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著者(和文)	平田路和
Author(English)	Michikazu Hirata
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論文要旨

THESIS SUMMARY

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要旨（英文800語程度）
Thesis Summary（approx.800 English Words ）

Formalization of probability theory plays important roles as a foundation of formal verification for systems with probabilistic behaviors such as probabilistic programs, stochastic processes, and machine learning algorithms. Although basic probability theory has already been formalized, existing libraries still do not cover advanced theories used in some latest research.

This thesis aims to formalize advanced topics of probability theory in the interactive theorem prover Isabelle/HOL. Especially, we focus on the basis of higher-order probability theory. We formalize the three theories: standard Borel spaces, the Lévy-Prokhorov metric, and quasi-Borel spaces.

In Chapter 2, we review Isabelle/HOL and basic mathematics used in this thesis.

The topic of Chapter 3 is standard Borel spaces. Standard Borel spaces are a certain class of measurable spaces. Standard Borel spaces are defined on Polish spaces, which are a class of topological spaces. Those spaces are often used in applied probability theory and statistics areas, including the theory of quasi-Borel spaces, because they have good properties: any Polish space is embedded into a compact space and any standard Borel space is either a countable discrete space or isomorphic to the set of real numbers (Kuratowski’s theorem). In Chapter 3, we formalize Polish spaces and standard Borel spaces. We also prove the useful properties. We finally apply the Kuratowski’s theorem to prove the disintegration theorem, which ensures the existence of a conditional probability kernel.

The topic of Chapter 4 is the Lévy-Prokhorov metric. Although this theory is independent of higher-order probability theory, the formalization of the theory includes important results such as the Riesz representation theorem and Prokhorov’s theorem. The Lévy-Prokhorov metric is a mathematical tool to analyze asymptotic behaviors of distributions or measures in terms of weak convergence. Such analysis is one of the important aspects of probability theory and a foundation of statistics because the knowledge on asymptotic behaviors provides insights of what will be likely to happen when we collect large data. In Chapter 4, we formalize the Lévy-Prokhorov metric and related notions. We first formalize the weak convergence including the Portmanteau theorem, equivalent conditions of the weak convergence, and the topology of weak convergence. We then formalize the Lévy-Prokhorov metric. We also formalize Prokhorov’s theorem using the Lévy-Prokhorov metric. In order to formalize Prokhorov’s theorem, we also prove (a special case of) Alaoglu’s theorem and the Riesz representation theorem. We finally show that the measurable space of finite measures on a standard Borel space is a standard Borel space. The measurable space of measures on some measurable space is used in stochastic processes and semantics of probabilistic programs. The measurable space of measures is defined independently of metrics or topologies. We prove that the measurable space of finite measures is induced by the Lévy-Prokhorov metric. Consequently, we obtain that the measurable space of finite measures on a standard Borel space is a standard Borel space.

The topic of Chapter 5 is quasi-Borel spaces. The theory of quasi-Borel spaces is a new denotational model for higher-order probabilistic programs. In general, measurable spaces and measurable functions are used for the semantics of probabilistic programs. However, there are no function spaces of measurable spaces with desired properties. Thus, the semantics based on measure theory is not suitable for higher-order probabilistic programs. Quasi-Borel spaces enable us to denote higher-order probabilistic programs because function spaces always exist and the s-finite measure monad on quasi-Borel spaces is used to denote the type of probability distributions. Although the construction of the s-finite measure monad is non-trivial, no paper provides its detailed proofs. In addition, the definition of the monad varies among prior studies. Those situations make it difficult to use the theory and check the correctness of the research results. In Chapter 5, we formalize the notion of quasi-Borel spaces and the s-finite measure monad on it. This is the first formalization of the theory of quasi-Borel spaces, to our best knowledge. We also formalize s-finite kernels, which are used to denote first-order probabilistic programs, because the s-finite measure monad depends on s-finite kernels. In addition to formalizing the s-finite measure monad, we also implement qbs prover, a proof automation for showing that the term is a member of the quasi-Borel space. In a typical situation of formal proofs in measure theory, we need to check measurability of functions to apply equations or theorems because most of the theorems require that functions are measurable. Working with quasi-Borel spaces also faces similar situations. The qbs prover reduces the cost of proof by automatically solving this kind of side conditions.

In Chapter 6, we apply our formalization to verify probabilistic programs. We demonstrate four example probabilistic programs including the Monte Carlo approximation and the Gaussian mean learning algorithm. We prove that the distribution of the average of samples converges in probability to the expected value in the Monte Carlo method example. In the example of the Gaussian mean learning algorithm, we prove convergence and stability under changes of priors. We also formalize differential privacy using quasi-Borel spaces.

In Chapter 7, we conclude our work. We also mention future works.

In Appendix, we formalize the coproduct of measures.