論文 / 著書情報
Article / Book Information

題目(和文)	有効場の理論によるトポロジカル物質の解析
Title(English)	Effective Field Theory to Topological Materials
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出典(和文)	学位:博士(理学), 学位授与機関:東京科学大学, 報告番号:甲第207号, 授与年月日:2025年3月26日, 学位の種別:課程博士, 審査員:西田 祐介,村上 修一,古賀 昌久,石塚 大晃,打田 正輝
Citation(English)	Degree:Doctor (Science), Conferring organization: Institute of Science Tokyo, Report number:甲第207号, Conferred date:2025/3/26, Degree Type:Course doctor, Examiner:,,,,
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Effective Field Theory to Topological Materials

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February 13, 2025

Abstract

Topological materials, characterized by their intrinsic topological invariants, exhibit intriguing response properties. While theoretical studies employing model computations have successfully elucidated the response behavior of individual topological materials, a comprehensive and universal understanding of these properties remains elusive. To address this gap, we aim to explore and establish universal response properties of topological materials through the framework of the effective field theory. The effective field theory provides a powerful approach to simplifying complex many-body problems by focusing only on the relevant degrees of freedom at a suitable energy scale. Its principal advantage lies in its ability to capture universal characteristics that transcend the details of the materials. In this thesis, we examine the universal response properties of the quantum Hall systems and the Weyl semimetals as two prominent classes of topological materials. The quantum Hall system is a relatively simple example of a topological material. Despite its seemingly straightforward setup, it exhibits a variety of Hall responses, including the Hall viscosity. Although numerous studies have explored individual quantum Hall states, a comprehensive and unified framework remains elusive. On the other hand, the Weyl semimetals offer an excellent platform for realizing relativistic physics in the context of condensed matter systems since the low-energy physics of the Weyl semimetals is governed by massless electrons. In particular, the axial anomaly emerges and gives rise to intriguing response behaviors. Investigating these responses is valuable not only for enhancing our understanding of the Weyl semimetals but also for deepening insights into other massless electronic systems, such as the neutron stars, the quark-gluon plasmas, and so on.

In the first part of the thesis, we explore the quantum Hall systems with Galilean invariance. To investigate all responses, including energy and momentum currents, we introduce external fields conjugate to these currents by placing the system on a non-relativistic curved spacetime, described by the Newton-Cartan geometry. This spacetime features three metrics corresponding to external fields conjugate to energy current, momentum density, and stress tensor. Electrons on a Newton-Cartan spacetime exhibit Milne symmetry in addition to the symmetries under U(1) gauge and general coordinate transformations. We construct the effective action based on these symmetries, imposing Milne invariance by dressing the external fields. Using a power counting scheme where the electromagnetic fields are of order of unity, we investigate nonlinear responses. The resulting action, up to next-to-next-to-leading order in the derivative expansion, includes four terms, such as the dressed Chern-Simons and Wen-Zee terms. From this, we compute the local currents induced by electromagnetic fields and identify the coefficients of these terms as the Hall conductivity, Hall viscosity, energy density, and energy magnetization. The dressing of external fields leads to universal relations between responses, with two key results: one shows the Hall conductivity determining the longitudinal conductivity at nonzero frequency, while the other shows the Hall viscosity contributing to nonlinear electrothermal conductivity at nonzero wave number.

In the second part of the thesis, we examine the Weyl semimetals using the low-energy effective field theory of massless Dirac fermions coupled to an axial gauge field, which describes the separation of the Weyl nodes. We compute the charge current density in linear order in both vector and axial gauge fields. Regularization via the Pauli-Villars method, introducing a ghost field with infinite mass, allows for the proper definition of superficially divergent integrals. The resulting current density contains terms dependent on temperature and chemical potential as well as terms independent of these variables. The latter is just the Chern-Simons current, whose correct form is obtained owing to the regularization. In the static limit, the chiral magnetic current vanishes, which is consistent with the vanishing of the chiral magnetic effect in equilibrium. In the uniform limit, a dynamical chiral magnetic current emerges, with a coefficient 2/3 smaller in magnitude and opposite in sign compared to the chiral magnetic current. We further investigate the current density driven by a uniform but time-dependent magnetic field. Our analysis shows that the total transported charge is independent of both temperature and chemical potential. Additionally, we find that a pulsed magnetic field gives rise to a temporal Friedel oscillation, whose amplitude is determined by the temperature and the chemical potential.

Lastly, we compute the electromagnetic linear responses of the Weyl semimetals using chiral kinetic theory under the relaxation time approximation that ensures local charge conservation. The resulting current density, including the Chern-Simons current, agrees with the result of the low-energy field theory in the long wavelength and low frequency limit. We couple the current density with Maxwell's equation to investigate the dispersion relation of collective excitations. At the linear order in wave vector, the dispersion relation is determined by the anomalous Hall, chiral magnetic, and Ohmic conductivities. Importantly, one dispersion relation exhibits a positive imaginary part, suggesting exponential growing modes akin to the chiral plasma instability, with propagation oriented along or opposite to the direction of the Weyl node separation.

List of Publications

Publications relevant to this thesis

1. Tatsuya Amitani and Yusuke Nishida,
   "Universal nonlinear responses of quantum Hall systems with Galilean invariance,"
   Phys. Rev. B **110**, 195132 (2024).

2. Tatsuya Amitani and Yusuke Nishida,
   "Dynamical chiral magnetic current and instability in Weyl semimetals,"
   Phys. Rev. B **107**, 014302 (2023).

Publication irrelevant to this thesis

1. Tatsuya Amitani and Yusuke Nishida,
   "Torsion-induced chiral magnetic current in equilibrium,"
   Ann. Phys. **448**, 169181 (2023).

Acknowledgment

First and foremost, I would like to express my gratitude to my supervisor, Prof. Yusuke Nishida, for his invaluable guidance, continuous support, and encouragement throughout my doctoral studies. His expertise, insights, and patience have been instrumental in the completion of this thesis. I am also profoundly grateful to the members of Nishida group and Condensed Matter Theory group at Institute of Science Tokyo. The stimulating discussions, collaboration, and camaraderie have made this journey both intellectually enriching and personally rewarding. Special thanks go to Dr. Keisuke Fujii, Dr. Ken Inayoshi, Mr. Nobuhiro Arai, Dr. Takuya Furusawa, Mr. Tomohiro Tanaka, and Dr. Yuma Nakanishi. I extend my thanks to Dr. Akira Furusaki and the member of Condensed Matter Theory Laboratory at RIKEN. I also wish to express my gratitude to RIKEN Junior Research Associate Program for the financial support, which made this research possible. Finally, my deepest appreciation goes to my family for their unconditional love, encouragement, and understanding. This thesis would not have been possible without the contributions and support of all these individuals and institutions.

Contents

Chapter 1

Introduction

In condensed matter physics, understanding the response properties of systems is crucial for unraveling the fundamental characteristics of materials. Among these, topological materials [1] have garnered significant attention due to their intriguing response properties. These materials are characterized by the topology of their electronic band structures, which fundamentally distinguishes them from conventional materials. The topological materials have the robustness against perturbations, which arises from their protection by topological invariants. Theoretical investigations using model computations have successfully elucidated the response behaviors of individual topological materials. However, a comprehensive and universal understanding of their response properties remains elusive in prior studies. In order to address this gap, this thesis aims to explore and establish the universal response properties of the topological materials.

To achieve this, we adopt an effective field theory approach [2], which aligns naturally with our objectives. An effective field theory provides a simplified framework by focusing on the relevant degrees of freedom within an appropriate energy scale, while abstracting away unnecessary microscopic details. This approach excels at capturing the universal properties of a system, making it a particularly valuable tool for studying complex many-body systems. A well-known example of an effective theory is hydrodynamics [3]. Hydrodynamics describes the dynamics of fluids at long wavelengths and low frequencies compared with a mean free path and a relaxation time of microscopic particles. Despite the diversity of fluids ranging from water to oil and other substances, they are all governed by a single universal framework: the Navier-Stokes equation. This equation encapsulates the essence of fluid dynamics, with the specific details of each system encoded in its parameters. Such universality is a hallmark of effective theories, demonstrating their ability to unify seemingly disparate phenomena under a common framework.

In this thesis, we specifically apply the effective field theory framework to two prominent examples of topological materials: quantum Hall systems [4] and Weyl semimetals [5]. These systems not only exhibit fascinating physical phenomena but also serve as worthwhile platforms for exploring the universal response properties. The quantum Hall systems display a wide range of intriguing Hall responses despite their seemingly simple setups. While extensive studies have been conducted on specific quantum Hall states, there remains a need for a comprehensive and unified formalism that encapsulates their diverse response behaviors within a single theoretical framework. On the other hand, the Weyl semimetals represent a distinct class of topological materials where relativistic physics emerges at low energy scales, such as the axial anomaly, as discussed later. Investigating their unique response properties will not only deepen our understanding of the Weyl semimetals but also shed light on their connections to other systems governed by relativistic principles. The following two sections of this chapter will delve into the details of these systems.

1.1 Quantum Hall effect

1.1.1 History

In 1980, the Hall conductivity of MOSFETs (metal-oxide-semiconductor field-effect transistors) was experimentally studied by von Klitzing et al. [6]. In the MOSFETs, the electrons can effectively be considered to exist in two spatial dimensions. By measuring the Hall conductivity at low temperatures under a strong magnetic field, they discovered that the Hall conductivity is quantized,

$$\sigma_H = \frac{e^2\nu}{2\pi\hbar}, \tag{1.1}$$

where ν is an integer. This phenomenon is known as the integer quantum Hall effect. The integer ν, referred to as the filling factor, corresponds to the number of filled Landau levels. Following this discovery, Tsui et al. observed the Hall conductivity of highly pure GaAs-AlAs heterojunctions [7], which can also be considered as two-dimensional systems. In this experiment, they observed that ν takes specific rational values, for example, $\frac{1}{3}$, $\frac{2}{5}$, and so on. This phenomenon is called the fractional quantum Hall effect.

The integer quantum Hall effect was first theoretically suggested by Ando et al. [8]. After the experimental observation of the integer quantum Hall effect, Laughlin provided the explanation of the quantization in a cylindrical system [9]. It was also shown that the filling number ν is related to a topological invariant, the first Chern number [10, 11]. Thus, the integer quantum Hall effect can be viewed as a simple example of topological phases.

In contrast, the theoretical explanation of the fractional quantum Hall effect is more challenging because the Coulomb interaction plays a crucial role. Laughlin successfully described the quantum Hall state with a filling number $\nu = 1/m$ ($m \in \mathbb{Z}$) by a trial wavefunction [12]. In addition to Laughlin's theory, various theories have been developed, such as the Chern-Simons theory (see, e.g., Refs. [13,14]), the composite fermion approach [15], and so on. The fractional quantum Hall effect remains an active area of research.

1.1.2 Hall viscosity

In addition to the Hall conductivity, quantum Hall systems exhibit other Hall responses, such as the thermal Hall effect [16]. Here, we focus on the Hall responses in a viscosity tensor, known as the Hall viscosity [17,18]. First, we review the concept of the viscosity [3]. We introduce the stress tensor T^{ij} and the displacement vector $\xi_i = x'_i - x_i$. Using the displacement vector, we also define the strain tensor $u_{ij} = \partial_i\xi_j + \partial_j\xi_i$. The stress tensor T^{ij} can be expanded with respect to u_{ij} as

$$T^{ij} \simeq P\delta^{ij} - \lambda^{ijkl}u_{kl} - \eta^{ijkl}\dot{u}_{kl}, \tag{1.2}$$

where P is the pressure, λ^{ijkl} denotes the elastic moduli, and η^{ijkl} represents the viscosity tensor. If the time-reversal symmetry is preserved, the viscosity tensor is symmetric $\eta^{ijkl} = \eta^{klij}$. On the other hand, if the time-reversal symmetry is broken, for instance, by a magnetic field, the viscosity tensor acquires an antisymmetric part $\eta_A^{ijkl} = -\eta_A^{klij}$ [17,18].

In two spatial dimensions, the antisymmetric part takes the form:

$$\eta_A^{ijkl} = -\frac{\eta_H}{2}(\epsilon^{ik}\delta^{jl} + \epsilon^{jk}\delta^{il} + (k \leftrightarrow l)). \tag{1.3}$$

The coefficient η_H is called the Hall viscosity or the odd viscosity.

The Hall viscosity has an intriguing property in quantum Hall systems. Specifically, it is quantized as [19, 20]

$$\eta_H = \frac{e\kappa B}{8\pi}, \tag{1.4}$$

where κ takes specific values and is given by the product of the filling number ν and the Wen-Zee shift $\mathcal{S}$ [21]. The Wen-Zee shift is defined as follows. First, we virtually place the system on a closed surface with genus g. Then, the relation between the total number of the electrons N and the total number of the magnetic fluxes N_ϕ is given by

$$N = \nu(N_\phi + \mathcal{S}(1 - \mathsf{g})). \tag{1.5}$$

For example, for the integer quantum Hall state with $\nu = N$, the shift is also given by $\mathcal{S} = N$, and for the Laughlin state with $\nu = 1/m$, the shift is given by $\mathcal{S} = m$ [22].

Since the Hall viscosity encodes topological information through the Wen-Zee shift, it has attracted much interest [22]. In particular, it has been shown that the Hall viscosity contributes to the Hall conductivity at nonzero wave number (originally derived in Refs. [25, 26], and subsequently derived by using various methods [27–30]). If we can observe the Hall viscosity, we can distinguish the different state with the same filling number. For example, for the Pfaffian and anti-Pfaffian states, which are candidates for the 5/2 states, the Hall viscosity take the opposite sign [20]. Beyond the quantum Hall systems, the Hall viscosity has been studied in other contexts, such as graphene systems [23] and active matter systems [24].

1.2 Weyl semimetals

1.2.1 Overview

The semimetals are medium states between insulators and metals, in which the conduction and valence bands slightly touch in the Brillouin zone. The semimetals whose band-touching is point-like are classified as Dirac or Weyl semimetals [5]. The crossing points are referred to as Dirac/Weyl nodes. The distinction between the Dirac and Weyl semimetals lies in their symmetry. The Dirac semimetals have both the time-reversal and inversion symmetries, leading to doubly degenerated crossing bands. In contrast, either the time-reversal or inversion symmetry is absent in Weyl semimetals. As a result, the degeneracy is lifted, and at least two Weyl nodes appear in the Brillouin zone [5]. The Weyl semimetal phases are robust against small perturbations. The Weyl nodes are protected by the topology of the band structure, with each node carrying a topological charge (the Chern number) ± 1 [5].

In the vicinity of the Weyl nodes, the dispersion relation is linear. Therefore, the low-energy physics of the Weyl semimetals is described by the Weyl fermions, whose Hamiltonian reads

$$H = \pm \boldsymbol{p} \cdot \boldsymbol{\sigma} \tag{1.6}$$

with the sign ± 1 denoting the chirality of the Weyl particles. The chirality corresponds to the topological charge of the Weyl nodes. According to the Nielsen-Ninomiya theorem [31, 32], the total chirality is zero in periodic systems. Therefore, the Weyl nodes always appear as pairs with the opposite chirality.

The Weyl semimetals were initially proposed as a gapless phase between trivial and topological insulator phases [33]. The name "Weyl semimetal" was introduced in Ref. [34]. Various systems are theoretically proposed as the Weyl semimetals, such as a $HgCr_2Se_4$ [35], a heterostructure of topological and normal insulators [36], and a $Hg_{1-x-y}Cd_xMn_yTe$ film [37]. In 2015, the inversion-breaking Weyl semimetal has been experimentally observed in TaAs [38], where the distinctive band structure and surface state of the Weyl semimetals have been observed.

1.2.2 Axial anomaly and chiral magnetic effect

The Weyl semimetals are excellent platforms for studying relativistic phenomena, such as the axial anomaly [39]. The anomaly is a phenomenon where symmetries existing at a classical level are broken by quantum effect. In (3+1)-dimensional massless electron systems, the vector and axial U(1) symmetries exist classically, but the latter is broken. This anomaly is called the axial anomaly. It results in a violation of the axial charge conservation law when the electric and magnetic fields are applied in parallel,

$$\partial_\mu J_5^\mu = \frac{e^3}{2\pi^2\hbar^2}\boldsymbol{E}\cdot\boldsymbol{B}. \tag{1.7}$$

The axial anomaly can be computed by the diagrammatic calculation [40, 41] or the Fujikawa method [42, 43]. Intuitively, it can also be understood through the relativistic Landau level [44], as we will review. The axial anomaly induces the transport phenomena, referred to as anomalous transport phenomena (see Refs. [45, 46] for reviews). For instance, the charge current is induced along the magnetic field in the presence of an imbalance between the number of right-handed and left-handed fermions,

$$\boldsymbol{J} = \frac{e^2\mu_5}{2\pi^2\hbar^2}\boldsymbol{B}, \tag{1.8}$$

where μ_5 is a difference of chemical potential between right-handed and left-handed fermions and is called an axial chemical potential. This effect is called the chiral magnetic effect [47,48]. The coefficient $1/2\pi^2$ is completely determined by the anomaly coefficient [49]. The chiral magnetic effect was originally discussed in the context of the quark-gluon plasmas created by the heavy-ion collisions. It is now actively studied even in the condensed matter physics [50].

The key to realizing the chiral magnetic effect is the axial chemical potential. Son and Spivak proposed that the axial chemical potential is induced as $\mu_5 \propto (\boldsymbol{B}\cdot\boldsymbol{E})$ in Weyl semimetals if we apply the electric and magnetic field in parallel [51]. Under this condition, the chiral magnetic conductivity exhibits a quadratic dependence on the magnetic field, leading to a negative magnetoresistance. Experiments on Weyl semimetals [52, 53] have indeed observed a large negative magnetoresistance, which is interpreted as a signature of the chiral magnetic effect. It is also theoretically proposed that the chiral magnetic effect can be induced in Weyl semimetals through alternative mechanisms, such as by distorting a crystal with strain [54, 55] or heating Weyl nodes unequally under a strong electric field [56].

1.3 Outline of this thesis

In this thesis, we analyze the quantum Hall systems and Weyl semimetals by the effective theories. For the quantum Hall systems, we construct a low-energy effective field theory under the assumption of the Galilean invariance and investigate various responses, including energy and nonlinear responses [Chapter 3]. For the Weyl semimetals, we begin with the already known effective theories and analyze their electromagnetic linear responses [Chapters 4 and 5].

We start with a review of effective theories of the quantum Hall systems and the Weyl semimetals in Chapter 2. We first review the effective actions, which describe the Hall effect and the Hall viscosity. We also review the derivation of the low-energy effective action of the Weyl semimetals. Additionally, we review the chiral kinetic theory, which incorporates the axial anomaly and serves as an alternative framework describing the Weyl semimetals.

In Chapter 3, we focus on the quantum Hall systems with the Galilean invariance. We construct the most general effective action which allows us to compute all responses, including energy responses and all orders in an electric field. From the obtained effective action, we compute electromagnetic nonlinear responses and reveal universal relations between different types of responses.

In Chapters 4 and 5, we turn our attention to the Weyl semimetals and investigate their electromagnetic linear responses. In Chapter 4, starting from the low-energy effective action of the Weyl semimetals reviewed in Chapter 2, we compute the charge current diagrammatically. We also employ the chiral kinetic theory with the relaxation time approximation in Chapter 5. Furthermore, we investigate the collective excitations by coupling the derived charge current with the Maxwell equations.

In Chapter 6, we summarize the findings of this thesis and provide an outlook on future research directions.

We set $\hbar = k_B = q(= -|e|) = 1$. The Minkowski metric is chosen with the mostly negative convention. In Chapter 3, where we deals with (2+1)-dimensional systems, Greek indices such as $\mu, \nu, \dots$ are valued at t, x, y, whereas Latin $a, b, \dots$ and $i, j, \dots$ are valued at x, y. In Chapters 4 and 5, which address (3+1)-dimensional systems, Greek indices such as $\mu, \nu, \dots$ are valued at t, x, y, z, whereas Latin $i, j, \dots$ are valued at x, y, z. We implicitly sum a pair of repeated indices regardless of their positions. The totally antisymmetric symbols are denoted as $\epsilon^{\lambda\rho\mu\nu}$, $\epsilon^{\lambda\mu\nu}$, and ϵ^{ab} with $\epsilon^{txyz} = \epsilon^{txy} = \epsilon^{xy} = 1$. The round (square) brackets enclosing two indices indicate their (anti-)symmetrization, such as $C_{(\mu\nu)} \equiv C_{\mu\nu} + C_{\nu\mu}$ and $C_{[\mu\nu]} \equiv C_{\mu\nu} - C_{\nu\mu}$.

Chapter 2

Review of low-energy effective theories

In this chapter, we review four topics regarding effective theories. We first introduce the Chern-Simons and Wen-Zee terms, both of which play crucial roles in quantum Hall systems, in Sec. 2.1. Next, we review the derivation of the low-energy effective field theory of Weyl semimetals from a lattice model in Sec 2.2. In Sec. 2.3, we discuss the anomaly inherent in the effective action of Weyl semimetals. As an alternative way to describe the Weyl semimetals, we also review the kinetic theory for Weyl fermions (the chiral kinetic theory) in Sec. 2.4.

2.1 Chern-Simons and Wen-Zee terms

2.1.1 Low-energy effective action

To begin, we define the effective action. In a quantum field theory, physical quantities are computed from the partition function defined by

$$\mathcal{Z}[A] = \int D\Psi e^{iS[\Psi;A]}, \tag{2.1}$$

where Ψ and A denote dynamical degrees of freedom and background fields, respectively. The effective action is then defined by

$$S_{\mathrm{eff}}[A] = -i \ln \mathcal{Z}[A]. \tag{2.2}$$

We now introduce a current J, which is conjugate to the external field A, via

$$\delta S = \int d^3x J\delta A. \tag{2.3}$$

It follows that J is obtained from

$$J = \frac{\delta S_{\mathrm{eff}}[A]}{\delta A}. \tag{2.4}$$

For instance, if we introduce a U(1) gauge field A_μ, which couples to the current density J^μ, then we have

$$J^\mu = \frac{\delta S_{\mathrm{eff}}[A_\mu]}{\delta A_\mu}. \tag{2.5}$$

Alternatively, if we couple a spatial metric h_{ij}, which couples to the stress tensor T^{ij}, we have

$$T^{ij} = -\frac{2}{\sqrt{h}} \frac{\delta S_{\text{eff}}[h_{ij}]}{\delta h_{ij}}, \tag{2.6}$$

where $h = \det h_{ij}$.

Therefore, to compute physical quantities in principle, one needs to obtain the effective action by integrating out the dynamical degrees of freedom. However, performing the path integral explicitly is generally impractical. Instead, an effective action is often constructed using symmetry principles [57]. This approach rests on the idea that the effective action should respect the same symmetries as the original action, allowing one to write down all possible terms consistent with those symmetries. Because there are infinitely many such terms, an additional organization scheme, such as a derivative expansion, is usually introduced to systematically constrain the form of the effective action.

2.1.2 Chern-Simons term

We derive the effective action that describes quantum Hall systems. To this end, we consider a free electronic system in two spatial dimensions, couples to an external U(1) gauge field. The microscopic action for this setup is given by

$$S[\Psi, \Psi^\dagger; A_\mu] = \int dt d^2x \left[\frac{i}{2} \Psi^\dagger \overleftrightarrow{D}_t \Psi - \frac{\delta^{ij}}{2m} D_i \Psi^\dagger D_j \Psi \right], \tag{2.7}$$

where we define the covariant derivative D_μ as $D_\mu \Psi = (\partial_\mu - iA_\mu)\Psi$, $D_i \Psi^\dagger = (\partial_\mu + iA_\mu)\Psi^\dagger$, and $\Psi^\dagger \overleftrightarrow{D}_\mu \Psi = \Psi^\dagger D_\mu \Psi - (D_\mu \Psi^\dagger)\Psi$. This action is invariant under the U(1) gauge transformation,

$$\Psi \to e^{i\chi}\Psi, \ A_\mu \to A_\mu + \partial_\mu \chi. \tag{2.8}$$

This action is also invariant under the spatial rotation,

$$\boldsymbol{x}' = R\boldsymbol{x}, \ R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.9}$$

The effective action $S_{\text{eff}} = S_{\text{eff}}[A_\mu]$ is given by the functional of A_μ. Then, we write down terms consistent with the U(1) gauge and rotational invariances by using A_μ. We can write a term as

$$S_{\text{CS}}[A_\mu] = \frac{\nu}{4\pi} \int dt d^2x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{2.10}$$

This term is called the Chern-Simons term [4]. At first glance, the presence of the gauge field in the Chern-Simons term seems to break U(1) gauge invariance. However, this term is actually gauge-invariant up to a total surface term, thereby preserving the underlying gauge symmetry overall:

$$\begin{aligned} S_{\text{CS}}[A_\mu] &\to S_{\text{CS}}[A_\mu] + \int dt d^2x \epsilon^{\mu\nu\lambda} \partial_\mu \chi \partial_\nu A_\lambda \\ &= S_{\text{CS}}[A_\mu] + \int dt d^2x \partial_\mu (\epsilon^{\mu\nu\lambda} \chi \partial_\nu A_\lambda). \end{aligned} \tag{2.11}$$

From this effective action, we can derive the following expression for the charge current:

$$J^i = \frac{\nu}{2\pi} \epsilon^{ij} E^j. \tag{2.12}$$

Hence, the Chern-Simons term encapsulates the quantum Hall effect described by Eq. (1.1).

2.1.3 Wen-Zee term

There is another Chern-Simons-like term responsible for the Hall viscosity. Since the Hall viscosity appears as a coefficient in the stress tensor, we can extract it by computing the stress tensor. To do this, we introduce a spatial metric h_{ij}. A microscopic action of free electrons with a metric h_{ij} is

$$S_0[\Psi, \Psi^\dagger; A_\mu, h_{ij}] = \int dtd^2x\sqrt{h}\left[\frac{i}{2}\Psi^\dagger \overleftrightarrow{D}_t\Psi - \frac{h^{ij}}{2m}D_i\Psi^\dagger D_j\Psi\right]. \tag{2.13}$$

If we take the flat space limit $h_{ij} \to \delta_{ij}$, this action reduces to Eq. (2.7).

Before constructing the effective action, we introduce a vielbein for our convenience. The vielbein $e^a{}_i$ is defined as

$$h_{ij} = e^a{}_i e^a{}_j, \tag{2.14}$$

where indices $a = x, y$ refer to a local orthogonal coordinate. This definition of the vielbein is not unique, because the metric remains invariant if we rotate the vielbein in the a space:

$$e^x{}_i \pm ie^y{}_i \to e^{\pm i\theta}(e^x{}_i \pm ie^y{}_i). \tag{2.15}$$

Such rotation is is known as an SO(2) local rotation. If we define the spin connection ω_μ as [25]

$$\omega_t = \frac{1}{2}\epsilon^{ab}e^{aj}\partial_t e^b{}_j, \tag{2.16}$$

$$\omega_i = \frac{1}{2}\epsilon^{ab}e^{aj}\nabla_i e^b{}_j, \tag{2.17}$$

they acts as an Abelian gauge field under the SO(2) local rotation $\omega_\mu \to \omega_\mu + \partial_\mu\theta$.

Using the spin connection and U(1) gauge field, we can construct the following term:

$$S_{\mathrm{WZ}} = \frac{\kappa}{4\pi}\int dtd^2x\epsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu A_\lambda, \tag{2.18}$$

which is invariant under the SO(2) transformation up to a surface term. This term is known as the Wen-Zee term [21]. Following Ref. [25], we observe that the coefficient κ is related to the Wen-Zee shift [21]. By performing integration by parts, one finds that the Wen-Zee term contains

$$\frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu A_\lambda = \frac{\kappa}{8\pi}\sqrt{h}A_t R + \cdots, \tag{2.19}$$

where R is the scalar curvature, defined by

$$R = \frac{2}{\sqrt{h}}\epsilon^{ij}\partial_i\omega_j. \tag{2.20}$$

Combined with the contribution from the Chern-Simons term, the total charge Q is given by

$$Q = \int d^2x\sqrt{h}J^t = \int d^2x\frac{\delta S_{\mathrm{eff}}[A_\mu, h_{ij}]}{\delta A_t} = \int d^2x\sqrt{h}\left(\frac{\nu}{2\pi}B + \frac{\kappa}{8\pi}R\right), \tag{2.21}$$

where $B = \epsilon^{ij}\partial_i A_j/\sqrt{h}$ is the magnetic field. On a closed surface where electrons reside, the spatial integral of the scalar curvature is proportional to the Euler characteristic:

$$\int d^2x\sqrt{h}R = 4\pi\chi. \tag{2.22}$$

Hence, the total charge can be written as

$$Q = \nu N_\phi + \kappa \frac{\chi}{2} = \nu N_\phi + \kappa(1 - \mathtt{g}), \tag{2.23}$$

where N_ϕ is the total number of magnetic fluxes and $\mathtt{g}$ is the genus. Compared with the definition of the Wen-Zee shift [21], we conclude $\kappa = \nu\mathcal{S}$. Moreover, the Wen-Zee term gives rise to the Hall viscosity [25]. From the Wen-Zee term, one can compute the stress tensor as

$$T^{ij} = -\frac{\kappa B}{8\pi}(\epsilon^{ik}\delta^{jl} + \epsilon^{jk}\delta^{il})\partial_t h_{kl} + \mathcal{O}\left(h^2\right). \tag{2.24}$$

In a curved spacetime, the strain rate tensor $\dot{u}_{ij}$ is defined by [58]

$$\dot{u}_{ij} = \frac{1}{2}[\nabla_i \dot{u}_j + \nabla_j \dot{u}_i + \partial_t h_{ij}]. \tag{2.25}$$

Thus, the coefficient of $\partial_t h_{kl}$ in Eq. (2.24) is identified as the viscosity tensor, leading to the Hall viscosity $\eta_H = \kappa B/8\pi$. Hence, the Wen-Zee term describes the Hall viscosity. The Wen-Zee term also plays an important role as well as the Chern-Simons term in our study.

2.2 Effective field theory of Weyl semimetals

2.2.1 Lattice model of Weyl semimetals

In this section, we review how to derive the low-energy effective action for Weyl semimetals, starting from a microscopic lattice model that hosts precisely two Weyl nodes. Such a Weyl semimetal phase has been proposed to occur in transition-metal-doped Bi_2Se_3 family [59]. Consequently, we begin our analysis with an effective lattice Hamiltonian originally introduced to describe three-dimensional topological insulators in the Bi_2Se_3 family [60, 61],

$$H_0 = \frac{1}{2}\sum_{\boldsymbol{x},j} \psi^\dagger(\boldsymbol{x} + \boldsymbol{a}_j)(it\alpha^j - r\beta)\psi(\boldsymbol{x}) + \frac{1}{2}\sum_{\boldsymbol{x}} \psi^\dagger(\boldsymbol{x})(m + 3r)\beta\psi(\boldsymbol{x}) + (h.c.), \tag{2.26}$$

where t, r, and m are the hopping parameters, and $\boldsymbol{a}_j$ with $j = x, y, z$ is the lattice basis vector. Here, α^i and β are the 4×4 matrices defined as

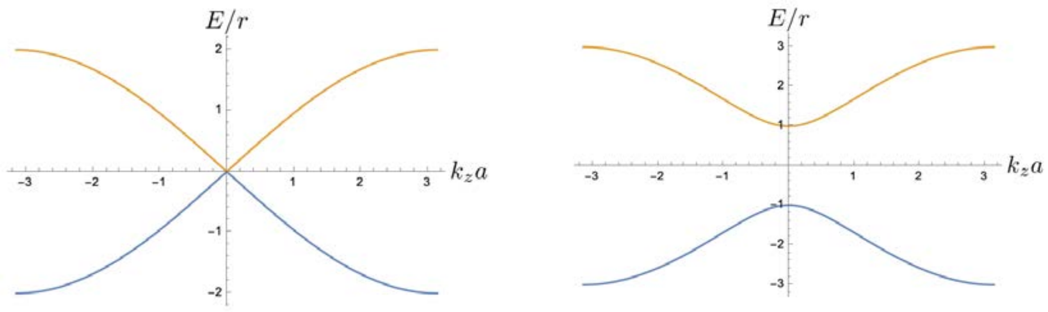
$$\beta = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix},\ \alpha^x = \begin{pmatrix} \sigma^x & 0 \\ 0 & \sigma^x \end{pmatrix},\ \alpha^y = \begin{pmatrix} \sigma^y & 0 \\ 0 & \sigma^y \end{pmatrix},\ \alpha^z = \begin{pmatrix} 0 & \sigma^z \\ \sigma^z & 0 \end{pmatrix}, \tag{2.27}$$

where σ^i are the Pauli matrices. These matrices satisfy the following relations,

$$\beta^2 = \mathbb{I}_2,\ \{\alpha_i, \alpha_j\} = 2\delta_{ij},\ \{\beta, \alpha_i\} = 0. \tag{2.28}$$

This model is also known as the Wilson fermion in the lattice QCD [62]. The Hamiltonian H_0 respects both time-reversal and inversion symmetries. Performing the Fourier transformation $\psi(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \psi(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, we obtain

$$H_0 = \sum_{\boldsymbol{k},j} \psi^\dagger(\boldsymbol{k})(t\sin(k_j a)\alpha^j - r\cos(k_j a)\beta)\psi(\boldsymbol{k}) + \sum_{\boldsymbol{k}} \psi^\dagger(\boldsymbol{k})(m + 3r)\beta\psi(\boldsymbol{k}), \tag{2.29}$$



(a) $m/r = 0$ (b) $m/r = 1$

Figure 2.1: The energy dispersion relation in k_z direction, where the vertical axis is E/r and the horizontal axis is $k_z a \in [-\pi, \pi]$. For simplicity, we set $t/r = 1$. In the left panel, $m/r = 0$, and the gap closes, indicating a Dirac semimetal phase. In contrast, the right panel, with $m/r = 1$, is gapped, corresponding to an insulator phase.

where a is a lattice constant. The Hamiltonian has two pairs of doubly degenerate energy eigenvalues, a degeneracy arising from the inversion and time-reversal symmetries of the Hamiltonian. Figure 2.1 illustrates the energy dispersion along the k_z direction at $(k_x, k_y) = (0, 0)$. When $m/r = -2$ or 0, a single Dirac cone appears in the Brillouin zone. Otherwise, the system becomes gapped, indicating an insulating phase.

To realize the Weyl semimetal phase, one must break at least one of the time-reversal or inversion symmetries [5]. We therefore add terms that explicitly break these symmetries. A term that breaks the time-reversal symmetry can be written as [63–65]

$$H_{b_j} = \sum_{\boldsymbol{x},j} \psi^\dagger(\boldsymbol{x}) b_j \alpha^j \alpha^5 \psi(\boldsymbol{x}) \tag{2.30}$$

with

$$\alpha^5 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}. \tag{2.31}$$

This term encapsulates the effect of doped magnetic impurities [36, 59]. Additionally, we break inversion symmetry by adding [54, 63]

$$H_{b_0} = \sum_{\boldsymbol{x}} \psi^\dagger(\boldsymbol{x}) b_0 \alpha^5 \psi(\boldsymbol{x}). \tag{2.32}$$

We plot the band structure of $H = H_0 + H_{b_j} + H_{b_0}$ when $(k_x, k_y) = (0, 0)$ and $\boldsymbol{b} = (0, 0, b_z)$ in Fig. 2.2. Consequently, a pair of Weyl nodes emerges, giving rise to a Weyl semimetal phase.

2.2.2 Low-energy effective action of Weyl semimetals

We now derive the low-energy effective action from the Hamiltonian H by following [54, 64, 65]. In matrix form, the Hamiltonian is given by

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} v\boldsymbol{p}_\perp \cdot \boldsymbol{\sigma} + (b_z + m(\boldsymbol{k}))\sigma^z & vp_z\sigma^z + b_0\mathbb{I}_2 \\ vp_z\sigma^z + b_0\mathbb{I}_2 & v\boldsymbol{p}_\perp \cdot \boldsymbol{\sigma} + (b_z - m(\boldsymbol{k}))\sigma^z \end{pmatrix}, \tag{2.33}$$

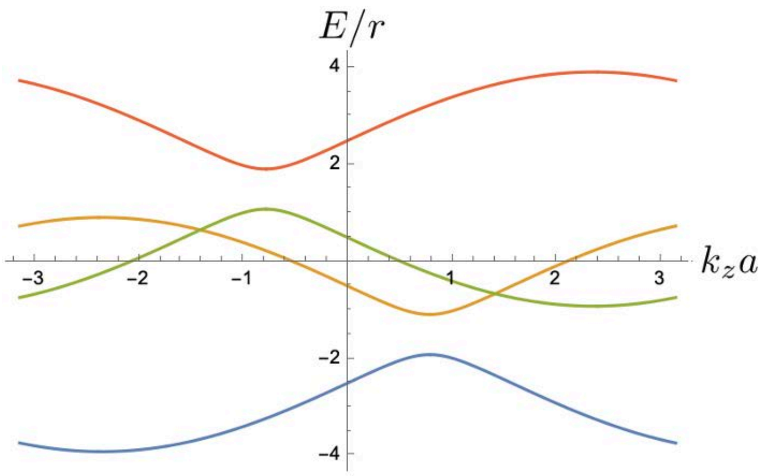


Figure 2.2: The energy dispersion relation of the Hamiltonian H with $(k_x, k_y) = (0, 0)$ and $\boldsymbol{b} = (0, 0, b_z)$. As in the previous figure, the vertical axis is E/r, and the horizontal axis is $k_z a \in [-\pi, \pi]$. We choose $m/r = 0$, $t/r = 1$, $b_0/r = 1$, and $b_z/r = 1.5$.

where we define $p_i = \sin(k_i a)/a$, $\boldsymbol{p}_\perp = (p_x, p_y)$, $v = ta$, and $m(\boldsymbol{k}) = m + 3r - r\sum_j \cos(k_j a)$. This Hamiltonian acts on the four-component spinor $\psi(\boldsymbol{k}) = (\psi_1(\boldsymbol{k}), \psi_2(\boldsymbol{k}))^T$.

We denote the position of the Weyl nodes as $(0, 0, \lambda_\pm)$ and assume they lie close to the origin of the Brillouin zone,

$$|\lambda_\pm| \ll \frac{\pi}{a}. \tag{2.34}$$

Expanding the Hamiltonian around $\boldsymbol{k} = (0, 0, 0)$, we obtain

$$\mathcal{H}(\boldsymbol{k}) \simeq \begin{pmatrix} v\boldsymbol{k}_\perp \cdot \boldsymbol{\sigma} + (b_z + m)\sigma^z & vk_z\sigma^z + b_0\mathbb{I}_2 \\ vk_z\sigma^z + b_0\mathbb{I}_2 & v\boldsymbol{k}_\perp \cdot \boldsymbol{\sigma} + (b_z - m)\sigma^z \end{pmatrix} \tag{2.35}$$

with $\boldsymbol{k}_\perp = (k_x, k_y)$. We further assume that the energy scale that we focus on is sufficiently smaller than $b_z + m$,

$$E \ll b_z + m. \tag{2.36}$$

In this limit, we have

$$\psi_1(\boldsymbol{k}) \simeq -\frac{vk_z\mathbb{I}_2 + b_0\sigma_z}{b_z + m}\psi_2(\boldsymbol{k}). \tag{2.37}$$

Consequently, the low-energy effective Hamiltonian becomes

$$H_{\text{eff}} = \sum_{\boldsymbol{k}} \psi_2^\dagger(\boldsymbol{k})\left[v\boldsymbol{k}_\perp \cdot \boldsymbol{\sigma} + \frac{1}{b_z + m}(b_z^2 - b_0^2 - m^2 - v^2k_z^2)\sigma^z - \frac{2vk_zb_0}{b_z + m}\mathbb{I}_2\right]\psi_2(\boldsymbol{k}). \tag{2.38}$$

The positions of the Weyl nodes denoted by $(0, 0, \lambda_\pm)$ are found by solving

$$b_z^2 - b_0^2 - m^2 - v^2k_z^2 = 0, \tag{2.39}$$

whose solutions are

$$\lambda_\pm = \pm\lambda, \tag{2.40}$$

$$\lambda = \pm\frac{\sqrt{b_z^2 - b_0^2 - m^2}}{v}. \tag{2.41}$$

The energies at the Weyl nodes are

$$E_{\pm}(0,0,\pm\lambda) = \mp 2b_0 \frac{\sqrt{b_z^2 - b_0^2 - m^2}}{b_z + m}. \tag{2.42}$$

Expanding around the two Weyl nodes $k_z = \pm\lambda + \delta k_z$, we obtain

$$\mathcal{H}_{\pm}(\boldsymbol{k}_\perp, \delta k_z) \simeq v\boldsymbol{k}_\perp \cdot \boldsymbol{\sigma} \pm \tilde{v}(\delta k_z \pm \lambda)\sigma^z \pm \frac{2v\lambda b_0}{b_z + m}\mathbb{I}_2 \tag{2.43}$$

with

$$\tilde{v} = \frac{2v\sqrt{b_z^2 - b_0^2 - m^2}}{b_z + m}. \tag{2.44}$$

Introducing the axial gauge field $A_5^\mu = (\frac{2v\lambda b_0}{b_z+m}, 0, 0, -\lambda)$ and the velocity matrix $V = \mathrm{diag}(v, v, \pm\tilde{v})$, we can write

$$\mathcal{H}_{\pm}(\boldsymbol{k}) = V_{ij}(k^i - A_5^i)\sigma^j \pm A_5^0. \tag{2.45}$$

This Hamiltonian describes the Weyl fermions with the anisotropic velocities [55,66] in the presence of an axial gauge field. In the isotropic limit ($V = \mathrm{diag}(v, v, v)$ or $-\mathrm{diag}(v, v, v)$), the Hamiltonian simplifies to

$$\mathcal{H}_\chi(\boldsymbol{k}) = \chi v(\boldsymbol{k} - \boldsymbol{A}_5) \cdot \boldsymbol{\sigma} + \chi A_5^0, \tag{2.46}$$

where $\chi = \pm 1$ is the chirality of the Weyl fermions. By combining both chiralities, $\mathcal{H}_{\mathrm{Weyl}} = \mathcal{H}_+ \oplus \mathcal{H}_-$, we see that the effective Hamiltonian for Weyl semimetals is equivalent to that of massless Dirac fermions coupled to an axial gauge field. Hence, the effective action for the Weyl semimetal can be written as

$$S_{\mathrm{Weyl}} = \int d^4x \bar{\psi}(i\partial\!\!\!/ - A\!\!\!/_5\gamma^5)\psi, \tag{2.47}$$

where $a\!\!\!/ = \gamma^\mu a_\mu$, and γ^μ are the gamma matrices. If we additionally include a U(1) gauge field, the effective action becomes

$$S_{\mathrm{Weyl}} = \int d^4x \bar{\psi}(i\partial\!\!\!/ - A\!\!\!/ - A\!\!\!/_5\gamma^5)\psi, \tag{2.48}$$

which is analogous to a Lorentz-violating QED action [67–69]. One can also derive this effective action from a lattice model composed of alternating layers of topological and normal insulators [67,70]. We note if we distort the Weyl semimetal described by H with the strain, we make the axial gauge fields space-dependent [54,64].

2.3 Axial anomaly

2.3.1 Axial anomaly from Landau levels

The action in Eq. (2.48) exhibits the axial anomaly. Before providing a formal derivation, we offer an intuitive explanation based on Landau levels [44].

The axial anomaly arises when both electric and magnetic fields are present. For simplicity, we choose $A_\mu^5 = (0, 0, 0, b_z)$ with constant b_z , and adopt the Landau gauge $A_y = B_z x$. In this setup, relativistic Landau levels form

$$E_0 = \chi p_z, \tag{2.49}$$

$$E_n = \pm\sqrt{p_z^2 + 2B_z n} \quad (n = 1, 2, ...), \tag{2.50}$$

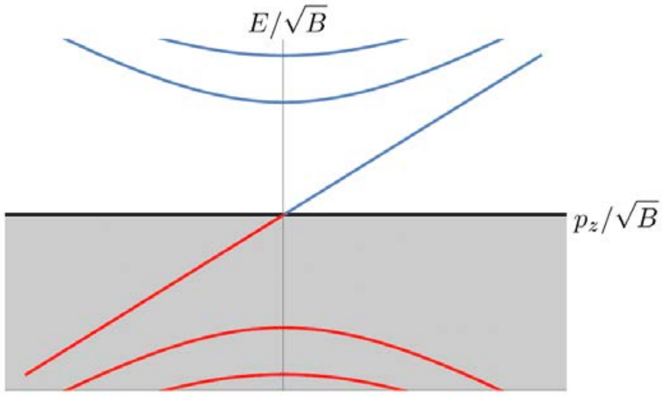


Figure 2.3: The relativistic Landau levels for $\chi = +1$. The vertical and horizontal axises are $E_n/\sqrt{B}$ and $p_z/\sqrt{B}$, respectively. The gray area is the Dirac sea, where the particles are occupied. The red lines express the occupied states.

where χ is the chirality of the electrons and $p_z = k_z - \chi b_z$. The spectrum is shown in Fig. 2.3. Since we choose the Landau gauge, each Landau level degenerates along the y direction, and the degeneracy per area is $B_z/2\pi$. In addition to the magnetic field, we apply the electric field in the z direction. In the zeroth Landau level E_0, the state above the Dirac sea is initially empty, allowing electrons from the Dirac sea to be promoted by the electric field. This corresponds to particle creation by the electric field. By contrast, electrons in the negative-index Landau levels (E_n with $n \leq -1$) cannot undergo such particle creation because all those states are already filled.

Let us calculate the rate of creation of these particles. The density of the states is given by $dn = dk_z/(2\pi)$. Meanwhile, the electric field changes the momentum at a rate $\dot{k}_z = E_z$. Including the degeneracy of each Landau levels, we obtain

$$\dot{n}_+ = \frac{E_z B_z}{4\pi^2}, \tag{2.51}$$

which can be generalized as

$$\dot{n}_+ = \frac{\boldsymbol{E} \cdot \boldsymbol{B}}{4\pi^2}. \tag{2.52}$$

This derivation applies to electrons with chirality $\chi = +1$. For $\chi = -1$, the electrons are annihilated instead of being created. Hence, the axial charge conservation law takes the form

$$\dot{n}_5 = \dot{n}_+ - \dot{n}_- = \frac{\boldsymbol{E} \cdot \boldsymbol{B}}{2\pi^2}, \tag{2.53}$$

which is precisely the axial anomaly Eq. (1.7).

2.3.2 Axial anomaly from Feynman diagrams

The axial anomaly can also be derived through a diagrammatic computation (originally established in Refs. [40, 41]). We begin by decomposing the action S_{Weyl} into its right-

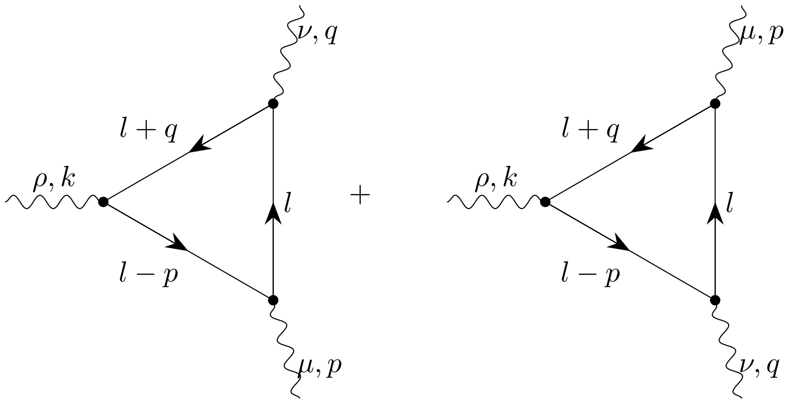


Figure 2.4: The Feynman diagrams corresponding to $T^{\mu\nu\rho}(p,q,k)$. The solid and wavy lines express the fermion and the external gauge field, respectively.

handed and left-handed parts:

$$S_{\mathrm{Weyl}} = \sum_{\chi=\pm} S_\chi, \tag{2.54}$$

$$S_\chi = \int d^4x \bar{\psi}_\chi (i\partial\!\!\!/ - A\!\!\!/_\chi)\psi_\chi, \tag{2.55}$$

where we define

$$\psi_\chi = P_\chi \psi, \tag{2.56}$$

$$A^\chi_\mu = A_\mu + \chi A^5_\mu, \tag{2.57}$$

$$P_\chi = \frac{1+\chi\gamma^5}{2}. \tag{2.58}$$

We consider the three-point function defined as

$$T^{\mu\nu\rho}_\chi(p,q,k) = -i\frac{\delta^3 \ln \mathcal{Z}_\chi}{\delta A^\chi_\mu(p)\delta A^\chi_\nu(q)\delta A^\chi_\rho(k)}, \tag{2.59}$$

which is expressed by the Feynman diagrams as Fig. 2.4. Here $\mathcal{Z}_\chi = \int D\psi D\bar{\psi}\exp[iS_\chi]$ is the partition function. In terms of the current density $J^\mu_\chi = -i(\delta \ln \mathcal{Z}_\chi/\delta A^\chi_\mu)$, the three-point function can be written as

$$T^{\mu\nu\rho}_\chi = \left\langle T J^\mu_\chi(p) J^\nu_\chi(q) J^\rho_\chi(k)\right\rangle, \tag{2.60}$$

where T is the time-ordering operator. Thus, one might naively expect the following Ward identities to hold:

$$p_\mu T^{\mu\nu\rho}_\chi = q_\nu T^{\mu\nu\rho}_\chi = k_\rho T^{\mu\nu\rho}_\chi \stackrel{?}{=} 0. \tag{2.61}$$

We will see, however, that a subtlety arises in regularizing the loop integral, which leads to the axial anomaly.

To check whether the Ward identities are satisfied, let us compute the three-point function explicitly. By the Feynman rules, we obtain

$$iT^{\mu\nu\rho}_\chi(p,q,k) = -\int \frac{d^4l}{(2\pi)^4}\mathrm{tr}\left[\frac{1}{l\!\!\!/ - p\!\!\!/}\gamma^\mu \frac{1}{l\!\!\!/}\gamma^\nu \frac{1}{l\!\!\!/ + q\!\!\!/}\gamma^\rho \frac{1+\chi\gamma^5}{2}\right] + (\mu, p \leftrightarrow \nu, q). \tag{2.62}$$

After evaluating the trace, one obtains

$$ik_\rho T_\chi^{\mu\nu\rho}(p,q,k) = 2\epsilon^{\mu\nu\rho\sigma}\int \frac{d^4l}{(2\pi)^4}\left[\frac{l_\rho(l-p)_\sigma}{l^2(l-p)^2} - \frac{l_\rho(l+q)_\sigma}{l^2(l+q)^2}\right] + (\mu, p \leftrightarrow \nu, q). \tag{2.63}$$

Taking into account the Lorentz invariance, the first (second) term in the square brackets should be proportional to $p_\rho p_\sigma$ ($q_\rho q_\sigma$), and thus vanish upon contraction with the antisymmetric tensor. As a result, we have

$$k_\rho T_\chi^{\mu\nu\rho}(p,q,k) = 0. \tag{2.64}$$

Next, we examine the other Ward identities. One finds similarly

$$p_\mu T_\chi^{\mu\nu\rho}(p,q,k) = 2\epsilon^{\mu\nu\rho\sigma}\int \frac{d^4l}{(2\pi)^4}\left[\frac{l_\rho q_\sigma}{l^2(l+q)^2} - \frac{(l-p)_\rho(p+q)_\sigma}{(l-p)^2(l+q)^2}\right] + (\mu, p \leftrightarrow \nu, q). \tag{2.65}$$

The first term again vanishes by the same argument. For the second term, a shift of the integration variable $l \to l+p$, suggests that the integrand would also vanish if we can freely shift the loop momentum. But because the integral is linearly divergent, we must be more careful about how we regulate it.

As discussed in Refs. [39,45,71], the final result of the integration depends on how one chooses the internal loop momentum l. Concretely, if we replace l by $l \to l + c(p-q) + d(p+q)$ with c, d being arbitrary real numbers, the Ward identities take the form [39,45,71]

$$ip_\mu T_\chi^{\mu\nu\rho}(p,q,k) = \chi\frac{1}{8\pi^2}(1-c)\epsilon^{\nu\rho\sigma\tau}q_\sigma k_\tau, \tag{2.66}$$

$$iq_\nu T_\chi^{\mu\nu\rho}(p,q,k) = \chi\frac{1}{8\pi^2}(1-c)\epsilon^{\mu\rho\sigma\tau}p_\sigma k_\tau, \tag{2.67}$$

$$ik_\rho T_\chi^{\mu\nu\rho}(p,q,k) = \chi\frac{1}{8\pi^2}2c\epsilon^{\mu\nu\sigma\tau}p_\sigma q_\tau. \tag{2.68}$$

The ambiguity of the Ward identity corresponds to how we choose the regularization scheme. Because no single value of c can make all of Eqs. (2.66)-(2.68) vanish simultaneously, at least one of the Ward identities must be broken. This breakdown of the classical Ward identities is the axial anomaly.

2.3.3 Consistent anomaly

One natural choice of c is $c = 1/3$, where we treat three identities equally. In this choice, the conservation law in terms of the current density and the gauge field becomes

$$\partial_\mu \mathcal{J}_\chi^\mu = \frac{\chi}{96\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^\chi F_{\rho\sigma}^\chi \tag{2.69}$$

with $F_{\mu\nu}^\chi = \partial_\mu A_\nu^\chi - \partial_\nu A_\mu^\chi$. This anomaly is referred to as the *consistent* anomaly [45]. We rewrite these conservation laws in terms of the vector and axial currents as

$$\partial_\mu \mathcal{J}^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}^5, \tag{2.70}$$

$$\partial_\mu \mathcal{J}_5^\mu = \frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}(F_{\mu\nu}F_{\rho\sigma} + F_{\mu\nu}^5 F_{\rho\sigma}^5), \tag{2.71}$$

where

$$\mathcal{J}^\mu = \mathcal{J}_+^\mu + \mathcal{J}_-^\mu,\ \mathcal{J}_5^\mu = \mathcal{J}_+^\mu - \mathcal{J}_-^\mu. \tag{2.72}$$

The lack of vector charge conservation is problematic. By following Ref. [72], we can remedy this by adding a local counterterm to the effective action:

$$S_{\text{eff}}[A, A_5] \to S_{\text{eff}}[A, A_5] + \frac{1}{12\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} A_\mu A^5_\nu F_{\rho\sigma}. \tag{2.73}$$

With this addition, the conservation laws become

$$\partial_\mu J^\mu = 0, \tag{2.74}$$

$$\partial_\mu J^\mu_5 = \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \left(F_{\mu\nu} F_{\rho\sigma} + \frac{1}{3} F^5_{\mu\nu} F^5_{\rho\sigma} \right). \tag{2.75}$$

The modified currents J^μ and J^μ_5 are called the consistent currents.

The term "consistent" reflects that this anomaly satisfies the Wess-Zumino consistency condition [45, 73]. Following Ref. [45], we introduce the gauge transformation operator,

$$\delta_\lambda = \int d^4x \partial_\mu \lambda \frac{\delta}{\delta A_\mu}. \tag{2.76}$$

The effective action is not gauge invariant because of the anomaly,

$$\delta_\lambda S_{\text{eff}}[A] = \mathcal{A}_\lambda. \tag{2.77}$$

The gauge transformation operators with different parameters obey the gauge algebra,

$$[\delta_\lambda, \delta_\sigma] = 0. \tag{2.78}$$

Then the anomaly $\mathcal{A}$ should fulfill

$$\delta_\lambda \mathcal{A}_\sigma - \delta_\sigma \mathcal{A}_\lambda = [\delta_\lambda, \delta_\sigma] S_{\text{eff}} = 0, \tag{2.79}$$

which is the Wess-Zumino consistency condition. Here, by replacing the parameter λ with a Grassmann number c, we introduce the BRST operator,

$$s = \int d^4x \partial_\mu c \frac{\delta}{\delta A_\mu}, \tag{2.80}$$

which satisfies $s^2 = 0$. Then, the anomaly can be expressed as

$$s S_{\text{eff}}[A] = \mathcal{A}. \tag{2.81}$$

This is analogous to the differential form, where the field strength F is the exterior derivative $F = dA$. The Wess-Zumino consistency condition requires that the anomaly A is a "closed one-form,"

$$s\mathcal{A} = 0. \tag{2.82}$$

The consistent anomaly is not gauge-invariant since the gauge transformation of the consistent current is

$$s J^\mu_\chi = \frac{\delta}{\delta A_\mu} \mathcal{A} = -\chi \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu c F^\chi_{\rho\sigma}. \tag{2.83}$$

This reflects the fact that under a gauge transformation, the consistent current itself shifts in a way that encodes the anomaly.

2.3.4 Covariant anomaly

Let us go back to the Ward identities Eqs. (2.66)-(2.68). We have another choice to push anomaly on the one vertex, i.e. $c = 1$. In this case, the conservation law is given by

$$\partial_\mu j^\mu_\chi = \frac{\chi}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^\chi_{\mu\nu}F^\chi_{\rho\sigma}. \tag{2.84}$$

which is called the *covariant* anomaly [45]. The covariant current is gauge-invariant, $sj^\mu_\pm = 0$. The conservation laws for the covariant current are given by

$$\partial_\mu j^\mu = \frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F^5_{\rho\sigma}, \tag{2.85}$$

$$\partial_\mu j^\mu_5 = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}(F_{\mu\nu}F_{\rho\sigma} + F^5_{\mu\nu}F^5_{\rho\sigma}). \tag{2.86}$$

The covariant currents are related to the consistent current by

$$J^\mu = j^\mu - \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}A^5_\nu F_{\rho\sigma}, \tag{2.87}$$

$$J^\mu_5 = j^\mu_5 - \frac{1}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}A^5_\nu F^5_{\rho\sigma}. \tag{2.88}$$

The difference between the consistent and covariant currents $\delta j^\mu = J^\mu - j^\mu$ given by

$$\delta j^\mu = -\frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}A^5_\nu F_{\rho\sigma} \tag{2.89}$$

is called the Bardeen-Zumino polynomial [74] or the Chern-Simons current [45, 75]. In Weyl semimetals, the Chern-Simons current plays a pivotal role in describing transport properties. From the perspective of Landau levels, one can regard the Chern-Simons current as the flow that emerges at the bottom of the Dirac sea, i.e., below a momentum cutoff Λ [45].

2.4 Chiral kinetic theory

The kinetic theory is a macroscopic, low-energy effective framework that describes non-equilibrium dynamics in a semiclassical manner. Because it is inherently semiclassical, quantum effects such as the axial anomaly are absent. In what follows, we review how to incorporate these anomalous effects and thereby construct a kinetic theory suitable for Weyl semimetals.

2.4.1 Kinetic equation with Berry curvature

In this section, we review the derivation of the chiral kinetic theory based on Refs. [76–78], (see also Ref. [79] for an alternative derivation). We begin with the Hamiltonian of the right-handed Weyl fermion in the presence of an external electromagnetic field,

$$H = \boldsymbol{\sigma}\cdot(\boldsymbol{p} - \boldsymbol{A}) + \Phi. \tag{2.90}$$

Here, $\boldsymbol{\sigma}\cdot\boldsymbol{p}$ can be diagonalized by a suitable unitary matrix $V_{\boldsymbol{p}}$ as

$$V^\dagger_{\boldsymbol{p}}\boldsymbol{\sigma}\cdot\boldsymbol{p}V_{\boldsymbol{p}} = |\boldsymbol{p}|\sigma_z. \tag{2.91}$$

We denote the eigenvector corresponding to the positive energy by $u_{\boldsymbol{p}}$,

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p}) u_{\boldsymbol{p}} = |\boldsymbol{p}| u_{\boldsymbol{p}}. \tag{2.92}$$

We consider the transition amplitude between the two states i and f. The transition matrix element can be written as a path integral,

$$\langle f | e^{iH(t_f - t_i)} | i \rangle = \int D\boldsymbol{x} D\boldsymbol{p} \mathcal{P} \exp\left[i \int_{t_i}^{t_f} (\boldsymbol{p} \cdot d\boldsymbol{x} - \boldsymbol{\sigma} \cdot (\boldsymbol{p} - \boldsymbol{A}) dt - \Phi dt) \right], \tag{2.93}$$

where $\mathcal{P}$ is the path-ordering operator. We insert the identity matrix $I = V_{\boldsymbol{p}} V_{\boldsymbol{p}}^\dagger$ into each step of the path integral, and then $\boldsymbol{\sigma} \cdot \boldsymbol{p}$ is diagonalized,

$$\langle f | e^{iH(t_f - t_i)} | i \rangle = \lim_{\Delta t \to 0} \int \prod_t d\boldsymbol{x}(t) d\boldsymbol{p}(t) e^{i\boldsymbol{p} \cdot \Delta \boldsymbol{x}} e^{-i|\boldsymbol{p}|\sigma_z \Delta t - i\Phi \Delta t} V_{\boldsymbol{p}}^\dagger e^{i\boldsymbol{\sigma} \cdot \boldsymbol{A} \Delta t} V_{\boldsymbol{p} - \Delta \boldsymbol{p}} \tag{2.94}$$

with $\Delta \boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p}' = \boldsymbol{p}(t) - \boldsymbol{p}(t - \Delta t)$. We now assume there is no transition between the particle and antiparticle states and neglect the off-diagonal component. We focus on the amplitude between the positive energy states. Then, we should evaluate

$$[V_{\boldsymbol{p}}^\dagger e^{i\boldsymbol{\sigma} \cdot \boldsymbol{A} \Delta t} V_{\boldsymbol{p} - \Delta \boldsymbol{p}}]_{++} = u_{\boldsymbol{p}}^\dagger e^{i\boldsymbol{\sigma} \cdot \boldsymbol{A} \Delta t} u_{\boldsymbol{p} - \Delta \boldsymbol{p}}. \tag{2.95}$$

Here, we have the Gordon-like identity [80],

$$u_{\boldsymbol{p}}^\dagger \boldsymbol{\sigma} u_{\boldsymbol{p}'} = \frac{1}{|\boldsymbol{p}| + |\boldsymbol{p}'|} [-i u_{\boldsymbol{p}}^\dagger (\Delta \boldsymbol{p} \times \boldsymbol{p}') u_{\boldsymbol{p}'} + (\boldsymbol{p} + \boldsymbol{p}') u_{\boldsymbol{p}}^\dagger u_{\boldsymbol{p}'}]. \tag{2.96}$$

By using this identity, we find

$$\begin{aligned} u_{\boldsymbol{p}}^\dagger e^{i\boldsymbol{\sigma} \cdot \boldsymbol{A} \Delta t} u_{\boldsymbol{p}'} &= u_{\boldsymbol{p}}^\dagger \exp\left[\frac{1}{|\boldsymbol{p}| + |\boldsymbol{p}'|} [(\Delta \boldsymbol{p} \times \boldsymbol{p}') \cdot \boldsymbol{A} + i(\boldsymbol{p} + \boldsymbol{p}') \cdot \boldsymbol{A}] \Delta t \right] u_{\boldsymbol{p}'} \\ &= u_{\boldsymbol{p}}^\dagger u_{\boldsymbol{p}'} \exp\left[\frac{(\Delta \boldsymbol{p} \times \hat{\boldsymbol{p}}')}{2|\boldsymbol{p}|} \cdot \boldsymbol{A} \Delta t + i \hat{\boldsymbol{p}} \cdot \boldsymbol{A} \Delta t \right] + \mathcal{O}\left(\Delta \boldsymbol{p}^2\right), \end{aligned} \tag{2.97}$$

where $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$. The second term in the square brackets in Eq. (2.97) combined with $e^{-i|\boldsymbol{p}|\Delta t}$ in Eq. (2.94) becomes $e^{-i|\boldsymbol{p} - \boldsymbol{A}|\Delta t}$ since $|\boldsymbol{p} - \boldsymbol{A}| \simeq \boldsymbol{p} - \hat{\boldsymbol{p}} \cdot \boldsymbol{A}$ up to $\mathcal{O}\left(\boldsymbol{A}^2\right)$. We can rewrite the first term in the square brackets in Eq. (2.97) as

$$\exp\left[i \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|} \Delta t \right]. \tag{2.98}$$

Lastly, we define the Berry connection $\boldsymbol{\mathcal{A}}$ as $u_{\boldsymbol{p}}^\dagger u_{\boldsymbol{p}'} = e^{-i\boldsymbol{\mathcal{A}} \cdot \Delta \boldsymbol{p}}$ [81], which can be regarded as a "vector potential" in momentum space. The field strength of the Berry connection is called the Berry curvature, whose concrete form is given by

$$\boldsymbol{\Omega} = \boldsymbol{\nabla}_{\boldsymbol{p}} \times \boldsymbol{\mathcal{A}} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}. \tag{2.99}$$

This formula shows the presence of the monopole at $\boldsymbol{p} = 0$ in momentum space,

$$\boldsymbol{\nabla}_{\boldsymbol{p}} \cdot \boldsymbol{\Omega} = 2\pi \delta(\boldsymbol{p}). \tag{2.100}$$

If we introduce the gauge-invariant momentum $\boldsymbol{P} = \boldsymbol{p} - \boldsymbol{A}$, we find

$$u_{\boldsymbol{p}}^\dagger u_{\boldsymbol{p}'} = (1 + \boldsymbol{\Omega} \cdot \boldsymbol{B}) e^{-i\boldsymbol{\mathcal{A}} \cdot \Delta \boldsymbol{P}}. \tag{2.101}$$

This factor $\sqrt{G} = (1+\boldsymbol{\Omega}\cdot\boldsymbol{B})$ acts like the determinant of the metric. Indeed, the invariant measure of the phase space with Berry curvature is given by $\sqrt{G}d^3\boldsymbol{x}d^3\boldsymbol{P}/(2\pi)^3$ [82].

The total effective action for a particle can be written as

$$S = \int dt[(\boldsymbol{p}+\boldsymbol{A})\cdot\dot{\boldsymbol{x}} - (\epsilon_{\boldsymbol{p}}+\Phi) - \boldsymbol{\mathcal{A}}\cdot\dot{\boldsymbol{p}}], \tag{2.102}$$

where the dispersion relation is

$$\epsilon_{\boldsymbol{p}} = |\boldsymbol{p}|(1-\boldsymbol{B}\cdot\boldsymbol{\Omega}). \tag{2.103}$$

The dispersion relation is modified by the magnetic moment coupling (the Zeeman effect). This modification is crucial for maintaining Lorentz invariance in the chiral kinetic theory [77, 78].

From this action, one can derive the equations of motion,

$$\dot{\boldsymbol{x}} = \boldsymbol{v}_{\boldsymbol{p}} + \dot{\boldsymbol{p}}\times\boldsymbol{\Omega}, \tag{2.104}$$

$$\dot{\boldsymbol{p}} = \tilde{\boldsymbol{E}} + \dot{\boldsymbol{x}}\times\boldsymbol{B}, \tag{2.105}$$

where $\boldsymbol{v}_{\boldsymbol{p}} = \partial\epsilon_{\boldsymbol{p}}/\partial\boldsymbol{p}$ is the quasiparticle velocity, and $\tilde{\boldsymbol{E}} = \boldsymbol{E} - \boldsymbol{\nabla}\epsilon_{\boldsymbol{p}}$. If the Berry curvature is absent, they reduce to the usual equations of motion. By substituting Eq. (2.105) into (2.104), we obtain

$$\sqrt{G}\dot{\boldsymbol{x}} = \boldsymbol{v}_{\boldsymbol{p}} + \tilde{\boldsymbol{E}}\times\boldsymbol{\Omega} + \boldsymbol{B}(\boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\Omega}), \tag{2.106}$$

$$\sqrt{G}\dot{\boldsymbol{p}} = \tilde{\boldsymbol{E}} + \boldsymbol{v}_{\boldsymbol{p}}\times\boldsymbol{B} + \boldsymbol{\Omega}(\tilde{\boldsymbol{E}}\cdot\boldsymbol{B}). \tag{2.107}$$

Once the equations of motion are known, one can formulate a kinetic equation by introducing the distribution function $f(t,\boldsymbol{x},\boldsymbol{p})$. The Boltzmann equation for right-handed particles takes the form

$$\frac{\partial f_R}{\partial t} + \frac{1}{\sqrt{G}}\left[\boldsymbol{v}_{\boldsymbol{p}} + \tilde{\boldsymbol{E}}\times\boldsymbol{\Omega} + \boldsymbol{B}(\boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\Omega})\right]\cdot\frac{\partial f_R}{\partial\boldsymbol{x}} + \frac{1}{\sqrt{G}}\left[\tilde{\boldsymbol{E}} + \boldsymbol{v}_{\boldsymbol{p}}\times\boldsymbol{B} + \boldsymbol{\Omega}(\tilde{\boldsymbol{E}}\cdot\boldsymbol{B})\right]\cdot\frac{\partial f_R}{\partial\boldsymbol{p}} = C[f_R], \tag{2.108}$$

where $C[f_R]$ is the collision term. For left-handed particles, we should flip the sign of the Berry curvature, $\boldsymbol{\Omega}\to-\boldsymbol{\Omega}$. For antiparticles, we should replace $A_\mu\to-A_\mu$ and $\boldsymbol{\Omega}\to-\boldsymbol{\Omega}$ in addition to the flip of the chemical potential in the distribution function $\mu_\chi\to-\mu_\chi$.

The charge and current densities are given by [77, 83]

$$\boldsymbol{\rho}_\chi = \sum_{p,a}\int\frac{d^3p}{(2\pi)^3}\sqrt{G}f_\chi, \tag{2.109}$$

$$\boldsymbol{j}_\chi = \sum_{p,a}\int\frac{d^3p}{(2\pi)^3}\left[(\boldsymbol{v}_{\boldsymbol{p}} + (\boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\Omega})\boldsymbol{B} + \tilde{\boldsymbol{E}}\times\boldsymbol{\Omega})f_\chi + \boldsymbol{\nabla}\times f_\chi\epsilon_{\boldsymbol{p}}\boldsymbol{\Omega}\right], \tag{2.110}$$

where $\sum_{p,a}$ denotes the summation over the contribution of the particles and antiparticles. For antiparticles, we should change the total sign of the integrand in addition to the above replacement. By applying the collisionless Boltzmann equation and using the definitions of the charge and current densities, we obtain the continuity equations for the electric and axial currents,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0, \tag{2.111}$$

$$\frac{\partial\rho_5}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_5 = \frac{1}{2\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}, \tag{2.112}$$

thereby reproducing the axial anomaly in the kinetic theory framework.

2.4.2 Consistent chiral kinetic theory

We consider the situation in which an axial gauge field $A^5_\mu(x)$ is present. Although such a field does not exist as a fundamental gauge field in high-energy physics, it can arise naturally in Weyl semimetals. In particular, the separation of the Weyl nodes can be represented by a constant axial gauge field, and a spacetime-dependent axial gauge field can emerge from strain [54, 64].

In this case, we should replace the electromagnetic fields in the Boltzmann equation and in the definition of the charge and current densities by the effective electromagnetic fields,

$$\boldsymbol{B} \to \boldsymbol{B}_\chi = \boldsymbol{B} + \chi \boldsymbol{B}_5, \tag{2.113}$$

$$\boldsymbol{E} \to \boldsymbol{E}_\chi = \boldsymbol{E} + \chi \boldsymbol{E}_5. \tag{2.114}$$

Then, the continuity equations are modified as [84]

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = \frac{1}{2\pi^2}(\boldsymbol{E} \cdot \boldsymbol{B}_5 + \boldsymbol{E}_5 \cdot \boldsymbol{B}), \tag{2.115}$$

$$\frac{\partial \rho_5}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_5 = \frac{1}{2\pi^2}(\boldsymbol{E} \cdot \boldsymbol{B} + \boldsymbol{E}_5 \cdot \boldsymbol{B}_5). \tag{2.116}$$

These equations coincide with the covariant conservation laws in Eq. (2.85) and (2.86), where the charge current is no longer conserved. Since the conserved consistent current is physical, we should add the Chern-Simons current [45, 75],

$$\delta j^\mu = -\frac{1}{4\pi^2} \epsilon^{\mu\nu\lambda\rho} A^5_\nu F_{\lambda\rho}. \tag{2.117}$$

The formulation of chiral kinetic theory that incorporates this prescription (i.e., adding the Chern-Simons current) is known as the consistent chiral kinetic theory [84].

2.4.3 Chiral magnetic effect

Chiral magnetic effect

Even in the absence of any spacetime dependence of the axial gauge field (so that total charge remains conserved), the Chern-Simons current still plays an essential role [45]. To illustrate this, let us compute the charge current in equilibrium under a small constant magnetic field. The distribution function for particles with the chirality χ is

$$f_\chi = n_F(\beta(\epsilon_{\boldsymbol{p}} - \mu_\chi)) \simeq n_F(\beta(|\boldsymbol{p}| - \mu_\chi)) - |\boldsymbol{p}|(\boldsymbol{B} \cdot \boldsymbol{\Omega}) n'_F(\beta(|\boldsymbol{p}| - \mu_\chi)) \tag{2.118}$$

with $n_F(x) = 1/(e^x + 1)$ being the Fermi-Dirac distribution function. The quasiparticle velocity is given by

$$\boldsymbol{v}_{\boldsymbol{p}} = \hat{\boldsymbol{p}} + 2\hat{\boldsymbol{p}}(\boldsymbol{B} \cdot \boldsymbol{\Omega}) - \boldsymbol{B}(\hat{\boldsymbol{p}} \cdot \boldsymbol{\Omega}). \tag{2.119}$$

Up to $\mathcal{O}(B^2)$, the current density of particle is given by

$$\begin{aligned} \boldsymbol{j}_{\chi,p} &= \int \frac{d^3p}{(2\pi)^3} \Bigg[\hat{\boldsymbol{p}} \left(n_F(\beta(|\boldsymbol{p}| - \mu_\chi)) - \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|} n'_F(\beta(|\boldsymbol{p}| - \mu_\chi)) \right) \\ &\quad + (2\hat{\boldsymbol{p}}(\boldsymbol{B} \cdot \boldsymbol{\Omega}) - \boldsymbol{B}(\hat{\boldsymbol{p}} \cdot \boldsymbol{\Omega})) n_F(\beta(|\boldsymbol{p}| - \mu_\chi)) + (\hat{\boldsymbol{p}} \cdot \boldsymbol{\Omega}) \boldsymbol{B} n_F(\beta(|\boldsymbol{p}| - \mu_\chi)) \Bigg] \\ &= \int \frac{d^3p}{(2\pi)^3} 3\hat{\boldsymbol{p}}(\boldsymbol{B} \cdot \boldsymbol{\Omega}) n_F(\beta(|\boldsymbol{p}| - \mu_\chi)). \end{aligned} \tag{2.120}$$

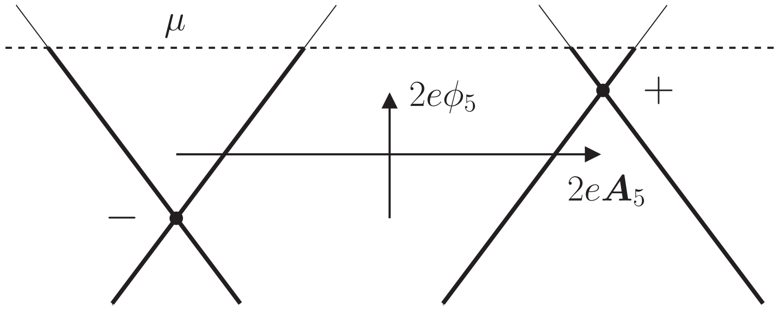


Figure 2.5: Two Weyl cones in equilibrium. This figure is cited from T. Amitani and Y. Nishida, "Dynamical chiral magnetic current and instability in Weyl semimetals," Phys. Rev. B **107**, 014302 (2023), © 2023 American Physical Society.

By performing the angular integration, we obtain

$$\boldsymbol{j}_{\chi,p} = \frac{\boldsymbol{B}}{4\pi^2}\int_0^\infty dp n_F(\beta(p-\mu_\chi)). \tag{2.121}$$

Therefore, the total charge current is

$$\begin{aligned}\boldsymbol{j} &= \frac{\boldsymbol{B}}{4\pi^2}\int_0^\infty dp[n_F(\beta(p-\mu_+)) - n_F(\beta(p+\mu_+)) - n_F(\beta(p-\mu_-)) + n_F(\beta(p+\mu_-))] \\ &= \frac{\mu_5}{2\pi^2}\boldsymbol{B},\end{aligned} \tag{2.122}$$

where $\mu_5 = (\mu_+ - \mu_-)/2$ is the axial chemical potential. This result is known as the chiral magnetic effect [48].

In Weyl semimetals in equilibrium where $\mu_5 = -A_0^5$ (see Fig. 2.5), the chiral magnetic current is given by

$$\boldsymbol{j} = -\frac{A_5^0}{2\pi^2}\boldsymbol{B}. \tag{2.123}$$

Hence, the chiral magnetic current is present even in equilibrium. However, such an equilibrium current is problematic, and the chiral magnetic current must vanish in equilibrium [45, 63, 85]. The resolution is to consider the consistent current rather than the covariant current [45]. The consistent current is obtained by adding the Chern-Simons current,

$$\boldsymbol{J} = \underbrace{-\frac{A_5^0}{2\pi^2}\boldsymbol{B}}_{\boldsymbol{j}} + \underbrace{\frac{A_5^0}{2\pi^2}\boldsymbol{B}}_{\delta\boldsymbol{j}} = 0. \tag{2.124}$$

Thus, the Chern-Simons current precisely cancels with the covariant current, causing the consistent chiral magnetic current to vanish in equilibrium. Therefore, including the Chern-Simons current is indispensable for a complete description of Weyl semimetals. The Chern-Simons current also contains the contribution of the anomalous Hall current [36,86] in Weyl semimetals,

$$\boldsymbol{J}_{\mathrm{AH}} = -\frac{1}{2\pi^2}\boldsymbol{A}_5 \times \boldsymbol{E}, \tag{2.125}$$

which does not appear in conventional chiral kinetic theory without the Chern-Simons current.

Chiral plasma instability

We also review the chiral plasma instability [87], which is a phenomenon closely related to the chiral magnetic effect, by following Ref. [88]. We couple the chiral magnetic current Eq. (2.122) with Maxwell's equation and investigate the behavior of the plasma. The Ampère's law with the chiral magnetic current is

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{\mu_5}{2\pi^2}\boldsymbol{B} + \frac{\partial \boldsymbol{E}}{\partial t}. \tag{2.126}$$

By taking the curl on both sides and using the Faraday's law, we find

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \frac{\mu_5}{2\pi^2}(\boldsymbol{\nabla} \times \boldsymbol{B}) - \frac{\partial^2 \boldsymbol{B}}{\partial t^2}. \tag{2.127}$$

For simplicity, we consider the following helical modes:

$$\boldsymbol{B}_{\lambda,k} = B_0(\hat{\boldsymbol{x}} + i\lambda\hat{\boldsymbol{y}})e^{-i\omega t + ikz}, \tag{2.128}$$

where $\lambda = \pm 1$ is the helicity of the mode. These modes are the eigenstates of the curl operator:

$$\boldsymbol{\nabla} \times \boldsymbol{B}_{\lambda,k} = \lambda k \boldsymbol{B}_{\lambda,k}. \tag{2.129}$$

By substituting these helical modes in Eq. (2.127) and solving the equation, we obtain the two solutions:

$$\omega = \pm\sqrt{k\left(k - \lambda\frac{\mu_5}{2\pi^2}\right)}. \tag{2.130}$$

We find one of the solutions with the helicity $\lambda = \mathrm{sign}(\mu_5)$ and small wave vector $k < \frac{\mu_5}{2\pi^2}$ is a positive imaginary number. This solution implies the existence of the exponentially growing mode, which is called the chiral plasma instability or the chiral magnetic instability. The chiral plasma instability can intuitively understood as follows. Initially, the presence of an axial chemical potential allows a magnetic field to induce a chiral magnetic current. In turn, this chiral magnetic current generates an additional magnetic field via Ampère's law. This induced magnetic field further enhances the chiral magnetic current, establishing a feedback loop that leads to exponential growth of the magnetic field. As the instability develops, it consumes the axial imbalance, thereby reducing μ_5 [88]. Eventually, once the axial chemical potential is depleted, the system stabilizes.

2.5 Summary

In this Chapter, we reviewed four topics related to our study. In Sec. 2.1, we reviewed the low-energy effective action of the quantum Hall effect. The Chern-Simons term was introduced, and we checked that it describes the Hall effect. We also introduced the Wen-Zee term by considering the system on a curved space. The Wen-Zee term describes the Hall viscosity, and its coefficient is related to the Wen-Zee shift. The Chern-Simons and Wen-Zee terms play an important role in our study of the quantum Hall effect (Chapter 3). In Sec. 2.2, we derived the low-energy effective action from the lattice model. The Weyl semimetal is realized by the topological insulator without time-reversal and inversion symmetries. The resulting action is the Weyl fermion with the axial gauge field. We also discussed the anomaly of the Weyl fermion with the axial gauge field in Sec. 2.3. There are covariant and consistent anomalies, and the conserved consistent current is physical. The difference between the covariant and consistent current is called the Chern-Simons current. We will discuss responses derived from the effective action in Chapter 4. Lastly, we reviewed the derivation of the chiral kinetic theory, which is a kinetic theory with the axial anomaly. By introducing the Berry curvature, we reproduced the covariant anomaly. Therefore, we should add the Chern-Simons current to the conventional chiral kinetic theory to obtain the physical current. We also checked that the Chern-Simons current is essential for the chiral magnetic current to vanish in equilibrium. We will employ the chiral kinetic theory in Chapter 5.

Chapter 3

Quantum Hall systems

In this chapter, we focus on the quantum Hall systems with the Galilean invariance. Starting from the microscopic action, we will construct the general effective action based on symmetries and compute various responses of the system. While similar attempts can be found in previous works [25, 27, 29] (see also Refs. [89, 90] for cases without Galilean invariance), we will extend these effort to allow for computation of an energy current and nonlinear responses to full orders in an electric field.

3.1 Microscopic action

We begin with two spatial dimensional electrons under an external electromagnetic field, whose microscopic action is given by

$$S = S_0 + S_{\text{int}} \tag{3.1}$$

with

$$S_0 = \int dt\, d^2x \left(\Psi^\dagger i \overleftrightarrow{D}_t \Psi - \frac{D_i \Psi^\dagger D_i \Psi}{2m} + \frac{gB}{4m} \Psi^\dagger \Psi \right) \tag{3.2}$$

and $\Psi^\dagger \overleftrightarrow{D}_\mu \Psi \equiv [\Psi^\dagger (D_\mu \Psi) - (D_\mu \Psi^\dagger)\Psi]/2$. Here, $D_\mu \Psi = (\partial_\mu - iA_\mu)\Psi$ and $D_\mu \Psi^\dagger = (\partial_\mu + iA_\mu)\Psi^\dagger$ are covariant derivatives, $\phi = -A_t$ and A_i are scalar and vector potentials, respectively, and g is the g-factor. The interaction term S_{int} is arbitrary as long as it is invariant under symmetries discussed in Sec. 3.4. We will discuss possible interaction terms in Sec. 3.5.

Since this action couple with the U(1) gauge field, we can compute the charge current by differentiating the action with respect to A_μ as we explained in Sec. 2.1. Additionally, we are interested in energy and momentum currents. Therefore, we need to couple the system with their external sources. For relativistic systems, this is achieved by placing the system on a curved spacetime, where a Riemann metric acts as an external source of the energy-momentum tensor. Similarly, for nonrelativistic systems, we should place the system on a nonrelativistic curved spacetime, which is described by the Newton-Cartan geometry. It is also possible to couple external fields conjugate to energy and momentum currents without relying on the Newton-Cartan geometry [91, 92]. However, the Newton-Cartan geometry provides the advantage of treating currents in a covariant manner [28]. Importantly, the choice of method for incorporating external fields does not affect our results in flat spacetime.

3.2 Newton-Cartan geometry

In this section, we review the Newton-Cartan geometry following Ref. [28, 30, 93, 94].

A Newton-Cartan spacetime is described by three metrics: a clock covector n_μ, a spatial metric $h^{\mu\nu}$, and a velocity vector v^μ, which are not independent. The clock covector n_μ is a zero eigenvector of $h^{\mu\nu}$:

$$h^{\mu\nu} n_\mu = 0, \tag{3.3}$$

and the velocity vector v^μ is normalized by n_μ:

$$n_\mu v^\mu = 1. \tag{3.4}$$

Since the spatial metric $h^{\mu\nu}$ does not have its inverse, we cannot define a spatial metric with lower indices $h_{\mu\nu}$ as in the Riemann geometry. Instead, $h_{\mu\nu}$ is defined to satisfy the following conditions,

$$h^{\mu\lambda} h_{\lambda\nu} = P^\mu_{\ \nu}, \tag{3.5}$$

$$h_{\mu\nu} v^\nu = 0, \tag{3.6}$$

where $P^\mu_{\ \nu} = \delta^\mu_{\ \nu} - v^\mu n_\nu$ is the spatial projector.

We introduce a covariant derivative ∇_μ with a connection $\Gamma^\lambda_{\ \mu\nu}$, which is determined as follows [93]. At first, we require the metric compatibility conditions:

$$\nabla_\lambda h^{\mu\nu} = 0, \tag{3.7}$$

$$\nabla_\nu n_\mu = 0. \tag{3.8}$$

Additionally, we impose the following condition,

$$h_{\lambda\alpha} T^\alpha_{\ \mu\nu} = 0, \tag{3.9}$$

where $T^\alpha_{\ \mu\nu} = \Gamma^\alpha_{\ [\mu\nu]}$ is a torsion tensor. This condition expresses that the spatial components of the torsion are zero. The connection is decomposed into components in parallel and perpendicular to n_μ:

$$\Gamma^\lambda_{\ \mu\nu} = v^\lambda \Gamma^v_{\mu\nu} + h^{\lambda\rho} \Gamma^h_{\rho\mu\nu}. \tag{3.10}$$

Then, the condition Eq. (3.9) reduces to

$$\Gamma^h_{\rho[\mu\nu]} = n_\rho A_{\mu\nu}, \tag{3.11}$$

where $A_{\mu\nu}$ is an arbitrary antisymmetric tensor, $A_{(\mu\nu)} = 0$. From Eqs. (3.7) and (3.8), we find

$$\Gamma^h_{\rho\mu\sigma} + \Gamma^h_{\sigma\mu\rho} = \partial_\mu h_{\rho\sigma} + n_\rho B_{\sigma\mu} + n_\sigma B_{\rho\mu}, \tag{3.12}$$

$$\Gamma^v_{\mu\nu} = \partial_\mu n_\nu, \tag{3.13}$$

where $B_{\mu\nu}$ is an arbitrary tensor. By cycling the indices of Eq. (3.12), we have

$$-\Gamma^h_{\mu\rho\sigma} - \Gamma^h_{\sigma\rho\mu} = -(\partial_\rho h_{\sigma\mu} + n_\sigma B_{\mu\rho} + n_\mu B_{\sigma\rho}), \tag{3.14}$$

$$\Gamma^h_{\rho\sigma\mu} + \Gamma^h_{\mu\sigma\rho} = \partial_\sigma h_{\mu\rho} + n_\mu B_{\rho\sigma} + n_\rho B_{\mu\sigma}. \tag{3.15}$$

By summing up Eqs. (3.12), (3.14), and (3.15) and using Eq (3.11), we obtain

$$\begin{aligned}\Gamma^h_{\rho(\mu\nu)} &= \partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} \\ &\quad + n_\rho B_{(\mu\nu)} + n_\mu (A_{\rho\nu} + B_{[\rho\nu]}) + n_\nu (A_{\rho\mu} + B_{[\rho\mu]}),\end{aligned} \tag{3.16}$$

which leads to

$$\begin{aligned}\Gamma^h_{\rho\mu\nu} &= \frac{1}{2}(\Gamma^h_{\rho(\mu\nu)} + \Gamma^h_{\rho[\mu\nu]}) \\ &= \frac{1}{2}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \\ &\quad + \frac{1}{2} n_\rho \left(A_{\mu\nu} + B_{(\mu\nu)}\right) + \frac{1}{2} n_\mu \left(A_{\rho\nu} + B_{[\rho\nu]}\right) + \frac{1}{2} n_\nu \left(A_{\rho\mu} + B_{[\rho\mu]}\right).\end{aligned} \tag{3.17}$$

From Eqs. (3.13) and (3.17), we obtain the explicit form of the connection,

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu + \frac{1}{2} h^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) + \frac{1}{2} h^{\lambda\rho} n_{(\mu} f_{\nu)\rho}, \tag{3.18}$$

where $f_{\mu\nu} = -(A_{\mu\nu} + B_{[\mu\nu]})$ is an antisymmetric tensor.

In general, we have degrees of freedom to add an arbitrary tensor to a connection. In this case, the combination of the first and second terms in Eq. (3.18) acts as a connection,[1] while the third term is a tensor. Here, we employ the minimal connection[2]

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu + \frac{1}{2} h^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}). \tag{3.23}$$

This connection has a nontrivial torsion,

$$T^\lambda{}_{\mu\nu} = v^\lambda \partial_{[\mu} n_{\nu]}, \tag{3.24}$$

which is called a temporal torsion since its spatial components are zero $h_{\rho\lambda} T^\lambda{}_{\mu\nu} = 0$. We also introduce a torsionless connection by subtracting the torsion tensor,

$$\mathring{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} v^\lambda \partial_{(\mu} n_{\nu)} + \frac{1}{2} h^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}). \tag{3.25}$$

[1] Under the general coordinate transformation $x^\mu \to y^\mu(x)$, the first and second terms transform as

$$v^\lambda \partial_\mu n_\nu \to \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} v^\gamma \partial_\alpha n_\beta + \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial^2 x^\beta}{\partial y^\mu \partial y^\nu} v^\gamma n_\beta, \tag{3.19}$$

$$\begin{aligned}\begin{Bmatrix} \lambda \\ \mu\ \nu \end{Bmatrix} &\to \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\begin{Bmatrix} \gamma \\ \alpha\ \beta \end{Bmatrix} + h_{\alpha\beta} h^{\gamma\delta} \delta^\beta{}_\delta \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu} \\ &= \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\begin{Bmatrix} \gamma \\ \alpha\ \beta \end{Bmatrix} + \frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu} - \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu} v^\gamma n_\alpha,\end{aligned} \tag{3.20}$$

where we define

$$\begin{Bmatrix} \lambda \\ \mu\ \nu \end{Bmatrix} = \frac{1}{2} h^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}). \tag{3.21}$$

Therefore, these combination transforms as a connection,

$$\Gamma^\lambda{}_{\mu\nu} \to \frac{\partial y^\lambda}{\partial x^\gamma}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\Gamma^\gamma{}_{\alpha\beta} + \frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu}. \tag{3.22}$$

[2] The ambiguity of the connection does not affect our final result because they can be absorbed into arbitrary coefficients in Eq. (3.93).

The covariant derivative with Γ has the following properties [28, 30]:

$$\nabla_\mu h^{\nu\lambda} = 0, \tag{3.26}$$
$$\nabla_\mu n_\nu = 0, \tag{3.27}$$
$$\nabla_\mu h_{\nu\lambda} = -\frac{1}{2}\tau_{\mu(\nu}n_{\lambda)}, \tag{3.28}$$
$$\nabla_\mu v^\nu = \frac{1}{2}\tau_{\mu\lambda}h^{\lambda\nu}, \tag{3.29}$$
$$v^\mu \nabla_\mu v^\nu = 0, \tag{3.30}$$
$$\nabla^{[\mu} v^{\nu]} = 0, \tag{3.31}$$

where we define $\tau_{\mu\nu} = v^\lambda \partial_\lambda h_{\mu\nu} + h_{\lambda\nu}\partial_\mu v^\lambda + h_{\mu\lambda}\partial_\nu v^\lambda$ and $\nabla^\mu = h^{\mu\nu}\nabla_\nu$.

3.3 Microscopic action on a Newton-Cartan spacetime

We now couple the action with Newton-Cartan metrics [28, 93, 94]. The microscopic action incorporating the Newton-Cartan metric should be constructed in a covariant manner, ensuring that all indices are fully contracted by other indices. Moreover, the action should reduce to its original form in the flat spacetime limit. A minimal action satisfying these conditions is given by

$$S_0 = \int d^3x \sqrt{\gamma}\left[v^\mu \Psi^\dagger i \overleftrightarrow{D}_\mu \Psi - \left(\frac{h^{\mu\nu}}{2m} + \frac{ig\,\varepsilon^{\lambda\mu\nu}n_\lambda}{4m}\right) D_\mu \Psi^\dagger D_\nu \Psi\right]. \tag{3.32}$$

Here the volume element is provided by $\sqrt{\gamma}\, d^3x$ with

$$\gamma = \det(\gamma_{\mu\nu}), \qquad \gamma_{\mu\nu} = n_\mu n_\nu + h_{\mu\nu}, \tag{3.33}$$

where $\gamma_{\mu\nu}$ has an inverse $(\gamma^{-1})^{\mu\nu} = v^\mu v^\nu + h^{\mu\nu}$, and the totally antisymmetric tensor reads $\varepsilon^{\lambda\mu\nu} = \epsilon^{\lambda\mu\nu}/\sqrt{\gamma}$. To recover the flat spacetime action Eq. (3.2), we should set $n_\mu = v^\mu = (1, 0, 0)$ and $h_{\mu\nu} = h^{\mu\nu} = \mathrm{diag}(0, 1, 1)$.

To confirm the Newton-Cartan metrics serve as external sources of energy and momentum currents, we vary the action with respect to the external fields $(A_\mu, n_\mu, v^\mu, h^{\mu\nu})$. Here we should be careful that we cannot vary the external fields arbitrary because of the constraints Eqs. (3.3) and (3.4). If we choose δA_μ and δn_μ to be arbitrary, the other variations are given by [28, 93, 94]

$$\delta v^\mu = -v^\mu v^\nu \delta n_\nu + P^\mu_\nu \delta \bar{v}^\nu, \tag{3.34}$$
$$\delta h^{\mu\nu} = -v^{(\mu} h^{\nu)\lambda}\delta n_\lambda + P^\mu_\rho P^\nu_\sigma \delta \bar{h}^{\rho\sigma}, \tag{3.35}$$
$$\delta h_{\mu\nu} = -n_{(\mu} h_{\nu)\lambda}\delta \bar{v}^\lambda - h_{\mu\rho}h_{\nu\sigma}\delta \bar{h}^{\rho\sigma}, \tag{3.36}$$

where δA_μ, δn_μ, $\delta \bar{v}^\mu$, and $\delta \bar{h}^{\mu\nu}$ are independent[3]. We now introduce currents conjugate to the external fields as

$$\delta S = \int d^3x \sqrt{\gamma}\left(\mathcal{J}^\mu \delta A_\mu - \mathcal{E}^\mu \delta n_\mu - \mathcal{P}_\mu \delta \bar{v}^\mu - \frac{1}{2}\mathcal{T}_{\mu\nu}\delta \bar{h}^{\mu\nu}\right), \tag{3.37}$$

[3]We can choose δv^μ or $\delta h^{\mu\nu}$ to be arbitrary. In Appendix A, we examine that our computations are not affected if we choose δv^μ arbitrarily.

where $\mathcal{J}^\mu$, $\mathcal{E}^\mu$, $\mathcal{P}_\mu$, and $\mathcal{T}_{\mu\nu}$ are to be identified as the electric current, energy current, momentum density, and stress tensor, respectively. Differentiations of S_0 in Eq. (3.32) are given by

$$\mathcal{J}^\mu = v^\mu \Psi^\dagger \Psi - \frac{h^{\mu\nu}}{m} \Psi^\dagger i \overleftrightarrow{D}_\nu \Psi + \frac{g\,\varepsilon^{\lambda\mu\nu} n_\lambda}{4m} \partial_\nu (\Psi^\dagger \Psi), \tag{3.38}$$

$$\mathcal{E}^\mu = \frac{v^\mu h^{\nu\lambda} - v^{(\nu} h^{\lambda)\mu}}{2m} D_\nu \Psi^\dagger D_\lambda \Psi + \frac{ig\,\varepsilon^{\mu\nu\lambda}}{4m} D_\nu \Psi^\dagger D_\lambda \Psi, \tag{3.39}$$

$$\mathcal{P}_\mu = -P^\nu_\mu \Psi^\dagger i \overleftrightarrow{D}_\nu \Psi, \tag{3.40}$$

$$\mathcal{T}_{\mu\nu} = h_{\mu\nu} \left(v^\lambda \Psi^\dagger i \overleftrightarrow{D}_\lambda \Psi - \frac{h^{\rho\sigma}}{2m} D_\rho \Psi^\dagger D_\sigma \Psi \right) + \frac{P^\rho_\mu P^\sigma_\nu}{2m} D_{(\rho} \Psi^\dagger D_{\sigma)} \Psi. \tag{3.41}$$

These expressions in a flat spacetime reduce to familiar forms of the currents (see, e.g., Ref. [95]) under the equations of motion for Ψ and $\Psi^\dagger$. This confirms that the Newton-Cartan metrics serve as external fields conjugate to the energy and momentum currents. We note ϕ_g in $n_\mu = (1+\phi_g, 0, 0)$ and $v^\mu = ((1+\phi_g)^{-1}, 0, 0)$ corresponds to Luttinger's gravitational potential conjugate to the energy density [96].

3.4 Symmetries

3.4.1 Milne boost

We now investigate the symmetries of the action Eq. (3.32). This action is invariant under both the U(1) gauge transformation,

$$\Psi \to e^{i\chi} \Psi, \ A_\mu \to A_\mu + \partial_\mu \chi, \tag{3.42}$$

and the general coordinate transformation,

$$x^\mu \to x'^\mu = x'^\mu(x), \tag{3.43}$$

where the fields with upper (lower) indices transform contravariantly (covariantly).

The constrains Eqs. (3.3) and (3.4) are preserved under a boost of the velocity vector:

$$v^\mu \to v^\mu + h^{\mu\nu} \psi_\nu, \tag{3.44}$$

$$h_{\mu\nu} \to h_{\mu\nu} - n_{(\mu} P^\lambda_{\nu)} \psi_\lambda + n_\mu n_\nu h^{\rho\sigma} \psi_\rho \psi_\sigma. \tag{3.45}$$

Under these transformations combined with

$$A_\mu \to A_\mu + m P^\nu_\mu \psi_\nu - \frac{m}{2} n_\mu h^{\rho\sigma} \psi_\rho \psi_\sigma + \frac{g}{4} n_\mu \varepsilon^{\nu\rho\sigma} \partial_\nu (n_\rho P^\lambda_\sigma \psi_\lambda), \tag{3.46}$$

the action Eq. (3.32) is invariant. This set of the transformations is called *Milne boost* [93, 94]. We note that γ is invariant under the Milne boost.[4]

[4]This invariance can be demonstrated for an infinitesimal Milne boost. Under the infinitesimal Milne boost, the inverse of $\gamma_{\mu\nu}$ transforms as

$$(\gamma^{-1})^{\mu\nu\prime} = (\gamma^{-1})^{\mu\nu} + v^{(\mu} h^{\nu)\lambda} \psi_\lambda, \tag{3.47}$$

which leads to

$$\frac{\partial (\gamma^{-1})^{\mu\nu\prime}}{\partial \psi_\lambda} = v^{(\mu} h^{\nu)\lambda}. \tag{3.48}$$

The transformation laws of the fields under the infinitesimal transformations are given as follows. For the infinitesimal U(1) transformation parameterized by χ, we have

$$\begin{aligned}&\delta_\chi n_\mu = 0, \quad \delta_\chi h^{\mu\nu} = 0, \quad \delta_\chi v^\mu = 0, \quad \delta_\chi h_{\mu\nu} = 0,\\ &\delta_\chi A_\mu = \partial_\mu \chi, \quad \delta_\chi \Psi = i\chi\Psi.\end{aligned} \tag{3.50}$$

For the infinitesimal general coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$, we have

$$\begin{aligned}\delta_\xi n_\mu &= -\xi^\nu \partial_\nu n_\mu - n_\nu \partial_\mu \xi^\nu,\\ \delta_\xi h^{\mu\nu} &= -\xi^\lambda \partial_\lambda h^{\mu\nu} + h^{\lambda\nu}\partial_\lambda \xi^\mu + h^{\mu\lambda}\partial_\lambda \xi^\nu,\\ \delta_\xi v^\mu &= -\xi^\nu \partial_\nu v^\mu + v^\nu \partial_\nu \xi^\mu,\\ \delta_\xi h_{\mu\nu} &= -\xi^\lambda \partial_\lambda h_{\mu\nu} - h_{\lambda\nu}\partial_\mu \xi^\lambda - h_{\mu\lambda}\partial_\nu \xi^\lambda,\\ \delta_\xi A_\mu &= -\xi^\nu \partial_\nu A_\mu - A_\nu \partial_\mu \xi^\nu,\\ \delta_\xi \Psi &= -\xi^\mu \partial_\mu \Psi.\end{aligned} \tag{3.51}$$

For the infinitesimal Milne boost parametrized by ψ_μ, we have

$$\begin{aligned}&\delta_\psi n_\mu = 0, \quad \delta_\psi h^{\mu\nu} = 0, \quad \delta_\psi v^\mu = h^{\mu\nu}\psi_\nu,\\ &\delta_\psi h_{\mu\nu} = -(n_\mu P^\lambda_{\ \nu} + n_\nu P^\lambda_{\ \mu})\psi_\lambda,\\ &\delta_\psi A_\mu = mP^\nu_{\ \mu}\psi_\nu + \frac{g}{4} n_\mu \varepsilon^{\nu\rho\sigma}\partial_\nu(n_\rho P^\lambda_\sigma \psi_\lambda),\\ &\delta_\psi \Psi = 0.\end{aligned} \tag{3.52}$$

The Galilean boost is a special case of these transformations in a flat spacetime:

$$\psi_i = V^i, \quad \xi^i = -V^i t, \quad \chi = \frac{1}{2} m V^2 t - m V^i x^i, \tag{3.53}$$

where V^i is the boost velocity. Therefore, the Milne boost can be understood as the local extension of the Galilean boost. While the Milne symmetry remains hidden in a Galilean-invariant flat spacetime, it becomes explicitly manifest in a Newton-Cartan spacetime. However, as we will see later, the constraints imposed by the Milne symmetry persist even in a flat spacetime, leading to nontrivial consequences. We also note the nonrelativistic general coordinate transformation [98] can also be reproduced by the combination of specific cases of the general coordinate and Milne transformations [93].

3.4.2 Ward identities

The symmetries of the action lead to the Ward identities (the conservation laws). The Ward identities can be derived by requiring $\delta S = 0$ under the transformations Eqs. (3.50)-(3.52), combined with the equations of motion for Ψ and $\Psi^\dagger$. The U(1) gauge invariance implies the charge conservation law [28, 93, 94],

$$(\nabla_\mu + B_\mu)\mathcal{J}^\mu = 0, \tag{3.54}$$

This implies

$$\gamma^{-1\prime} \frac{\partial \gamma'}{\partial \psi_\lambda} = \gamma'_{\mu\nu} \frac{\partial (\gamma^{-1})^{\mu\nu\prime}}{\partial \psi_\lambda} = 0. \tag{3.49}$$

Thus, γ is invariant under the infinitesimal Milne boost, $\partial_{\psi_\lambda}\gamma' = 0$.

where $B_\mu = T^\lambda_{\ \mu\lambda}$. The general coordinate invariance leads to the energy and momentum conservation laws [28, 93, 94],

$$(\nabla_\mu + B_\mu)\mathcal{E}^\mu = F_{\mu\nu}\mathcal{J}^\mu v^\nu - G_{\mu\nu}\mathcal{E}^\mu v^\nu - \frac{1}{2}(\nabla^\mu v^\nu + \nabla^\nu v^\mu)P^\rho_{\ \mu}P^\sigma_{\ \nu}\mathcal{T}_{\rho\sigma}, \tag{3.55}$$

$$(\nabla^\nu v^\mu)P^\rho_{\ \mu}\mathcal{P}_\rho + \nabla_\mu(v^\mu P^\nu_{\ \rho}\mathcal{P}^\rho) + (\nabla_\mu + B_\mu)(P^\rho_{\ \mu}P^\sigma_{\ \nu}\mathcal{T}_{\rho\sigma}) = -F_\mu^{\ \ \nu}\mathcal{J}^\mu + G_\mu^{\ \ \nu}\mathcal{E}^\mu, \tag{3.56}$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ and $G_{\mu\nu} = \partial_{[\mu}n_{\nu]}$. The Milne invariance establishes a relationship between the charge current and the momentum density [93, 94],

$$mP^\lambda_{\ \mu}\mathcal{J}^\mu - \frac{g}{4}\epsilon^{\nu\rho\sigma}\partial_\nu(\mathcal{J}^\mu n_\mu)n_\rho P^\lambda_{\ \sigma} = h^{\mu\lambda}\mathcal{P}_\mu. \tag{3.57}$$

In a flat spacetime, these conservation laws reduce to the familiar forms,

$$\partial_\mu\mathcal{J}^\mu = 0, \tag{3.58}$$

$$\partial_\mu\mathcal{E}^\mu = \mathcal{J}^i F_{it}, \tag{3.59}$$

$$\partial_t\mathcal{P}_i + \partial_j\mathcal{T}_{ij} = \mathcal{J}^\mu F_{i\mu}, \tag{3.60}$$

$$\mathcal{P}_i = m\mathcal{J}^i - \frac{g}{4}\epsilon^{ij}\partial_j\mathcal{J}^t. \tag{3.61}$$

3.5 Interaction

The various interaction can be incorporated into the theory while preserving the U(1) gauge, general coordinate, and Milne invariance. For instance, the Coulomb interaction, which propagates in three spatial dimensions, can be introduced via an auxiliary field a_0 [22],

$$S_{\text{int}} = \int d^3x\sqrt{\gamma}a_0\Psi^\dagger\Psi + 2\pi\varepsilon_0\int d^3x dz\sqrt{\gamma}[h^{\mu\nu}\partial_\mu a_0\partial_\nu a_0 + (\partial_z a_0)^2], \tag{3.62}$$

where ε_0 is the dielectric constant. The interaction by the power law potentials can similarly be introducing by employing auxiliary fields that live in higher dimensions [97]. As another example, an attractive interaction of range m_ϕ^{-1} can be incorporated as [22, 98],

$$S_{\text{int}} = \int d^3x\sqrt{\gamma}\left(\lambda\Psi^\dagger\Psi\phi - \frac{1}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m_\phi^2}{2}\phi^2\right). \tag{3.63}$$

Therefore, the above discussion on symmetries in Sec. 3.4 remains valid for interacting electrons. This ensures that our framework is available for the fractional quantum Hall states.

3.6 Preparation to construct effective action

In this section, we make some preparations to construct the effective action of quantum Hall systems which respects the U(1) gauge, general coordinate, and Milne invariance.

3.6.1 Power counting scheme

To systematically construct the effective action, we introduce a power counting scheme based on the assumption that external fields vary slowly over spacetime. This allows us to perform a derivative expansion, treating spacetime derivatives as small expansion parameters compared with the energy gap:

$$\partial_\mu \sim \mathcal{O}(\partial)\,. \tag{3.64}$$

In the quantum Hall systems, the energy gap is characterized by the cyclotron frequency, $\omega_c = B/m$. If we assume the magnetic field is $B \sim 1\,\mathrm{T}$ and use the mass of electrons in vacuum, the cyclotron frequency is of the order of $10^{11}\,\mathrm{s}^{-1}$. Therefore, our theory remains valid in the regime, where $\omega \ll \omega_c \sim 10^{11}\,\mathrm{s}^{-1}$.

The electromagnetic and Newton-Cartan fields are assumed to be order of unity,

$$F_{\mu\nu},\ n_\mu,\ v^\mu,\ h^{\mu\nu} \sim \mathcal{O}(1)\,. \tag{3.65}$$

Since the electromagnetic field $F_{\mu\nu}$ involves derivatives of the gauge field, the U(1) gauge field is regarded as

$$A_\mu \sim \mathcal{O}\left(\partial^{-1}\right). \tag{3.66}$$

3.6.2 Spin connection

To facilitate our calculations, we introduce the vielbein and spin connection, as outlined in Sec. 2.1.3. The vielbein is define as

$$h^{\mu\nu} = e^{a\mu} e^{a\nu}, \quad n_\mu e^{a\mu} = 0. \tag{3.67}$$

The spatial metric is invariant under the SO(2) rotation,

$$e^{x\mu} \pm i e^{y\mu} \to e^{\pm i\theta}(e^{x\mu} \pm i e^{y\mu}). \tag{3.68}$$

The spin connection is defined as

$$\omega_\mu = \frac{1}{2}\epsilon^{ab} h_{\lambda\nu} e^{a\lambda} \nabla_\mu e^{b\nu}, \tag{3.69}$$

where ∇_μ is the covariant derivative with the connection Eq. (3.23). Under a local SO(2) rotation of the vielbein, the spin connection transforms as an Abelian gauge field:

$$\omega_\mu \to \omega_\mu + \partial_\mu \theta. \tag{3.70}$$

3.6.3 Milne invariant objects

To construct the Milne invariant effective action, it is inconvenient that the external fields $v^\mu, h_{\mu\nu}, A_\mu$ transform under the Milne boost. Then, we modify the external fields to be invariant under the Milne boost. To achieve this, we introduce a Milne invariant vector u^μ, which is normalized as $n_\mu u^\mu = 1$. The corresponding covector $u_\mu = h_{\mu\nu} u^\nu$ and the scalar product $u^2 = h_{\mu\nu} u^\mu u^\nu$ are not Milne invariant:

$$u^\mu \to u^\mu, \tag{3.71}$$

$$u_\mu \to u_\mu - P^\nu_{\ \mu} + n_\mu h^{\nu\lambda}(\psi_\nu \psi_\lambda - u_\nu \psi_\lambda), \tag{3.72}$$

$$u^2 \to u^2 + h^{\mu\nu}(\psi_\mu \psi_\nu - 2u_\mu \psi_\nu). \tag{3.73}$$

Using u_μ and u^2, we construct Milne invariant objects [94],

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - n_{(\mu} u_{\nu)} + n_\mu n_\nu u^2, \tag{3.74}$$

$$\tilde{A}_\mu = A_\mu + m u_\mu - \frac{m}{2} n_\mu u^2 + \frac{g}{4} n_\mu \varepsilon^{\nu\rho\sigma} \partial_\nu (n_\rho u_\sigma). \tag{3.75}$$

The connection and spin connection are also not Milne invariant. To make them invariant, we should replace $(v^\mu, h_{\mu\nu})$ with $(u^\mu, \tilde{h}_{\mu\nu})$,

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = u^\lambda \partial_\mu n_\nu + \frac{1}{2} h^{\lambda\rho}(\partial_\mu \tilde{h}_{\rho\nu} + \partial_\nu \tilde{h}_{\mu\rho} - \partial_\rho \tilde{h}_{\mu\nu}), \tag{3.76}$$

$$\tilde{\omega}_\mu = \frac{1}{2}\epsilon^{ab}\tilde{h}_{\lambda\nu} e^{a\lambda}\tilde{\nabla}_\mu e^{b\nu}, \tag{3.77}$$

where $\tilde{\nabla}_\mu$ is the covariant derivative with $\tilde{\Gamma}^\lambda{}_{\mu\nu}$.

The properties of the Newton-Cartan geometry remain valid when replacing $(h^{\mu\nu}, n_\mu, v^\mu, \nabla_\mu)$ with $(h^{\mu\nu}, n_\mu, u^\mu, \tilde{\nabla}_\mu)$ [94]:

$$n_\mu u^\mu = 1, \tag{3.78}$$

$$\tilde{h}_{\mu\nu} u^\nu = 0, \tag{3.79}$$

$$\tilde{h}_{\mu\lambda} h^{\lambda\nu} = \tilde{P}^\nu{}_\mu = \delta^\nu{}_\mu - u^\nu n_\mu, \tag{3.80}$$

$$\tilde{\nabla}_\lambda h^{\mu\nu} = 0, \tag{3.81}$$

$$\tilde{\nabla}_\nu n_\mu = 0, \tag{3.82}$$

$$\tilde{\nabla}_\lambda \tilde{h}_{\mu\nu} = -\tilde{\tau}_{\lambda(\mu} n_{\nu)}, \tag{3.83}$$

$$\tilde{\nabla}_\nu u^\mu = \frac{1}{2}\tilde{\tau}_{\nu\alpha} h^{\alpha\mu}, \tag{3.84}$$

$$u^\lambda \tilde{\nabla}_\lambda u^\mu = 0, \tag{3.85}$$

$$\tilde{\nabla}^{[\mu} u^{\nu]} = 0. \tag{3.86}$$

We note $\tilde{\gamma} = \gamma$ and $\tilde{\Gamma}^\lambda{}_{\mu\lambda} = \Gamma^\lambda{}_{\mu\lambda}$.

We construct u^μ order by order in derivative expansion,

$$u^\mu = u^\mu_{(0)} + u^\mu_{(1)} + \cdots, \tag{3.87}$$

where $u^\mu_{(n)} \sim \mathcal{O}(\partial^n)$. The leading term is uniquely determined by the symmetries and is given by

$$u^\mu_{(0)} = \frac{\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda}{B}, \tag{3.88}$$

with $B = \varepsilon^{\mu\nu\lambda} n_\mu \partial_\nu A_\lambda$. Under the Milne boost, the U(1) gauge field transforms as $A_\mu \to A_\mu + \mathcal{O}(1)$, which leads to $\partial_\mu A_\nu \to \partial_\mu A_\nu + \mathcal{O}(\partial)$. Thus, $u^\mu_{(0)}$ is Milne invariant up to $\mathcal{O}(\partial)$. The vector $u^\mu_{(0)}$ corresponds to the drift velocity of the quantum Hall systems in a flat spacetime. The drift velocity also appeared in Refs. [29, 99, 100] to serve similar roles.

Although we cannot determine the next-to-leading term $u^\mu_{(1)}$ uniquely, it does not appear in our effective action as we will show later.

3.7 Construction of effective action

We construct the effective action up to $\mathcal{O}(\partial^2)$ based on the symmetries. We summarize available data for constructing effective action in Table. 3.1, where we define $\tilde{B} = \varepsilon^{\mu\nu\lambda} n_\mu \partial_\nu \tilde{A}_\lambda$, $G_{\mu\nu} = \partial_{[\mu} n_{\nu]}$, $G^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu n_\lambda$, $G = \varepsilon^{\mu\nu\lambda} n_\mu \partial_\nu n_\lambda$, and $\tilde{B}_\mu = u^\lambda G_{\mu\lambda}$. From these data, scalars up to $\mathcal{O}(\partial^2)$ can be constructed as in Table. 3.2. This table contains scalars that are not independent or are zero up to $\mathcal{O}(\partial^2)$. By the evaluation presented in Appendix B, Table. 3.2 is simplified as Table. 3.3. In this table, we have not

Leading order	Data
∂^{-1}	$\tilde{A}_\mu$
∂^0	$\tilde{F}_{\mu\nu}$, $h^{\mu\nu}$, $\tilde{h}_{\mu\nu}$, n_μ, u^μ, $\varepsilon^{\mu\nu\lambda}$
∂^1	$\tilde{\nabla}_\mu$, $\tilde{\omega}_\mu$, $\tilde{B}_\mu$, $G_{\mu\nu}$, G^μ, G

Table 3.1: Available data for constructing effective action.

Leading order	Scalar
∂^0	$\tilde{B}$, $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$, $\tilde{h}_{\mu\rho}\varepsilon^{\mu\nu\lambda}\tilde{F}_{\nu\lambda}\varepsilon^{\rho\sigma\tau}\tilde{F}_{\sigma\tau}$
∂^1	$(\tilde{\nabla}_\mu\varepsilon^{\mu\nu\lambda})\tilde{F}_{\nu\lambda}$, $\varepsilon^{\mu\nu\lambda}(\tilde{\nabla}_\mu\tilde{F}_{\nu\lambda})$, $\varepsilon^{\mu\nu\lambda}\tilde{F}_{\nu\lambda}\partial_\mu\tilde{B}$, $h^{\mu\nu}(\tilde{\nabla}_\mu u^\lambda)\tilde{F}_{\nu\lambda}$, $h^{\mu\nu}u^\lambda(\tilde{\nabla}_\mu\tilde{F}_{\nu\lambda})$, $h^{\mu\nu}u^\lambda\tilde{F}_{\nu\lambda}\partial_\mu\tilde{B}$, $h^{\mu\nu}(\tilde{\nabla}_\mu u^\lambda)\tilde{h}_{\nu\lambda}$, $h^{\mu\nu}u^\lambda(\tilde{\nabla}_\mu\tilde{h}_{\nu\lambda})$, $h^{\mu\nu}u^\lambda\tilde{h}_{\nu\lambda}\partial_\mu\tilde{B}$ $\tilde{\nabla}_\mu u^\mu$, $u^\mu\partial_\mu\tilde{B}$, $\varepsilon^{\mu\nu\lambda}\tilde{B}_\mu\tilde{F}_{\nu\lambda}$, $h^{\mu\nu}u^\lambda\tilde{B}_\mu\tilde{F}_{\nu\lambda}$, $\tilde{B}_\mu u^\mu$, $G_{\mu\nu}\tilde{F}^{\mu\nu}$, $G^\mu u^\nu\tilde{F}_{\mu\nu}$, $G^\mu u^\nu\tilde{h}_{\mu\nu}$, G

Table 3.2: Scalars constructed from the data.

yet considered total derivative terms, which are the surface term in the action. In general, a total derivative term is written as

$$\partial_\mu(\sqrt{\gamma}V^\mu) = \sqrt{\gamma}(\nabla_\mu + B_\mu)V^\mu = \sqrt{\gamma}(\tilde{\nabla}_\mu + \tilde{B}_\mu)V^\mu, \tag{3.89}$$

where V^μ is a vector. Using this formula, we find that $\tilde{\nabla}_\mu u^\mu$ and $u^\mu\partial_\mu\tilde{B}$ are not independent because they differ by a total derivative term:

$$\begin{aligned}(\text{surface}) &= \sqrt{\gamma}(\tilde{\nabla}_\mu + \tilde{B}_\mu)(f(\tilde{B})u^\mu)\\ &= \sqrt{\gamma}f'(\tilde{B})u^\mu\partial_\mu\tilde{B} + \sqrt{\gamma}f(\tilde{B})\tilde{\nabla}_\mu u^\mu + \sqrt{\gamma}f(\tilde{B})u^\mu\tilde{B}_\mu\\ &= \sqrt{\gamma}f'(\tilde{B})u^\mu\partial_\mu\tilde{B} + \sqrt{\gamma}f(\tilde{B})\tilde{\nabla}_\mu u^\mu,\end{aligned} \tag{3.90}$$

where we use $u^\mu\tilde{B}_\mu = 0$. Moreover, we observe that $u^\mu\partial_\mu\tilde{B}$ itself is a surface term:

$$\begin{aligned}(\text{surface}) &= \partial_\mu(\sqrt{\gamma}f(\tilde{B})\tilde{B}u^\mu)\\ &= \partial_\mu(\sqrt{\gamma}f(B)Bu^\mu_{(0)}) + \mathcal{O}\left(\partial^2\right)\\ &= \partial_\mu(f(B)\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda) + \mathcal{O}\left(\partial^2\right)\\ &= \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\partial_\mu f(B) + \mathcal{O}\left(\partial^2\right)\\ &= \sqrt{\gamma}Bf'(B)u^\mu_{(0)}\partial_\mu B + \mathcal{O}\left(\partial^2\right)\\ &= \sqrt{\gamma}Bf'(B)u^\mu\partial_\mu\tilde{B} + \mathcal{O}\left(\partial^2\right).\end{aligned} \tag{3.91}$$

We also have the Chern-Simons and Wen-Zee terms as we reviewed in Chapter 2. They should be modified to be Milne invariant as $\varepsilon^{\mu\nu\lambda}\tilde{A}_\mu\partial_\nu\tilde{A}_\lambda$ and $\varepsilon^{\mu\nu\lambda}\tilde{\omega}_\mu\partial_\nu\tilde{A}_\lambda$.

From the above evaluation, we finally find that the scalars we can use in the effective action are

$$\varepsilon^{\mu\nu\lambda}\tilde{A}_\mu\partial_\nu\tilde{A}_\lambda, \quad \varepsilon^{\mu\nu\lambda}\tilde{\omega}_\mu\partial_\nu\tilde{A}_\lambda, \quad \tilde{B}, \quad G = \varepsilon^{\mu\nu\lambda}n_\mu\partial_\nu n_\lambda. \tag{3.92}$$

Leading order	Scalar
∂^0	$\tilde{B}$
∂^1	$\tilde{\nabla}_\mu u^\mu$, $u^\mu \partial_\mu \tilde{B}$, G

Table 3.3: Non-vanishing and independent scalars constructed from the data.

Therefore, the effective action which respects U(1) gauge, general coordinate, and Milne invariance up to $\mathcal{O}\left(\partial^2\right)$ is given by

$$S_{\text{eff}} = \int d^3x\sqrt{\gamma}\left[\frac{\nu}{4\pi}\varepsilon^{\mu\nu\lambda}\tilde{A}_\mu\partial_\nu\tilde{A}_\lambda + \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\lambda}\tilde{\omega}_\mu\partial_\nu\tilde{A}_\lambda - \mathcal{E}(\tilde{B}) - \mathcal{M}(\tilde{B})\varepsilon^{\lambda\mu\nu}n_\lambda\partial_\mu n_\nu + O(\partial^2)\right], \tag{3.93}$$

where ν and κ are arbitrary constants, and $\mathcal{E}(\tilde{B})$ and $\mathcal{M}(\tilde{B})$ are arbitrary functions of $\tilde{B}$. Their physical meanings implied already by their symbols will be identified later, whose expressions for free fermions are also presented in Appendix C.

Since the Chern-Simons term is $\mathcal{O}\left(\partial^{-1}\right)$ at its leading order, $u^\mu_{(1)}$ could potentially contribute to the effective action at $\mathcal{O}(\partial)$. Its contribution is proportional to

$$m(u^{(1)}_\mu - u^{(1)}_\rho u^\rho_{(0)}n_\mu)\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = mB(u^{(1)}_\mu - u^{(1)}_\rho u^\rho_{(0)}n_\mu)u^\mu_{(0)} = 0. \tag{3.94}$$

Thus, $u^\mu_{(1)}$ does not contribute to the effective action.

The Wen-Zee term is expressed in terms of the vielbein rather than the spatial metric. Here, the variation of the vielbein constrained by Eq. (3.67) is provided by

$$\delta e^{a\mu} = -v^\mu e^{a\nu}\delta n_\nu + P^\mu_\nu \delta\bar{e}^{a\nu}, \tag{3.95}$$

where $\delta\bar{e}^{a\mu}$ relates to $\delta\bar{h}^{\mu\nu} = e^{a(\mu}\delta\bar{e}^{a\nu)}$. The stress tensor is then obtained from

$$\mathcal{T}_{\mu\nu} = -\frac{h_{\mu\lambda}e^{a\lambda}}{\sqrt{\gamma}}\frac{\delta S_{\text{eff}}}{\delta\bar{e}^{a\nu}}, \tag{3.96}$$

which is guaranteed to be symmetric in $\mu \leftrightarrow \nu$ by the invariance of the effective action under the local rotation in Eq. (3.68). We note that the Wen-Zee action can also be expressed with the spatial metric as shown in Appendix D, although its gauge invariance up to a surface term is obscured.

3.8 Responses

We can compute the charge, energy, and momentum currents straightforwardly by differentiating the effective action (3.93) according to Eqs. (3.37) and (3.96) and using formulae presented in Appendix E. Below, we present their expressions in flat spacetime for two cases: equilibrium and out-of-equilibrium. General expressions are available in Appendix F.

In the equilibrium case, there is no electric field $E_i = \partial_{[i}A_{t]} = 0$, and a static and inhomogeneous magnetic field $B = \partial_{[x}A_{y]}$ is applied. The charge current is given by

$$\mathcal{J}^t = \frac{\nu}{2\pi}B + \partial_i\left[\left\{m\mathcal{E}''(B) + \frac{\nu g}{8\pi} - \frac{\kappa}{8\pi}\right\}\frac{\partial_i B}{B}\right] + \mathcal{O}\left(\partial^3\right), \tag{3.97}$$

$$\mathcal{J}^i = -\epsilon^{ij}\partial_j\mathcal{E}'(B) + \mathcal{O}\left(\partial^3\right). \tag{3.98}$$

The energy current is

$$\mathcal{E}^t = \mathcal{E}(B) + \mathcal{O}\left(\partial^2\right), \tag{3.99}$$

$$\mathcal{E}^i = \epsilon^{ij}\partial_j\mathcal{M}(B) + \mathcal{O}\left(\partial^2\right). \tag{3.100}$$

The stress tensor is

$$\mathcal{T}_{ij} = \delta_{ij}[B\mathcal{E}'(B) - \mathcal{E}(B)] + \mathcal{O}\left(\partial^2\right). \tag{3.101}$$

The momentum density is provided by Eq. (3.61). From Eq. (3.99), $\mathcal{E}(B)$ is identified as the energy density. We also identify $\mathcal{M}(B)$ as the energy magnetization from Eq. (3.100), whereas $-\mathcal{E}'(B)$ is identified as the charge magnetization from Eq. (3.98). We find the internal pressure (the diagonal elements of the stress tensor) is given by $P(B) = B\mathcal{E}'(B) - \mathcal{E}(B)$, which is consistent with the formula in Refs. [26, 101]. Since κ represents the Hall viscosity as we will show later, the formula Eq. (3.97) provides the relationship between the charge density and the Hall viscosity. A similar formula is found in Ref. [30].

In the out-of-equilibrium case, a spacetime-dependent electric field and a constant magnetic field are applied. First, the stress tensor is

$$\begin{aligned}\mathcal{T}_{ij} = \frac{\nu m}{2\pi}&\left[\frac{\epsilon^{ik}\epsilon^{jl}E_kE_l}{B}\left(1 + m\frac{\partial_nE_n}{B^2}\right) + m\epsilon^{(ik}E_k\frac{B\partial_tE_{j)} - \frac{1}{2}\epsilon^{j)l}\partial_lE^2}{B^3} - \frac{g}{4m}\delta_{ij}\partial_kE_k\right]\\ &+ \frac{\kappa}{8\pi}[\partial_{(i}E_{j)} - \delta_{ij}\partial_kE_k] + \delta_{ij}[B\mathcal{E}'(B) - \mathcal{E}(B) - \mathcal{E}''(B)m\partial_kE_k] + \mathcal{O}\left(\partial^2\right).\end{aligned} \tag{3.102}$$

The second term (the term with κ) comes from the Wen-Zee term and can be expressed as

$$\mathcal{T}_{ij}^{\mathrm{WZ}} = -\frac{\kappa B}{8\pi}\frac{1}{2}[\epsilon^{ik}\delta^{jl} + \epsilon^{jk}\delta^{il} + (k \leftrightarrow l)]\partial_ku_l, \tag{3.103}$$

where $u^i = \epsilon^{ij}E_j/B$ is the drift velocity. From the definition of the Hall viscosity Eq. (1.3), we find

$$\eta_H = \frac{\kappa B}{8\pi}. \tag{3.104}$$

The charge current is

$$\mathcal{J}^t = \frac{\nu}{2\pi}\left[B - m\frac{\partial_iE_i}{B} + m^2\partial_i\left(\frac{\frac{1}{2}\partial_iE^2 - E_i\partial_jE_j}{B^3}\right)\right] + \mathcal{O}\left(\partial^3\right), \tag{3.105}$$

$$\begin{aligned}\mathcal{J}^i = \frac{\nu}{2\pi}&\left[\epsilon^{ij}E_j + m\frac{B\partial_tE_i - \frac{1}{2}\epsilon^{ij}\partial_jE^2}{B^2} + \frac{g}{4}\frac{\epsilon^{ij}\partial_j\partial_kE_k}{B}\right.\\ &\left.-m^2\partial_t\left(\frac{\epsilon^{ij}B\partial_tE_j + \frac{1}{2}\partial_iE^2 - E_i\partial_jE_j}{B^3}\right) + m^2\epsilon^{ij}\partial_j\left(E_k\frac{\epsilon^{kl}B\partial_tE_l + \frac{1}{2}\partial_kE^2 - E_k\partial_lE_l}{B^4}\right)\right]\\ &-\frac{\kappa}{8\pi}\frac{\epsilon^{ij}\partial_j\partial_kE_k}{B} + \mathcal{E}''(B)m\frac{\epsilon^{ij}\partial_j\partial_kE_k}{B} + \mathcal{O}\left(\partial^3\right).\end{aligned} \tag{3.106}$$

The linear response in the charge current is

$$\begin{aligned}\mathcal{J}^i = &\frac{\nu}{2\pi}\left(\epsilon^{ij}E_j + m\frac{\partial_tE_i}{B} - m^2\frac{\epsilon^{ij}\partial_t^2E_j}{B^2}\right)\\ &+ \left[m\mathcal{E}''(B) + \frac{\nu g}{8\pi} - \frac{\kappa}{8\pi}\right]\frac{\epsilon^{ij}\partial_j\partial_kE_k}{B} + \mathcal{O}\left(E^2\right) + \mathcal{O}\left(\partial^3\right).\end{aligned} \tag{3.107}$$

The second line of Eq. (3.107) shows the Hall viscosity contributing to the Hall conductivity at nonzero wave number [25–30].

The first term in the first line allows us to identify $\nu/2\pi$ as the Hall conductivity,

$$\sigma_H = \frac{\nu}{2\pi}. \tag{3.108}$$

The second and third terms in the first line show that the Hall conductivity contributes to other responses, such as the longitudinal response at nonzero frequency. This can be understood as follows. We consider the equation of motion for a single electron under a dynamical but homogeneous electric field and a constant magnetic field,

$$m\dot{u}^i = E_i + \epsilon^{ij} u^j B. \tag{3.109}$$

This equation can be solved as

$$\begin{aligned} u^i &= \frac{1}{B^2 + m^2\partial_t^2}\left(B\epsilon^{ij}E_j + m\partial_t E_i\right) \\ &= \frac{1}{B}\epsilon^{ij}E_j + \frac{m}{B^2}\partial_t E_i + \frac{m^2}{B^3}\epsilon^{ij}\partial_t^2 E_j + \mathcal{O}\left(\partial_t^3\right). \end{aligned} \tag{3.110}$$

The charge current is obtained by multiplying the charge density $\nu B/2\pi$,

$$\begin{aligned} \mathcal{J}^i &= \frac{\nu B}{2\pi} u^i \\ &= \frac{\nu}{2\pi}\left(\epsilon^{ij}E_j + m\frac{\partial_t E_i}{B} - m^2\frac{\epsilon^{ij}\partial_t^2 E_j}{B^2}\right) + \mathcal{O}\left(\partial_t^3\right), \end{aligned} \tag{3.111}$$

which is consistent with the first line of Eq. (3.107). Although this discussion is for free electrons, our result is also applicable to interacting electrons, which is consistent with Kohn's theorem [102]. The same result is reported in Ref [26], which is derived by the Kubo formula.

The energy current is found to be

$$\mathcal{E}^t = \frac{\nu m}{2\pi}\left(\frac{E^2}{2B} - mE_i\frac{\epsilon^{ij}B\partial_t E_j + \frac{1}{2}\partial_i E^2 - \frac{1}{2}E_i\partial_j E_j}{B^3}\right) + \mathcal{E}(B) - \mathcal{E}'(B)m\frac{\partial_i E_i}{B} + \mathcal{O}\left(\partial^2\right), \tag{3.112}$$

$$\begin{aligned} \mathcal{E}^i = \frac{\nu m}{2\pi}&\left[\frac{\epsilon^{ij}E_j}{B}\left(\frac{E^2}{2B} - mE_k\frac{\epsilon^{kl}B\partial_t E_l + \frac{1}{2}\partial_k E^2 - E_k\partial_l E_l}{B^3} - \frac{g}{4m}\partial_k E_k\right)\right. \\ &\left.+ \frac{m}{2}E^2\left(\frac{B\partial_t E_i - \frac{1}{2}\epsilon^{ij}\partial_j E^2}{B^4}\right)\right] - \frac{\kappa}{8\pi}\frac{\epsilon^{ij}E_k\partial_k E_j + \epsilon^{jk}E_j\partial_k E_i}{B} \\ &+ \mathcal{E}'(B)\left(\epsilon^{ij}E_j + m\frac{B\partial_t E_i - \frac{1}{2}\epsilon^{ij}\partial_j E^2}{B^2}\right) - \mathcal{E}''(B)m\frac{\epsilon^{ij}E_j\partial_k E_k}{B} + \mathcal{O}\left(\partial^2\right). \end{aligned} \tag{3.113}$$

The second term in the second line (the term with κ) of $\mathcal{E}^i$ shows that the Hall viscosity contributes to the nonlinear electrothermal response at nonzero wave number. This is a main result of our work.

We note that the terms without derivatives in the currents are consistent with the ideal hydrodynamics (see, e.g., Ref. [95]), where

$$\mathcal{J}^i_{\text{hydro}} = \mathcal{J}^t u^i, \tag{3.114}$$

$$\mathcal{E}^t_{\text{hydro}} = \mathcal{E}(B) + \frac{m}{2} J^t u^2, \tag{3.115}$$

$$\mathcal{E}^i_{\text{hydro}} = [P(B) + \mathcal{E}^t] u^i, \tag{3.116}$$

$$\mathcal{T}^{ij}_{\text{hydro}} = P(B)\delta^{ij} + m\mathcal{J}^t u^i u^j \tag{3.117}$$

hold. We also note that the terms with κ arising from the Wen-Zee term are consistent with the parity-violating hydrodynamics in two dimensions [94, 103, 104],

$$\mathcal{E}^i_{\text{WZ}} = \mathcal{T}^{ij}_{\text{WZ}} u^j. \tag{3.118}$$

3.9 Summary

Let us summarize this chapter. We studied quantum Hall systems with Galilean invariance and their charge, energy, and momentum currents induced by electromagnetic fields with an effective field theory rooted entirely in symmetries. The resulting local currents are completely determined by the Hall conductivity and viscosity and the energy density $\mathcal{E}(B)$ and magnetization $\mathcal{M}(B)$ up to the next-to-next-to-leading orders in the derivative expansion. We found universal relations among distinct kinds of responses.

To highlight our key findings, we consider the case when an electric field is applied in x direction under a constant magnetic field. Then, the longitudinal conductivity at nonzero frequency is determined by the Hall conductivity according to

$$\mathcal{J}^x|_{\mathcal{O}(\partial E)} = \frac{\sigma_H m}{B} \partial_t E_x = \frac{\sigma_H}{\omega_c} \partial_t E_x. \tag{3.119}$$

If we assume an electric field with a frequency of a few hertz, this frequency is significantly smaller than the cyclotron frequency, $\omega_c \sim 10^{11}\,\text{Hz}$. Therefore, this effect cannot be observed in such a situation.

Furthermore, the Hall viscosity contributes not only to the Hall conductivity at nonzero wave number as

$$\mathcal{J}^y|_{\mathcal{O}(\partial^2 E)} = -\left[m\mathcal{E}''(B) + \frac{g}{4}\sigma_H - \frac{\eta_H}{B}\right] \frac{\partial_x^2 E_x}{B}, \tag{3.120}$$

but also to the nonlinear electrothermal conductivity at nonzero wave number as

$$\mathcal{E}^y|_{\mathcal{O}(\partial E^2)} = \left[\frac{m\mathcal{E}'(B)}{B} + m\mathcal{E}''(B) + \frac{g}{4}\sigma_H + \frac{\eta_H}{B}\right] \frac{\partial_x E_x^2}{2B}. \tag{3.121}$$

The result Eq. (3.121) can be anticipated from Eq. (3.120) and the conservation law Eq. (3.59) except for the first term $m\mathcal{E}'(B)/B$.

Chapter 4

Weyl semimetals from low-energy effective field theory

In this chapter, we investigate the electromagnetic linear responses of Weyl semimetals. Starting from the low-energy effective action Eq. (2.48), we will compute the expectation value of the current density at the linear order both in electromagnetic and axial gauge fields in Sec. 4.1. In Sec. 4.2, we will investigate the property of the current density driven by time-dependent magnetic fields. We will summarize this chapter in Sec. 4.3.

4.1 Current density from field-theoretical computation

As we reviewed in Sec. 2.2, the effective action of the Weyl semimetals is given by [67, 70]

$$S_{\text{Weyl}} = \int d^4x \bar{\psi}(x)(i\partial\!\!\!/ - A\!\!\!/(x) - A\!\!\!/_5\gamma^5)\psi(x) \tag{4.1}$$

with $A_\mu = (\phi/v, -\boldsymbol{A})$ and $A_{5\mu} = (\phi_5/v, -\boldsymbol{A}_5)$. As discussed in Chap. 2, this effective theory is valid in the regime $\omega \ll \frac{v}{a}$, where a is the lattice constant. In general, the lattice constant of a Weyl semimetal is several angstroms, and the Fermi velocity is of the order of $10^5\,\mathrm{m/s}$ [5]. This gives $\omega \ll 10^{15}\,\mathrm{s}^{-1} \sim 10^2\,\mathrm{meV}$.

Starting from this effective action, we compute the current density in the absence of the axial imbalance, $\mu_5 = -\phi_5$, as illustrated in Fig. 2.5. Moreover, we do not consider the effect of the collision. The current density is expressed as

$$\begin{aligned} J^\mu(x) &= \left\langle \bar{\psi}(x)\gamma^\mu\psi(x) \right\rangle \\ &= -\lim_{x'\to x} \mathrm{tr}[\gamma^\mu G(x-x')], \end{aligned} \tag{4.2}$$

where $G(x-x')$ is the full Green function under the vector and axial gauge fields:

$$G(x-x') = \frac{i}{i\partial\!\!\!/ - A\!\!\!/(x) - A\!\!\!/_5\gamma^5}\delta(x-x'). \tag{4.3}$$

We are interested in the linear order both in A_μ and $A_{5\mu}$. Then, we expand the current density as

$$J^\mu(x) \simeq -i \lim_{x'\to x} \mathrm{tr}\left[\gamma^\mu \frac{1}{i\partial\!\!\!/} A\!\!\!/_5\gamma^5 \frac{1}{i\partial\!\!\!/} A\!\!\!/(x) \frac{1}{i\partial\!\!\!/}\delta(x-x') + \gamma^\mu \frac{1}{i\partial\!\!\!/} A\!\!\!/(x) \frac{1}{i\partial\!\!\!/} A\!\!\!/_5\gamma^5 \frac{1}{i\partial\!\!\!/}\delta(x-x')\right]. \tag{4.4}$$

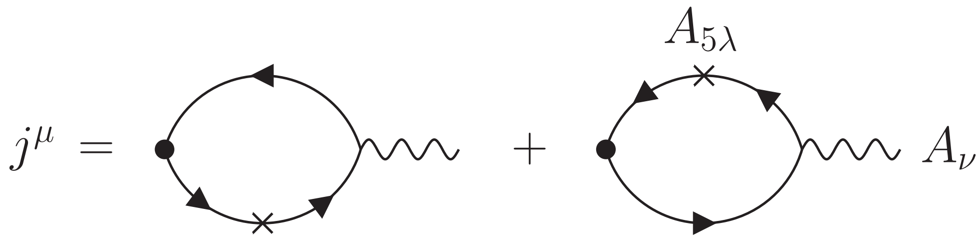


Figure 4.1: Feynman diagrams contributing to the current density in Eq. (4.4). This figure is cited from T. Amitani and Y. Nishida, "Dynamical chiral magnetic current and instability in Weyl semimetals," Phys. Rev. B **107**, 014302 (2023), © 2023 American Physical Society.

In terms of the Feynman diagram, the current density is represented as shown in Fig. 4.1.

We perform the Fourier transformation of the delta function and the gauge field as

$$
\begin{aligned}
J^\mu(x) &= -i \lim_{x\to x'} \int_{l,k} \mathrm{tr}\left[\gamma^\mu \frac{1}{i\not{\partial}} \not{A}_5 \gamma^5 \frac{1}{i\not{\partial}} \not{A}(k) e^{-ik\cdot x} \frac{1}{i\not{\partial}} e^{-il\cdot(x-x')} + \gamma^\mu \frac{1}{i\not{\partial}} \not{A}(k) e^{-ik\cdot x} \frac{1}{i\not{\partial}} \not{A}_5 \gamma^5 \frac{1}{i\not{\partial}} e^{-il\cdot(x-x')}\right] \\
&= -i \int_{l,k} \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}} \gamma^\lambda \gamma^5 \frac{1}{\not{l}+\not{k}} \gamma^\nu \frac{1}{\not{l}} + \gamma^\mu \frac{1}{\not{l}+\not{k}} \gamma^\nu \frac{1}{\not{l}} \gamma^\lambda \gamma^5 \frac{1}{\not{l}}\right] A_\nu(k) A_{5\lambda} e^{-ik\cdot x}, \quad (4.5)
\end{aligned}
$$

where $\int_l = \int \frac{d^4 l}{(2\pi)^4}$. Since the l-integral is linearly divergent at first glance, we should regularize the integral.[1] We now employ the Pauli-Villars regularization, where a ghost field with infinite mass is introduced [105]. In this regularization, we replace the integral as

$$
\begin{aligned}
&\int_l \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}} \gamma^\lambda \gamma^5 \frac{1}{\not{l}+\not{k}} \gamma^\nu \frac{1}{\not{l}} + \gamma^\mu \frac{1}{\not{l}+\not{k}} \gamma^\nu \frac{1}{\not{l}} \gamma^\lambda \gamma^5 \frac{1}{\not{l}}\right] \\
&\to \int_l \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}-m} \gamma^\lambda \gamma^5 \frac{1}{\not{l}+\not{k}-m} \gamma^\nu \frac{1}{\not{l}-m} + \gamma^\mu \frac{1}{\not{l}+\not{k}-m} \gamma^\nu \frac{1}{\not{l}-m} \gamma^\lambda \gamma^5 \frac{1}{\not{l}-m}\right]_{\mathrm{reg}}
\end{aligned}
\quad (4.6)
$$

with

$$
\int_l [f(l;m)]_{\mathrm{reg}} = \lim_{M\to\infty} \int_l [f(l;0) - f(l;M)], \quad (4.7)
$$

where the limit is taken after the integration [39]. Then, the regularized current density in Fourier spacetime is given by

$$
\begin{aligned}
J^\mu(k) = &-i \int_k \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}-m} \gamma^\lambda \gamma^5 \frac{1}{\not{l}+\not{k}-m} \gamma^\nu \frac{1}{\not{l}-m}\right]_{\mathrm{reg}} A_\nu(k) A_{5\lambda} \\
&-i \int_k \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}-m} \gamma^\nu \frac{1}{\not{l}-m} \gamma^\lambda \gamma^5 \frac{1}{\not{l}-m}\right]_{\mathrm{reg}} A_\nu(k) A_{5\lambda}. \quad (4.8)
\end{aligned}
$$

Intuitively, the Pauli-Villars contribution corresponds to the contribution from the bottom of the band, where the right-handed and left-handed bands merge. Indeed, in the presence of Pauli-Villars regularization, the right-handed and left-handed contributions cannot be computed separately.

[1] In fact, the integral is not divergent, but the regularization is essential, as will be shown later.

In order to obtain the current density at finite temperature and density, we employ the imaginary time formalism [106] by replacing

$$l_0 \to i\nu_n + \mu, \tag{4.9}$$

$$k_0 \to i\omega_n, \tag{4.10}$$

$$\int \frac{dl_0}{2\pi} \to iT \sum_n, \tag{4.11}$$

where $\nu_n = (2n+1)\pi T$ and $\omega_n = 2n\pi T$ are the fermionic and bosonic Matsubara frequencies, respectively. After the frequency summation, ω_n is analytically continued into $i\omega_n \to \omega^+ = \omega + i0^+$.

After the trace computation, Matsubara frequency summation, and angular integration (see Appendix. G for details), we find

$$J^\mu(k) = -\frac{v}{2\pi^2}\epsilon^{\mu\lambda\kappa\nu}A_{5\lambda}ik_\kappa A_\nu(k) + \frac{1}{2\pi^2}\int_0^\infty d\varepsilon\, N'_+(\varepsilon)f_\varepsilon(\omega^+, |\boldsymbol{k}|)\delta^\mu{}_i\phi_5 B^i(k) \tag{4.12}$$

with

$$N_+(\varepsilon) = n_F(\beta(\varepsilon-\mu)) + n_F(\beta(\varepsilon+\mu)), \tag{4.13}$$

$$f_\varepsilon(\omega,k) = \frac{\omega^2-(vk)^2}{4(vk)^2}\left[\frac{\omega}{vk}\ln\left(\frac{(\omega+vk)^2[(\omega-vk)^2-4\varepsilon^2]}{(\omega-vk)^2[(\omega+vk)^2-4\varepsilon^2]}\right) + \frac{2\varepsilon}{vk}\ln\left(\frac{\omega^2-(vk-2\varepsilon)^2}{\omega^2-(vk+2\varepsilon)^2}\right)\right]. \tag{4.14}$$

While the massless Dirac fermion contributes to both terms on the right-hand side of Eq. (4.12), the Pauli-Villars ghost contributes only to the first term. The first term is just the Chern-Simons current (Bardeen-Zumino polynomial), which we introduced in Sec. 2.3. Its correct form is unavailable without the regularization, which is consistent with the observation in a lattice model of Weyl semimetals [107].

4.2 Dynamical chiral magnetic current

The limits $\omega \to 0$ and $k \to 0$ of $f_\varepsilon(\omega,k)$ do not commute as observed in Refs. [108, 109]. First, we take the static limit, where $k \to 0$ is taken after $\omega \to 0$:

$$\begin{aligned}\lim_{k\to0}\lim_{\omega\to0} f_\varepsilon(\omega,k) &= \lim_{k\to0}\frac{\varepsilon}{2vk}\ln\left(\frac{(vk+2\varepsilon)^2}{(vk-2\varepsilon)^2}\right)\\ &= 1,\end{aligned} \tag{4.15}$$

which leads to

$$\boldsymbol{J}_{\text{static}} = -\frac{1}{2\pi^2}\boldsymbol{A}_5 \times \boldsymbol{E}. \tag{4.16}$$

In this case, only the anomalous Hall current survives, and the chiral magnetic current vanishes in equilibrium.

Next, we take the uniform limit, where $\omega \to 0$ is taken after $k \to 0$:

$$\begin{aligned}\lim_{\omega\to0}\lim_{k\to0} f_\varepsilon(\omega,k) &= \lim_{\omega\to0}\frac{4\varepsilon^2(4\varepsilon^2-3\omega^2)}{3(4\varepsilon^2-\omega^2)^2}\\ &= \frac{1}{3},\end{aligned} \tag{4.17}$$

which leads to

$$\boldsymbol{J}_{\text{uniform}} = -\frac{1}{2\pi^2}\boldsymbol{A}_5 \times \boldsymbol{E} + \frac{1}{3\pi^2}\phi_5 \boldsymbol{B}. \tag{4.18}$$

In this case, the current density along the magnetic field is driven. This current is referred to as the dynamical chiral magnetic current or the gyrotropic magnetic current [110, 111]. Compared with the chiral magnetic current Eq. (2.122), the coefficient is smaller in magnitude by 2/3 and opposite in sign for $\mu_5 = -\phi_5$. In this effect, a dynamical magnetic field is applied, and then the system is driven out of equilibrium. In contrast, a static magnetic field is applied to non-equilibrium systems with $\mu_5 \neq 0$ and $\phi_5 = 0$ in the chiral magnetic effect. We emphasize that the dynamical chiral magnetic effect occurs even when $\mu_5 = 0$ if ϕ_5 is finite, while the static chiral magnetic effect requires the axial chemical poteintial. This indicate that the mechanism driving the generation of the dynamical chiral magnetic current differs from that of the static chiral magnetic current. We note that the dynamical chiral magnetic current is also reported in cubic noncentrosymmetric superconductors, and it leads the negative refractive index [112].

We further analyze the dynamical chiral magnetic current. We now apply a uniform but time-dependent magnetic field $\boldsymbol{B}(t)$. In this case, the current density is given by

$$\boldsymbol{J}(\omega) = \frac{1}{2\pi^2}\phi_5 \boldsymbol{B}(\omega)\left[1 + \int_0^\infty d\varepsilon\, N'_+(\varepsilon)\frac{4\varepsilon^2(4\varepsilon^2 - 3\omega^{+2})}{3(4\varepsilon^2 - \omega^{+2})^2}\right], \tag{4.19}$$

in addition to the anomalous Hall current. The total transported charge density is

$$\int_{-\infty}^{\infty} dt \boldsymbol{J}(t) = \frac{\phi_5}{3\pi^2}\int_{-\infty}^{\infty} dt \boldsymbol{B}(t), \tag{4.20}$$

which is independent of both temperature and chemical potential. In particular, we apply the pulsed magnetic field $\boldsymbol{B} = \boldsymbol{b}\delta(t)$. The resulting current density is

$$\boldsymbol{J}(t) = \frac{1}{2\pi^2}\phi_5 \boldsymbol{b}\left[\delta(t) + \int_0^\infty d\varepsilon\, N'_+(\varepsilon)\frac{4\varepsilon}{3}\left[\sin(2\varepsilon t) + \varepsilon t\cos(2\varepsilon t)\right]\Theta(t)\right]. \tag{4.21}$$

We show the time evolution of the current density for various values of μ/T in Fig. 4.2. We find the temporal Friedel oscillation. Roughly speaking, the oscillation frequency is determined by the typical energy scale $\varepsilon \sim \max(\mu, T)$. The integral $\int_{t>0} dt J(t) = -\phi_5 b/(6\pi^2)$ is independent of T and μ.

Lastly, we estimate a magnitude of this dynamical chiral magnetic current as $|\boldsymbol{J}| \sim e^3\phi_5|\boldsymbol{B}|/(2\pi^2\hbar^2)$. Here, $e\phi_5$ typically has a magnitude of the order of 1 meV [54]. If we assume $|\boldsymbol{B}| \sim 1\,\mathrm{T}$, the estimated current density is approximately of the order of $10^7\,\mathrm{A{\cdot}m^{-2}}$.

4.3 Summary

In this chapter, we field-theoretically computed the current density, which is linear both in A_μ and $A_{5\mu}$. To properly define the divergent integrals encountered in the calculation, we employed the Pauli-Villars regularization. This is the novelty of this study. The regularization is essential to obtain the correct form of the Chern-Simons current. The $\omega \to 0$ and $\boldsymbol{k} \to 0$ limits of the resulting current density does not commute. In the static limit, we confirmed that the chiral magnetic current is absent, which is consistent with the vanishing of the chiral magnetic effect in equilibrium. By taking the uniform limit,

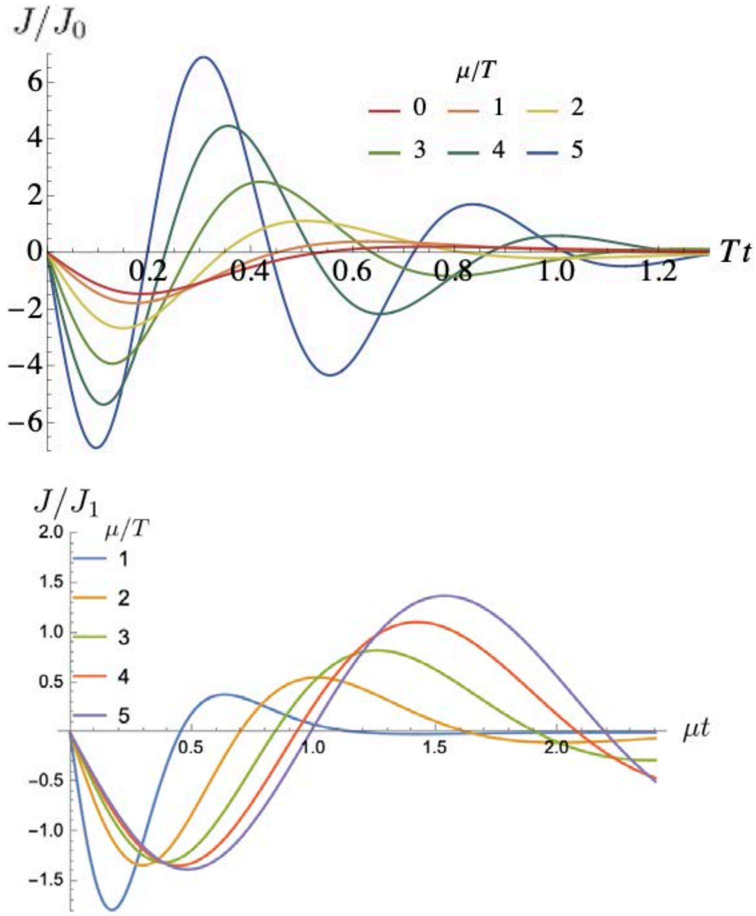


Figure 4.2: $J(t) = \hat{\boldsymbol{b}} \cdot \boldsymbol{J}(t)$ at $t > 0$ resulting from Eq. (4.21). In the top panel, $J(t)$ is shown in units of $J_0 = \phi_5 bT/(2\pi^2)$ as a function of Tt for $\mu/T = 0, 1, \ldots, 5$. In this case, the amplitude increases as μ/T increases. In the bottom panel, $J(t)$ is shown in units of $J_1 = \phi_5 b\mu/(2\pi^2)$ as a function of μt for $\mu/T = 1, \ldots, 5$.

we showed the existence of the dynamical chiral magnetic current, which is driven by a time-dependent magnetic field in its direction. The dynamical chiral magnetic current is driven even when $\mu_5 = 0$, and its mechanism differs from that of the static chiral magnetic current. We analyzed the properties of the dynamical chiral magnetic current. In particular, we found that the total transported charge is universal for a uniform field in the sense of its independence from temperature and chemical potential. Moreover, the temporal Friedel oscillation is observed when we apply the pulsed magnetic field.

Chapter 5

Weyl semimetals from chiral kinetic theory

In this chapter, we analyze the electromagnetic linear responses of the Weyl semimetals using the framework of the chiral kinetic theory. In Sec. 5.1, we discuss the relaxation time approximation that ensures the charge conservation. Using this approximation, we compute the charge and current densities in Sec. 5.2. In Sec. 5.3, we couple the resulting current with Maxwell's equation and investigate the collective excitations. Finally, we summarize this chapter in Sec. 5.4.

5.1 Relaxation time approximation

We recall the chiral kinetic equation reviewed in Sec. 2.4 [76, 79]:

$$\begin{aligned}\frac{\partial f_{p,\chi}}{\partial t} &+ \frac{1}{\sqrt{G}}\left[\boldsymbol{v}_{\boldsymbol{p}} + \tilde{\boldsymbol{E}} \times \boldsymbol{\Omega} + \boldsymbol{B}(\boldsymbol{v}_{\boldsymbol{p}} \cdot \boldsymbol{\Omega})\right] \cdot \frac{\partial f_{p,\chi}}{\partial \boldsymbol{x}} \\ &+ \frac{1}{\sqrt{G}}\left[\tilde{\boldsymbol{E}} + \boldsymbol{v}_{\boldsymbol{p}} \times \boldsymbol{B} + \boldsymbol{\Omega}(\tilde{\boldsymbol{E}} \cdot \boldsymbol{B})\right] \cdot \frac{\partial f_{p,\chi}}{\partial \boldsymbol{p}} = C_{p,\chi}[f_{p,\chi}, f_{a,\chi}], \qquad (5.1)\end{aligned}$$

where the subscript p (a) denotes the (anti)particles. We expand the distribution function as

$$f_{\lambda,\chi}(t, \boldsymbol{x}, \boldsymbol{p}) = n_F(\beta(\epsilon_{\boldsymbol{p}} - \lambda\mu_\chi)) + \delta f_{\lambda,\chi}(t, x, \boldsymbol{p}), \qquad (5.2)$$

$$\delta f_{\lambda,\chi}(t, \boldsymbol{x}, \boldsymbol{p}) = -h_{\lambda,\chi}(t, \boldsymbol{x}, \boldsymbol{p}) n_F'(\beta(vp - \lambda\mu_\chi)), \qquad (5.3)$$

where $\delta f_{\lambda,\chi}(t, x, \boldsymbol{p})$ is the perturbation at the linear order in the electromagnetic field: $\delta f_{\lambda,\chi}(t, x, \boldsymbol{p}) \sim \mathcal{O}(\boldsymbol{E}, \boldsymbol{B})$. We now employ the relaxation time approximation. The minimal approximation

$$C_{\lambda,\chi}[f_{p,\chi}, f_{a,\chi}] = -\frac{\delta f_{\lambda,\chi}(t, \boldsymbol{x}, \boldsymbol{p})}{\tau} \qquad (5.4)$$

violates the local charge conservation [113, 114]. Instead, we adopt the modified collision term:

$$C_{\lambda,\chi}[f_{p,\chi}, f_{a,\chi}] = \frac{1}{\tau} n_F'(\beta(vp - \lambda\mu_\chi))(h_{\lambda,\chi}(t, \boldsymbol{x}, \boldsymbol{p}) - \lambda\delta\mu(t, \boldsymbol{x})), \qquad (5.5)$$

where $\delta\mu(t,\boldsymbol{x})$ is a local shift of the chemical potential. We determine $\delta\mu(t,\boldsymbol{x})$ to ensure the charge conservation [113]. At the linear order in the electromagnetic field, the Boltzmann equation is simplified to

$$\left(\frac{\partial}{\partial t}+\frac{1}{\tau}+v\hat{\boldsymbol{p}}\cdot\frac{\partial}{\partial\boldsymbol{x}}\right)h_{\lambda,\chi}(t,\boldsymbol{x},\boldsymbol{p})=\left(\boldsymbol{E}\cdot\hat{\boldsymbol{p}}-\frac{\partial}{\partial t}p(\boldsymbol{B}\cdot\boldsymbol{\Omega})\right)+\frac{\lambda\delta\mu(t,\boldsymbol{x})}{\tau}. \tag{5.6}$$

By the Fourier transformation, we find

$$h_{\lambda,\chi}(\omega,\boldsymbol{k},\boldsymbol{p})=\frac{\boldsymbol{E}\cdot\hat{\boldsymbol{p}}+(i\omega)p(\boldsymbol{B}\cdot\boldsymbol{\Omega})+\lambda\delta\mu(\omega,\boldsymbol{k})/\tau}{-i\omega+1/\tau+iv\hat{\boldsymbol{p}}\cdot\boldsymbol{k}}. \tag{5.7}$$

The condition for the local charge conservation requires [113, 114]

$$\int_{\boldsymbol{p}}\sqrt{G}(C_{p,\chi}[f_{p,\chi},f_{a,\chi}]-C_{a,\chi}[f_{p,\chi},f_{a,\chi}])=0, \tag{5.8}$$

which leads to the expression for $\delta\mu(\omega,\boldsymbol{k})$ as

$$\delta\mu(\omega,\boldsymbol{k})=\frac{1}{X}\int_{\boldsymbol{p}}\sqrt{G}[h_{p,\chi}(\omega,\boldsymbol{k},\boldsymbol{p})n_F'(\beta(vp-\mu_\chi))-h_{a,\chi}(\omega,\boldsymbol{k},\boldsymbol{p})n_F'(\beta(vp+\mu_\chi))] \tag{5.9}$$

with

$$X=\frac{\mu_\chi^2}{2\pi^2}+\frac{T^2}{6}. \tag{5.10}$$

By substituting $h_{\lambda,\chi}(\omega,\boldsymbol{k},\boldsymbol{\omega})$, we obtain

$$\delta\mu(\omega,\boldsymbol{k})=-\frac{1}{X}\int_{\boldsymbol{p}}\left[\frac{\boldsymbol{E}\cdot\hat{\boldsymbol{p}}+\delta\mu(\omega,\boldsymbol{k})/\tau}{-i\omega+1/\tau+iv\hat{\boldsymbol{p}}\cdot\boldsymbol{k}}(n_F'(\beta(vp-\mu_\chi))+n_F'(\beta(vp+\mu_\chi)))\right], \tag{5.11}$$

where we use

$$\int_{\hat{\boldsymbol{p}}}\frac{\boldsymbol{B}\cdot\hat{\boldsymbol{p}}}{-i\omega+1/\tau+iv\hat{\boldsymbol{p}}\cdot\boldsymbol{k}}=0, \tag{5.12}$$

which can be shown by using $i\boldsymbol{k}\cdot\boldsymbol{B}=0$ and Eq. (H.4). By performing $\boldsymbol{p}$-integration, we finally obtain the expression of $\delta\mu(\omega,\boldsymbol{k})$:

$$\delta\mu(\omega,\boldsymbol{k})=\frac{1}{1-\frac{i}{2\tau v|\boldsymbol{k}|}G(\omega_\tau,|\boldsymbol{k}|)}\frac{iv\boldsymbol{E}\cdot\hat{\boldsymbol{k}}}{2|\boldsymbol{k}|^2}(\omega_\tau G(\omega_\tau,|\boldsymbol{k}|)-2v|\boldsymbol{k}|), \tag{5.13}$$

where $\omega_\tau=\omega+i/\tau$ and

$$G(\omega,k)=\ln\left(\frac{\omega+vk}{\omega-vk}\right). \tag{5.14}$$

5.2 Charge and current densities

As reviewed in Sec. 2.4, the charge and current densities are given by [77, 83]

$$\rho_\chi=\sum_{p,a}\int\frac{d^3p}{(2\pi)^3}\sqrt{G}f_\chi, \tag{5.15}$$

$$\boldsymbol{j}_\chi=\sum_{p,a}\int\frac{d^3p}{(2\pi)^3}\left[(\boldsymbol{v}_{\boldsymbol{p}}+(\boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\Omega})\boldsymbol{B}+\tilde{\boldsymbol{E}}\times\boldsymbol{\Omega})f_\chi+\boldsymbol{\nabla}\times f_\chi\epsilon_{\boldsymbol{p}}\boldsymbol{\Omega}\right]. \tag{5.16}$$

In the previous section, we computed the distribution function under the relaxation time approximation:

$$f_{\lambda,\chi}(\omega,\boldsymbol{k},\boldsymbol{p}) = n_F(\beta(\epsilon_{\boldsymbol{p}}-\lambda\mu_\chi)) - \frac{\boldsymbol{E}\cdot\hat{\boldsymbol{p}} + (i\omega)p(\boldsymbol{B}\cdot\boldsymbol{\Omega}) + \lambda\delta\mu(\omega,\boldsymbol{k})/\tau}{-i\omega + 1/\tau + iv\hat{\boldsymbol{p}}\cdot\boldsymbol{k}} n_F'(\beta(vp-\lambda\mu_\chi)). \tag{5.17}$$

By using the integration formulae presented in Appendix H, we find the charge density is given by

$$\rho(\omega,\boldsymbol{k}) = \rho_0 + \rho_1(\omega,\boldsymbol{k}), \tag{5.18}$$

$$\rho_0 = \frac{\pi^2T^2+\mu^2+3\mu_5^2}{3\pi^2v^3}, \tag{5.19}$$

$$\rho_1(\omega,\boldsymbol{k}) = 3\epsilon_0\Omega_e^2 \frac{1}{1-\frac{i}{2\tau vk}G(\omega_\tau,|\boldsymbol{k}|)} \frac{\boldsymbol{E}\cdot\boldsymbol{k}}{2i\omega_\tau^2}(g_1(\omega_\tau,|\boldsymbol{k}|)-g_3(\omega_\tau,|\boldsymbol{k}|)), \tag{5.20}$$

and the current density is

$$\begin{aligned}\boldsymbol{j}(\omega,\boldsymbol{k}) = {}& \frac{\mu_5}{2\pi^2}\boldsymbol{B} + \frac{\mu_5}{2\pi^2}g_1(\omega_\tau,|\boldsymbol{k}|)\frac{\omega}{\omega_\tau}\boldsymbol{B} + \frac{v}{8\pi^2}\left[\frac{2}{3}+g_1(\omega_\tau,|\boldsymbol{k}|)\frac{\omega}{\omega_\tau}\right]i\boldsymbol{k}\times\boldsymbol{B} \\ &+ \frac{3\epsilon_0\Omega_e^2}{2}\left[g_1(\omega_\tau,|\boldsymbol{k}|)\frac{\boldsymbol{E}}{i\omega_\tau} - g_3(\omega_\tau,|\boldsymbol{k}|)\frac{\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\boldsymbol{E})}{i\omega_\tau}\right] \\ &- \frac{3\epsilon_0\Omega_e^2}{2}\frac{v^2\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})}{2\tau\omega_\tau^4}\frac{1}{1-\frac{i}{2\tau vk}G(\omega_\tau,|\boldsymbol{k}|)}(g_1(\omega_\tau,|\boldsymbol{k}|)-g_3(\omega_\tau,|\boldsymbol{k}|)). \end{aligned}\tag{5.21}$$

Here, $\Omega_e^2 = (\pi^2T^2/3+\mu^2+\mu_5^2)/(3\pi^2\epsilon_0 v)$ is the plasma frequency, and we define

$$g_n(\omega,k) = \frac{\omega}{vk}\left[\frac{n\omega^2-(vk)^2}{2(vk)^2}G(\omega,k) - \frac{n\omega}{vk}\right]. \tag{5.22}$$

We can check the charge and current densities satisfy the conservation law:

$$-i\omega\rho_1(\omega,\boldsymbol{k}) + i\boldsymbol{k}\cdot\boldsymbol{j}(\omega,\boldsymbol{k}) = 0. \tag{5.23}$$

By following the prescription of the consistent chiral kinetic theory, we should add the Chern-Simons current δj^μ, which is given by

$$\delta\rho = \frac{1}{2\pi^2}\boldsymbol{A}_5\cdot\boldsymbol{B}, \tag{5.24}$$

$$\delta\boldsymbol{j} = -\frac{1}{2\pi^2}\boldsymbol{A}_5\times\boldsymbol{E} + \frac{\phi_5}{2\pi^2}\boldsymbol{B}. \tag{5.25}$$

Unlike in Chapter. 4, $J^\mu = j^\mu + \delta j^\mu$ includes the contribution of collisions through the relaxation time τ and a finite axial imbalance, where $\mu_5+\phi_5 \neq 0$.

We compare the results of Chapters 4 and 5. In Chapter 4, we computed the current density at the linear order both in A_μ and $A_{5\mu}$:

$$\boldsymbol{J}(k) = -\frac{1}{2\pi^2}\boldsymbol{A}_5\times\boldsymbol{E} + \frac{\phi_5}{2\pi^2}\boldsymbol{B} + \frac{1}{2\pi^2}\int_0^\infty d\varepsilon\, N_+'(\varepsilon)f_\varepsilon(\omega^+,|\boldsymbol{k}|)\phi_5\boldsymbol{B}. \tag{5.26}$$

The corresponding result from the consistent chiral kinetic theory is

$$\boldsymbol{J}(k)|_{\mathcal{O}(AA_5)} = -\frac{1}{2\pi^2}\boldsymbol{A}_5\times\boldsymbol{E} + \frac{\phi_5+\mu_5}{2\pi^2}\boldsymbol{B} + \frac{\mu_5}{2\pi^2}g_1(\omega_\tau,|\boldsymbol{k}|)\frac{\omega}{\omega_\tau}\boldsymbol{B}. \tag{5.27}$$

To evaluate the consistency of these two results, we recall the kinetic theory is valid when the external fields vary sufficiently slowly compared to μ, T. The particles and antiparticles have the typical energy $\varepsilon \sim T, \mu$. Therefore, the kinetic theory assumes $\omega, v|\boldsymbol{k}| \ll \varepsilon$. In equilibrium $\mu_5 = -\phi_5$ and the collisionless regime $1/\tau = 0^+$, the current density Eq. (5.26) reduces to Eq. (5.27) since

$$\lim_{\varepsilon\to\infty} f_\varepsilon(\omega, k) = 1 + g_1(\omega, k) \tag{5.28}$$

holds. Therefore, our two approaches are indeed consistent.

In the next section, we will investigate the collective excitation in long wavelengths regime. As its preparation, we examine the uniform limit, $\boldsymbol{k} \to 0$. In this limit, the current density simplifies to

$$\lim_{\boldsymbol{k}\to 0} \boldsymbol{J}(k) = -\sigma_H \hat{\boldsymbol{A}}_5 \times \boldsymbol{E} + \left[\sigma_M - \frac{\mu_5}{3\pi^2}\frac{\omega\tau}{\omega\tau + i}\right]\boldsymbol{B} + \sigma_E \frac{i}{\omega\tau + i}\boldsymbol{E}, \tag{5.29}$$

where we use

$$\lim_{k\to 0} g_n(\omega, k) = \frac{n-3}{3} + \mathcal{O}\left((vk/\omega)^2\right), \tag{5.30}$$

and $\sigma_H = |\boldsymbol{A}_5|/(2\pi^2)$, $\sigma_M = (\phi_5 + \mu_5)/(2\pi^2)$, and $\sigma_E = \epsilon_0 \Omega_e^2 \tau$ are the anomalous Hall, chiral magnetic, and Ohmic conductivities, respectively. The Ohmic conductivity takes the Drude form. The chiral magnetic conductivity remains finite only out of equilibrium $\phi_5 \neq -\mu_5$. The anomalous Hall conductivity is independent of the relaxation time. The dynamical chiral magnetic current is found to be valid for $\omega\tau \gg 1$ but suppressed by the dissipation for $\omega\tau \ll 1$.

5.3 Chiral magnetic instability

The dynamics of the electromagnetic field is governed by Maxwell's equation,

$$\boldsymbol{\nabla} \times \boldsymbol{B}(x) - \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}(x)}{\partial t} = \mu_0 \boldsymbol{J}(x), \tag{5.31}$$

which is coupled with the current density in Eq. (5.29). We now adopt the temporal gauge $\phi(x) = 0$ and assume a plane wave $\boldsymbol{A}(x) = \tilde{\boldsymbol{A}} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$. The left-hand side is the second order in derivative with respect to $\boldsymbol{A}(x)$, while the right-hand side is the first order. Thus, the right-hand side must vanish by itself at $\mathcal{O}(\omega, \boldsymbol{k})$:

$$\boldsymbol{J}(\omega, \boldsymbol{k}) = 0. \tag{5.32}$$

At $|\omega|,\ v|\boldsymbol{k}|, \ll 1/\tau$, this equation can be written as

$$-i\omega \hat{\boldsymbol{A}}_5 \times \boldsymbol{A} + \sigma_M i\boldsymbol{k} \times \boldsymbol{A} + \sigma_E i\omega \boldsymbol{A} = 0. \tag{5.33}$$

The characteristic equation is given by

$$\det \begin{pmatrix} \omega\sigma_E & \omega\sigma_H \hat{A}_{5z} - k_z\sigma_M & -(\omega\sigma_H \hat{A}_{5y} - k_y\sigma_M) \\ -(\omega\sigma_H \hat{A}_{5z} - k_z\sigma_M) & \omega\sigma_E & \omega\sigma_H \hat{A}_{5x} - k_x\sigma_M \\ \omega\sigma_H \hat{A}_{5y} - k_y\sigma_M & -(\omega\sigma_H \hat{A}_{5x} - k_x\sigma_M) & \omega\sigma_E \end{pmatrix} = 0, \tag{5.34}$$

which is simplified to

$$\sigma_E(\sigma_E^2 + \sigma_H^2)\omega^3 - 2\hat{\boldsymbol{A}}_5 \cdot \boldsymbol{k}\sigma_E\sigma_H\omega^2 + |\boldsymbol{k}|^2\sigma_E\sigma_M^2\omega = 0. \tag{5.35}$$

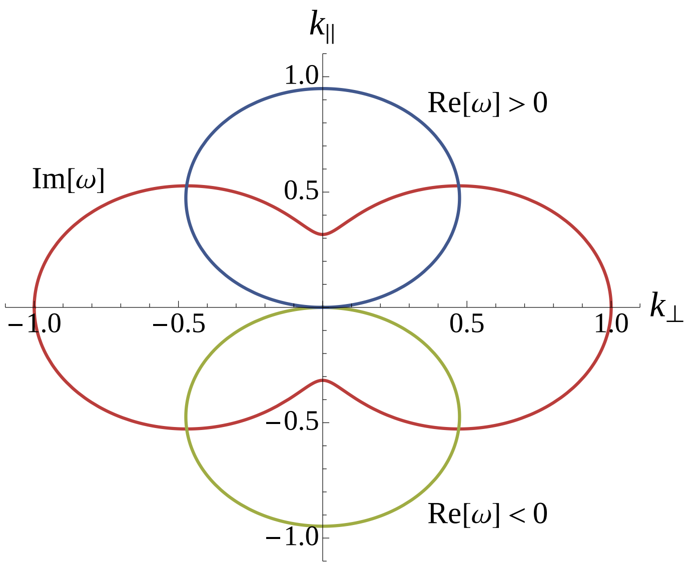


Figure 5.1: Directional dependence of real and imaginary parts of $\omega(k_\perp, k_\|)$ resulting from Eq. (5.36) for $\sigma_M > 0$ and $\sigma_H = 3\sigma_E$. Their magnitudes in units of $\omega_0 \equiv \sigma_M|\boldsymbol{k}|/\sqrt{\sigma_H^2+\sigma_E^2}$ are represented by distances from the origin in the direction of $(k_\perp, k_\|)$. This figure is cited from T. Amitani and Y. Nishida, "Dynamical chiral magnetic current and instability in Weyl semimetals," Phys. Rev. B **107**, 014302 (2023), © 2023 American Physical Society.

There are two nontrivial solutions:

$$\omega = \frac{\sigma_H \sigma_M k_\| \pm i\sigma_M\sqrt{\sigma_E^2 k_\|^2 + (\sigma_H^2+\sigma_E^2)k_\perp^2}}{\sigma_H^2+\sigma_E^2} \tag{5.36}$$

with $k_\| = \boldsymbol{k}\cdot\hat{\boldsymbol{A}}_5$ and $k_\perp^2 = |\boldsymbol{k}|^2 - k_\|^2$. The dispersion relations are determined by the conductivities. Importantly, one of them has a positive imaginary part. The positive imaginary part implies the exponential growth of electromagnetic fields.

The imaginary part shows the anisotropy, which depends on the direction of $\boldsymbol{k}$ relative to $\boldsymbol{A}_5$ as illustrated in Fig. 5.1. The imaginary part is maximal when $k_\| = 0$, where the real part vanishes. In this case, the unstable modes do not propagate. When $k_\| \neq 0$, these modes become propagating waves. Notably, when $k_\perp = 0$, where the imaginary part is minimal, the unstable modes are circularly polarized. The sign of the real part depends on $\sigma_M k_\|$ so that the propagation of unstable modes is toward $\pm\hat{\boldsymbol{k}}$ for $\pm\sigma_M k_\| > 0$ and thus oriented to the direction of $\sigma_M \boldsymbol{A}_5$. Both chiral magnetic and anomalous Hall conductivities are essential for the propagating wave. This instability of the electromagnetic fields is analogous to the chiral plasma instability [87]. However, there are differences between our instability and the chiral plasma instability: the latter was predicted at $\omega \sim \boldsymbol{k}^2$ in the absence of dissipation. The resulting exponential growth of electromagnetic fields is considered to reduce Weyl node populations toward equilibrium $\mu_5 \to -\phi_5$ [87, 115, 116], so as to gradually attenuate our chiral magnetic instability. We note that the collective excitations are diffusive in $\mu_5 = -\phi_5$, i.e., no instability occurs in this case since the chiral plasma instability requires the axial imbalance for its growth [87, 115, 116].[1]

[1] We also note that the instability by the dynamical chiral magnetic current can be predicted even when $\mu_5 = -\phi_5$ if we use an incorrect dispersion relation $\varepsilon_{\boldsymbol{p}} = |\boldsymbol{p}|$. Therefore, the correction to the energy dispersion relation $-p(\boldsymbol{B}\cdot\boldsymbol{\Omega})$ is essential.

5.4 Summary

We first investigate the electromagnetic linear responses in Weyl semimetals from the framework of the consistent chiral kinetic theory. To consider the effect of the collision, we employ the modified relaxation time approximation, which ensure the charge conservation. The resulting current density is consistent with that obtained from the field-theoretical computation. We also studied dispersion relations of collective excitations coupled with Maxwell electromagnetic fields at low frequencies and long wavelengths. The dispersion relations are determined only by electric, chiral magnetic, and anomalous Hall conductivities, which predict unstable modes when Weyl node populations deviate from equilibrium. If the wave vectors are perpendicular to the direction of Weyl node separation, the unstable modes are nonpropagating and otherwise become propagating waves. Their propagation is not radial but oriented to or opposite to the direction of Weyl node separation, depending on the sign of the chiral magnetic conductivity. Therefore, whereas the instability is caused by the chiral magnetic effect, the anomalous Hall effect causes propagation. As an observable signature of the instability, the anomalous reflectance of the surface of the Weyl semimetal was proposed [117]. It was found that the reflectance of the Weyl semimetal with pumped axial charge can exceed unity. However, the practical observation of this phenomenon remains challenging. Therefore, future work is needed to propose alternative setups to observe this instability.

Chapter 6

Summary and outlook

Summary

In this thesis, we applied the effective field theory approach to the quantum Hall systems and the Weyl semimetals. The effective field theory captures the universal properties of a system while abstracting away unnecessary details. To investigate the universal nature of these materials, we analyzed their electromagnetic responses.

In Chapter 2, we reviewed four topics related to effective field theory. First, we discussed the Chern-Simons and Wen-Zee terms in the quantum Hall systems. The former relates to the Hall conductivity, while the latter relates to the Hall viscosity. Next, we derived the low-energy effective action of the Weyl semimetals starting from a lattice model. The derived action corresponds to that of a massless Dirac fermion coupled to an axial gauge field. We then explained how such systems exhibit the axial anomaly. Additionally, we reviewed the consistent chiral kinetic theory, an alternative framework for describing the Weyl semimetals.

In Chapter 3 with Appendix A-F, we constructed the effective action of the quantum Hall systems with the Galilean invariance base on symmetries. To compute all currents including the energy current, we consider the system on a Newton-Cartan spacetime. This framework introduces three metrics corresponding to the external fields conjugate to the energy current density, momentum density, and stress tensor. The quantum Hall system on a Newton-Cartan spacetime has the U(1) gauge, general coordinate, and Milne invariance, the last of which can be regarded as the generalization of the Galilean invariance. To enforce the Milne invariance, we dressed the external fields. The resulting action is Eq. (3.93), which contains four unknown coefficients. Using this effective action, we computed the electromagnetic responses. We first identified four coefficients from the obtained current densities: the Hall conductivity, Hall viscosity, energy density, and energy magnetization. We also provided explicit expressions for the energy density and energy magnetization in the case of free electrons. We then found the relations among different kinds of the responses. Notably, two key findings emerged. First, the longitudinal conductivity at nonzero frequency is determined by the Hall conductivity [Eq. (3.120)], which is consistent with the result from the equation of motion of a single electron and Kohn's theorem. Second, the Hall viscosity contributes to the nonlinear electrothermal conductivity at nonzero wave number [Eq. (3.121)]. This result aligns with the energy conservation law and the parity-violating hydrodynamics in two spatial dimensions.

In Chapter 4 with Appendix G, we investigated the electromagnetic linear responses of the Weyl semimetals by using the low-energy effective action Eq. (4.1). We employed the Pauli-Villars regularization to define the superficially divergent integral. The resulting current density [Eq. (4.12)] comprises terms dependent on temperature and chemical potential as well as terms independent of these variables. The latter is consistent with the Chern-Simons current. We confirmed that the chiral magnetic current vanishes in equilibrium by taking the static limit of the current density. On the other hand, in the uniform limit, we found the dynamic magnetic field drives the current density in its direction (the dynamical chiral magnetic current). We analyzed the properties of the dynamical chiral magnetic current under the uniform magnetic field. In particular, we applied the pulsed magnetic field and observed the temporal Friedel oscillation.

In Chapter 5 with Appendix H, we computed the electromagnetic linear responses by using the consistent chiral kinetic theory. We adopted the modified relaxation time approximation to ensure the local charge conservation. The resulting charge and current densities [Eqs. (5.18) and (5.21)] are consistent with those obtained from the field-theoretical computation in Chapter 4. By coupling the resulting current with Maxwell's equation, we investigated the collective excitations. In the long wavelength regime, the plasmon dispersion relation is determined only by the chiral magnetic, anomalous Hall, and Ohmic conductivities [Eq. (5.36)]. The dispersion relation has a positive imaginary part, indicating the existence of the exponentially growing modes. It is an analogy of the chiral plasma instability although there are some differences. If the wave vectors are perpendicular to the direction of the Weyl node separation, the modes do not propagate. Otherwise, the modes become propagating waves. The propagation of the unstable modes is anisotropic, and the direction is parallel or antiparallel to the Weyl node separation. The growth of these modes requires an axial imbalance, which subsequently reduces the Weyl node populations toward equilibrium ($\mu_5 \to -\phi_5$), eventually stabilizing the system.

Outlook

There are several intriguing prospects for extending our studies. In Chapter 3, we derived the effective action of the quantum Hall systems with Galilean invariance, grounded in the symmetries of the microscopic action. An exciting avenue for future exploration would be to determine whether the same effective action can emerge from the non-relativistic limit of the effective action of relativistic quantum Hall systems. The relativistic counterpart has already been constructed in Ref. [118], providing a strong foundation for such an investigation. However, the process of taking the non-relativistic limit is nuanced, as there are multiple formalisms available, such as those proposed in Refs. [99,119]. Identifying the most appropriate framework for reproducing the non-relativistic action is far from trivial. If we succeed in identifying a suitable limiting procedure, the implications would extend beyond the quantum Hall systems. For instance, the methodology could be applied to derive non-relativistic hydrodynamics from its relativistic counterpart, thereby broadening our understanding of fluid dynamics in non-relativistic regimes.

Another direction worth pursuing is the extension of our studies in Chapters 4 and 5, where we focused on the electromagnetic responses of the Weyl semimetals, to include responses induced by other types of external fields. For example, as discussed in Sec.2.2, strain in the Weyl semimetals generates a spacetime-dependent axial gauge field. Furthermore, it has been established that a constant axial magnetic field $\boldsymbol{B}_5$ can induce a current density: $\boldsymbol{j} \propto \mu \boldsymbol{B}_5$ [120]. Extending this framework to the case of a spacetime-dependent $\boldsymbol{B}_5$ would provide new insights into the interplay between strain-induced gauge fields and electromagnetic responses in the Weyl semimetals. The strain in the Weyl semimetals induces not only charge currents but also stress, leading to intriguing phenomena such as the anomalous Hall viscosity [64]. However, existing evaluations of this viscosity lack rigor, necessitating a precise computation. To address this, a field-theoretical calculation of the stress tensor under axial gauge fields is essential. Such an approach would deepen our understanding of the stress-strain responses in the Weyl semimetals, including the anomalous Hall viscosity.

In Sec.5.4, we highlighted the need for future work to propose experimental setups capable of observing signatures of the chiral magnetic instability. As an initial step, a promising approach would involve numerical simulations to study the growth of electromagnetic fields by solving the set of Maxwell's equations and the kinetic equations, similar to the methodologies used in the context of supernovae [121]. Beyond the chiral magnetic instability, exploring other potential instabilities in the Weyl semimetals offers a rich field of inquiry. For instance, the anomalous Hall instability [122], which is the instability of Alfvén waves in the presence of chiral magnetic and anomalous Hall currents, could be particularly relevant. This instability is predicted under conditions analogous to those we study, suggesting its occurrence in the Weyl semimetals is plausible. Another interesting possibility is the chiral magnetovortical instability [123], a plasma instability driven by the interplay of chiral magnetic and vortical currents. While the realization of this instability in the Weyl semimetals might be challenging due to the need for a vorticity, a detailed evaluation of its feasibility in such systems is worthwhile. Understanding these instabilities deepens our comprehension of the dynamical properties of the Weyl semimetals.

Lastly, we will be able to integrate our understanding of the quantum Hall systems and the Weyl semimetals to explore the quantum Hall effect in the Weyl semimetals. The quantum Hall effective action for two-dimensional Dirac semimetals has been constructed based on symmetries [124]. Building on this, we could similarly derive the quantum Hall effective action for the Weyl semimetals. This effective action is expected to include terms that describe the anomalous Hall viscosity. Thus, this approach would enable the investigation of the universal properties of the anomalous Hall viscosity, advancing theoretical understanding. Moreover, extending this analysis from two to three dimensions would provide deeper insights.

In conclusion, this thesis has uncovered universal physical phenomena in both the quantum Hall systems and the Weyl semimetals. Thanks to the universality inherent in the effective field theory approach, these findings are broadly applicable, extending beyond specific material details. Notably, the formalism employed for the quantum Hall systems relies entirely on symmetries, suggesting that this methodology can be extended to other topological materials. While the approach employed for the Weyl semimetals is somewhat more restricted, the Weyl semimetals themselves offer potential applications. As mentioned in Chapter. 1, the Weyl semimetals serve as excellent platforms for realizing relativistic effects that are challenging to achieve in high-energy physics experiments. Consequently, further exploration of universal phenomena in the Weyl semimetals is worthwhile, and achieving their experimental detection is an important objective for future research. We hope that this thesis will not only foster a more comprehensive understanding of the topological materials, but also provide deeper insights into the high-energy physics.

Appendix A

Another choice of variation

In this appendix, we choose δv^μ to be arbitrary instead of δn_μ. The other variations are given by

$$\delta n_\mu = -n_\mu n_\nu \delta v^\nu + P^\nu_\mu \delta\bar{n}_\nu, \tag{A.1}$$

$$\delta h^{\mu\nu} = -v^{(\mu} h^{\nu)\lambda} \delta\bar{n}_\lambda + P^\mu_\rho P^\nu_\sigma \delta\bar{h}^{\rho\sigma}, \tag{A.2}$$

$$\delta h_{\mu\nu} = -n_{(\mu} h_{\nu)\lambda} \delta v^\lambda - h_{\mu\rho} h_{\nu\sigma} \delta\bar{h}^{\rho\sigma}, \tag{A.3}$$

where $\delta\bar{n}_\mu$, δv^μ, and $\delta\bar{h}^{\mu\nu}$ are independent. We introduce the current densities conjugate to the external fields as

$$\delta S = \int d^3x \sqrt{\gamma} \left(\mathcal{J}^\mu \delta A_\mu - \mathcal{E}'^\mu \delta\bar{n}_\mu - \mathcal{P}'_\mu \delta v^\mu - \frac{1}{2} \mathcal{T}_{\mu\nu} \delta\bar{h}^{\mu\nu} \right), \tag{A.4}$$

where the action is given by

$$S_0 = \int d^3x \sqrt{\gamma} \left[v^\mu \Psi^\dagger i \overleftrightarrow{D}_\mu \Psi - \left(\frac{h^{\mu\nu}}{2m} + \frac{ig\, \varepsilon^{\lambda\mu\nu} n_\lambda}{4m} \right) D_\mu \Psi^\dagger D_\nu \Psi \right]. \tag{A.5}$$

We now compute $\mathcal{E}'^\mu$ and $\mathcal{P}'_\mu$. The volume element varies as

$$\begin{aligned} \delta\sqrt{\gamma} &= \frac{\sqrt{\gamma}}{2} (\gamma^{-1})^{\mu\nu} \delta\gamma_{\mu\nu} \\ &= -\sqrt{\gamma} \left[n_\mu \delta v^\mu + \frac{1}{2} h_{\mu\nu} \delta\bar{h}^{\mu\nu} \right]. \end{aligned} \tag{A.6}$$

Then, we obtain

$$\begin{aligned} \mathcal{E}'^\mu &= -\frac{1}{\sqrt{\gamma}} \frac{\delta S_0}{\delta\bar{n}_\mu} \\ &= -\frac{v^{(\rho} h^{\sigma)\mu}}{2m} D_\rho \Psi^\dagger D_\sigma \Psi + \frac{ig}{4m} P^\mu_\lambda \varepsilon^{\lambda\rho\sigma} D_\rho \Psi^\dagger D_\sigma \Psi \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} \mathcal{P}'_\mu &= -\frac{1}{\sqrt{\gamma}} \frac{\delta S_0}{\delta v^\mu} \\ &= -P^\nu_\mu \Psi^\dagger i D_\nu \Psi - n_\mu \left(\frac{h^{\rho\sigma}}{2m} D_\rho \Psi^\dagger D_\sigma \Psi + \frac{ig\varepsilon^{\lambda\rho\sigma}}{4m} n_\lambda D_\rho \Psi^\dagger D_\sigma \Psi \right). \end{aligned} \tag{A.8}$$

Compared with the expressions in Eqs. (3.39) and (3.40), we find the energy current density is given by

$$\mathcal{E}^{\mu} = \mathcal{E}'^{\mu} - v^{\mu} v^{\nu} \mathcal{P}'_{\nu}, \tag{A.9}$$

and the momentum density is given by

$$\mathcal{P}_{\mu} = P_{\mu}^{\nu} \mathcal{P}'_{\nu}. \tag{A.10}$$

Therefore, $\mathcal{E}'^{\mu}$ and $\mathcal{P}'_{\mu}$ themselves are not the energy current and momentum densities, respectively. Thus, the choice in this appendix is not efficient since we should compute both $\mathcal{E}'^{\mu}$ and $\mathcal{P}'_{\mu}$ to obtain the energy current density.

Appendix B

Evaluation of scalars to construct effective action

This appendix summarizes the steps to simplify Table 3.2 into Table 3.3 by examining the dependence and order of scalars.

- Dependence of $\tilde{B}$ and $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$:

$$\tilde{B}^2 = \frac{1}{2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{B.1}$$

- Higher-order nature of $\tilde{h}_{\mu\rho}\varepsilon^{\mu\nu\lambda}\tilde{F}_{\nu\lambda}\varepsilon^{\rho\sigma\tau}\tilde{F}_{\sigma\tau}$:

$$\begin{aligned}
&\tilde{h}_{\mu\rho}\varepsilon^{\mu\nu\lambda}\tilde{F}_{\nu\lambda}\varepsilon^{\rho\sigma\tau}\tilde{F}_{\sigma\tau} \\
&= h^{(0)}_{\mu\nu}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}\varepsilon^{\rho\sigma\tau}F_{\sigma\tau} + h^{(1)}_{\mu\nu}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}\varepsilon^{\rho\sigma\tau}F_{\sigma\tau} + 2h^{(0)}_{\mu\nu}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}\varepsilon^{\rho\sigma\tau}F^{(1)}_{\sigma\tau} + \mathcal{O}\left(\partial^2\right) \\
&= 4B^2h^{(0)}_{\mu\nu}u^{\mu}_{(0)}u^{\nu}_{(0)} + 4B^2h^{(1)}_{\mu\nu}u^{\mu}_{(0)}u^{\nu}_{(0)} + 4Bh^{(0)}_{\mu\rho}u^{\mu}_{(0)}\varepsilon^{\rho\sigma\tau}F^{(1)}_{\sigma\tau} + \mathcal{O}\left(\partial^2\right) \\
&= \mathcal{O}\left(\partial^2\right),
\end{aligned} \tag{B.2}$$

where we define

$$h^{(0)}_{\mu\nu} = h_{\mu\nu} - n_{(\mu}u^{(0)}_{\nu)} + n_\mu n_\nu u^2_{(0)}, \tag{B.3}$$

$$h^{(1)}_{\mu\nu} = -n_{(\mu}u^{(1)}_{\nu)} + 2n_\mu n_\nu u^{(0)}_\rho u^\rho_{(1)}, \tag{B.4}$$

$$F^{(1)}_{\mu\nu} = \partial_{[\mu}A^{(0)}_{\nu]}, \tag{B.5}$$

$$A^{(0)}_\mu = mu^{(0)}_\mu - \frac{m}{2}n_\mu u^2_{(0)}, \tag{B.6}$$

and use $h^{(0)}_{\mu\nu}u^{\mu}_{(0)} = h^{(1)}_{\mu\nu}u^{\mu}_{(0)}u^{\nu}_{(0)} = 0$.

- Vanishing of $(\tilde{\nabla}_\mu\varepsilon^{\mu\nu\lambda})\tilde{F}_{\nu\lambda} = 0$: since $\tilde{\nabla}_\mu\varepsilon^{\mu\nu\lambda} = 0$, this term is identically zero.

- Dependence of $\varepsilon^{\mu\nu\lambda}(\tilde{\nabla}_\mu\tilde{F}_{\nu\lambda})$, $u^\mu\partial_\mu\tilde{B}$, and $\tilde{\nabla}_\mu u^\mu$:

$$\begin{aligned}
f(\tilde{B})\varepsilon^{\mu\nu\lambda}(\tilde{\nabla}_\mu\tilde{F}_{\nu\lambda}) &= f(B)\varepsilon^{\mu\nu\lambda}(\tilde{\nabla}_\mu F_{\nu\lambda}) + \mathcal{O}\left(\partial^2\right) \\
&= f(B)\tilde{\nabla}_\mu(\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}) + \mathcal{O}\left(\partial^2\right) \\
&= 2f(B)\tilde{\nabla}_\mu(Bu^\mu_{(0)}) + \mathcal{O}\left(\partial^2\right) \\
&= 2f(B)u^\mu_{(0)}\partial_\mu B + 2Bf(B)\tilde{\nabla}_\mu u^\mu_{(0)} + \mathcal{O}\left(\partial^2\right) \\
&= 2f(\tilde{B})u^\mu\partial_\mu\tilde{B} + 2\tilde{B}f(\tilde{B})\tilde{\nabla}_\mu u^\mu + \mathcal{O}\left(\partial^2\right),
\end{aligned} \tag{B.7}$$

where f is an arbitrary function.

- Dependence of $\varepsilon^{\mu\nu\lambda}\tilde{F}_{\nu\lambda}\partial_\mu\tilde{B}$ and $u^\mu\partial_\mu\tilde{B}$:

$$\begin{aligned}\varepsilon^{\mu\nu\lambda}\tilde{F}_{\nu\lambda}\partial_\mu\tilde{B} &= \varepsilon^{\mu\nu\lambda}F_{\nu\lambda}\partial_\mu B + \mathcal{O}\left(\partial^2\right)\\ &= 2Bu^\mu_{(0)}\partial_\mu B + \mathcal{O}\left(\partial^2\right)\\ &= 2\tilde{B}u^\mu\partial_\mu\tilde{B} + \mathcal{O}\left(\partial^2\right).\end{aligned} \tag{B.8}$$

- Vanishing of $h^{\mu\nu}(\tilde{\nabla}_\mu u^\lambda)\tilde{F}_{\nu\lambda}$: since $\tilde{\nabla}^{[\nu}u^{\lambda]} = 0$, this term vanishes.
- Higher-order nature of $h^{\mu\nu}u^\lambda(\tilde{\nabla}_\mu\tilde{F}_{\nu\lambda})$:

$$\begin{aligned}h^{\mu\nu}u^\lambda(\tilde{\nabla}_\mu\tilde{F}_{\nu\lambda}) &= h^{\mu\nu}u^\lambda_{(0)}(\tilde{\nabla}_\mu F_{\nu\lambda}) + \mathcal{O}\left(\partial^2\right)\\ &= h^{\mu\nu}\tilde{\nabla}_\mu(u^\lambda_{(0)}F_{\nu\lambda}) - h^{\mu\nu}(\tilde{\nabla}_\mu u^\lambda)F_{\nu\lambda} + \mathcal{O}\left(\partial^2\right)\\ &= \mathcal{O}\left(\partial^2\right),\end{aligned} \tag{B.9}$$

  where we use $u^\lambda_{(0)}F_{\nu\lambda} = 0$.

- Higher-order nature of $h^{\mu\nu}u^\lambda\tilde{F}_{\nu\lambda}\partial_\mu\tilde{B}$:

$$\begin{aligned}h^{\mu\nu}u^\lambda\tilde{F}_{\nu\lambda}\partial_\mu\tilde{B} &= h^{\mu\nu}u^\lambda_{(0)}F_{\nu\lambda}\partial_\mu\tilde{B} + \mathcal{O}\left(\partial^2\right)\\ &= \mathcal{O}\left(\partial^2\right),\end{aligned} \tag{B.10}$$

  where we use $u^\lambda_{(0)}F_{\nu\lambda} = 0$.

- Dependence of $h^{\mu\nu}(\tilde{\nabla}_\mu u^\lambda)\tilde{h}_{\nu\lambda}$ and $\tilde{\nabla}_\mu u^\mu$:

$$\begin{aligned}h^{\mu\nu}(\tilde{\nabla}_\mu u^\lambda)\tilde{h}_{\nu\lambda} &= \tilde{P}^\mu_\lambda(\tilde{\nabla}_\mu u^\lambda)\\ &= \tilde{\nabla}_\mu u^\mu - u^\mu n_\lambda\tilde{\nabla}_\mu u^\lambda\\ &= \tilde{\nabla}_\mu u^\mu + u^\mu u^\lambda\tilde{\nabla}_\mu n_\lambda\\ &= \tilde{\nabla}_\mu u^\mu.\end{aligned} \tag{B.11}$$

- Dependence of $h^{\mu\nu}u^\lambda(\tilde{\nabla}_\mu\tilde{h}_{\nu\lambda})$ and $\tilde{\nabla}_\mu u^\mu$:

$$\begin{aligned}h^{\mu\nu}u^\lambda(\tilde{\nabla}_\mu\tilde{h}_{\nu\lambda}) &= u^\lambda(\tilde{\nabla}_\mu h^{\mu\nu}\tilde{h}_{\nu\lambda})\\ &= -u^\lambda(\tilde{\nabla}_\mu n_\lambda u^\mu)\\ &= -\tilde{\nabla}_\mu u^\mu.\end{aligned} \tag{B.12}$$

- Vanishing of $h^{\mu\nu}u^\lambda\tilde{h}_{\nu\lambda}\partial_\mu\tilde{B}$: since $u^\lambda\tilde{h}_{\nu\lambda} = 0$, this term is identically zero.
- Dependence of $\varepsilon^{\mu\nu\lambda}\tilde{B}_\mu\tilde{F}_{\nu\lambda}$ and $B_\mu u^\mu$:

$$\begin{aligned}\varepsilon^{\mu\nu\lambda}\tilde{B}_\mu\tilde{F}_{\nu\lambda} &= \varepsilon^{\mu\nu\lambda}\tilde{B}_\mu F_{\nu\lambda} + \mathcal{O}\left(\partial^2\right)\\ &= 2Bu^\mu_{(0)}\tilde{B}_\mu + \mathcal{O}\left(\partial^2\right)\\ &= 2Bu^\mu\tilde{B}_\mu + \mathcal{O}\left(\partial^2\right).\end{aligned} \tag{B.13}$$

- Higher-order nature of $h^{\mu\nu}u^\lambda\tilde{B}_\mu\tilde{F}_{\nu\lambda}$:

$$h^{\mu\nu}u^\lambda\tilde{B}_\mu\tilde{F}_{\nu\lambda} = h^{\mu\nu}u^\lambda_{(0)}\tilde{B}_\mu F_{\nu\lambda} + \mathcal{O}\left(\partial^2\right) = \mathcal{O}\left(\partial^2\right), \tag{B.14}$$

  where we use $u^\lambda_{(0)}F_{\nu\lambda} = 0$.

- Vanishing of $\tilde{B}_\mu u^\mu = u^\nu G_{\mu\nu} u^\nu$: this term is identically zero since $G_{\mu\nu}$ is anti-symmetric under $\mu \leftrightarrow \nu$.
- Dependence of $G_{\mu\nu}\tilde{F}^{\mu\nu}$ and G:

$$BG = \frac{1}{2} G_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{B.15}$$

- Higher-order nature of $G^\mu u^\nu \tilde{F}_{\mu\nu}$:

$$G^\mu u^\nu \tilde{F}_{\mu\nu} = G^\mu u^\nu_{(0)} F_{\mu\nu} + \mathcal{O}\left(\partial^2\right) = \mathcal{O}\left(\partial^2\right), \tag{B.16}$$

  where we use $u^\nu_{(0)} F_{\mu\nu} = 0$.
- Vanishing of $G^\mu u^\nu \tilde{h}_{\mu\nu}$: this term vanishes since $u^\nu \tilde{h}_{\mu\nu} = 0$.

Appendix C

Free electrons

We compute functional forms of $\mathcal{E}(\tilde{B})$ and $\mathcal{M}(\tilde{B})$ in this appendix. The effective action Eq. (3.93) contains four unknown coefficients: ν, κ, $\mathcal{E}(\tilde{B})$, and $\mathcal{M}(\tilde{B})$. While ν and κ are constants and independent of interactions, $\mathcal{E}(\tilde{B})$ and $\mathcal{M}(\tilde{B})$ depends on interactions. Although the functional forms of $\mathcal{E}(\tilde{B})$ and $\mathcal{M}(\tilde{B})$ are not universal, we can compute them for free electrons (the integer quantum Hall systems).

We consider an electron under a constant magnetic field B and nontrivial metrics

$$n_\mu = (1, -C_i), \tag{C.1}$$

$$h^{\mu\nu} = \begin{pmatrix} C^2 & C_i \\ C_j & \delta^{ij} \end{pmatrix}, \tag{C.2}$$

$$v^\mu = (1, 0, 0), \tag{C.3}$$

$$h_{\mu\nu} = \mathrm{diag}(0, 1, 1). \tag{C.4}$$

In this setup, the energy density is given by

$$\mathcal{E} = n_\mu \mathcal{E}^\mu = \mathcal{E}(B) - 2\mathcal{M}(B)\epsilon^{ij}\partial_i C_j + \mathcal{O}\left(\partial^2\right). \tag{C.5}$$

Then, we employ the perturbation theory in C_i and compute the energy eigenvalue up to $\mathcal{O}\left(C^2, \partial^2 C\right)$. The free Hamiltonian under the same background fields is given by

$$H = H_0 + H_1 + \mathcal{O}\left(n_i^2\right), \tag{C.6}$$

$$H_0 = \int d^2x \left[\frac{D_i\Psi^\dagger D_i\Psi}{2m} - \frac{gB}{4m}\Psi^\dagger\Psi\right], \tag{C.7}$$

$$H_1 = -\int d^2x C_i \mathcal{E}^i, \tag{C.8}$$

where

$$\mathcal{E}^i = -\frac{D_{(t}\Psi^\dagger D_{i)}\Psi}{2m} - \frac{ig\,\epsilon^{ij}}{4m} D_{[t}\Psi^\dagger D_{j]}\Psi \tag{C.9}$$

is the energy current density. We expand the field operators as

$$\Psi(\boldsymbol{x}) = \sum_{n,m} \psi_{n,k}(\boldsymbol{x})\alpha_{n,k}, \tag{C.10}$$

$$\Psi^\dagger(\boldsymbol{x}) = \sum_{n,k} \psi^*_{n,k}(\boldsymbol{x})\alpha^\dagger_{n,k}, \tag{C.11}$$

where $\alpha^{(\dagger)}_{n,k}$ is the annihilation (creation) operator of electrons with the (n,k) state, which satisfy

$$\{\alpha_{n,k}, \alpha^{\dagger}_{n',k'}\} = \delta_{nn'}\delta_{kk'}. \tag{C.12}$$

The expansion coefficient $\psi_{n,k}(\boldsymbol{x})$ is the one-body wavefunction,

$$\psi_{n,k}(\boldsymbol{x}) = \langle \boldsymbol{x}|n,k\rangle\,, \tag{C.13}$$

where n and k are the Landau level and intra-Landau quantum numbers, respectively. We now introduce the complex coordinates,

$$z = x + iy,\ \bar{z} = x - iy, \tag{C.14}$$

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y),\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y), \tag{C.15}$$

$$A_z = A_x + iA_y,\ A_{\bar{z}} = A_x - iA_y, \tag{C.16}$$

$$C_z = C_x + iC_y,\ C_{\bar{z}} = C_x - iC_y. \tag{C.17}$$

The ladder operators between Landau levels can be written as

$$a^{\dagger} = i\sqrt{2}l_B D_z = \frac{i}{\sqrt{2}}l_B(D_x - iD_y), \tag{C.18}$$

$$a = i\sqrt{2}l_B D_{\bar{z}} = \frac{i}{\sqrt{2}}l_B(D_x + iD_y). \tag{C.19}$$

The ladder operators intra Landau levels are

$$b^{\dagger} = -i\sqrt{2}l_B D_{\bar{z}} + \frac{i}{\sqrt{2}l_B}z, \tag{C.20}$$

$$b = -i\sqrt{2}l_B D_z - \frac{i}{\sqrt{2}l_B}\bar{z}. \tag{C.21}$$

These operators satisfy the commutation relations $[a, a^{\dagger}] = [b, b^{\dagger}] = 1$. In this coordinates, the Hamiltonian can be written as

$$H_0 = \int d^2x \left[\frac{1}{m}(D_z\Psi^{\dagger}D_{\bar{z}}\Psi + D_{\bar{z}}\Psi^{\dagger}D_z\Psi) - \frac{gB}{4m}\Psi^{\dagger}\Psi\right], \tag{C.22}$$

$$H_1 = -\int d^2x \left[\frac{C_z}{2m^2 i}(D_z\Psi^{\dagger}(D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi) + \frac{C_{\bar{z}}}{2m^2 i}(D_{\bar{z}}\Psi^{\dagger}(D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi)\right. \tag{C.23}$$

$$- \frac{C_z}{2m^2 i}((D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi^{\dagger}D_z\Psi) - \frac{C_{\bar{z}}}{2m^2 i}((D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi^{\dagger}D_{\bar{z}}\Psi) \tag{C.24}$$

$$- \frac{igC_z}{4m^2}(D_z\Psi^{\dagger}(D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi) + \frac{igC_{\bar{z}}}{4m^2}(D_{\bar{z}}\Psi^{\dagger}(D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi) \tag{C.25}$$

$$- \frac{igC_z}{4m^2}((D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi^{\dagger}D_z\Psi) + \frac{igC_{\bar{z}}}{4m^2}((D_zD_{\bar{z}} + D_{\bar{z}}D_z)\Psi^{\dagger}D_{\bar{z}}\Psi) \tag{C.26}$$

$$+ \frac{gBC_z}{8m^2 i}(D_z\Psi^{\dagger}\Psi - \Psi^{\dagger}D_z\Psi) + \frac{gBC_{\bar{z}}}{8m^2 i}(D_{\bar{z}}\Psi^{\dagger}\Psi - \Psi^{\dagger}D_{\bar{z}}\Psi) \tag{C.27}$$

$$\left.- \frac{ig^2B}{16m^2}C_z(D_z\Psi^{\dagger}\Psi + \Psi^{\dagger}D_z\Psi) + \frac{ig^2B}{16m^2}C_{\bar{z}}(D_{\bar{z}}\Psi^{\dagger}\Psi + \Psi^{\dagger}D_{\bar{z}}\Psi)\right], \tag{C.28}$$

where we also use the equations of motion for Ψ and $\Psi^{\dagger}$. Here, $C_{z(\bar{z})}$ is expanded as

$$C_z = C_z^0 + z\partial_z C_z^0 + \bar{z}\partial_{\bar{z}} C_z^0, \tag{C.29}$$

$$C_{\bar{z}} = C_{\bar{z}}^0 + z\partial_z C_{\bar{z}}^0 + \bar{z}\partial_{\bar{z}} C_{\bar{z}}^0, \tag{C.30}$$

where the subscript 0 means we substitute $(z, \bar{z}) = (0, 0)$ after the differentiation.

Using the ladder operators, we compute the covariant derivatives of the wavefunctions as

$$D_z \psi_{n,k}(\boldsymbol{x}) = -\frac{i}{\sqrt{2} l_B} \sqrt{n+1} \psi_{n+1,k}(\boldsymbol{x}), \tag{C.31}$$

$$D_{\bar{z}} \psi_{n,k}(\boldsymbol{x}) = -\frac{i}{\sqrt{2} l_B} \sqrt{n} \psi_{n-1,k}(\boldsymbol{x}). \tag{C.32}$$

We also find

$$z \psi_{n,k} = -i\sqrt{2} l_B (\sqrt{n} \psi_{n-1,k} + \sqrt{k+1} \psi_{n,k+1}), \tag{C.33}$$

$$\bar{z} \psi_{n,k} = i\sqrt{2} l_B (\sqrt{n+1} \psi_{n+1,k} + \sqrt{k} \psi_{n,k-1}). \tag{C.34}$$

Then, the zeroth-order Hamiltonian H_0 is written as

$$H_0 = \sum_{n,k} \frac{B}{m} \left(n + \frac{1}{2} - \frac{g}{4} \right) \alpha^\dagger_{n,k} \alpha_{n,k}, \tag{C.35}$$

while the first-order Hamiltonian H_1 is

$$H_1 = \sum_{n,k} \left[\left(-\frac{B}{4m^2}(2n+1)^2 + \frac{gB}{4m^2}(2n+1) - \frac{g^2 B}{16m^2} \right) i(\partial_z C^0_z - \partial_{\bar{z}} C^0_{\bar{z}}) \right] \alpha^\dagger_{n,k} \alpha_{n,k} + \text{(off-diagonal part)}. \tag{C.36}$$

Here the off-diagonal part expresses the terms proportional to $\alpha^\dagger_{n',k'} \alpha_{n,k}$ with $(n, k) \neq (n', k')$. These terms do not contribute to the final computation of energy eigenvalues in the first-order perturbation theory as shown later.

We now employ the perturbation theory for a one-body state $|n, k\rangle$. The zeroth-order energy eigenvalue for the n-th Landau level is given by

$$\begin{aligned} E^{(0)}_n &= \langle n, k| H_0 |n, k\rangle \\ &= \langle \text{vac}| \alpha_{n,k} H_0 \alpha^\dagger_{n,k} |\text{vac}\rangle \\ &= \frac{B}{m} \left(n + \frac{1}{2} - \frac{g}{4} \right). \end{aligned} \tag{C.37}$$

Since the system is degenerated, the first-order correction of the energy eigenvalue for the n-th Landau level is obtained by diagonalizing the matrix

$$V_{n;k_1 k_2} = \langle n, k_1| H_1 |n, k_2\rangle . \tag{C.38}$$

First, we show the off-diagonal part does not contribute to this matrix:

$$\begin{aligned} \langle n, k_1| H^{\text{off}}_1 |n, k_2\rangle &\propto \langle \text{vac}| \alpha_{n,k_1} \alpha^\dagger_{m,k} \alpha_{m',k'} \alpha^\dagger_{n,k_2} |\text{vac}\rangle \\ &= \langle \text{vac}| \delta_{n,m} \delta_{k,k_1} \delta_{n,m'} \delta_{k',k_2} |\text{vac}\rangle && \text{(C.39)} \\ &= 0, && \text{(C.40)} \end{aligned}$$

where we use Eq. (C.12), $\alpha_{n,k} |\text{vac}\rangle = 0$, and $m \neq m'$. Therefore, we find

$$V_{n;k_1 k_2} = -\left(\frac{B}{4m^2}(2n+1)^2 - \frac{gB}{4m^2}(2n+1) + \frac{g^2 B}{16m^2} \right) G \delta_{k_1,k_2} \tag{C.41}$$

with $G = -\epsilon^{ij}\partial_i C_j = i(\partial_z C_z - \partial_{\bar{z}} C_{\bar{z}})$. Since this matrix is already diagonal, the first-order correction is directly given by

$$E_n^{(1)} = -\left(\frac{B}{4m^2}(2n+1)^2 - \frac{gB}{4m^2}(2n+1) + \frac{g^2 B}{16m^2}\right)G. \tag{C.42}$$

The total energy density $\mathcal{E}$ is calculated by summing up the contributions from all occupied Landau levels up to the filling factor ν:

$$\begin{aligned} \mathcal{E} &= \frac{B}{2\pi}\sum_{n=0}^{\nu-1}(E_n^{(0)} + E_n^{(1)}) \\ &= \frac{B^2}{4\pi m}\nu^2 - \frac{gB^2}{8\pi m}\nu - \left(\frac{B^2}{24\pi m^2}(4\nu^3 - \nu) - \frac{gB^2}{8\pi m^2}\nu^2 + \frac{g^2 B^2}{32\pi m^2}\nu\right)G, \end{aligned} \tag{C.43}$$

where $B/2\pi$ accounts for the degeneracy of each Landau level. Compared with Eq. (C.5), we finally find

$$\mathcal{E}(B) = \frac{B^2}{4\pi m}\nu^2 - \frac{gB^2}{8\pi m}\nu, \tag{C.44}$$

$$\mathcal{M}(B) = -\frac{1}{2}\left(\frac{B^2}{24\pi m^2}(4\nu^3 - \nu) - \frac{gB^2}{8\pi m^2}\nu^2 + \frac{g^2 B^2}{32\pi m^2}\nu\right). \tag{C.45}$$

Appendix D

Wen-Zee term

In this appendix, we rewrite the Wen-Zee term in terms of the spatial metric. To do this, we recall the torsionless connection Eq. (3.25),

$$\mathring{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2}v^{\lambda}\partial_{(\mu}n_{\nu)} + \frac{1}{2}h^{\lambda\rho}(\partial_{\mu}h_{\rho\nu} + \partial_{\nu}h_{\mu\rho} - \partial_{\rho}h_{\mu\nu}). \tag{D.1}$$

The spin connection Eq. (3.69) can be expressed as

$$\omega_{\mu} = \frac{1}{2}\epsilon^{ab}h_{\lambda\nu}e^{a\lambda}\mathring{\nabla}_{\mu}e^{b\nu}, \tag{D.2}$$

where $\mathring{\nabla}_{\mu}$ is the covariant derivative with the connection $\mathring{\Gamma}^{\lambda}{}_{\mu\nu}$. Here the anti-symmetrized derivative of the spin connection is

$$\partial_{[\mu}\omega_{\nu]} = \mathring{\nabla}_{[\mu}\omega_{\nu]} \tag{D.3}$$

$$= \frac{1}{2}\epsilon^{ab}e^{a}{}_{\sigma}\mathring{\nabla}_{[\mu}\mathring{\nabla}_{\nu]}e^{b\sigma} + \frac{1}{2}\epsilon^{ab}e^{a}{}_{\sigma}(\mathring{\nabla}_{[\mu}v^{\sigma}\mathring{\nabla}_{\nu]}n_{\tau})e^{b\tau} + \frac{1}{2}\epsilon^{ab}(\mathring{\nabla}_{[\mu}e^{a}{}_{\sigma})P^{\sigma}_{\tau}(\mathring{\nabla}_{\nu]}e^{b\tau}). \tag{D.4}$$

By employing $\nabla_{\lambda}h^{\mu\nu} = 0$ and $e^{a\lambda}e^{b}{}_{\lambda} = \delta^{ab}$, the last term is found to vanish,

$$\frac{1}{2}\epsilon^{ab}(\mathring{\nabla}_{[\mu}e^{a}{}_{\sigma})P^{\sigma}_{\tau}(\mathring{\nabla}_{\nu]}e^{b\tau}) = \frac{1}{2}\epsilon^{ab}(e^{c\sigma}\nabla_{[\mu}e^{a}{}_{\sigma})(e^{c\tau}\nabla_{\nu]}e^{b}{}_{\tau}) = 0, \tag{D.5}$$

because $e^{a\lambda}\nabla_{\mu}e^{b}{}_{\lambda} = -e^{b\lambda}\nabla_{\mu}e^{a}{}_{\lambda}$ vanishes for $a = b$. We now define the Riemann curvature of the torsionless connection $\mathring{\Gamma}^{\lambda}{}_{\mu\nu}$ as

$$\mathring{R}^{\sigma}{}_{\tau\mu\nu} = \partial_{[\mu}\mathring{\Gamma}^{\sigma}_{\nu]\tau} + \mathring{\Gamma}^{\sigma}_{[\mu\rho}\mathring{\Gamma}^{\rho}_{\nu]\tau}. \tag{D.6}$$

Since the Riemann curvature is also given by the commutation relation of the covariant derivatives as $\mathring{\nabla}_{[\mu}\mathring{\nabla}_{\nu]}e^{b\sigma} = \mathring{R}^{\sigma}{}_{\tau\mu\nu}e^{b\tau}$, we obtain

$$\partial_{[\mu}\omega_{\nu]} = \frac{1}{2}\varepsilon^{\kappa\rho\tau}n_{\kappa}h_{\rho\sigma}(\mathring{R}^{\sigma}{}_{\tau\mu\nu} + \mathring{\nabla}_{[\mu}v^{\sigma}\mathring{\nabla}_{\nu]}n_{\tau}), \tag{D.7}$$

where we also use $\epsilon^{ab}e^{a\rho}e^{b\tau} = \varepsilon^{\kappa\rho\tau}n_{\kappa}$.

Therefore, the Wen-Zee term turns out to be expressed as

$$\varepsilon^{\lambda\mu\nu} A_\lambda \partial_\mu \omega_\nu = A_\mu J^\mu_{\text{Euler}}, \tag{D.8}$$

$$J^\lambda_{\text{Euler}} = \frac{1}{4} \varepsilon^{\lambda\mu\nu} \varepsilon^{\kappa\rho\tau} n_\kappa h_{\rho\sigma} (\mathring{R}^\sigma{}_{\tau\mu\nu} + 2\mathring{\nabla}_\mu v^\sigma \mathring{\nabla}_\nu n_\tau), \tag{D.9}$$

where J^λ_{Euler} is a Newton-Cartan analog of the Euler current introduced to describe relativistic quantum Hall systems [118, 125]. The Milne invariant expression is given by replacing v^μ, $h_{\mu\nu}$, and A_μ with u^μ, $\tilde{h}_{\mu\nu}$, and $\tilde{A}_\mu$, respectively. We can check that this Euler-like current is indeed conserved,

$$(\nabla_\lambda + B_\lambda) J^\lambda_{\text{Euler}} = 0. \tag{D.10}$$

Appendix E

Formulae of variations

In this appendix, we present the formulae of the variation to derive the current densities from the effective action. Under the variations of the external fields,

$$\delta A_\mu, \tag{E.1}$$
$$\delta n_\mu, \tag{E.2}$$
$$\delta v^\mu = -v^\mu v^\nu \delta n_\nu + P^\mu_\nu \delta \bar{v}^\nu, \tag{E.3}$$
$$\delta h^{\mu\nu} = -v^{(\mu} h^{\nu)\lambda} \delta n_\lambda + P^\mu_\rho P^\nu_\sigma \delta \bar{h}^{\rho\sigma}, \tag{E.4}$$
$$\delta h_{\mu\nu} = -n_{(\mu} h_{\nu)\lambda} \delta \bar{v}^\lambda - h_{\mu\rho} h_{\nu\sigma} \delta \bar{h}^{\rho\sigma}, \tag{E.5}$$
$$\delta e^{a\mu} = -v^\mu e^{a\lambda} \delta n_\lambda + P^\mu_\rho \delta \bar{e}^{a\rho}, \tag{E.6}$$
$$\delta \bar{h}^{\mu\nu} = e^{a\mu} \delta \bar{e}^{a\nu} + e^{a\nu} \delta \bar{e}^{a\mu}, \tag{E.7}$$

we have the following formulae:

$$\delta\sqrt{\gamma} = \sqrt{\gamma}\left(v^\rho \delta n_\rho - \frac{1}{2} h_{\rho\sigma} \delta \bar{h}^{\rho\sigma}\right), \tag{E.8}$$

$$\delta\varepsilon^{\mu\nu\lambda} = -\varepsilon^{\mu\nu\lambda}\left(v^\rho \delta n_\rho - \frac{1}{2} h_{\rho\sigma} \delta \bar{h}^{\rho\sigma}\right), \tag{E.9}$$

$$\delta B = -B\left(v^\rho \delta n_\rho - \frac{1}{2} h_{\rho\sigma} \delta \bar{h}^{\rho\sigma}\right) + \varepsilon^{\mu\nu\lambda} \delta n_\mu \partial_\nu A_\lambda + \varepsilon^{\mu\nu\lambda} n_\mu \partial_\nu \delta A_\lambda, \tag{E.10}$$

$$\delta u^\lambda = \frac{\varepsilon^{\mu\nu\lambda} - u^\mu n_\rho \varepsilon^{\rho\nu\lambda}}{B} \partial_\nu \delta A_\lambda - u^\mu u^\nu \delta n_\rho, \tag{E.11}$$

$$\delta u_\mu = -(n_\mu u_\nu + h_{\mu\nu})\delta \bar{v}^\nu - h_{\mu\nu} u_\rho \delta \bar{h}^{\nu\rho} + \frac{h_{\mu\rho} - u_\mu n_\rho}{B} \varepsilon^{\rho\nu\lambda} \partial_\nu \delta A_\lambda - u_\mu u^\rho \delta n_\rho, \tag{E.12}$$

$$\delta u^2 = -2u_\rho \delta \bar{v}^\rho - u_\rho u_\sigma \delta \bar{h}^{\rho\sigma} + \frac{2u_\rho - 2u^2 n_\rho}{B} \varepsilon^{\rho\nu\lambda} \partial_\nu \delta A_\lambda - 2u^2 u^\rho \delta n_\rho. \tag{E.13}$$

Appendix F

Full expressions of currents

In this appendix, we present the full expressions of the currents.

F.1 Charge current

The charge current from the Chern-Simons term is given by

$$
\begin{aligned}
\mathcal{J}^{\mu}_{\rm CS} = \frac{1}{\sqrt{\gamma}}\frac{\nu}{2\pi}\bigg(& \epsilon^{\mu\nu\lambda}\partial_\nu(A_\lambda + mu^{(0)}_\lambda - \frac{1}{2}mu^2_{(0)}n_\lambda) \\
& + m\epsilon^{\gamma\nu\lambda}\partial_\beta(h_{\gamma\rho}\frac{1}{B^2}\varepsilon^{\rho\sigma\kappa}\partial_\sigma A_\kappa\varepsilon^{\alpha\beta\mu}n_\alpha\partial_\nu(A_\lambda + mu^{(0)}_\lambda - \frac{1}{2}mu^2_{(0)}n_\lambda)) \\
& - m\epsilon^{\kappa\nu\lambda}\partial_\sigma(h_{\kappa\rho}\frac{1}{B}\varepsilon^{\rho\sigma\mu}\partial_\nu(A_\lambda + mu^{(0)}_\lambda - \frac{1}{2}mu^2_{(0)}n_\lambda)) \\
& - m\epsilon^{\gamma\nu\lambda}\partial_\beta(h_{\rho\tau}u^\tau_{(0)}n_\gamma\frac{1}{B^2}\varepsilon^{\rho\sigma\kappa}\partial_\sigma A_\kappa\varepsilon^{\alpha\beta\mu}n_\alpha\partial_\nu(A_\lambda + mu^{(0)}_\lambda - \frac{1}{2}mu^2_{(0)}n_\lambda)) \\
& + m\epsilon^{\kappa\nu\lambda}\partial_\sigma(\frac{1}{B}\varepsilon^{\rho\sigma\mu}h_{\rho\tau}u^\tau_{(0)}n_\kappa\partial_\nu(A_\lambda + mu^{(0)}_\lambda - \frac{1}{2}mu^2_{(0)}n_\lambda)) \\
& - \frac{g}{4}\partial_\rho\bigg(\partial_\alpha(\sqrt{\gamma}B\varepsilon^{\alpha\beta\gamma})n_\beta h_{\gamma\delta}\frac{1}{B}u^\delta_{(0)}\varepsilon^{\kappa\rho\mu}n_\kappa\bigg) \\
& + \frac{g}{4}\partial_\rho\bigg(\partial_\alpha(\sqrt{\gamma}B\varepsilon^{\alpha\beta\gamma})n_\beta h_{\gamma\delta}\frac{1}{B}\varepsilon^{\delta\rho\mu}\bigg) + \frac{g}{4}\epsilon^{\mu\nu\lambda}\partial_\nu(n_\lambda\varepsilon^{\alpha\beta\gamma}\partial_\alpha(n_\beta u_\gamma))\bigg),
\end{aligned}
\tag{F.1}
$$

that from the $\mathcal{E}$ term is given by

$$
\begin{aligned}
\mathcal{J}^{\mu}_{\mathcal{E}} = -\frac{1}{\sqrt{\gamma}}(& -\partial_\beta(\mathcal{E}'(B)\epsilon^{\alpha\beta\mu}n_\alpha) - m\partial_\beta(\mathcal{E}''(B)\varepsilon^{\alpha\beta\mu}n_\alpha(\epsilon^{\rho\nu\lambda}n_\rho\partial_\nu u^{(0)}_\lambda - \frac{1}{2}u^2_{(0)}\epsilon^{\rho\nu\lambda}n_\mu\partial_\nu n_\lambda) \\
& - m\partial_\beta(\mathcal{E}'(B)\epsilon^{\sigma\nu\lambda}\partial_\nu n_\sigma h_{\lambda\rho}\frac{1}{B}u^\rho_{(0)}\varepsilon^{\alpha\beta\mu}n_\alpha) + m\partial_\sigma(\mathcal{E}'(B)\epsilon^{\tau\nu\lambda}\partial_\nu n_\tau h_{\lambda\rho}\frac{1}{B}\varepsilon^{\rho\sigma\mu}) \\
& - m\partial_\beta(\mathcal{E}''(B)\epsilon^{\sigma\nu\lambda}n_\sigma\partial_\nu Bh_{\lambda\rho}\frac{1}{B}u^\rho_{(0)}\varepsilon^{\alpha\beta\mu}n_\alpha) + m\partial_\sigma(\mathcal{E}''(B)\epsilon^{\tau\nu\lambda}n_\tau\partial_\nu Bh_{\lambda\rho}\frac{1}{B}\varepsilon^{\rho\sigma\mu}) \\
& - m\partial_\beta(\epsilon^{\sigma\nu\lambda}n_\sigma\partial_\nu n_\lambda\mathcal{E}'(B)h_{\rho\tau}u^\tau_{(0)}\frac{1}{B}u^\rho_{(0)}\varepsilon^{\alpha\beta\mu}n_\alpha) + m\partial_\sigma(\epsilon^{\zeta\nu\lambda}n_\zeta\partial_\nu n_\lambda\mathcal{E}'(B)h_{\rho\tau}u^\tau_{(0)}\frac{1}{B}\varepsilon^{\rho\sigma\mu}))),
\end{aligned}
\tag{F.2}
$$

that from the Wen-Zee term is given by

$$\mathcal{J}^\mu_{\rm WZ} = \frac{1}{\sqrt{\gamma}}\frac{\kappa}{4\pi}\bigg(\frac{1}{2}\partial_\beta(\partial_\gamma(\epsilon^{\gamma\nu\lambda}\partial_\nu A_\lambda \epsilon^{ab}e^{a\eta})e^b{}_\rho \frac{1}{B}u^\rho_{(0)}\varepsilon^{\alpha\beta\mu}n_\alpha n_\eta) - \frac{1}{2}\partial_\sigma(\partial_\kappa(\epsilon^{\kappa\nu\lambda}\partial_\nu A_\lambda \epsilon^{ab}e^{a\eta})e^b{}_\rho \frac{1}{B}\varepsilon^{\rho\sigma\mu}n_\eta)$$
$$- \frac{1}{2}\epsilon^{\gamma\nu\lambda}\partial_\beta(\partial_\eta(\partial_\nu A_\lambda \epsilon^{ab}e^{a\eta}e^{b\rho})h_{\gamma\zeta}\frac{1}{B}u^\zeta_{(0)}\varepsilon^{\alpha\beta\mu}n_\alpha n_\rho) + \frac{1}{2}\partial_\sigma(\partial_\eta(\epsilon^{\kappa\nu\lambda}\partial_\nu A_\lambda\epsilon^{ab}e^{a\eta}e^{b\rho})h_{\kappa\zeta}\frac{1}{B}\varepsilon^{\zeta\sigma\mu}n_\rho)$$
$$- \frac{1}{2}\epsilon^{\gamma\nu\lambda}\partial_\beta(\partial_\eta(\partial_\nu A_\lambda \epsilon^{ab}e^{a\eta}e^{b\rho})h_{\rho\zeta}\frac{1}{B}u^\zeta_{(0)}\varepsilon^{\alpha\beta\mu}n_\alpha n_\gamma) + \frac{1}{2}\partial_\sigma(\partial_\eta(\epsilon^{\kappa\nu\lambda}\partial_\nu A_\lambda\epsilon^{ab}e^{a\eta}e^{b\rho})h_{\rho\zeta}\frac{1}{B}\varepsilon^{\zeta\sigma\mu}n_\kappa)$$
$$+ \epsilon^{\gamma\nu\lambda}\partial_\beta(\partial_\eta(\partial_\nu A_\lambda \epsilon^{ab}e^{a\eta}e^{b\rho})h_{\tau\zeta}u^\tau_{(0)}\frac{1}{B}u^\zeta_{(0)}\varepsilon^{\alpha\beta\mu}n_\alpha n_\gamma n_\rho)$$
$$- \partial_\sigma(\partial_\eta(\epsilon^{\kappa\nu\lambda}\partial_\nu A_\lambda\epsilon^{ab}e^{a\eta}e^{b\rho})h_{\tau\zeta}u^\tau_{(0)}\frac{1}{B}\varepsilon^{\zeta\sigma\mu}n_\kappa n_\rho) - \epsilon^{\lambda\nu\mu}\partial_\nu\tilde{\omega}_\lambda\bigg), \tag{F.3}$$

and that from the $\mathcal{M}$ term is given by

$$\mathcal{J}^\mu_{\mathcal{M}} = -\frac{1}{\sqrt{\gamma}}(-\partial_\beta(\mathcal{M}'(B)\varepsilon^{\alpha\beta\mu}n_\alpha\epsilon^{\rho\nu\lambda}n_\rho\partial_\nu n_\lambda)). \tag{F.4}$$

F.2 Energy current

The energy currents are given by

$$\mathcal{E}^\mu_{\rm CS} = -\frac{1}{\sqrt{\gamma}}\frac{\nu}{2\pi}\bigg(((-\epsilon^{\rho\nu\lambda}mu^{(0)}_\rho u^\mu_{(0)} + \epsilon^{\rho\nu\lambda}mu^2_{(0)}u^\mu_{(0)}n_\rho - \epsilon^{\mu\nu\lambda}\frac{1}{2}mu^2_{(0)})\partial_\nu(A_\lambda + mu^{(0)}_\lambda - \frac{1}{2}mu^2_{(0)}n_\lambda)$$
$$- \frac{g}{4}\bigg(\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\varepsilon^{\alpha\beta\gamma}\partial_\alpha(n_\beta u^{(0)}_\gamma) - \sqrt{\gamma}Bv^\mu\varepsilon^{\alpha\beta\gamma}\partial_\alpha(n_\beta u^{(0)}_\gamma) - \partial_\alpha(\sqrt{\gamma}B\varepsilon^{\alpha\mu\gamma})u^{(0)}_\gamma$$
$$+ \partial_\alpha(\sqrt{\gamma}B\varepsilon^{\alpha\beta\gamma})n_\beta u^{(0)}_\gamma u^\mu_{(0)}\bigg)\bigg), \tag{F.5}$$

$$\mathcal{E}^\mu_{\mathcal{E}} = \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}v^\mu\mathcal{E}(B) + (\sqrt{\gamma}\mathcal{E}'(B) + m\mathcal{E}''(B)(\epsilon^{\alpha\nu\lambda}n_\alpha\partial_\nu u^{(0)}_\lambda - \frac{1}{2}u^2_{(0)}\epsilon^{\alpha\nu\lambda}n_\alpha\partial_\nu n_\lambda))B(u^\mu_{(0)} - v^\mu)$$
$$+ m\mathcal{E}'(B)\epsilon^{\mu\nu\lambda}\partial_\nu u^{(0)}_\lambda + m\partial_\nu(\mathcal{E}'(B)\varepsilon^{\alpha\nu\lambda}n_\alpha)u^{(0)}_\lambda u^\mu_{(0)}$$
$$+ m\mathcal{E}'(B)u^2_{(0)}u^\mu_{(0)}\epsilon^{\alpha\nu\lambda}n_\alpha\partial_\nu n_\lambda - \frac{1}{2}m\mathcal{E}'(B)u^2_{(0)}\epsilon^{\mu\nu\lambda}\partial_\nu n_\lambda + \frac{1}{2}m\partial_\nu(\mathcal{E}'(B)u^2_{(0)}\epsilon^{\lambda\nu\mu}n_\lambda)), \tag{F.6}$$

$$\mathcal{E}^\mu_{\rm WZ} = -\frac{1}{\sqrt{\gamma}}\frac{\kappa}{4\pi}\bigg(- \epsilon^{\sigma\alpha\beta}\partial_\alpha A_\beta\frac{1}{2}\epsilon^{ab}v^\nu e^{a\mu}\partial_\sigma\tilde{e}^b{}_\nu + \partial_\sigma(\epsilon^{\sigma\alpha\beta}\partial_\alpha A_\beta\frac{1}{2}\epsilon^{ab}e^{a\nu})e^b{}_\rho(-u^\rho_{(0)}u^\mu_{(0)}n_\nu + u^\rho_{(0)}\delta^\mu_\nu)$$
$$+ \epsilon^{\sigma\alpha\beta}\partial_\alpha A_\beta\frac{1}{2}\epsilon^{ab}v^\nu e^{a\mu}e^{b\rho}\partial_\nu\tilde{h}_{\sigma\rho} + \epsilon^{\sigma\alpha\beta}\partial_\alpha A_\beta\frac{1}{2}\epsilon^{ab}e^{a\nu}v^\rho e^{b\mu}\partial_\nu\tilde{h}_{\sigma\rho}$$
$$+ \partial_\nu(\epsilon^{\sigma\alpha\beta}\partial_\alpha A_\beta\frac{1}{2}\epsilon^{ab}e^{a\nu}e^{b\rho})(-\delta^\mu_\sigma u_\rho - n_\sigma(-u^{(0)}_\mu u^\tau_{(0)}\delta^\mu_\tau) - \delta^\mu_\rho u_\sigma - n_\rho(-u^{(0)}_\sigma u^\tau_{(0)}\delta^\mu_\tau)$$
$$+ 2(-u^2_{(0)}u^\tau_{(0)}\delta^\mu_\tau)n_\sigma n_\rho + u^2\delta^\mu_\sigma n_\rho + u^2 n_\sigma\delta^\mu_\rho)\bigg), \tag{F.7}$$

$$\mathcal{E}^\mu_{\mathcal{M}} = \frac{1}{\sqrt{\gamma}}(\mathcal{M}'(B)B(u^\mu_{(0)} - v^\mu)\epsilon^{\alpha\nu\lambda}n_\alpha\partial_\nu n_\lambda + \mathcal{M}(B)\epsilon^{\mu\nu\lambda}\partial_\nu n_\lambda - \partial_\nu(\mathcal{M}(B)\epsilon^{\lambda\nu\mu}n_\lambda)). \tag{F.8}$$

F.3 Stress tensor

The stress tensors are given by

$$\begin{aligned}\mathcal{T}^{\mathrm{CS}}_{\rho\sigma} = \frac{1}{\sqrt{\gamma}}\frac{\nu m}{2\pi}e^{a}{}_{\sigma}\Bigg(&\epsilon^{\mu\nu\lambda}(u^{(0)}_{\rho}e^{a}{}_{\mu} + h_{\mu\rho}e^{a}{}_{\alpha}u^{\alpha}_{(0)} - u^{(0)}_{\rho}u^{\alpha}_{(0)}e^{a}{}_{\alpha}n_{\mu})\partial_{\nu}(A_{\lambda} + mu^{(0)}_{\lambda} - \frac{1}{2}mu^{2}_{(0)}n_{\lambda})\\ &- \frac{g}{4}\Bigg(\sqrt{\gamma}Be^{a}{}_{\rho}\varepsilon^{\alpha\beta\gamma}\partial_{\alpha}(n_{\beta}h_{\gamma\delta}u^{\delta}_{(0)}) + \partial_{\alpha}(\sqrt{\gamma}B\varepsilon^{\alpha\beta\gamma})n_{\beta}(h_{\gamma\rho}e^{a}{}_{\delta} + h_{\delta\rho}e^{a}{}_{\gamma})u^{\delta}_{(0)}\Bigg)\Bigg),\end{aligned} \tag{F.9}$$

$$\begin{aligned}\mathcal{T}^{\mathcal{E}}_{\rho\sigma} = \frac{1}{\sqrt{\gamma}}e^{a}{}_{\sigma}(&-\sqrt{\gamma}\mathcal{E}(B)e^{a}{}_{\rho} + \sqrt{\gamma}\mathcal{E}'(B)Be^{a}{}_{\rho}\\ &+ m\mathcal{E}''(B)Be^{a}{}_{\rho}(\epsilon^{\mu\nu\lambda}n_{\mu}\partial_{\nu}u^{(0)}_{\lambda} - \frac{1}{2}u^{2}_{(0)}\epsilon^{\mu\nu\lambda}n_{\mu}\partial_{\nu}n_{\lambda})\\ &+ (m\partial_{\nu}(\mathcal{E}'(B)\epsilon^{\mu\nu\zeta}n_{\mu})u^{\eta}_{(0)} + \frac{1}{2}m\mathcal{E}'(B)u^{\zeta}_{(0)}u^{\eta}_{(0)}\epsilon^{\mu\nu\lambda}n_{\mu}\partial_{\nu}n_{\lambda})(h_{\zeta\rho}e^{a}{}_{\eta} + h_{\eta\rho}e^{a}{}_{\zeta}),\end{aligned} \tag{F.10}$$

$$\begin{aligned}\mathcal{T}^{\mathrm{WZ}}_{\rho\sigma} = -\frac{1}{\sqrt{\gamma}}\frac{\kappa}{8\pi}e^{c}{}_{\sigma}\Bigg(&\epsilon^{\mu\alpha\beta}\partial_{\alpha}A_{\beta}\epsilon^{cb}P^{\gamma}{}_{\rho}\partial_{\mu}\tilde{e}^{b}{}_{\gamma} - \partial_{\mu}(\epsilon^{\mu\alpha\beta}\partial_{\alpha}A_{\beta}\epsilon^{ab}e^{a\gamma})(-e^{b}{}_{\rho}e^{c}{}_{\gamma} + e^{b}{}_{\rho}e^{c}{}_{\sigma}u^{\sigma}_{(0)}n_{\gamma})\\ &- \epsilon^{\mu\alpha\beta}\partial_{\alpha}A_{\beta}\epsilon^{cb}P^{\nu}_{\rho}e^{b\gamma}\partial_{\nu}\tilde{h}_{\mu\gamma} - \epsilon^{\mu\alpha\beta}\partial_{\alpha}A_{\beta}\epsilon^{ac}e^{a\nu}P^{\gamma}{}_{\rho}\partial_{\nu}\tilde{h}_{\mu\gamma}\\ &+ \partial_{\nu}(\epsilon^{\mu\alpha\beta}\partial_{\alpha}A_{\beta}\epsilon^{ab}e^{a\nu}e^{b\gamma})(-h_{\mu\rho}e^{c}{}_{\gamma} - h_{\gamma\rho}e^{c}{}_{\mu} + n_{\mu}(u^{(0)}_{\rho}e^{c}{}_{\gamma} + h_{\rho\gamma}e^{c}{}_{\tau}u^{\tau}_{(0)}) + n_{\gamma}(u^{(0)}_{\rho}e^{c}{}_{\mu} + h_{\mu\rho}e^{c}{}_{\tau}u^{\tau}_{(0)})\\ &- 2u^{(0)}_{\rho}u^{\tau}_{(0)}e^{c}{}_{\tau}n_{\mu}n_{\rho})\Bigg),\end{aligned} \tag{F.11}$$

$$\mathcal{T}^{\mathcal{M}}_{\rho\sigma} = \frac{1}{\sqrt{\gamma}}\mathcal{M}'(B)Be^{a}{}_{\sigma}e^{a}{}_{\rho}\epsilon^{\mu\nu\lambda}n_{\mu}\partial_{\nu}n_{\lambda}. \tag{F.12}$$

F.4 Momentum Density

The momentum density is given according to Eq. (3.57).

Appendix G

Derivation of current

In this appendix, we derive Eq. (4.12) from the following expression (Eq. (4.8)):

$$\begin{aligned}
J^\mu(k) = &-i\int_l \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}-m}\gamma^\lambda\gamma^5\frac{1}{\not{l}+\not{k}-m}\gamma^\nu\frac{1}{\not{l}-m}\right]_{\mathrm{reg}} A_\nu(k)A_{5\lambda} \\
&-i\int_l \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}-m}\gamma^\nu\frac{1}{\not{l}-m}\gamma^\lambda\gamma^5\frac{1}{\not{l}-m}\right]_{\mathrm{reg}} A_\nu(k)A_{5\lambda} \\
= &-i\int_l \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}-m}\gamma^\lambda\gamma^5\frac{1}{\not{l}-m}\gamma^\nu\frac{1}{\not{l}-\not{k}-m}\right]_{\mathrm{reg}} A_\nu(k)A_{5\lambda} \\
&-i\int_l \mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}-m}\gamma^\nu\frac{1}{\not{l}-m}\gamma^\lambda\gamma^5\frac{1}{\not{l}-m}\right]_{\mathrm{reg}} A_\nu(k)A_{5\lambda}.
\end{aligned} \tag{G.1}$$

For simplicity, we set $v = 1$ throughout this appendix.

The trace of gamma matrices is computed as follows:

$$\begin{aligned}
&\mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}-m}\gamma^\lambda\gamma^5\frac{1}{\not{l}-m}\gamma^\nu\frac{1}{\not{l}-\not{k}-m}\right] \\
&= \frac{4i}{((l-k)^2-m^2)(l^2-m^2)^2}[(l^2-m^2)(l-k)_\alpha\epsilon^{\mu\nu\lambda\alpha} - 2m^2k_\alpha\epsilon^{\mu\nu\lambda\alpha} + 2l^\lambda l_\alpha k_\beta\epsilon^{\mu\nu\alpha\beta}],
\end{aligned} \tag{G.2}$$

$$\begin{aligned}
&\mathrm{tr}\left[\gamma^\mu \frac{1}{\not{l}+\not{k}-m}\gamma^\nu\frac{1}{\not{l}-m}\gamma^\lambda\gamma^5\frac{1}{\not{l}-m}\right] \\
&= -\frac{4i}{((l+k)^2-m^2)(l^2-m^2)^2}[(l^2-m^2)(l+k)_\alpha\epsilon^{\mu\nu\lambda\alpha} + 2m^2k_\alpha\epsilon^{\mu\nu\lambda\alpha} - 2l^\lambda l_\alpha k_\beta\epsilon^{\mu\nu\alpha\beta}].
\end{aligned} \tag{G.3}$$

The current density can now be expressed as

$$\begin{aligned}
J^\mu(k) = -i\int_l \Bigg[&\frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}[(l^2-m^2)(l+k)_\alpha\epsilon^{\mu\nu\lambda\alpha} + 2m^2k_\alpha\epsilon^{\mu\nu\lambda\alpha} - 2l^\lambda l_\alpha k_\beta\epsilon^{\mu\nu\alpha\beta}] \\
&- (k\to -k)\Bigg]_{\mathrm{reg}} A_\nu(k)A_{5\lambda}.
\end{aligned} \tag{G.4}$$

Then, we divide the current into two components:

$$J^\mu(k) = J^\mu_{\mathrm{I}}(k) + J^\mu_{\mathrm{II}}(k) \tag{G.5}$$

with

$$J_{\mathrm{I}}^{\mu}(k) = -i\int_l \left[\frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}[(l^2-m^2)(l+k)_\alpha\epsilon^{\mu\nu 0\alpha} + 2m^2k_\alpha\epsilon^{\mu\nu 0\alpha} - 2l^0 l_\alpha k_\beta\epsilon^{\mu\nu\alpha\beta}]\right.$$
$$\left. - (k\to -k)\right]_{\mathrm{reg}} A_\nu(k)A_{5,0}, \tag{G.6}$$

$$J_{\mathrm{II}}^{\mu}(k) = -i\int_l \left[\frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}[(l^2-m^2)(l+k)_\alpha\epsilon^{\mu\nu i\alpha} + 2m^2k_\alpha\epsilon^{\mu\nu i\alpha} - 2l^i l_\alpha k_\beta\epsilon^{\mu\nu\alpha\beta}]\right.$$
$$\left. - (k\to -k)\right]_{\mathrm{reg}} A_\nu(k)A_{5i}. \tag{G.7}$$

G.1 Matsubara frequency summation formulae

Before computing the current density, we present the formulae for the Matsubara frequency summations. In general, we can rewrite the Matsubara frequency summation as a contour integral:

$$T\sum_{l_0} f(l_0) = \oint_C \frac{dz}{2\pi i} f(z)n_F(\beta(z-\mu)) = \sum_{z_n}\operatorname*{Res}_{z=z_n}(f(z)n_F(\beta(z-\mu))), \tag{G.8}$$

where z_n are the poles of $f(z)$. By using this formula, we obtain

$$\begin{aligned}
&T\sum_{l_0}\frac{1}{((l+k)^2-m^2)(l^2-m^2)} \\
&= \frac{n_F(\beta(E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} - \frac{n_F(\beta(-E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}((k_0-E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} \\
&\quad + \frac{n_F(\beta(E_{\boldsymbol{l+k}}-\mu))}{2E_{\boldsymbol{l+k}}((k_0-E_{\boldsymbol{l+k}})^2-E_{\boldsymbol{l+k}}^2)} - \frac{n_F(\beta(-E_{\boldsymbol{l+k}}-\mu))}{2E_{\boldsymbol{l+k}}((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)},
\end{aligned} \tag{G.9}$$

$$\begin{aligned}
&T\sum_{l_0}\frac{1}{((l+k)^2-m^2)(l^2-m^2)^2} \\
&= -\frac{n_F(\beta(E_{\boldsymbol{l}}-\mu))}{4E_{\boldsymbol{l}}^3((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} + \frac{n_F(\beta(-E_{\boldsymbol{l}}-\mu))}{4E_{\boldsymbol{l}}^3((k_0-E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} \\
&\quad - \frac{(k_0+E_{\boldsymbol{l}})n_F(\beta(E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}^2((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)^2} - \frac{(k_0-E_{\boldsymbol{l}})n_F(\beta(-E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}^2((k_0-E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)^2} \\
&\quad + \frac{n_F'(\beta(E_{\boldsymbol{l}}-\mu))}{4E_{\boldsymbol{l}}^2((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} + \frac{n_F'(\beta(-E_{\boldsymbol{l}}-\mu))}{4E_{\boldsymbol{l}}^2((k_0-E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} \\
&\quad + \frac{n_F(\beta(E_{\boldsymbol{l+k}}-\mu))}{2E_{\boldsymbol{l+k}}((k_0-E_{\boldsymbol{l+k}})^2-E_{\boldsymbol{l}}^2)^2} - \frac{n_F(\beta(-E_{\boldsymbol{l+k}}-\mu))}{2E_{\boldsymbol{l+k}}((k_0+E_{\boldsymbol{l+k}})^2-E_{\boldsymbol{l}}^2)^2},
\end{aligned} \tag{G.10}$$

$$\begin{aligned}
&T\sum_{l_0}\frac{l_0}{((l_0+k_0)^2-E_{\boldsymbol{l+k}}^2)(l_0^2-E_{\boldsymbol{l}}^2)} \\
&= \frac{E_{\boldsymbol{l}}n_F(\beta(E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} + \frac{E_{\boldsymbol{l}}n_F(\beta(-E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}((k_0-E_{\boldsymbol{l}})^2-E_{\boldsymbol{l+k}}^2)} \\
&\quad + \frac{-(k_0-E_{\boldsymbol{l+k}})n_F(\beta(E_{\boldsymbol{l+k}}-\mu))}{2E_{\boldsymbol{l+k}}((k_0-E_{\boldsymbol{l+k}})^2-E_{\boldsymbol{l}}^2)} + \frac{(k_0+E_{\boldsymbol{l+k}})n_F(\beta(-E_{\boldsymbol{l+k}}-\mu))}{2E_{\boldsymbol{l+k}}((k_0+E_{\boldsymbol{l+k}})^2-E_{\boldsymbol{l}}^2)},
\end{aligned} \tag{G.11}$$

$$
\begin{aligned}
&T\sum_{l_0}\frac{l_0}{((l+k)^2-m^2)(l^2-m^2)^2}\\
&=-\frac{(k_0+E_{\boldsymbol{l}})n_F(\beta(E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l}+\boldsymbol{k}}^2)^2}+\frac{(k_0-E_{\boldsymbol{l}})n_F(\beta(-E_{\boldsymbol{l}}-\mu))}{2E_{\boldsymbol{l}}((k_0-E_{\boldsymbol{l}})^2-E_{\boldsymbol{l}+\boldsymbol{k}}^2)^2}\\
&\quad+\frac{n_F'(\beta(E_{\boldsymbol{l}}-\mu))}{4E_{\boldsymbol{l}}((k_0+E_{\boldsymbol{l}})^2-E_{\boldsymbol{l}+\boldsymbol{k}}^2)}-\frac{n_F'(\beta(-E_{\boldsymbol{l}}-\mu))}{4E_{\boldsymbol{l}}((k_0-E_{\boldsymbol{l}})^2-E_{\boldsymbol{l}+\boldsymbol{k}}^2)}\\
&\quad-\frac{(k_0-E_{\boldsymbol{l}+\boldsymbol{k}})n_F(\beta(E_{\boldsymbol{l}+\boldsymbol{k}}-\mu))}{2E_{\boldsymbol{l}+\boldsymbol{k}}((k_0-E_{\boldsymbol{l}+\boldsymbol{k}})^2-E_{\boldsymbol{l}}^2)^2}+\frac{(k_0+E_{\boldsymbol{l}+\boldsymbol{k}})n_F(\beta(-E_{\boldsymbol{l}+\boldsymbol{k}}-\mu))}{2E_{\boldsymbol{l}+\boldsymbol{k}}((k_0+E_{\boldsymbol{l}+\boldsymbol{k}})^2-E_{\boldsymbol{l}}^2)^2}. \qquad \text{(G.12)}
\end{aligned}
$$

Here, we define $E_{\boldsymbol{l}}=\sqrt{|\boldsymbol{l}|^2+m^2}$ and $k_0=i\omega_n$, the latter of which is analytically continued to ω^+.

G.2 Computation of $J^\mu_{\mathbf{I}}(k)$

We now compute $J^\mu_{\mathrm{I}}(k)$, which can be written as

$$
\begin{aligned}
J^\mu_{\mathrm{I}}(k)&=-i\int_l\Bigg[\frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}[(l^2-m^2)(l-k)_k-2|\boldsymbol{l}|^2k_k+2l_0l_kk_0]\\
&\quad-(k\to-k)\Bigg]_{\mathrm{reg}}\epsilon^{\mu\nu0k}A_\nu(k)A_{5,0}\\
&=T\sum_{l_0}\int_{\boldsymbol{l}}\Bigg[\frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}[(l^2-m^2)(l-k)_k-2|\boldsymbol{l}|^2k_k+2l_0l_kk_0]\\
&\quad-(k\to-k)\Bigg]_{\mathrm{reg}}\epsilon^{\mu\nu0k}A_\nu(k)A_{5,0}, \qquad \text{(G.13)}
\end{aligned}
$$

where $\int_{\boldsymbol{l}}=\int\frac{d\boldsymbol{l}}{(2\pi)^3}$. After the Matsubara frequency summation and straightforward calculations, we find

$$
J^\mu_{\mathrm{I}}(k)=J^\mu_{\mathrm{I,vac}}(k)+J^\mu_{\mathrm{I,mat}}(k), \qquad \text{(G.14)}
$$

$$
J^\mu_{\mathrm{I,vac}}(k)=-4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{m^2}{2E_{\boldsymbol{l}}^3(\Omega_u^2-E_{\boldsymbol{l}+\boldsymbol{k}}^2)}+\frac{um^2k_0}{E_{\boldsymbol{l}}^2(\Omega_u^2-E_{\boldsymbol{l}+\boldsymbol{k}}^2)^2}\right]_{\mathrm{reg}}\epsilon^{\mu\nu0k}k_kA_\nu(p)A_{50}, \qquad \text{(G.15)}
$$

$$
J^\mu_{\mathrm{I,mat}}(k)=-4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{-E_{\boldsymbol{l}}k_k+ul_kk_0}{2E_{\boldsymbol{l}}(\Omega_u^2-E_{\boldsymbol{l}+\boldsymbol{k}}^2)}N_+'(E_{\boldsymbol{l}})\right]_{\mathrm{reg}}\epsilon^{\mu\nu0k}A_\nu(p)A_{50}, \qquad \text{(G.16)}
$$

where we define $N_+(k)=n_F(\beta(k-\mu))+n_F(\beta(k+\mu))$ and $\Omega_u=k_0+uE_{\boldsymbol{l}}$ with $u=\pm$. Here, $J^\mu_{\mathrm{I,vac}}(k)$ has only the Pauli-Villars contribution while the Pauli-Villars contribution of $J^\mu_{\mathrm{I,mat}}(k)$ is exponentially suppressed by the Fermi-Dirac distribution function with infinite energy.

To proceed with the computation, we first rescale $|\boldsymbol{l}|$ as $|\boldsymbol{l}|\to|\boldsymbol{l}|/M$. Next, we take the $M\to\infty$ limit. Under this limit, the integrand reduces to a simpler form, allowing us to perform the integration analytically. After these steps, we finally obtain

$$J^\mu_{\rm I}(\omega,\boldsymbol{k}) = -\frac{1}{2\pi^2}\epsilon^{0k\mu\nu}(ik_k)A_\nu(\omega,\boldsymbol{k})A_{50} \\ -\frac{1}{2\pi^2}\int_0^\infty dl N'_+(l)\frac{k_0^2-k^2}{4k^3}\left[k_0\ln\left(\frac{(k_0+k)^2((k_0-k)^2-4l^2)}{(k_0-k)^2((k_0+k)^2-4l^2)}\right)\right. \\ \left.+2l\ln\left(\frac{k_0^2-(k-2l)^2}{k_0^2-(k+2l)^2}\right)\right]\epsilon^{0k\mu\nu}(ik_k)A_\nu(\omega,\boldsymbol{k})A_{50} \tag{G.17}$$

with $k_0 = \omega^+$ and $k = |\boldsymbol{k}|$.

G.3 Computation of $J^\mu_{\rm II}(k)$

Next, we compute $J^\mu_{\rm II}(k)$, which can be divided into four components:

$$J^\mu_{\rm II}(k) = J^\mu_{\text{II-1}}(k) + J^\mu_{\text{II-2}}(k) + J^\mu_{\text{II-3}}(k) + J^\mu_{\text{II-4}}(k), \tag{G.18}$$

$$J^\mu_{\text{II-1}}(k) = -i\int_l \frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}((l^2-m^2)(l+k)_0 + 2m^2k_0)\epsilon^{\mu\nu l0}A_\nu(k)A_{5l} - (k\to -k), \tag{G.19}$$

$$J^\mu_{\text{II-2}}(k) = -i\int_l \frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}((l^2-m^2)(l+k)_k + 2m^2k_k)\epsilon^{\mu\nu lk}A_\nu(k)A_{5l} - (k\to -k), \tag{G.20}$$

$$J^\mu_{\text{II-3}}(k) = -i\int_l \frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}(-2l^l l_0 k_\beta)\epsilon^{\mu\nu 0\beta}A_\nu(k)A_{5l} - (k\to -k), \tag{G.21}$$

$$J^\mu_{\text{II-4}}(k) = -i\int_l \frac{-4i}{((l+k)^2-m^2)(l^2-m^2)^2}(-2l^l l_k k_\beta)\epsilon^{\mu\nu k\beta}A_\nu(k)A_{5l} - (k\to -k). \tag{G.22}$$

After the Matsubara frequency summation and straightforward calculations, we find

$$J^\mu_{\text{II-}n}(k) = J^\mu_{\text{II-}n,\text{vac}}(k) + J^\mu_{\text{II-}n,\text{mat}}(k) \tag{G.23}$$

with

$$J^\mu_{\text{II-1,vac}}(k) = 4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{|\boldsymbol{l}|^2k_0}{2E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})} - \frac{um^2k_0^2}{E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}\right]_{\text{reg}}\epsilon^{\mu\nu l0}A_\nu(k)A_{5l}, \tag{G.24}$$

$$J^\mu_{\text{II-1,mat}}(k) = -4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{k_0N_+(E_{\boldsymbol{l}})}{2E_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})}\right]_{\text{reg}}\epsilon^{\mu\nu l0}A_\nu(k)A_{5l}, \tag{G.25}$$

$$J^\mu_{\text{II-2,vac}}(k) = 4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{|\boldsymbol{l}|^2}{2E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})} - \frac{um^2k_0}{E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}\right]_{\text{reg}}\epsilon^{\mu\nu lk}k_kA_\nu(k)A_{5l}, \tag{G.26}$$

$$J^\mu_{\text{II-2,mat}}(k) = -4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{|\boldsymbol{l}|^2N_+(E_{\boldsymbol{l}})}{2E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})}\right]_{\text{reg}}\epsilon^{\mu\nu lk}k_kA_\nu(k)A_{5l}, \tag{G.27}$$

$$J^\mu_{\text{II-3,vac}}(k) = -4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{\Omega_u}{E_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}\right]_{\text{reg}}|\boldsymbol{k}|\epsilon^{\mu\nu0k}\hat{k}^lk_kA_\nu(k)A_{5l}, \tag{G.28}$$

$$J^\mu_{\text{II-3,mat}}(k) = -4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[-\frac{\Omega_uN_+(E_{\boldsymbol{l}})}{E_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}|\boldsymbol{k}| - \frac{x|\boldsymbol{l}|N_+(E_{\boldsymbol{l}})}{2E_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})}\right]_{\text{reg}}\epsilon^{\mu\nu0k}\hat{k}^lk_kA_\nu(k)A_{5l}, \tag{G.29}$$

$$\begin{aligned}J^\mu_{\text{II-4,vac}}(k) = &-4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{1}{4E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})} + \frac{uk_0}{2E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}\right]_{\text{reg}}(1-x^2)|\boldsymbol{l}|^2\epsilon^{\mu\nu l\beta}k_\beta A_\nu(k)A_{5l}\\ &-4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\left(\frac{1}{4E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})} + \frac{uk_0}{2E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}\right)(1-3x^2)|\boldsymbol{l}|^2\right.\\ &\left.+\frac{1}{E_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}(2|\boldsymbol{l}||\boldsymbol{k}|x - |\boldsymbol{k}|^2)\right]_{\text{reg}}\hat{k}^l\hat{k}_k\epsilon^{\mu\nu k0}k_0A_\nu(k)A_{5l},\end{aligned} \tag{G.30}$$

$$\begin{aligned}J^\mu_{\text{II-4,mat}}(k) = &-4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[-\frac{N_+(E_{\boldsymbol{l}})}{4E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})} - \frac{uk_0N_+(E_{\boldsymbol{l}})}{2E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2} + \frac{N'_+(E_{\boldsymbol{l}})}{4E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})}\right]_{\text{reg}}\\ &\times(1-x^2)|\boldsymbol{k}|^2\epsilon^{\mu\nu l\beta}k_\beta A_\nu(k)A_{5l}\\ &-4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\left(-\frac{N_+(E_{\boldsymbol{l}})}{4E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})} - \frac{uk_0N_+(E_{\boldsymbol{l}})}{2E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2} + \frac{N'_+(E_{\boldsymbol{l}})}{4E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}\right)(1-3x^2)|\boldsymbol{k}|^2\right.\\ &\left.-\frac{N_+(E_{\boldsymbol{l}})}{E_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}(2|\boldsymbol{l}||\boldsymbol{k}|x + |\boldsymbol{k}|^2)\right]_{\text{reg}}\hat{k}^l\hat{k}_k\epsilon^{\mu\nu k0}k_0A_\nu(k)A_{5l}.\end{aligned} \tag{G.31}$$

Here, we define $x = \cos\theta$, where θ is the angle between $\boldsymbol{l}$ and $\boldsymbol{k}$. Similar to $J^\mu_{\text{I,mat}}$, the contribution from the Pauli-Villars ghost in $J^\mu_{\text{II-}n\text{,mat}}$ vanishes due to the suppression by the Fermi-Dirac distribution function. From

From Eqs. (G.24)-(G.27), we obtain

$$J^\mu_{\text{II-1,vac}}(k) + J^\mu_{\text{II-2,vac}}(k) = 4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{|\boldsymbol{l}|^2}{2E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})} - \frac{um^2k_0}{E^2_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})^2}\right]_{\text{reg}}\epsilon^{\mu\nu l\beta}k_\rho A_\nu(k)A_{5l}, \tag{G.32}$$

$$J^\mu_{\text{II-1,mat}}(k) + J^\mu_{\text{II-2,mat}}(k) = -4i\sum_{u=\pm}\int_{\boldsymbol{l}}\left[\frac{|\boldsymbol{l}|^2N_+(E_{\boldsymbol{l}})}{2E^3_{\boldsymbol{l}}(\Omega_u^2 - E^2_{\boldsymbol{l}+\boldsymbol{k}})}\right]_{\text{reg}}\epsilon^{\mu\nu l\beta}k_\rho A_\nu(k)A_{5l} \tag{G.33}$$

Then, we can divide terms in $J^\mu_{\text{II}}(k)$ into two part: one is proportional to $\epsilon^{\mu\nu l\beta}k_\rho A_\nu(k)A_{5l}$ and the other is proportional to $\hat{k}^l\hat{k}_k\epsilon^{\mu\nu0k}A_\nu(k)A_{5l}$. We denote the former as $J^\mu_{\text{II}'}(k)$ and the latter as $J^\mu_{\text{II}''}(k)$. Following the same steps as in the computation of J^μ_{I}, we obtain

$$J^\mu_{\text{II}',\text{vac}}(k) = \frac{1}{\pi^2}\Bigg[\underbrace{-\frac{k^2-k_0^2}{2k^3}\left(k - k_0\tanh^{-1}\left(\frac{k}{k_0}\right)\right)}_{\text{D}} \underbrace{-\frac{k_0}{2k^3}\left(kk_0 - (k^2-k_0^2)\tanh^{-1}\left(\frac{k}{k_0}\right)\right)}_{\text{PV}}\Bigg]\epsilon^{\mu\nu l\beta}k_\rho A_\nu(k)A_{5l}$$
$$= -\frac{1}{2\pi^2}\epsilon^{\mu\nu l\beta}(ik_\beta)A_\nu(k)A_{5l}, \tag{G.34}$$

$$J^\mu_{\text{II}',\text{vac}}(k) = 0, \tag{G.35}$$

$$J^\mu_{\text{II}'',\text{vac}}(k) = -\frac{1}{\pi^2}\Bigg[\underbrace{-\frac{k^2-k_0^2}{2k^4}\left(3kk_0 - (k^2-3k_0^2)\tanh^{-1}\left(\frac{k}{k_0}\right)\right)}_{\text{D}} \underbrace{-\frac{k^2-k_0^2}{2k^4}\left(3kk_0 - (k^2-3k_0^2)\tanh^{-1}\left(\frac{k}{k_0}\right)\right)}_{\text{PV}}\Bigg]\hat{k}^l(ik_k)\epsilon^{\mu\nu 0k}A_\nu(k)A_{5l}$$
$$= 0, \tag{G.36}$$

and

$$J^\mu_{\text{II}'',\text{vac}}(k) = 0, \tag{G.37}$$

where D and PV denote the contribution from the massless Dirac fermion and Pauli-Villars ghost, respectively. Then, we finally find

$$J^\mu_{\text{II},\text{vac}} = -\frac{1}{2\pi^2}(ik_\beta)\epsilon^{\mu\nu l\beta}A_\nu(k)A_{5l}, \tag{G.38}$$
$$J^\mu_{\text{II},\text{mat}} = 0. \tag{G.39}$$

G.4 Result

From the above calculations, we find that the current density is given by

$$\begin{aligned} J^\mu(\omega,\boldsymbol{k}) = &-\frac{1}{2\pi^2}\epsilon^{\mu\nu\lambda\rho}A_{5\nu}(ik_\lambda)A_\rho(\omega,\boldsymbol{k}) \\ &-\frac{1}{2\pi^2}\int_0^\infty dl N'_+(l)\frac{k_0^2-k^2}{4k^3}\left[k_0\ln\left(\frac{(k_0+k)^2((k_0-k)^2-4l^2)}{(k_0-k)^2((k_0+k)^2-4l^2)}\right)\right. \\ &\left.+2l\ln\left(\frac{k_0^2-(k-2l)^2}{k_0^2-(k+2l)^2}\right)\right]\epsilon^{0k\mu\nu}(ik_k)A_\nu(\omega,\boldsymbol{k})A_{50}. \end{aligned} \tag{G.40}$$

This expression is consistent with Eq. (4.12) under utilizing $\epsilon^{0\beta\mu\nu}(ik_\beta)A_\nu(\omega,\boldsymbol{k}) = -\delta^\mu{}_i B^i(\omega,\boldsymbol{k})$ and restoring the Fermi velocity v.

Appendix H

Integration formulae

In this appendix, we present the integration formulae used in the computation of the charge and current densities within the framework of the chiral kinetic theory.

H.1 Angular integral

The following angular integration formulae are used:

$$\frac{1}{4\pi}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\hat{p}^i = 0, \tag{H.1}$$

$$\frac{1}{4\pi}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\hat{p}^i\hat{p}^j = \frac{1}{3}\delta^{ij}, \tag{H.2}$$

$$\frac{1}{4\pi}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\frac{1}{\omega - v\hat{\boldsymbol{p}}\cdot\boldsymbol{k}} = \frac{1}{2v|\boldsymbol{k}|}G(\omega,|\boldsymbol{k}|), \tag{H.3}$$

$$\frac{1}{4\pi}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\frac{\hat{p}^i}{\omega - v\hat{\boldsymbol{p}}\cdot\boldsymbol{k}} = \frac{1}{\omega}[g_1(\omega,|\boldsymbol{k}|) - g_3(\omega,|\boldsymbol{k}|)]\hat{k}^i, \tag{H.4}$$

$$\frac{1}{4\pi}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\frac{\hat{p}^i\hat{p}^j}{\omega - v\hat{\boldsymbol{p}}\cdot\boldsymbol{k}} = \frac{1}{2}[g_3(\omega,|\boldsymbol{k}|)\hat{k}^i\hat{k}^j - g_1(\omega,|\boldsymbol{k}|)\delta^{ij}]. \tag{H.5}$$

H.2 Radial integral

The radial integration formulae are

$$\int_0^{\infty}dp n_F(\beta(p-\mu)) = T\ln(1+e^{\mu/T}), \tag{H.6}$$

$$\begin{aligned}\int_0^{\infty}dp[n_F(\beta(p-\mu)) - n_F(\beta(p+\mu))] &= T\ln(1+e^{\mu/T}) - T\ln(1+e^{-\mu/T}) \\ &= \mu,\end{aligned} \tag{H.7}$$

$$\int_0^{\infty}dpp[n_F(\beta(p-\mu)) + n_F(\beta(p+\mu))] = \frac{\pi^2T^2}{6} + \frac{\mu^2}{2}, \tag{H.8}$$

$$\int_0^{\infty}dpp^2[n_F(\beta(p-\mu)) - n_F(\beta(p+\mu))] = \frac{2}{3}(\pi^2T^2+\mu^2)\mu, \tag{H.9}$$

$$\begin{aligned}\int_0^{\infty}dpp^2[n_F'(\beta(p-\mu)) + n_F'(\beta(p+\mu))] &= -\int_0^{\infty}dpp[n_F(\beta(p-\mu)) + n_F(\beta(p+\mu))] \\ &= -\left(\frac{\pi^2T^2}{6} + \frac{\mu^2}{2}\right).\end{aligned} \tag{H.10}$$

We note the following general formula [126]:

$$\int_0^\infty dpp^n[n_F(\beta(p-\mu)) - (-1)^n n_F(\beta(p+\mu))] = \frac{(2\pi iT)^{n+1}}{n+1} B_{n+1}\left(\frac{1}{2} + \frac{\mu}{2\pi iT}\right), \quad \text{(H.11)}$$

where $B_n(x)$ is the Bernoulli polynomial:

$$B_0(x) = 1, \quad \text{(H.12)}$$

$$B_1(x) = x - \frac{1}{2}, \quad \text{(H.13)}$$

$$B_2(x) = x^2 - x + \frac{1}{6}. \quad \text{(H.14)}$$

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