and a reasonable compositeness at the same time. On the contrary, we observed that, in the same model, increasing the binding energy almost always results in a larger compositeness. If the model starts from $\breve{X} = 1$, we always have $\breve{X} > 1$ at the deuteron property we want to achieve. As a result, we start to suspect that this outcome may actually be a property of the formalism instead of the absence of a good interaction model.

CHAPTER 6

Problem with the Formalism

In the previous chapter, we performed some numerical calculations and started to suspect that the unphysical compositeness is a result of the formalism instead of the interaction model. In this chapter, we will make a formal statement of this problem from a theoretical perspective.

This statement is made through proof by contradiction. We will first discuss the positivity of elementariness Z as defined in the theory. Then, we will perform a perturbative calculation of compositeness $\breve{X}$ under the surjective interpretation, which indicates that compositeness $\breve{X}$ increases with the increase of binding energy.

1. Positivity of Elementariness

In this section, we will prove that elementariness Z is positive by definition, no matter whether the interaction originates from a one-body, two-body, or three-body state. We would like to note that, in the current formalism, the state should be transformed into interaction so that the elementariness can be calculated. As a result, the methodology we choose is to calculate the elementariness by the interaction generated by states.

1.1. One-body State. The one-body state is an elementary state by definition. The interaction caused by including this state into the integrated potential $v^{\rm int}$ is shown in Eq.(66). We may notice that the factor $(g_{(0)})^2$ is fixed as positive. As a result, the elementariness calculated through Eq.(76) will always be positive. This is a natural outcome because the state $|\psi_0\rangle$ is a part of the Hilbert space, so that the norm of this state shall be positive.

1.2. Two-body state. To perform the calculation on the two-body state, we shall recall the multi-channel formalism of compositeness, especially Eq.(115). We may consider a scenario where a scattering channel N is to be reduced into the integrated potential of channel 0. We assume that v_{0N} and v_{NN} both are constant. We have,

$$\begin{aligned}\frac{\partial(w_{00}-v_{00})}{\partial E} &= v_{0N}^2\frac{\partial}{\partial E}\frac{G_N(E)}{1-v_{NN}G_N(E)}\\ &= v_{0N}^2\frac{1}{(1-v_{NN}G_N)^2}\frac{\partial G_N}{\partial E}.\end{aligned} \tag{143}$$

We can find that the only part that we have not discussed about the sign is $\frac{\partial G_N}{\partial E}$. From the optical theorem, we know that the loop function is limited and $\frac{\partial G_N}{\partial E}$ has to take a negative sign. Alternatively, we can calculate its sign from the mathematical from of the loop function,

$$G(E) = -\int_{s_+}^{\infty} ds' \frac{\rho(s')}{(s'-E^2)}, \tag{144}$$

$$G'(E) = -\int_{s_+}^{\infty} ds' \frac{\rho(s')}{(s'-E^2)^2}(2E) < 0, \tag{145}$$

where $\rho(s)$ is the phase space, and G is a real function below the scattering threshold.

As a result, $\frac{\partial(w_{00}-v_{00})}{\partial E}$ is negative, and elementariness Z will be positive, regardless of the interaction that is used in the model.

1.3. Three-body State. Similar to the two-body state scenario, Eq.(143) is still satisfied for three-body states, but the form of the loop function G is different. Physically, we may think of the one pion exchange procedure as a $NN\pi$ state, and this state is to be introduced into the NN scattering channel as an integrated interaction. In this subsection, we will perform a non-relativistic calculation, and the mass of the particles will be excluded from the energy. According to Ref.[48], we may introduce three-body reduced mass as,

$$\mu_{NN} = \frac{m_N}{2}, \tag{146}$$

$$\mu_{\pi} = \frac{m_{\pi}(m_N+m_N)}{m_N+m_N+m_{\pi}} = \frac{2m_{\pi}m_N}{2m_N+m_{\pi}}, \tag{147}$$

where m_N is the mass of necleon and m_{π} is mass of pion. With the reduced mass, we can write the three-body loop function as,

$$\begin{aligned} G(E) &= \int \frac{d^3p_{\pi}}{(2\pi)^3}\frac{d^3q_{NN}}{(2\pi)^3}\frac{1}{E+i\epsilon-\frac{p_{\pi}^2}{2\mu_{\pi}}-\frac{q_{NN}^2}{2\mu_{NN}}} \\ &= \int \frac{d^3p_{\pi}}{(2\pi)^3}[-\frac{2\mu_{NN}\Lambda_{NN}}{2\pi^2}-\frac{2i\mu_{NN}}{4\pi}\sqrt{2\mu_{NN}(E-\frac{p_{\pi}^2}{2\mu_{\pi}})}], \end{aligned} \tag{148}$$

where Λ_{NN} is the cutoff scale when calculating the two-nucleon loop function G_{NN}, and p_{π} is the momentum of the pion in the three-body center of mass frame. In this calculation, we have utilized the form of the loop function of Eq.(129) in Ref.[41]. However, we added a constant term to it to make it have a similar form as a more proper loop function and avoid generating the problem in sign convention. Regardless, since it is a constant term, it will not influence the calculation of $\frac{\partial G}{\partial E}$. Further, since the loop function is largely constrained by the optical theorem, we do not lose generality by taking a specific form of loop function.

We will only discuss the derivative of Eq.(148) instead of making a full evaluation,

$$\frac{dG}{dE} = -\frac{\mu_{NN}^{3/2}}{8\sqrt{2}\pi^2}\int_0^{\infty} dp_{\pi}p_{\pi}^2\frac{1}{\sqrt{-E+\frac{p_{\pi}^2}{2\mu_{\pi}}}}. \tag{149}$$

In our energy of interest, which is below the threshold, $E<0$, so we naturally have $\frac{dG}{dE}$ in the three-body state case.

1.4. Discussion. We have proved that the 1,2,3-body states, when introduced into the scattering process as a part of the integrated interaction, will not generate negative elementariness Z. In other words, we have proved the positivity of elementariness Z. This outcome is in close relationship with the nature of the Hilbert space as an inner product space. Even if the states are reduced, they shall still have a positive norm, so that the elementariness generated by them should not be negative. As a result, we normally expect that the evaluation $\breve{Z}$ under the surjective

interpretation should also be positive. However, we will give a calculation that is not the case in perturbative calculation in the next section.

2. Negative Elementariness Outcome with the Surjective Interpretation

From the numerical outcome in chapter 5, we are motivated to perform a more theoretical calculation on the value of compositeness $\breve{X}$ based on the surjective interpretation.

Let us consider a single channel scenario, where the integrated potential v_0 is set as a constant. We assume it has a bound state with binding energy M_0, and the compositeness will be calculated as $\breve{X}_0 = 1$ according to Eq.(124) and (125). We then perform perturbation to it, adding an energy-dependent part $\tilde{v}$, so that the full interaction becomes,

$$v = v_0 + \lambda\tilde{v}, \tag{150}$$

where λ is the perturbation order factor. The binding energy and compositeness will be changed accordingly. Formally, we can write

$$M = M_0 + \lambda M_1 + \lambda^2 M_2 + \dots, \tag{151}$$

$$\breve{X} = \breve{X}_0 + \lambda\breve{X}_1 + \lambda^2\breve{X}_2 + \dots, \tag{152}$$

and our goal is to calculate $\breve{X}_1$ in connection to M_1 to justify our guess that the surjective interpretation causes the outcome that increasing the binding energy also makes compositeness $\breve{X}$ increase.

For the simplicity of notation, we would like to define a convention that is only effective in this chapter. For any function F,

$$F' := \frac{\partial F}{\partial E}|_{E=M_0}, \tag{153}$$

$$\dot{F} := \frac{\partial F}{\partial E}|_{E=M}. \tag{154}$$

The same goes for higher order derivatives, that apostrophe means evaluated at $E = M_0$ and dot means evaluated at $E = M$.

2.1. Non-perturbative Calculations as Preparation. We will first do some preparation work before moving to the main part of the calculation. The T-matrix elements before and after perturbation can be written as,

$$t_0 = \frac{1}{1 - v_0 G(E)} v_0, \tag{155}$$

$$t = \frac{1}{1 - v(E)G(E)} v(E), \tag{156}$$

where t_0 is the T-matrix element before perturbation, t is the T-matrix element after perturbation, and $G(E)$ is the Green's function of H_0 whose form was discussed in Eq.(70). From the setting of the problem, we know that they have a pole as $E = M_0$ and $E = M$ respectively. Without losing generality, we can express them in the form of,

$$t_0 \equiv \frac{N_0(E)}{E - M_0}, \tag{157}$$

$$t \equiv \frac{N(E)}{E - M}. \tag{158}$$

For any form of t_0 and t, we always have a formal expression for $N_0(E)$ and $N(E)$ which are the numerators of the T-matrix. To connect t_0 and t, we can introduce a self-energy term $\Sigma(E)$,

$$t = \frac{N_0(E)}{E - M_0 + \Sigma(E)}, \tag{159}$$

which always have a formal solution for determined t and $N_0(E)$. By combining Eqs.(158)(159), we have,

$$N(E) = N_0(E)\frac{E - M}{E - M_0 + \Sigma(E)}, \tag{160}$$

in this way, t and t_0 are connected through the relation of $N(E)$ and $N_0(E)$.

However, the self-energy term $\Sigma(E)$ does not have a more direct expression in this connection, so we would like to give one. By combining Eq.(155) with Eq.(157), Eq.(156) with Eq.(159), we can obtain

$$E - M_0 = N_0(E)(\frac{1}{v_0} - G(E)), \tag{161}$$

$$E - M_0 + \Sigma(E) = N_0(E)(\frac{1}{v} - G(E)). \tag{162}$$

We can combine them and obtain a expression for $\Sigma(E)$,

$$\Sigma(E) = N_0(E)(\frac{1}{v_0 + \tilde{v}(E)} - \frac{1}{v_0}). \tag{163}$$

For the convenience of later calculation, we would like to pay special attention to $N(E)$, as we can find that both the numerator and denominator are zero at $E = M$. We can utilize L'Hospital's rule and obtain,

$$\begin{aligned} N(M) &= \lim_{E\to M} \frac{(E - M)N_0(E)}{E - M_0 + \Sigma(E)} \\ &= \lim_{E\to M} \frac{N_0(E) + (E - M)\dot{N}_0(E)}{1 + \dot{\Sigma}(E)} \\ &= \frac{N_0(M)}{1 + \dot{\Sigma}(E)}. \end{aligned} \tag{164}$$

In the same manner, we would like to calculate $N_0(M_0)$ and N_0'. From Eq.(157), and utilizing L'Hospital's rule, we have,

$$N_0(M_0) = \lim_{E\to M_0} \frac{E - M_0}{\frac{1}{v_0} - G(E)} = -\frac{1}{G'}. \tag{165}$$

Still from Eq.(157), we can calculate the derivative of $N_0(E)$,

$$\begin{aligned} \frac{\partial N_0(E)}{\partial E} &= \frac{(\frac{1}{v_0} - G) - (E - M_0)(-\frac{\partial G}{\partial E})}{(\frac{1}{v_0} - G)^2} \\ &= \frac{1 + N_0(E)\frac{\partial G}{\partial E}}{\frac{1}{v_0} - G}. \end{aligned} \tag{166}$$

When evaluated at $E = M_0$, we have to utilize L'Hospital's rule again,

$$N_0' = \lim_{E\to M_0} \frac{\frac{\partial N_0(E)}{\partial E}\frac{\partial G}{\partial E} + N_0(E)\frac{\partial^2 G}{\partial E^2}}{-\frac{\partial G}{\partial E}} = -N_0' - N_0(M_0)\frac{G''}{G'}. \tag{167}$$

From it, we can connect the left hand side and right hand side, which gives us,

$$N_0' = -\frac{1}{2}N_0(M_0)\frac{G''}{G'}. \tag{168}$$

2.2. Perturbative Calculation. The perturbative calculation here will only be given at the first order. However, some of the calculations require us to perform up to the second order. As the first step of the perturbative calculation, we will start by calculating the modification on the bound state energy. Consider the bound state condition for both v_0 and v as in Eq.(69),

$$\frac{1}{v_0} - G(M_0) = 0, \tag{169}$$

$$\frac{1}{v(M)} - G(M) = 0. \tag{170}$$

We finally need to expand Eq.(170), and we will expand $\frac{1}{v(M)}$ and $G(M)$ separately. As a preparation we first give an expansion of $\tilde{v}(M)$

$$\begin{aligned}\tilde{v}(M) &\simeq \tilde{v}(M_0) + (M - M_0)\tilde{v}' + \frac{1}{2}(M - M_0)^2\tilde{v}'' \\ &\simeq \tilde{v}(M_0) + \lambda M_1\tilde{v}' + \lambda^2(M_2\tilde{v}' + \frac{1}{2}M_1^2\tilde{v}'').\end{aligned} \tag{171}$$

Based on it, we can calculate $\frac{1}{v(M)}$ up to the second order,

$$\begin{aligned}\frac{1}{v_0 + \lambda\tilde{v}(M)} &\simeq \frac{1}{v_0}[1 - \lambda\frac{\tilde{v}(M)}{v_0} + \lambda^2(\frac{\tilde{v}(M)}{v_0})^2] \\ &\simeq \frac{1}{v_0}[1 - \lambda\frac{\tilde{v}(M_0)}{v_0} + \lambda^2[-\frac{M_1\tilde{v}'}{v_0} + (\frac{\tilde{v}(M_0)}{v_0})^2]]\end{aligned} \tag{172}$$

The remaining part we need to calculate is $G(M)$,

$$\begin{aligned}G(M) &\simeq G(M_0) + (M - M_0)G' + \frac{1}{2}(M - M_0)^2G'' \\ &\simeq G(M_0) + \lambda M_1G' + \lambda^2(M_2G' + \frac{1}{2}M_1^2G'')\end{aligned} \tag{173}$$

Gathering Eqs.(171)(172)(173), we can compare each order of λ between Eq.(169) and Eq.(170),

$$\frac{1}{v_0} = G(M_0), \tag{174}$$

$$-\frac{\tilde{v}(M_0)}{v_0^2} = M_1G', \tag{175}$$

$$-\frac{M_1\tilde{v}'}{v_0^2} + \frac{\tilde{v}^2(M_0)}{v_0^3} = M_2G' + \frac{1}{2}M_1^2G''. \tag{176}$$

As a result, we can obtain the modification to bound state energy up to the second order,

$$M_1 = -\frac{\tilde{v}(M_0)}{v_0^2G'}, \tag{177}$$

$$M_2 = -\frac{M_1\tilde{v}'}{v_0^2G'} + \frac{\tilde{v}^2(M_0)}{v_0^3G'} - \frac{1}{2}M_1^2G'' = (\frac{\tilde{v}'}{\tilde{v}(M_0)} + v_0G' - \frac{G''}{2G'})M_1^2. \tag{178}$$

We may look at the calculation from another prospect. The calculation of compositeness based on the surjective interpretation in Eq.(124) can be expressed in a more appropriate manner for this calculation,

$$\breve{X} = -N(M)\dot{G}. \tag{179}$$

Inserting Eq.(164) into Eq.(179), $\breve{X}$ can be calculated up to the first order as,

$$\breve{X} \simeq -N_0(M)\frac{1}{1+\dot{\Sigma}}G'(1+\lambda\frac{G''}{G'}M_1). \tag{180}$$

From it, we find that there are two parts, $N_0(M)$ and $(1+\dot{\Sigma})^{-1}$, yet to be calculated. They both need to be calculated perturbatively, and we will tackle them separately.

First, we will calculate $N_0(M)$ up to the first order. From Eq.(157),

$$\begin{aligned}
N_0(M) &= \frac{M-M_0}{\frac{1}{v_0}-G(M)} = \frac{M-M_0}{\frac{1}{v_0}-\frac{1}{v(M)}} \\
&\simeq \frac{\lambda M_1+\lambda^2 M_2}{\frac{1}{v_0}-\frac{1}{v_0}[1-\lambda\frac{\tilde{v}(M)}{v_0}+\lambda^2(\frac{\tilde{v}(M)}{v_0})^2]} \\
&\simeq \frac{M_1+\lambda M_2}{\frac{1}{v_0}[\frac{\tilde{v}(M)}{v_0}-\lambda(\frac{\tilde{v}(M)}{v_0})^2]} \\
&\simeq \frac{M_1+\lambda M_2}{\frac{\tilde{v}(M)}{v_0^2}}[1+\lambda(\frac{\tilde{v}(M)}{v_0})] \\
&\simeq \frac{M_1+\lambda M_2}{\tilde{v}(M_0)+\lambda M_1\tilde{v}'}v_0^2[1+\lambda(\frac{\tilde{v}(M)}{v_0})] \\
&\simeq v_0^2 M_1(1+\lambda\frac{M_2}{M_1})\frac{1}{\tilde{v}(M_0)}(1-\lambda\frac{M_1\tilde{v}'}{\tilde{v}(M_0)})[1+\lambda(\frac{\tilde{v}(M_0)}{v_0})] \\
&\simeq \frac{v_0^2 M_1}{\tilde{v}(M_0)}[1+\lambda(\frac{M_2}{M_1}-\frac{M_1\tilde{v}'}{\tilde{v}(M_0)}+\frac{\tilde{v}(M_0)}{v_0})].
\end{aligned} \tag{181}$$

By utilizing Eqs.(177)(178)(165), we can write,

$$\begin{aligned}
N_0(M) &= N_0(M_0)[1+\lambda(G'v_0M_1-\frac{1}{2}M_1\frac{G''}{G'}+\frac{\tilde{v}(M_0)}{v_0})] \\
&= N_0(M_0)[1-\lambda\frac{G''}{2G'}M_1].
\end{aligned} \tag{182}$$

It is worth noting that, in this calculation, even though the outcome is at the first order, the second order has to be utilized, because the order parameter can cancel between the numerator and denominator.

Before calculating $(1+\dot{\Sigma})^{-1}$, we would like to calculate Σ up to the first order. Utilizing Eq.(163),

$$\begin{aligned}
\Sigma &\simeq N_0(E)[\frac{1}{v_0}(1-\lambda\frac{\tilde{v}(E)}{v_0})-\frac{1}{v_0}] \\
&= -\lambda N_0(E)\frac{\tilde{v}(E)}{v_0^2}.
\end{aligned} \tag{183}$$

As a result,

$$\begin{aligned}(1+\dot{\Sigma})^{-1} &\simeq 1-\dot{\Sigma} \\ &\simeq 1+\lambda\frac{\dot{N}_0\tilde{v}(M)+N_0(M)\dot{\tilde{v}}}{v_0^2} \\ &\simeq 1+\lambda\frac{N_0'\tilde{v}(M_0)+N_0(M_0)\tilde{v}'}{v_0^2} \\ &\simeq 1+\lambda\frac{N_0'\tilde{v}(M_0)+N_0(M_0)\tilde{v}'}{v_0^2}. \end{aligned} \tag{184}$$

Utilizing Eqs.(165)(168), we have,

$$\begin{aligned}(1+\dot{\Sigma})^{-1} &\simeq 1+\lambda N_0(M_0)\frac{-\frac{G''}{2G'}\tilde{v}(M_0)+\tilde{v}'}{v_0^2}. \\ &= 1+\lambda(-\frac{1}{G'})\frac{-\frac{G''}{2G'}\tilde{v}(M_0)+\tilde{v}'}{v_0^2}. \\ &= 1-\lambda(\frac{G''}{2G'}M_1+\frac{\tilde{v}'}{v_0^2G'}). \end{aligned} \tag{185}$$

We have performed all the preparations for the final calculation. By inserting Eqs.(182)(185) into Eq.(180), we will be able to calculate $\breve{X}$ at the first order,

$$\begin{aligned}\breve{X} &\simeq -N_0(M_0)G'[1-\lambda\frac{G''}{2G'}M_1][1-\lambda(\frac{G''}{2G'}M_1+\frac{\tilde{v}'}{v_0^2G'})][1+\lambda\frac{G''}{G'}M_1] \\ &\simeq -N_0(M_0)G'[1-\lambda\frac{G''}{2G'}M_1-\lambda(\frac{G''}{2G'}M_1+\frac{\tilde{v}'}{v_0^2G'})+\lambda\frac{G''}{G'}M_1] \\ &= -N_0(M_0)G'(1-\lambda\frac{\tilde{v}'}{v_0^2G'}). \end{aligned} \tag{186}$$

Consider the calculation of compositeness based on the surjective interpretation in Eq.(124) and the bound state energy modification as first order in Eq.(177), $\breve{X}$ can be expressed as,

$$\breve{X} \simeq \breve{X}_0(1+\lambda\frac{\tilde{v}'}{\tilde{v}(M_0)}M_1), \tag{187}$$

so that

$$\breve{X}_1 = \breve{X}_0\frac{\tilde{v}'}{\tilde{v}(M_0)}M_1. \tag{188}$$

2.3. Physical Significance. Recall the point we would like to make, which is we need to examine the sign of $\breve{X}_1$. $\breve{X}_0$ is positive by definition and related to the structure of the Hilbert space, and M_1 is related to how the pole energy is modified, which should be left untouched, so we would like to examine the sign of $\frac{\tilde{v}'}{\tilde{v}(M_0)}$.

In the regime where E is close to the scattering threshold M_{th}, we may have a look at the possible form of $\tilde{v}(E)$,

$$\tilde{v}(E) = \sum_{n=1}^{\infty} a_n(E-M_{th})^n, \tag{189}$$

where we have ignored the constant term. Even if such a term exists, it will alter the scattering length in the same way as discussed in chapter 5. In a real-world

calculation, both v_0 and v should have a realistic scattering length, so that $\tilde{v}$ should not change it, and a constant term is ruled out.

We may consider the derivative of $\tilde{v}(E)$ order by order, with each order expressed as $\tilde{v}_n(E)$,

$$\tilde{v}_n(E) = a_n(E - M_{th})^n, \tag{190}$$

so that,

$$\tilde{v}_n'(E) = na_n(E - M_{th})^{n-1}, \tag{191}$$

and

$$\frac{\tilde{v}_n'}{\tilde{v}(M_0)} = \frac{n}{M_0 - M_{th}}. \tag{192}$$

Since it is a bound state with $(M_0 - M_{th}) < 0$, we can find that, instead of $\frac{\tilde{v}'}{\tilde{v}(M_0)} < 0$ in a strict sense, but $\frac{\tilde{v}_n'}{\tilde{v}(M_0)_n} < 0$ order by order in an expansion with $(M_0 - M_{th})$.

It is possible that $\frac{\tilde{v}'}{\tilde{v}(M_0)}$ is positive even if $\frac{\tilde{v}_n'}{\tilde{v}(M_0)_n} < 0$ at any order. However, as long as the expansion in Eq.(189) is convergent, we should have $\frac{\partial\tilde{v}(E)}{\partial E}\frac{1}{\tilde{v}(E)} < 0$ when $(E - M_{th}) \to 0$ for a physically reasonable $\tilde{v}$. If we still have $\frac{\tilde{v}'}{\tilde{v}(M_0)} > 0$, then it means $\frac{\partial\tilde{v}(E)}{\partial E}\frac{1}{\tilde{v}(E)}$ inverted sign between M_0 and M_{th}. Assuming both $\tilde{v}$ and $\frac{\partial\tilde{v}(E)}{\partial E}$ are continuous, $\tilde{v}$ should have a maximum or minimum between M_0 and M_{th}. If we look at the bound state condition Eq.(69) which can also be expressed as $v - \frac{1}{G} = 0$, assuming $\frac{1}{G}$ variates slowly when compared with v, $v - \frac{1}{G}$ should also have a maximum or minimum between M_0 and M_{th}. As a result, $v - \frac{1}{G} = 0$ is likely to have two solutions, which generates one more bound state, making it not a physical choice of $\tilde{v}$.

As a result, we may conclude that $\frac{\tilde{v}'}{\tilde{v}(M_0)}$ is likely to be negative, and,

$$sign(\breve{X}_1) = -sign(M_1). \tag{193}$$

Physically, we can interpret this outcome as follows: If adding an energy-dependent modification $\tilde{v}$ to the interaction, which increases binding energy without altering the calculated value of scattering length, to a constant interaction v_0, it is likely the compositeness calculated from the surjective interpretation, $\breve{X}$, is enhanced from unity. As a result, $\breve{Z}$ will be negative.

3. Discussion

In this chapter, we have shown the positivity of Z, and the calculated negative $\breve{Z}$ with the surjective interpretation. As a result, we argue that the surjective interpretation may not be appropriate for the states in which the scattering length indicates a lower binding energy than the experiment, like in the deuteron. As a result, $\breve{X}$ and $\breve{Z}$ in Eqs.(124)(125) may not be appropriate alternatives to X and Z as in the original definition.

3.1. Complication of Phenomenology. The reason for this difficulty is that the knowledge of the true Hilbert space is not available to us. The only knowledge we have when performing the calculation of compositeness is the effective interaction. As a result, any state shall finally show up as a part of the effective interaction. In reality, the primary goal of the effective interaction is to reproduce the physical

observables into account, and the underlying Hilbert space structure is ignored, allowing the possibility of introducing a negative norm state.

The surjective interpretation is a very good way to interpret the interaction. It attributes the energy dependency to other states, which aligns with our understanding that the Hamiltonian should be energy-independent to be physical.

To intuitively understand this matter, let us consider a single channel model. Even though in Eq.(66), the sign of numerator of the second term is fixed, which subtlely enforces a positive norm state of $|\psi_0\rangle$, $v^{\rm int}$ does not have to take that form. From a purely phenomenological point of view, as long as the observables are reproduced, $v^{\rm int}$ may take a form as,

$$v^{\rm int}(E) = v_0 - \frac{g^2}{E - E_\psi}, \tag{194}$$

where v_0 and g are real constants. As a result, it will correspond to a negative norm state, and finally negative elementariness $\breve{Z}$ when calculated from the surjective interpretation. Even though it feels unnatural, we believe that introducing intrinsically energy-dependent interaction $v(E)$ is required to avoid negative elementariness in the deuteron, and $\breve{Z}$ can not be understood interchangeably with Z.

3.2. Origin of Intrinsic Energy Dependency. In the deuteron, at the leading order of the Weinberg program [41], the interaction only has a constant term and a one pion exchange potential(OPEP) term. Since the constant term can not generate elementariness, it is the OPEP term being interpreted as a negative norm state, and giving the negative elementariness. Further, the OPEP interaction is always attractive, which will generate negative elementariness according to our calculation, giving one more evidence for our argument. We may interpret this outcome in two ways.

3.2.1. *Quantum Field Theory.* Let us recall that in the Weinberg Program [41], where the interaction is derived from Chiral Perturbation Theory (ChPT), while the scattering process is dictated by the Lippmann-Schwinger equation, i.e. Quantum Mechanics (QM). It is possible that attempting to represent the influence of interactions like OPEP will be interpreted as a negative norm state, because QM resides in the Hilbert space, and the extra pion in QFT resides in the Fock space.

This argument can be supported by the outcome in Ref.[49]. The authors discussed the differences between a pion-ful theory and a pion-less effective theory. They have shown that a pion-less effective theory will contain an energy-dependent interaction, which provides a positive effective range. This interaction will be interpreted as a negative norm state during the calculation of compositeness and elementariness with the surjective interpretation. Besides, this energy-dependent interaction in pion-less effective theory is caused by integrating out the pion degree of freedom from a pion-ful theory. As a result, we can argue that the existence of negative elementariness as well as a negative norm state may be a consequence of defining compositeness based on QM.

On the other hand, if we attempt to define compositeness in a QFT manner, we will encounter the problem that the free states are not really free, but instead dressed, making it relatively difficult to make a sound definition of compositeness. This problem is also discussed in Ref.[21] which argues that a satisfactory definition of compositeness based on QFT cannot be achieved.

3.2.2. *Effective Range Model.* Regardless of the difference in research area, a similar outcome was reported in Ref.[50]. Under a local Hamiltonian with the Effective Range Model, it is reported that as long as the effective range is positive, we have to include negative-norm states in the space to reproduce the physical observables. It is the same scenario in our research about the deuteron, and negative norm states (or negative elementariness) show up for the same reason.

In nuclear and hadron physics, there is also phenomenological study on NN scattering that reproduced the phase shift by including a negative norm state.[51]

In a phenomenological calculation, it may be acceptable to incorporate negative norm states to cater to physical observations. However, since the study of compositeness is strongly correlated to the structure of the Hilbert space, we believe it cannot be a solution. Further, we understand compositeness and elementariness in a probability way, and accepting negative elementariness will necessarily force us to find an alternative interpretation of the quantities, which will finally diminish the motivation of the research of compositeness as a whole.

Further, since the weak-binding limit in Refs.[5, 9] is closely related to the effective range model, we suspect that the calculation based on weak-binding limit may not be appropriate in certain cases, including the deuteron in the same manner.

CHAPTER 7

Possible Solution: Interactioness

In this chapter, we will give a straightforward expansion to the theory by accepting intrinsic energy dependency in interaction $v(E)$.

1. Loss of Model-Independency

The side effect of rejecting the surjective interaction and accepting intrinsically energy-dependent $v(E)$ is the loss of a model-independent way of calculation.

As can be seen from Eqs.(76)(125),

$$Z = -|c|^2[G\frac{d(v^{\text{int}} - v)}{dE}G]_{E=M_\Psi} \neq -|g|^2[G\frac{dv^{\text{int}}}{dE}G]_{E=M_\Psi} = \breve{Z}, \tag{195}$$

the calculation of Z through $\breve{Z}$ is no longer acceptable, and there is inevitably arbitrarity in separating between X and Z in Eq.(77). As $c = g$ is no longer available, and c is not an observable, even if we have v^{int} from the experiment, we still have to separate v from v^{int} to calculate c as well as X and Z.

On the other hand, we would like to note that the normalization from the Lippmann-Schwinger equation (122) is not influenced.

2. Interactioness

As a straightforward extension to the calculation based on the surjective interpretation, we would observe the form of Eqs.(77)(122). Due to the structure of the Hilbert space, we have,

$$[\frac{dG}{dE}]_{E=M_\Psi} < 0, \tag{196}$$

$$[\frac{d(v^{\text{int}} - v)}{dE}]_{E=M_\Psi} < 0, \tag{197}$$

and we may try to recover a form similar to Eq.(77) from Eq.(122). The new set of quantities are notated with an acute symbol.

$$\acute{X} = -|g|^2[\frac{dG}{dE}]_{E=M_\Psi} = \breve{X}, \tag{198}$$

$$\acute{Y} = -|g|^2[\frac{dv}{dE}]_{E=M_\Psi}, \tag{199}$$

$$\acute{Z} = -|g|^2[\frac{d(v^{\text{int}} - v)}{dE}]_{E=M_\Psi}, \tag{200}$$

where $\acute{Y}$ is called interactioness. From Eq.(122), we can find that they are subject to normalization

$$\acute{X} + \acute{Y} + \acute{Z} = 1. \tag{201}$$

Further, both $\acute{X}$ and $\acute{Z}$ are positive, while it is possible that $\acute{Y}$ is negative. In this way, we can retain the physical meaning of both $\acute{X}$ and $\acute{Z}$.

This set of quantities is chosen to have a strong connection with the original formalism. Even though we no longer assert $c = g$, we can find that $\acute{X} = \breve{X}$ and $\acute{Z} = \frac{|g|^2}{|c|^2} Z$ which retains most of the physical meaning as well as the positivity of elementariness. Finally, if $[\frac{dv}{dE}]_{E=M_\Psi} \to 0$, this set of quantities can smoothly connect to the calculations by surjective interpretation $\breve{X}$ and $\breve{Z}$.

From here, we may take the word "interpretation" into a more generalized form. It is a way of separating v from v^{int}, or how we view the model space. The surjective interpretation is one of the infinite number of interpretations, where no energy dependency is interpreted as a part of v. The other extreme of the interpretation is that any energy dependency is intrinsic, then $\acute{Z}$ is always zero.

As we can see, this set of quantities still requires us to have a full understanding of the underlying interaction, and we are relatively free when choosing among the interpretations. This is not always possible. For example, the interactions in the exotic hadrons are not clear, even though compositeness is mostly used upon them. Besides, this can introduce some arbitrarity and diminish the meaning of this calculation, i.e. one can always find an interpretation that gives whatever value of compositeness they want.

3. Deuteron Revisited

Practically, one may want to attribute interactioness $\acute{Y}$ as a part of the compositeness. The deuteron can be an example of it. When choosing the leading order Chiral Perturbation Theory (ChPT) interaction, there is only a constant term and a one pion exchange potential (OPEP) term in the interaction. From the calculation in chapter 5, we have calculated $\breve{X} = 1.870$ in this model. As a result, we can conclude that,

$$\acute{X} = 1.870, \tag{202}$$

$$\acute{Y} = -0.870, \tag{203}$$

$$\acute{Z} = 0. \tag{204}$$

By taking this way of attributing, $\acute{X} + \acute{Y}$ is unity, so that the deuteron is a composite particle.

If we think of the case of the deuteron physically, OPEP should be considered as contributing to compositeness. It is an attractive interaction mediated by the exchange of a pion, and both of the nucleons can be identified separately. An extreme case of this is the Coulomb force, where the photon is exchanged. Consider two particles only interacting through Coulomb force, like the proton and electron in the hydrogen atom, since no strong interaction is taking place, we shall finally consider this state as fully composite. The same shall go for OPEP interaction, and we may even postulate that a single particle exchange process should be considered as a part of the interactioness, which will finally be attributed to compositeness as long as the particle exchanged has relatively low energy.

This discussion can be generalized from the deuteron to other particles with $\breve{X} > 1$. In chapter 6, we have found that $\breve{X} > 1$ can happen in some particles, while $\acute{X}$ and $\acute{Z}$ are always positive for their physical meaning. As a result, we are left with the only choice by the normalization to make $\acute{Y}$ negative, regardless of how the details of the model is chosen. This will lead us to a conclusion that these particles,

like the deuteron, possibly is related with an attractive particle exchange interaction, and shall be considered a composite particle. This aligns with the discussion in Ref.[31] which quotes that "the true token that the deuteron is composite is an effective range small and positive rather than large and negative" from S. Weinberg, as the positive effective range correlates to an attractive energy-dependent part in the interaction.

CHAPTER 8

Energy Scale Decomposition as Compositeness

In chapter 7, we discussed a possible solution by introducing interactioness Y. However, such a solution will require an exact knowledge of the underlying interaction model of the scattering process. This attribution may not be reliable enough in the calculation of exotic hadrons with effective theories. Besides, the power of compositeness largely resides in the existence of the weak-binding limit and a psuedo-model-independent calculation, which was lost in the solution with Y.

In this chapter, we attempt to give a new definition of compositeness and elementariness, which are limited between zero and one by definition.

This research is performed in collaboration with Prof. Daisuke Jido, Dr. Kotaro Murakami, and Prof. Makoto Oka (in alphabetical order by last name).

1. Dual Theory Approach

In this section, we will give a brief review of Refs.[4, 5]. We will call this approach the dual theory approach of formalizing compositeness.

1.1. Definitions. We will define two theories I and II. Theory I is described by a Hamiltonian H^{I}, which can be divided into the free part H_0^{I} and interaction V^{I}. Similarly, Theory II is described by H^{II}, and separated into H_0^{II} and V^{II},

$$H^{\mathrm{I}} = H_0^{\mathrm{I}} + V^{\mathrm{I}}, \tag{205}$$

$$H^{\mathrm{II}} = H_0^{\mathrm{II}} + V^{\mathrm{II}}. \tag{206}$$

H_0^{I} has only continuum scattering states $|E, n\rangle$ as its eigenstates, while H_0^{II} has one more discrete elementary source state $|0\rangle$ than H^{I}. Formally,

$$H_0^{\mathrm{I}} |E, n\rangle = E |E, n\rangle , \tag{207}$$

$$H_0^{\mathrm{II}} |E, n\rangle = E |E, n\rangle , \tag{208}$$

$$H_0^{\mathrm{II}} |0\rangle = E_0 |0\rangle , \tag{209}$$

where the states have normalizations,

$$\langle E', n'|E, n\rangle = \delta_{n',n}\delta(E' - E), \tag{210}$$

$$\langle 0|0\rangle = 1, \tag{211}$$

$$\langle E, n|0\rangle = 0. \tag{212}$$

In this way, the two theories will have different Hilbert space configurations, which can be explicitly written as

$$\mathcal{H}^{\mathrm{I}} = \{|E, n\rangle\}, \tag{213}$$

$$\mathcal{H}^{\mathrm{II}} = \{|E, n\rangle\} \oplus \{|0\rangle\}, \tag{214}$$

where $\mathcal{H}^{\mathrm{I}}$ is the Hilbert space of theory I, and $\mathcal{H}^{\mathrm{II}}$ is the Hilbert space of theory II.

Following the flow of the original paper, a mapping between these two theories is prescribed. This prescription is defined in a way that they finally shall yield the equivalent scattering properties at $|W| \ll |E_0|$, where W is the kinetic energy. The justification of this prescription will be discussed in subsection 1.2.

The prescription is not unique but rather depends on the choice of the composite source states $|\Gamma\rangle$ and $|\bar{\Gamma}\rangle$.[1] The state $|\Gamma\rangle$ and $|\bar{\Gamma}\rangle$ are linear combinations of $|E,n\rangle$ and $\langle E,n|$ respectively, so that they reside in Hilbert space $\mathcal{H}^{\mathrm{I}}$. We first define a reduced interaction V^R which acts on Hilbert space $\mathcal{H}^{\mathrm{I}}$,

$$V^R = V^{\mathrm{I}} - V^{\mathrm{I}} |\Gamma\rangle \langle\bar{\Gamma}| V^{\mathrm{I}}. \tag{215}$$

With the aid of V^R, we may give the prescription,

$$\langle E', n'|V^{\mathrm{II}}|E,n\rangle = \langle E', n'|V^R|E,n\rangle\,, \tag{216}$$

$$\langle E', n'|V^{\mathrm{II}}|0\rangle = (-E_0/N)^{1/2} \langle E', n'|V^R|\Gamma\rangle\,, \tag{217}$$

$$\langle 0|V^{\mathrm{II}}|E', n'\rangle = (-E_0/N)^{1/2} \langle\bar{\Gamma}|V^R|E,n\rangle\,, \tag{218}$$

$$\langle 0|V^{\mathrm{II}}|0\rangle = (-E_0/N) \langle\bar{\Gamma}|V^R|\Gamma\rangle\,, \tag{219}$$

where N is renormalization factor defined as,

$$N = 1 - \langle\bar{\Gamma}|V^{\mathrm{I}}|\Gamma\rangle\,. \tag{220}$$

We would like to emphasize that V^{I} and V^R are in Hilbert space $\mathcal{H}^{\mathrm{I}}$, while V^{II} is in Hilbert space $\mathcal{H}^{\mathrm{II}}$. The prescription is given in a fashion of matrix elements, which are scalars, instead of any matrix representation to avoid confusion. Regardless of this difference in Hilbert space, it is still possible to make a connection between these two through a substitution, performed matrix element by matrix element

$$V^{\mathrm{II}} \Rightarrow V^R, \tag{221}$$

$$|0\rangle \Rightarrow (-E_0/N)^{1/2} |\Gamma\rangle\,, \tag{222}$$

$$\langle 0| \Rightarrow (-E_0/N)^{1/2} \langle\bar{\Gamma}|\,. \tag{223}$$

This substitution relationship will be utilized in later calculations.

1.2. Equivalence Theorem. Physically, we would require the two theories, I and II, are equivalent under the scattering experiments with asymptotic free states $|E,n\rangle$ in the limit of $|E_0| \to \infty$. This subsection is devoted to the justification of this property.

The scattering properties are given by the T-matrix, which can be calculated through the Lippmann-Schwinger equation for both theories,

$$T^{\mathrm{I}}(W) = V^{\mathrm{I}} + V^{\mathrm{I}} \frac{1}{W - H_0^{\mathrm{I}}} T^{\mathrm{I}}(W), \tag{224}$$

$$T^{\mathrm{II}}(W) = V^{\mathrm{II}} + V^{\mathrm{II}} \frac{1}{W - H_0^{\mathrm{II}}} T^{\mathrm{II}}(W), \tag{225}$$

This T-matrices can be connected to S matrix through

$$S_{n'n} = \delta_{n',n} - 2\pi i \langle E, n'|T(E + i\epsilon)|E,n\rangle \tag{226}$$

[1]The states $|\Gamma\rangle$ and $|\bar{\Gamma}\rangle$ are called "bare vertices" in the original paper, but we decided to rename them so that the physical significance can manifest.

Formally, the equivalence can be expressed as

$$\langle E', n'|T^{\rm II}(W)|E, n\rangle \to \langle E', n'|T^{\rm I}(W)|E, n\rangle \tag{227}$$

at $|E_0| \to \infty$. We will calculate the T-matrix from both theories and make a comparison to justify this equivalence.

1.2.1. *Theory* I. Through the Schmidt method, the solution of the Lippmann-Schwinger equation of theory $H^{\rm I}$ can be written as,

$$T^{\rm I}(W) = T^R(W) + N^{-2}T^R(W)\,|\Gamma\rangle\,\Delta^{\rm I}(W)\,\langle\bar{\Gamma}|\,T^R(W), \tag{228}$$

$$T^R(W) = V^R + V^R\frac{1}{W - H_0^{\rm I}}T^R(W), \tag{229}$$

where the "propagator" is

$$\Delta^{\rm I}(W) = [1 - J^{\rm I}(W)]^{-1}, \tag{230}$$

$$J^{\rm I}(W) = N^{-2}\,\langle\bar{\Gamma}|V^R\frac{1}{W - H_0^{\rm I}}T^R(W)|\Gamma\rangle \tag{231}$$

$$= 1 - N^{-1} + N^{-2}\,\langle\bar{\Gamma}|T^R(W)|\Gamma\rangle\,. \tag{232}$$

For the convenience of later discussion, we also give an extended form of Eq.(229),

$$T^R(W) = V^R + \sum_n \int dE\, V^R\,|E, n\rangle\,\frac{1}{W - E}\,\langle E, n|\,T^R(W) \tag{233}$$

1.2.2. *Theory* II. In this theory, we may find a similar expression of the Lippmann-Schwinger equation as Eq.(228) by separating the elementary source state $|0\rangle$ from the continuum states $|E, n\rangle$. From Eq.(225), we may write,

$$T^{\rm II}(W) = T^P(W) + T^P(W)\,|0\rangle\,\Delta^{\rm II}(W)\,\langle 0|\,T^P(W), \tag{234}$$

$$T^P(W) = V^{\rm II} + \sum_n \int dE\, V^{\rm II}\,|E, n\rangle\,\frac{1}{W - E}\,\langle E, n|\,T^P(W), \tag{235}$$

where the complete propagator of state $|0\rangle$, $\Delta^{\rm II}$, is written as

$$\Delta^{\rm II}(W) = [W - E_0 - \Pi^{\rm II}]^{-1}, \tag{236}$$

$$\Pi^{\rm II} = \langle 0|T^P(W)|0\rangle\,. \tag{237}$$

Physically, the "proper" T-matrix, T^P, is the sum of graphs that do not arise from the exchange of $|0\rangle$ particle, i.e. from a single vertex or the exchange of continuum states. $\Pi^{\rm II}$ is the proper self-energy insertion.

1.2.3. *Comparison.* By comparing Eq.(233) and Eq.(235), we find that they take a very similar form. Utilizing the substitutions in Eqs.(221)-(223) matrix element by matrix element, we can make a connection between the matrix elements of $T^R(W)$ and $T^P(W)$,

$$\langle E', n'|T^P(W)|E, n\rangle = \langle E', n'|T^R(W)|E, n\rangle\,, \tag{238}$$

$$\langle 0|T^P(W)|E, n\rangle = (-E_0/N)^{1/2}\,\langle\bar{\Gamma}|T^R(W)|E, n\rangle\,, \tag{239}$$

$$\langle E', n'|T^P(W)|0\rangle = (-E_0/N)^{1/2}\,\langle E', n'|T^R(W)|\Gamma\rangle\,, \tag{240}$$

$$\langle 0|T^P(W)|0\rangle = (-E_0/N)\,\langle\bar{\Gamma}|T^R(W)|\Gamma\rangle\,. \tag{241}$$

To justify the proposition in Eq.(227), we will express $\langle E', n'|T^{\rm II}(W)|E, n\rangle$ in terms of the physical quantities in theory $H^{\rm I}$.

By inserting Eqs.(238)-(241) into Eq.(234), we have,

$$\langle E', n'|T^{\mathrm{II}}(W)|E, n\rangle = \langle E', n'|T^{R}(W)|E, n\rangle - (E_0/N)\Delta^{\mathrm{II}} \langle E', n'|T^{R}(W)|\Gamma\rangle \langle\bar{\Gamma}|T^{R}(W)|E, n\rangle . \quad (242)$$

By inserting Eqs.(241) into Eq.(237), and compare with Eq.(232), the proper self-energy insertion of theory H^{II} can be expressed as,

$$\Pi^{\mathrm{II}}(W) = -E_0[1 - N + NJ^{\mathrm{I}}(W)]. \quad (243)$$

As a result, with Eq.(236),

$$\Delta^{\mathrm{II}}(W) = (-E_0N)^{-1}[1 - (W/E_0N) - J^{\mathrm{I}}(W)]^{-1}. \quad (244)$$

Combining Eq.(242) and Eq.(244), we have,

$$\langle E', n'|T^{\mathrm{II}}(W)|E, n\rangle = \langle E', n'|T^{R}(W)|E, n\rangle + N^{-2}[1 - (W/E_0N) - J^{\mathrm{I}}(W)]^{-1} \times \langle E', n'|T^{R}(W)|\Gamma\rangle \langle\bar{\Gamma}|T^{R}(W)|E, n\rangle . \quad (245)$$

On the other hand, from Eqs.(228) and (230), the matrix elements of T-matrix in theory H^{I} can be expressed as,

$$\langle E', n'|T^{\mathrm{I}}(W)|E, n\rangle = \langle E', n'|T^{R}(W)|E, n\rangle + N^{-2}[1 - J^{\mathrm{I}}(W)]^{-1} \times \langle E', n'|T^{R}(W)|\Gamma\rangle \langle\bar{\Gamma}|T^{R}(W)|E, n\rangle . \quad (246)$$

By comparing Eq.(245) and Eq.(246), we can conclude that the scattering experiments of theory H^{I} and H^{II} with states $|E, n\rangle$ will give the same result as long as $|E_0| \to \infty$. We may also express this conclusion in another way that the two theories are equivalent under scattering experiments as long as $W \ll E_0$. As a result, adding an elementary source state $|0\rangle$ with a reasonably large self-energy $|E_0|$ under this prescription will not influence the low energy scattering properties. This discussion can also be generalized to any number of elementary source states.

1.3. Physical Significance. In this subsection, we will justify that the composite source state $|\Gamma\rangle$ is related to a composite particle, while the elementary source state $|0\rangle$ is related to an elementary one, so that this prescription can have physical significance in a real-world scenario where we attempt to find if a particle is composite or elementary.

1.3.1. *The Nature Chooses Theory* I. In the scenario where the state of interest is composite, i.e. the bound state is generated by V^I, which causes the Lippmann-Schwinger equation to diverge. In this scenario, we may assume the elementary source state $|0\rangle$ is irrelevant.

Recall the reduced interaction V^R discussed in subsection 1.2, we will use the same equation under a different meaning, dissecting the interaction V^I into two parts V^R and V^S,

$$V^{\mathrm{I}} = V^R + V^S, \quad (247)$$

$$V^S = V^{\mathrm{I}} |\Gamma\rangle \langle\bar{\Gamma}| V^{\mathrm{I}}, \quad (248)$$

where $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$ are chosen so that V^S is a good enough approximation of V^{I}. As a result, the bound state origins from Δ^{I}, and T^R do not contain the pole we are

interested in, because V^R is too weak to generate it. The details of this dissecting process are discussed in Ref.[52].

Consider that $\Delta^{\rm I}$ has a pole at $E=-B$ with a residue of η. At $W\to -B$, Eq.(228) yields,

$$T^{\rm I}(W)\to \frac{\eta N^{-2}T^R(-B)\left|\Gamma\right\rangle\left\langle\bar{\Gamma}\right|T^R(-B)}{W+B}. \tag{249}$$

On the other hand, Eq.(224) can have a formal solution,

$$T^{\rm I}=\frac{W-H_0^{\rm I}}{W-H^{\rm I}}V^{\rm I}. \tag{250}$$

We may define the bound state as $|B\rangle$, which is an eigenstate of $H^{\rm I}$,

$$H^{\rm I}\left|B\right\rangle=-B\left|B\right\rangle, \tag{251}$$

$$\langle B|B\rangle=1, \tag{252}$$

so that the bound state can be formally written as,

$$T^{\rm I}(W)\to \frac{[H_0^{\rm I}+B]\left|B\right\rangle\left\langle B\right|[H_0^{\rm I}+B]}{W+B}. \tag{253}$$

Through comparing Eq.(249) and Eq.(253), we can relate the bound state $|B\rangle$ and $\langle B|$ with state $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$,

$$|B\rangle=-\eta^{1/2}N^{-1}[H_0^{\rm I}+B]^{-1}T^R(-B)\left|\Gamma\right\rangle, \tag{254}$$

$$\langle B|=-\eta^{1/2}N^{-1}\left\langle\bar{\Gamma}\right|T^R(-B)[H_0^{\rm I}+B]^{-1}. \tag{255}$$

More explicitly, we can write wave function of the bound state as,

$$\langle E,n|B\rangle=-\frac{\eta^{1/2}N^{-1}}{E+B}\left\langle E,n\right|T^R(-B)\left|\Gamma\right\rangle. \tag{256}$$

We can see that with $\dfrac{1}{E+B}$ explicitly generating the pole, $|\Gamma\rangle$ is strongly related to $|B\rangle$ especially at $E\to -B$. Further, due to the completeness of Hilbert space $\mathcal{H}^{\rm I}$, the bound state is fully a composite state as expected,

$$1=\sum_n\int dE|\left\langle E,n|B\right\rangle|^2. \tag{257}$$

1.3.2. *The Nature Chooses Theory* II. In the scenario where the state of interest is elementary, i.e. V^I is too weak to generate any pole in T^P, as well as in T^R, which is equivalent to T^P in Hilbert space $\mathcal{H}^{\rm I}$. Instead, the inclusion of elementary source state $|0\rangle$ causes $\Delta^{\rm II}$ to diverge, and generating the state. From Eq.(236), we have,

$$B+E_0+\Pi^{\rm II}(-B)=0. \tag{258}$$

Consider that $\Delta^{\rm II}$ has a pole at $E=-B$ with a residue of Z. At $W\to -B$, Eq.(234) yiels,

$$T^{\rm II}(W)\to \frac{Z\,T^P(-B)\left|0\right\rangle\left\langle 0\right|T^P(-B)}{W+B}. \tag{259}$$

On the other hand, following the same procedure of getting Eq.(253), and utilizing the same definition of bound state (while $|B\rangle$ is defined as the bound state of

$H^{\rm II}$),

$$T^{\rm II}(W) \to \frac{[H_0^{\rm II}+B]\,|B\rangle\,\langle B|\,[H_0^{\rm II}+B]}{W+B}. \tag{260}$$

Comparing Eq.(249) and Eq.(253), we have

$$|B\rangle = -Z^{1/2}[H_0^{\rm II}+B]^{-1}T^P(-B)\,|\Gamma\rangle\,, \tag{261}$$

$$\langle B| = -Z^{1/2}\,\langle 0|\,T^P(-B)[H_0^{\rm II}+B]^{-1}. \tag{262}$$

More explicitly, we can write wave function of the bound state, but we need to consider the elementary state $|0\rangle$ in this scenario,

$$\langle E,n|B\rangle = -\frac{Z^{1/2}}{E+B}\,\langle E,n|\,T^P(-B)\,|0\rangle\,, \tag{263}$$

$$\langle 0|B\rangle = -\frac{Z^{1/2}}{E_0+B}\,\langle 0|\,T^P(-B)\,|0\rangle\,. \tag{264}$$

Applying Eq.(258) and Eq.(237) to Eq.(264),

$$\langle 0|B\rangle = Z^{1/2}, \tag{265}$$

Z is the same elementariness from previous chapters, which follows the same completeness condition,

$$1 = Z + \sum_n \int dE|\,\langle E,n|B\rangle\,|^2. \tag{266}$$

We would also like to point out the mathematical similarities that $|0\rangle$ and $|\Gamma\rangle$ bear by comparing among Eqs.(256)(263)(264).

1.4. Discussion. In this section, we would like to discuss what this approach tells us.

Weinberg constructed two theories, one with only scattering states $|E,n\rangle$ as eigenstates of the free Hamiltonian, and the other contains one more elementary state $|0\rangle$. These two theories can only be distinguished at high energy regions. The composite source states $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$ are related to the bound state if the nature chooses to generate the bound state through the interaction among $|E,n\rangle$. On the other hand, the elementary source state $|0\rangle$ is related to the bound state if the nature chooses to generate the bound state with the inclusion of $|0\rangle$.

In a practical scenario, where we try to distinguish between theory I and II, we may expect that the elementary source state, as well as elementariness, are mainly contributed from the high energy sector of the scattering. We need to note that, this is an expectation that reverses the order of reasoning from the approach of Weinberg and shall not be taken as a hard criterion. However, this is one of the major motivations for our new definition of compositeness.

Finally, we would like to discuss if elementariness and compositeness are observables. Even though the calculations of compositeness in Eq.(257) and Eq.(266) indicate that both of the quantities are observables, we may find that to acquire such a conclusion, we have to assume that the nature chooses a specific theory. However, solely from this calculation, it is impossible to certainly distinguish which theory it is, and thus they are not observables even though they both look like so.

2. New Definition of Compositeness

At a conceptual level, the study of compositeness is the attempt to distinguish whether a pole from scattering experiment $\langle E'n'|T|E,n\rangle$ is composite or elemental. When approaching this problem from the perspective of hadron physics, it finally will lead to the problem of "what is a particle", as every hadronic state is, at the very end, a composite state of quarks and gluons.

We may, however, disregard such notation in a way as if the incident states are the only states of physical existence, while the dynamics are encapsulated in the interaction potential. This approach utilizes the Lippmann-Schwinger to account for the dynamics, as is discussed in previous chapters. Even though we are able to avoid encountering the problem of "what is a particle", the problem of "what is the origin of the interaction" inevitably arises, as the states that do not exist in this approach have to manifest as interactions. The surjective interpretation is one way to interpret the origin of the interaction, which fails by relating to negative norm states.[27] On the other hand, the solution that includes a third quantity of interactioness discussed in chapter 7 has to invoke some interpretation of the interaction or even an arbitrary choice. Regardless of how educated the guess we make, we have to make attribution for each separate pole, and the inherent arbitrarity will eventually undermine the attempt as a whole.

As a result, we are motivated to give a new definition of compositeness from a different approach.

2.1. Motivations. Ref.[21] discussed what a satisfactory definition of compositeness is. The three necessary requirements are as follows,

Ontological Requirement 1. *For X to be a composite object made of A and B we need to be able to refer to A and B, in a well-defined way, whilst X is in existence.*

Ontological Requirement 2. *If A is made of B then B is not made of A - that is, compositeness is not reflexive.*

Ontological Requirement 3. *Reality cannot depend on arbitrary choices.*

It is argued that no satisfactory, exact (non-perturbative), and rigorous definition of compositeness exists compatible with QFT, while a non-relativistic quantum mechanical formalism is possible. However, we have found that the attempt to attribute the interaction has to invoke arbitrary choices of the interpretation, even though it automatically satisfies requirements 1 and 2. We believe it is not because of any philosophical statement, but just that attributing different parts of hadron interactions inherently requires QFT. For example, in the Weinberg program [41], an effective theory that utilizes the Lippmann-Schwinger equation so that it is quantum mechanical, the interaction kernel is determined by Chiral perturbation theory, which is an effective field theory. We believe that trying to fully answer the same question even in a fully quantum mechanical framework may still be difficult.

Further, we may take a look at the example when the authors of Ref.[21] argued that a quantum mechanical formalism of compositeness is possible. The example given is the hydrogen atom. However, we argue the interaction is elastic by definition, so that the compositeness can be easily discussed, or even predefined as one in the word elastic. From here, we may acquire some key elements required for the construction of a new definition of compositeness.

Physical Requirement 1. *Every state X bound by the elastic interaction of particles A and B is, or close to, fully composite, like those bound by the electromagnetic force.*

Physical Requirement 2. *Elastic interaction is characterized by a positive energy scale Λ as its upper limit.*

Physical Requirement 3. *The dynamics are formulated quantum mechanically, in a way of the Lippmann-Schwinger equation.*

These requirements are, as the Ontological requirements, necessary conditions for a satisfactory definition of compositeness. We may check if the composite states satisfying the physical requirements also satisfy the ontological requirements.

For a composite state of A and B under elastic interaction below Λ as prescribed in the physical requirements, the particle by definition satisfies ontological requirement 1, as the word "elastic" dictates. Further, as the dynamics are formulated quantum mechanically, there is no creation of any particle, so that ontological requirement 2 is satisfied. However, we may find the arbitrarity of energy scale Λ in physical requirement 2 violates ontological requirement 3.

We argue that this scale is physically meaningful, and is necessary in this theory. For example, Hagedorn temperature is an important scale, which indicates the upper limit of temperature (or energy) where hadronic matter can exist. We believe that it is unrealistic to totally remove arbitrarity, and overly utilizing skepticism will ultimately make any physical problem metaphysical. Further, we argue that this arbitrarity of Λ is weaker than the one required in the method of attributing interactions particle by particle, because we can determine a much lower number of scales than the number of bound states, and scale is a single variable instead of interactions being functions.

2.2. Definition. [2] Here, we give the formal definition of compositeness based on the decomposition with the energy scale Λ.

Consider the full Hamiltonian H, it can be separated in two ways. In the first way, a "free" Hamiltonian H_0 with eigenstates $|E, n\rangle$, and the interaction V which generates the bound state of interest in T-matrix T. In the second way, an isolated Hamiltonian H^γ which defines state, and the interactions between them, V^γ,

$$H = H_0 + V = H^\gamma + V^\gamma. \tag{267}$$

H^γ has eigenstates $|\epsilon, n\rangle$ which form a complete set in its Hilbert space, then the new compositeness χ and elementariness ζ are defined as,

$$1 = \chi + \zeta, \tag{268}$$

$$\chi = \int_0^\Lambda d\epsilon |\langle \epsilon, n|B\rangle|^2, \tag{269}$$

$$\zeta = \int_\Lambda^\infty d\epsilon |\langle \epsilon, n|B\rangle|^2. \tag{270}$$

[2]Due to the increasing number of variables, we would like to clarify our naming convention. The namings on the new definition are made on the Greek letter counterparts of the original definition. For example, letter e origins from ϵ, letter x origins from χ, and letter z origins from ζ.

The choice of H^γ shall be made in a way that the scattering properties of the free theory and the isolated theory shall have the same asymptotic behaviors at low energy,

$$(\frac{d\sigma}{d\Omega})^\gamma \sim (\frac{d\sigma}{d\Omega}), \text{when } W \to 0, \tag{271}$$

However, T^γ does not contain the bound state of interest.

We need to note that this calculation only assumes the existence of the scattering states $|E, n\rangle$ so that the calculation is performed as if theory I is taking place. As a result, we can have the normalization Eq.(268).

2.3. Physical Significance. In this subsection, the physical significance of this definition will be discussed.

2.3.1. *What is Separated?* First, we discuss the scenario where the nature chooses theory I, i.e. the bound state is generated by the interaction between $|E, n\rangle$. As mentioned above, to define H^γ, we need to remove the pole from the T-matrix. This process is physically the same as that described in Eq.(248). The isolated potential V^γ has the same meaning as V^S above, and we may relate them through,

$$H^\gamma = H_0^{\mathrm{I}} + V^R. \tag{272}$$

In this way, we essentially remove the inelastic part of the interaction, while the elastic part (or asymptotic part) V^R stays inside H^γ. We need to note that this separation is not an arbitrary one, even though there is a range of choices that can be made.

Second, we consider the scenario where the nature chooses theory II. Recall the substitution relationship between the two theories Eqs.(221)-(223), we argue that the mathematical aspects in Theory I still hold. However, it is the influence of the elementary state $|0\rangle$ being removed.

As a result, we conclude that it is the strongly interacting part that contributes the most to the bound state, no matter which theory the nature chooses, being defined as V^γ, and the part that is not strongly interacting inside defined as H^γ. We call the eigenstates of H^γ, $|\epsilon, n\rangle$, dressed states. We may interpret the states being dressed by the interaction with other particles only in the sense of QFT, but we still use this name because there is only elastic scattering happening inside H^γ (under the basis of $|E, n\rangle$), so the "identity" of the particles shall still remain.

2.3.2. *Physical Interpretations.* As the inelastic part of interaction, regardless if it originates from interaction or $|0\rangle$, is separated as V^γ, we would like to discuss how this new definition should be interpreted.

Since the elementary source state $|0\rangle$ is not observed, we are always limited inside theory I to perform the calculation. However, we argue that the difference between theory I and theory II is not important in hadron physics, so we can always interpret the high energy part as elementariness ζ. It is because the high energy deep inelastic scattering is inherently based on QCD, and describing such interactions through hadron may be inappropriate. We argue that, by definition, any interaction that cannot be interpreted in a hadronic way shall be considered as elementariness.

A possible objection is that we may also be cutting off some probability coming from elastic scattering. However, we argue that, with appropriate fitting, those contributions shall be minimal. Further, we at least can interpret this quantity as a lower bound of the "true" compositeness, so that if the calculation provides $\chi > 0.5$, we will be able to conclude that such particle is dominated by the scattering state.

2.3.3. *Why Integrate over* $|\epsilon, n\rangle$*?* Please recall ontological requirement 3. Even though we do not strictly eliminate every instance of arbitrarity (Λ is still inside the theory), separating H_0 from H may introduce too much arbitrarity, as the state and Hamiltonian are much harder to define than a scale Λ. We expect, by hiding H_0 from the formalism, the arbitrarity can be negated.

Besides, by doing so, the usefulness of this formalism can be extended much more freely into any other interaction, with just a change of scale Λ.

Finally, we expect that, if we were able to find a satisfiable choice of H^γ and V^γ, the "leakage" between χ and ζ shall be minimal.

3. Calculations

In this section, we will discuss how the definition can be performed in real particles, especially how the energy scale Λ is determined. An analytical discussion of 1D delta well potential will also be given.

3.1. Determine Λ. Up to now, we have only discussed the theoretical constructions of this new definition. We still need to find a way to determine Λ.

Practically, the energy scale Λ can be determined as the energy range where the scattering properties can be reproduced. Eq.(271) only indicates the low energy scattering properties shall be the same, while to perform real calculations, especially the numerical calculations, we need it to be more concrete. Formally,

$$(\frac{d\sigma}{d\Omega})^\gamma \sim (\frac{d\sigma}{d\Omega}), \text{when } W < \Lambda, \tag{273}$$

so that we do not need to worry about the definition of "low energy", and the energy scale Λ can also be determined in a real calculation.

3.2. Analytical Calculation of 1D Delta Well. This model has exactly one bound state, and the other eigenstates are linear combinations of plane waves. We can reverse the sign of the delta function, making it a delta function wall, and recover the same scattering properties.[3] Since we are able to recover the scattering properties of any given state, we are able to take $\Lambda \to +\infty$. Thus, we can conclude that $\chi \to 1^-$, and this state is a composite state.

This outcome is in accordance with the evaluation based on the surjective interpretation Eq.(124) when we have $v^{\text{int}} = const.$ The Fourier transformation of a delta function is constant in the momentum (or energy) space, and a constant potential will always give $\breve{X} = 1$.

[3]The phase of the reflected and transmitted part of the wave function are also reversed, but the scattering cross section remains the same, so they can not be distinguished experimentally.

CHAPTER 9

Summary and Discussion

In this thesis, we studied the formalism of compositeness from a theoretical approach. The motivation of this study is that, even though compositeness has been utilized broadly in exotic hadrons, different calculations have always been giving unrealistic outcomes for the deuteron since S. Weinberg provided the idea in Ref.[5]. Compositeness should normally be interpreted as probability, but it is calculated as larger than one in the deuteron, causing much difficulty in the consistency of the theory as a whole.

To approach this problem, we constructed the theoretical basis of an energy-dependent version of compositeness formalism, which can be reduced into the formalism in T. Sekihara's paper Ref.[9] by taking the surjective interpretation. In Ref.[9], the weak-binding limit is derived under the surjective interpretation, which gives the same form as Ref.[5]. As a result, we have good reason to believe that this line of calculation is ultimately based on the surjective interpretation.

With the numerical calculations of the deuteron compositeness performed on several models, we found that none of them can reproduce reasonable deuteron properties and have physical compositeness value at the same time. This can be viewed as an example that goes along with the line of the previous calculations, including Refs.[9, 11, 17] etc. Further, since the deuteron pole sits close to the scattering threshold, the interaction is largely bound by the scattering properties, so we believe it is more likely to be caused by the formalism rather than the lack of a good interaction potential.

We then move on to formally state this problem. Due to the structure of the Hilbert space, states shall always have positive norms, so elementariness should always be positive, i.e. compositeness shall not exceed unity for a bound state. On the other hand, the perturbative calculation of compositeness based on the surjective interpretation from the Lippmann-Schwinger equation shows that attractive energy-dependent interaction is likely to generate negative elementariness. As a result, we are able to conclude that the problem lies in the surjective interpretation.

The power of the surjective interpretation is that it promises a model-independent calculation of compositeness, and it allows us to ignore the details of the interaction model. With the weak-binding limit, one is able to calculate compositeness with only the above threshold scattering properties, giving it great usability in phenomenology. However, in this paper, we concluded that the surjective interpretation may not be a proper assumption to take for some particles, including the deuteron. As a result, when applying the surjective interpretation to exotic hadrons, one must be aware of the possibility that the surjective interpretation is not appropriate.

We give two possible solutions to this problem.

The first solution is to fully disregard the surjective interpretation and define interactioness $\acute{Y}$ to account for the explicit energy dependency. The advantage of this solution is that it bears close mathematical relationship with the calculation

under the surjective interpretation. On the other hand, it has the great drawback of being model-dependent at its core. Even though it may be fine for particles like the deuteron, being model-dependent limits its usage on exotic hadrons, as one may be able to have any compositeness value they wants. We revisited the deuteron with this approach and argued that exchanging a particle with reasonably low energy shall be considered as a part of compositeness.

As another solution, we proposed a new definition of compositeness. Instead of relying on the structure of the whole Hilbert space to limit compositeness and elementariness between zero and one, we assume only the existence of the scattering state on which the experiment is performed and a pole in the T-matrix. This, as we call it, is the energy scale decomposition definition of compositeness. This new definition is motivated by the dual theory approach in Ref.[4], the philosophical discussions in Ref.[21], and our understanding that the pion exchange process should be counted as a part of compositeness. We give one extreme case where this definition can give the same outcome as the surjective interpretation in a one-dimensional delta well.

This new definition is made based on the logic that compositeness is, at the end of the day, a metric to interpret the structure of possibly composite particles. Since compositeness is not understood in the same way as, for example, the binding energy which is directly observed, we shall be very careful when introducing arbitrarity, which can possibly kill the purpose if there is too much freedom. We also argue that any hadron is a composite particle of quarks and gluons, and at the same time a hydrogen atom can be viewed as elementary if viewed from a much larger length scale (or lower energy scale). As a result, we believe that compositeness shall be understood as a way of scale decomposition, and we choose energy scale decomposition to have a clearer connection with the Lippmann-Schwinger equation.

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