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Popularity on Variations of Multidimensional Roommate Games



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Abstract

We study the variations of the Multidimensional Stable Roommates Problem with popularity as equilibrium concept. The goal of these problems is to find a partitioning of agents that is stable or optimal according to the given equilibrium concept. The Multidimensional Stable Roommates Problem is a coalition formation game, and it is known that determining the existence of a popular partitioning of agents for an instance of the Stable Roommates Problem is NP-hard.

Coalition formation games find their applications in swarm robotics, user-server assignment, school choice, and hospitals/residents appointment.

In this dissertation, we study two variants of the Multidimensional Stable Roommates Problem, namely the Roommate Diversity Problem and 3D-Euclidean Stable Roommates Problem, with popularity as equilibrium concept.

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Chapter 1

Introduction

The formation of stable or optimal coalitions in multi-agent systems is a computational problem that has several variations with different conditions on the coalition size and preferences of the agents.

The study of computational coalition formation problems began in 1962 with Gale and Shapley [29], who introduced the Stable Marriage Problem in which two disjoint sets, usually referred to as men and women, are to be stably matched according to their preferences. A matching is stable if no unmatched pair prefers each other over their assigned partners in the matching. Gale and Shapley also provided an algorithm that computes a stable matching, which is guaranteed to exist in the Stable Marriage Problem, in polynomial time. This algorithm is known as the Gale-Shapley algorithm. Since the initial work by Gale and Shapley, coalition formation problems have been studied in depth [16, 39, 42, 53, 35, 43].

The Stable Marriage Problem applies to various real-world scenarios such as matching students to universities [66], allocating residents to hospital posts [54, 55], school choice [1], and kidney exchange [57, 56].

Additionally, Gale and Shapley introduced the Stable Roommates Problem, which is essentially the Stable Marriage Problem but with one set. Unlike the Stable Marriage Problem, a stable matching is not guaranteed to exist in the Stable Roommates Problem. Irving [41] constructed a polynomial time algorithm that returns a stable matching, if one exists.

Both the Stable Marriage Problem and the Stable Roommates Problem consider coalitions of size two. Knuth [44] initiated the study of three- or multidimensional variants of these problems. Note that each coalition is required to be of equal size. We study two variants of the Multidimensional Stable Roommates Problem.

There are various notions that define the stability or optimality of a partitioning of agents. The most commonly used notion of stability is core stability, where no set of agents prefer each other over the agents in its assigned coalition. This notion of stability is equivalent to the one used in the work by Gale and Shapley [29].

Alcalde [3] introduced the concept of exchange stability, where no pair of agents would prefer to exchange their coalitions over staying in their assigned coalition. Though this notion of stability is more appropriate in certain cases, like core stability, an exchange stable outcome is not guaranteed to exist in the Stable Roommates Problem.

The notion of envy-freeness can be viewed as a “single-sided” version of exchange stability. That is, no agent wants to take the place of another agent. The notion was introduced by Varian [64].

A commonly used by economists notion of optimality is pareto optimality. This notion of optimality was introduced by, and named after, the Italian civil engineer and economist Vilfredo Pareto [49]. A partitioning of agents is pareto optimal, if there is no other partitioning of agents such that none of the agents are worse off and some of the agents are better off.

The notion of popularity, which blends elements of both stability and optimality, was introduced by Gärdenfors [31] in 1975. A partitioning of agents is popular if there does not exist another partitioning in which more agents are better off than worse off. Furthermore, a partitioning of agents is strictly popular if it is preferred by a majority of agents over any other partitioning. A probability distribution p over all possible partitionings of agents is mixed popular if the partitioning returned by p is expected to be at least as popular as the partitioning returned by any other probability distribution. Popular matchings have been an exciting area of research in the last few years [47]. Cseh [19] has recently provided a survey on popular matchings.

Computing a popular partition in a stable roommates game is NP-hard, even if the preferences are strict [21, 28, 34]. One method of obtaining tractability results from intractable coalition formation problems is to put restrictions on the agents' preferences for which the associated computational problems become tractable. This approach has been successfully applied in the Stable Roommates Problem [2, 7, 8, 13, 18, 20]. The two variants of the Multidimensional Stable Roommates Problem we study, called the Roommate Diversity Problem and the 3D-Euclidean Stable Roommates Problem, fall into the same type of approach.

1.1 Roommate Diversity Problem

Boehmer and Elkind [11] recently introduced a restricted variant of the Multidimensional Stable Roommates Problem called the Roommate Diversity Problem: each agent belongs to one of two types (e.g., red and blue), and the agents' preferences over the rooms solely depend on the fraction of agents of their own type among their roommates. Ties are allowed in the preferences of the agents.

The roommate diversity model captures important aspects of several real-world coalition formation scenarios, such as seating arrangements at events, and splitting students into teams for group projects. In the latter scenario, a local student may have difficulty communicating in English and an international student may not be able to speak the local language well. Thus, the preference of a student may solely depend on the fraction of international group members, with the ranking of fractions depending on their desire to learn the other language and their required ease of communication.

The Roommate Diversity Problem has been studied with the notions envy-freeness, exchange stability, core stability, and Pareto optimality. For most of these notions, computing a stable or optimal partitioning in a roommate diversity game is NP-hard or co-NP-hard. On the other hand, for some of the stability notions the problem is FPT when using the room size as parameter [11].

We show that popularity on the Roommate Diversity Problem with the room size fixed to 2 is tractable (Theorem 3.1.2). Particularly, a popular par-

tioning of agents is guaranteed to exist and can be computed in polynomial time. Additionally, a mixed popular partitioning of agents is guaranteed to exist in any roommate diversity game (Theorem 3.3.1). By contrast, when no restrictions are placed on the room size in a roommate diversity game, a popular partitioning may not exist (Theorem 3.4.7) and the problem becomes intractable. Our intractability results are summarized as follows:

- Determining the existence of a popular partitioning is co-NP-hard, even if the agents' preferences are trichotomous (Theorem 3.4.17).
- Determining the existence of a strictly popular partitioning is co-NP-hard, even if the agents' preferences are dichotomous (Theorem 3.2.2).
- A mixed popular outcome for a roommate diversity game is not computable in polynomial time, even if the agents' preferences are dichotomous, unless $P=NP$ (Theorem 3.3.6).

Our hardness proofs for general room size are inspired by the construction in the work by Boehmer and Elkind [11, Theorem 7.3] in combination with the work by Brandt and Bullinger [12, Section 4.2.2]. Additionally, we use the work by Brandt and Bullinger [12, Proposition 1] to show that a mixed popular partitioning of agents is guaranteed to exist in any roommate diversity game.

1.2 Geometric Multidimensional Stable Roommates

Arkin et al. [5] introduced a restricted variant of the Stable Roommates Problem called the Geometric Stable Roommates: each agent is associated with a point in d -dimensional metric space \mathbb{R}^d and an agent prefers to be matched with an agent located close to it over an agent that is located far from it.

Chen and Roy [17] investigated a restricted variant of the Geometric Stable Roommates, called the Euclidean d -Dimensional Stable Roommates (Euclid- d -SR) problem, where every agent is associated with a point in 2-dimensional Euclidean space. The agents are partitioned into rooms of size

d. Chen and Roy [17] found that determining the existence of a core stable outcome for $d \geq 3$ is NP-complete for the Euclid- d -SR problem.

The Euclidean d -Dimensional model captures important aspects of real-world coalition formation scenarios, such as swarm robotics. A robot is a physical object, thus it has coordinates in space associated with it. When forming coalitions of robots to perform a task, it is desirable for the robots to be located close to each other as they will be able to group faster and start the task earlier.

We study a modified version of the Euclid- d -SR problem, called the 3D-Euclidean Multidimensional Stable Roommates problem, where we consider 3-dimensional space instead of 2-dimensional, with the notion of popularity. We show that determining the existence of a strictly popular outcome in a 3D-Euclidean Multidimensional Stable Roommates game with room size 3 is co-NP-hard (Theorem 4.4.8). Our hardness proofs are inspired by the work by Chen and Roy [17, Section 3] in combination with the work by Brandt and Bullinger [12, Section 4.2.2].

1.3 Related Work

The diversity preferences were originally introduced by Brederick et al. [14] in the context of Hedonic Games. The term “hedonic” was coined by Drèze and Greenberg [6, 26, 51], who referred to the dependence of an agent’s utility solely on the members of their coalition the “hedonic aspect”.

A hedonic game is essentially the same as a multidimensional stable roommates game without conditions on the coalition size. That is, whereas each coalition of a multidimensional stable roommates game is required to be of size $s \in \mathbb{N}$, the coalitions of a hedonic game do not need to be the same size.

This relaxation on the coalition size has several real-life applications such as scheduling group activities [23], clustering in social networks [48], research team formation [4], distributed task allocation [58], and formation of coalition governments [15].

The Hedonic Games with diversity preferences have been extensively studied by Brederick et al. [14], Boehmer and Elkind [10], and Darmann

[22]. Galian et. al. [30] analyzed the parameterized complexity of hedonic diversity games using a combination of the parameters: number of agent types (colors); maximum coalition size; maximum number of coalitions; and number of agent (preference) types.

With additive preferences each agent has a function that assigns a value to every other agent. The value of a coalition is the summation over values of the members in its coalition. Additive preferences are often used to describe consumer behavior [52, 65, 59–61, 46, 45, 36–38, 33, 40, 9]. Huang [39] demonstrated that determining the existence of a core stable solution for the 3-dimensional Stable Roommates with additive preferences is NP-hard.

The preferences are metric, when every agent has a point in metric space and their preference only depends on its distance to the members of the coalition. Metric preferences was introduced by Arkin et. al. [5]. Deineko and Woeginger [24] strengthened Huang’s results by demonstrating that determining the existence of a core stable solution for the 3-dimensional Stable Roommates with metric preferences is NP-hard.

Note that Euclidean preferences are metric preferences and metric preferences are additive. Thus, our hardness results will hold for the more general cases.

Chapter 2

Preliminaries

In this section, we explain the definitions and notation used in this dissertation. For the Roommate Diversity Problem and 3D-Euclidean Stable Roommates Problem, definitions and notations specific to each problem are provided in their respective sections. The notation relating to both problems are described in Section 2.1, which defines the Multidimensional Stable Roommates Problem. Additionally, as popularity is also used for both problems, we describe the equilibrium concept in its own section.

For the Roommate Diversity Problem, we slightly extend the notation and definitions related to the Roommate Diversity Problem used in the work by Boehmer and Elkind [11, Section 2 and Theorem 7.3]. For our hardness results, we construct polynomial-time reductions from the Exact Cover by 3-Sets problem [32].

For the 3D-Euclidean Stable Roommates problem, we use the notation and definitions related to the Euclidean d -Dimensional Stable Roommates problem described in the work by Chen and Roy [17, Section 2]. For our hardness results, we construct polynomial-time reductions from the Planar and Cubic Exact Cover by 3-Sets (PC-X3C) problem [50], which is a restricted variant of the Exact Cover by 3 Sets problem.

Additionally, we also use the notation and definitions related to popularity used in the work by Brandt and Bullinger [12, Section 3].

To ensure that this dissertation is self-contained, we include the relevant existing definitions in this section.

For $s \in \mathbb{N}, t \in \mathbb{N}^+$, we define the integer sets $[t] = \{1, \dots, t\}$ and $[s, t] = \{s, \dots, t\}$.

2.1 Multidimensional Stable Roommates Problem

Though the Multidimensional Stable Roommates Problem is not extensively discussed in this dissertation, the definitions and notation relating to this problem are used in both the Roommate Diversity Problem and 3D-Euclidean Stable Roommates problem. Therefore, the Multidimensional Stable Roommates Problem is described in this section.

Let N be a set of agents, $s \in \mathbb{N}$ the room size, and $\mathcal{N}_a = \{C \subseteq N \mid |C| = s \wedge a \in C\}$ be the family of subsets of N with cardinality s that contain a . A s -sized subset of N is also called a room. We have that $|N| = k \cdot s$ for some $k \in \mathbb{N}$. That is, the number of rooms is k . For each agent $a \in N$, let \succsim_a be a complete and transitive weak order over the set \mathcal{N}_a .

For an agent $a \in N$, the preference relation \succsim_a represents the preferences of a over the rooms in \mathcal{N}_a . For example, given two rooms $S, T \in \mathcal{N}_a$, $S \succsim_a T$ means that agent a likes being in a room S at least as much as being in a room T . We write $S \succ_a T$ and say that agent a strictly prefers S over T if $S \succsim_a T$ and $T \not\sucsim_a S$. Additionally, we say that agent a weakly prefers S over T if $S \succsim_a T$. If agent a weakly prefers S over T and T over S , we write $S \sim_a T$ and say that a is indifferent between S and T .

An instance of the Multidimensional Stable Roommates Problem is called a game G . The game G is a triple $(N, s, (\succsim_a)_{a \in N})$.

An outcome π of G is a partitioning of the agents N into k rooms of size s , i.e., $\pi = \{C_1, \dots, C_k\}$ such that $|C_i| = s$ for each $i \in [k]$, $\bigcup_{i=1}^k C_i = N$, and for $1 \leq i < j \leq k$ we have $C_i \cap C_j = \emptyset$. Let $\pi(a)$ denote the room in π that contains agent $a \in N$, i.e., for $a \in N$, we have $\pi(a) = C_i$ where $i \in [k]$ such that $a \in C_i$.

2.2 Roommate Diversity Problem

A roommate diversity game is a quadruple $G = (R, B, s, (\succsim_a)_{a \in R \cup B})$ with room size s and agent set $N = R \cup B$ where $|N| = k \cdot s$ for some $k \in \mathbb{N}$. The preference relation \succsim_a of each agent $a \in N$ is a complete and transitive weak order over the set $D = \{\frac{j}{s} | j \in [0, s]\}$. We call the agents in R red agents and the agents in B blue agents.

For a room $C \subseteq N$, let $\theta(C)$ denote the fraction of red agents in C , i.e., $\theta(C) = \frac{|C \cap R|}{|C|}$. We say that C has/is of fraction $\theta(C)$ or the fraction of C is $\theta(C)$.

For an agent $a \in N$, the preference relation \succsim_a represents the preferences of a over the fraction of red agents in its room. For example, $\frac{2}{s} \succsim_a \frac{3}{s}$ means that agent a likes being in a room with 2 red agents at least as much as being in a room with 3 red agents. Note that a red agent cannot be in a room with fraction $\frac{0}{s}$ and a blue agent cannot be in a room with fraction $\frac{s}{s}$. Thus, discarding these ‘impossible’ fractions in their preference relation does not impact our results.

The preference relation of an agent a is said to be dichotomous if there exists a partitioning of D into two sets D_a^+ and D_a^- such that for all $d^+ \in D_a^+$, $d^- \in D_a^-$ it holds that $d^+ \succ_a d^-$, for all $d_1^+, d_2^+ \in D_a^+$ it holds that $d_1^+ \sim_a d_2^+$, and for all $d_1^-, d_2^- \in D_a^-$ it holds that $d_1^- \sim_a d_2^-$. We say that agent a approves of the fractions in D_a^+ and disapproves of the fractions in D_a^- .

The preference relation of an agent a is said to be trichotomous if there exists a partitioning of D into three sets D_a^+ , D_a^n , and D_a^- such that for all $d^+ \in D_a^+$, $d^n \in D_a^n$, and $d^- \in D_a^-$ it holds that $d^+ \succ_a d^n$ and $d^n \succ_a d^-$. Additionally, for all $d_1^+, d_2^+ \in D_a^+$ it holds that $d_1^+ \sim_a d_2^+$, for all $d_1^n, d_2^n \in D_a^n$ it holds that $d_1^n \sim_a d_2^n$, and for all $d_1^-, d_2^- \in D_a^-$ it holds that $d_1^- \sim_a d_2^-$. We say that agent a approves of the fractions in D_a^+ , is neutral about the fraction in D_a^n , and disapproves of the fractions in D_a^- .

We shall use tables to define the tri- and dichotomous preferences. The first column specifies an agent a . The second column denotes the set D_a^+ . In the example table below, we have that $D_a^+ = \{f_1, \dots, f_l\}$. The sets D_a^n and D_a^- are defined analogously in the third and fourth column respectively. In the case of dichotomous preferences, the third column defining the set D_a^n is

discarded. The final column may contain additional comments regarding the range of a variable.

Agent	Preference Profile			
	D_a^+	D_a^n	D_a^-	
a	$\{f_1, \dots, f_l\}$	$\{f_{l+1}, \dots, f_m\}$	$\{f_{m+1}, \dots, f_n\}$	“comments”

For an outcome π , let D_π^+ denote the agents that approve of the fraction of its room in π , i.e., $D_\pi^+ = \{a \in N \mid \theta(\pi(a)) \in D_a^+\}$. We define D_π^n and D_π^- analogously.

2.3 3D-Euclidean Multidimensional Stable Roommates

Let ϵ denote a small fractional value such that $0 < \epsilon < 0.001$, and let $d = \sqrt{1 - (\frac{\epsilon}{2})^2}$.

A 3D-Euclidean Multidimensional Stable Roommates (3D-EuclidSR) game G is a triple (N, E, s) , where the set $N = [s \cdot k]$ represents the set of agents, the embedding $E : N \rightarrow \mathbb{R}^3$ denotes the position of an agent in 3-dimensional Euclidean space, and $s \in \mathbb{N}^+$ is the room size. For an agent $a \in N$, let $E(a)_x$, $E(a)_y$, and $E(a)_z$ denote the x , y , and z coordinates of agent a in E , respectively.

For $p, q \in N$, let $\delta(p, q) = \sqrt{\sum_{i \in \{x, y, z\}} (E(p)_i - E(q)_i)^2}$ denote the Euclidean distance between agents p and q . Let $a \in N$ be an agent and $R \in \mathcal{N}_a$ be a room such that $a \in R$. We overload the notation by defining $\delta(a, R) = \sum_{a' \in R} \delta(a, a')$.

The preference of an agent solely depends on its distance to its roommates. Specifically, an agent prefers being in a room with roommates that are positioned close to themselves. Formally, let $a \in N$ be an agent, and $R, T \in \mathcal{N}_a$ be two rooms such that $a \in R$ and $a \in T$. We have that $R \succsim_a T \iff \delta(a, R) \leq \delta(a, T)$.

2.4 Popularity

For outcomes π, π' , let $N(\pi, \pi')$ be the set of agents who prefer π over π' , i.e., $N(\pi, \pi') = \{a \in N \mid \pi \succ_a \pi'\}$. For any subset $N' \subseteq N$ and outcomes π, π' , let $\phi_{N'}(\pi, \pi') = |N(\pi, \pi') \cap N'| - |N(\pi', \pi) \cap N'|$. We call $\phi_{N'}(\pi, \pi')$ the popularity margin on N' with respect to π and π' . We define the popularity margin of π and π' as $\phi_N(\pi, \pi')$ and denote it by $\phi(\pi, \pi')$.

An outcome π is more popular than outcome π' if $\phi(\pi, \pi') > 0$. An outcome π is popular if for any outcome π' we have $\phi(\pi, \pi') \geq 0$, i.e., no outcome is more popular than π . An outcome π is strictly popular if for any other outcome $\pi' \neq \pi$ we have $\phi(\pi, \pi') > 0$, i.e., π is more popular than any other outcome. Note that there can be at most one strictly popular outcome.

We define a mixed outcome $p = \{(\pi_1, p_1), \dots, (\pi_t, p_t)\}$ to be a set of pairs, where for each $i \in [t]$, π_i is an outcome of a roommate diversity game and (p_1, \dots, p_t) is a probability distribution. For mixed outcome $p = \{(\pi_1, p_1), \dots, (\pi_t, p_t)\}$ and $q = \{(\sigma_1, q_1), \dots, (\sigma_u, q_u)\}$, we define the popularity margin of p and q to be their expected popularity margin, i.e., $\phi(p, q) = \sum_{i=1}^t \sum_{j=1}^u p_i q_j \phi(\pi_i, \sigma_j)$. A mixed outcome p is popular if for any mixed outcome q we have $\phi(p, q) \geq 0$.

2.5 Set Cover Problems

For our hardness results, we construct polynomial-time reductions from NP-complete Set Cover problems [32, 50]. These are the Exact Cover by 3-Sets Problem and its restricted planar and cubic variant.

2.5.1 Exact Cover by 3-Sets Problem

For our reductions for the Roommate Diversity Problem, we use the Exact Cover by 3-Sets problem, which has been proven to be NP-complete by Garey and Johnson [32]. We abbreviate Exact Cover by 3-Sets with X3C.

Let $X = [m]$, where $m \in \mathbb{N}^+$ and $m \bmod 3 = 0$, and let $C = \{A_1, \dots, A_q\}$ be a collection of 3-element subsets of X , i.e., for each $j \in [q]$ we have $A_j \subseteq X$ and $|A_j| = 3$. An instance of the X3C problem is a tuple (X, C) and asks:

does there exist a subset $C' \subseteq C$ such that every element of X occurs in exactly one member of C' ? That is, does there exist a subset $C' \subseteq C$ such that

$$\bigcup_{S \in C'} S = X \wedge \forall_{S, S' \in C', S \neq S'} : S \cap S' = \emptyset?$$

We call such a C' a solution of (X, C) .

We require the following definition for our reduction. For $i \in X$, let $J^i = \{j_1^i, \dots, j_{m_i}^i\}$ be the set of indices of the 3-sets in C that contain i , i.e., $j \in J^i \iff i \in A_j$ (or equivalently $J^i = \{j \in [q] \mid i \in A_j\}$).

2.5.2 Planar and Cubic Exact Cover by 3-Sets

For our reductions for the 3D-Euclidean Multidimensional Stable Roommates Problem, we use the Planar and Cubic Exact Cover by 3-Sets problem, which is a restricted version of the Exact Cover by 3-Sets problem. We abbreviate planar and cubic exact cover by 3-sets with PC-X3C. Moore and Robson [50] have proven that this restricted version remains NP-complete.

Let $X = [m]$, where $m \in \mathbb{N}^+$ and $m \bmod 3 = 0$, and let $C = \{A_1, \dots, A_q\}$ be a collection of 3-element subsets of X . An instance of the X3C problem is a tuple $I = (X, C)$, which asks: does there exist $S \subseteq C$ such that S partitions X ? We call such an S a solution of I .

The associated graph of I , denoted by $G(I)$, is a bipartite graph $(U \cup W, E)$ where $U = \{u_i \mid i \in X\}$, $W = \{w_j \mid A_j \in C\}$, and an edge $\{u_i, w_j\} \in E$ if and only if $i \in A_j$. We call vertices in U and W the element-vertices and set-vertices, respectively.

Instance I is an instance of the PC-X3C problem if each element $x \in X$ occurs in exactly three sets of C and the associated graph is planar. Note that for a valid instance of the PC-X3C problem, we have that $|X| \geq 6$. Additionally, we have that $|X| = |C|$.

Chapter 3

Roommate Diversity Problem

We start with the discussion on our results of the Roommate Diversity Problem. These results are summarized in Section 1.1, the relevant definitions and notation can be found in Sections 2.1 and 2.2, and the NP-complete X3C problem [32] that we use for our reductions is described in Section 2.5.1.

3.1 Room Size Two

For a roommate diversity game $G = (R, B, 2, (\succsim_a)_{a \in R \cup B})$ with room size 2, a room $S \subseteq R \cup B$, can have exactly one of 3 possible fractions, i.e., $\theta(S) \in \{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}\}$. We refer to a room with a fraction of $\frac{0}{2}$ as a pure blue room and a room with a fraction of $\frac{2}{2}$ as a pure red room. A room with fraction $\frac{1}{2}$ is called a mixed room.

As mentioned in Chapter 2, we can discard the ‘impossible’ fractions from the preference relations of the agents. Thus, the relevant fractions for a red agent are $\frac{1}{2}$ and $\frac{2}{2}$, while for a blue agent, they are $\frac{0}{2}$ and $\frac{1}{2}$. Let us call an agent that is in a room with one of their most preferred fractions happy. Otherwise, we call the agent sad. That is, an agent $a \in R \cup B$ is happy in outcome π if for each $f \in \{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}\}$ we have $\theta(\pi(a)) \succsim_a f$. An agent $a \in R \cup B$ is sad in outcome π if there exists $f \in \{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}\}$ such that $\theta(\pi(a)) \prec_a f$.

A red agent r can only have one of 3 possible preference relations, namely $\frac{1}{2} \succ_r \frac{2}{2}$, $\frac{1}{2} \prec_r \frac{2}{2}$, or $\frac{1}{2} \sim_r \frac{2}{2}$. We call a red agent r with preference relation $\frac{1}{2} \succ_r \frac{2}{2}$, $\frac{1}{2} \prec_r \frac{2}{2}$, or $\frac{1}{2} \sim_r \frac{2}{2}$ a mixed, pure, or indifferent red agent respectively.

We define mixed, pure, and indifferent blue agents using fractions $\frac{0}{2}$ and $\frac{1}{2}$ analogously.

Let us define the set of pure red agents $R^p = \{r \in R | \frac{2}{2} \succ_r \frac{1}{2}\}$, the set of mixed red agents $R^m = \{r \in R | \frac{2}{2} \prec_r \frac{1}{2}\}$, and the set of indifferent red agents $R^i = \{r \in R | \frac{1}{2} \sim_r \frac{2}{2}\}$. We define the set of pure blue agents B^p , the set of mixed blue agents B^m , and the set of indifferent blue agents B^i analogously. Note that $R = R^p \cup R^m \cup R^i$ and $B = B^p \cup B^m \cup B^i$.

We demonstrate that a popular outcome is guaranteed to exist in the roommate diversity game G and can be computed in polynomial time by reducing G to the maximum weight perfect matching problem.

Let us define an undirected weighted complete graph $G_m = (R \cup B, E)$ where for each pair of distinct agents $a, b \in R \cup B$, we use $w(a, b)$ to denote the weight of the edge $(a, b) \in E$, where

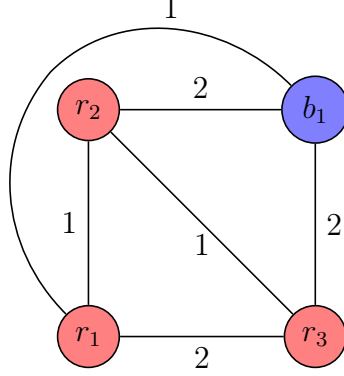
$$w(a, b) = \begin{cases} 2 & \{a, b\} \text{ is a room with exactly 2 happy agents;} \\ 1 & \{a, b\} \text{ is a room with exactly 1 happy agent;} \\ 0 & \{a, b\} \text{ is a room with exactly 0 happy agents.} \end{cases}$$

For example, let us consider the roommate diversity game $G' = \{R', B', 2, (\succ_a)_{a \in R' \cup B'}\}$, where $R' = \{r_1, r_2, r_3\}$ and $B' = \{b_1\}$ with the preference profiles

$$\begin{array}{c} \frac{2}{2} \succ_{r_1} \frac{1}{2} \\ \frac{2}{2} \prec_{r_2} \frac{1}{2} \\ \frac{2}{2} \sim_{r_3} \frac{1}{2} \\ \frac{1}{2} \succ_{b_1} \frac{0}{2}. \end{array}$$

The corresponding undirected weighted clique graph G'_m can be found in Fig. 3.1.

Since the room size is 2, $|R \cup B|$ must be even. For the weighted graph G_m , let $w(M)$ be the weight of a perfect matching $M \subseteq E$. Given a perfect

Figure 3.1: Example undirected weighted clique graph G'_m .

matching M of G_m , we define the points $p_M(a)$ for an agent $a \in R \cup B$ as follows: For each $(a, b) \in M$,

1. if $w(a, b) = 2$, then $p_M(a) = p_M(b) = 1$;
2. (a) if $w(a, b) = 1$ where a is happy and b is sad, then $p_M(a) = 1$ and $p_M(b) = 0$;
- (b) if $w(a, b) = 1$ where a is sad and b is happy, then $p_M(a) = 0$ and $p_M(b) = 1$;
3. if $w(a, b) = 0$, then $p_M(a) = p_M(b) = 0$.

We have that for any perfect matching M of G_m ,

$$w(M) = \sum_{(a,b) \in M} w(a, b) = \sum_{a \in R \cup B} p_M(a). \quad (3.1)$$

Lemma 3.1.1. *For the weighted graph $G_m = (R \cup B, E)$ as defined above, the maximum weight perfect matching M_* of G_m is popular.*

Proof. Fix an arbitrary perfect matching M of G_m . It is immediate that $w(M_*) \geq w(M)$. For the maximum weight perfect matching M_* of G_m , we define H_{M_*} and S_{M_*} as

$$H_{M_*} = \{a \in R \cup B | p_{M_*}(a) = 1\}; \quad S_{M_*} = \{a \in R \cup B | p_{M_*}(a) = 0\},$$

and for the perfect matching M of G_m , we also define H_M and S_M by

$$H_M = \{a \in R \cup B | p_M(a) = 1\}; \quad S_M = \{a \in R \cup B | p_M(a) = 0\}.$$

From Eq. (3.1) and the fact that $w(M_*) \geq w(M)$, it follows that

$$|H_{M_*}| = w(M_*) \geq w(M) = |H_M|. \quad (3.2)$$

We have that $\phi(M_*, M) = |H_{M_*}| - |H_{M_*} \cap H_M|$ and $\phi(M, M_*) = |H_M| - |H_{M_*} \cap H_M|$. Thus, from Eq. (3.2), we have that $\phi(M_*, M) - \phi(M, M_*) = |H_{M_*}| - |H_M| \geq 0$. \square

Theorem 3.1.2. *Let $G = (R, B, 2, (\succ_a)_{a \in R \cup B})$ be a roommate diversity game with room size 2. We can find a popular outcome π in polynomial time.*

Proof. We already know that for any (integer) weighted G_m , a maximum weight perfect matching can be found in polynomial time [27]. Thus, using the existing polynomial time algorithm and Lemma 3.1.1, for Roommate Diversity Problem with room size 2, a popular outcome always exists and it can be computed in polynomial time. \square

3.2 Strict Popularity

In this subsection, we show that determining the existence of a strictly popular outcome in a roommate diversity game is co-NP-hard, even if the preferences are dichotomous. We shall construct a roommate diversity game $G = (R, B, s, (\succ_a)_{a \in R \cup B})$ with dichotomous preferences from an X3C instance (X, C) such that there exists a solution $C' \subseteq C$ that partitions X if and only if no strictly popular outcome exists for G .

3.2.1 Instance of the Roommate Diversity Problem

We set the room size $s = 5(q + 1) + 1 + m = 5q + 6 + m$. The set of red agents is given by $R = R^{set} \cup R^{mon} \cup R^{red}$, where

$$\begin{aligned}
\text{Set Agents } R^{set} &= \{r_i | i \in X\} = \{r_1, \dots, r_m\} \\
\text{Redundant Agents } R^{red} &= R_1^{red} \cup \dots \cup R_q^{red} \\
&\quad \text{with } R_j^{red} = \{r_j^1, \dots, r_j^{5j-2}\} \text{ for } j \in [q] \\
\text{Monolith Agents } R^{mon} &= \left\{ r_{mon}^1, \dots, r_{mon}^{5(q+1)+1} \right\}
\end{aligned}$$

The set of blue agents is given by $B = B^{even} \cup B^{mon} \cup B^{add} \cup B^{fill}$, where

$$\begin{aligned}
\text{Filling Agents } B^{fill} &= B_1^{fill} \cup \dots \cup B_q^{fill} \\
&\quad \text{with } B_j^{fill} = \left\{ b_j^1, \dots, b_j^{s-(5j-2)-3} \right\} \text{ for } j \in [q] \\
\text{Additional Agents } B^{add} &= B_1^{add} \cup \dots \cup B_q^{add} \\
&\quad \text{with } B_j^{add} = \left\{ \tilde{b}_j^1, \tilde{b}_j^2, \tilde{b}_j^3 \right\} \text{ for } j \in [q] \\
\text{Monolith Agents } B^{mon} &= \left\{ b_{mon}^1, \dots, b_{mon}^{s-(5(q+1)+1)} \right\} \\
\text{Evening Agents } B^{even} &= \left\{ b_{even}^1, \dots, b_{even}^{5(q+1)+1} \right\}
\end{aligned}$$

The preference profiles of the agents are defined in Table 3.1.

Agent	Preference Profile		
	D_a^+	D_a^-	
$a = r_i \in R^{set}$	$\left\{ \frac{5j_i^i+1}{s}, \dots, \frac{5j_{m_i}^i+1}{s} \right\} \cup \left\{ \frac{5(q+1)+1+m}{s} \right\}$	$D \setminus D_a^+$	$j \in [q]$
$a = r_j^p \in R_j^{red}$	$\left\{ \frac{5j+1}{s}, \frac{5j-2}{s} \right\}$	$D \setminus D_a^+$	
$a = r_{mon}^p \in R^{mon}$	$\left\{ \frac{5(q+1)+1+m}{s}, \frac{5(q+1)+1}{s} \right\}$	$D \setminus D_a^+$	
$a = b_j^p \in B_j^{fill}$	$\left\{ \frac{5j+1}{s}, \frac{5j-2}{s} \right\}$	$D \setminus D_a^+$	$j \in [q]$
$a = \tilde{b}_j^p \in B_j^{add}$	$\left\{ \frac{5j-2}{s}, 0 \right\}$	$D \setminus D_a^+$	$j \in [q]$
$a = b_{mon}^p \in B^{mon}$	$\left\{ \frac{5(q+1)+1}{s}, 0 \right\}$	$D \setminus D_a^+$	
$a = b_{even}^p \in B^{even}$	$\{0\}$	$D \setminus D_a^+$	

Table 3.1: Preference profile $(\succsim_a)_{a \in RUB}$.

3.2.2 Predefined Outcomes

We define the monolithic outcome π_{mon} and for a solution $C' \subseteq C$ for (X, C) the reduced outcome $\pi_{C'}$. The number of reduced outcomes is equivalent to

the number of solutions for (X, C) . Note that no reduced outcomes exists, if (X, C) has no solution. In these outcomes every agent is in a room with a fraction that it approves of.

Observe that an outcome π in which every agent is in a room with a fraction that it approves of is popular, i.e., for any agent $a \in R \cup B$, if $\theta(\pi(a)) \in D_a^+$, then π is popular. In fact, such an outcome is more popular than any outcome that does not assign every agent to a room with a fraction that it approves of.

We show that the monolithic outcome π_{mon} and the reduced outcomes are the only outcomes that assign every agent to a room with a fraction that it approves. This implies that the monolithic outcome π_{mon} and the reduced outcomes are more than any other outcome. If (X, C) has no solution, then π_{mon} is the only outcome that is more popular than any other outcome. In this case π_{mon} is strictly popular. If (X, C) has a solution, then we have multiple popular outcomes, namely the monolithic outcome and the reduced outcomes. In this case we do not have a strictly popular outcome.

Monolithic Outcome.

First we define the rooms that contain red agents. Let

$$P_j = B_j^{add} \cup R_j^{red} \cup B_j^{fill} \text{ for } j \in [q]; \quad P_{q+1} = R^{set} \cup R^{mon}.$$

Let $\pi_R = \{P_j | j \in [q + 1]\}$. Additionally, let us define the set B_r of remaining blue agents as the blue agents that are not contained in any room P_j . That is,

$$B_r = B \setminus \bigcup_{j=1}^{q+1} P_j = B^{mon} \cup B^{even}.$$

Note that $|B^{mon}| + |B^{even}| = s - (5(q + 1) + 1) + (5(q + 1) + 1) = s$. We define π_B to be the partition that only contains the room B_r . Finally, we define the monolithic outcome to be $\pi_{mon} = \pi_R \cup \pi_B$. Note that every agent is assigned a room by π_{mon} with a fraction that it approves of, i.e., for every agent $a \in R \cup B$ we have $\theta(\pi_{mon}(a)) \in D_a^+$. Additionally, a monolithic outcome always exists.

Observe that since every agent is assigned a room by π_{mon} with a fraction that it approves of and π_{mon} always exists, any other popular outcome must also assign every agent to a room that it approves of. Therefore, any popular outcome is more popular than any non-popular outcome.

Reduced Outcome.

Let the solution $C' \subseteq C$ partition X . For every 3-element subset $A_j \in C$, we define the rooms that contain red agents. For $j \in [q]$, let

$$P'_j = \begin{cases} \{r_i \in R^{set} | i \in A_j\} \cup R_j^{red} \cup B_j^{fill} & , \text{ if } A_j \in C' \\ B_j^{add} \cup R_j^{red} \cup B_j^{fill} & , \text{ if } A_j \notin C' \end{cases}$$

$$P'_{q+1} = R^{mon} \cup B^{mon}.$$

Let $\pi_R = \{P'_j | j \in [q+1]\}$. Additionally, let us define the set B_r of remaining blue agents as the blue agents that are not contained in any P'_j . That is,

$$B_r = B \setminus \bigcup_{j=1}^{q+1} P'_j = B^{even} \cup \bigcup_{A_j \in C'} B_j^{add}.$$

Since $|\bigcup_{A_j \in C'} B_j^{add}| = \frac{m}{3} \cdot 3 = m$ and $|B^{even}| = 5(q+1) + 1$, we have $|B_r| = 5(q+1) + 1 + m = s$. We define π_B to be the partition only containing the coalition B_r . Finally, we define the reduced outcome to be $\pi_{C'} = \pi_R \cup \pi_B$.

Note that every agent is assigned a room by $\pi_{C'}$ with a fraction that it approves of, i.e., for every agent $a \in R \cup B$ we have $\theta(\pi_{C'}(a)) \in D_a^+$. Additionally, a reduced outcome exists if and only if (X, C) has a solution.

Example.

Let us consider an example X3C instance (X, C) and its corresponding roommate diversity game G to demonstrate the construction.

Let (X, C) be an X3C instance, where $X = \{1, 2, 3, 4, 5, 6\}$ and $C = \{\{1, 2, 3\}, \{2, 3, 4\}, \{4, 5, 6\}\} = \{C_1, C_2, C_3\}$. For the corresponding roommate diversity game $G = \{R, B, 27, (\succ_a)_{a \in R \cup B}\}$, we visualize the red agent sets in Fig. 3.2 and the blue agent sets in Fig. 3.3.

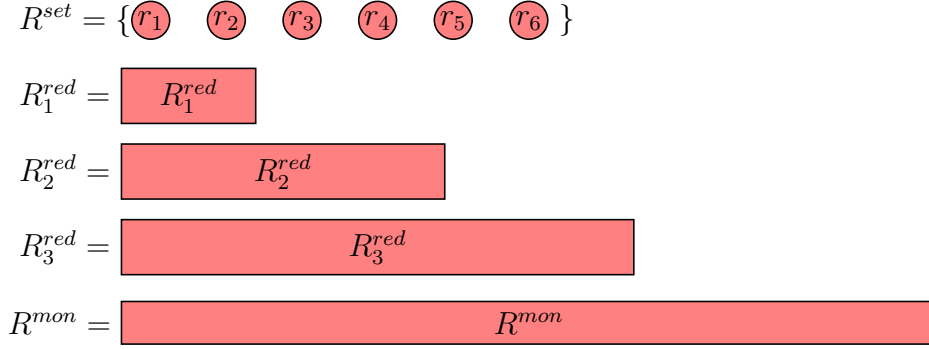


Figure 3.2: Visualization of the red agent sets of G , where $R = R^{set} \cup R_1^{red} \cup R_2^{red} \cup R_3^{red} \cup R^{mon}$.

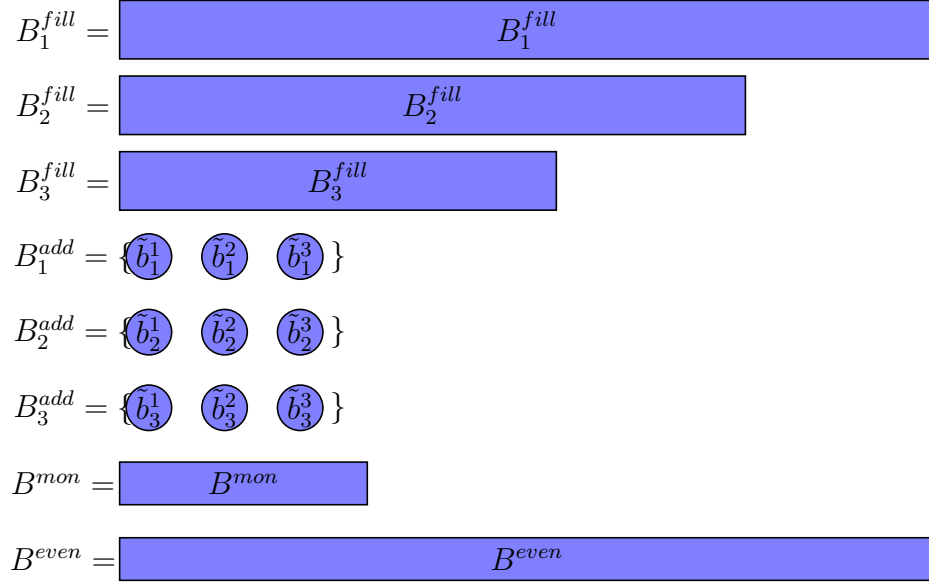
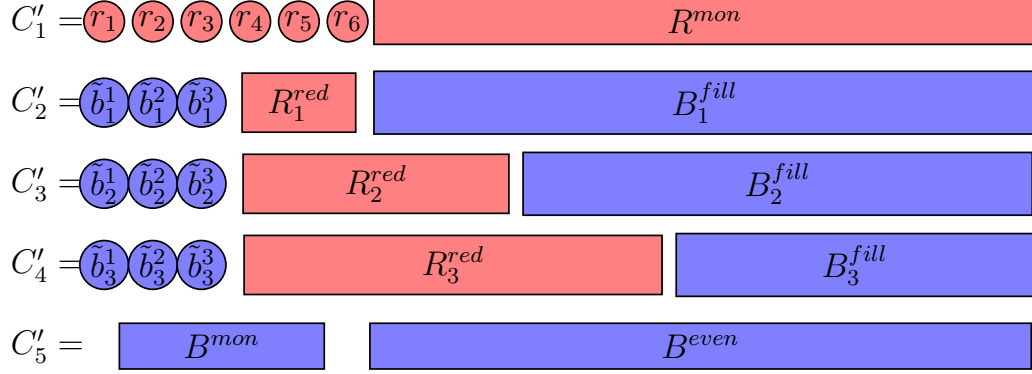
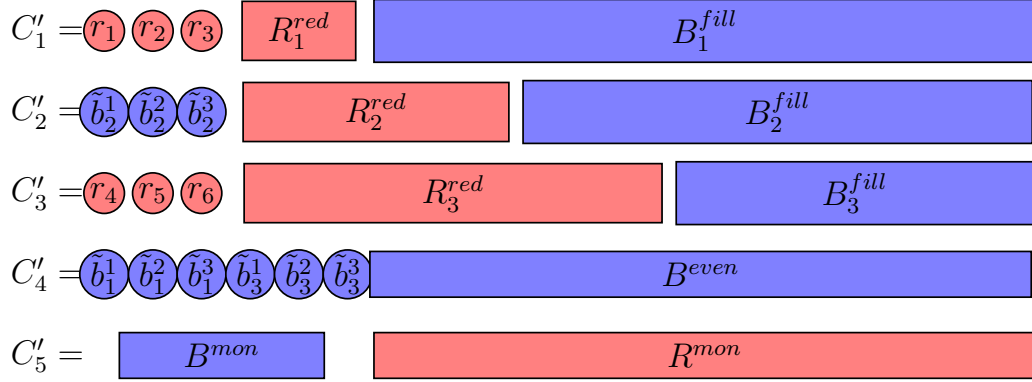


Figure 3.3: Visualization of the blue agent sets of G , where $B = B_1^{fill} \cup B_2^{fill} \cup B_3^{fill} \cup B_1^{add} \cup B_2^{add} \cup B_3^{add} \cup B^{mon} \cup B^{even}$.

An illustration of the monolithic outcome π_{mon} can be found in Fig. 3.4. Additionally, an illustration of the reduced outcome $\pi_{C'}$, where $C' = \{\{1, 2, 3\}, \{4, 5, 6\}\} = \{C_1, C_3\}$ can be found in Fig. 3.5.

Figure 3.4: Visualization the rooms of the monolithic outcome π_{mon} .Figure 3.5: Visualization the reduced outcome $\pi_{C'}$, where $C' = \{\{1, 2, 3\}, \{4, 5, 6\}\} = \{C_1, C_3\}$.

3.2.3 Hardness

We demonstrate co-NP-hardness by showing that the monolithic outcome π_{mon} is the only popular outcome and therefore strictly popular, if (X, C) has no solution. Otherwise we have multiple popular outcomes, which are exactly the monolithic outcome π_{mon} and all the reduced outcomes $\pi_{C'}$, where subset $C' \subseteq C$ is a solution of (X, C) .

Lemma 3.2.1. *Let G be a roommate diversity game constructed as in Section 3.2.1 and π be an outcome of G such that every agent is in a room with a*

fraction that it approves of. We have that π is either the monolithic outcome π_{mon} or a reduced outcome $\pi_{C'}$, where $C' \subseteq C$ partitions X .

Proof. Let $r_i \in R^{set}$ be an arbitrary set agent. As r_i must be assigned a room in π with a fraction that it approves of, we have that $\theta(\pi(r_i)) = 1$ or $\theta(\pi(r_i)) = \frac{5j+1}{s}$, where $j \in J^i$. Let us consider the following cases.

1. $\theta(\pi(r_i)) = 1$.

Since there are exactly s red agents that approve of fraction 1, namely the agents in $R^{set} \cup R^{mon}$, these agents must be contained in the same room, i.e., $R^{set} \cup R^{mon} \in \pi$. A redundant agent $r_{j'}^p \in R^{red}$ cannot be in a room with fraction $\frac{5j'+1}{s}$ as there are only $s - 3$ remaining agents that approve of that fraction. Thus, any redundant agent $r_{j'}^p$ must be in a room with fraction $\frac{5j'-2}{s}$. The rooms must be $R_{j'}^{red} \cup B_{j'}^{fill} \cup B_{j'}^{add}$ for $j' \in [q]$ as these are exactly the agents that approve of fraction $\frac{5j'-2}{s}$, i.e. for $j' \in [q]$ we have $R_{j'}^{red} \cup B_{j'}^{fill} \cup B_{j'}^{add} \in \pi$. We have s remaining blue agents, namely $B^{even} \cup B^{mon}$, that must belong to the same room, i.e., $B^{even} \cup B^{mon} \in \pi$. Thus, π must be π_{mon} in this case.

2. $\theta(\pi(r_i)) = \frac{5j+1}{s}$.

There are exactly s red agents, including r_i , that approve of fraction 1. Since $\theta(\pi(r_i)) \neq 1$ and π is an outcome such that every agent is in a room with a fraction that it approves of, the outcome π cannot contain a room with only red agents. Thus, every set agent $r_{i'} \in R^{set}$ must be in a room with fraction $\frac{5j'+1}{s}$ such that $i' \in A_{j'}$. There are exactly 3 set agents that approve of $\frac{5j''+1}{s}$ for each $j'' \in [q]$. Thus, for some solution $C' \subseteq C$ of (X, C) , every set agent must be in a room with the shape $\{r_i | i \in A_j\} \cup R_j^{red} \cup B_j^{fill}$ where $A_j \in C'$, i.e., for each $A_j \in C'$ we have $\{r_i | i \in A_j\} \cup R_j^{red} \cup B_j^{fill} \in \pi$.

The remaining redundant agents $r_{j'''}^p \in R^{red}$ must be in a room with fraction $\frac{5j'''-2}{s}$ as only $s - 3$ remaining agents approve of fraction $\frac{5j'''-2}{s}$. The rooms must be $R_{j'''}^{red} \cup B_{j'''}^{fill} \cup B_{j'''}^{add}$ for $A_{j'''} \notin C'$, i.e., for each $A_{j'''} \notin C'$ we have $R_{j'''}^{red} \cup B_{j'''}^{fill} \cup B_{j'''}^{add} \in \pi$. Since there is no room with only red agents, the red monolith agents must be in a room with the blue monolith agents, i.e., $R^{mon} \cup B^{mon} \in \pi$. We have s remaining blue

agents, namely $B^{even} \cup \bigcup_{A_j \in C'} B_j^{add}$, that must belong to the same room, i.e.,

$$B^{even} \cup \bigcup_{A_j \in C'} B_j^{add} \in \pi.$$

Thus, π must be $\pi_{C'}$ in this case.

□

Theorem 3.2.2. *Determining whether a strict popular outcome exists in a roommate diversity game is co-NP-hard, even if the preferences are dichotomous.*

Proof. Let us define the family of solution sets $\mathcal{C} = \{C' \subseteq C \mid C' \text{ is a solution of } (X, C)\}$. By Lemma 3.2.1 and the observation in Section 3.2.2, we have that $\mathcal{P} = \{\pi_{C'} \mid C' \in \mathcal{C}\} \cup \{\pi_{mon}\}$ is the collection of all popular outcomes.

From Lemma 3.2.1 we have that if (X, C) has no solution, then $|\mathcal{P}| = 1$. Therefore, by the observation in Section 3.2.2, we have a strictly popular outcome, namely π_{mon} . Otherwise we have that $|\mathcal{P}| > 1$ and therefore no outcome is strictly popular.

Thus, we have a reduction from any X3C instance (X, C) to a roommate diversity game G with dichotomous preferences such that (X, C) has a solution if and only if G has no strictly popular outcome. Since the X3C problem is NP-complete, determining whether a strict popular outcome exists in a roommate diversity game is co-NP-hard, even if the preferences are dichotomous. □

3.3 Mixed Popularity

In this subsection, we show that a mixed popular outcome is guaranteed to exist. However, under the assumption that $P \neq NP$, a mixed popular outcome cannot be computed in polynomial time, even if the preferences are dichotomous.

To show the guaranteed existence, we use the minimax theorem [62]. To show hardness, we construct a reduction from the X3C problem to the Roommate Diversity Problem. The reduction is similar to the reduction in Section 3.2.

Theorem 3.3.1. *Any roommate diversity game is guaranteed to have a mixed popular outcome.*

Proof. Every roommate diversity game can be viewed as a finite two-player symmetric zero-sum game where the rows and columns of the game matrix are indexed by all possible outcomes π_1, \dots, π_t and the entry at (i, j) of the game matrix has value $\phi(\pi_i, \pi_j)$. By the minimax theorem [62], we have that the value of this game is 0. Therefore, any maximin strategy is popular. \square

We change the reduction in Section 3.2 by doubling the agents, changing the room size and preference profile accordingly, and adding new agents. These additions ensure that the monolithic outcome is strictly popular if (X, C) has no solution and that the monolithic outcome is not popular if (X, C) has a solution.

3.3.1 Instance of the Roommate Diversity Problem

We set the room size to $s = 2(5(q+2) + 1 + m) + 6$. The set of red agents is given by $R = R^{set} \cup \tilde{R}^{set} \cup \hat{R}^{set} \cup R^{mon} \cup R^{red}$, where

$$\begin{aligned}
\text{Set Agents} & R^{set} = \{r_i | i \in X\} = \{r_1, \dots, r_m\} \\
\text{Copy Set Agents} & \tilde{R}^{set} = \{\tilde{r}_i | i \in X\} = \{\tilde{r}_1, \dots, \tilde{r}_m\} \\
\text{Auxiliary Set Agents} & \hat{R}^{set} = \{\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5, \hat{r}_6\} \\
\text{Redundant Agents} & R^{red} = R_1^{red} \cup \dots \cup R_{q+1}^{red} \\
& \text{with } R_j^{red} = \left\{ r_j^1, \dots, r_j^{2(5j-2)} \right\} \text{ for } j \in [q+1] \\
\text{Monolith Agents} & R^{mon} = \left\{ r_{mon}^1, \dots, r_{mon}^{2(5(q+2)+1)} \right\}
\end{aligned}$$

The set of blue agents is given by $B = B^{even} \cup B^{mon} \cup B^{add} \cup B^{fill}$, where

$$\begin{aligned}
\text{Filling Agents} & B^{fill} = B_1^{fill} \cup \dots \cup B_{q+1}^{fill} \\
& \text{with } B_j^{fill} = \left\{ b_j^1, \dots, b_j^{s-2(5j-2)-6} \right\} \text{ for } j \in [q+1] \\
\text{Additional Agents} & B^{add} = B_1^{add} \cup \dots \cup B_{q+1}^{add} \\
& \text{with } B_j^{add} = \left\{ \tilde{b}_j^1, \tilde{b}_j^2, \tilde{b}_j^3, \tilde{b}_j^4, \tilde{b}_j^5, \tilde{b}_j^6 \right\} \text{ for } j \in [q+1] \\
\text{Monolith Agents} & B^{mon} = \left\{ b_{mon}^1, \dots, b_{mon}^{s-2(5(q+2)+1)} \right\} \\
\text{Evening Agents} & B^{even} = \left\{ b_{even}^1, \dots, b_{even}^{2(5(q+2)+1)} \right\}
\end{aligned}$$

The preference profiles of the agents are defined in Table 3.2.

Agent	Preference Profile		
	D_a^+	D_a^-	
$a = r_i \in R^{set}$	$\left\{ \frac{2(5j_1^i+1)}{s}, \dots, \frac{2(5j_{m_i}^i+1)}{s} \right\} \cup \{1\}$	$D \setminus D_a^+$	
$a = \tilde{r}_i \in \tilde{R}^{set}$	$\left\{ \frac{2(5j_1^i+1)}{s}, \dots, \frac{2(5j_{m_i}^i+1)}{s} \right\} \cup \{1\}$	$D \setminus D_a^+$	
$a \in \{\hat{r}_1, \dots, \hat{r}_5\}$	$\left\{ \frac{2(5(q+1)+1)}{s}, 1 \right\}$	$D \setminus D_a^+$	
$a = \hat{r}_6$	$\left\{ \frac{2(5(q+1)+1)}{s} \right\}$	$D \setminus D_a^+$	
$a = r_j^p \in R_j^{red}$	$\left\{ \frac{2(5j+1)}{s}, \frac{2(5j-2)}{s} \right\}$	$D \setminus D_a^+$	$j \in [q+1]$
$a = r_{mon}^p \in R^{mon}$	$\left\{ 1, \frac{2(5(q+2)+1)}{s} \right\}$	$D \setminus D_a^+$	
$a = b_j^p \in B_j^{fill}$	$\left\{ \frac{2(5j+1)}{s}, \frac{2(5j-2)}{s} \right\}$	$D \setminus D_a^+$	$j \in [q+1]$
$a = \tilde{b}_j^p \in B_j^{add}$	$\left\{ \frac{2(5j-2)}{s}, 0 \right\}$	$D \setminus D_a^+$	$j \in [q+1]$
$a = b_{mon}^p \in B^{mon}$	$\left\{ \frac{2(5(q+2)+1)}{s}, 0 \right\}$	$D \setminus D_a^+$	
$a = b_{even}^p \in B^{even}$	$\{0\}$	$D \setminus D_a^+$	

Table 3.2: Preference profile $(\succsim_a)_{a \in R \cup B}$.

3.3.2 Predefined Outcomes

We define the monolithic outcome π_{mon} and for a solution $C' \subseteq C$ for (X, C) the reduced outcome $\pi_{C'}$ in a similar manner as in Section 3.2.2. In outcome π_{mon} exactly 1 agent is in a room that it disapproves of. In outcome $\pi_{C'}$ all agents are in a room that it approves of.

Monolithic Outcome.

First we define the rooms that contain red agents.

$$P_j = B_j^{add} \cup R_j^{red} \cup B_j^{fill} \text{ for } j \in [q+1]; \quad P_{q+2} = R^{set} \cup \tilde{R}^{set} \cup \hat{R}^{set} \cup R^{mon}.$$

Let π_R and π_B be defined in a similar manner as in Section 3.2.2, i.e., $\pi_R = \bigcup_{j \in [q+2]} \{P_j\}$ and π_B contains 1 room with the blue agents not contained

in any P_j , where $j \in [q+2]$. We define the monolithic outcome to be $\pi_{mon} = \pi_R \cup \pi_B$. Note that agent \hat{r}_6 is the only agent that is assigned a room by π_{mon} with a fraction that it does not approve of. A monolithic outcome always exists.

Reduced Outcome.

Let $C' \subseteq C$ be a solution of (X, C) , i.e., C' partitions X . For every 3-element subset $A_j \in C$, we define the rooms that contain red agents. For $j \in [q]$, let

$$\begin{aligned} P'_j &= \begin{cases} \{r_i \in R^{set} | i \in A_j\} \cup \{\tilde{r}_i \in \tilde{R}^{set} | i \in A_j\} \cup R_j^{red} \cup B_j^{fill} & , \text{ if } A_j \in C' \\ B_j^{add} \cup R_j^{red} \cup B_j^{fill} & , \text{ if } A_j \notin C' \end{cases} \\ P'_{q+1} &= \tilde{R}^{set} \cup R_{q+1}^{red} \cup B_{q+1}^{fill}; \\ P'_{q+2} &= R^{mon} \cup B^{mon}. \end{aligned}$$

Let π_R and π_B be defined in a similar manner as in Section 3.2.2, i.e., $\pi_R = \bigcup_{j \in [q+2]} \{P'_j\}$ and π_B contains 1 room with the blue agents not contained in any P'_j , where $j \in [q+2]$. We define the reduced outcome to be $\pi_{C'} = \pi_R \cup \pi_B$. Note that every agent is assigned a room by $\pi_{C'}$ with a fraction that it approves of. A reduced outcome exists if and only if (X, C) has a solution.

3.3.3 Hardness

We demonstrate that it is hard to compute a mixed popular solution by showing that π_{mon} is strictly popular, if (X, C) has no solution. Otherwise π_{mon} is not popular. If π_{mon} is strictly popular, then the only mixed popular outcome is $p = \{(\pi_{mon}, 1)\}$. If π_{mon} is not popular, then $p = \{(\pi_{mon}, 1)\}$ cannot be a mixed popular outcome.

Remark 3.3.2. If outcome π has a room that consists of only red agents and assigns exactly 1 agent to a room that it disapproves of, then π must be the monolithic outcome. That is, for any π , if $|D_\pi^-| = 1$ and there exists a room $S \in \pi$ such that $\theta(S) = 1$, then $\pi = \pi_{mon}$.

Proof. Let π be an arbitrary outcome that contains a room that consists of only red agents and assigns exactly 1 agent to a room with a fraction that it

disapproves of, i.e., for outcome π we have $|D_\pi^-| = 1$ and there exists a room $S \in \pi$ such that $\theta(S) = 1$.

Let $S \in \pi$ be the room that only consists of red agents. We have exactly $s - 1$ red agents, namely $R^{set} \cup \tilde{R}^{set} \cup R^{mon} \cup \{\hat{r}_1, \dots, \hat{r}_5\}$, that approve of fraction 1. Thus, the room S must contain $R^{set} \cup \tilde{R}^{set} \cup R^{mon} \cup \{\hat{r}_1, \dots, \hat{r}_5\}$, otherwise there are at least 2 red agents in S that do not approve of fraction 1. Let r denote the red agent in S that disapproves of fraction 1. We can write $R^{set} \cup \tilde{R}^{set} \cup R^{mon} \cup \{\hat{r}_1, \dots, \hat{r}_5\} \cup \{r\} \in \pi$.

Since $|D_\pi^-| = 1$, $r \in D_\pi^-$, and $r \in S$, any agent in a room in $\pi \setminus \{S\}$ must approve of the fraction of its assigned room. Note that $r \in \{\hat{r}_6\} \cup R^{red}$ as these are the red agents that disapprove of fraction 1. Consider the following cases regarding agent r .

1. $r \in R^{red}$.

Then we have that $\hat{r}_6 \notin S$ and $\hat{r}_6 \in D_\pi^+$. Let $S' \in \pi$ denote the room that contains \hat{r}_6 . Thus, we have that $\theta(S') = \frac{2(5(q+1)+1)}{s}$. We have exactly $2(5(q+1)+1)$ red agents that approve of fraction $\frac{2(5(q+1)+1)}{s}$, namely $\hat{R}^{set} \cup R_{q+1}^{red}$. However 5 agents from \hat{R}^{set} are contained in S , thus S' must contain at least 5 agents that disapprove of fraction $\frac{2(5(q+1)+1)}{s}$. This contradicts $|D_\pi^-| = 1$, therefore this case cannot occur.

2. $r = \hat{r}_6$.

Then for a redundant red agent $r_j^p \in R^{red}$ we have that $r_j^p \in D_\pi^+$. Let $S' \in \pi$ denote the room that contains r_j^p . We have that either $\theta(S') = \frac{2(5j+1)}{s}$ or $\theta(S') = \frac{2(5j-2)}{s}$.

There are exactly $2(5j+1)$ red agents, namely $R_j^{red} \cup \{r_i, \tilde{r}_i | i \in A_j\}$ or $R_j^{red} \cup \hat{R}^{set}$, that approve of fraction $\frac{2(5j+1)}{s}$ of which 6 are contained in room S . Thus, if $\theta(S') = \frac{2(5j+1)}{s}$, then S' must contain 6 red agents that disapprove of fraction $\frac{2(5j+1)}{s}$. This contradicts $|D_\pi^-| = 1$, therefore $\theta(S') = \frac{2(5j-2)}{s}$.

There are exactly $2(5j-2)$ red agents that approve of fraction $\frac{2(5j-2)}{s}$, namely R_j^{red} . Additionally, there are exactly $s - 2(5j-2)$ blue agents that approve of fraction $\frac{2(5j-2)}{s}$, namely $B_j^{add} \cup B_j^{fill}$. Thus, S' must contain exactly $R_j^{red} \cup B_j^{add} \cup B_j^{fill}$, as otherwise S' must contain an

agent that disapprove of fraction $\frac{2(5j-2)}{s}$. Thus, for any $j \in [q+1]$, we have that $R_j^{red} \cup B_j^{add} \cup B_j^{fill} \in \pi$.

Since $r = \hat{r}_6$, we also have that $R^{set} \cup \tilde{R}^{set} \cup R^{mon} \cup \hat{R}^{set} \in \pi$.

There are exactly s remaining blue agents, namely $B^{mon} \cup B^{even}$. Thus, these remaining blue agents must belong to the same room, i.e., $B^{mon} \cup B^{even} \in \pi$. Thus, π must be π_{mon} .

□

Lemma 3.3.3. *If (X, C) has no solution, then for any outcome π such that $\pi \neq \pi_{mon}$ has at least 2 agents assigned to a room with a fraction that it does not approve of. That is, for any π such that $\pi \neq \pi_{mon}$, we have $|D_\pi^-| \geq 2$.*

Proof. To derive a contradiction, assume that there exists an outcome π such that $\pi \neq \pi_{mon}$ that assigns fewer than 2 agents to a room with a fraction that it disapproves of, i.e., assume that there exists an outcome $\pi \neq \pi_{mon}$ such that $|D_\pi^-| < 2$. We have the following cases.

1. $|D_\pi^-| = 0$.

As we have exactly $s - 1$ red agents that approve of fraction 1, the outcome π cannot have a room that only consists of red agents. Thus, a set agent $r_i \in R^{set}$ must be in a room with fraction $\frac{2(5j+1)}{s}$, where $j \in J^i$. Let S denote the room that contains r_i .

There are exactly $2(5j+1)$ red agents that approve of fraction $\frac{2(5j+1)}{s}$, namely $R_j^{red} \cup \{r_i, \tilde{r}_i | i \in A_j\}$. Additionally, there are exactly $s - 2(5j+1)$ blue agents that approve of fraction $\frac{2(5j+1)}{s}$, namely B_j^{fill} . Thus, S must contain the agents $\{r_i, \tilde{r}_i | i \in A_j\} \cup R_j^{red} \cup B_j^{fill}$ where $A_j \in C$, i.e., $\{r_i, \tilde{r}_i | i \in A_j\} \cup R_j^{red} \cup B_j^{fill} \in \pi$.

Let \mathcal{S} denote the set of rooms in π that contain set agents, i.e., $\mathcal{S} = \{S' \in \pi | S' \cap R^{set} \neq \emptyset\}$. Let C' be the family of set agent index sets of the rooms in \mathcal{S} , i.e., $C' = \{\{i | r_i \in S'\} | S' \in \mathcal{S}\}$.

Since for every set agent $r_{i'} \in R^{set}$, the room in π that contains $r_{i'}$ must be of form $\{r_{i'}, \tilde{r}_{i'} | i' \in A_j\} \cup R_j^{red} \cup B_j^{fill}$, we have that C' must be a solution for (X, C) . This contradicts (X, C) not having a solution.

2. $|D_\pi^-| = 1$.

As $\pi \neq \pi_{mon}$, by Remark 3.3.2 we have that π does not contain a room with only red agents. Let us denote the agent in D_π^- by a . We have the following 2 cases regarding a .

2.1. $a \notin R^{set} \cup \tilde{R}^{set}$.

Let $S \in \pi$ denote the room that contains a . Let us consider the following cases regarding S .

2.1.1. $S \cap (R^{set} \cup \tilde{R}^{set} \cup \hat{R}^{set}) \neq \emptyset$.

Then we have that $\theta(S) = \frac{2(5j+1)}{s}$, where $j \in [q+1]$.

Consider the following cases regarding a .

2.1.1.1. $a \in R$.

There are exactly $2(5j+1)$ red agents that approve of fraction $\frac{2(5j+1)}{s}$, namely $\{r_i, \tilde{r}_i | i \in A_j\} \cup R_j^{red}$ or $\hat{R}^{set} \cup R_j^{red}$.

Let us denote the set of red agents that approve of fraction $\frac{2(5j+1)}{s}$ by S^r , i.e. $S^r = \left\{ r \in R \mid \frac{2(5j+1)}{s} \in D_a^+ \right\}$.

Room S must contain exactly $2(5j+1) - 1$ agents from S^r . Let $r \in S^r$ be the agent not contained in S and let $S' \in \pi$ denote the room that contains r . We have that $r \in D_\pi^+$, thus either $\theta(S') = \frac{2(5j+1)}{s}$ or $\theta(S') = \frac{2(5j-2)}{s}$.

If $\theta(S') = \frac{2(5j+1)}{s}$, then S' must contain $2(5j+1) - 1$ red agents that disapprove of fraction $\frac{2(5j+1)}{s}$. There are a total of $2(5j+1)$ red agents that approve of fraction $\frac{2(5j+1)}{s}$ and $2(5j+1) - 1$ of which must be contained in S . This would contradict $|D_\pi^-| = 1$.

If $\theta(S') = \frac{2(5j-2)}{s}$, then S' must contain $2(5j-2) - 1$ red agents that disapprove of fraction $\frac{2(5j-2)}{s}$. There are a total of $2(5j-2)$ red agents that approve of fraction $\frac{2(5j-2)}{s}$, namely R_j^{red} , and $2(5j-2) - 1$ of which must be contained in S . This would contradict $|D_\pi^-| = 1$.

Thus, this case cannot occur.

2.1.1.2. $a \in B$.

Let us denote the set of blue agents that approve of fraction

$\frac{2(5j+1)}{s}$ by S^b , i.e. $S^b = \left\{ b \in B \mid \frac{2(5j+1)}{s} \in D_a^+ \right\} = B_j^{fill}$. We have that $|B_j^{fill}| = s - 2(5j - 2) - 6$.

Room S must contain exactly $s - 2(5j - 2) - 7$ agents from B_j^{fill} . Let $b \in B_j^{fill}$ be the agent not contained in S and let $S' \in \pi$ denote the room that contains b . We have that $b \in D_\pi^+$, thus either $\theta(S') = \frac{2(5j+1)}{s}$ or $\theta(S') = \frac{2(5j-2)}{s}$. If $\theta(S') = \frac{2(5j+1)}{s}$, then S' must contain $2(5j+1)$ red agents that disapprove of fraction $\frac{2(5j+1)}{s}$. There are a total of $2(5j+1)$ red agents that approve of fraction $\frac{2(5j+1)}{s}$ and all $2(5j+1)$ of them must be contained in S . This would contradict $|D_\pi^-| = 1$.

If $\theta(S') = \frac{2(5j-2)}{s}$, then S' must contain $2(5j-2)$ red agents that disapprove of fraction $\frac{2(5j-2)}{s}$. There are a total of $2(5j-2)$ red agents that approve of fraction $\frac{2(5j-2)}{s}$, namely R_j^{red} , and all $2(5j-2)$ of them must be contained in S . This would contradict $|D_\pi^-| = 1$.

Thus, this case cannot occur.

2.1.2. $S \cap (R^{set} \cup \tilde{R}^{set} \cup \hat{R}^{set}) = \emptyset$.

Then every agent in a room that contains a set agent must approve of the fraction of their room, i.e., for any $a' \in R \cup B$, if $\pi(a') \cap R^{set} \neq \emptyset$, then $a' \in D_\pi^+$. We have a similar situation as in Case 1.

Every set agent $r_i \in R^{set}$ must be in a room with fraction $\frac{2(5j+1)}{s}$, where $j \in J^i$. Let $S' \in \pi$ denote the room that contains r_i . Since every agent in S' must approve of fraction $\frac{2(5j+1)}{s}$, room S' must contain exactly $\{r_i, \tilde{r}_i \mid i \in A_j\} \cup R_j^{red} \cup B_j^{fill}$ where $A_j \in C$, i.e., $\{r_i, \tilde{r}_i \mid i \in A_j\} \cup R_j^{red} \cup B_j^{fill} \in \pi$.

Let \mathcal{S} denote the set of rooms in π that contain set agents, i.e., $\mathcal{S} = \{S'' \in \pi \mid S'' \cap R^{set} \neq \emptyset\}$. Let C' be the family of set agent index sets of the rooms in \mathcal{S} , i.e., $C' = \{\{i \mid r_i \in S'\} \mid S' \in \mathcal{S}\}$.

Since for every set agent $r_{i'} \in R^{set}$, the room in π that contains $r_{i'}$ must be of shape $\{r_{i'}, \tilde{r}_{i'} \mid i' \in A_{j'}\} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, we have

that C' must be a solution for (X, C) . This contradicts (X, C) not having a solution.

2.2. $a \in R^{set} \cup \tilde{R}^{set}$.

W.l.o.g. assume that $a \in R^{set}$. As a is a set agent, we can write $a = r_i$. Since $|D_\pi^-| = 1$, we have that \tilde{r}_i must be in a room S that it approves of, i.e., $\tilde{r}_i \in D_\pi^+$. Let us write $\theta(S) = \frac{2(5j+1)}{s}$, where $j \in J^i$. There are exactly s agents that approve of fraction $\frac{2(5j+1)}{s}$, namely $\{r_{i'}, \tilde{r}_{i'} | i' \in A_j\} \cup R_j^{red}$ which includes r_i . Since $r_i \in D_\pi^-$, we have that r_i cannot be in S . Thus, S must contain a red agent r , where $r \neq r_i$, that disapproves of the fraction of fraction $\frac{2(5j+1)}{s}$. Therefore, we have at least 2 agents in D_π^- , i.e., $r_i, r \in D_\pi^-$. This contradicts $|D_\pi^-| = 1$.

□

Lemma 3.3.4. *If (X, C) has no solution, then π_{mon} is strictly popular.*

Proof. Let π be an arbitrary outcome such that $\pi \neq \pi_{mon}$. By Lemma 3.3.3 we have that $|D_\pi^-| \geq 2$. Consider the following 2 cases.

1. $\hat{r}_6 \in D_\pi^-$.

Then we have $N(\pi, \pi_{mon}) = \emptyset$. Since $|D_\pi^-| \geq 2$, there exists an agent $a \in D_\pi^-$ such that $a \neq \hat{r}_6$. We have that $a \in D_{\pi_{mon}}^+$, thus $|N(\pi_{mon}, \pi)| \geq 1$. Therefore, $\phi(\pi_{mon}, \pi) \geq 1$, i.e., π_{mon} is more popular than π .

2. $\hat{r}_6 \notin D_\pi^-$.

Then we have $N(\pi, \pi_{mon}) = \{\hat{r}_6\}$. Since $|D_\pi^-| \geq 2$, there exist agents $a_1, a_2 \in D_\pi^-$ such that $\hat{r}_6 \neq a_1$ and $\hat{r}_6 \neq a_2$. We have that $a_1, a_2 \in D_{\pi_{mon}}^+$, thus $|N(\pi_{mon}, \pi)| \geq 2$. Therefore, $\phi(\pi_{mon}, \pi) \geq 1$, i.e., π_{mon} is more popular than π .

□

Remark 3.3.5. If (X, C) has some solution $C' \subseteq C$, then π_{mon} is not popular. This is due to reduced outcome $\pi_{C'}$ being more popular than π_{mon} .

Proof. The reduced outcome $\pi_{C'}$ is more popular than π_{mon} . All the agents are assigned a room with a fraction that it approves of by $\pi_{C'}$, i.e., for any

$a \in R \cup B$, $\theta(\pi_{C'}(a)) \in D_a^+$. Thus, we have that $N(\pi_{mon}, \pi_{C'}) = \emptyset$ as no agent can be improved. However, agent \hat{r}_6 is better off in $\pi_{C'}$, i.e., $\hat{r}_6 \in D_{\pi_{mon}}^-$ and $\hat{r}_6 \in D_{\pi_{C'}}^+$. Therefore, $N(\pi_{C'}, \pi_{mon}) = \{\hat{r}_6\}$ which means that $\pi_{C'}$ is more popular than π_{mon} . Thus, π_{mon} is not popular. \square

Theorem 3.3.6. *A mixed popular outcome for a roommate diversity game cannot be computed in polynomial time, even if the preferences are dichotomous, unless $P=NP$.*

Proof. From Lemma 3.3.4 we have that π_{mon} is strictly popular, if (X, C) has no solution. From Remark 3.3.5 we have that π_{mon} is not popular, if (X, C) has a solution. Thus, if we were able to compute a mixed popular outcome in polynomial time, then we would be able to verify whether (X, C) has a solution in polynomial time by checking whether the computation yields mixed outcome $\{(\pi_{mon}, 1)\}$. If $\{(\pi_{mon}, 1)\}$ is the mixed popular outcome, then (X, C) has no solution. Otherwise (X, C) has a solution. As the X3C problem is NP-complete, this would imply that $P=NP$. \square

3.4 Popularity

In this section, we show that a popular outcome is not guaranteed to exist in a roommate diversity game. We provide a roommate diversity game in which no popular outcome exist. To show that determining the existence of a popular outcome is co-NP-Hard, we present a reduction from the X3C problem. The reduction is similar to those in the previous sections.

Let $R = \{r_1, r_2, r_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5, b_6\}$. Consider the roommate diversity game $\bar{G} = (R, B, 3, (\succ_a)_{a \in R \cup B})$ with their trichotomous preference profiles described in Table 3.3.

Let us define the rooms

$$P_1 = \{r_1, \hat{b}_1, \hat{b}_2\}; \quad P_2 = \{r_2, r_3, \hat{b}_3\}; \quad P_3 = \{b_5, b_6, \hat{b}_4\},$$

where $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4 \in \{b_1, b_2, b_3, b_4\}$. An outcome π_{top} is called a top-type outcome if we can write $\pi_{top} = \{P_1, P_2, P_3\}$.

Agent	Preference Profile		
	D_a^+	D_a^n	D_a^-
$a = r_1$ $a \in \{r_2, r_3\}$	$\left\{\frac{1}{3}\right\}$ $\left\{\frac{2}{3}\right\}$		$\left\{\frac{2}{3}, \frac{3}{3}\right\}$ $\left\{\frac{1}{3}, \frac{3}{3}\right\}$
$a \in \{b_1, b_2, b_3, b_4\}$ $a \in \{b_5, b_6\}$	$\left\{\frac{1}{3}\right\}$ $\left\{\frac{0}{3}\right\}$	$\left\{\frac{2}{3}\right\}$	$\left\{\frac{0}{3}\right\}$ $\left\{\frac{1}{3}, \frac{2}{3}\right\}$

Table 3.3: Preference profile $(\succsim_a)_{a \in RUB}$.

Lemma 3.4.1. *For any outcome π of \overline{G} , we have that there are at least 2 agents in a room with a fraction that it does not approve of. That is, for any outcome π of \overline{G} , $|D_\pi^n \cup D_\pi^-| \geq 2$.*

Proof. To derive a contradiction, assume that there exists an outcome π of \overline{G} such that $|D_\pi^n \cup D_\pi^-| < 2$. Consider the following cases.

1. $|D_\pi^n \cup D_\pi^-| = 0$.

Then $b_1, b_2, b_3, b_4 \in D_\pi^+$, therefore each of b_1, b_2, b_3, b_4 must be in a room with fraction $\frac{1}{3}$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each of b_1, b_2, b_3, b_4 is contained in a room with fraction $\frac{1}{3}$. There is exactly 1 red agent that approves of fraction $\frac{1}{3}$, namely r_1 . Thus, S_1 or S_2 must contain a red agent r that does not approve of fraction $\frac{1}{3}$. Thus, $|D_\pi^n \cup D_\pi^-| \geq 1$ in this case as $r \in D_\pi^-$. This contradicts $|D_\pi^n \cup D_\pi^-| = 0$.

2. $|D_\pi^n \cup D_\pi^-| = 1$.

Let us write $D_\pi^n \cup D_\pi^- = \{a\}$. Consider the following cases regarding a .

- 2.1. $a = r_1$.

Then we have that $b_1, b_2, b_3, b_4 \in D_\pi^+$, therefore each of b_1, b_2, b_3, b_4 must be in a room with fraction $\frac{1}{3}$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each of b_1, b_2, b_3, b_4 is contained in a room with fraction $\frac{1}{3}$. There is exactly 1 red agent that approves of fraction $\frac{1}{3}$, namely r_1 . Thus, S_1 and S_2 both must contain a red agent $r, r' \in R$ that do not approve of fraction $\frac{1}{3}$. Thus, $|D_\pi^n \cup D_\pi^-| \geq 2$ in this case as $r, r' \in D_\pi^-$. This contradicts $|D_\pi^n \cup D_\pi^-| = 1$.

2.2. $a \in \{r_2, r_3\}$.

W.l.o.g. assume that $a = r_2$. We have that $r_3 \in D_\pi^+$, thus the room S that contains r_3 must be of fraction $\frac{2}{3}$. There are exactly 2 agents that approve of fraction $\frac{2}{3}$, namely r_2 and r_3 . Since $r_2 \notin S$, as $r_2 \in D_\pi^-$, the room S must contain a red agent $r \neq r_2$ such that $r \in D_\pi^-$. Thus, $|D_\pi^n \cup D_\pi^-| \geq 2$ in this case as $r_2, r \in D_\pi^-$. This contradicts $|D_\pi^n \cup D_\pi^-| = 1$.

2.3. $a \in \{b_1, b_2, b_3, b_4\}$.

Then we have for each agent $b \in \{b_1, b_2, b_3, b_4\} \setminus \{a\}$, we must have that $b \in D_\pi^+$, therefore $\theta(\pi(b)) = \frac{1}{3}$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each agent in $\{b_1, b_2, b_3, b_4\} \setminus \{a\}$ is contained in a room with fraction $\frac{1}{3}$. There is exactly 1 red agent that approves of fraction $\frac{1}{3}$, namely r_1 . Thus, S_1 or S_2 must contain a red agent r that does not approve of fraction $\frac{1}{3}$. Note that $a \neq r$ as a must be a blue agent. Thus, $|D_\pi^n \cup D_\pi^-| \geq 2$ in this case as $r \in D_\pi^-$ and $a \in D_\pi^n \cup D_\pi^-$. This contradicts $|D_\pi^n \cup D_\pi^-| = 1$.

2.4. $a \in \{b_5, b_6\}$.

Then $b_1, b_2, b_3, b_4 \in D_\pi^+$, therefore each of b_1, b_2, b_3, b_4 must be in a room with fraction $\frac{1}{3}$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each of b_1, b_2, b_3, b_4 is contained in a room with fraction $\frac{1}{3}$. There is exactly 1 red agent that approves of fraction $\frac{1}{3}$, namely r_1 . Thus, S_1 or S_2 must contain a red agent r that does not approve of fraction $\frac{1}{3}$. Note that $r \neq a$ as a must be a blue agent. Thus, $|D_\pi^n \cup D_\pi^-| \geq 2$ in this case as $r \in D_\pi^-$ and $a \in D_\pi^n \cup D_\pi^-$. This contradicts $|D_\pi^n \cup D_\pi^-| = 1$.

□

Lemma 3.4.2. *For any outcome π of \overline{G} , if $|D_\pi^n| = 1$ and $|D_\pi^-| = 1$, then π is a top-type outcome.*

Proof. Let π be an outcome such that $|D_\pi^n| = 1$ and $|D_\pi^-| = 1$. Let us write $D_\pi^n = \{a^n\}$ and $|D_\pi^-| = \{a^-\}$. Since the agents in $\{b_1, b_2, b_3, b_4\}$ are the only agents with a non-empty corresponding D_a^n , we have that $a^n \in \{b_1, b_2, b_3, b_4\}$.

To derive a contradiction, assume that $a^- \notin \{b_1, b_2, b_3, b_4\}$. Then we have that $a^- \in \{r_1, r_2, r_3, b_5, b_6\}$. Let us consider the following cases.

1. $a^- = r_1$.

Then we have that each agent $b \in \{b_1, b_2, b_3, b_4\} \setminus \{a^n\}$ must be in a room that it approves of, i.e., $b \in D_\pi^+$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each agent in $\{b_1, b_2, b_3, b_4\} \setminus \{a^n\}$ is contained in a room with fraction $\frac{1}{3}$. There is exactly 1 red agent that approves of fraction $\frac{1}{3}$, namely r_1 . Thus, S_1 and S_2 both must contain a red agent $r, r' \in R$ that do not approve of fraction $\frac{1}{3}$. Thus, $|D_\pi^-| \geq 2$ in this case as $r, r' \in D_\pi^-$. This contradicts $|D_\pi^-| = 1$.

2. $a^- \in \{r_2, r_3\}$.

W.l.o.g. assume that $a = r_2$. We have that $r_3 \in D_\pi^+$, thus the room S that contains r_3 must be of fraction $\frac{2}{3}$. There are exactly 2 agents that approve of fraction $\frac{2}{3}$, namely r_2 and r_3 . Since $r_2 \notin S$, as $r_2 \in D_\pi^-$, the room S must contain a red agent $r \neq r_2$ such that $r \in D_\pi^-$. Thus, $|D_\pi^-| \geq 2$ in this case as $r_2, r \in D_\pi^-$. This contradicts $|D_\pi^-| = 1$.

3. $a^- \in \{b_5, b_6\}$.

Then $b_1, b_2, b_3, b_4 \in D_\pi^+$, therefore each of b_1, b_2, b_3, b_4 must be in a room with fraction $\frac{1}{3}$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each of b_1, b_2, b_3, b_4 is contained in a room with fraction $\frac{1}{3}$. There is exactly 1 red agent that approves of fraction $\frac{1}{3}$, namely r_1 . Thus, S_1 or S_2 must contain a red agent r that does not approve of fraction $\frac{1}{3}$. Note that $r \neq a^-$ as a^- must be a blue agent. Thus, $|D_\pi^-| \geq 2$ in this case as $r, a^- \in D_\pi^-$. This contradicts $|D_\pi^-| = 1$.

□

Remark 3.4.3. For any outcome π of \overline{G} , we have that $|D_\pi^-| \geq 1$.

Proof. Assume that $|D_\pi^-| = 0$, then we have that $b_5, b_6 \in D_\pi^+$. Therefore, there must exist a room S with fraction $\frac{0}{3}$. There are exactly 2 blue agents that approve of fraction $\frac{0}{3}$, the remaining blue agents disapprove of fraction $\frac{0}{3}$. Thus, S must contain a blue agent that disapproves of the fraction of its assigned room. Therefore, $|D_\pi^-| = 0$ cannot hold. □

Remark 3.4.4. For any outcome π of \overline{G} , if $|D_\pi^n \cup D_\pi^-| = 2$, then $D_\pi^n \cup D_\pi^- \subseteq \{b_1, b_2, b_3, b_4\}$.

Proof. Let π be an outcome of \overline{G} such that $|D_\pi^n \cup D_\pi^-| = 2$ and let us write $D_\pi^n \cup D_\pi^- = \{a_1, a_2\}$. To derive a contradiction, assume that $D_\pi^n \cup D_\pi^- \not\subseteq \{b_1, b_2, b_3, b_4\}$. Consider the following cases.

1. $|(D_\pi^n \cup D_\pi^-) \cap \{b_1, b_2, b_3, b_4\}| = 1$.

W.l.o.g. assume that $b_4 \in D_\pi^n \cup D_\pi^-$. We have that $b_1, b_2, b_3 \in D_\pi^+$, therefore each of b_1, b_2, b_3 must be in a room with fraction $\frac{1}{3}$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each of b_1, b_2, b_3 is contained in a room with fraction $\frac{1}{3}$.

One of S_1 or S_2 must contain the red agent $r \in \{r_2, r_3\}$. Additionally one of S_1 or S_2 must contain the blue agent $b \in \{b_5, b_6\}$. Neither agent r nor agent b approves of fraction $\frac{1}{3}$, thus we have that $b_4, r, b \in D_\pi^n \cup D_\pi^-$. This contradicts $|D_\pi^n \cup D_\pi^-| = 2$. Therefore, this case cannot occur.

2. $|(D_\pi^n \cup D_\pi^-) \cap \{b_1, b_2, b_3, b_4\}| = 0$.

Then $b_1, b_2, b_3, b_4 \in D_\pi^+$, therefore each of b_1, b_2, b_3, b_4 must be in a room with fraction $\frac{1}{3}$. We require at least 2 rooms S_1, S_2 with fraction $\frac{1}{3}$ so that each of b_1, b_2, b_3, b_4 is contained in a room with fraction $\frac{1}{3}$.

Assume that neither S_1 nor S_2 contain r_1 . We have that $D_\pi^n \cup D_\pi^- = \{r_2, r_3\}$, thus $r_1 \in D_\pi^+$. The room S_3 that contains r_1 must contain 2 blue agents b_5, b_6 which disapprove of fraction $\frac{1}{3}$. We would have that $D_\pi^n \cup D_\pi^- = \{r_2, r_3, b_5, b_6\}$, which contradicts $|D_\pi^n \cup D_\pi^-| = 2$. Thus, one of S_1 or S_2 must contain r_1 .

W.l.o.g. assume that r_2 is also contained in S_1 or S_2 . We have 3 remaining agents, namely r_3, b_5, b_6 , thus they must belong in the same room. All 3 agents disapprove of fraction $\frac{1}{3}$, thus we have that $|D_\pi^n \cup D_\pi^-| \geq 3$. This contradicts $|D_\pi^n \cup D_\pi^-| = 2$, thus this case cannot occur.

□

Lemma 3.4.5. *Let π be an outcome that is not a top-type outcome. There exist a top-type outcome π' that is more popular than π .*

Proof. By Lemma 3.4.1 we have that $|D_\pi^n \cup D_\pi^-| \geq 2$. By Lemma 3.4.2 we have that $|D_\pi^n| \neq 1$ or $|D_\pi^-| \neq 1$. Let us consider the following cases.

1. $|D_\pi^-| \neq 1$.

By Remark 3.4.3 and case assumption we have that $|D_\pi^-| \geq 2$. Consider the following cases.

1.1. $|D_\pi^-| = 2$.

Consider the following cases.

1.1.1. $|D_\pi^n| = 0$.

By Remark 3.4.4 we have that $D_\pi^- \subseteq \{b_1, b_2, b_3, b_4\}$. Let us write $D_\pi^- = \{a_1, a_2\}$. Since $a_1, a_2 \in \{b_1, b_2, b_3, b_4\}$, there exists a top-type outcome π' such that $D_{\pi'}^- = \{a_1\}$ and $D_{\pi'}^n = \{a_2\}$. Thus, we have that $\phi(\pi', \pi) = 1$.

1.1.2. $|D_\pi^n| \geq 1$.

Let us write $D_\pi^- = \{a_1, a_2\}$ and $a_3 \in D_\pi^n$. We have that $a_3 \in \{b_1, b_2, b_3, b_4\}$, since the agents in $\{b_1, b_2, b_3, b_4\}$ are the only agents with a corresponding non-empty D_a^n .

Let π' be a top-type outcome such that $\{a_3\} = D_{\pi'}^n$ and $\{a^-\} = D_{\pi'}^-$. Note that $N(\pi, \pi') \subseteq \{a^-\}$. Consider the following cases.

1.1.2.1. $N(\pi, \pi') = \emptyset$.

Then we have that $a^- \in D_\pi^-$. W.l.o.g. assume that $a_1 = a^-$. We have that $a_2 \in D_{\pi'}^+$, thus $a_2 \in N(\pi', \pi)$. Thus, we have that $\phi(\pi', \pi) > 0$, i.e., π' is more popular than π .

1.1.2.2. $N(\pi, \pi') = \{a^-\}$.

Then we have that $a^- \in D_\pi^n \cup D_\pi^+$ and $a_1, a_2 \in D_{\pi'}^+$. Thus, $a_1, a_2 \in N(\pi', \pi)$. Therefore, we have that $\phi(\pi', \pi) > 0$, i.e., π' is more popular than π .

1.2. $|D_\pi^-| > 2$.

Let us write $a_1, a_2, a_3 \in D_\pi^-$. Let π' be an arbitrary top-type outcome with $\{a^n\} = D_{\pi'}^n$ and $\{a^-\} = D_{\pi'}^-$. Note that $N(\pi, \pi') \subseteq \{a^n, a^-\}$. Consider the following cases.

1.2.1. $a^- \in D_\pi^-$.

W.l.o.g. assume that $a_1 = a^-$. In this case, we have that $N(\pi, \pi') \subseteq \{a^n\}$. We have that $a_2, a_3 \in N(\pi', \pi)$. Thus, $\phi(\pi', \pi) > 0$.

1.2.2. $a^- \notin D_\pi^-$.

In this case, we have that $N(\pi, \pi') \subseteq \{a^-, a^n\}$. We have that $a_1, a_2, a_3 \in N(\pi', \pi)$. Thus, $\phi(\pi', \pi) > 0$.

2. $|D_\pi^n| \neq 1$.

Consider the following cases.

2.1. $|D_\pi^n| = 0$.

By Lemma 3.4.1, we have that $|D_\pi^-| \geq 2$. This case is proven similarly to case 1.1.1. as we have the exact same case assumption, i.e., $|D_\pi^n| = 0$ and $|D_\pi^-| \geq 2$.

2.2. $|D_\pi^n| \geq 2$.

Let us write $a_1, a_2 \in D_\pi^n$. We have that $\{a_1, a_2\} \subseteq \{b_1, b_2, b_3, b_4\}$, since the agents in $\{b_1, b_2, b_3, b_4\}$ are the only agents with a corresponding non-empty D_a^n . By Remark 3.4.3, we have that $|D_\pi^-| \geq 1$. Let us write $a_3 \in D_\pi^-$. Let us consider the following cases regarding a_3 .

2.2.1. $a_3 \in \{b_1, b_2, b_3, b_4\}$.

By the definition of top-type outcome, we can construct a top-type outcome π' such that $D_{\pi'}^- = \{a_3\}$, $D_{\pi'}^n = \{a_2\}$, and $a_1 \in D_{\pi'}^+$.

We have that $N(\pi, \pi') = \emptyset$ and $|N(\pi', \pi)| \geq 1$. Therefore, $\phi(\pi', \pi) > 0$.

2.2.2. $a_3 \notin \{b_1, b_2, b_3, b_4\}$.

By the definition of top-type outcome, we can construct a top-type outcome π' such that $D_{\pi'}^- = \{a^-\}$, $D_{\pi'}^n = \{a_1\}$, and $a_2, a_3 \in D_{\pi'}^+$.

We have that $N(\pi, \pi') \subseteq \{a^-\}$. Additionally we have that $a_2, a_3 \in N(\pi', \pi)$, therefore $\phi(\pi', \pi) > 0$.

□

Lemma 3.4.6. *A top-type outcome π is not popular.*

Proof. Let π be a top-type outcome. We can write

$$\pi = \{P_1, P_2, P_3\} = \left\{ \left\{ r_1, \hat{b}_1, \hat{b}_2 \right\}, \left\{ r_2, r_3, \hat{b}_3 \right\}, \left\{ b_5, b_6, \hat{b}_4 \right\} \right\},$$

where $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4 \in \{b_1, b_2, b_3, b_4\}$. Let us construct outcome π' as follows:

$$\pi' = \left\{ P_1 \setminus \left\{ \hat{b}_2 \right\} \cup \left\{ \hat{b}_3 \right\}, P_2 \setminus \left\{ \hat{b}_3 \right\} \cup \left\{ \hat{b}_4 \right\}, P_3 \setminus \left\{ \hat{b}_4 \right\} \cup \left\{ \hat{b}_2 \right\} \right\}.$$

We have that $N(\pi, \pi') = \left\{ \hat{b}_2 \right\}$ and $N(\pi', \pi) = \left\{ \hat{b}_3, \hat{b}_4 \right\}$. Thus, π' is more popular than π . \square

Theorem 3.4.7. *A popular outcome is not guaranteed to exist in a roommate diversity game \overline{G} .*

Proof. From Lemma 3.4.5 and 3.4.6 we have that no popular outcome exists in \overline{G} . Thus, not every roommate diversity game permits a popular outcome. \square

The idea of the co-NP-hardness reduction is to combine the reduction of Section 3.3 with the proof of a popular outcome not being guaranteed to exist. In this reduction, we have that a popular outcome exists if and only if (X, C) has no solution.

3.4.1 Instance of the Roommate Diversity Problem

We set the room size to $s = 2(5(q+4)+1) + 2m + 3 = 10q + 45 + 2m$. The set of red agents is given by $R = \hat{R}^{set} \cup R^{set} \cup \tilde{R}^{set} \cup R^{mon} \cup R^{red}$, where

$$\begin{aligned} \text{Circular Set Agents} & \quad \hat{R}^{set} = \{\hat{r}_1, \hat{r}_2, \hat{r}_3\} \\ \text{Circular Redundant Agents} & \quad R_j^{red} = \{r_j^1, \dots, r_j^{5j-2}\} \text{ for } j \in [3] \\ \text{Set Agents} & \quad R^{set} = \{r_1, \dots, r_m\} \\ \text{Copy Set Agents} & \quad \tilde{R}^{set} = \{\tilde{r}_1, \dots, \tilde{r}_m\} \\ \text{Redundant Agents} & \quad R^{red} = R_1^{red} \cup \dots \cup R_{q+3}^{red} \\ & \quad \text{with } R_j^{red} = \{r_j^1, \dots, r_j^{2(5j-2)}\} \text{ for } j \in [4, q+3] \\ \text{Monolith Agents} & \quad R^{mon} = \{r_{mon}^1, \dots, r_{mon}^{s-2m-3}\} \end{aligned}$$

The set of blue agents is given by $B = B^{even} \cup B^{mon} \cup B^{add} \cup B^{fill}$, where

$$\begin{aligned}
\text{Circular Filling Agents } B_j^{fill} &= \{b_j^1, \dots, b_j^{s-(5j-2)-1}\} \text{ for } j \in [3] \\
\text{Circular Additional Agents } B_j^{add} &= \{\tilde{b}_j\} \text{ for } j \in [3] \\
\text{Filling Agents } B^{fill} &= B_1^{fill} \cup \dots \cup B_{q+3}^{fill} \\
&\text{with } B_j^{fill} = \{b_j^1, \dots, b_j^{s-2(5j-2)-6}\} \text{ for } j \in [4, q+3] \\
\text{Additional Agents } B^{add} &= B_1^{add} \cup \dots \cup B_{q+3}^{add} \\
&\text{with } B_j^{add} = \{\tilde{b}_j^1, \tilde{b}_j^2, \tilde{b}_j^3, \tilde{b}_j^4, \tilde{b}_j^5, \tilde{b}_j^6\} \text{ for } j \in [4, q+3] \\
\text{Monolith Agents } B^{mon} &= \{b_{mon}^1, \dots, b_{mon}^{2m+3}\} \\
\text{Evening Agents } B^{even} &= \{b_{even}^1, \dots, b_{even}^{s-2m-3}\}
\end{aligned}$$

The preference profiles of the agents are defined in Table 3.4.

Agent	Preference Profile			
	D_a^+	D_a^n	D_a^-	
$a \in \{\hat{r}_1, \hat{r}_2\}$	$\left\{\frac{5 \cdot (3) - 1}{s}\right\}$	$\left\{\frac{5 \cdot (2) - 1}{s}\right\}$	$D \setminus (D_a^+ \cup D_a^n)$	$j \in [2]$
$a \in \hat{r}_3$	$\left\{\frac{5 \cdot (3) - 1}{s}, 1\right\}$	$\left\{\frac{5 \cdot (2) - 1}{s}\right\}$	$D \setminus (D_a^+ \cup D_a^n)$	
$a = r_j^p \in R_j^{red}$	$\left\{\frac{5j-1}{s}, \frac{5j-2}{s}\right\}$		$D \setminus D_a^+$	
$a = r_3^p \in R_3^{red}$	$\left\{\frac{5 \cdot (3) - 1}{s}, \frac{5 \cdot (3) - 2}{s}\right\}$	$\left\{\frac{5 \cdot (2) - 1}{s}\right\}$	$D \setminus (D_a^+ \cup D_a^n)$	
$a = r_i \in R^{set}$	$\left\{\frac{2(5\tilde{j}_i+1)}{s}, \dots, \frac{2(5\tilde{j}_{m_i}+1)}{s}\right\} \cup \{1\}$		$D \setminus D_a^+$	$\tilde{j}_p^i = j_p^i + 3$
$a = \tilde{r}_i \in \tilde{R}^{set}$	$\left\{\frac{2(5\tilde{j}_i+1)}{s}, \dots, \frac{2(5\tilde{j}_{m_i}+1)}{s}\right\} \cup \{1\}$		$D \setminus D_a^+$	$\tilde{j}_p^i = j_p^i + 3$
$a = r_j^p \in R_j^{red}$	$\left\{\frac{2(5j+1)}{s}, \frac{2(5j-2)}{s}\right\}$		$D \setminus D_a^+$	$j \in [4, q+3]$
$a = r_{mon}^p \in R^{mon}$	$\left\{1, \frac{s-2m-3}{s}\right\}$		$D \setminus D_a^+$	
$a = b_j^p \in B_j^{fill}$	$\left\{\frac{5j-1}{s}, \frac{5j-2}{s}\right\}$		$D \setminus D_a^+$	$j \in [3]$
$a = \tilde{b}_j \in B_j^{add}$	$\left\{\frac{5j-2}{s}, 0\right\}$		$D \setminus D_a^+$	$j \in [3]$
$a = b_j^p \in B_j^{fill}$	$\left\{\frac{2(5j+1)}{s}, \frac{2(5j-2)}{s}\right\}$		$D \setminus D_a^+$	$j \in [4, q+3]$
$a = \tilde{b}_j^p \in B_j^{add}$	$\left\{\frac{2(5j-2)}{s}, 0\right\}$		$D \setminus D_a^+$	$j \in [4, q+3]$
$a = b_{mon}^p \in B^{mon}$	$\left\{\frac{s-2m-3}{s}, 0\right\}$		$D \setminus D_a^+$	
$a = b_{even}^p \in B^{even}$	$\{0\}$		$D \setminus D_a^+$	

Table 3.4: Preference profile $(\succsim_a)_{a \in RUB}$.

3.4.2 Predefined Outcomes

Monolithic Outcome.

First we define the rooms that contain red agents. Let

$$P_j = B_j^{add} \cup R_j^{red} \cup B_j^{fill} \text{ for } j \in [q+3]; \quad P_{q+4} = R^{set} \cup \tilde{R}^{set} \cup \hat{R}^{set} \cup R^{mon}.$$

Let π_R and π_B be defined in a similar manner as in Section 3.2.2, i.e., $\pi_R = \bigcup_{j \in [q+4]} \{P_j\}$ and π_B contains 1 room with the blue agents not contained in any P_j , where $j \in [q+4]$. We define the monolithic outcome to be $\pi_{mon} = \pi_R \cup \pi_B$. Only agents \hat{r}_1, \hat{r}_2 disapprove of the fraction of its assigned room. The others approve of the fraction of their assigned room. A monolithic outcome always exists.

Reduced-type Outcome.

Let the solution $C' \subseteq C$ partition X and consider an arbitrary subset $\{a_1, \dots, a_5\} \subseteq \hat{R}^{set} \cup R_3^{red}$. We define the rooms that contain red agents. For $j \in [q]$ and for $j' \in [2]$, let

$$\begin{aligned} P_{j'} &= \{a_{j'}\} \cup R_{j'}^{red} \cup B_{j'}^{fill} \\ P_3 &= \{a_3, a_4, a_5\} \cup (R_3^{red} \setminus \{a_1, \dots, a_5\}) \cup B_3^{fill}; \\ P'_{j+3} &= \begin{cases} \{r_i \in R^{set} | i \in A_j\} \cup \{\tilde{r}_i \in \tilde{R}^{set} | i \in A_j\} \cup R_{j+3}^{red} \cup B_{j+3}^{fill} & , \text{ if } A_j \in C' \\ B_{j+3}^{add} \cup R_{j+3}^{red} \cup B_{j+3}^{fill} & , \text{ if } A_j \notin C' \end{cases} \\ P'_{q+4} &= R^{mon} \cup B^{mon}. \end{aligned}$$

Let π_R and π_B be defined in a similar manner as in Section 3.2.2, i.e., $\pi_R = \bigcup_{j \in [q+4]} \{P'_j\}$ and π_B contains 1 room with the blue agents not contained in any P'_j , where $j \in [q+4]$. We define the reduced-type outcome to be $\pi_{C'} = \pi_R \cup \pi_B$. We have that $D_{\pi_{C'}}^n = \{a_2\}$ and $D_{\pi_{C'}}^- = \{a_1\}$. A reduced-type outcome exists if and only if (X, C) has a solution.

3.4.3 Hardness

We demonstrate co-NP-hardness of determining the existence of a popular outcome in a roommate diversity game by applying similar proof strategies as in our previous proofs. In our reduction, the roommate diversity game has a popular outcome if and only if (X, C) has no solution.

(X, C) has NO solution.

Lemma 3.4.8. *Let π be an outcome of the roommate diversity game G such that it contains a room with only red agents. If $|D_\pi^-| = 2$ and $|D_\pi^n| = 0$, then $\pi = \pi_{mon}$.*

Proof. Let π be an outcome of the roommate diversity game G such that it contains a room S^r with only red agents and $|D_\pi^-| = 2$ and $|D_\pi^n| = 0$. Since there are exactly $s - 2$ red agents that approve of fraction 1, S^r must contain \hat{r}_3 , R^{set} , \tilde{R}^{set} , R^{mon} , and 2 other red agents that disapprove of fraction 1. Thus, we have that $|S^r \cap D_\pi^-| = 2$, i.e., all the agents that disapprove of the fraction of their room are contained in S^r .

Since all the blue agents must be in a room in which every agent approves of the fraction of their assigned room, i.e. $\forall_{b \in B} : \forall_{a \in \pi(b)} : \theta(\pi(a)) \in D_a^+$, for $j \in [3, q + 3]$ we must have $B_j^{add} \cup R_j^{red} \cup B_j^{fill} \in \pi$, as $b_j^p \in B_j^{fill}$ cannot be in a room with fraction $\frac{2(5j+1)}{s}$, since there are strictly less than s remaining agents that approve of fraction $\frac{2(5j+1)}{s}$.

Note that S^r cannot contain redundant agents from R_j^{red} , where $j \in [2]$, as otherwise a redundant agent r_j^p not in S^r , i.e. $r_j^p \in R_j^{red} \setminus S^r$, cannot be in a room that only contains agents that approve of the fraction of their assigned room. This would mean that we have $|D_\pi^-| > 2$. The only other remaining red agents are \hat{r}_1 and \hat{r}_2 , thus they must be contained in S^r . Therefore, $S^r = \hat{R}^{set} \cup R^{set} \cup \tilde{R}^{set} \cup R^{mon} \in \pi$.

Since all the remaining agents must be in a room with a fraction that they approve of, for $j \in [2]$ we must have $B_j^{add} \cup R_j^{red} \cup B_j^{fill} \in \pi$, and $B^{even} \cup B^{mon}$. Thus, we have that $\pi = \pi_{mon}$. \square

Lemma 3.4.9. *If (X, C) has no solution, then for any outcome π of G has at least 2 agents in a room that it disapproves of. That is, for any π , $|D_\pi^-| \geq 2$.*

Proof. To derive a contradiction, assume that there exists an outcome π that assigns fewer than 2 agents to a room that it disapproves of, i.e., assume that there exists an outcome π such that $|D_\pi^-| < 2$. Thus, we have the following 2 cases.

1. $|D_\pi^-| = 0$.

As we have exactly $s - 2$ red agents that approve of fraction 1, the outcome π cannot have a room that only consists of red agents. Thus, every set agent $r_i \in R^{set}$ must be in a room with fraction $\frac{2(5j+1)}{s}$, where $j - 3 \in J^i$. However, this means that we can extract a solution C' for (X, C) from π . This contradicts (X, C) not having a solution.

2. $|D_\pi^-| = 1$.

As we have exactly $s - 2$ red agents that approve of fraction 1 we have that π does not contain a room with only red agents. Let us denote the agent in D_π^- by a . We have the following 2 cases regarding a .

- 2.1. $a \notin R^{set} \cup \tilde{R}^{set}$.

Then we have a similar situation as in Case 1. Every set agent $r_i \in R^{set}$ must be in a room with fraction $\frac{2(5j+1)}{s}$, where $j - 3 \in J^i$. This means that we can extract a solution C' for (X, C) from π . Thus, this case contradicts (X, C) not having a solution.

- 2.2. $a \in R^{set} \cup \tilde{R}^{set}$.

W.l.o.g. assume that $a \in R^{set}$. As a is a set agent, we can write $a = r_i$. Since there is exactly 1 agent in a room with a fraction that it disapproves of, \tilde{r}_i must be in a room S with a fraction that it approves of. Let us write $\theta(S) = \frac{2(5j+1)}{s}$, where $j - 3 \in J^i$. There are exactly s agents that approve of fraction $\frac{2(5j+1)}{s}$, including r_i . Since $r_i \in D_\pi^-$, we have that r_i cannot be in S . Thus, S must contain a red agent other than r_i that disapproves of the fraction of its room. Therefore, we have at least 2 agents in D_π^- . This contradicts $|D_\pi^-| = 1$.

□

Lemma 3.4.10. *If (X, C) has no solution, then the monolithic outcome π_{mon} is popular in the roommate diversity game G , i.e., for any outcome π we have that $|N(\pi_{mon}, \pi)| \geq |N(\pi, \pi_{mon})|$.*

Proof. Since $D_{\pi_{mon}}^- = \{\hat{r}_1, \hat{r}_2\}$ and $D_{\pi_{mon}}^n = \emptyset$, for any outcome $\pi \neq \pi_{mon}$ we have that $N(\pi, \pi_{mon}) \subseteq \{\hat{r}_1, \hat{r}_2\}$. Let us consider the following cases regarding $N(\pi, \pi_{mon})$.

1. $N(\pi, \pi_{mon}) = \emptyset$.

Then $|N(\pi_{mon}, \pi)| \geq |N(\pi, \pi_{mon})|$ trivially holds as $|N(\pi, \pi_{mon})| = 0$.

2. $N(\pi, \pi_{mon}) = \{\hat{r}_1\}$.

Then we have that $\hat{r}_1 \in D_{\pi}^+ \cup D_{\pi}^n$ and $\hat{r}_2 \in D_{\pi}^-$. By Lemma 3.4.9, we have that $|D_{\pi}^-| \geq 2$. Therefore, there exists $a \in D_{\pi}^-$ such that $a \neq \hat{r}_2$. Since $a \in D_{\pi}^-$ we also have $a \neq \hat{r}_1$. Thus, $a \in D_{\pi_{mon}}^+$ as $D_{\pi_{mon}}^n = \emptyset$ and $D_{\pi_{mon}}^- = \{\hat{r}_1, \hat{r}_2\}$. Since $a \in D_{\pi_{mon}}^+$ and $a \in D_{\pi}^-$, we have $a \in N(\pi_{mon}, \pi)$. Therefore, $|N(\pi_{mon}, \pi)| \geq 1 = |N(\pi, \pi_{mon})|$.

3. $N(\pi, \pi_{mon}) = \{\hat{r}_2\}$.

Analogous to case 2. We have that $\hat{r}_2 \in D_{\pi}^+ \cup D_{\pi}^n$ and $\hat{r}_1 \in D_{\pi}^-$. By Lemma 3.4.9, we have that $|D_{\pi}^-| \geq 2$. Therefore, there exists $a \in D_{\pi}^-$ such that $a \neq \hat{r}_1$. Since $a \in D_{\pi}^-$ we also have $a \neq \hat{r}_2$. Thus, $a \in D_{\pi_{mon}}^+$ as $D_{\pi_{mon}}^n = \emptyset$ and $D_{\pi_{mon}}^- = \{\hat{r}_1, \hat{r}_2\}$. Since $a \in D_{\pi_{mon}}^+$ and $a \in D_{\pi}^-$, we have $a \in N(\pi_{mon}, \pi)$. Therefore, $|N(\pi_{mon}, \pi)| \geq 1 = |N(\pi, \pi_{mon})|$.

4. $N(\pi, \pi_{mon}) = \{\hat{r}_1, \hat{r}_2\}$.

Then we have that $\hat{r}_1, \hat{r}_2 \in D_{\pi}^+ \cup D_{\pi}^n$. By Lemma 3.4.9, we have that $|D_{\pi}^-| \geq 2$. Therefore, there exist $a_1, a_2 \in D_{\pi}^-$ such that $a_1, a_2 \notin \{\hat{r}_1, \hat{r}_2\}$. Thus, $a_1, a_2 \in D_{\pi_{mon}}^+$ as $D_{\pi_{mon}}^n = \emptyset$ and $D_{\pi_{mon}}^- = \{\hat{r}_1, \hat{r}_2\}$. Since $a_1, a_2 \in D_{\pi_{mon}}^+$ and $a_1, a_2 \in D_{\pi}^-$, we have $a_1, a_2 \in N(\pi_{mon}, \pi)$. Therefore, $|N(\pi_{mon}, \pi)| \geq 2 = |N(\pi, \pi_{mon})|$.

□

(X, C) has a solution.

Lemma 3.4.11. *Every outcome π has at least 2 agents not in D_{π}^+ .*

Proof. Let π be an arbitrary outcome. If π contains a room that consists only of red agents, then $|D_\pi^- \cup D_\pi^n| \geq 2$ trivially holds for π as there are exactly $s - 2$ agents that approve of fraction 1. Thus, for the remainder of the proof, we shall assume that π does not contain a room that only consists of red agents.

Additionally, if π does not contain a room with fraction $\frac{5 \cdot (3) - 1}{s}$, then the circular set agents $\hat{r}_1, \hat{r}_2, \hat{r}_3$ must be in $D_\pi^n \cup D_\pi^-$ as we assume that π has no room consisting of only red agents and $\hat{r}_1, \hat{r}_2, \hat{r}_3$ only approve of fraction $\frac{5 \cdot (3) - 1}{s}$. Thus, π must contain a room with fraction $\frac{5 \cdot (3) - 1}{s}$.

Consider all the agents that approve of fraction $\frac{5 \cdot (3) - 1}{s}$. That is, let us consider the set

$$\left\{ a \in R \cup B \mid \frac{5 \cdot (3) - 1}{s} \in D_a^+ \right\} = \hat{R}^{set} \cup R_3^{red} \cup B_3^{fill}.$$

There are a total of $s + 2$ agents that approve of fraction $\frac{5 \cdot (3) - 1}{s}$. Let us denote this set of agents by $H_1 = \hat{R}^{set} \cup R_3^{red} \cup B_3^{fill}$. Note that for each agent a in H_1 their corresponding $D_a^+ \setminus \{1\}$ set is a subset of $\left\{ \frac{5 \cdot (3) - 1}{s}, \frac{5 \cdot (3) - 2}{s} \right\}$. That is,

$$\forall a \in H_1 : D_a^+ \setminus \{1\} \subseteq \left\{ \frac{5 \cdot (3) - 1}{s}, \frac{5 \cdot (3) - 2}{s} \right\}.$$

Let us also consider all the agents that approve of fraction $\frac{5 \cdot (3) - 2}{s}$. That is, let us consider the set

$$\left\{ a \in R \cup B \mid \frac{5 \cdot (3) - 2}{s} \in D_a^+ \right\} = \{\tilde{b}_3\} \cup R_3^{red} \cup B_3^{fill}.$$

There are a total of s agents that approve of fraction $\frac{5 \cdot (3) - 2}{s}$. Let us denote this set of agents by $H_2 = \{\tilde{b}_3\} \cup R_3^{red} \cup B_3^{fill}$. Note that $H_1 \cap H_2 = R_3^{red} \cup B_3^{fill}$ and $|H_1 \cap H_2| = s - 1$.

Let $S \in \pi$ be the room such that $\theta(\pi(S)) = \frac{5 \cdot (3) - 1}{s}$. To derive a contradiction, we assume that $|D_\pi^n \cup D_\pi^-| < 2$. Let us consider the following cases regarding room S .

1. $|S \cap (D_\pi^n \cup D_\pi^-)| = 1$.

Let $a \in S$ be such that $\frac{5 \cdot (3) - 1}{s} \notin D_a^+$, i.e., let $a \in S \cap (D_\pi^n \cup D_\pi^-)$. Then

we have that $S \setminus \{a\} \subseteq H_1$. Note that $|H_1 \setminus S| = |H_1| - |S \setminus \{a\}| = 3$ and $|H_2| - |H_1 \cap H_2| = 1 \leq |H_2 \setminus S| \leq 4 = |H_2| - |S \setminus \{a\}| + |\{\hat{r}_1, \hat{r}_2, \hat{r}_3\}|$.

Since $a \in D_\pi^n \cup D_\pi^-$, for any agent $a' \in H_1 \setminus S$ we have that $a' \in D_\pi^+$. As π does not contain a room that consists of only red agents and $a' \in H_1$, we have that $\theta(\pi(a')) \in \left\{ \frac{5 \cdot (3) - 1}{s}, \frac{5 \cdot (3) - 2}{s} \right\}$. For an arbitrary agent $a' \in H_1 \setminus S$, let us consider the following cases.

$$1.1. \theta(\pi(a')) = \frac{5 \cdot (3) - 1}{s}.$$

We have at least $s - |H_1 \setminus S| = s - 3 \geq 2$ agents in room $\pi(a')$ that do not approve of fraction $\frac{5 \cdot (3) - 1}{s}$. This is due to the fact that the only remaining agents outside of S that approve of fraction $\frac{5 \cdot (3) - 1}{s}$, are the agents in $H_1 \setminus S$. Thus, in this case, we have that $|D_\pi^n \cup D_\pi^-| \geq 2$.

$$1.2. \theta(\pi(a')) = \frac{5 \cdot (3) - 2}{s}.$$

We have at least $s - |H_2 \setminus S| \geq s - 4 > 2$ agents in room $\pi(a')$ that do not approve of fraction $\frac{5 \cdot (3) - 2}{s}$. This is due to the fact that the only remaining agents outside of S that approve of fraction $\frac{5 \cdot (3) - 2}{s}$, are the agents in $H_2 \setminus S$. Thus, in this case, we have that $|D_\pi^n \cup D_\pi^-| \geq 2$.

$$2. |S \cap (D_\pi^n \cup D_\pi^-)| = 0.$$

Then we have that $S \subseteq H_1$. Note that $|H_1 \setminus S| = |H_1| - |S| = 2$ and $|H_2| - |H_1 \cap H_2| = 1 \leq |H_2 \setminus S| \leq 3 = |H_2| - |S| + |\{\hat{r}_1, \hat{r}_2, \hat{r}_3\}|$.

Let us denote $H_1 \setminus S = \{a_1, a_2\}$. Since we assume that $|D_\pi^n \cup D_\pi^-| < 2$, at least one of a_1 and a_2 must be in D_π^+ . That is, $a_1 \in D_\pi^+ \vee a_2 \in D_\pi^+$.

Let $i \in [2]$ such that $a_i \in D_\pi^+$. Then we have that $\theta(\pi(a_i)) \in \left\{ \frac{5 \cdot (3) - 1}{s}, \frac{5 \cdot (3) - 2}{s} \right\}$, since $a_i \in H_1$ and π has no room that consists of only red agents. Consider the following cases.

$$2.1. \theta(\pi(a_i)) = \frac{5 \cdot (3) - 1}{s}.$$

We have at least $s - |H_1 \setminus S| = s - 2 > 2$ agents in room $\pi(a_i)$ that do not approve of fraction $\frac{5 \cdot (3) - 1}{s}$. This is due to the fact that the only remaining agents outside of S that approve of fraction

$\frac{5 \cdot (3) - 1}{s}$ are the agents in $H_1 \setminus S$. Thus, in this case, we have that $|D_\pi^n \cup D_\pi^-| \geq 2$.

2.2. $\theta(\pi(a_i)) = \frac{5 \cdot (3) - 2}{s}$.

We have at least $s - |H_2 \setminus S| \geq s - 3 > 2$ agents in room $\pi(a_i)$ that do not approve of fraction $\frac{5 \cdot (3) - 2}{s}$. This is due to the fact that the only remaining agents outside of S that approve of fraction $\frac{5 \cdot (3) - 2}{s}$ are the agents in $H_2 \setminus S$. Thus, in this case, we have that $|D_\pi^n \cup D_\pi^-| \geq 2$.

□

Lemma 3.4.12. *If for an outcome π we have that $|D_\pi^n \cup D_\pi^-| = 2$, then $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$.*

Proof. Let π be an outcome such that $|D_\pi^n \cup D_\pi^-| = 2$. If π contains a room consisting only of red agents, then by Lemma 3.4.8 $\pi = \pi_{mon}$. By definition of π_{mon} , we have $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$. Therefore, the lemma trivially holds, when π contains a room consisting of only red agents. Thus, for the remainder of the proof, we shall consider the case where π does not contain a room that consists of only red agents.

To derive a contradiction, let us assume that $D_\pi^n \cup D_\pi^- \not\subseteq \hat{R}^{set} \cup R_3^{red}$. This means that at most 1 agent from $\hat{R}^{set} \cup R_3^{red}$ is contained in $D_\pi^n \cup D_\pi^-$. Thus, we can consider the following cases.

1. $|(D_\pi^n \cup D_\pi^-) \cap (\hat{R}^{set} \cup R_3^{red})| = 1$.

Let $\{a\} = (D_\pi^n \cup D_\pi^-) \cap (\hat{R}^{set} \cup R_3^{red})$. The remaining agents in $(\hat{R}^{set} \cup R_3^{red}) \setminus \{a\}$ must be contained in D_π^+ . Since π does not contain a room consisting of only red agents, the remaining agents in $(\hat{R}^{set} \cup R_3^{red}) \setminus \{a\}$ must be in a room with fraction $\frac{5 \cdot 3 - 1}{s}$ or $\frac{5 \cdot 3 - 2}{s}$.

There are 15 agents in $(\hat{R}^{set} \cup R_3^{red}) \setminus \{a\}$. The remaining red agents that approve of fraction $\frac{5 \cdot 3 - 1}{s}$ and/or $\frac{5 \cdot 3 - 2}{s}$ are exactly $(\hat{R}^{set} \cup R_3^{red}) \setminus \{a\}$. Thus, the agents $(\hat{R}^{set} \cup R_3^{red}) \setminus \{a\}$ must be contained in at least 2 separate rooms S_1, S_2 with fraction $\frac{5 \cdot 3 - 1}{s}$ or $\frac{5 \cdot 3 - 2}{s}$. The blue agents that approve of fraction $\frac{5 \cdot 3 - 1}{s}$ or $\frac{5 \cdot 3 - 2}{s}$ are exactly the agents in $B_3^{fill} \cup \{\tilde{b}_3\}$ of which there are exactly $s - (5 \cdot (3) - 2)$. Thus, we have at least

$s - 5 \cdot 3 - 1$ blue agents in $S_1 \cup S_2$ such that they are contained in D_π^- . Therefore, this case contradicts our assumption that $|D_\pi^n \cup D_\pi^-| = 2$.

$$2. |(D_\pi^n \cup D_\pi^-) \cap (\hat{R}^{set} \cup R_3^{red})| = 0.$$

Then all the agents in $\hat{R}^{set} \cup R_3^{red}$ must be contained in D_π^+ . Since π does not contain a room consisting of only red agents, the remaining agents of $\hat{R}^{set} \cup R_3^{red}$ must be in a room with fraction $\frac{5 \cdot 3 - 1}{s}$ or $\frac{5 \cdot 3 - 2}{s}$.

There are 16 remaining agents in $\hat{R}^{set} \cup R_3^{red}$. The remaining red agents that approve of fraction $\frac{5 \cdot 3 - 1}{s}$ and/or $\frac{5 \cdot 3 - 2}{s}$ are exactly $\hat{R}^{set} \cup R_3^{red}$. Thus, the agents $\hat{R}^{set} \cup R_3^{red}$ must be contained in at least 2 separate rooms S_1, S_2 with fraction $\frac{5 \cdot 3 - 1}{s}$ or $\frac{5 \cdot 3 - 2}{s}$. The blue agents that approve of fraction $\frac{5 \cdot 3 - 1}{s}$ or $\frac{5 \cdot 3 - 2}{s}$ are exactly the agents in $B_3^{fill} \cup \{\tilde{b}_3\}$ of which there are exactly $s - (5 \cdot (3) - 2)$. Thus, we have at least $s - 5 \cdot 3 - 1$ blue agents in $S_1 \cup S_2$ such that they are contained in D_π^- . Therefore, this case contradicts our assumption that $|D_\pi^n \cup D_\pi^-| = 2$.

Thus, our assumption that $D_\pi^n \cup D_\pi^- \not\subseteq \hat{R}^{set} \cup R_3^{red}$ cannot hold. Therefore, $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$. \square

Lemma 3.4.13. *Every outcome π has at least 1 agents in D_π^- .*

Proof. Let π be an arbitrary outcome. By Lemma 3.4.11, we have that there are at least 2 agents in $D_\pi^n \cup D_\pi^-$.

To derive a contradiction assume that there exists an outcome π such that $D_\pi^n \cup D_\pi^- = D_\pi^n$. That is, we assume that $D_\pi^- = \emptyset$. Note that π cannot contain a room that only consists of red agents as there are exactly $s - 2$ agents that do not disapprove of fraction 1. By construction of the roommate diversity game, we have that $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, as these are the only agents with non-empty D_a^n sets. The only fraction in the set D_a^n is $\frac{5 \cdot (2) - 1}{s}$. Note that $2 \leq |D_\pi^n \cup D_\pi^-| = |D_\pi^n| \leq 5 \cdot (3) + 1 = 16$.

Consider all the agents that do not disapprove of fraction $\frac{5 \cdot (2) - 1}{s}$. That is, let us consider the set

$$\left\{ a \in R \cup B \mid \frac{5 \cdot (2) - 1}{s} \notin D_a^- \right\} = \hat{R}^{set} \cup R_2^{red} \cup R_3^{red} \cup B_2^{fill}.$$

Let us consider the following cases regarding the cardinality of $D_\pi^n \cap \hat{R}^{set}$.

1. $|D_\pi^n \cap \hat{R}^{set}| = 3$.

Note that $\hat{r}_1, \hat{r}_2, \hat{r}_3$ must be in the same room S . If they are in 2 separate rooms, we require $2 \cdot (s - (5 \cdot 2 - 1))$ blue agents that do not disapprove of fraction $\frac{5 \cdot (2) - 1}{s}$ and we only have $s - (5 \cdot 2 - 1)$ blue agents, namely B_2^{fill} . Similarly, if they are in 3 separate rooms, we require $3 \cdot (s - (5 \cdot 2 - 1))$ blue agents that do not disapprove of fraction $\frac{5 \cdot (2) - 1}{s}$.

Since $D_\pi^- = \emptyset$ and $\theta(\pi(S)) = \frac{5 \cdot (2) - 1}{s}$, all the blue agents in B_2^{fill} must be contained in S .

Since S contains $5 \cdot (2) - 1$ red agents, 3 of which are $\hat{r}_1, \hat{r}_2, \hat{r}_3$, there must exist a circular agent $r_2^i \in R_2^{red}$, where $1 \leq i \leq 5 \cdot 2 - 2$ such that $r_2^i \notin S$. Let S' denote the room in π that contains r_2^i . Since $D_\pi^- = \emptyset$, we have that $r_2^i \in D_\pi^+$. Thus, $\theta(S') = \frac{5 \cdot 2 - 1}{s}$ or $\theta(S') = \frac{5 \cdot 2 - 2}{s}$. There are exactly $s - (5 \cdot 2 - 2)$ blue agents that do not disapprove of fraction $\frac{5 \cdot 2 - 1}{s}$ and/or $\frac{5 \cdot 2 - 2}{s}$, namely $B_2^{fill} \cup B_2^{add}$. Since $B_2^{fill} \subseteq S$, there is at least 1 blue agent in S' that is contained in D_π^- , which contradicts our assumption that $D_\pi^- = \emptyset$.

2. $|D_\pi^n \cap \hat{R}^{set}| = 2$.

W.l.o.g. let $\hat{r}_1, \hat{r}_2 \in D_\pi^n$, thus $\hat{r}_3 \in D_\pi^+$. Note that \hat{r}_1, \hat{r}_2 must be in the same room S . If they are in 2 separate rooms, we require $2 \cdot (s - (5 \cdot 2 - 1))$ blue agents that do not disapprove of fraction $\frac{5 \cdot (2) - 1}{s}$ and we only have $s - (5 \cdot 2 - 1)$ blue agents, namely B_2^{fill} .

Since $D_\pi^- = \emptyset$ and $\theta(\pi(S)) = \frac{5 \cdot (2) - 1}{s}$, all the blue agents in B_2^{fill} must be contained in S .

Since S contains $5 \cdot (2) - 1$ red agents, 2 of which are \hat{r}_1, \hat{r}_2 , there must exist a circular agent $r_2^i \in R_2^{red}$, where $1 \leq i \leq 5 \cdot 2 - 2$ such that $r_2^i \notin S$. Let S' denote the room in π that contains r_2^i . Since $D_\pi^- = \emptyset$, we have that $r_2^i \in D_\pi^+$. Thus, $\theta(S') = \frac{5 \cdot 2 - 1}{s}$ or $\theta(S') = \frac{5 \cdot 2 - 2}{s}$. There are exactly $s - (5 \cdot 2 - 2)$ blue agents that do not disapprove of fraction $\frac{5 \cdot 2 - 1}{s}$ and/or $\frac{5 \cdot 2 - 2}{s}$, namely $B_2^{fill} \cup B_2^{add}$. Since $B_2^{fill} \subseteq S$, there is at least 1 blue agent in S' that is contained in D_π^- , which contradicts our assumption that $D_\pi^- = \emptyset$.

3. $|D_\pi^n \cap \hat{R}^{set}| = 1$.

W.l.o.g. let $\hat{r}_1 \in D_\pi^n$, thus $\hat{r}_2, \hat{r}_3 \in D_\pi^+$. Note that \hat{r}_2, \hat{r}_3 must be in the same room S . If they are in 2 separate rooms, we require $2 \cdot (s - (5 \cdot 3 - 1))$ blue agents that do not disapprove of fraction $\frac{5 \cdot (3) - 1}{s}$ and we only have $s - (5 \cdot 3 - 1)$ blue agents, namely B_3^{fill} .

Since $D_\pi^- = \emptyset$ and $\theta(\pi(S)) = \frac{5 \cdot (3) - 1}{s}$, all the blue agents in B_3^{fill} must be contained in S .

Since S contains $5 \cdot (3) - 1$ red agents, 2 of which are \hat{r}_2, \hat{r}_3 and the remaining $5 \cdot (3) - 1 - 2$ from R_3^{red} , there must exist a circular agent $r_3^i \in R_3^{red}$, where $1 \leq i \leq 5 \cdot 3 - 2$ such that $r_3^i \notin S$. Let S' denote the room in π that contains r_3^i . Since $D_\pi^- = \emptyset$, we have that $r_3^i \in D_\pi^+ \cup D_\pi^n$. Thus, $\theta(S') = \frac{5 \cdot 3 - 1}{s}$, $\theta(S') = \frac{5 \cdot 3 - 2}{s}$ or $\theta(S') = \frac{5 \cdot 2 - 1}{s}$. The blue agents that do not disapprove of fraction $\frac{5 \cdot 3 - 1}{s}$ and/or $\frac{5 \cdot 3 - 2}{s}$ are exactly $B_3^{fill} \cup B_3^{add}$ of which all but 1 are contained in S . Thus, we have that $\theta(S') = \frac{5 \cdot 2 - 1}{s}$, otherwise D_π^- cannot be empty.

Since $D_\pi^- = \emptyset$ and $\theta(\pi(S')) = \frac{5 \cdot (2) - 1}{s}$, all the blue agents in B_2^{fill} must be contained in S' .

Since S' must contain $5 \cdot (2) - 1$ red agents from \hat{r}_1, R_3^{red} and R_2^{red} , as these are the only remaining red agents that do not disapprove of fraction $\frac{5 \cdot (2) - 1}{s}$, there must exist an agent $r \in \{\hat{r}_1\} \cup R_3^{red} \cup R_2^{red}$ such that $r \notin S'$. Let S'' denote the room in π that contains r . Since $D_\pi^- = \emptyset$, we have that $r \in D_\pi^+ \cup D_\pi^n$. Thus, $\theta(S'') = \frac{5 \cdot 2 - 1}{s}$, $\theta(S'') = \frac{5 \cdot 2 - 2}{s}$, $\theta(S'') = \frac{5 \cdot 3 - 1}{s}$, or $\theta(S'') = \frac{5 \cdot 3 - 2}{s}$. The blue agents that do not disapprove of fraction $\frac{5 \cdot 3 - 1}{s}$ and/or $\frac{5 \cdot 3 - 2}{s}$ are exactly in $B_3^{fill} \cup B_3^{add}$ of which all but 1 are contained in S . The blue agents that do not disapprove of fraction $\frac{5 \cdot 2 - 1}{s}$ and/or $\frac{5 \cdot 2 - 2}{s}$ are exactly $B_2^{fill} \cup B_2^{add}$ of which all but 1 are contained in S' . Thus, at least 1 blue agent in S'' must be contained in D_π^- , which contradicts our assumption that $D_\pi^- = \emptyset$.

4. $|D_\pi^n \cap \hat{R}^{set}| = 0$.

Note that $\hat{r}_1, \hat{r}_2, \hat{r}_3 \in D_\pi^+$ and that $\hat{r}_1, \hat{r}_2, \hat{r}_3$ must be in the same room S . If they are in 2 separate rooms, we require $2 \cdot (s - (5 \cdot 3 - 1))$ blue agents that do not disapprove of fraction $\frac{5 \cdot (3) - 1}{s}$ and we only have $s - (5 \cdot 3 - 1)$

blue agents, namely B_3^{fill} . Similarly, if they are in 3 separate rooms, we require $3 \cdot (s - (5 \cdot 3 - 1))$ blue agents that do not disapprove of fraction $\frac{5 \cdot (3) - 1}{s}$.

Since $D_\pi^- = \emptyset$ and $\theta(\pi(S)) = \frac{5 \cdot (3) - 1}{s}$, all the blue agents in B_3^{fill} must be contained in S .

Since S contains $5 \cdot (3) - 1$ red agents, 3 of which are $\hat{r}_1, \hat{r}_2, \hat{r}_3$ and the remaining $5 \cdot (3) - 1 - 3$ from R_3^{red} , there must exist a circular agent $r_3^i \in R_3^{red}$, where $1 \leq i \leq 5 \cdot 3 - 2$ such that $r_3^i \notin S$. Let S' denote the room in π that contains r_3^i . Since $D_\pi^- = \emptyset$, we have that $r_3^i \in D_\pi^+ \cup D_\pi^n$. Thus, $\theta(S') = \frac{5 \cdot 3 - 1}{s}$, $\theta(S') = \frac{5 \cdot 3 - 2}{s}$ or $\theta(S') = \frac{5 \cdot 2 - 1}{s}$. The blue agents that do not disapprove of fraction $\frac{5 \cdot 3 - 1}{s}$ and/or $\frac{5 \cdot 3 - 2}{s}$ are exactly $B_3^{fill} \cup B_3^{add}$ of which all but 1 are contained in S . Thus, we have that $\theta(S') = \frac{5 \cdot 2 - 1}{s}$, otherwise D_π^- cannot be empty.

Since $D_\pi^- = \emptyset$ and $\theta(\pi(S')) = \frac{5 \cdot (2) - 1}{s}$, all the blue agents in B_2^{fill} must be contained in S' .

Since S' must contain $5 \cdot (2) - 1$ red agents from R_3^{red} and R_2^{red} , as these are the only remaining red agents that do not disapprove of fraction $\frac{5 \cdot (2) - 1}{s}$, there must exist an agent $r_j^k \in R_3^{red} \cup R_2^{red}$ where $2 \leq j \leq 3$ and $1 \leq k \leq 5 \cdot j - 2$ such that $r_j^k \notin S'$. Let S'' denote the room in π that contains r_j^k . Since $D_\pi^- = \emptyset$, we have that $r_j^k \in D_\pi^+ \cup D_\pi^n$. Thus, $\theta(S'') = \frac{5 \cdot 2 - 1}{s}$, $\theta(S'') = \frac{5 \cdot 2 - 2}{s}$, $\theta(S'') = \frac{5 \cdot 3 - 1}{s}$, or $\theta(S'') = \frac{5 \cdot 3 - 2}{s}$. The blue agents that do not disapprove of fraction $\frac{5 \cdot 3 - 1}{s}$ and/or $\frac{5 \cdot 3 - 2}{s}$ are exactly $B_3^{fill} \cup B_3^{add}$ of which all but 1 are contained in S . The blue agents that do not disapprove of fraction $\frac{5 \cdot 2 - 1}{s}$ and/or $\frac{5 \cdot 2 - 2}{s}$ are exactly $B_2^{fill} \cup B_2^{add}$ of which all but 1 are contained in S' . Thus, at least 1 blue agent in S'' must be contained in D_π^- , which contradicts our assumption that $D_\pi^- = \emptyset$.

□

Lemma 3.4.14. *Let π be an outcome of the roommate diversity game and assume that (X, C) has a solution. If for outcome π it holds that $|D_\pi^n| = 1$ and $|D_\pi^-| = 1$, then π must be a reduced-type outcome.*

Proof. Let π be an outcome such that $|D_\pi^n|=1$ and $|D_\pi^-|=1$. By Lemma 3.4.12, we have that $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$. Lemma 3.4.12 also implies that $B \subseteq D_\pi^+$. Let us denote the agents in D_π^n and D_π^- by r^n and r^- respectively. Note that π cannot contain a room that only consists of red agents, as there are exactly $s-2$ red agents that do not disapprove of fraction 1. Let $C' \subseteq C$ denote some solution of (X, C) . We shall show that the following 7 properties, which show that there is a one-to-one correspondence between the rooms in π and the rooms in $\pi_{C'}$, hold for π .

1. $R^{mon} \cup B^{mon} \in \pi$.

Let $S^{mon} \in \pi$ denote a room that contains some monolith red agent. Since $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, all the monolith red agents R^{mon} and monolith blue agents B^{mon} must be contained in D_π^+ . Since π cannot contain a room consisting of only red agents, any monolith red agent must be in a room of fraction $\frac{s-2m-3}{s}$. The only agents that approve of fraction $\frac{s-2m-3}{s}$ are exactly the monolith red agents R^{mon} and monolith blue agents B^{mon} , any other agent disapproves of that fraction.

There are exactly $s-2m-3$ monolith red agents. Thus, we have that $R^{mon} \subseteq S^{mon}$. If there exists a monolith red agent $r_{mon}^p \in R^{mon}$ such that $r_{mon}^p \notin S^{mon}$, we would require 2 rooms with fraction $\frac{s-2m-3}{s}$. This would contradict $|D_\pi^-|=1$ as we would have at least $s-2m-3$ red agents in D_π^- .

There are exactly $2m+3$ monolith blue agents. Thus, we have that $B^{mon} \subseteq S^{mon}$. If there exists a monolith blue agent $b_{mon}^p \in B^{mon}$ such that $b_{mon}^p \notin S^{mon}$, there must be a blue agent $b \in S^{mon}$ that disapproves of fraction $\frac{s-2m-3}{s}$. This would contradict $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, since we would have $b \in D_\pi^-$.

Thus, the agents in R^{mon} and B^{mon} must belong to the same room, i.e., $S^{mon} = R^{mon} \cup B^{mon} \in \pi$.

2. $\{r_i, \tilde{r}_i\}_{i \in A_{j-3}} \cup R_j^{red} \cup B_j^{fill} \in \pi$ for each $A_{j-3} \in C'$.

Since π does not contain a room that only consists of red agents and $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, any set agents $r_{i'} \in R^{set}$ must be in a room with fraction $\frac{2(5 \cdot j' + 1)}{s}$ where $i' \in A_{j'-3}$. There are exactly s agents that

approve of fraction $\frac{2(5 \cdot j' + 1)}{s}$, namely the agents in $\{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, any other agent disapproves of that fraction. Since $|D_\pi^-| = 1$, the room $\pi(r_{i'})$ must contain at least $s - 1$ agents from $\{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill}$.

Let us assume that there exists an agent $a \in \{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill}$ such that $a \notin \pi(r_{i'})$. Note that under this assumption, the agent $r^- \in D_\pi^-$ must be contained in room $\pi(r_{i'})$. Therefore, we have that $a \in D_\pi^+$. Consider the following cases.

2.1. $a \in \{r_i, \tilde{r}_i\}_{i \in A_{j'-3}}$.

W.l.o.g. let us write $a = r_{i''}$. Since π does not contain a room that only consists of red agents, we must have $\theta(\pi(a)) = \frac{2(5 \cdot j'' + 1)}{s}$ where $i'' \in A_{j''-3}$. There are exactly s agents that approve of fraction $\frac{2(5 \cdot j'' + 1)}{s}$, namely the agents in $\{r_i, \tilde{r}_i\}_{i \in A_{j''-3}} \cup R_{j''}^{red} \cup B_{j''}^{fill}$, any other agent disapproves of that fraction.

Note that $\tilde{r}_{i''} \in \{r_i, \tilde{r}_i\}_{i \in A_{j''-3}}$ and $\tilde{r}_{i''} \in \pi(r_{i'})$. Since $a \notin \pi(r_{i'})$, we have that $\tilde{r}_{i''} \notin \pi(a)$. Thus, the room $\pi(a)$ contains at most $s - 1$ agents from $\{r_i, \tilde{r}_i\}_{i \in A_{j''-3}} \cup R_{j''}^{red} \cup B_{j''}^{fill}$. Therefore, room $\pi(a)$ contains at least 1 agent that disapproves of fraction $\frac{2(5 \cdot j'' + 1)}{s}$. Since both room $\pi(r_{i'})$ and room $\pi(a)$ contain an agent that disapproves of the fraction of its assigned room, we have that $|D_\pi^-| \geq 2$. This contradicts $|D_\pi^-| = 1$, thus this case cannot occur.

2.2. $a \in R_{j'}^{red} \vee a \in B_{j'}^{fill}$.

We have that $\theta(\pi(a)) = \frac{2(5 \cdot j' + 1)}{s}$ or $\theta(\pi(a)) = \frac{2(5 \cdot j' - 2)}{s}$.

There are exactly s agents that approve of fraction $\frac{2(5 \cdot j' + 1)}{s}$, namely the agents in $\{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, any other agent disapproves of that fraction.

There are exactly s agents that approve of fraction $\frac{2(5 \cdot j' - 2)}{s}$, namely the agents in $B_{j'}^{add} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, any other agent disapproves of that fraction.

Since $R_{j'}^{red} \cup B_{j'}^{fill} \setminus \{a\} \subseteq \pi(r_{i'})$, the room $\pi(a)$ must contain at least $|R_{j'}^{red} \cup B_{j'}^{fill} \setminus \{a\}| = s - 7 > 2$ agents that disapprove of the

fraction $\theta(\pi(a))$. This contradicts $|D_\pi^-|=1$, thus this case cannot occur.

Since there does not exist an agent $a \in \{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill}$ such that $a \notin \pi(r_i)$, we have that $\{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill} \in \pi$.

Let A denote the collection of 3-sets $A_{j-3} \in C$ such that $\{r_i, \tilde{r}_i\}_{i \in A_{j-3}} \cup R_j^{red} \cup B_j^{fill} \in \pi$, i.e., $A = \left\{ A_{j-3} \in C \mid \{r_i, \tilde{r}_i\}_{i \in A_{j-3}} \cup R_j^{red} \cup B_j^{fill} \in \pi \right\}$. In a valid outcome π , every set agent is assigned to exactly 1 room. Therefore, no two 3-sets in A share an element, i.e., for any $A_i, A_j \in A$ such that $i \neq j$, we have that $A_i \cap A_j = \emptyset$.

From the above, we have that A must be a solution of (X, C) . Therefore, we have some solution $A = C' \subseteq C$ of (X, C) where for each $A_{j-3} \in C'$ it holds that $\{r_i, \tilde{r}_i\}_{i \in A_{j-3}} \cup R_j^{red} \cup B_j^{fill} \in \pi$.

3. $B_j^{add} \cup R_j^{red} \cup B_j^{fill} \in \pi$ for each $A_{j-3} \notin C'$.

Since $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, any redundant agent $r_{j'}^p \in R_{j'}^{red}$, where $A_{j'-3} \notin C'$, must be in a room with fraction $\frac{2(5 \cdot j' + 1)}{s}$ or $\frac{2(5 \cdot j' - 2)}{s}$.

There are exactly s agents that approve of fraction $\frac{2(5 \cdot j' + 1)}{s}$, namely the agents in $\{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, any other agent disapproves of that fraction.

There are exactly s agents that approve of fraction $\frac{2(5 \cdot j' - 2)}{s}$, namely the agents in $B_{j'}^{add} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, any other agent disapproves of that fraction.

By property 2, all the set agents $r_{i'} \in R^{set}$, $\tilde{r}_{i'} \in \tilde{R}^{set}$ must be in the room $\{r_i, \tilde{r}_i\}_{i \in A_{j''-3}} \cup R_{j''}^{red} \cup B_{j''}^{fill} \in \pi$ where $i' \in A_{j''-3}$ and $A_{j''-3} \in C'$. Thus, there are exactly $2(5j' - 2)$ remaining red agents that approve of fraction $\frac{2(5 \cdot j' + 1)}{s}$. Therefore, room $\pi(r_{j'}^p)$ cannot have fraction $\frac{2(5 \cdot j' + 1)}{s}$, as otherwise we would have at least 3 agents in D_π^- . Therefore, room $\pi(r_{j'}^p)$ must be of fraction $\frac{2(5 \cdot j' - 2)}{s}$.

Since $|D_\pi^-|=1$, the room $\pi(r_{j'}^p)$ contains at least $s - 1$ agents from $R_{j'}^{red} \cup B_{j'}^{fill} \cup B_{j'}^{add}$. Assume that there exists an agent $a \in R_{j'}^{red} \cup B_{j'}^{fill} \cup B_{j'}^{add}$ such that $a \notin \pi(r_{j'}^p)$. Note that under this assumption, the agent $r^- \in D_\pi^-$ must be contained in room $\pi(r_{j'}^p)$. Therefore, we have that

$a \in D_\pi^+$. Additionally, since $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, we have that a is a red agent. Thus, we have that $a \in R_{j'}^{red}$. Since $a \in R_{j'}^{red}$ and $a \in D_\pi^+$, we have that $\theta(\pi(a)) = \frac{2(5 \cdot j' + 1)}{s}$ or $\theta(\pi(a)) = \frac{2(5 \cdot j' - 2)}{s}$.

There are exactly s agents that approve of fraction $\frac{2(5 \cdot j' + 1)}{s}$, namely the agents in $\{r_i, \tilde{r}_i\}_{i \in A_{j'-3}} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, any other agent disapproves of that fraction.

There are exactly s agents that approve of fraction $\frac{2(5 \cdot j' - 2)}{s}$, namely the agents in $B_{j'}^{add} \cup R_{j'}^{red} \cup B_{j'}^{fill}$, any other agent disapproves of that fraction.

Since $R_{j'}^{red} \cup B_{j'}^{fill} \setminus \{a\} \subseteq \pi(r_{j'}^p)$, the room $\pi(a)$ must contain at least $|R_{j'}^{red} \cup B_{j'}^{fill} \setminus \{a\}| = s - 7 > 2$ agents that disapprove of the fraction $\theta(\pi(a))$. This contradicts $|D_\pi^-| = 1$, thus there cannot exist an agent $a \in R_{j'}^{red} \cup B_{j'}^{fill} \cup B_{j'}^{add}$ such that $a \notin \pi(r_{j'}^p)$. Therefore, we have that $R_{j'}^{red} \cup B_{j'}^{fill} \cup B_{j'}^{add} \in \pi$.

The above holds for any redundant agent $r_j^p \in R_j^{red}$ where $A_{j-3} \notin C'$. Thus, we have that for each $A_{j-3} \notin C'$ $B_j^{add} \cup R_j^{red} \cup B_j^{fill} \in \pi$.

4. $\{r^n\} \cup R_2^{red} \cup B_2^{fill} \in \pi$.

Let us define the room $S^n = \pi(r^n)$. We have that $\theta(S^n) = \frac{5 \cdot 2 - 1}{s}$ as it is the only fraction contained in any D_a^n . Since $r^- \in D_\pi^-$ and $r^- \in \hat{R}^{set} \cup R_3^{red}$, we have that $r^- \notin S^n$. If $r^- \in S^n$, we would have $r^- \in D_\pi^n$. Thus, the remaining agents in S^n must be contained in D_π^+ .

The red agents that approve of fraction $\frac{5 \cdot 2 - 1}{s}$ are exactly the agents in R_2^{red} of which there are $5 \cdot 2 - 2$, any other red agent does not approve of that fraction. Thus, the red agents in S^n are exactly r^n and R_2^{red} .

The blue agents that approve of fraction $\frac{5 \cdot 2 - 1}{s}$ are exactly $B_2^{fill} \cup B_2^{add}$ of which there are $s - (5 \cdot 2 - 2)$, any other blue agent disapproves of that fraction.

Assume that $\{\tilde{b}_2\} = B_2^{add} \subseteq S^n$. This means that there exists a blue agent $b \in B_2^{fill}$ such that $b \notin S^n$. Since $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, we have that $b \in D_\pi^+$. Therefore, we have $\theta(\pi(b)) = \frac{5 \cdot 2 - 1}{s}$ or $\theta(\pi(b)) = \frac{5 \cdot 2 - 2}{s}$.

There are exactly $s - 1$ agents that approve of fraction $\frac{5 \cdot 2 - 1}{s}$, namely the agents in $R_2^{red} \cup B_2^{fill}$, any other agent does not approve of that fraction.

There are exactly s agents that approve of fraction $\frac{5 \cdot 2 - 2}{s}$, namely the agents in $B_2^{add} \cup R_2^{red} \cup B_2^{fill}$, any other agent disapproves of that fraction.

Since $R_2^{red} \cup B_2^{fill} \setminus \{b\} \subseteq S^n$, the room $\pi(b)$ must contain at least $|R_2^{red} \cup B_2^{fill} \setminus \{b\}| = s - 2 > 2$ agents that do not approve of the fraction $\theta(\pi(b))$. This contradicts $|D_\pi^-| = 1$ and $|D_\pi^n| = 1$, thus $\{\tilde{b}_2\} = B_a^{add} \not\subseteq S^n$.

Therefore, the blue agents in room S^n must be exactly the agents in B_2^{fill} and we have $\{r^n\} \cup R_2^{red} \cup B_2^{fill} \in \pi$.

5. $(\hat{R}^{set} \cup R_3^{red}) \setminus \{r^n, r^-\} \cup B_3^{fill} \in \pi$.

Let $\hat{r}_j \in \hat{R}^{set}$ where $1 \leq j \leq 3$ such that $\hat{r}_j \in D_\pi^+$ and let S^j denote the room in π that contains \hat{r}_j . Since $\hat{r}_j \in D_\pi^+$, we have that $\theta(S^j) = \frac{5 \cdot 3 - 1}{s}$. Note that S^j cannot contain the agents r^n and r^- as they approve of fraction $\frac{5 \cdot 3 - 1}{s}$. Thus, every agent in S^j must approve of fraction $\frac{5 \cdot 3 - 1}{s}$.

The red agents that approve of fraction $\frac{5 \cdot 3 - 1}{s}$ that may be contained in S^j are exactly $(\hat{R}^{set} \cup R_3^{red}) \setminus \{r^n, r^-\}$. There are $5 \cdot 3 - 1$ agents in $(\hat{R}^{set} \cup R_3^{red}) \setminus \{r^n, r^-\}$. Thus, the red agents in S^j must be exactly $(\hat{R}^{set} \cup R_3^{red}) \setminus \{r^n, r^-\}$.

The blue agents that approve of fraction $\frac{5 \cdot 3 - 1}{s}$ are exactly $B_3^{fill} \cup B_3^{add}$ of which there are $s - (5 \cdot 3 - 2)$.

Assume that $\{\tilde{b}_3\} = B_3^{add} \subseteq S^j$. This means that there exists a blue agent $b \in B_3^{fill}$ such that $b \notin S^j$. Since $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, we have that $b \in D_\pi^+$. Therefore, we have $\theta(\pi(b)) = \frac{5 \cdot 3 - 1}{s}$ or $\theta(\pi(b)) = \frac{5 \cdot 3 - 2}{s}$.

There are exactly $s + 2$ agents that approve of fraction $\frac{5 \cdot 3 - 1}{s}$, namely the agents in $\hat{R}^{set} \cup R_3^{red} \cup B_3^{fill}$, any other agent disapproves of that fraction.

There are exactly s agents that approve of fraction $\frac{5 \cdot 3 - 2}{s}$, namely the agents in $B_3^{add} \cup R_3^{red} \cup B_3^{fill}$, any other agent disapproves of that fraction.

Since $(\hat{R}^{set} \cup R_3^{red}) \setminus \{r^n, r^-\} \cup B_3^{fill} \setminus \{b\} \subseteq S^j$, the room $\pi(b)$ must have at least $|(\hat{R}^{set} \cup R_3^{red}) \setminus \{r^n, r^-\} \cup B_3^{fill} \setminus \{b\}| - 2 = s - 5 > 2$ agents

that disapprove of the fraction $\theta(\pi(b))$. This contradicts $|D_\pi^-|=1$, thus $\{\tilde{b}_3\} = B_3^{add} \notin S^j$.

Therefore, the blue agents in room S^n must be exactly B_3^{fill} and we have $(\hat{R}^{set} \cup R_3^{red}) \setminus \{r^n, r^-\} \cup B_3^{fill} \in \pi$.

$$6. B^{even} \cup \bigcup_{j \in [3]} B_j^{add} \cup \bigcup_{A_{j-3} \in C'} B_j^{add} \in \pi.$$

Since $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, we have that $B^{even} \subseteq D_\pi^+$. Thus, any evening agent $b \in B^{even}$ must be in a room of fraction 0.

There are exactly $s + 6q + 3$ agents that approve of fraction 0, namely the agents in $B^{even} \cup B^{mon} \cup B^{add}$, any other agent disapproves of that fraction. Since $s < s + 6q + 3 < 2s$, we have at most 1 room with fraction 0. Otherwise we have at least $2s - (s + 6q + 3) = 4q + 42 + 2m$ agents in D_π^- which would contradict $|D_\pi^-|=1$. Thus, every evening agent must be assigned to the same room, i.e., for any $b', b'' \in B^{even}$ we have $\pi(b') = \pi(b'')$.

By Property 1 the room of evening agent b cannot contain agents from B^{mon} . By Property 3 the room of evening agent b cannot contain agents from $\bigcup_{A_j \notin C'} B_j^{add}$. Since $D_\pi^n \cup D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$, no blue agent may be assigned to a room by π with a fraction that it disapproves of. Therefore, we have $\pi(b) \subseteq B^{even} \cup \bigcup_{j \in [3]} B_j^{add} \cup \bigcup_{A_{j-3} \in C'} B_j^{add}$. Since there are $s - 2m - 3$ evening agents, $\pi(b)$ must contain exactly $2m + 3$ agents from $\bigcup_{j \in [3]} B_j^{add} \cup \bigcup_{A_{j-3} \in C'} B_j^{add}$. Note that $|\bigcup_{j \in [3]} B_j^{add}| = 3$ and $|\bigcup_{A_j \in C'} B_j^{add}| = \frac{m}{3} \cdot 6 = 2m$. Thus, we have that $B^{even} \cup \bigcup_{j \in [3]} B_j^{add} \cup \bigcup_{A_{j-3} \in C'} B_j^{add} \in \pi$.

$$7. \{r^-\} \cup R_1^{red} \cup B_1^{fill} \in \pi.$$

By Properties 1-6, we have exactly s remaining agents of which their assigned room has not been specified yet. These agents are $\{r^-\} \cup R_1^{red} \cup B_1^{fill}$, where $r^- \in \hat{R}^{set} \cup R_3^{red}$. Thus, we have that $\{r^-\} \cup R_1^{red} \cup B_1^{fill} \in \pi$.

By Properties 1-7, we have that π must be a reduced-type outcome. \square

Lemma 3.4.15. *For any non-reduced-type outcome π , there exists some reduced-type outcome π' that is strictly more popular than π .*

Proof. Let π be an arbitrary non-reduced-type outcome. By Lemma 3.4.14 we have that $|D_\pi^n| \neq 1 \vee |D_\pi^-| \neq 1$. Consider the following cases.

1. $|D_\pi^-| \neq 1$.

By Lemma 3.4.13 and case assumption, we have that $|D_\pi^-| \geq 2$. Consider the following cases.

1.1. $|D_\pi^-| = 2$.

Consider the following cases.

1.1.1. $|D_\pi^n| = 0$.

By Lemma 3.4.12, we have that $D_\pi^- \subseteq \hat{R}^{set} \cup R_3^{red}$. Let us write $D_\pi^- = \{a_1, a_2\}$. Since $a_1, a_2 \in \hat{R}^{set} \cup R_3^{red}$, there exists a reduced-type outcome π' such that $D_{\pi'}^- = \{a_1\}$ and $D_{\pi'}^n = \{a_2\}$. Thus, we have that $\phi(\pi', \pi) = 1$.

1.1.2. $|D_\pi^n| \geq 1$.

Let us write $D_\pi^- = \{a_1, a_2\}$ and $a_3 \in D_\pi^n$. We have that $a_3 \in \hat{R}^{set} \cup R_3^{red}$, since the agents in $\hat{R}^{set} \cup R_3^{red}$ are the only agents with a corresponding non-empty D_a^n .

Let π' be a reduced-type outcome such that $\{a_3\} = D_{\pi'}^n$ and $\{a^-\} = D_{\pi'}^-$. Note that $N(\pi, \pi') \subseteq \{a^-\}$. Consider the following cases.

1.1.2.1. $N(\pi, \pi') = \emptyset$.

Then we have that $a^- \in D_\pi^-$. W.l.o.g. assume that $a_1 = a^-$. We have that $a_2 \in D_{\pi'}^+$, thus $a_2 \in N(\pi', \pi)$. Thus, we have that $\phi(\pi', \pi) > 0$, i.e., π' is more popular than π .

1.1.2.2. $N(\pi, \pi') = \{a^-\}$.

Then we have that $a^- \in D_\pi^n \cup D_\pi^+$ and $a_1, a_2 \in D_{\pi'}^+$. Thus, $a_1, a_2 \in N(\pi', \pi)$. Therefore, we have that $\phi(\pi', \pi) > 0$, i.e., π' is more popular than π .

1.2. $|D_\pi^-| > 2$.

Let us write $a_1, a_2, a_3 \in D_\pi^-$. Let π' be an arbitrary reduced-type outcome with $\{a^n\} = D_{\pi'}^n$ and $\{a^-\} = D_{\pi'}^-$. Note that $N(\pi, \pi') \subseteq \{a^n, a^-\}$. Consider the following cases.

1.2.1. $a^- \in D_\pi^-$.

W.l.o.g. assume that $a_1 = a^-$. In this case, we have that $N(\pi, \pi') \subseteq \{a^n\}$. We have that $a_2, a_3 \in N(\pi', \pi)$. Thus, $\phi(\pi', \pi) > 0$.

1.2.2. $a^- \notin D_\pi^-$.

In this case, we have that $N(\pi, \pi') \subseteq \{a^-, a^n\}$. We have that $a_1, a_2, a_3 \in N(\pi', \pi)$. Thus, $\phi(\pi', \pi) > 0$.

2. $|D_\pi^n| \neq 1$.

Consider the following cases.

2.1. $|D_\pi^n| = 0$.

By Lemma 3.4.11, we have that $|D_\pi^-| \geq 2$. This case is proven similarly to case 1.1.1. as we have the exact same case assumption, i.e., $|D_\pi^n| = 0$ and $|D_\pi^-| \geq 2$.

2.2. $|D_\pi^n| \geq 2$.

Let us write $a_1, a_2 \in D_\pi^n$. We have that $\{a_1, a_2\} \subseteq \hat{R}^{set} \cup R_3^{red}$, since the agents in $\hat{R}^{set} \cup R_3^{red}$ are the only agents with a corresponding non-empty D_a^n . By Lemma 3.4.13, we have that $|D_\pi^-| \geq 1$. Let us write $a_3 \in D_\pi^-$. Let us consider the following cases regarding a_3 .

2.2.1. $a_3 \in \hat{R}^{set} \cup R_3^{red}$.

By the definition of reduced-type outcome, we can construct a reduced-type outcome π' such that $D_{\pi'}^- = \{a_3\}$, $D_{\pi'}^n = \{a_2\}$, and $a_1 \in D_{\pi'}^+$.

We have that $N(\pi, \pi') = \emptyset$ and $|N(\pi', \pi)| \geq 1$. Therefore, $\phi(\pi', \pi) > 0$.

2.2.2. $a_3 \notin \hat{R}^{set} \cup R_3^{red}$.

By the definition of reduced-type outcome, we can construct a reduced-type outcome π' such that $D_{\pi'}^- = \{a^-\}$, $D_{\pi'}^n = \{a_1\}$, and $a_2, a_3 \in D_{\pi'}^+$.

We have that $N(\pi, \pi') \subseteq \{a^-\}$. Additionally we have that $a_2, a_3 \in N(\pi', \pi)$, therefore $\phi(\pi', \pi) > 0$.

□

Lemma 3.4.16. *A reduced-type outcome π is not popular.*

Proof. Let π be some reduced-type outcome. Since π is a reduced-type outcome, we can write $\{a_j\} \cup R_j^{red} \cup B_j^{fill} \in \pi$ for $j \in [2]$, and $\{a_3, a_4, a_5\} \cup (R_3^{red} \setminus \{a_1, \dots, a_5\}) \cup B_3^{fill} \in \pi$, where $\{a_1, \dots, a_5\} \subseteq \hat{R}^{set} \cup R_3^{red}$.

Let us construct an outcome π' as follows:

$$\begin{aligned} \pi' = \pi \setminus & \left\{ \{a_1\} \cup R_1^{red} \cup B_1^{fill} \right\} \setminus \left\{ \{a_2\} \cup R_2^{red} \cup B_2^{fill} \right\} \\ & \setminus \left\{ \{a_3, a_4, a_5\} \cup (R_3^{red} \setminus \{a_1, \dots, a_5\}) \cup B_3^{fill} \right\} \\ & \cup \left\{ \{a_3\} \cup R_1^{red} \cup B_1^{fill} \right\} \cup \left\{ \{a_1\} \cup R_2^{red} \cup B_2^{fill} \right\} \\ & \cup \left\{ \{a_2, a_4, a_5\} \cup (R_3^{red} \setminus \{a_1, \dots, a_5\}) \cup B_3^{fill} \right\}. \end{aligned}$$

That is, we “rotate” the agents a_1, a_2, a_3 in π to obtain π' . We have that $N(\pi', \pi) = \{a_1, a_2\}$ and $N(\pi, \pi') = \{a_3\}$. Thus, π' is more popular than π . Note that π' itself is also a reduced-type outcome. \square

Theorem 3.4.17. *Determining whether a popular outcome exists in a roommate diversity game is co-NP-hard, even if the preferences are trichotomous.*

Proof. From Lemma 3.4.10, if (X, C) has no solution, then we have a popular outcome, namely π_{mon} . From Lemma 3.4.15 and 3.4.16, if (X, C) has a solution, then we have no popular outcome. Therefore, determining the existence of a popular outcome for a Roommate diversity Game is co-NP-hard. \square

Chapter 4

3D-Euclidean Stable Roommates Problem

We conclude with the discussion on our result of the 3D-Euclidean Stable Roommates Problem. This result is mentioned in Section 1.2, the relevant definitions and notation can be found in Sections 2.1 and 2.3, and the NP-complete PC-X3C problem [50] that we use for our reduction is described in Section 2.5.2.

In this chapter, we show that determining the existence of a strictly popular outcome is co-NP-hard, when the room size is fixed to 3. We shall construct a 3D-EuclidSR game $G = (N, E, 3)$ from a PC-X3C instance $I = (X, C)$ such that there exists a solution $S \subseteq C$ that partitions X if and only if no strictly popular outcome exists for G .

4.1 Construction

The 3D-EuclidSR game G we construct consists of 3 parts: the bottom layer, the top layer, and the ascending layer. We shall construct separate sets of agents V_b, V_t, V_a , and embeddings E^b, E^t, E^a for the bottom layer, the top layer, and the ascending layer, respectively. Note that $N = V_b \cup V_t \cup V_a$ and

for each $a \in N$ we have

$$E = \begin{cases} E^b(a) & , \text{ if } a \in V_b \\ E^t(a) & , \text{ if } a \in V_t . \\ E^a(a) & , \text{ if } a \in V_a \end{cases}$$

4.1.1 Construction of the Bottom Layer

For the bottom layer, we use the fact that the associated graph $G(I)$ is planar and cubic. By Valiant [63], $G(I)$ admits a specific planar embedding in \mathbb{Z}^2 , called orthogonal drawing, which maps each vertex $v \in U \cup W$ to an integer grid point and each edge to a chain of non-overlapping horizontal and vertical segments along the grid (except at the endpoints). A horizontal segment is a collection of consecutive horizontal edges of the grid \mathbb{Z}^2 . A vertical segment is defined analogously.

We use the following more specific restricted orthogonal drawing $E^{G(I)}$ for $G(I)$:

Proposition 4.1.1. ([25]) *A planar graph with maximum vertex degree three can be embedded, in polynomial time, in the grid \mathbb{Z}^2 such that its vertices are at the integer grid points and its edges are drawn using at most one horizontal and one vertical segment in the grid.*

The intersection point of the horizontal and vertical segments is called a bending point.

The planar graph in Proposition 4.1.1 can be constructed without any empty vertical nor empty horizontal grid lines that intersect the graph. By empty, we mean that the grid line does not intersect any vertices. If there is an empty vertical or empty horizontal grid line, the graph can be compressed by shifting the vertices to populate the empty space. See Fig. 4.1 for an illustration. Note that a grid line cannot only intersect with an edge of the graph, as otherwise it would have at least 2 bending points, thus have more than one horizontal or one vertical segment. We shall assume that our orthogonal graph drawing $E^{G(I)}$ does not have any empty vertical nor empty horizontal grid lines that intersect the graph.

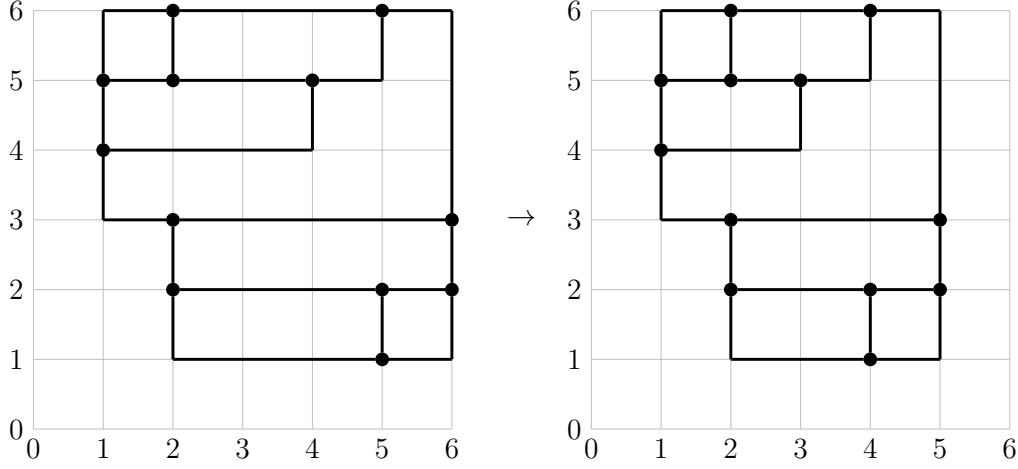


Figure 4.1: Example 3-regular orthogonal graph drawing, where each edge has at most one vertical and horizontal segment. The vertical grid line $x = 3$, which intersects the graph, is initially empty.

In the orthogonal drawing $E^{G(I)}$, the edges of the grid \mathbb{Z}^2 are set to length 10. Since in $E^{G(I)}$ there are no empty vertical nor empty horizontal grid lines that intersect the graph, the height and width of the graph in the embedding is at most $10 \cdot (|X| + |C|)$. See Fig. 4.2 for an example orthogonal drawing.

For each element-vertex $u_i \in U$, we create an agent u_i for V_b , which we call an element-agent, with the same xy -coordinates as the element-vertex u_i and z -coordinate 0 for embedding E^b .

Additionally, for each edge $\{u_i, w_j\}$ in $G(I)$, if $\{u_i, w_j\}$ has a bending point in $E^{G(I)}$, we create an agent b_j^i for V_b , which we call a bending point agent, with the same xy -coordinates as the bending point and z -coordinate 0 for embedding E^b . Let $B = \{b_j^i | \{u_i, w_j\} \text{ in } G(I) \wedge \{u_i, w_j\} \text{ has bending point in } E^{G(I)}\}$ denote the set of all bending point agents.

Finally, for each set-vertex $w_j \in W$ we create three agents w_j^i for V_b , where $i \in C_j$, which form an equilateral triangle with edge length 1. Specifically, for a set-vertex $w_j \in W$, we can assume w.l.o.g. that it has three connecting edges going upwards, leftward, and rightward connecting to element-vertices u_i, u_p , and u_q , respectively. If this is not the case, we can rotate the coordinate system and apply the same reasoning. We create three set-agents $w_j^i, w_j^p, w_j^q \in V_b$ to

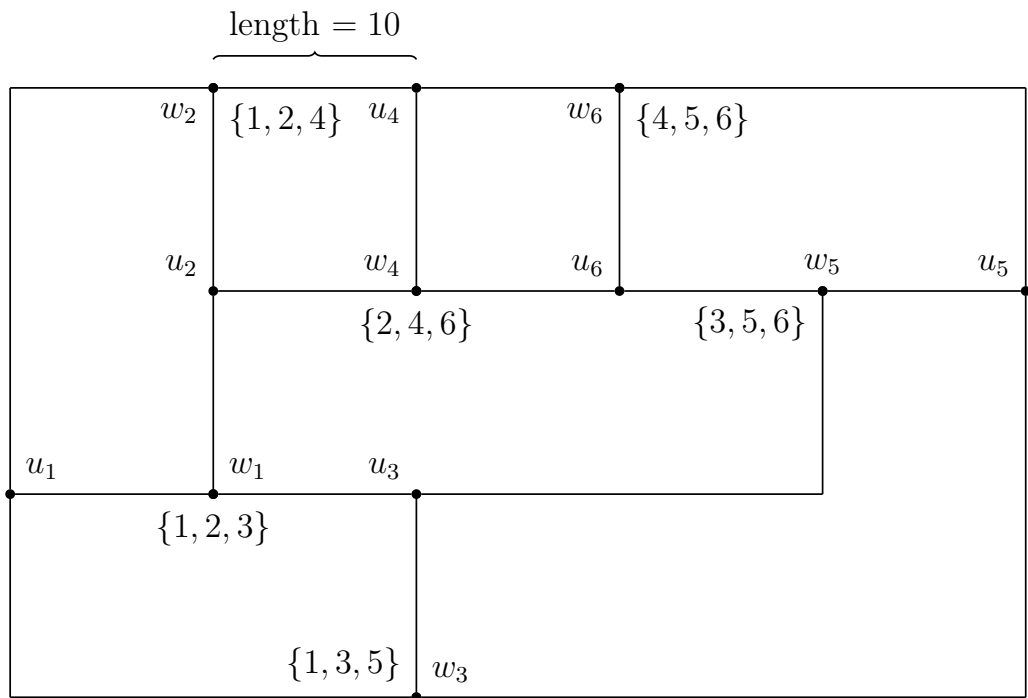


Figure 4.2: Example orthogonal drawing $E^{G(I)}$, where
 $I = (X, C)$,
 $X = \{1, 2, 3, 4, 5, 6\}$,
 $C = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\}\}$.

replace w_j . The set-agents w_j^i, w_j^p, w_j^q are embedded such that they are on the segment of the upward, leftward, and rightward edge, respectively, and are of equidistance 1 to each other. As previous agents, each of these set-agents have z -coordinate 0. Let $W' = \{w_j^i, w_j^p, w_j^q | C_j = \{i, p, q\} \in C\}$ be the set of set-agents. See Fig. 4.3 for an illustration.

Let us modify $G(I)$ to create the graph $G'(I) = (V', E')$. Let V' be the vertices consisting of all element-agents, set-agents, and bending point agents, i.e., $V' = U \cup B \cup W'$. For each edge $\{u_i, w_j\}$ in $G(I)$, if $\{u_i, w_j\}$ has a bending point in $E^{G(I)}$, then we create the edges $\{u_i, b_j^i\}, \{b_j^i, w_j^i\}$ and place them in E' . Otherwise we create the edge $\{u_i, w_j^i\}$ and place it in E' . Let us consider the embedding $E^{G'(I)} : V' \rightarrow \mathbb{Z}^2$, that maps the vertices in V' to the same x, y -coordinates that embedding E^b assigns. That is, for each $v \in V'$ we have $E^{G'(I)}(v) = (E_x^b(v), E_y^b(v))$. See Fig. 4.4 for an example.

Note that for each edge in E' its segment in embedding $E^{G'(I)}$ is either horizontal or vertical. We replace the segment in $E^{G'(I)}$ of each edge in $G'(I)$ with a chain of copies of three agents. This chain ensures that either all three agents w_j^i for $i \in C_j$ are matched in the same triple (indicating that the corresponding set is in the solution) or none of them is matched in the same triple (indicating that the corresponding set is not in the solution).

For an edge $e \in E'$, let s_e denote its corresponding segment in $E^{G'(I)}$. Let $S_{G'(I)} = \{s_e | e \in E'\}$ denote the set of all segments of the edges in $E^{G'(I)}$ and for a segment $s \in S_{G'(I)}$ let $l(s)$ denote the length of segment s .

For each segment $s \in S_{G'(I)}$ we create $\hat{n} = \left\lceil \frac{l(s)}{d} \right\rceil$ copies of the triple $A_s[z] = \{\alpha_s[z], \beta_s[z], \gamma_s[z]\}$, where $z \in [\hat{n}]$, for V_b . We embed the agents $A_s[z]$,

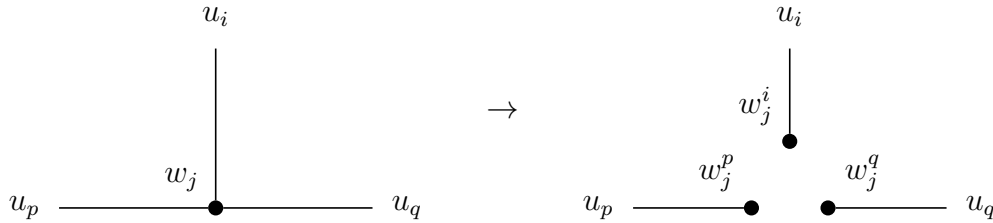


Figure 4.3: Gadget for set-vertex w_j , where $C_j = \{i, p, q\}$ and w_j^i, w_j^p, w_j^q form equilateral triangle with edge length 1.

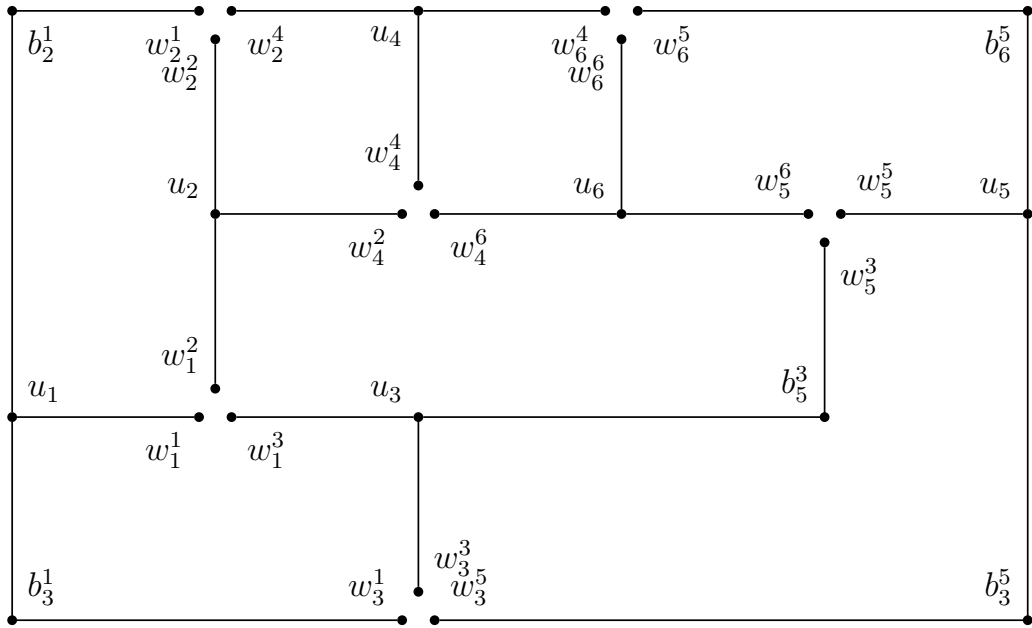


Figure 4.4: Example embedding $E^{G'(I)}$, where
 $I = (X, C)$,
 $X = \{1, 2, 3, 4, 5, 6\}$,
 $C = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\}\}$.
 Note that the distances between w_j^i, w_j^p, w_j^q are not to scale.

where $z \in [\hat{n}]$, around segment s to connect the two vertices at the endpoints of s in embedding $E^{G'(I)}$. For convenience, when using \hat{n} , we refer to the constant corresponding to segment s which should be clear from the context. Let $f, t \in V'$ denote the vertices at the endpoints of segment s . We merge vertex t and $\gamma_s[\hat{n}]$ into one vertex, while referring to it by either name. For convenience, we shall use $\gamma_s[0]$ to refer to f . The agents will be embedded such that the following hold for each $z \in [\hat{n}]$:

- The distance between agents $\alpha_s[z]$ and $\beta_s[z]$ is ϵ .
- The distance between agents $\alpha_s[z]$ (resp. $\beta_s[z]$) and $\gamma_s[z]$ is 1.
- The distance between agents $\alpha_s[z]$ (resp. $\beta_s[z]$) and $\gamma_s[z - 1]$ is 1.

This embedding ensures that for a strictly popular outcome that either all $A_s[z]$, where $z \in [\hat{n} - 1]$, or all $\gamma_s[z - 1], \alpha_s[z], \beta_s[z]$ must be matched together.

As the length of the chain $A_s[z]$, where $z \in [\hat{n}]$, is slightly longer than $l(s)$, the chain $A_s[z]$ is ‘bent’ into the xy -plane in the negative z -direction in embedding E^b . The agents in $A_s[0]$ and $A_s[\hat{n}]$ have the closest z -coordinate to 0. The agents in $A_s[\lceil \hat{n}/2 \rceil]$ and $A_s[\lfloor \hat{n}/2 \rfloor]$ have the furthest z -coordinate from 0. For each $z \in [\hat{n}]$, the vertices $\alpha_s[z]$ and $\beta_s[z]$ have the same z -coordinate. See Figs. 4.5, 4.6 and 4.7 for an illustration.

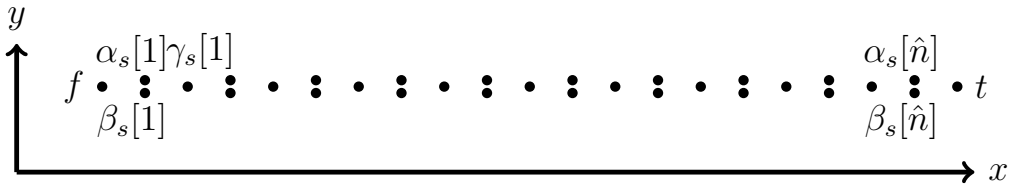


Figure 4.5: Projection of example chain $A_s[z]$, where $z \in [\hat{n}]$, of segment s , where t, f are the endpoints of s , on the xy -plane.

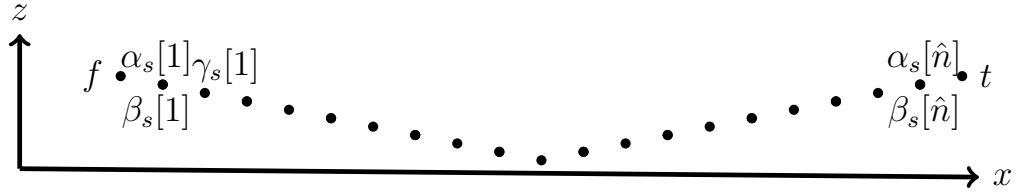


Figure 4.6: Projection of example chain $A_s[z]$, where $z \in [\hat{n}]$, of segment s , where t, f are the endpoints of s , on the xz -plane.

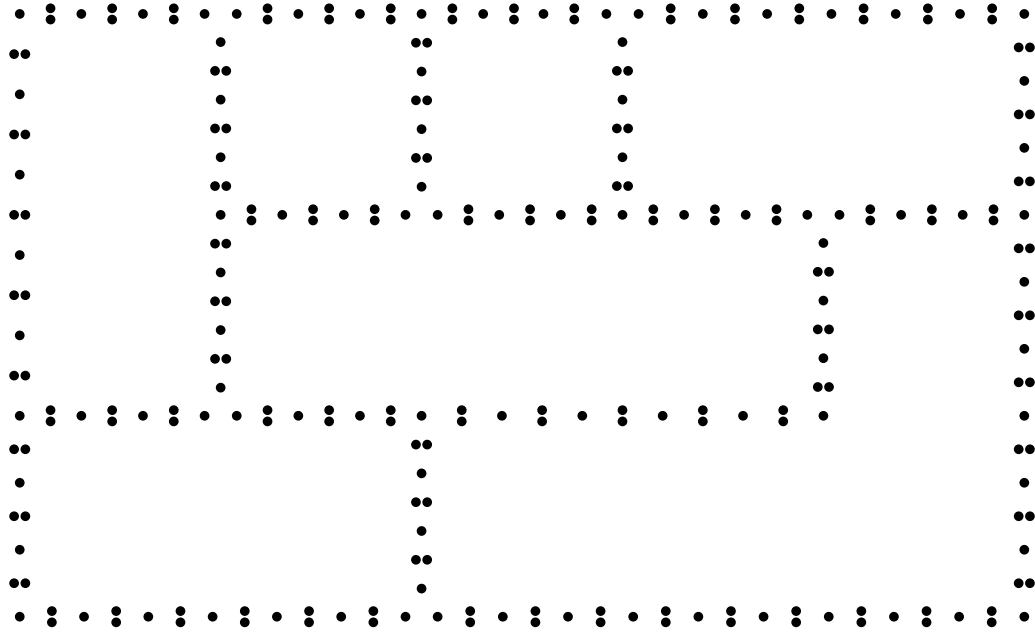


Figure 4.7: Segments of embedding $E^{G'(I)}$ of Fig. 4.4 replaced with chains. The number of agents in the chains are reduced for clarity.

In conclusion, for the bottom layer we have that

$$V_b = U \cup B \cup W' \cup \bigcup_{s \in S_{G'(I)}, z \in [\hat{n}]} A_s[z].$$

These agents in $\bigcup_{s \in S_{G'(I)}, z \in [\hat{n}]} A_s[z]$ are embedded in E^b as defined above and

for each agent $a \in U \cup B \cup W'$, we have that $E^b(a) = (E_x^{G'(I)}, E_y^{G'(I)}, 0)$.

4.1.2 Construction of the Top Layer

In top layer, the embedding of the agents is shaped somewhat similarly to a snowflake. It consists of a triple of agents $d_{1,0}^1, d_{2,0}^1, d_{3,0}^1$, which we call center agents, placed in the center in embedding E^t that form an equilateral triangle with edge length 1. Out each of these center agents, a binary tree-like structure grows. In this structure, the parents and their two children also form an equilateral triangle. The edge length of the equilateral triangle at depth 0 is large, but decreases exponentially as the depth increases. The total depth depends on the size of X . Finally, the edges are replaced with chains similar to the chains in the bottom layer. The constructed top layer will be placed on the xy -plane with z -coordinate $10|X|+10$ directly above the bottom layer to create distance from the bottom layer. In the construction of the top layer, the embedding of the edges will not have any bending points.

We shall construct an initial snowflake graph $G_{snow} = (V_{snow}, E_{snow})$ with embedding $E^{G_{snow}} : V_{snow} \rightarrow \mathbb{R}^2$. Let $k \in \mathbb{R}$ be such that $|X| = 3 \cdot 2^k$. The binary tree structure attached to a center vertices $d_{1,0}^1, d_{2,0}^1, d_{3,0}^1 \in V_{snow}$ will have depth $\lfloor k \rfloor$. We shall label the vertices in the binary tree structure attached to center vertex $d_{i,0}^1$ at depth j by $d_{i,j}^l \in V_{snow}$, where $l \in [j]$. The edge length of the equilateral triangle that a vertex at depth j forms with its children is $10(|X|+|C|) + 4^{\lfloor k \rfloor - j + 2}$. See Fig. 4.8 for an example initial snowflake graph and embedding.

We create a balanced binary tree gadget that has depth 2. a vertex at depth j forms an equilateral triangle with edge length $10(|X|+|C|) + 4^{2-j}$. The vertices in the balanced binary tree gadget will follow the same naming convention as the aforementioned vertices in the binary tree structure in the

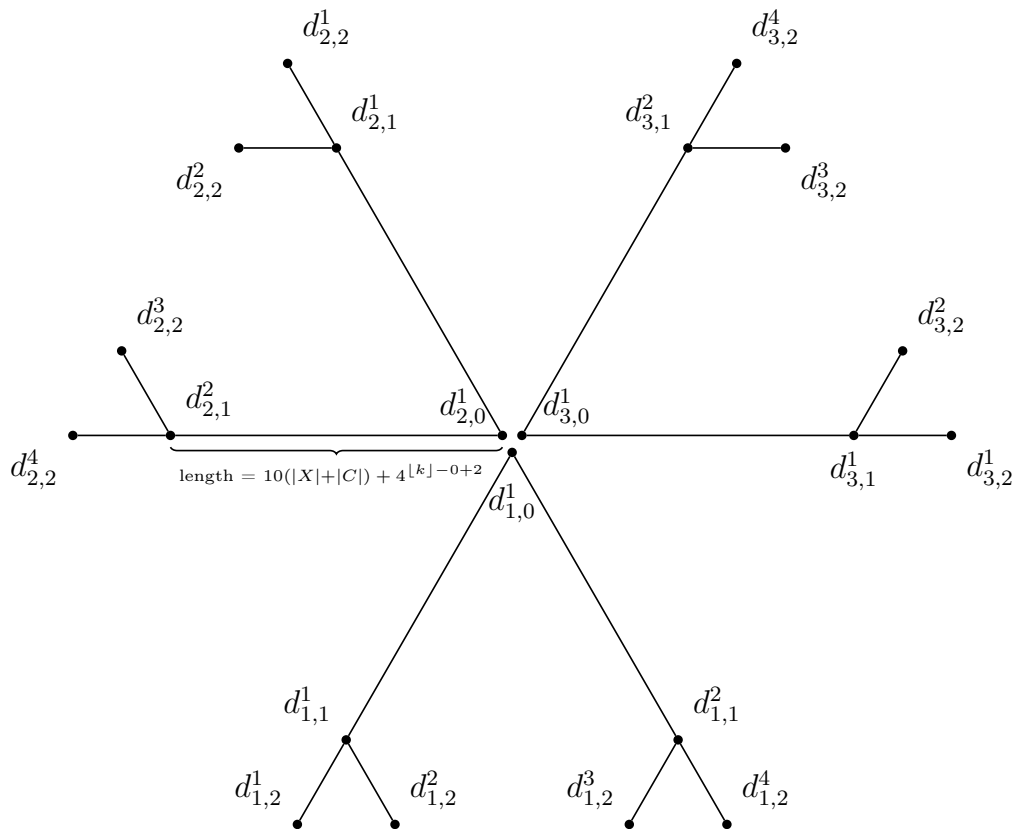


Figure 4.8: Initial snowflake graph G_{snow} and its embedding $E^{G_{snow}}$ where $|X|=12$. Note that $\lfloor k \rfloor = 2$.

initial snowflake graph. See Fig. 4.9 for a figure of the balanced binary tree gadget.

If $k \notin \mathbb{N}$, we have that the number of leaves in G_{snow} is smaller than $|X|$. We replace $\frac{|X|-3 \cdot 2^{\lfloor k \rfloor}}{3}$ leaves with the balanced binary tree gadget to create the unbalanced snowflake graph $G'_{snow} = (V'_{snow}, E'_{snow})$. Let $d_{i,j}^l$ be a leaf that is replaced by the balanced binary tree gadget. We label the root vertex of that balanced binary tree gadget with $d_{i,j}^l$.

For the embedding $E^{G'_{snow}} : V'_{snow} \rightarrow \mathbb{R}^2$, the balanced binary tree gadgets are embedded in the same manner as the binary tree structures in $E^{G_{snow}}$. This replacement ensures that the number of leaves in G'_{snow} is equivalent to $|X|$. Note that it is always possible to replace (a part of) the leaves of G_{snow} with the balanced binary tree gadget to achieve the desired number of leaves in G'_{snow} . See Fig. 4.10 for an example graph and embedding.

We modify the snowflake graph by replacing each internal vertex with 3 vertices that form an equilateral triangle with edge length 1. One these vertices will be placed on the exact same location as the original internal vertex. The other 2 vertices will be placed on the edges between the internal vertex and its children. The vertices placed on the edges are connected to the child at the end of its corresponding edge. See Figs. 4.11 and 4.12 for an illustration of the gadget.

Let $G_{triple} = (V_{triple}, E_{triple})$ denote the graph after applying the internal vertex gadget to graph G'_{snow} and embedding $E^{G_{triple}} : V_{triple} \rightarrow \mathbb{R}^2$ be as described above.

Let $d_{i,j}^l$ be a leaf of one of the trees in G'_{snow} . For convenience, we shall also use the label $d_{i,j}^{l,1}$ refer to the same leaf.

Let $l(e)$, where $e \in E_{triple}$, denote the length of e in embedding $E^{G_{triple}}$. The final step for the top layer is to replace each edge e in G_{triple} with $\hat{n} = \left\lceil \frac{l(e)}{d} \right\rceil$ copies of the triple $A_e[z] = \{\alpha_e[z], \beta_e[z], \gamma_e[z]\}$, where $z \in [\hat{n}]$. These triples are embedded such that they have the same distance properties as they have in the bottom layer. These properties are:

- The distance between agents $\alpha_s[z]$ and $\beta_s[z]$ is ϵ .
- The distance between agents $\alpha_s[z]$ (resp. $\beta_s[z]$) and $\gamma_s[z]$ is 1.
- The distance between agents $\alpha_s[z]$ (resp. $\beta_s[z]$) and $\gamma_s[z-1]$ is 1.

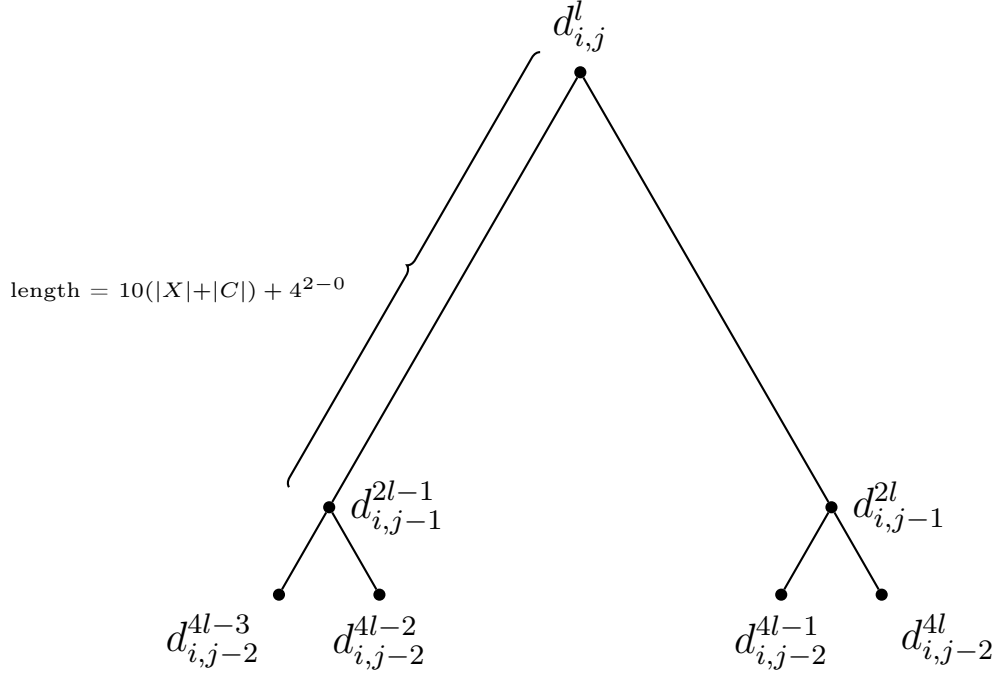


Figure 4.9: balanced binary tree gadget.

The embedding of the chain $A_s[z]$ is almost the exact same as in the bottom layer. Instead of ‘bending’ the chain $A_s[z]$ in the negative direction, in the top layer the chain is ‘bent’ in the positive z -direction in embedding E^t . See Figs. 4.13, 4.14 and 4.15 for an example.

In conclusion, for the top layer we have that $V_t = V_{triple} \cup \bigcup_{e \in E_{triple}, z \in [\hat{n}]} A_s[z]$. As mentioned in the beginning of this section, the top layer is placed on the xy -plane with z -coordinate $10|X|+10$. Thus, for each agent $a \in V_{triple}$, we have that $E^t(a) = (E_x^{G_{triple}}, E_y^{G_{triple}}, 10|X|+10)$. The agents in $\bigcup_{e \in E_{triple}, z \in [\hat{n}]} A_s[z]$ are embedded in E^t as defined above.

4.1.3 Construction of the Ascending Layer

The purpose of the ascending layer is to connect the bottom layer with the top layer. In the construction of the ascending layer, the embedding of the edges will not have any bending points.

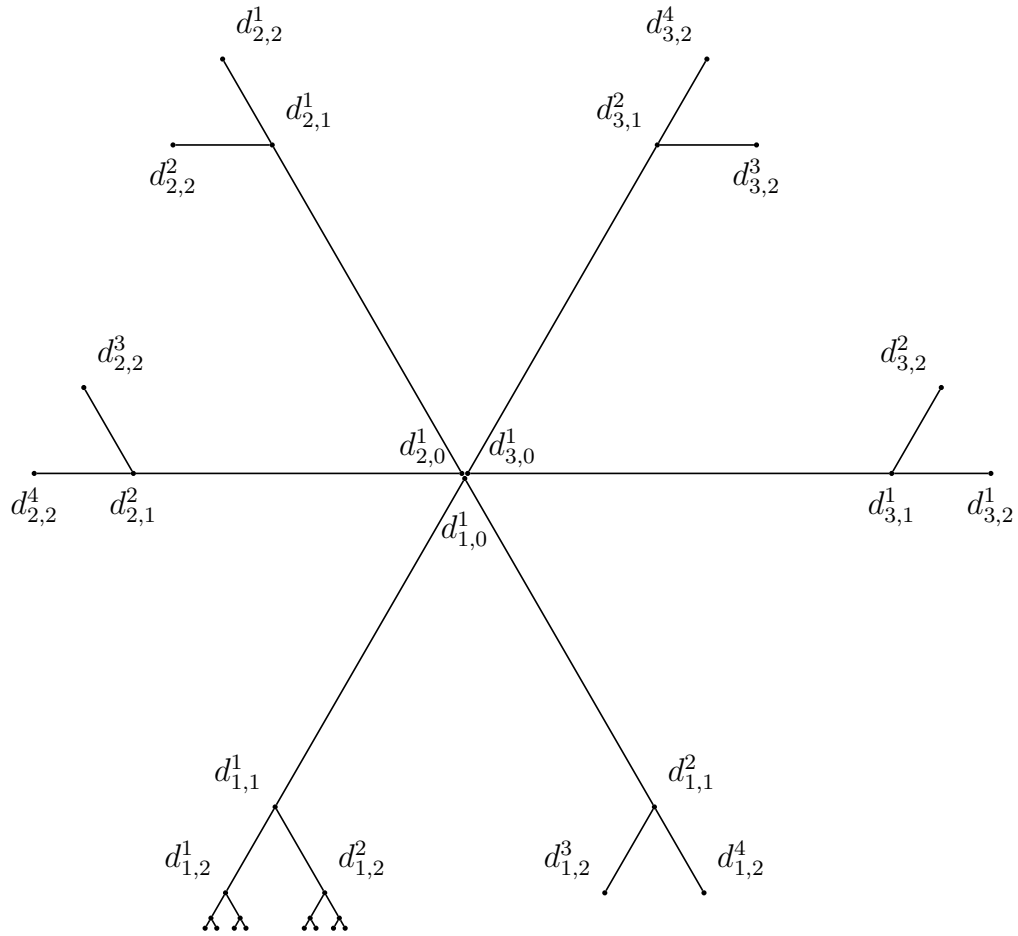


Figure 4.10: unbalanced snowflake graph G'_{snow} and its embedding $E^{G'_{snow}}$ where $|X|=21$. Note that $\lfloor k \rfloor = 2$.

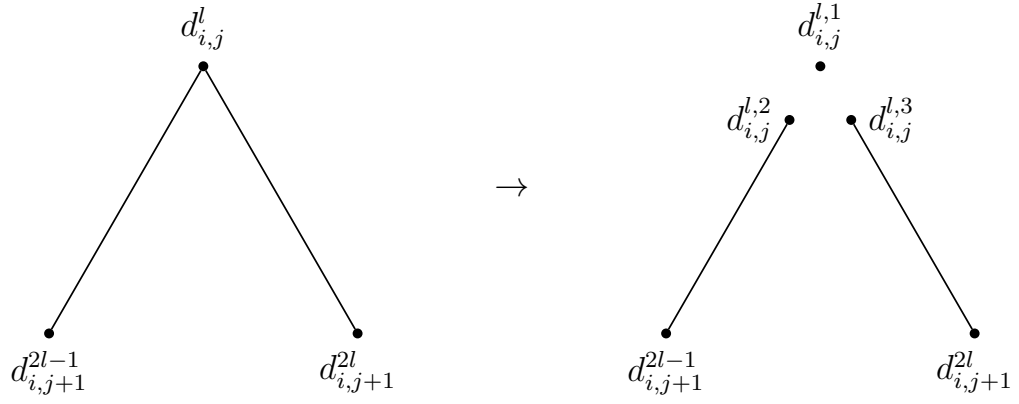


Figure 4.11: internal vertex gadget.

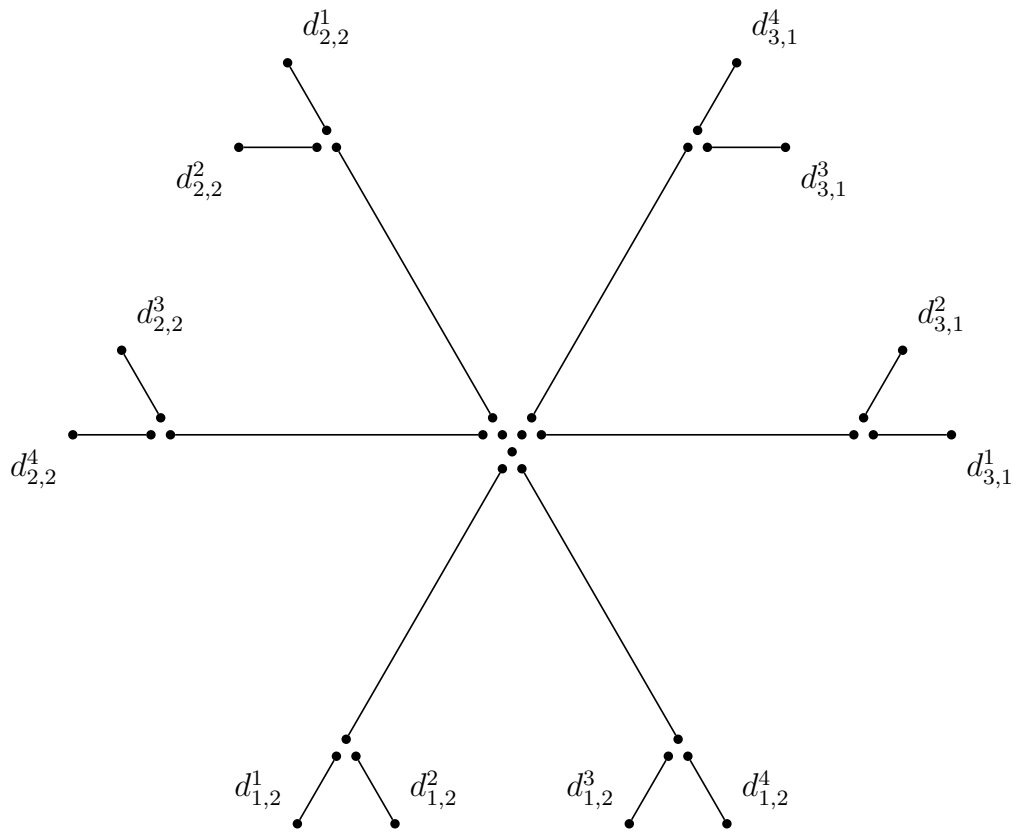


Figure 4.12: Graph G_{triple} , which is internal vertex gadget applied to G_{snow} of Fig. 4.9.

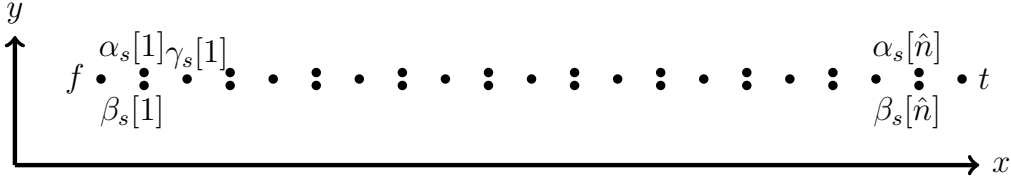


Figure 4.13: Projection of example chain $A_s[z]$, where $z \in [\hat{n}]$, of edge e , where t, f are the endpoints of e , on the xy -plane.

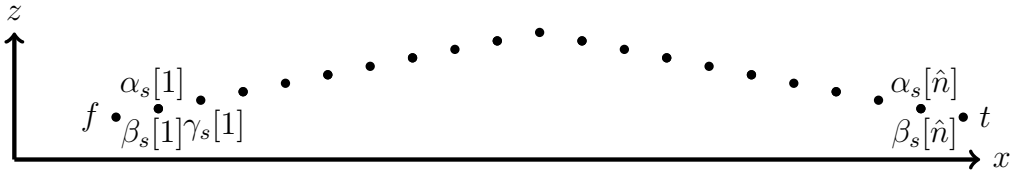


Figure 4.14: Projection of example chain $A_s[z]$, where $z \in [\hat{n}]$, of edge e , where t, f are the endpoints of e , on the xz -plane.

Let us arbitrarily label the leaves in the top layer as l_1, \dots, l_m . We shall connect vertex agent u_i from the bottom layer to the leaf l_i of the top layer, where $i \in [m]$.

We shall create a graph $G_{asc} = (V_{asc}, E_{asc})$ and embedding $E^{G_{asc}} : V_{asc} \rightarrow \mathbb{R}^3$. For each $u_i \in U$, let us create 2 auxiliary vertices u'_i, l'_i . The vertex set V_{asc} consists of all element vertices from the bottom layer, the leaves from the top layer and the aforementioned auxiliary vertices. That is, $V_{asc} = U \cup \{l_i, u'_i, l'_i | i \in [m]\}$. In embedding $E^{G_{asc}}$, the element vertices and leaves are embedded in the same way as in embedding E^b and E^t , respectively. That is, for $i \in [m]$, $E^{G_{asc}}(u_i) = E^b(u_i)$ and $E^{G_{asc}}(l_i) = E^t(l_i)$.

The auxiliary vertex u'_i is placed directly above vertex u_i at height $10i$ in embedding $E^{G_{asc}}$. That is, $E^{G_{asc}}(u'_i) = (E_x^b(u_i), E_y^b(u_i), 10i)$. The auxiliary vertex l'_i is placed directly below vertex l_i also at height $10i$ in embedding $E^{G_{asc}}$. That is, $E^{G_{asc}}(l'_i) = (E_x^t(l_i), E_y^t(l_i), 10i)$.

The edges in E_{asc} consists of $\{u_i, u'_i\}$, $\{u'_i, l'_i\}$, and $\{l'_i, l_i\}$, for each $i \in [m]$. See Fig. 4.16 for an illustration.

Let $l(e)$, where $e \in E_{asc}$, denote the length of e in embedding $E^{G_{asc}}$. The final step for the ascending layer is to replace edge e in G_{asc} with $\hat{n} = \left\lceil \frac{l(e)}{d} \right\rceil$

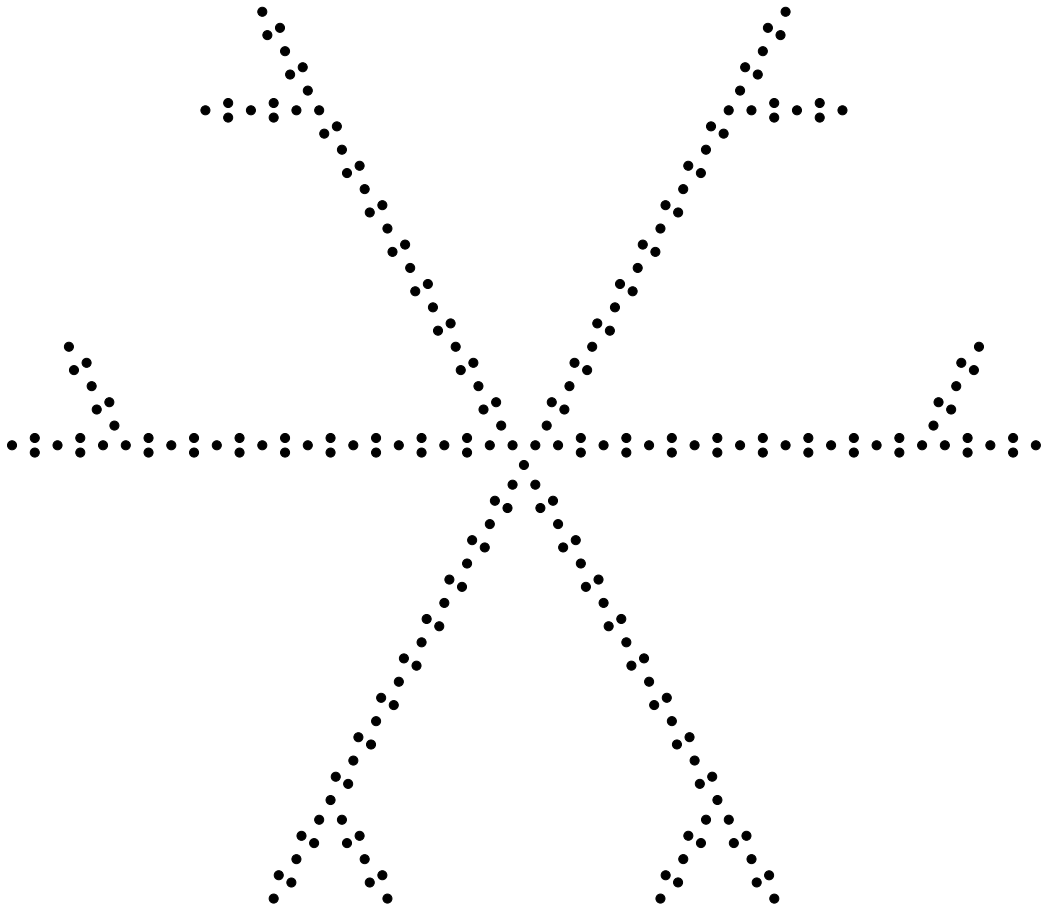


Figure 4.15: The edges of Fig. 4.12 replaced with the chains. The number of agents and the distances between the agents are exaggerated for clarity.

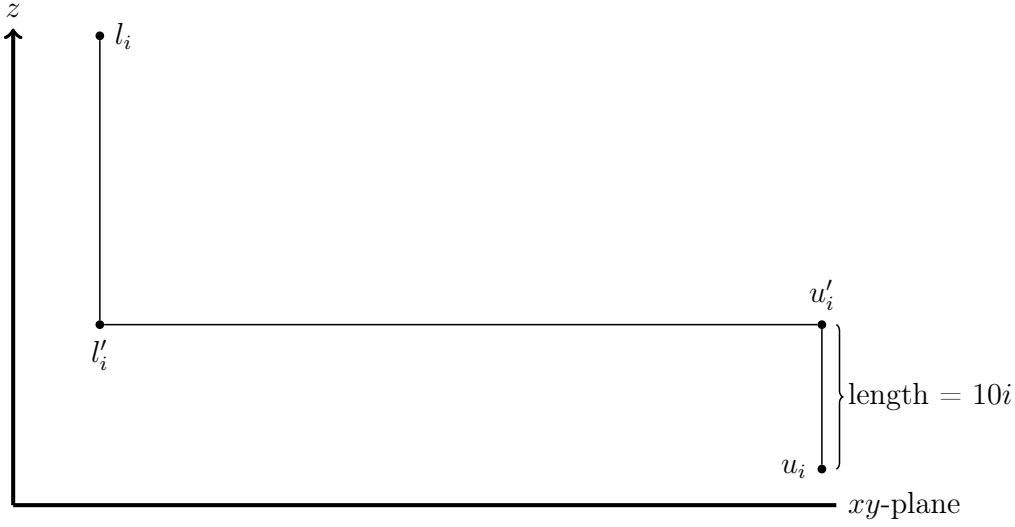


Figure 4.16: Initial ascending graph with element vertex u_i and leaf l_i .

copies of the triple $A_e[z] = \{\alpha_e[z], \beta_e[z], \gamma_e[z]\}$, where $z \in [\hat{n}]$. These triples are embedded such that they have the same distance properties as in the bottom and top layer.

As in the bottom layer and the top layer the chain A_e is longer than edge e . We shall define 3 distinct bending directions for the edges $\{u_i, u'_i\}$, $\{u'_i, l'_i\}$, and $\{l'_i, l_i\}$ in embedding E^a . The chain $A_{\{u_i, u'_i\}}[z]$ will be bent in the $u'_i - l'_i$ direction. The chain $A_{\{u'_i, l'_i\}}[z]$ will be bent in the negative z -direction. The chain $A_{\{l'_i, l_i\}}[z]$ will be bent in the $l'_i - u'_i$ direction. See Fig. 4.17 for an illustration.

In conclusion, for the ascending layer we have that

$$V_a = V_{asc} \cup \bigcup_{e \in E_{asc}, z \in [\hat{n}]} A_e[z].$$

These agents are embedded as described above.

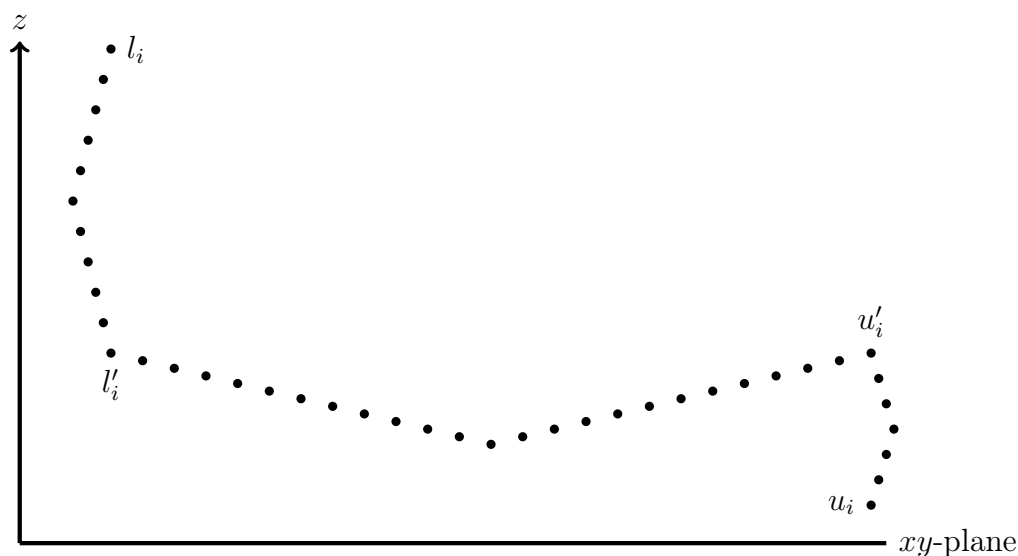


Figure 4.17: Replacing the edges in Fig. 4.16 with the chains $A_e[z]$, where $e \in \{\{u_i, u'_i\}, \{u'_i, l'_i\}, \{l'_i, l_i\}\}$.

4.2 Instance size

We shall demonstrate that the size of the constructed 3D-EuclidSR game G is polynomial in the size of the X3C instance I . This is shown by determining the order of the number of agents in each layer of G separately.

4.2.1 Instance size of the Bottom Layer

We compute the bound on the number of agents in the bottom layer by determining the maximum length of a segment and the total number of segments in $G'(I)$ with embedding $E^{G'(I)}$.

The length of a segment in embedding $E^{G'(I)}$ is at most $10 \cdot (|X| + |C|)$, which is the maximum width and height of the graph in the embedding. Each segment s is replaced with an agent triple chain $A_s[z]$ of which the amount of is linear in the length of the segment. Note that the number of segments is at most $2 \cdot |E'| = 2 \cdot 3 \cdot |X|$. Additionally, there are element agents, set agents and bending point agents. The number of bending point agents is at most the number of element agents. Note that $|X| = |C|$. Thus, the order of the

number of agents is

$$\begin{aligned}
& \mathcal{O}(2 \cdot 3 \cdot |X| \cdot (10 \cdot (|X| + |C|)) + |X| + |C| + |X|) \\
&= \mathcal{O}(60|X|^2 + 60|X||C| + 2|X| + |C|) \\
&= \mathcal{O}(|X|^2 + |X||C|) \\
&= \mathcal{O}(|X|^2).
\end{aligned}$$

4.2.2 Instance size of the Top Layer

The bound on the number of agents in top layer be computed by analyzing the binary tree structures and balanced binary tree gadgets of G'_{snow} .

The depth of the tree is $\lfloor k \rfloor + 2$. The edge length between depth j and $j + 1$ is $10(|X| + |C|) + 4^{\lfloor k \rfloor - j + 2}$. The number of edges between depth j and $j + 1$ is 2^{j+1} . There are 2^j internal vertices at depth j . We have that $|X| \geq 3 \cdot 2^{\lfloor k \rfloor} \iff \lg \frac{|X|}{3} \geq \lfloor k \rfloor$.

Each edge $e \in E_{triple}$ is replaced with an agent triple chain A_e . The amount of agents in chain A_e is linear in the length of the edge e . Additionally, each internal vertex in the tree is replaced by a triple. Thus, the total amount of agents is in the order of

$$\begin{aligned}
& \mathcal{O}\left(3 \cdot \left(\sum_{j=0}^{\lfloor k \rfloor + 1} 2^{j+1} \cdot (10(|X| + |C|) + 4^{\lfloor k \rfloor - j + 2})\right) + 3 \cdot \sum_{j=0}^{\lfloor k \rfloor + 1} 3 \cdot 2^j\right) \\
&= \mathcal{O}\left(3 \cdot \left(\sum_{j=0}^{\lfloor k \rfloor + 1} 2^{j+1} \cdot 20|X| + 2^{2\lfloor k \rfloor - j + 5}\right) + 9 \cdot (2^{\lfloor k \rfloor + 2} - 1)\right) \\
&= \mathcal{O}\left(60|X| \cdot \left(\sum_{j=0}^{\lfloor k \rfloor + 1} 2^{j+1}\right) + 3 \cdot \left(\sum_{j=0}^{\lfloor k \rfloor + 1} 2^{2\lfloor k \rfloor - j + 5}\right) + 9 \cdot (2^{\lfloor k \rfloor + 2} - 1)\right) \\
&= \mathcal{O}\left(120|X| \cdot \left(\sum_{j=0}^{\lfloor k \rfloor + 1} 2^j\right) + 3 \cdot 2^{2\lfloor k \rfloor + 5} \cdot \left(\sum_{j=0}^{\lfloor k \rfloor + 1} \frac{1}{2^j}\right) + 9 \cdot (2^{\lfloor k \rfloor + 2} - 1)\right) \\
&= \mathcal{O}\left(120|X| \cdot (2^{\lfloor k \rfloor + 2} - 1) + 3 \cdot 2^{2\lfloor k \rfloor + 5} \cdot \left(2 - \frac{1}{2^{\lfloor k \rfloor + 1}}\right) + 9 \cdot (2^{\lfloor k \rfloor + 2} - 1)\right) \\
&= \mathcal{O}\left(480|X| \cdot 2^{\lfloor k \rfloor} - 120|X| + 3 \cdot 2^5 \cdot (2^{\lfloor k \rfloor})^2 \cdot \left(2 - \frac{1}{2 \cdot 2^{\lfloor k \rfloor}}\right) + 36 \cdot 2^{\lfloor k \rfloor} - 9\right)
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{O}(480|X| \cdot \frac{|X|}{3} - 120|X| + 3 \cdot 2^5 \cdot (\frac{|X|}{3})^2 \cdot (2 - \frac{1}{2 \cdot \frac{|X|}{3}}) + 36 \cdot \frac{|X|}{3} - 9) \\
&= \mathcal{O}(|X|^2).
\end{aligned}$$

4.2.3 Instance size of the Ascending Layer

We bound the number of agents in the ascending layer by counting the number and lengths of edges in the initial ascending graph.

There are 2 edges in E_{asc} parallel to the z -axis of length $10i$, where $i \in [|X|]$. Additionally, there are $|X|$ edges in E_{asc} parallel to the xy -plane, of length at most the width of the top layer. The width of the top layer is $\sum_{j=0}^{\lfloor k \rfloor + 1} 10(|X| + |C|) + 4^{\lfloor k \rfloor - j + 2}$.

Each edge $e \in E_{asc}$ is replaced with an agent triple chain A_e . The amount of agents in chain A_e is linear in the length of edge e . Thus, the total amount of agents is in the order of

$$\begin{aligned}
&\mathcal{O}\left(\sum_{i=1}^{|X|} 2 \cdot 10i + |X| \cdot \left(\sum_{j=0}^{\lfloor k \rfloor + 1} 10(|X| + |C|) + 4^{\lfloor k \rfloor - j + 2}\right)\right) \\
&= \mathcal{O}\left(20 \cdot \left(\sum_{i=1}^{|X|} i\right) + |X| \cdot \left(\left(\sum_{j=0}^{\lfloor k \rfloor + 1} 20|X|\right) + \left(\sum_{j=0}^{\lfloor k \rfloor + 1} 4^{\lfloor k \rfloor - j + 2}\right)\right)\right) \\
&= \mathcal{O}\left(\frac{20|X|(|X|+1)}{2} + (\lfloor k \rfloor + 2) \cdot 20|X|^2 + |X| \cdot 4^{\lfloor k \rfloor + 2} \left(\sum_{j=0}^{\lfloor k \rfloor + 1} \frac{1}{4^j}\right)\right) \\
&= \mathcal{O}\left(\frac{20|X|^2 + 20|X|}{2} + (\lfloor k \rfloor + 2) \cdot 20|X|^2 + |X| \cdot (4 \cdot 2^{\lfloor k \rfloor})^2 \cdot \left(\frac{1 - \frac{1}{4}^{\lfloor k \rfloor + 2}}{1 - \frac{1}{4}}\right)\right) \\
&= \mathcal{O}\left(\frac{20|X|^2 + 20|X|}{2} + (\lfloor k \rfloor + 2) \cdot 20|X|^2 + |X| \cdot \left(4 \cdot \frac{|X|}{3}\right)^2 \cdot \left(\frac{4(1 - (2^4 \cdot (2^{\lfloor k \rfloor})^2)^{-1})}{3}\right)\right) \\
&= \mathcal{O}\left(\frac{20|X|^2 + 20|X|}{2} + (\lfloor k \rfloor + 2) \cdot 20|X|^2 + \frac{16|X|^3}{9} \cdot \left(\frac{4(1 - (2^4 \cdot (\frac{|X|}{3})^2)^{-1})}{3}\right)\right) \\
&= \mathcal{O}\left(\frac{20|X|^2 + 20|X|}{2} + (\lfloor k \rfloor + 2) \cdot 20|X|^2 + \frac{64|X|^3}{27} \cdot \left(1 - \frac{9}{16|X|^2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{O}\left(\frac{20|X|^2+20|X|}{2} + \left(\lg\left(\frac{|X|}{3}\right) + 2\right) \cdot 20|X|^2 + \frac{64|X|^3}{27} - \frac{576|X|}{432}\right) \\
&= \mathcal{O}(|X|^3).
\end{aligned}$$

4.3 Predefined Outcomes

We define the permanent popular outcome π_{pp} and for a solution $C' \subseteq C$ for I the reduced outcome $\pi_{C'}$. The number of reduced outcomes is equivalent to the number of solutions for I . Note that no reduced outcomes exists, if I has no solution. In these outcomes every agent is in a room that minimizes the distance to its roommates. That is, for any agent a and any room R such that $a \in R$, we have that $\delta(a, \pi_{pp}(a)) \leq \delta(a, R)$ and $\delta(a, \pi_{C'}(a)) \leq \delta(a, R)$.

Observe that an outcome π in which every agent is in a room, that minimizes the distance to its roommates, is popular. In fact, such an outcome is more popular than any outcome that does not assign every agent to a room that minimizes the distance to its roommates.

We show that the permanent popular outcome π_{pp} and the reduced outcomes are the only outcomes that assign every agent to a room that minimizes the distance to its roommates. This implies that the permanent popular outcome π_{pp} and the reduced outcomes are more than any other outcome. If I has no solution, then π_{pp} is the only outcome that is more popular than any other outcome. In this case π_{pp} is strictly popular. If I has a solution, then we have multiple popular outcomes, namely the permanent popular outcome and the reduced outcomes. In this case we do not have a strictly popular outcome.

4.3.1 Permanent popular outcome

We shall define the permanent popular outcome π_{pp} in this section. In this outcome every agent is assigned a room with its closest neighbors. That is, for any outcome π and an arbitrary agent a , we have that $\delta(a, \pi_{pp}(a)) \leq \delta(a, \pi(a))$ and $\delta(a, \pi_S(a)) \leq \delta(a, \pi(a))$. Additionally, a permanent popular outcome always exists independent of I having a solution.

We shall describe how the agents are assigned to the rooms separately for each layer to construct the permanent popular outcome π_{pp} .

Bottom layer.

We shall first construct the rooms of the bottom layer. Let u_i be an element-agent and t be a set-agent or bending point agent such that $\{u_i, t\} \in E'$. For the chain $A_{\{u_i, t\}}$, w.l.o.g. assume that $t = \gamma_{\{u_i, t\}}[\hat{n}]$. We shall create the rooms $\{\alpha_{\{u_i, t\}}[z], \beta_{\{u_i, t\}}[z], \gamma_{\{u_i, t\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

Let w_j^i be a set-agent and f be an element-agent or bending point agent such that $\{f, w_j^i\} \in E'$. For the chain $A_{\{f, w_j^i\}}$, w.l.o.g. assume that $f = \gamma_{\{f, w_j^i\}}[0]$ and $w_j^i = \gamma_{\{f, w_j^i\}}[\hat{n}]$. We shall create the rooms $\{\alpha_{\{u_i, t\}}[z], \beta_{\{u_i, t\}}[z], \gamma_{\{u_i, t\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

See Fig. 4.18 for an illustration.

Ascending layer.

For the ascending layer, let u_i be an element-agent with corresponding leaf l_i and auxiliary agents u'_i, l'_i . For chain $A_{\{u_i, u'_i\}}$, w.l.o.g. assume that $u_i = \gamma_{\{u_i, u'_i\}}[0]$ and $u'_i = \gamma_{\{u_i, u'_i\}}[\hat{n}]$. We shall create the rooms $\{\gamma_{\{u_i, u'_i\}}[z - 1], \alpha_{\{u_i, u'_i\}}[z], \beta_{\{u_i, u'_i\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

For chain $A_{\{u'_i, l'_i\}}$, w.l.o.g. assume that $u'_i = \gamma_{\{u'_i, l'_i\}}[0]$ and $l'_i = \gamma_{\{u'_i, l'_i\}}[\hat{n}]$. We shall create the rooms $\{\gamma_{\{u'_i, l'_i\}}[z - 1], \alpha_{\{u'_i, l'_i\}}[z], \beta_{\{u'_i, l'_i\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

For chain $A_{\{l'_i, l_i\}}$, w.l.o.g. assume that $l'_i = \gamma_{\{l'_i, l_i\}}[0]$ and $l_i = \gamma_{\{l'_i, l_i\}}[\hat{n}]$. We shall create the rooms $\{\gamma_{\{l'_i, l_i\}}[z - 1], \alpha_{\{l'_i, l_i\}}[z], \beta_{\{l'_i, l_i\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

See Fig. 4.19 for an illustration.

Top layer.

Finally, for the top layer, the rooms depend on the parity of $[k]$. The construction of the rooms starts at the leaves and ‘‘crawls up’’ towards the center agents. Let us first consider the case where $[k]$ is odd.

Let $j \in \{j' \in [k] | j' \bmod 2 = 1\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i,j}^{l',1}$ and $d_{i,j-1}^{l',p}$, where $l' \in \{2l - 1, 2l\}$ and $p \in \{2, 3\}$. W.l.o.g. assume that $d_{i,j}^{l',1} = \gamma_e[0]$. We shall create the rooms $\{\gamma_e[z - 1], \alpha_e[z], \beta_e[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$ and room $\{d_{i,j-1}^{l',1}, d_{i,j-1}^{l',2}, d_{i,j-1}^{l',3}\}$.

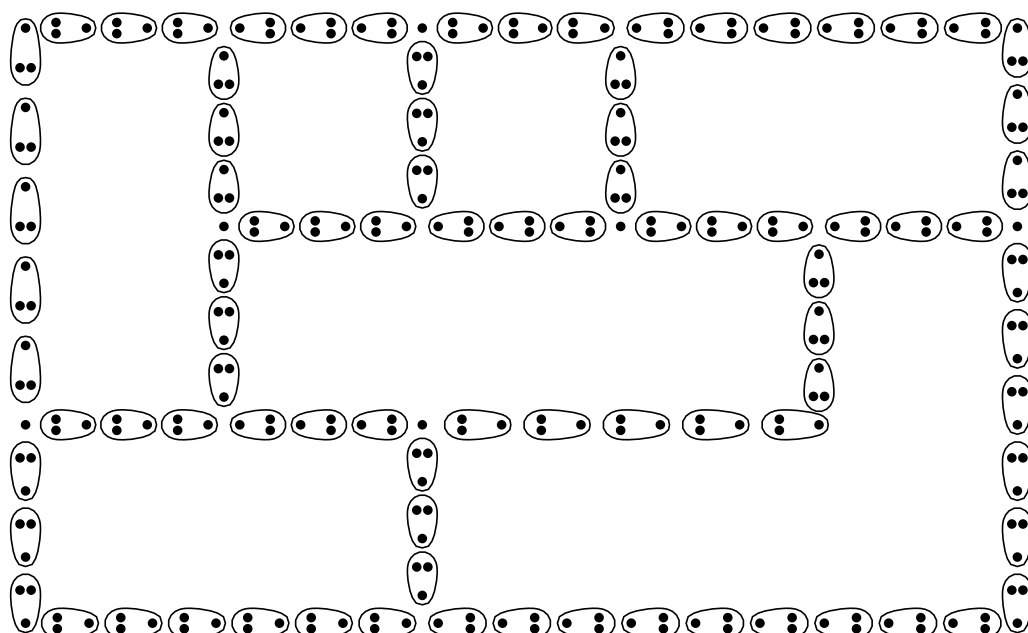


Figure 4.18: Illustration of the rooms in the bottom layer from Fig. 4.7 in the permanent popular outcome.

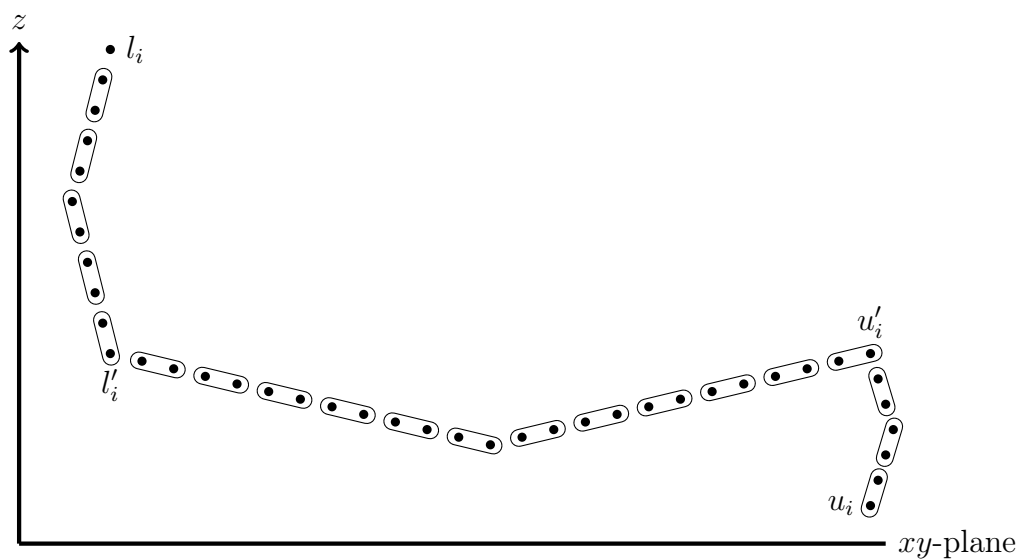


Figure 4.19: Illustration of the rooms in the ascending layer in the permanent popular outcome.

Let $j \in \{j' \in [k-1] \mid j' \bmod 2 = 1\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i,j}^{l,p}$ and $d_{i,j+1}^{l',1}$, where $l' \in \{2l-1, 2l\}$ and $p \in \{2, 3\}$. W.l.o.g. assume that $d_{i,j}^{l,p} = \gamma_e[0]$. We shall create the rooms $\{\gamma_e[z-1], \alpha_e[z], \beta_e[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

Let us now consider the case where $[k]$ is even. First we create room $\{d_{1,0}^{1,1}, d_{2,0}^{1,1}, d_{3,0}^{1,1}\}$.

Let $j \in \{j' \in [k] \mid j' \bmod 2 = 0\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i,j}^{l',1}$ and $d_{i,j-1}^{l,p}$, where $l' \in \{2l-1, 2l\}$ and $p \in \{2, 3\}$. W.l.o.g. assume that $d_{i,j}^{l',1} = \gamma_e[0]$. We shall create the rooms $\{\gamma_e[z-1], \alpha_e[z], \beta_e[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$ and room $\{d_{i,j-1}^{l',1}, d_{i,j-1}^{l',2}, d_{i,j-1}^{l',3}\}$.

Let $j \in \{j' \in [0, k-1] \mid j' \bmod 2 = 0\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i,j}^{l,p}$ and $d_{i,j+1}^{l',1}$, where $l' \in \{2l-1, 2l\}$ and $p \in \{2, 3\}$. W.l.o.g. assume that $d_{i,j}^{l,p} = \gamma_e[0]$. We shall create the rooms $\{\gamma_e[z-1], \alpha_e[z], \beta_e[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$. See Fig. 4.20 for an illustration.

4.3.2 Reduced outcome

In this section, we shall define the reduced outcome π_S , where $S \subseteq C$ is a solution of I . Like the permanent popular outcome, the reduced outcome will assign every agent to a room with its closest neighbors. Additionally, a reduced outcome exists if and only if I has a solution.

We shall describe how the agents are assigned to the rooms separately for each layer to construct the reduced outcome π_S .

Bottom layer.

For the bottom layer, we shall use solution S to assign every agent in the bottom layer to a room. These rooms only contain agents from the bottom layer.

Let $C_j \in S$ and $i \in C_j$. First, for each $C_j \in S$, we shall create the room $\{w_j^i \mid i \in C_j\}$.

We shall first consider the chains from the element-agent u_i towards the set-agent w_j^i corresponding to the 3-set $C_j \in S$ that contains the element i in the solution. Let t be a set-agent or bending point agent such that

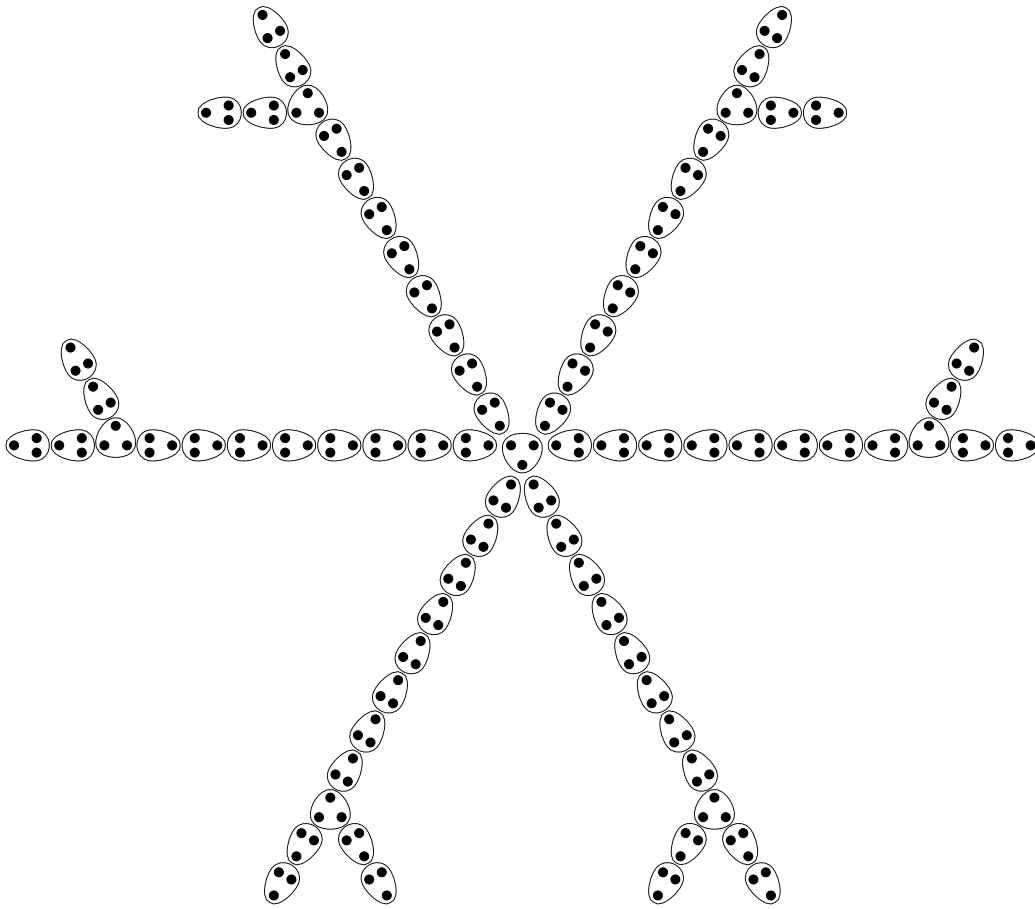


Figure 4.20: Illustration of the rooms in the top layer in the permanent popular outcome.

$\{u_i, t\} \in E'$. For chain $A_{\{u_i, t\}}$, assume w.l.o.g. that $u_i = \gamma_{\{u_i, t\}}[0]$. We shall create the rooms $\{\alpha_{\{u_i, t\}}[z], \beta_{\{u_i, t\}}[z], \gamma_{\{u_i, t\}}[z - 1]\}$ for $z \in [\hat{n}]$.

Let f be an element-agent or bending point agent such that $\{f, w_j^i\} \in E'$. For the chain $A_{\{f, w_j^i\}}$, assume w.l.o.g. that $f = \gamma_{\{f, w_j^i\}}[0]$. We shall create the rooms $\{\alpha_{\{f, w_j^i\}}[z], \beta_{\{f, w_j^i\}}[z], \gamma_{\{f, w_j^i\}}[z - 1]\}$ for $z \in [\hat{n}]$.

Let $C_{j'} \in C$ be a 3-set such that $C_{j'} \notin S$. We shall consider the chains from the element-agent u_i , where $i \in C_{j'}$, towards the set-agent $w_{j'}^i$. Let t be a set-agent or bending point agent such that $\{u_i, t\} \in E'$. For chain $A_{\{u_i, t\}}$, assume w.l.o.g. that $t = \gamma_{\{u_i, t\}}[\hat{n}]$. We shall create the rooms $\{\alpha_{\{u_i, t\}}[z], \beta_{\{u_i, t\}}[z], \gamma_{\{u_i, t\}}[z]\}$ for $z \in [\hat{n}]$.

Let f be an element-agent or bending point agent such that $\{f, w_{j'}^i\} \in E'$. For the chain $A_{\{f, w_{j'}^i\}}$, assume w.l.o.g. that $w_{j'}^i = \gamma_{\{f, w_{j'}^i\}}[\hat{n}]$. We shall create the rooms $\{\alpha_{\{f, w_{j'}^i\}}[z], \beta_{\{f, w_{j'}^i\}}[z], \gamma_{\{f, w_{j'}^i\}}[z]\}$ for $z \in [\hat{n}]$.

See Fig. 4.21 for an illustration.

Ascending layer.

For the ascending layer, let u_i be an element-agent with corresponding leaf l_i and auxiliary agents u'_i, l'_i . For chain $A_{\{u_i, u'_i\}}$, w.l.o.g. assume that $u_i = \gamma_{\{u_i, u'_i\}}[0]$ and $u'_i = \gamma_{\{u_i, u'_i\}}[\hat{n}]$. We shall create the rooms $\{\alpha_{\{u_i, u'_i\}}[z], \beta_{\{u_i, u'_i\}}[z], \gamma_{\{u_i, u'_i\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

For chain $A_{\{u'_i, l'_i\}}$, w.l.o.g. assume that $u'_i = \gamma_{\{u'_i, l'_i\}}[0]$ and $l'_i = \gamma_{\{u'_i, l'_i\}}[\hat{n}]$. We shall create the rooms $\{\alpha_{\{u'_i, l'_i\}}[z], \beta_{\{u'_i, l'_i\}}[z], \gamma_{\{u'_i, l'_i\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

For chain $A_{\{l'_i, l_i\}}$, w.l.o.g. assume that $l'_i = \gamma_{\{l'_i, l_i\}}[0]$ and $l_i = \gamma_{\{l'_i, l_i\}}[\hat{n}]$. We shall create the rooms $\{\alpha_{\{l'_i, l_i\}}[z], \beta_{\{l'_i, l_i\}}[z], \gamma_{\{l'_i, l_i\}}[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$.

See Fig. 4.22 for an illustration

Top layer.

Similar to the permanent popular outcome, the rooms of the top layer in the reduced outcome depends on the parity of $[k]$. Let us first consider the case where $[k]$ is even.

Let $j \in \{j' \in [k] \mid j' \bmod 2 = 0\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i, j}^{l', 1}$ and $d_{i, j-1}^{l', p}$, where $l' \in \{2l - 1, 2l\}$ and $p \in \{2, 3\}$.

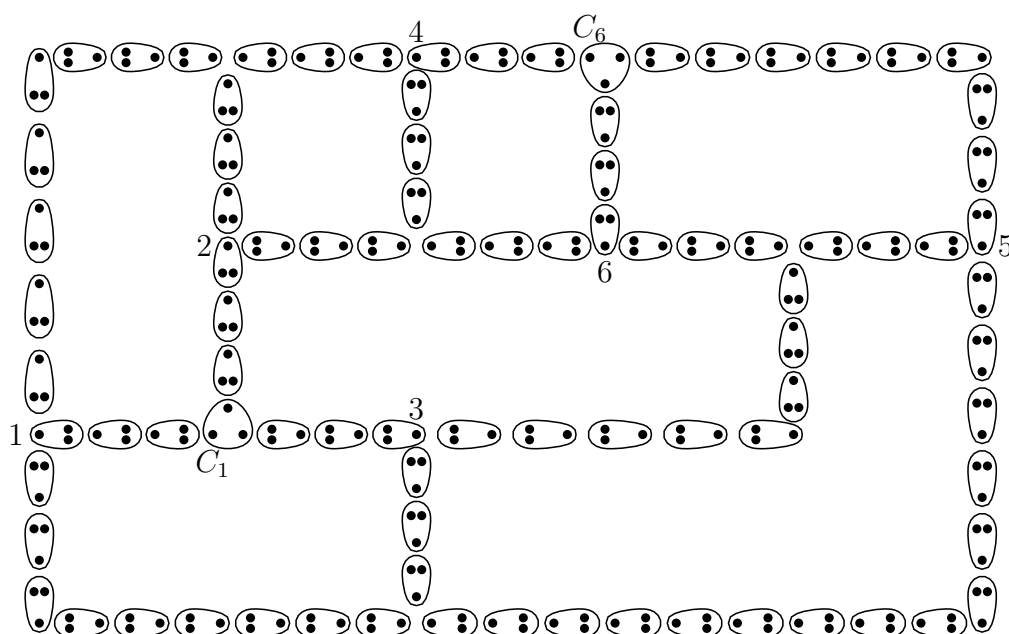


Figure 4.21: Illustration of the rooms in the bottom layer from Fig. 4.7 in the reduced outcome, where the solution is $S = \{\{1, 2, 3\}, \{4, 5, 6\}\} = \{C_1, C_6\}$.

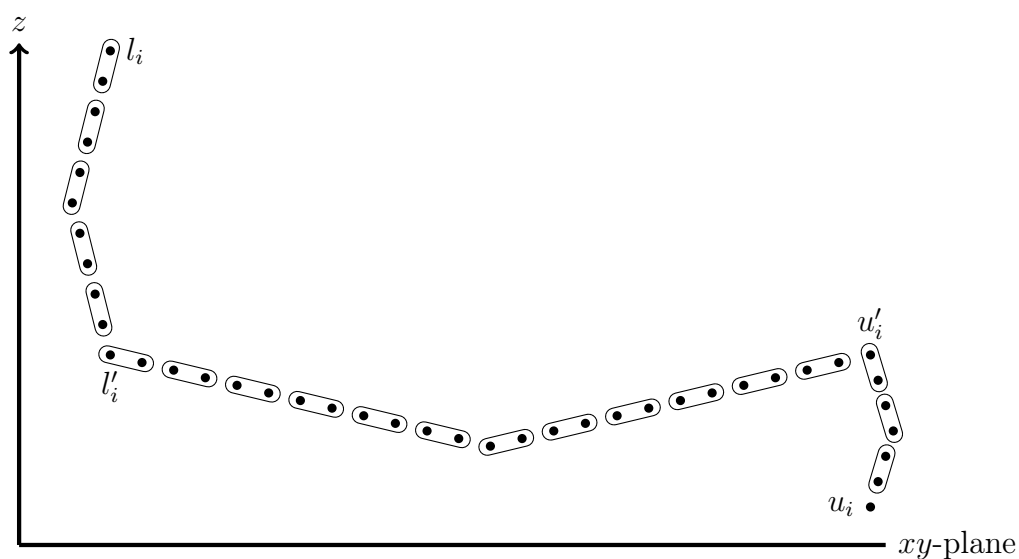


Figure 4.22: Illustration of the rooms in the ascending layer in the reduced outcome.

W.l.o.g. assume that $d_{i,j}^{l',1} = \gamma_e[0]$ and $d_{i,j-1}^{l',p} = \gamma_e[\hat{n}]$. We shall create the rooms $\{\alpha_e[z], \beta_e[z], \gamma_e[z]\} \in \pi_S$ for $z \in [\hat{n}]$.

Let $j \in \{j' \in [0, k-1] \mid j' \bmod 2 = 0\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i,j}^{l',p}$ and $d_{i,j+1}^{l',1}$, where $l' \in \{2l-1, 2l\}$ and $p \in \{2, 3\}$. W.l.o.g. assume that $d_{i,j+1}^{l',1} = \gamma_e[0]$. We shall create the rooms $\{\gamma_e[z-1], \alpha_e[z], \beta_e[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$ and room $\{d_{i,j}^{l',1}, d_{i,j}^{l',2}, d_{i,j}^{l',3}\}$.

Let us now consider the case where $\lfloor k \rfloor$ is odd. First we create room $\{d_{1,0}^{1,1}, d_{2,0}^{1,1}, d_{3,0}^{1,1}\}$.

Let $j \in \{j' \in [k] \mid j' \bmod 2 = 1\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i,j}^{l',1}$ and $d_{i,j-1}^{l',p}$, where $l' \in \{2l-1, 2l\}$ and $p \in \{2, 3\}$. W.l.o.g. assume that $d_{i,j-1}^{l',p} = \gamma_e[\hat{n}]$. We shall create the rooms $\{\alpha_e[z], \beta_e[z], \gamma_e[z]\} \in \pi_S$ for $z \in [\hat{n}]$.

Let $j \in \{j' \in [k-1] \mid j' \bmod 2 = 1\}$ be chosen arbitrarily. Let e be an edge with endpoints $d_{i,j}^{l',p}$ and $d_{i,j+1}^{l',1}$, where $l' \in \{2l-1, 2l\}$ and $p \in \{2, 3\}$. W.l.o.g. assume that $d_{i,j+1}^{l',1} = \gamma_e[0]$. We shall create the rooms $\{\gamma_e[z-1], \alpha_e[z], \beta_e[z]\} \in \pi_{pp}$ for $z \in [\hat{n}]$ and room $\{d_{i,j}^{l',1}, d_{i,j}^{l',2}, d_{i,j}^{l',3}\}$.

See Fig. 4.23 for an illustration.

4.4 Hardness

We shall show that determining the existence of a strictly popular outcome for a 3D-EuclidSR game G is co-NP-hard. The co-NP-hardness is demonstrated by proving that the original PC-X3C instance $I = (X, C)$ has a solution $S \subseteq C$ if and only if G has no strictly popular outcome.

Remark 4.4.1. Any edge in the construction in Section 4.1.1 has length at least 9.

Remark 4.4.2. Let π be an arbitrary outcome of G . For an agent $a \in \{\alpha_e[z], \beta_e[z] \mid z \in [\hat{n}]\}$, we have that $\delta(a, \pi(a)) \geq 1 + \epsilon$. This follows from the construction in Section 4.1.1

Remark 4.4.3. For any agent $a \notin \{\alpha_e[z], \beta_e[z] \mid z \in [\hat{n}]\}$, we have that $\delta(a, \pi(a)) \geq 2$. This follows from the construction in Section 4.1.1

Remark 4.4.4. Let A_e be an arbitrary chain from the construction in Section 4.1.1 and let π be an arbitrary outcome of G . For an agent $a \in$

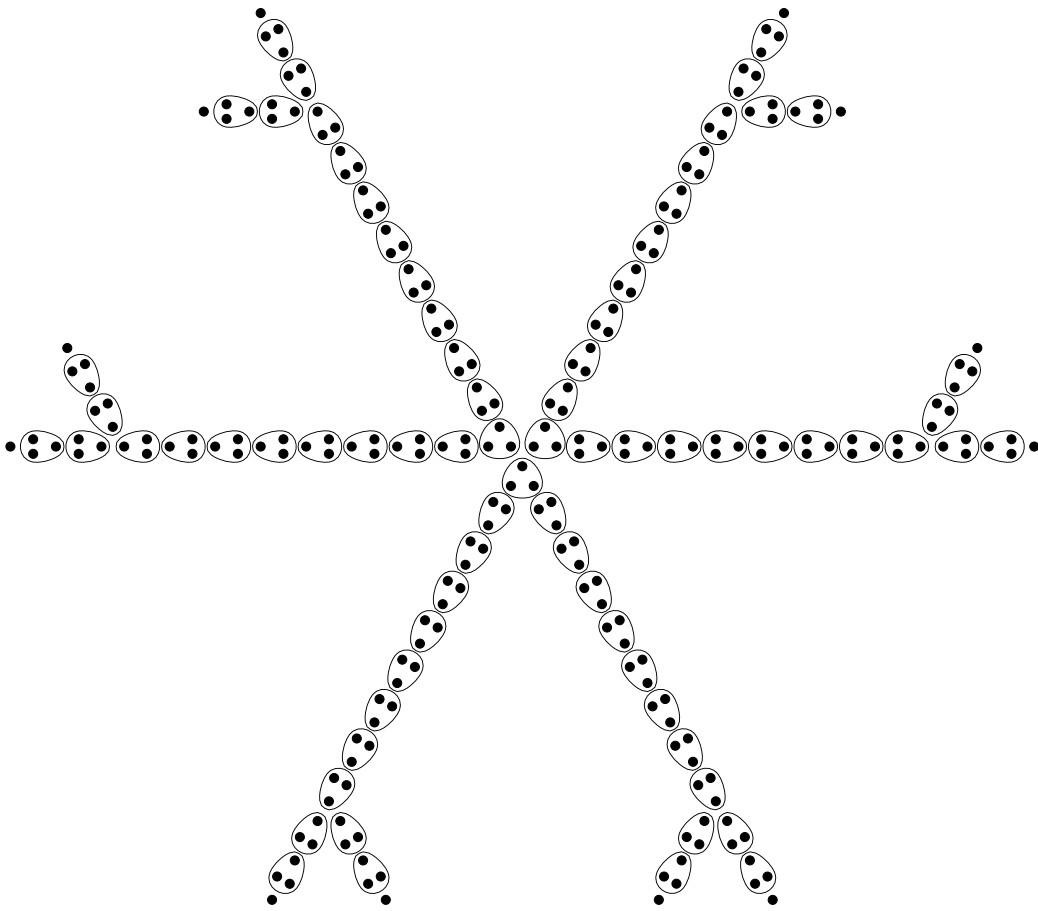


Figure 4.23: Illustration of the rooms in the top layer in the reduced outcomes.

$\{\alpha_e[z], \beta_e[z] | z \in [\hat{n}]\}$, we have that $\delta(a, \pi(a)) = 1 + \epsilon$ if and only if $\pi(a) = \{\alpha_e[z], \beta_e[z], \gamma_e[z]\}$ or $\pi(a) = \{\alpha_e[z], \beta_e[z], \gamma_e[z - 1]\}$.

That is, the most preferred room assignment of $\alpha_e[z]$ (resp. $\beta_e[z]$) is to be put together with $\beta_e[z]$ (resp. $\alpha_e[z]$) and $\gamma_e[z']$, where $z' = z$ or $z' = z - 1$.

Remark 4.4.5. Let A_e be an arbitrary chain from the construction in Section 4.1.1 and let π be an arbitrary outcome of G . For any agent $a \notin \{\alpha_e[z], \beta_e[z] | z \in [\hat{n}]\}$ we have that $\delta(a, \pi(a)) = 2$ if and only if $\pi(a) = \{\alpha_e[z], \beta_e[z], \gamma_e[z]\}$ or $\pi(a) = \{\alpha_e[z], \beta_e[z], \gamma_e[z - 1]\}$ or $\pi(a) = \{w_j^{i_1}, w_j^{i_2}, w_j^{i_3}\}$ or $\pi(a) = \{d_{i,j}^{l,1}, d_{i,j}^{l,2}, d_{i,j}^{l,3}\}$ or $\pi(a) = \{d_{1,0}^{1,1}, d_{1,0}^{1,2}, d_{1,0}^{1,3}\}$.

That is, the most preferred room assignment of $\gamma_e[z]$ is to be put together with $\alpha_e[z]$ and $\beta_e[z]$ or, if $z > 1$, with $\alpha_e[z]$ and $\beta_e[z - 1]$.

In the bottom layer, we have an additional most preferred room in the case where $\gamma_e[z]$ is a set-agent. This room consists of the set-agents that form an equilateral triangle, i.e., $\{w_j^{i_1}, w_j^{i_2}, w_j^{i_3}\}$.

In the top layer, we also have an additional most preferred room for $\gamma_e[z]$. In the case that $\gamma_e[z]$ is an agent from the internal vertex gadget, the additional most preferred room consist of the agents that form an equilateral triangle with $\gamma_e[z]$, i.e., $\{d_{i,j}^{l,1}, d_{i,j}^{l,2}, d_{i,j}^{l,3}\}$.

Finally, the most preferred rooms of a center agent a consists of the other center agents with which it forms an equilateral triangle, i.e., $\{d_{1,0}^{1,1}, d_{1,0}^{1,2}, d_{1,0}^{1,3}\}$, or the vertices from the internal vertex gadget with which it also forms an equilateral triangle, i.e., $\{d_{i,j}^{l,1}, d_{i,j}^{l,2}, d_{i,j}^{l,3}\}$.

Lemma 4.4.6. *Let π be an outcome of G that assigns all agents to its most preferred room and assume that $[k]$ is even. If $\{d_{1,0}^{1,1}, d_{1,0}^{1,2}, d_{1,0}^{1,3}\} \in \pi$, then $\pi = \pi_{pp}$. Otherwise $\pi = \pi_S$ for some solution S of I .*

Proof. This follows from the above remarks. □

Lemma 4.4.7. *Let π be an outcome of G that assigns all agents to its most preferred room and assume that $[k]$ is odd. If $\{d_{1,0}^{1,1}, d_{1,0}^{1,2}, d_{1,0}^{1,3}\} \in \pi$, then $\pi = \pi_S$ for some solution S of I . Otherwise $\pi = \pi_{pp}$.*

Proof. This follows from the above remarks. □

From the above remarks, we have that if an outcome π assigns all agents to its most preferred room, then $\pi = \pi_{pp}$ or $\pi = \pi_S$. Note that an outcome that assigns all agents to its most preferred room is popular.

Since π_{pp} exists independent of I having a solution, any popular outcome of G must assign all agents to its most preferred room.

Thus, if I has no solution, then π_{pp} is the only popular outcome and therefore a strictly popular outcome. Otherwise, we have multiple popular outcomes and therefore no strictly popular outcome. Therefore, the 3D-EuclidSR problem with the room size fixed to 3 is co-NP-hard.

Theorem 4.4.8. *Determining whether a strict popular outcome exists in an instance of the 3D-EuclidSR problem with the room size $s = 3$ is co-NP-hard.*

Proof. Follows from Lemmas 4.4.6 and 4.4.7. □

Chapter 5

Conclusion

We have demonstrated that determining the existence of a (strictly) popular outcome for a roommate diversity game is co-NP-hard and that computing a mixed popular outcome is not possible in polynomial time, unless $P=NP$. Even when the preferences are tri- or dichotomous, the problem remains intractable. As we have only demonstrated hardness, a potential avenue for future research would be to demonstrate completeness for a certain complexity class. We conjecture that the problem is Π_2^P -complete. Furthermore, we can consider additional restrictions, such as only allowing strict preferences or popularity with dichotomous preferences, to strengthen the hardness results or perhaps obtain tractability results.

A popular outcome is guaranteed to exist in a roommate diversity game, when the room size is fixed to 2. Additionally, it is possible to compute a popular outcome in polynomial time. As the problem becomes tractable when fixing the room size to 2, it may be possible to construct a fixed-parameter tractable algorithm with the room size as parameter that determines the existence of a (strictly) popular outcome in a roommate diversity game. This is also left for future research.

We have also demonstrated that determining the existence of a strictly popular outcome for 3D-EuclidSR game is co-NP-hard when the room size is fixed to 3.

We believe that our construction can be modified to show that the problem remains hard, when the room size is greater than 3. Additionally, similar to our

results for the Roommate Diversity Problem, we believe that our construction can be modified to show that determining the existence of a popular outcome is co-NP-hard and that, under the assumption that $P \neq NP$, a mixed popular outcome cannot be computed in polynomial time. These are left for future research.

The complexity of computing and determining the existence of a popular outcome, when the room size is fixed to 2 is also yet unknown. We believe that the problem becomes tractable in this case.

Finally, it would be interesting to investigate whether it would be possible to obtain a stronger result by showing that determining the existence of a strictly popular outcome remains co-NP-hard, when the agents are assigned a point in 2-dimensional euclidean space, i.e., popularity on the Euclid- d -SR problem. It is also possible that the problem becomes tractable in this case.

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