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Ph.D. Thesis

Integrals of motion in Toda field theories and the ODE/IM correspondence

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Abstract

Integrable models represent a class of mathematical and physical systems characterized by their exact solvability due to a large number of conserved quantities, known as integrals of motion (IoMs). These models can be precisely analyzed using the Quantum Inverse Scattering Method (QISM), with the Yang-Baxter equation governing the R-matrix forming the foundation of this approach. Integrable models play a pivotal role in quantum mechanics, statistical physics, and quantum field theory, offering profound insights into the dynamics of strongly interacting quantum systems. In particular, integrability in quantum field theory enables exact solutions for energy spectra and scattering data, providing essential tools for understanding these systems. The Sine-Gordon model and its generalization, the Toda field theories, are key examples of such models and form the central focus of this thesis.

This research investigates quantum integrable models through a classical lens, leveraging the ODE/IM correspondence. Initially discovered as a connection between the spectral analysis of ordinary differential equations (ODEs) and the functional relations of integrable models, this correspondence has since been extended to continuous systems. In field theory, the most notable instance is the relationship between the Schrödinger equation on the complex plane and the Sine-Gordon model. Further developments revealed that the Schrödinger equation arises naturally from the classical Sine-Gordon equation, and this framework has been generalized to affine Toda field theories with broader Lie algebraic structures. The correspondence is established through ψ-systems and Bethe ansatz equations, with solutions to the linear problem of the Sine-Gordon equation shown to be proportional to the IoMs of quantum Liouville theory in its ground state.

Building on this foundation, the thesis presents two main research projects. First, we establish a connection between the classical Toda field equations and certain (pseudo) ordinary differential equations. Second, we explore the correspondence between the IoMs of quantum Toda field theories and the WKB integrals derived from the aforementioned ODEs. By synthesizing these findings, we extend the known ODE/IM correspondence into a broader framework, unifying classical and quantum correspondences for Toda field theories and KdV hierarchies.

Contents

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Chapter 1

Introduction

Integrable models

Integrable models form a unique and elegant class of physical systems characterized by their solvability and rich mathematical structure. In essence, a system is considered integrable if it possesses a sufficient number of conserved quantities to make the dynamics highly constrained or completely solvable. The formalism underlying integrable models provides an exact framework for understanding their behavior and solutions. There are many ways to define integrability, reflecting the diverse nature of the integrable systems. One of the most widely used definitions in classical mechanics is Liouville integrability [1]. A system is said to be Liouville integrable if it possesses as many independent conserved quantities in involution as degrees of freedom. In the Hamiltonian formalism, this implies that the Poisson brackets between any two conserved quantities vanish:

$$\{H_i, H_j\} = 0,$$

where H_i $(i = 0, 1, \dots)$ are conserved quantities. Liouville's theorem further states that such a system can be solved exactly by canonical transformation to action-angle variables, where the dynamics become trivial linear motion on a torus. This definition allows us to capture the essence of classical integrability.

In classical integrable systems, the key to showing the integrability lies in the Lax formalism [2]. A classical system is considered integrable if its dynamics can be expressed in terms of a pair of matrices, (L, M) whose entries are functions of dynamical variables, known as the Lax pair, such that the time evolution of L is governed by the equation

$$\frac{dL}{dt} = [M, L].$$

The eigenvalues of the Lax operator L are constants of motion, and this structure ensures integrability. Typically, L depends on a spectral parameter λ, introducing a connection to complex geometry. The eigenvalue problem for L defines the spectral curve, a Riemann surface that encodes the integrable structure geometrically. This curve provides a compact way to represent the conserved quantities and serves as a foundation for constructing explicit solutions.

The classical r-matrix formalism further enriches this picture [3]. It specifies the Poisson brackets of the entries of L, ensuring the integrability of the system. The r-matrix itself satisfies the classical Yang-Baxter equation, which governs the algebraic structure of the interactions. These conserved quantities can be systematically derived from the Hamiltonian formalism, where the integrals of motion generated by the Hamiltonians commute under the Poisson bracket. The solutions to classical integrable systems often exhibit remarkable features like solitons, which emerge from the underlying integrable structure.

Quantum integrable systems extend these ideas into the quantum realm, where integrability is defined by the existence of a large number of commuting operators. The Quantum Inverse Scattering Method (QISM) provides the formal framework for this extension [4]. At its heart is the Yang-Baxter equation, a consistency condition for the R-matrix, which encodes the interaction rules of the system:

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu).$$

The Yang-Baxter equation ensures that the R-matrix satisfies a form of associativity, guaranteeing the integrability of the quantum system [5–7]. Central to QISM is the transfer matrix, which encodes the quantum analogue of the Lax operator [8]. The transfer matrix generates a family of commuting operators, corresponding to conserved quantities of the quantum system. The eigenstates and eigenvalues of the transfer matrix are typically obtained using the Bethe ansatz, a powerful method that reduces the quantum problem to solving a set of algebraic equations [4]. The Bethe roots, solutions to these equations, determine the spectrum of the model. Quantum integrable systems like the Heisenberg spin chain, and the quantum Toda lattice are solvable within this framework. Finally, in the thermodynamic limit, quantum integrable systems are described by the Thermodynamic Bethe Ansatz (TBA). This approach provides a powerful way to understand the macroscopic properties of the system, including its energy spectrum, entropy and thermodynamic behavior, by translating the microscopic Bethe ansatz equations into integral equations. The TBA formalism connects the microscopic quantities like scattering matrix to their macroscopic statistical mechanics, enabling computation of quantities like free energy and entropy.

The connection between classical and quantum integrable systems is particularly striking in the semiclassical limit. In this regime, the quantum R-matrix reduces to the classical r-matrix, and the quantum commutators converge to their classical counterparts. This correspondence bridges the two realms, highlighting the universality of integrable structures.

Integrable models are not only mathematically intriguing but also profoundly important in theoretical physics, appearing across a wide spectrum of disciplines with concrete examples that highlight their versatility. In statistical mechanics, the Ising model serves as a cornerstone for understanding ferromagnetic systems and phase transitions [9, 10], while the six-vertex model captures the intricate behavior of ice-type systems and related critical phenomena [11–14]. Similarly, the Kondo problem explores the quantum dynamics of magnetic impurities in metals [15], further demonstrating the breadth of integrable models in this field.

In nonlinear wave phenomena, integrable systems like the nonlinear Schrödinger equation models stable wave packets in optics [16], and the Kortweg-de Vries (KdV) equation describes shallow water waves with solitonic solutions [17]. The quantization of the KdV equation is realized in [18–20] with the quantum group, where the R-matrix and the transfer matrix are successfully constructed. The KdV equation can be further understood as part of a broader family of integrable systems known as the KdV hierarchy. This generalization is achieved through the Drinfeld-Sokolov reduction [21], which connects integrable systems to the mathematical structure of affine Lie algebras [22]. The reduction provides a systematic method to derive entire hierarchies of integrable equations by imposing constraints on a loop algebra and reducing the dynamics to a smaller, consistent system. In the quantum realm, the Drinfeld-Sokolov reduction acquires additional significance [23–25]. Its quantum version is also equivalent to the BRST formalism, which systematically handles constraints by introducing auxiliary ghost fields and constructing a BRST operator [26, 27].

Quantum field theory also benefits immensely from integrable models, particularly in two dimensions, where the interplay between conformal field theories (CFTs) and integrable field theories is a central theme. In 2D CFT, integrability often emerges from the infinite-dimensional symmetry provided by the Virasoro algebra or its extensions, the W-algebras [28]. This symmetry structure allows for the exact solution of certain models and the computation of quantities like correlation functions and spectra. A prominent example is Liouville theory, a paradigmatic integrable 2D CFT with applications in quantum gravity, string theory, and statistical mechanics [29, 30]. Its exact solvability is rooted in its rich symmetry structure and its role as a conformal field theory with a central charge parameter. Liouville theory's integrability is evident in its infinite conserved charges. Another example is Toda field theory [30, 31], an integrable field theory based on higher W-algebras, which generalizes Liouville theory. Besides CFT, there are massive two-dimensional integrable field theories that often arise as perturbations of 2D CFTs by relevant fields, keeping the integrability and exact solutions. These connections are exemplified by models like the Sine-Gordon model [30, 32], which can be understood as a deformation of the Liouville theory, and the affine Toda field theories [30, 33], which extend this framework by incorporating higher symmetries associated with Lie algebras.

In the context of supersymmetric gauge theories, the connection between 2D integrable systems such as the Toda lattice or Calogero-Moser model and $\mathcal{N} = 2$ supersymmetric gauge theories has emerged as a particularly rich field of study [34, 35]. These integrable systems are closely tied to the Seiberg-Witten description of the moduli space of vacua, where the conserved charges of the integrable model correspond to the BPS spectra of the gauge theory. Nekrasov extended this correspondence to the quantum regime by introducing the Ω-background deformation, where the two-dimensional twisted superpotential becomes the Yang-Yang function of quantum integrable systems, such as the quantum Calogero-Moser model or quantum Toda lattice [36]. A deeper understanding of this correspondence is made possible by exploring its geometry, particularly through Hitchin systems. These arise naturally in the compactification from 6D $\mathcal{N} = 2$ superconformal $(2, 0)$ field theories to 4D $\mathcal{N} = 2$ field theories whose BPS equations are well known to be the Hitchin equations [37]. Hitchin systems can be viewed as integrable systems defined on the moduli space of Higgs bundles over a Riemann surface [38]. Their spectral curve plays a central role, coinciding with the Seiberg-Witten curve that describes the low-energy dynamics of the gauge theory. The discovery provides a unifying framework that connects integrability and gauge theory.

Despite their simplicity, these models capture essential aspects of physical phenomena, providing not only exact solutions to ideal physical problems but also a framework to explore the underlying principles of complex systems. Their presence across disciplines underscores their foundational importance in modern physics.

The ODE/IM correspondence

Building on the examples of integrable models discussed in the previous section, we turn to a fascinating connection between certain ordinary differential equations (ODEs) and two-dimensional quantum integrable models (IMs). This connection, known as the ODE/IM correspondence, is a central theme of this thesis.

The ODE/IM correspondence was first proposed by P.Dorey and R.Tateo [39] in 1999. They discovered that functional relations such as the **T**-**Q** relation, the fusion hierarchy (**T**-system), and the quantum Wronskian relation (**Q**-system), which appear in certain 2D quantum integrable models, match the spectral determinants of specific Schrödinger-type ODEs with anharmonic potentials.

On the integrable model side, the **T**-operator represents the transfer matrix, serving as the generating function of integrals of motion (IoMs), while the **Q**-operator acts as an auxiliary tool for defining Bethe ansatz equations. The zeros of the **Q**-functions correspond to the roots of these equations. Furthermore, the fusion hierarchy leads directly to the **Y**-system, which in turn generates the Thermodynamic Bethe Ansatz equations [40]. On the ODE side, the spectral determinant is a function whose zeros coincide with the eigenvalues of the ODE's spectral problem. Here, the **T**-operator corresponds to the spectral determinant of solutions subdominant in different sectors at infinity, while the **Q**-operator relates to the spectral determinant of solutions near the origin and infinity. Notably, the Y-functions, which satisfy the TBA equations in integrable theories, correspond to the Borel resummation of the WKB periods of the ODE solutions [41].

Shortly after Dorey and Tateo's discovery, these observations were rigorously proved and extended by V. Bazhanov, S. Lukyanov, and A. Zamolodchikov in the context of the quantum KdV equation [42], which incorporates an angular momentum term into the Schrödinger equation potential. Since then, the ODE/IM correspondence has found applications in diverse areas of physics, including condensed matter [43,44], PT-symmetric quantum mechanics [45], boundary CFT [46] and non-compact sigma models [47].

The KdV equation is intimately connected to the Sine-Gordon model [21, 48], motivating further exploration of the correspondence in massive integrable models [49]. These advances laid the groundwork for new methods and tools in 2D integrable field theories. Additionally, extensions of the correspondence through Bethe ansatz and the **Q**-operator have been developed [45, 50–53]. This work culminated in a generalized ODE/IM correspondence for massive models associated with affine Lie algebras − the Toda field theories [54–57].

In these massive integrable models, linear problems play a crucial role. The ODEs there can be reformulated into linear problems, serving as spectral equations satisfied by the Lax operators of classical integrable models. This reformulation links the ODE/IM correspondence to the broader quantum/classical correspondence. Furthermore, the **Q**-operator can be explicitly constructed using solutions to the linear problem [54, 55]. This approach motivates a detailed analysis of the WKB approximation, which involves diagonalizing matrix-valued operators using gauge transformations. For $A_r^{(1)}$ systems, this procedure aligns with the treatment of higher-order ODEs and can be extended to other affine Lie algebras. Interestingly, the diagonalization approach is closely tied to the Drinfeld-Sokolov construction of infinite conserved charges in generalized KdV hierarchies [21]. The WKB expansion reveals that the structure of non-trivial terms in the WKB periods corresponds to the classical conserved charges of these hierarchies.

Beyond these structural insights, the ODE/IM correspondence manifests directly through connections between WKB integrals and quantum local IoMs in the vacuum [43,49,58]. In

CFTs with extended simple-laced W-symmetry, quantum IoMs can be derived via quantum Drinfeld-Sokolov reduction, while WKB integrals emerge from integrating WKB solutions for $A_r^{(1)}$ and $D_r^{(1)}$ linear problems along the Pochhammer contour. Interestingly, the parameter pairing derived from these local IoMs aligns precisely with that obtained from the Bethe ansatz ($\mathbf{Q}$-systems), offering a broader framework for understanding and reorganizing the ODE/IM correspondence.

Outline of the Thesis

This thesis is based on two research papers [58, 59] carried out by the author during his doctoral program. Their results are summarized in Chapter 4 after the two review chapters 2 and 3.

Chapter 2 introduces the foundational framework of quantum integrable models. We begin with the six-vertex model to explain the transfer matrix $\mathbf{T}$, the Bethe ansatz, and the associated functional relations. These concepts are then extended to the KdV hierarchies and to Toda field theories. Finally, we will derive two non-linear integral equations: the Thermodynamic Bethe Ansatz equation and the Destri-de Vega equation from the functional relations above.

Chapter 3 reviews the ODE/IM correspondence, highlighting functional relations and non-linear integral equations on the ODE side. We will see how the functional relations introduced in Chapter 2 arise in the spectral analysis of ordinary differential equations. This chapter concludes with a dictionary summarizing the general correspondence.

Chapter 4 explores the WKB solutions of linear problems in classical integrable models, showing how these solutions for certain (pseudo-)ODEs relate to the classical conserved charges in Toda field theories. The connections between classical and quantum integrals of motion (IoMs) are also clarified.

Chapter 2

Quantum integrable models in the ODE/IM correspondence

The ODE/IM correspondence [39, 42] reveals a profound and surprising connection between the spectral analysis of ordinary differential equations (ODEs) and the functional approach to two-dimensional quantum integrable models (IMs). This chapter introduces the essential quantum integrable models relevant to the ODE/IM correspondence.

Quantum IMs arise naturally in both quantum mechanics and quantum field theory. A key example in quantum mechanics is the six-vertex model and its quantum counterpart, the XXZ spin chain. Taking the continuum limit of the XXZ model leads to two-dimensional conformal field theory (CFT) associated with the Lie algebra A_1. Here, we explore the fundamental structures underlying integrability, including the transfer matrix $\mathbf{T}$, which enables partition function computations, the auxiliary operator $\mathbf{Q}$, which reformulates the Bethe ansatz equations, and the $\mathbf{Y}$-systems, which form the foundation for constructing Thermodynamic Bethe Ansatz equations.

On the field theory side, the continuum limit of the XXZ model corresponds to a CFT governed by the quantum Korteweg–de Vries (qKdV) equation [60]. This connection extends to Toda field theories [48], a fascinating class of integrable field theories characterized by an underlying Lie algebra $\mathfrak{g}$. Scattering processes in Toda theories are purely elastic and governed by the Yang-Baxter equation, a hallmark of integrability, due to the existence of infinitely many conserved integrals of motion.

When the Lie algebra $\mathfrak{g}$ is $su(2)$, Toda field theory reduces to the celebrated two-dimensional Liouville CFT. For more general Lie algebras, Toda theory corresponds to conformal field theories (CFTs) based on W-algebras (WCFTs), which significantly expand the framework of traditional CFTs.

In this chapter, we first introduce the functional relations in the six-vertex model and its continuum limit in Section 2.1. This includes the **T**-, **Q**-, and **Y**-operators, their interrelations, the continuum limit, and their connection to the Thermodynamic Bethe Ansatz equations. This part follows the review [61]. Next, we review how these functional relations emerge in integrable field theories, including the KdV hierarchies and Toda field theories [18–20], in Sections 2.2 and 2.3. We begin by discussing how integrability influences scattering processes and its relationship with the Yang-Baxter equation and Bethe ansatz equations. Afterward, we focus on the conformal properties of Toda field theories and demonstrate how quantum local integrals of motion are derived. Since integrals of motion are central to this thesis, we also illustrate how functional operators (**T**-, **Q**-, and **Y**-operators) relate to integrals of motion [18, 61] in Section 2.3.4.

2.1 The six-vertex model and its functional relations

The six-vertex model is a famous example of an integrable lattice model, celebrated for its mathematical structure and physical applications. The model is defined on an $N \times N'$ lattice with periodic boundary conditions in both directions, creating a toroidal structure. Each horizontal or vertical link of the lattice is assigned a spin variable that can take one of two possible values. For convenience, these spins are represented as arrows pointing either right or left (on horizontal links) or up or down (on vertical links), as illustrated in the figure below.

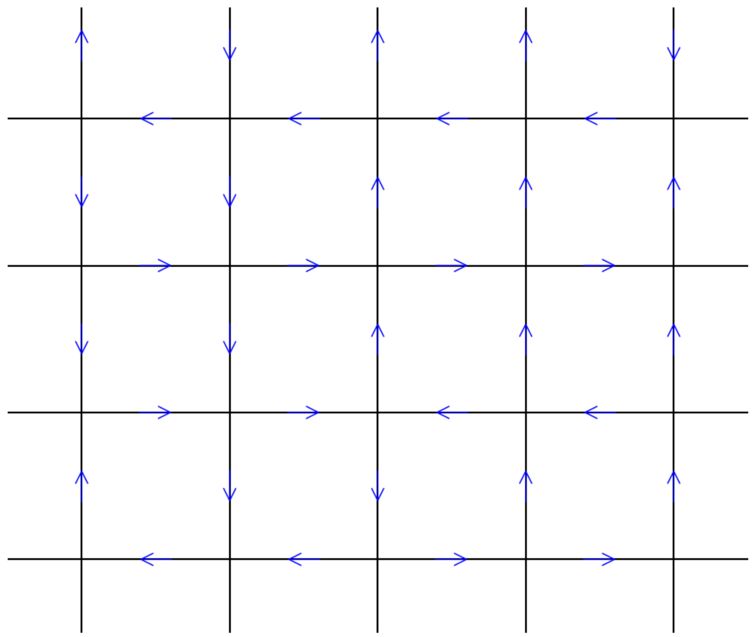


Figure 2.1.1: Configurations of spins in the six-vertex model

A defining feature of the six-vertex model is its strict constraint on allowed configurations. Only those arrangements of arrows that conserve the flux of spins through each vertex are permitted, resulting in exactly six possible configurations around any given vertex. To differentiate these configurations, we assign a Boltzmann weight W to each, which encodes their relative probabilities. These weights depend on two parameters: the spectral parameter λ and the anisotropy μ. The assignments are as follows:

$$
\begin{aligned}
W\begin{bmatrix} & \uparrow & \\ \rightarrow & & \rightarrow \\ & \uparrow & \end{bmatrix} &= W\begin{bmatrix} & \downarrow & \\ \leftarrow & & \leftarrow \\ & \downarrow & \end{bmatrix} = a(\lambda,\mu) = \sin(\mu + i\lambda),\\
W\begin{bmatrix} & \downarrow & \\ \rightarrow & & \rightarrow \\ & \downarrow & \end{bmatrix} &= W\begin{bmatrix} & \uparrow & \\ \leftarrow & & \leftarrow \\ & \uparrow & \end{bmatrix} = b(\lambda,\mu) = \sin(\mu - i\lambda),\\
W\begin{bmatrix} & \uparrow & \\ \rightarrow & & \leftarrow \\ & \downarrow & \end{bmatrix} &= W\begin{bmatrix} & \downarrow & \\ \leftarrow & & \rightarrow \\ & \uparrow & \end{bmatrix} = c(\lambda,\mu) = \sin(2\mu).
\end{aligned}
\tag{2.1.1}
$$

The parameter μ, which measures the degree of anisotropy, is typically fixed in physical discussions, making the Boltzmann weights functions of the spectral parameter λ. Additionally, we impose a zero-field condition: the Boltzmann weights remain unchanged under a global reversal of all arrows.

With these preparations, we can define the partition function, a central quantity that captures the statistical mechanics of the system:

$$Z = \sum_{\{\sigma\}} \prod_{\text{vertices}} W, \tag{2.1.2}$$

where $\{\sigma\}$ denotes all possible spin configurations. Remarkably, the six-vertex model is integrable, meaning its partition function can be computed exactly using the transfer matrix formalism.

To introduce the transfer matrix, we first consider an N-component vector $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_N)$. The transfer matrix T is then defined as follows:

$$T_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}'}(\lambda) = \sum_{\beta_i} W \begin{bmatrix} & \alpha_1' & \\ \beta_1 & & \beta_2 \\ & \alpha_1 & \end{bmatrix} W \begin{bmatrix} & \alpha_2' & \\ \beta_2 & & \beta_3 \\ & \alpha_2 & \end{bmatrix} \ldots W \begin{bmatrix} & \alpha_N' & \\ \beta_N & & \beta_1 \\ & \alpha_N & \end{bmatrix}. \tag{2.1.3}$$

This transfer matrix is a $2^N \times 2^N$ operator, and the partition function is obtained by tracing over its N'-fold product:

$$Z = \operatorname{Tr}\left[T^{N'}\right]. \tag{2.1.4}$$

Interestingly, the six-vertex model can be mapped to the XXZ spin chain, a fundamental quantum model in condensed matter physics. The XXZ Hamiltonian is given by:

$$H_{XXZ} = -\frac{1}{2} \sum_{j=1}^{N} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \cos(2\mu) \sigma_j^z \sigma_{j+1}^z \right), \tag{2.1.5}$$

where σ_j^α are the Pauli matrices at site j. The Hamiltonian is related to the transfer matrix $T(\lambda)$ via:

$$H_{XXZ} = -i \sin(2\mu) \frac{d}{d\lambda} \ln T(\lambda) \Big|_{\lambda=-2\mu} - \frac{1}{2} \cos(2\mu) I^{\otimes N}. \tag{2.1.6}$$

Finally, the diagonalization of the transfer matrix, essential for solving the model, is made possible by the **Bethe ansatz**. This technique involves:

- Proposing an eigenvector form for the transfer matrix, parameterized by Bethe roots $\{\lambda_1, \ldots, \lambda_n\}$;
- Ensuring the Bethe roots satisfy the Bethe ansatz equations (BAEs).

The BAEs are a set of n coupled equations whose solutions yield the eigenvalues of the transfer matrix. Notably, in the thermodynamic limit ($N \to \infty$), the ground-state configuration corresponds to $N/2$ real roots, densely packed near the origin. This structure plays a crucial role in understanding the macroscopic behavior of the six-vertex model and its relationship to other integrable systems.

Furthermore, one can generalize the procedures above by adding a twist on one single column of lattice, say the N-th one:

$$W\begin{bmatrix} & \alpha_N' & \\ \beta_N & & \beta_1 \\ & \alpha_N & \end{bmatrix} \to e^{i\phi\beta_1} W\begin{bmatrix} & \alpha_N' & \\ \beta_N & & \beta_1 \\ & \alpha_N & \end{bmatrix}$$

with $\beta_1 = \pm 1$ for different alignments. The BAEs then become

$$\prod_{j=1}^{n} \frac{\sinh\left(\lambda_j - \lambda_k + 2\mathrm{i}\mu\right)}{\sinh\left(\lambda_j - \lambda_k - 2\mathrm{i}\mu\right)} = -e^{-2i\phi}\left[\frac{a\left(\lambda_k, \mu\right)}{b\left(\lambda_k, \mu\right)}\right]^N \quad ; \quad k = 1, \ldots, n. \tag{2.1.7}$$

and the eigenvalues of the transfer matrix T become

$$t\left(\lambda \mid \{\lambda_j\}\right) = e^{-i\phi}[a(\lambda, \mu)]^N \prod_{j=1}^{n} g\left(\lambda_j - \lambda\right) + e^{i\phi}[b(\lambda, \mu)]^N \prod_{j=1}^{n} g\left(\lambda - \lambda_j\right) \tag{2.1.8}$$

The eigenvalues on the ground state (2.1.8) are derived by the algebraic Bethe ansatz [62], where the lattice structure is summarized into a Yang-Baxter diagram consistent with the scattering process in integrable field theories [63].

2.1.1 Baxter's T-Q Relation

So far, we have introduced the structure of the transfer matrix $T(\lambda)$ from the Bethe ansatz. It is also possible to define a q-function to generalize the BAEs. Let us assume there exists an auxiliary function $q(\lambda)$ which is entire and also $i\pi$ periodic satisfying

$$t(\lambda, \phi)q(\lambda, \phi) = e^{-i\phi}a^N(\lambda, \mu)q(\lambda + 2i\mu, \phi) + e^{i\phi}b^N(\lambda, \mu)q(\lambda - 2i\mu, \phi). \tag{2.1.9}$$

This T-Q relation is not the common one where both $T(\lambda)$ and $Q(\lambda)$ should be operators. The q-function here can also be viewed as the eigenvalue of the Q-operator defined in the next section.

The BAE can be extracted from the T-Q relation as follows. Suppose that the zeros of $q(\lambda)$ are $\{\lambda_i\}$, $1 < i < n$. It is possible to set $q(\lambda) = \prod_{l=1}^{n} \sinh(\lambda - \lambda_l)$ which does not break the $i\pi$ period. Then we can see that $q(\lambda)$ is fixed by the set $\{\lambda_i\}$. Especially when $\lambda = \lambda_i$, the T-Q relation (2.1.9) becomes

$$\frac{q(\lambda_i - 2i\mu)}{q(\lambda_i + 2i\mu)} = \prod_{l=1}^{n} \frac{\sinh(\lambda_i - \lambda_l + 2i\mu)}{\sinh(\lambda_i - \lambda_l - 2i\mu)} = -e^{-2i\phi} \frac{a^N(\lambda_i, \mu)}{b^N(\lambda_i, \mu)}, \tag{2.1.10}$$

since $t(\lambda)$ is an entire function, $t(\lambda_i)$ is non-singular. These are precisely the BAEs (2.1.7) with the zeros of q-functions being the Bethe roots.

2.1.2 Quantum Wronskian

Besides Baxter's T-Q relation, there is another important functional relation for the q-function with different phases first discovered in [19]. First, we assume the following reversal identity for the ground state eigenvalue of the transfer matrix $T(\lambda, \phi)$:

$$t_0(\lambda, \phi) = t_0(\lambda, -\phi) \equiv t_0(\lambda, |\phi|),$$

which can be realized if all the Boltzmann weights are invariant under simultaneous reversal of all the spins. Then, the following two T-Q relations hold simultaneously:

$$\begin{aligned} t_0(\lambda, |\phi|)\widetilde{q}_0(\lambda, \phi) &= a^N(\lambda, \mu)\widetilde{q}_0(\lambda + 2i\mu, \phi) + b^N(\lambda, \mu)\widetilde{q}_0(\lambda - 2i\mu, \phi), \\ t_0(\lambda, |\phi|)\widetilde{q}_0(\lambda, -\phi) &= a^N(\lambda, \mu)\widetilde{q}_0(\lambda + 2i\mu, -\phi) + b^N(\lambda, \mu)\widetilde{q}_0(\lambda - 2i\mu, -\phi), \end{aligned} \tag{2.1.11}$$

where we redefine the function $q(\nu)$ as

$$\widetilde{q}_0(\lambda, \phi) := e^{-\frac{\lambda\phi}{2\mu}} q_0(\lambda, \phi), \tag{2.1.12}$$

and $q_0(\lambda, \phi)$ represents the ground state eigenvalue of the Q-operator. Here we can see that both $\widetilde{q}_0(\lambda, \phi)$ and $\widetilde{q}_0(\lambda, -\phi)$ solve the equation

$$t_0(\lambda, |\phi|)\widetilde{q}(\lambda) = a^N(\lambda, \mu)\widetilde{q}(\lambda + 2i\mu) + b^N(\lambda, \mu)\widetilde{q}(\lambda - 2i\mu), \tag{2.1.13}$$

which is a finite-difference analogue of a second-order differential equation with two linearly independent solutions, namely $\widetilde{q}_0(\lambda, \phi)$ and $\widetilde{q}_0(\lambda, -\phi)$. It is thus natural to consider the Wronskian

$$\Delta(\lambda) := \widetilde{q}_0(\lambda + i\mu, -\phi)\widetilde{q}_0(\lambda - i\mu, \phi) - \widetilde{q}_0(\lambda + i\mu, \phi)\widetilde{q}_0(\lambda - i\mu, -\phi) \tag{2.1.14}$$

for a second-order differential equation. One can see

$$0 = a^N(\lambda, \mu)\Delta(\lambda + i\mu) - b^N(\lambda, \mu)\Delta(\lambda - i\mu). \tag{2.1.15}$$

Since $a(\lambda, \mu) = \sin(\mu + i\lambda)$, $b(\lambda, \mu) = \sin(\mu - i\lambda)$, the function $\mathcal{W}(\lambda) := \Delta(\lambda)/\sinh^N(\lambda)$ is periodic with period $2i\mu$. However, according to Eq.(2.1.11) and the periodicity of q-function, $\mathcal{W}(\lambda)$ also has the period $2i\pi$. Above all, $\mathcal{W}(\lambda)$ should be constant for all values of μ unless μ/π is rational. Evaluating $\mathcal{W}(\lambda)$ at $\lambda \to \infty$ gives an identity,

$$e^{-i\phi}q_0(\lambda + i\mu, \phi)q_0(\lambda - i\mu, -\phi) - e^{i\phi}q_0(\lambda + i\mu, -\phi)q_0(\lambda - i\mu, \phi) = -2i\sin(\phi)\sinh^N(\lambda) \tag{2.1.16}$$

2.1.3 The fusion hierarchy

Now let us show a new functional relation after multiplying the two equations (2.1.11) by $\widetilde{q}_0(\lambda + 2i\mu, -\phi)$ and $\widetilde{q}_0(\lambda + 2i\mu, \phi)$ respectively.

$$\begin{aligned} t_0(\lambda, |\phi|) &= -a^N(\lambda, \mu)\frac{\widetilde{q}_0(\lambda + 2i\mu, -\phi)\widetilde{q}_0(\lambda - 2i\mu, \phi) - \widetilde{q}_0(\lambda + 2i\mu, \phi)\widetilde{q}_0(\lambda - 2i\mu, -\phi)}{\Delta(\lambda - i\mu)} \\ &= \frac{-i}{2\sin\phi}\left(\widetilde{q}_0(\lambda + 2i\mu, -\phi)\widetilde{q}_0(\lambda - 2i\mu, \phi) - \widetilde{q}_0(\lambda + 2i\mu, \phi)\widetilde{q}_0(\lambda - 2i\mu, -\phi)\right), \end{aligned} \tag{2.1.17}$$

where the limitation of the Wronskian (2.1.16) has been used. This relation indicates that the quantum Wronskian fits into a hierarchy of relations. Define

$$\vec{q}^{(k)} := \frac{1}{\sqrt{-2i\sin\phi}}\left(e^{-ik\phi/2}q_0(\lambda - ik\widetilde{\mu}, \phi), e^{ik\phi/2}q_0(\lambda - ik\widetilde{\mu}, -\phi)\right)^T \tag{2.1.18}$$

with $\widetilde{\mu} = -\mu + \pi/2$ and the general Wronskian

$$\mathcal{W}[k, -k](\lambda) := \det\left(\vec{q}^{(k)}, \vec{q}^{(-k)}\right). \tag{2.1.19}$$

Then, if we set

$$t^{(k/2)}(\lambda) \equiv \mathcal{W}[k+1,-k-1](\lambda) \quad , \quad \forall k=-1,0,1,\ldots, \tag{2.1.20}$$

it is not difficult to see that

$$t^{(-1/2)}(\lambda)=0, \quad t^{(0)}(\lambda)=[i\cosh(\lambda)]^N, \quad t^{(1/2)}(\lambda)=t_0(\lambda). \tag{2.1.21}$$

We can further use the Plücker-type relation

$$\det(\vec{a}_0,\vec{a}_1)\det\left(\vec{b}_0,\vec{b}_1\right)=\det\left(\vec{b}_0,\vec{a}_1\right)\det\left(\vec{a}_0,\vec{b}_1\right)+\det\left(\vec{b}_1,\vec{a}_1\right)\det\left(\vec{b}_0,\vec{a}_0\right), \tag{2.1.22}$$

and the property

$$\mathcal{W}[k+a,-k+a](\lambda)=\mathcal{W}[k,-k](\lambda-ia\widetilde{\mu}) \tag{2.1.23}$$

to show the following fusion hierarchies

$$\begin{aligned}
&t^{(m)}(\lambda-i\tilde{\mu})t^{(m)}(\lambda+i\tilde{\mu})= \\
&=t^{(0)}(\lambda-i(2m+1)\tilde{\mu})t^{(0)}(\lambda+i(2m+1)\tilde{\mu})+t^{(m-1/2)}(\lambda)t^{(m+1/2)}(\lambda), \\
&t^{(1/2)}(\lambda)t^{(m)}(\lambda-i(2m+1)\tilde{\mu})= \\
&=t^{(0)}(\lambda-i\tilde{\mu})t^{(m+1/2)}(\lambda-2im\tilde{\mu})+t^{(0)}(\lambda+i\tilde{\mu})t^{(m-1/2)}(\lambda-i(2m+2)\tilde{\mu}),
\end{aligned} \tag{2.1.24}$$

where the index m can take the half-integers. Actually they can also be obtained by a process known as 'fusion' of the transfer matrix T without the auxiliary q-function [40].

In general, the fusion hierarchies (2.1.24) are an infinite set of functional relations. However, it is truncated to a finite set of functional equations (a $\mathbf{T}$-system) when μ/π is rational. To show this, let us set, for example, $\mu=\pi M/(2M+2)$ with $2M\in\mathbb{Z}^+$ and $\phi=\pi/(2M+2)$. Due to the $i\pi$ periodicity of $q(\lambda)$, we have

$$t^{(M+1/2)}(\lambda)=0, \tag{2.1.25}$$

and, by comparing the form of $\vec{q}^{2M+1}$ and $\vec{q}^{-2M-1}$ with $\vec{q}^{1}$ and $\vec{q}^{-1}$

$$t^{(M)}(\lambda)=t^{(0)}(\lambda). \tag{2.1.26}$$

It is also easy to check that the relations (2.1.24), (2.1.25) and (2.1.26) imply the symmetry

$$t^{(m)}(\lambda)=t^{(M-m)}(\lambda) \quad , \quad m=0,1/2,\ldots M/2 \tag{2.1.27}$$

The truncation is important because it provides us with a closed set of functional relations that can be converted into integral equations (the thermodynamic Bethe ansatz (TBA) equations) under suitable analyticity properties. Before deriving the TBA equations, we need first figure out how the six-vertex model transforms under the continuum limit.

2.1.4 The continuum limit

The phase transition of the lattice model is always an interesting topic. For the six-vertex model, it happens in the thermodynamic limit when the number of sites of the lattice $N \to \infty$ while the lattice spacing $d \to 0$ to keep the width of the lattice $L = Nd$ finite. This process is accompanied by renormalization, which implies that the limiting theory can be studied by the techniques in quantum field theory.

The six-vertex model lies in the region $0 < \mu < \pi/2$, at the phase transition of the eight-vertex model, and so is well-suited to the taking of this limit. The logarithm of the ground-state eigenvalue of the transfer matrix T is a good quantity to detect the universal behavior under the large-N limit:

$$\ln t_0(N) = -fN + \frac{\pi c_{\text{eff}}}{6N} + \ldots \tag{2.1.28}$$

The constant f is the large-N limit of the free energy per site and the second term is a consequence of the scaling symmetry characteristic of (second-order) phase transitions. If we replace the size N with the width L, the rescaling free energy can be defined as

$$F \equiv -\ln t_0(L) - fL \tag{2.1.29}$$

The '...' terms in Eq.(2.1.28) are all vanished when $d \to 0$ with L held fixed and in this limit

$$F(L) = -\frac{\pi c_{\text{eff}}}{6L} \tag{2.1.30}$$

This is the expected behavior of the free energy for a conformal field theory (CFT) on an infinite cylinder with circumference L. The effective central charge c_{eff} coincides with the standard Virasoro central charge c in the case of unitary theories. The effective central charge of the six-vertex model and the XXZ spin chain is $c_{\text{eff}} = 1$ in the periodic case and

$$c_{\text{eff}} = 1 - \frac{6\phi^2}{\pi(\pi - 2\eta)} < 1 \tag{2.1.31}$$

in the twisted cases [64, 65]. Finally, here we only discussed the free energy in the ground state of the model. For all states in the continuum limit of the model, see [66, 67].

It is well-known that the thermodynamic limit leads to a simplified form of the T-Q relation and Bethe ansatz equation.

$$E_i' = e^{2\lambda_i}, \quad \omega = -e^{-2i\mu} = e^{2i\tilde{\mu}}, \tag{2.1.32}$$

the BAE become

$$\prod_{l=1}^{n}\left(\frac{E_l' - \omega^2 E_i'}{E_l' - \omega^{-2}E_i'}\right) = -\omega^{2n-N}e^{-i2\phi}\left(\frac{1+\omega E_i'}{1+\omega^{-1}E_i'}\right)^N, \quad i = 1\ldots n. \tag{2.1.33}$$

Focusing on the ground state, we have $N/2$ Bethe roots on the real axis, implying that each E_i' is real and positive and thus eliminates the factor ω^{2n-N}. As $N \to \infty$, the number of these roots diverges, but the behavior of the 'edge roots'—those at the extreme left or right of the real axis—simplifies the analysis. For $\mu > \pi/4$[1], the left edge root behaves as

$$\lambda_{\text{edge}} \to -\frac{2\eta}{\pi}\log N, \tag{2.1.34}$$

while the right edge root behaves the same way, given the symmetry

$$q_0(-\lambda, \phi) = q_0(\lambda, -\phi) \leftrightarrow \lambda_i(\phi) = -\lambda_{N/2+1-i}(-\phi). \tag{2.1.35}$$

Thus one can simplify the BAE into

$$\prod_{l=1}^{\infty}\left(\frac{E_l - \omega^2 E_i}{E_l - \omega^{-2}E_i}\right) = -e^{-2i\phi}, \quad i = 1\ldots\infty \tag{2.1.36}$$

by substituting each E_i' by $E_i N^{-4\mu/\pi}$ when $N \to \infty$ and E_i keeps finite. After applying the techniques above to the functions $q_0(\lambda)$ and $t_0(\lambda)$, one can find the following modification:

q_0-function

$$q_0(\lambda) \to q_0(E) := \lim_{N\to\infty}\left[e^{N\lambda/2}q_0(\lambda)\right]_{\lambda=\frac{1}{2}\ln(EN^{-4\mu/\pi})} = \prod_{l=1}^{\infty}\left(1 - \frac{E}{E_l}\right). \tag{2.1.37}$$

T-Q relation

$$t_0(E)q_0(E) = e^{i\phi}q_0\left(\omega^2 E\right) + e^{-i\phi}q_0\left(\omega^{-2}E\right), \tag{2.1.38}$$

[1] For $\mu \leq \pi/4$, the product must be regulated to keep convergence in the large N limit.

the fusion hierarchies

$$\begin{aligned} t^{(m)}\left(\omega^{-1}E\right)t^{(m)}(\omega E) &= 1 + t^{(m-1/2)}(E)t^{(m+1/2)}(E) \\ t^{(1/2)}(E)t^{(m)}\left(\omega^{2m+1}E\right) &= t^{(m+1/2)}\left(\omega^{2m}E\right) + t^{(m-1/2)}\left(\omega^{2m+2}E\right). \end{aligned} \tag{2.1.39}$$

the truncation

$$t^{(m)}\left(\omega^{-1}E\right)t^{(m)}(\omega E) = 1 + \prod_{j=1/2}^{(h-1)/2}\left(t^{(j)}(E)\right)^{G_{2j,2m}} \quad , \quad m = 1/2, 1, \ldots, (h-1)/2 \tag{2.1.40}$$

for $\mu = \pi M/(2M+2)$ and $\phi = \pi/(2M+2)$. Here $h = 2M$, $\omega = e^{\pi i/(M+1)}$ and G_{ab} is the incidence matrix of the Lie algebra A_{h-1}. See Appendix C for more details on Lie algebras.

2.1.5 Thermodynamic Bethe Ansatz equations

After taking the continuum limit, the lattice model transitions into a quantum field theory. In certain $(1+1)$-dimensional integrable field theories, the Thermodynamic Bethe Ansatz (TBA) equation can be derived from the quantization condition of the scattering process in the thermodynamic limit [30, 68]. However, instead of deriving the TBA equation directly from quantum field theory, we obtain it here from the T-system (2.1.40), which emerges after truncation, focusing exclusively on simple-laced Lie algebras of ADE type. Further details can be found in [68, 69].

To proceed, let us introduce a more general notation commonly used in integrable models: defining $T_a(E) = t^{(a/2)}(E)$ transforms the truncated equation (2.1.40) into

$$T_a\left(\omega^{-1}E\right)T_a(\omega E) = 1 + \prod_{b=1}^{r}\left(T_b(E)\right)^{G_{ab}}, \quad a = 1, \ldots, r. \tag{2.1.41}$$

Next, we define the Y-functions as

$$Y_a(E) \equiv \prod_{b=1}^{r} T_b(E)^{G_{ab}}, \quad a = 1, \ldots, r, \tag{2.1.42}$$

which leads to the relation

$$T_b\left(\omega^{-1}E\right)T_b(\omega E) = 1 + Y_b(E). \tag{2.1.43}$$

By taking a suitable product over b, we obtain the set of equations:

$$Y_a(\omega E) Y_a\left(\omega^{-1} E\right)=\prod_{b=1}^r\left(1+Y_b(E)\right)^{G_{ab}}, \tag{2.1.44}$$

which is known as the **Y**-system.

To connect with quantum field theory, we introduce the parametrization $E=\exp(\theta/\eta)$, where θ is the rapidity in a $(1+1)$-dimensional relativistic quantum field theory, and the parameter η is given by $\eta=(M+1)/hM$. Under this transformation, the Y-system takes the form

$$Y_a\left(\theta+i\frac{\pi}{h}\right) Y_a\left(\theta-i\frac{\pi}{h}\right)=\prod_{b=1}^{h-1}\left(1+Y_b(\theta)\right)^{G_{ab}}. \tag{2.1.45}$$

Since the Y-functions depend on E, they inherit periodicity in θ with period $2\pi i\eta$. Notably, in the case of the A_{h-1} Lie algebra, this system coincides with the one derived in [68] for certain integrable quantum field theories exhibiting $\mathbb{Z}_h$ symmetry.

In order to obtain the integral equations, we need to define the pseudoenergies

$$\varepsilon_a(\theta)=\ln Y_a(\theta), \tag{2.1.46}$$

which allows us to rewrite the Y-system into

$$\varepsilon_a\left(\theta+i\frac{\pi}{h}\right)+\varepsilon_a\left(\theta-i\frac{\pi}{h}\right)-\sum_{b=1}^r G_{ab}\varepsilon_b(\theta)=\sum_{b=1}^r G_{ab}L_b(\theta) \tag{2.1.47}$$

with

$$L_a(\theta)=\ln\left(1+e^{-\varepsilon_a(\theta)}\right). \tag{2.1.48}$$

This equation has many solutions and some extra constraints are necessary to look for the desired one.

- First, $T_a(E)$ are regular at $E=0$, the functions $Y_a(\theta)$ approach constant values as $\theta\to-\infty$. These constants must solve the stationary (θ-independent) version of the Y-system (2.1.45). For the A_{h-1} Lie algebra, the solution has the following limit

$$\mathcal{Y}_a=\lim_{\theta\to-\infty} e^{\varepsilon_a(\theta)}=\frac{\sin\left(\frac{a\pi}{h+2}\right)\sin\left(\frac{(a+2)\pi}{h+2}\right)}{\sin^2\left(\frac{\pi}{h+2}\right)}. \tag{2.1.49}$$

- Next, turning to large E-behavior, the Y-function has the following limit [70]

$$\ln Y_a(E) \propto E^\eta, \quad |E| \to \infty, \quad |\arg(E)| < \pi - \delta, \tag{2.1.50}$$

  where the parameter δ is an arbitrarily small positive number. After introducing the constant $m_0 L$, it can be rewritten into

$$\varepsilon_a(\theta) \underset{\mathbb{Re}\,\theta\to\infty}{\sim} m_0 L e^\theta, \quad |\mathbb{Im}\, m\theta| < \pi\frac{h+2}{h} - \delta, \tag{2.1.51}$$

- Finally, from the large-θ limit of Eq.(2.1.47), where the right-hand side vanishes, shows that

$$m_a L = \frac{b_0}{2}\sin\left(\frac{\pi a}{h}\right), \quad b_0 \in \mathbb{R}. \tag{2.1.52}$$

  Here the trigonometric function on the RHS is the component of the Perron-Frobenius eigenvector of the incidence matrix G for the A_{h-1} Lie algebra.

Based on the constraints above, it is convenient to regularize the pseudoenergies as

$$f_a(\theta) = \varepsilon_a(\theta) - m_a L e^\theta \tag{2.1.53}$$

with the boundary strip $|\mathbb{Im}| < \pi/h$. It is easy to see that f satisfies the same equation (2.1.47) as ϵ. After the regularization, take their Fourier transform

$$\widetilde{f}(k) = \mathcal{F}[f(\theta)] = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} d\theta f(\theta) e^{-ik\theta+\epsilon\theta}, \tag{2.1.54}$$

which allows us to rewrite the equation (2.1.47) into

$$\sum_{b=1}^{r}\left(2\delta_{ab}\cosh\left(\frac{\pi k}{h}\right) - G_{ab}\right)\widetilde{f}_b(k) = \sum_{b=1}^{r} G_{ab}\widetilde{L}_b(k), \tag{2.1.55}$$

and $\widetilde{L}_a(k) = \mathcal{F}[L_a(\theta)]$. Finally, solving for $\widetilde{f}_a(k)$ and transforming back to the θ-space yields the TBA equations

$$\varepsilon_a(\theta) = m_a L e^\theta - \frac{1}{2\pi}\sum_{b=1}^{r}\int_{-\infty}^{\infty}\phi_{ab}(\theta-\theta')\, L_b(\theta')\, d\theta' \tag{2.1.56}$$

with the Fourier image of the function

$$\widetilde{\phi}_{ab}(k) = -2\pi\sum_{c=1}^{r}\left(2\delta_{ac}\cosh\left(\frac{\pi k}{h}\right) - G_{ac}\right)^{-1} G_{cb}. \tag{2.1.57}$$

The ϕ-function can also be exactly computed in terms of elementary functions and the factorized scattering matrix

$$\phi_{ab}(\theta) = -i\frac{d}{d\theta}\ln S_{ab}(\theta), \tag{2.1.58}$$

which can be exactly solved in integrable field theories. Especially, for A_{h-1} Lie algebra, the explicit result is given by

$$S_{a,b}(\theta) = \prod_{\substack{x=|a-b|+1 \\ \text{secp}\,2}}^{a+b-1} \{x\}, \quad a,b = 1,\dots,h-1, \tag{2.1.59}$$

and

$$\{x\} \equiv (x-1)(x+1), \quad (x) \equiv \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi x}{2h}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi x}{2h}\right)}. \tag{2.1.60}$$

As we have mentioned at the beginning of the section, the TBA equations can be interpreted in the context of the Thermodynamic Bethe Ansatz, where the elastic scattering among $(h-1)$-particles is assumed. Furthermore, in this picture, the constant m_0 in Eq.(2.1.51) is the mass of the particle, and the constant L is the circumference of an infinite cylinder on which the $(1+1)$d integrable theory is defined [30].

Integrals of motion generated by T-functions

Once we solve the $\epsilon(\theta)$, it is also possible to recover the T-functions. Let us begin with Eq.(2.1.43). Dividing through by $Y_a(E)$ and taking logarithms,

$$\ln T_a\left(\theta + i\frac{\pi}{h}\right) + \ln T_a\left(\theta - i\frac{\pi}{h}\right) - \sum_{b=1}^{r} G_{ab}\ln T_b(\theta) = L_a(\theta) \tag{2.1.61}$$

According to Eq.(2.1.43) and the limit (2.1.49), the functions $\ln T_a(\theta) - m_a L e^{\theta}/(2\cos(\pi/h))$ are bounded in the same analyticity strip as the $f_a(\theta)$ in Eq.(2.1.53), and can be found by a Fourier transformation:

$$\ln T_a(\theta) = \frac{m_a L}{2\cos(\pi/h)}e^{\theta} - \frac{1}{2\pi}\sum_{b=1}^{r}\int_{-\infty}^{\infty}\psi_{ab}(\theta-\theta')\,L_b(\theta')\,d\theta' \tag{2.1.62}$$

where the Fourier transform of the kernel $\psi_{ab}(\theta)$ is

$$\widetilde{\psi}_{ab}(k) = -2\pi\left(2\delta_{ab}\cosh\left(\frac{\pi k}{h}\right) - G_{ab}\right)^{-1}. \tag{2.1.63}$$

Similar with the TBA equations, ψ_{ab} can also be calculated by

$$\psi_{ab}(\theta) = -i\frac{d}{d\theta}\ln\left(\prod_{\substack{|a-b|+1\\ \text{step 2}}}^{a+b-1}(x)\right), \quad a,b = 1,\dots,h-1, \tag{2.1.64}$$

Finally, the equation (2.1.62) provides a method to calculate the ground-state eigenvalues of the local integrals of motion on a cylinder of circumference L. When $|\theta|\to\infty$, the relevant asymptotic expansion of $\ln T_1)$ has the form

$$\ln T_1(\theta) \sim \frac{m_1 L}{2\cos(\pi/h)}e^{\theta} - \sum_{n=1}^{\infty} C_n I_{2n}^{\text{vac}} e^{(1-2n)\theta}, \tag{2.1.65}$$

where I_{2n} are the eigenvalues of the local integrals of motion on a cylinder and the first-order coefficient $C_1 = 4\sin(\pi/h)/m_1$. Especially, for the A_r case, one can find [61]

$$\ln T_1(\theta) - \frac{m_1 L}{2\cos(\pi/h)}e^{\theta} = \frac{1}{\pi}\sum_{b=1}^{r}\int_{-\infty}^{\infty}\sin(\pi b/h)e^{-\theta+\theta'}L_b(\theta')\,d\theta' + \dots. \tag{2.1.66}$$

Compare with the expansion (2.1.65)

$$\begin{aligned} I_2^{\text{vac}} &= -\frac{1}{4\pi}\sum_{a=1}^{r}\int_{-\infty}^{\infty} m_1\frac{\sin(\pi a/h)}{\sin(\pi/h)}e^{\theta}L_a(\theta)d\theta \\ &= -\frac{1}{4\pi}\sum_{a=1}^{r}\int_{-\infty}^{\infty} m_a e^{\theta}L_a(\theta)d\theta, \end{aligned} \tag{2.1.67}$$

where L_a is defined in (2.1.48). Since $I_2^{\text{vac}} = F(L) = -\pi c_{\text{eff}}/(6L)$, the effective central charge can be given by

$$c_{\text{eff}} = \frac{3}{2\pi^2}\sum_{a=1}^{r}\int_{-\infty}^{\infty} d\theta m_a L e^{\theta}L_a(\theta) = \frac{hM-1}{hM+2}, \tag{2.1.68}$$

where the integral is calculated exactly as a sum of Rogers dilogarithm function. When $h=2$, it matches the result (2.1.31) and the truncation condition $\mu = \pi M/(2M+2)$ and $\phi = \pi/(2M+2)$.

2.2 A continuous model: the Kortweg-de Vries equation

So far, we have shown that the six-vertex model will become a unitary CFT after phase transition. It is also possible to derive $\mathbf{T}$, $\mathbf{Q}$ operators directly using field-theoretic methods [18–20]. The beginning point is the quantum Korteweg–de Vries (qKdV) equation. It emerges as a quantum counterpart to the classical KdV equation

$$u_t + 6uu_v + u_{vvv} = 0, \tag{2.2.1}$$

a foundational model in mathematical physics describing nonlinear waves in shallow water whose quantum analog finds relevance in a two-dimensional CFT with the central charge

$$c = 1 - \left(\beta - \frac{1}{\beta}\right)^2 < 1 \quad , \quad 0 < \beta < 1, \tag{2.2.2}$$

which can be unitary or non-unitary depending on the choice of β.

The qKdV equation is deeply related to another famous 2d CFT − Liouville theory. Their relation is shown by the Miura transformation which will be introduced in the next section. More fundamental detail on 2d CFT can be found in Appendix.B.

2.2.1 The conformal structure in the quantum KdV equation

The conformal symmetry in CFT is generated by the energy-momentum (EM tensor. Here we begin the theory on a cylinder where the EM tensor is given by

$$T(u) = -\frac{1}{\beta^2} : \phi'(u)^2 : -\frac{(1-\beta^2)}{\beta^2}\phi''(u) - \frac{1}{24}, \tag{2.2.3}$$

where : : is the normal ordering on a cylinder[2], and β is the coupling constant. The quantum **integrals of motion** can be found by the requirement of the involution

$$[\mathbf{I}_{2k-1}, \mathbf{I}_{2l-1}] = 0, \tag{2.2.4}$$

but the lowest terms are given by

$$\begin{aligned}
\mathbf{I}_1 &= \int_0^{2\pi} \frac{du}{2\pi} T(u), \\
\mathbf{I}_3 &= \int_0^{2\pi} \frac{du}{2\pi} : T^2(u) :, \\
\mathbf{I}_5 &= \int_0^{2\pi} \frac{du}{2\pi} \left[: T^3(u) : + \frac{c+2}{12} : (T'(u))^2 : \right], \\
&\dots
\end{aligned} \tag{2.2.5}$$

[2]The detail about the normal ordering will be given in the next section 2.3.4.

In the conformal field theory on a cylinder, the mode expansion of the EM tensor is given by

$$T(u) = -\frac{c}{24} + \sum_{-\infty}^{+\infty} L_{-n} e^{inu}, \tag{2.2.6}$$

where the operators L_n satisfy Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(n^3 - n\right)\delta_{n+m,0}, \tag{2.2.7}$$

and c is the central charge. Especially, for the KdV equation on a cylinder, it is given by Eq.(2.2.2). Similarly, the mode expansion of the free field operator ϕ in the EM tensor (2.2.3) is given by

$$\phi(u) = iQ + Pu + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{inu}, \tag{2.2.8}$$

where the mode operators Q, P and a_n satisfy the Heisenberg algebra

$$[Q, P] = \frac{i}{2}\beta^2; \qquad [a_n, a_m] = \frac{n}{2}\beta^2 \delta_{n+m,0}. \tag{2.2.9}$$

To give the continous Baxter's $\mathbf{T}$ operator, we need first construct the monodromy matrix [18]

$$\mathbf{M}_j(\lambda) = \pi_j \left[e^{2\pi i p H} \mathcal{P} \exp\left(\lambda \int_0^{2\pi} du' (: e^{-2\phi(u')} : q^{\frac{H}{2}} E + : e^{2\phi(u')} : q^{-\frac{H}{2}} F) \right) \Psi_0 \right], \tag{2.2.10}$$

where E,F and H are the generators of q-deformed quantum algebra $U_q(sl(2))$ with $q = e^{i\pi\beta^2}$

$$[H, E] = 2E, \qquad [H, F] = -2E, \qquad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{2.2.11}$$

The transfer matrix is then given by

$$\mathbf{T}_j(\lambda, p) = \mathrm{Tr}\, \mathbf{M}_j(\lambda, p) \tag{2.2.12}$$

with the involution condition

$$[\mathbf{T}_i(\lambda), \mathbf{T}_j(\lambda')] = 0. \tag{2.2.13}$$

Since the dimension of the representation π_j is $2j+1$, $\mathbf{T}_0$ is the identity operator and $\mathbf{T}_{\frac{1}{2}}$ is the common $\mathbf{T}$ operator introduced in the six-vertex model. We can also define the auxiliary $\mathbf{Q}$ operators. Follow the step in [19]. Let $\mathcal{E}_\pm$ and $\mathcal{H}$ be operators which satisfy the commutation relations as follows

$$q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}}, \qquad [\mathcal{H}, \mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm. \tag{2.2.14}$$

This is the so-called q-oscillator algebra. Let ρ be any representation of the algebra such that the trace

$$Z(\rho) = \mathrm{tr}_\rho[e^{2\pi i p\mathcal{H}}] \tag{2.2.15}$$

exists for complex p belonging to the upper half plane, $\Im m\ p > 0$. Then one can define two operators

$$\mathbf{A}_\pm(\lambda) = Z^{-1}(\pm P)\,\mathrm{tr}_\rho\left[e^{\pm 2\pi i P\mathcal{H}}\mathcal{P}\exp\left(\lambda\int_0^{2\pi} dx(:e^{-2\phi(x)}: q^{\pm\frac{1}{2}\mathcal{H}}\mathcal{E}_\pm + :e^{+2\phi(x)}: q^{\mp\frac{1}{2}\mathcal{H}}\mathcal{E}_\mp)\right)\right].$$

The operators $\mathbf{Q}_\pm(\lambda, p)$ are defined by a shift of $\mathbf{A}_\pm$,

$$\mathbf{Q}_\pm(\lambda, p) = \lambda^{\pm\frac{2P}{\beta^2}}\mathbf{A}_\pm(\lambda, p). \tag{2.2.16}$$

There are two important properties has been proved in [20]

$$[\mathbf{Q}_\pm(\lambda), \mathbf{Q}_\pm(\lambda')] = [\mathbf{Q}_\pm(\lambda), \mathbf{T}(\lambda')] = 0. \tag{2.2.17}$$

and the continuous version of $\mathbf{TQ}$ relations

$$\mathbf{T}(\lambda)\mathbf{Q}_\pm(\lambda) = \mathbf{Q}_\pm(q\lambda) + \mathbf{Q}_\pm(q^{-1}\lambda). \tag{2.2.18}$$

The first property guarantees that $\mathbf{T}$ and $\mathbf{Q}$ can be diagonalized simultaneously, allowing the $\mathbf{TQ}$ relations to be expressed in an eigenvalue form. In contrast to the lattice case, where the eigenvectors arise from the Bethe ansatz, the eigenvectors in CFT are constructed via a highest weight representation. Specifically, starting from the Heisenberg algebra (2.2.9), we define the Fock space $\mathcal{F}_p$ generated by the highest weight vector $|p\rangle$, which satisfies

$$P\,|p\rangle = p\,|p\rangle\,; \qquad a_n\,|p\rangle = 0 \quad \text{for} \quad n > 0. \tag{2.2.19}$$

Follow the convention in string theory, let us refer p as momentum parameter. It is also possible to construct them from the Virasoro algebra.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}, \tag{2.2.20}$$

where c is the central charge. The highest weight and the corresponding space $\mathcal{V}_{\Delta_a}$ obtained by acting L_{-n}'s on the highest weight state satisfying

$$L_0 \left|\Delta\right\rangle = \Delta \left|\Delta\right\rangle ; \qquad L_n \left|\Delta\right\rangle = 0 \quad \text{for} \quad n > 0. \tag{2.2.21}$$

Actually, these two states are closely related. For generic c and p, the space $\mathcal{F}_p$ is isomorphic to the space $\mathcal{V}_{\Delta_a}$ with

$$\Delta = (\frac{p}{\beta})^2 + \frac{c-1}{24}. \tag{2.2.22}$$

Now let us define the eigenvalue for $\mathbf{T}$ and $\mathbf{Q}$ operators

$$T(\lambda, p) = \langle p|\mathbf{T}(\lambda, p)|p\rangle, \quad Q_{\pm}(\lambda) = \langle p|\, \lambda^{\mp\frac{2P}{\beta^2}} \mathbf{Q}_{\pm}(\lambda)\, |p\rangle\,, \tag{2.2.23}$$

then in eigenvalue form, Eq.(2.2.18) becomes

$$T(\lambda)Q_{\pm}(\lambda) = e^{\mp 2\pi i p}Q_{\pm}(q\lambda) + e^{\pm 2\pi i p}Q_{\pm}(q^{-1}\lambda). \tag{2.2.24}$$

If we set

$$\mu = \pi(1-\beta^2)/2, \quad p = \frac{\phi}{2\pi} \tag{2.2.25}$$

which connects the six-vertex anisotropy μ and the coupling constant β, the twist parameter ϕ and the momentum p. It leads to $q^2 = \omega^2$ and the BLZ and continuum six-vertex **TQ** relations match perfectly.

Furthermore, if the coupling constant β belongs to the Semi-classical Domain

$$0 < \beta^2 < \frac{1}{2}, \tag{2.2.26}$$

and the momentum parameter is real, we can make the following assumptions [19]:

- **Analyticity.** The functions $Q_+(\lambda)$ and $T(\lambda)$ are entire functions of the complex variable λ^2

- **Location of zeroes.** Zeroes of the function $Q_+(\lambda)$ in the λ-plane are either real or occur in complex conjugated pairs. For the vacuum eigenvalues, all the zeroes are real and if $2p > -\beta^2$, they are all positive.

- **Asymptotic behavior.** The leading asymptotic behavior of $Q_{\pm}(\lambda)$ for large λ is

$$\log \mathbf{A}_{\pm}(\lambda) \sim M\,(-\lambda)^{\frac{1}{2-2\beta^2}}\,, \quad \lambda \to -\infty. \tag{2.2.27}$$

Based on the assumptions above, the eigenvalue $Q_+(\lambda)$ can be written as a convergent product over its zeros $\{\lambda_k\}$:

$$Q_+(\lambda) = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_k}\right) \quad , \quad (Q_+(0) = 1)\,. \tag{2.2.28}$$

Then, from the **TQ** relation and the entirety in λ of the eigenvalues, we obtain a set of Bethe ansatz equations:

$$\prod_{l=1}^{\infty}\left(\frac{\lambda_l - q^2\lambda_i}{\lambda_l - q^{-2}\lambda_i}\right) = -e^{4\pi i p} \quad , \quad i = 1\ldots\infty. \tag{2.2.29}$$

The other elements can also be directly generalized in the continuum context.

- **the fusion hierarchies**

  $$\mathbf{T}_j(q\lambda)\mathbf{T}_j\left(q^{-1}\lambda\right) = 1 + \mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda) \quad , \quad j = \frac{1}{2}, 1, \frac{3}{2}\ldots. \tag{2.2.30}$$

  At rational values of the parameter β^2 (recall that $\mu = \pi(1-\beta^2)/2$), the hierarchy truncates to a finite set of operators. For a non-unitary CFT on a cylinder with periodic boundary conditions, the effective central charge is given by

  $$c_{\text{eff}} = c - 24\Delta_{\text{min}}, \tag{2.2.31}$$

  where Δ_{min} is the lowest conformal weight in the spectrum. Hence, $c_{\text{eff}} = 1$ corresponds to the effective central charge of the untwisted six-vertex model. For a single p sector, the effective central charge associated with the highest-weight state $|p\rangle$ is

  $$c_{\text{eff}}^{(p)} = c - 24\Delta_p = 1 - 24\left(\frac{p}{\beta}\right)^2, \tag{2.2.32}$$

  which matches the effective central charge of the six-vertex model with a boundary twist given by the parameter ϕ.

- **the quantum Wronskian relation**

  The operators $\mathbf{T}_j$ can also be given directly in terms of the $\mathbf{Q}$'s:

  $$2i\sin(2\pi\mathbf{P})\mathbf{T}_j(\lambda) = \mathbf{Q}_+\left(q^{2j+1}\lambda\right)\mathbf{Q}_-\left(q^{-2j-1}\lambda\right) - \mathbf{Q}_+\left(q^{-2j-1}\lambda\right)\mathbf{Q}_-\left(q^{2j+1}\lambda\right), \tag{2.2.33}$$

which for $j = 0$ ($T_0 = \mathbf{I}$) and evaluating on eigenstate $|p\rangle$, we find the continuum version of the quantum Wronskian

$$q^{\frac{2p}{\beta^2}} Q_+(q\lambda) Q_-\left(q^{-1}\lambda\right) - q^{\frac{-2p}{\beta^2}} Q_+\left(q^{-1}\lambda\right) Q_-(q\lambda) = 2i\sin(2\pi p), \tag{2.2.34}$$

from which we can see $Q_-(s,p) = Q_+(s,-p)$.

2.2.2 Destri-de Vega equations from the BAEs

This section will derive the Destri-de Vega (DDV) equations or Nonlinear Integral Equations (NLIEs) from the Bethe ansatz equations satisfied by $Q_+(\lambda)$. For convenience, we suppress the index "+" in $Q_+(\lambda)$. Recall that the Bethe ansatz equation (2.2.29) can be written as

$$a(\lambda_i) = -1, \quad i = 1 \ldots \infty \tag{2.2.35}$$

when introducing the function

$$a(\lambda) = e^{4\pi i p} \frac{Q(\lambda q)}{Q\left(\lambda q^{-1}\right)} = e^{-4\pi i p} \prod_{l=1}^{\infty} \left(\frac{\lambda_l - q^2\lambda}{\lambda_l - q^{-2}\lambda} \right). \tag{2.2.36}$$

Actually, the zeros of $a(\lambda)+1$ are precisely the zeros of $Q(\lambda)$, the Bethe roots λ_k, together with the zeros of $T(\lambda)$. This thesis only considers the ground state situation where all the λ_k are real and positive according to the assumptions in Section 2.2.1, and a more general situation can be found in [19]. Based on the three assumptions above, one can find the following property for a-function:

- For real values of p. the location of zeros of Q function implies

$$a(\lambda)^* = a\left(\lambda^*\right)^{-1}, \tag{2.2.37}$$

where the star denotes the complex conjugation.

- From the asymptotic behavior of Q functions, one can see $a(\lambda)$ at large λ becomes

$$\log a(\lambda) \sim -2im\cos(\pi\xi/2)\lambda^{\frac{\xi+1}{2}}, \quad \lambda \to \infty, \quad |\arg\lambda| < 2\pi\beta^2, \tag{2.2.38}$$

where the parameter ξ satisfies $\beta^2 = \xi/(1+\xi)$.

- For small λ,

$$a(\lambda) = 4\pi i p + O\left(\lambda^2\right), \quad \lambda \to 0. \tag{2.2.39}$$

With these properties, let us now take the logarithm of Eq.(2.2.36).

$$\log a(\lambda) - 4\pi i p = \sum_{k=0}^{\infty} F\left(\lambda \lambda_i^{-1}\right), \tag{2.2.40}$$

where

$$F(\lambda) = \log \frac{1-\lambda q^2}{1-\lambda q^{-2}}. \tag{2.2.41}$$

Applying Cauchy's theorem, the infinite sum can be given by the contour integral

$$\sum_{i=0}^{\infty} F\left(\lambda \lambda_k^{-1}\right) = \int_C \frac{d\mu}{2\pi i} F\left(\lambda \mu^{-1}\right) \partial_\mu \log(1 + a(\mu)) \tag{2.2.42}$$

due to the singularity of $\partial_\mu \log(1 + a(\mu))$ at λ_i. The contour C goes from $+\infty$ to zero above the positive real axis, then winds around zero and returns to infinity below the positive real axis on the λ plane. Introducing new variables θ, θ' and θ_k as

$$\lambda = e^{2\theta(1-\beta^2)}, \quad \mu = e^{2\theta'(1-\beta^2)}, \quad \lambda_i = e^{2\theta_i(1-\beta^2)} \tag{2.2.43}$$

Integrating by parts we have the DDV equation for the ground state

$$\begin{aligned} \log a(\theta) - 4\pi i p = & - 2i \int_{-\infty}^{\infty} d\theta' R\left(\theta - \theta'\right) \mathbb{Im} \log\left(1 + a\left(\theta' - i0\right)\right) \\ & + \int_{-\infty}^{\infty} d\theta' R\left(\theta - \theta'\right) \log a\left(\theta'\right), \end{aligned} \tag{2.2.44}$$

where we have used $a(\lambda)^* = a\left(\lambda^*\right)^{-1}$ from the Bethe ansatz equations, and

$$R(\theta) = \frac{i(1-\beta^2)}{2\pi} \lambda \partial_\lambda F(\lambda). \tag{2.2.45}$$

With the convolution form

$$A * B(\theta) = \int_{-\infty}^{\infty} d\theta' A\left(\theta - \theta'\right) B\left(\theta'\right), \tag{2.2.46}$$

the integral equation becomes

$$(\delta(\theta) - R(\theta)) * \log a(\theta) = 4\pi i p + f(\theta) - 2iR * \mathbb{Im} \log(1 + a(\theta - i0)), \tag{2.2.47}$$

It is a simple method to solve the convolution equation using Fourier transforms.

$$(1-\widetilde{R}(k))\mathcal{F}[\ln a](k) = -8\pi^2 p\delta(k) - 2i\widetilde{R}(k)\mathbb{Im}\mathcal{F}[\ln(1+a)](k), \tag{2.2.48}$$

where the Fourier transforms are defined by

$$\widetilde{f}(k) = \mathcal{F}[f(\theta)] = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} d\theta f(\theta)e^{-ik\theta+\epsilon\theta}. \tag{2.2.49}$$

Then apply $(1-\widetilde{R}(k))^{-1}$ on both sides and take the inverse Fourier transform to find

$$i\log a(\theta) = -\frac{2\pi p}{\beta^2} + 2m\cos\frac{\pi\xi}{2}e^{\theta} + i\sum_a{}' \log S(\theta-\theta_a) - 2G*\mathbb{Im}\log(1+a(\theta-i0)) \tag{2.2.50}$$

where

$$S(\theta) = \exp\left\{-i\int_0^\infty \frac{d\nu}{\nu}\sin(\nu\theta)\frac{\sinh(\pi\nu(1+\xi)/2)}{\cosh(\pi\nu/2)\sinh(\pi\nu\xi/2)}\right\}, \tag{2.2.51}$$

$$G(\theta) = \delta(\theta) + \frac{1}{2\pi i}\partial_\theta \log S(\theta),$$

and $m\cos(\pi\xi/2)$ is a real constant fixed by the large $\theta(\lambda)$ limit. This equation holds for all θ within the strip $|\mathbb{Im}\theta| < \min(\pi, \pi\beta^2/(\beta^2-1))$ since the kernel has poles at $\theta = \pm i\pi$ and $\pm i\pi\beta^2/(\beta^2-1)$. The analytical continuation can be found in [71]. Finally, the function $S(\theta)$ is the scattering amplitude for the Sine-Gordon model as we will introduce in the next section.

Integrals of motion generated from Q-operators

In the section 2.1.5, we have shown how to generate local IoMs from the T-functions and the TBA equations. Although they are derived in the six-vertex model, the generalization to the quantum KdV hierarchies is direct. After the replacement from E to λ and Eq.(2.2.25), one can obtain

$$\log \mathbf{T}(\lambda) \simeq m\lambda^{\frac{1}{1-\beta^2}}\mathbf{I} - \sum_{n=1}^{\infty} C_n \lambda^{\frac{1-2n}{1-\beta^2}}\mathbf{I}_{2n}, \tag{2.2.52}$$

where $\{\mathbf{I}_{2n}\}$ is the basic set of commutative local IoMs given in Eq.(2.2.5). Similar to the $\mathbf{T}$ operator, the auxilary $\mathbf{Q}$ also generates a set of IoMs

$$\log \mathbf{Q}(\lambda) = -\sum_{n=1}^{\infty}(\beta^{-2}\Gamma\left(1-\beta^2\right)\lambda)^{2n}\mathbf{H}_n \tag{2.2.53}$$

Different from the local IoMs $\{\mathbf{I}_{2n}\}$'s, these $\{\mathbf{H}_n\}$'s are the non-local IoMs. Consider the ground state eigenvalue of $Q(\lambda)$ at $p \to \infty$.

$$\log Q^{(vac)}(\lambda)\big|_{p\to+\infty} = -\sum_{n=1}^{\infty} y^{2n} H_n^{(vac)}\Bigg|_{p\to\infty},$$

First, given a solution to the DDV equation, the $\log Q(\lambda)$ can be calculated as [72]

$$\log Q(\lambda) = -i\int_{C_\nu} d\nu \frac{g(\nu)}{\cosh \pi\nu/2 \sinh \pi\nu\xi/2} \left(-\lambda^2\right)^{i\nu(1+\xi)/2}, \tag{2.2.54}$$

where the function $g(\nu)$ is defined as

$$g(\nu) = \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} \mathbb{Im} \log(1 + a(\theta - i0))e^{-i\nu\theta}, \tag{2.2.55}$$

and the integration contour goes along the line $(-1-\epsilon)$ with ϵ some small positive number. In the ground state, it leads to

$$\log Q^{(vac)}(\lambda)\big|_{p\to+\infty} \sim$$
$$-\frac{p}{2\pi^{\frac{3}{2}}\xi}\int_{C_\nu} \frac{d\nu}{\nu^2} \Gamma(1 - i\nu(1+\xi)/2)\Gamma(1 + i\nu\xi/2)\Gamma\left(-\frac{1}{2} + i\nu/2\right) e^{i\delta\nu} \left(-\frac{\lambda^2}{\lambda_0^2}\right)^{i\nu(1+\xi)/2}, \tag{2.2.56}$$

from which we can extract the non-local IoMs at large p, especially for the first order,

$$H_1^{(vac)}\Big|_{p\to+\infty} \sim \frac{\beta^4\Gamma\left(\beta^2\right)\Gamma\left(1/2 - \beta^2\right)}{2\sqrt{\pi}} p^{2\beta^2-1}. \tag{2.2.57}$$

Other higher orders at large p and the exact H_1 can be found in [19]. However, the exact form of the non-local IoMs is still a open question.

Finally, although different from the TBA equation from which the whole set of local IoMs can be directly calculated, the first-order term, namely the free energy or the effective central charge can be obtained directly from the DDV equation.

$$c_{\text{eff}} = -\frac{6im\cos\left(\frac{\xi}{2}\right)}{\pi^2}\int_{\mathcal{C}} d\theta\ \mathbb{Im}\ e^{\theta}\log\left(1 + a\left(\theta - i0\right)\right) = 1 - 24\frac{p^2}{\beta^2}, \tag{2.2.58}$$

which matches the result (2.1.68) obtained in the TBA equation once we set the correspondence between the continuum parameters and lattice parameters

$$\beta^2 = 1 - \frac{2}{\pi}\eta, \quad p = \frac{\phi}{2\pi} \tag{2.2.59}$$

In the next chapter, we will see how the WKB analysis of the ordinary differential equations obtains the same Bethe ansatz equations and DDV equations.

2.3 The Toda field theories

In this section, we introduce another famous integrable field theory—Toda field theory—which is deeply related to the KdV hierarchies, see Appendix A for more details. Unlike the local IoMs generated from T-functions and Q-operators in Sections 2.1.5 and 2.2.2 where most local IoMs are only obtainable in numerical form, the local IoMs in Toda field theory (as well as in the KdV hierarchies) can also be computed from the involution condition (2.2.4) using the operator product expansion in 2d CFT. Although it is not an efficient method, it still allows us to compute the analytical results. This will be the main part of this section and will appear again from a different viewpoint in the ODE/IM correspondence in Chapter 4. Before we turn to it, let us first briefly introduce integrable field theories.

2.3.1 Introduction to integrable field theories

Similar to any other integrable model, an integrable quantum field theory is characterized by an infinite number of conserved charges. So far, a non-trivial integrable quantum field theory can only occur in $(1+1)$ dimensions. Recall that in classical mechanics, the existence of a sufficient number of integrals of motion allows us to find the exact solution of the equations of motion. Similarly, if there are an infinite number of conservation laws to match the infinite degrees of freedom in a quantum field theory, then we can derive its exact solution, including the S-matrix for scattering processes, correlation functions, thermodynamics, and so on.

The Scattering Matrix in $(1+1)$D Integrable Field Theories

In (1+1)D integrable field theories, the collision among particles is purely elastic scattering so that there is no particle creation or annihilation and the S-matrix is simplified. Multi-particle scattering can be decomposed into a sequence of two-body scatterings which satisfy the Yang-Baxter equation

$$S_{12}(\theta_1-\theta_2)S_{13}(\theta_1-\theta_3)S_{23}(\theta_2-\theta_3) = S_{23}(\theta_2-\theta_3)S_{13}(\theta_1-\theta_3)S_{12}(\theta_1-\theta_2), \tag{2.3.1}$$

where $\{S_{ij}\}$'s are two-body scattering matrix between ith and jth particle, and θ is the rapidity of the particle. The Yang-Baxter equation can be described by the figure below.

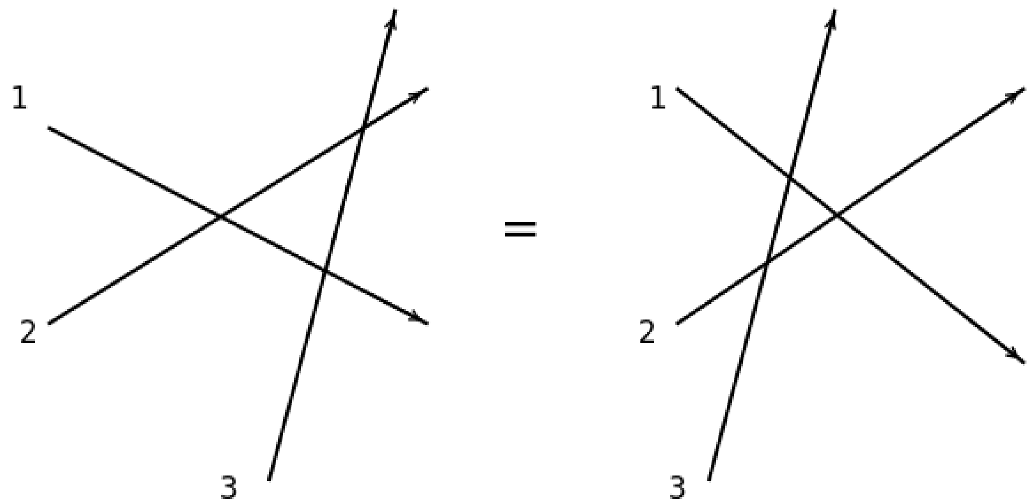


Figure 2.3.1: YBE

After combining the Yang-Baxter equation and the constraints including unitarity, crossing symmetry, and analytic structure, it is possible to solve the scattering matrix $S(\theta)$ exactly. This is a direct consequence of integrability. The scattering matrix of Sine-Gordon model can be found in [30].

The exact scattering matrix also satisfies the Bethe ansatz equation introduced in previous section 2.1 and can be used to study the thermal properties of $(1+1)$D integrable field theories. It leads to a set of nonlinear integral equations − the thermodynamics Bethe ansatz (TBA) equations as introduced in Section 2.1.5. Here we briefly introduce the procedure.

The Bethe ansatz

Consider a $(1 + 1)$-dimensional integrable theory defined on a circle of circumference L. Suppose that the spectrum consists of a set of particles A_a $(a = 1, 2, \ldots, n)$ with masses m_a, and that their scattering amplitudes are purely diagonal[3] and characterized by their phase shifts $\delta_{ab} = -i \log S_{ab}(\theta)$. In the configuration space of the N-particle state, which is divided into $N!$ domains ordered by $x_{i_1} \ll x_{i_2} \ll \cdots \ll x_{i_N}$, the wavefunction is

[3]Here, 'diagonal' means there is no reflection part.

particularly simple, being given by plane waves:

$$\Psi\big(x_{i_1}, x_{i_2}, \ldots, x_{i_N}\big) \;=\; \prod_{k=1}^{N} e^{i\, p_{i_k}\, x_{i_k}}, \tag{2.3.2}$$

together with the periodic boundary condition for bosonic particles.[4]

From the definition of rapidity, the energy and momentum of these states are

$$E \;=\; \sum_{i=1}^{N} m_i \cosh\theta_i, \quad p \;=\; \sum_{i=1}^{N} m_i \sinh\theta_i. \tag{2.3.3}$$

Notice that exchanging two particles sends one domain into another, and each such exchange multiplies the wavefunction by the corresponding scattering amplitude. Hence, the quantization condition for the momenta $\{p_i\}$ is

$$e^{ip_iL} \prod_{j\neq i}^{N} S\big(\theta_i - \theta_j\big) \;=\; 1, \quad i = 1, 2, \ldots, N, \tag{2.3.4}$$

or, in terms of rapidities,

$$m_i\, L\, \sinh\theta_i \;+\; \sum_{j\neq i}^{N} \delta_{ij}\big(\theta_i - \theta_j\big) \;=\; 2\pi\, n_i, \quad n_i \,\in\, \mathbf{Z}. \tag{2.3.5}$$

Finally, the wavefunction must be symmetric under the exchange of two identical bosons with the same rapidity, which leads to

$$S_{aa}(0) \;=\; 1, \tag{2.3.6}$$

i.e. the Bethe ansatz equation. The state parametrized by

$$\big|\, n_1, \theta_1;\, n_2, \theta_2;\, \ldots;\, n_N, \theta_N \,\big\rangle \tag{2.3.7}$$

is thus interpreted as a Bethe ansatz state in the context of quantum field theory.

The quantization conditions (2.3.5) simplify in the thermodynamic limit, where $L \to \infty$ and the total number of particles $N \to \infty$ in such a way that N/L remains finite. This analysis ultimately leads to the TBA equation once the free energy is calculated [30].

[4] We only focus on the bosonic case; a similar conclusion holds for the fermionic case.

2.3.2 The Toda field theories

From now on, we will deal with some specific $(1+1)$D integrable field theories. Let us now choose the complex coordinates to describe the analytic and anti-analytic indices of tensor quantities, the conservation law of a current with components (T_{s+1}, Θ_{s-1}) is given by

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}, \tag{2.3.8}$$

and the local integrals of motion are

$$\mathbf{I}_s = \oint \left[T_{s+1} dz + \Theta_{s-1} d\bar{z} \right]. \tag{2.3.9}$$

The integer index s is the spin of the local IoMs. When $s = 1$, T_2 is namely the energy-momentum tensor and $\mathbf{I}_1$ is the momentum in $(1+1)$ dimensions. Recall that two-dimensional conformal field theory can be split into holomorphic and anti-holomorphic parts and T_{s+1} is a purely holomorphic field with $\partial_{\bar{z}} T_{s+1} = 0$. Hence, all two-dimensional conformal theories have infinite conservation laws and can be considered integrable models. Let us show a simple example: the Liouville field theory, a famous 2d CFT well-known in cosmology [73]. The action for the Liouville field theory is

$$S[\phi] = \int \mathrm{d}^2 z \left[\frac{1}{2} \partial_z \phi \cdot \partial_{\bar{z}} \phi + \left(\frac{m^2}{\beta} \right) \exp\left(2\beta\phi\right) \right], \tag{2.3.10}$$

where $\phi(z)$ is a scalar field, m is the mass parameter and β is the coupling constant. Although the Liouville action is formally invariant under conformal transformations, its correct quantization requires the introduction of a charge at infinity $\alpha_0 = \beta - 1/\beta$. See Appendix B.2 for more detail. It modifies the quantum version of the EM tensor into

$$T(z) = -\frac{1}{2} \circ\!\!\circ\, (\partial_z \phi)^2(z) \,\circ\!\!\circ - i\alpha_0 \cdot \partial_z^2 \phi(z). \tag{2.3.11}$$

The Liouville theory can be generalized into the Toda field theory with a simple Lie algebra $\mathfrak{g}$ whose actions are given by

$$S[\phi] = \int \mathrm{d}^2 z \left[\frac{1}{2} \partial_z \phi \cdot \partial_{\bar{z}} \phi + \left(\frac{m^2}{\beta} \right) \sum_{i=1}^{r} \exp\left(\beta\alpha_i \cdot \phi\right) \right], \tag{2.3.12}$$

where $\phi(z) = (\phi_1(z), \ldots, \phi_r(z))$ becomes vector of r scalar fields corresponding to the rank r of $\mathfrak{g}$. α_i $(i = 1, \ldots, r)$ are the simple roots of $\mathfrak{g}$ introduced in Appendix C. We denote the coroots of α_i by $\alpha_i^\vee = 2\alpha_i/|\alpha_i|^2$. The EM tensor now becomes

$$T(z) = -\frac{1}{2} \mathop{:}\partial\phi \cdot \partial\phi\mathop{:}(z) - \left(\beta\rho - \frac{1}{\beta}\rho^\vee\right) \cdot i\partial^2\phi(z), \tag{2.3.13}$$

where $\rho(\rho^\vee)$ is Weyl vecotr(co-Weyl vector).

Before ending this subsection, let us make two more comments. First, the Toda field theory is deeply related to the KdV hierarchies in Section 2.2 by the (quantum) Drinfeld-Sokolov reduction [21]. Both of them approach the same conformal field theory with the same central charge [24, 25, 48]. A brief introduction on Drinfeld-Sokolov reduction is given in Appendix A. Finally, the conformal symmetry of Toda field theory can be broken while keeping the integrability by inserting a massive part corresponding to an affine root [30,74,75]. The Toda field theory becomes the so-called affine Toda field theory. For example, the Liouville theory will deform into the Sine-Gordon model. The linear problem of affine Toda field theory also plays an important role in the ODE/IM correspondence as we will see in Chapter 4.

2.3.3 Conformal field theory with W-symmetry

In this section, we will review quantum Toda field theory as a two-dimensional CFT with W-symmetry and its free field representation. For more details about the W algebra, see Appendix C and a review [48]. We will then compute the quantum IoMs on the cylinder.

Let us begin with a CFT on a complex plane. Set $(z, \bar{z})$ to be a complex coordinate on the plane. A CFT is completely characterized by a set of operators (*e.g.* the energy-momentum tensor and W-currents) and their operator product expansions (OPE). An operator $A(z, \bar{z})$ in CFT is decomposed into the holomorphic part $A(z)$ and the anti-holomorphic part $\bar{A}(\bar{z})$. We focus on the holomorphic part.

We will define the W-algebra by the free field realizations. For A_r and D_r type W-algebras, their free field realization can be obtained by the quantum Miura transformation [24,25]. The EM tensor and higher spin W currents are obtained from polynomials of the scalar field $\phi(z)$ and their derivatives. These are also regarded as quantized symmetry of non-affine Toda field theories of (2.3.12). The OPE of $\phi_i(z)$ is defined by

$$\phi_i(z)\phi_j(w) = -\delta_{ij}\log(z-w) + \cdots . \tag{2.3.14}$$

The mode expansion of $\partial\phi_j(z)$

$$i\partial\phi_j(z) = \sum_{n\in\mathbb{Z}} a_n^j z^{-n-1}, \tag{2.3.15}$$

leads to the commutation relations of a_n^j: $[a_n^i, a_m^j] = n\delta_{n+m,0}\delta^{ij}$. The EM tensor in the $W\mathfrak{g}$-algebra with simply-laced Lie algebra $\mathfrak{g}$ is defined by

$$T(z) = -\frac{1}{2} \mathbin{\text{:}} (\partial_z\phi)^2(z) \mathbin{\text{:}} - i\alpha_0\rho\cdot\partial_z^2\phi(z), \tag{2.3.16}$$

whose central charge is given by

$$c = r - 12\alpha_0^2\rho^2. \tag{2.3.17}$$

Here the coupling constant $\alpha_0 = \beta - 1/\beta$. We also define a vertex operator

$$V_\Lambda(z) = \mathbin{\text{:}} e^{i\Lambda\cdot\phi(z)} \mathbin{\text{:}}, \tag{2.3.18}$$

which is a primary field with conformal weight $\Delta_\Lambda = \frac{1}{2}\Lambda(\Lambda + 2\alpha_0\rho)$ and the Λ is a momentum parameter appearing in the mode expansion of the scalar field ϕ on a cylinder.

Free field realization of WA_r algebra

The WA_r algebra is generated by spin $2, \dots, r+1$ currents which can be found by the quantum Miura transformation [24]. It is convenient to first define the representation. Let $\{\epsilon_1, \dots, \epsilon_{r+1}\}$ be the weights of this representation. They satisfy the conditions:

$$\begin{aligned} \epsilon_i\cdot\epsilon_j = \delta_{ij} - \frac{1}{r+1}, \quad \sum_{i=1}^{r+1}\epsilon_i = 0, \\ \sum_{i<j}\epsilon_i\otimes\epsilon_j = -\frac{1}{2}\mathbb{1}, \quad \sum_{i=1}^{r+1} i\epsilon_i = -\rho^\vee. \end{aligned} \tag{2.3.19}$$

Then the quantum Miura transformation is a method to obtain their free field realization:

$$(\alpha_0\partial_z)^{r+1} - \sum_{k=2}^{r+1}\tilde{W}_k(z)\,(\alpha_0\partial_z)^{h-k} = \mathbin{\text{:}} (\alpha_0\partial_z - \epsilon_1\cdot i\partial_z\phi)\dots(\alpha_0\partial_z - \epsilon_{r+1}\cdot i\partial_z\phi) \mathbin{\text{:}} (z)\ . \tag{2.3.20}$$

Here $h = r + 1$. The current $\tilde{W}_2(z)$ is the energy-momentum tensor $T(z)$, which leads to the central charge

$$c = r - r(r+1)(r+2)\alpha_0^2. \tag{2.3.21}$$

Their OPE can be computed from the OPE (2.3.14). Especially for $r = 1$, it leads to the one in the KdV equation (2.2.2) and the EM tensor becomes the one (2.2.3) after a coordinate transformation from the complex plane to the cylinder. Applying the differential operator (2.3.20) to the vertex operator $V_\Lambda(z)$, one obtains the formula for conformal weight of $\tilde{W}_k(z)$ [24]:

$$\tilde{\Delta}_k = (-1)^{k-1} \sum_{i_1 < i_2 < \cdots < i_k} \prod_{j=1}^{k} \left[p_{i_j} + \left((k-j) - \frac{1}{2}(r + 2 - 2i_j) \right) \alpha_0 \right], \tag{2.3.22}$$

where the new parameter p is given by a shift of Λ

$$p = \Lambda + \alpha_0 \rho, \quad p_i = (\epsilon_i, p). \tag{2.3.23}$$

The conformal weights $\tilde{\Delta}_i$ can be expressed in terms of a symmetric polynomial

$$\sigma_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq r+1} p_{i_1} p_{i_2} \dots p_{i_k}. \tag{2.3.24}$$

The first two are given by

$$\tilde{\Delta}_2 = -\sigma_2 - \frac{1}{4} \binom{r+2}{3} \alpha_0^2, \quad \tilde{\Delta}_3 = \sigma_3 + (r-1)\alpha_0 \sigma_2 + \binom{r+2}{4} \alpha_0^3. \tag{2.3.25}$$

Usually, the W-currents obtained from the quantum Miura transformation (2.3.20) are not primary. But we can reconstruct these fields into primary ones denoted by $W_k(z)$. See Appendix B.4.1 for more detail.

Free field realization of WD_r algebra

The WD_r algebra contains spin $2, 4, \dots, 2r-2$ currents $\{W_2, W_4, \dots, W_{2r-2}\}$ and the spin r current R_r. The free field realization of the spin r current is defined by [76]:

$$R_r(z) = \,:\!(\alpha_0 \partial_z - i\epsilon_1 \cdot \partial_z \phi(z)) \dots (\alpha_0 \partial_z - i\epsilon_n \cdot \partial_z \phi(z))\!:\, \cdot 1. \tag{2.3.26}$$

Here $\epsilon_1, \ldots, \epsilon_r$ are the orthonormal basis of $\mathbb{R}^r$ with $\epsilon_i \cdot \epsilon_j = \delta_{ij}$. The other W-currents are determined by the OPE between R_r's:

$$R_r(z)R_r(w) = \frac{A_r}{(z-w)^{2r}} + \sum_{k=1}^{r-1} \frac{A_{r-k}}{(z-w)^{2(r-k)}} \left(\tilde{W}_{2k}(z) + \tilde{W}_{2k}(w)\right), \tag{2.3.27}$$

where

$$A_k = \prod_{j=1}^{k-1} (1 - 2j(2j+1)\alpha_0^2). \tag{2.3.28}$$

The energy-momentum tensor $T(z)$ is given by $\tilde{W}_2(z)$:

$$\tilde{W}_2(z) = -\frac{1}{2} \mathbin{\circ\kern-0.3em\circ} (\partial_z\phi \cdot \partial_z\phi) \mathbin{\circ\kern-0.3em\circ} (z) - i\alpha_0 \rho \cdot \partial_z^2 \phi(z) \tag{2.3.29}$$

with $\rho = \sum_{i=1}^{r} (r-i)\epsilon_i$. The central charge is

$$c = r - r(2r-1)(2r-2)\alpha_0^2. \tag{2.3.30}$$

The free field realization of $\tilde{W}_{2k}$ can also be obtained from (2.3.27), which are non-primary, while R_r is a primary field. Here we will construct $\tilde{W}_4$ and $\tilde{W}_6$ recursively as follows. First, we note that WD_r CFT is obtained by adding a free boson to WD_{r-1}:

$$WD_r = \{\tilde{\phi}_r\} \oplus WD_{r-1} \tag{2.3.31}$$

with $\tilde{\phi}_i = \epsilon_{r+1-i} \cdot \phi$. Let $W_2^{(r)}, \ldots, W_{2r-2}^{(r)}, R_r^{(r)}$ be generators of the WD_r algebra. Then $R_r^{(r)}$ is related to $R_{r-1}^{(r-1)}$ via

$$R_r^{(r)} = - \mathbin{\circ\kern-0.3em\circ} \tilde{p}_r R_{r-1}^{(r-1)} \mathbin{\circ\kern-0.3em\circ} + \alpha_0 \partial R_{r-1}^{(r-1)}, \quad \tilde{p}_r = -i\partial\tilde{\phi}_r. \tag{2.3.32}$$

This implies that the OPE $R_r^{(r)}(z)R_r^{(r)}(w)$ is obtained from $R_{r-1}^{(r-1)}(z)R_{r-1}^{(r-1)}(w)$, which leads to the recursion relations for $W_{2k}^{(r)}$. An explicit calculation and further primary-field construction are shown in the Appendix B.4.2. There is no explicit formula for conformal weight Δ_i in WD_r algebras [76], but we can still summarize some lowest-order ones as follows:

$$\tilde{\Delta}_2 = \frac{1}{2}\sigma_1 - \frac{r(2r-2)(2r-1)}{24}\alpha_0^2, \tag{2.3.33}$$

$$\begin{aligned} \tilde{\Delta}_4 = {} & \frac{1}{2}\sigma_2 - \left(\frac{1}{12}(r-7)(r-2)(2r-3)\alpha_0^2 + \frac{1}{4}\right)\sigma_1 + \frac{r(2r-1)(r-1)}{24}\alpha_0^2 \\ & + \frac{1}{720}(r-2)(r-1)r(2r-3)(2r-1)(5r-71)\alpha_0^4, \end{aligned} \tag{2.3.34}$$

where σ_i is defined by

$$\sigma_k = \sum_{1\le i_1<i_2<\cdots<i_k\le r} p_{i_1}^2 p_{i_2}^2 \ldots p_{i_k}^2. \tag{2.3.35}$$

The conformal weight for the extra generator R_r is

$$\Delta_r' = p_1 p_2 \cdots p_r. \tag{2.3.36}$$

2.3.4 Local IoMs and their vacuum eigenvalues on a cylinder

The IoMs for relevant perturbation of CFT were studied in [60, 75, 77, 78]. The conserved currents that correspond to the generalized KdV flows are constructed by the OPE between the W-currents [79]. Toda field theories, as integrable models, possess an infinite number of IoMs.

On the complex plane, the IoMs can be given by the contour integral around the origin

$$\mathbf{M}_k = \frac{1}{2\pi i} \oint dz \; J_k(z). \tag{2.3.37}$$

Here $J_k(z)$ ($k \in \mathbf{Z}$) are the conserved currents, which can be constructed from normal ordered products of the EM tensor T, higher-spin W fields, and their derivatives. $J_k(z)$ are uniquely determined up to the total derivative terms by the requirement of mutual commutativity condition:

$$[\mathbf{M}_i, \mathbf{M}_j] = 0, \quad \text{for } i, j \in \mathbf{Z}. \tag{2.3.38}$$

The explicit expressions of currents for Virasoro minimal models can be found in [77].

The IoMs obtained on a complex plane can be transformed into the ones on a cylinder by conformal transformation [18], which are the ones defined in the KdV hierarchies in Section 2.2. The zero mode of the conserved currents is necessary to see the correspondence between the ODE and the CFT. Due to the difference in the normal ordering prescriptions between the complex plane and cylinder, the zero modes on the cylinder have non-trivial corrections. Some technical details for the Virasoso CFT have been developed in [80, 81]. Here, we will find some formulas of the zero modes for the IoMs in WA_r and WD_r CFTs.

Let us begin with the conformal transformation defined by

$$z = e^u, \tag{2.3.39}$$

where u is the coordinate on a cylinder, and z is the one on a complex plane. Under the conformal transformation, a primary field $A(z) = \sum_n A_n z^{-n-\Delta_A}$ with conformal weight Δ_A on the complex plane transforms as

$$\hat{A}(u) = \left(\frac{dz}{du}\right)^{\Delta_A} A(z) = z^{\Delta_A} A(z) = \sum_n A_n z^{-n}. \tag{2.3.40}$$

For a non-primary field, the transformation rule is written as

$$\hat{A}(u) = z^{\Delta_A} A(z) + \delta A(z) =: \sum_n \hat{A}_n z^{-n}, \tag{2.3.41}$$

where $\delta A(z)$ is a contribution from the non-primary nature of the operator. For the energy-momentum tensor $T(z)$, one finds

$$\hat{T}(u) = z^2 T(z) - \frac{c}{24}, \tag{2.3.42}$$

with $\delta T(z) = -\frac{c}{24}$ from the Schwarzian derivative. We define $A_R(z) = A(z) + z^{-\Delta_A}\delta A(z)$ for later convenience.

Under the conformal transformation (2.3.39), a conserved current $J(z)$ is transformed into $\hat{J}(u)$. The terms in $\hat{J}(u)$ consist of the normal ordered product among the generators of the chiral algebra on the cylinder, where the normal ordered product of operators $\hat{A}$ and $\hat{B}$ is defined by the time-ordered product $T(\hat{A}(u)\hat{B}(v))$ for the time coordinates Re u and Re v on the cylinder. Then the normal ordered product $:\hat{A}\hat{B}:(v)$ on the cylinder is defined by

$$:\hat{A}\hat{B}:(v) = \frac{1}{2\pi i}\oint_v du \frac{T(\hat{A}(u)\hat{B}(v))}{u-v}, \tag{2.3.43}$$

which differs from the normal ordered product (B.1.24) based on the radial ordering on the complex plane. Under the conformal transformation (2.3.39) and (2.3.41), the normal ordered product on the cylinder is expressed as

$$:\hat{A}\hat{B}:(v) = \frac{1}{2\pi i}\oint_w \frac{dz}{z} \frac{z^{\Delta_A} w^{\Delta_B} R(A_R(z)B_R(w))}{\log\frac{z}{w}}. \tag{2.3.44}$$

The normal ordering (2.3.44) can be written in terms of the normal ordered products on the complex plane, which can be calculated even for higher-spin cases with programming.

As we will see later, this reformulation allows us to express the vacuum eigenvalues of the local IoMs in terms of conformal weights. Especially, we found a new formula for the normal ordering $:\hat{W}_3\hat{W}_3:(v)$ with this technique. See Appendix B.3 for more detail.

Based on the normal ordering (2.3.44) and the technique in Appendix B.3, we can obtain the conserved currents on the cylinder which consist of the EM tensor $\hat{T}(v)$, higher-spin currents $\hat{W}_k(v)$ and their normal orderings. Substituting the mode expansions of the W-currents, we find the mode expansion of the conserved currents and finally the IoMs on the cylinder after contour integral. Details are explained in Appendix B.3.

Integrals of motion in WA_r CFT Let us first begin with the conserved currents $\hat{J}_k(v)$ $(k=2,3,\dots)$ in the CFT with WA_r-algebra symmetry, where $\hat{J}_{[1+(r+1)i]}(v)$ does not exist for $i \in \mathbf{Z}$. From the OPE analysis, the first four currents are found to be

$$
\begin{aligned}
&\hat{J}_2(v) = \hat{T}(v), \\
&\hat{J}_3(v) = \hat{W}_3(v), \\
&\hat{J}_4(v) = \hat{W}_4 + a_1 :\hat{T}\hat{T}:(v), \\
&\hat{J}_5(v) = \hat{W}_5 + b_1 :\hat{T}\hat{W}_3:(v),
\end{aligned}
\tag{2.3.45}
$$

where a_1 and b_1 can be inferred from low-rank results $(r=4,5,6,7,8)$

$$
\begin{aligned}
a_1 &= \frac{3(r-4)\left[(r+2)c+8r^2-18r+4\right]}{2(5c+22)(r-2)(r-1)r}, \\
b_1 &= \frac{8(r-5)\left[(r+3)c+15r^2-33r+6\right]}{(7c+114)(r-2)(r-1)r}.
\end{aligned}
\tag{2.3.46}
$$

The 6th IoM takes the general form:

$$
\hat{J}_6 = \hat{W}_6 + c_1 :\hat{T}\hat{W}_4: + c_2 :\hat{W}_3\hat{W}_3: + c_3 :\hat{T}(:\hat{T}\hat{T}:): + c_4 :\partial\hat{T}\partial\hat{T}:, \tag{2.3.47}
$$

where c_i's are some constants. The computation of the coefficients requires more computational effort. Here we present examples of rank $r=1,2,3$. For $r=4$, $\hat{J}_6$ does not exist.

- $r=1$

$$
\hat{J}_6 =:\hat{T}(:\hat{T}\hat{T}:): - \frac{c+2}{12} :\partial\hat{T}\partial\hat{T}: . \tag{2.3.48}
$$

- $r=2$

$$\hat{J}_6 =: \hat{W}_3\hat{W}_3 : +\frac{1}{9} : \hat{T}(:\hat{T}\hat{T}:) : +\frac{10-c}{288} : \partial\hat{T}\partial\hat{T} : . \tag{2.3.49}$$

- $r=3$

$$\hat{J}_6 =: \hat{T}\hat{W}_4 : + : \hat{W}_3\hat{W}_3 : +\frac{7c+114}{20(5c+22)} : \hat{T}(:\hat{T}\hat{T}:) : -\frac{3c^2+44c-164}{240(5c+22)} : \partial\hat{T}\partial\hat{T} : . \tag{2.3.50}$$

We note that the conserved currents on the cylinder can also be directly calculated by the technique of the screening operators. The first three orders have been given in [25]. The expressions (2.3.45) actually share the same form as the ones on a complex plane.

The conserved charges are defined by the integral of the conserved current on the circle $u = i\sigma$ $(0 \leq \sigma \leq 2\pi)$:

$$\mathbf{I}_s = \int_0^{2\pi} \frac{d\sigma}{2\pi} \hat{J}_s(i\sigma). \tag{2.3.51}$$

Substituting the mode expansions $\hat{T}(u) = \sum_n \hat{L}_n e^{-nu}$ and $\hat{W}_s(u) = \sum_n (\hat{W}_s)e^{-nu}$, one finds that $\mathbf{I}_s$ is expressed in terms of the modes[5]. In particular, when we evaluate the eigenvalue of $\mathbf{I}_s$ on the highest weight state $|\{\Delta_s\}\rangle$ corresponding to the primary field $\Phi(z)$ satisfying

$$(\hat{W}_s)_0|\{\Delta_s\}\rangle = \Delta_s|\{\Delta_s\}\rangle. \tag{2.3.52}$$

The eigenvalues depend on the data Δ_s. Alternatively, we can evaluate the eigenvalue of $\mathbf{I}_s$ on the state $|p_i\rangle$ in the momentum basis:

$$(\epsilon_i \cdot \hat{a})_0|p_i\rangle = p_i|p_i\rangle, \tag{2.3.53}$$

where $\hat{a}_0$ is the zero mode of $\phi(u)$ based on Eq.(2.3.41), and $p_i = \Lambda_i + (\epsilon_i, \alpha_0\rho)$ as we have introduced in Eq.(2.3.23). Next, we evaluate the eigenvalue of the IoMs on the state $|\{\Delta_s\}\rangle$:

$$\mathbf{I}_s|\{\Delta_s\}\rangle = I_s|\{\Delta_s\}\rangle. \tag{2.3.54}$$

[5]Usually, $\mathbf{I}_s$ is labelled by the subscript $s-1$ in the literature [18].

For I_s in (2.3.45), the eigenvalues are

$$I_2 = \Delta_2 - \frac{c}{24}, \tag{2.3.55}$$

$$I_3 = \Delta_3, \tag{2.3.56}$$

$$I_4 = \Delta_4 + a_1\left(\Delta_2^2 - \frac{c+2}{12}\Delta_2 + \frac{5c^2+22c}{2880}\right), \tag{2.3.57}$$

$$I_5 = \Delta_5 + b_1\left(\Delta_3\Delta_2 - \frac{6+c}{24}\Delta_3\right). \tag{2.3.58}$$

For degree 6, we will show the eigenvalue for $r = 1, 2, 3$, where $r = 1, 2$ cases have been known in previous literature [18, 42] :

- $r = 1$

$$I_6 = \Delta_2^3 - \frac{c+4}{8}\Delta_2^2 + \frac{(c+2)(3c+20)}{576}\Delta_2 - \frac{c(3c+14)(68+7c)}{290304}. \tag{2.3.59}$$

- $r = 2$

$$I_6 = \Delta_3^2 + \frac{1}{9}\Delta_2^3 - \frac{1}{72}(c+8)\Delta_2^2 + \frac{c+2}{1728}(c+15)\Delta_2 - \frac{c(c+23)}{870912}(7c+30). \tag{2.3.60}$$

- $r = 3$

$$I_6 = \Delta_3^2 - \frac{c+12}{12}\Delta_4 + 2\Delta_4\Delta_2 + \frac{114+7c}{20(22+5c)}\Delta_2^3 + y_1\Delta_2^2 + y_2\Delta_2 + y_3, \tag{2.3.61}$$

where

$$y_1 = -\frac{2264+554c+21c^2}{248(22+5c)}, \tag{2.3.62}$$

$$y_2 = -\frac{(c+2)(20808+3754c+105c^2)}{57600(22+5c)}, \tag{2.3.63}$$

$$y_3 = -\frac{c(116+3c)(54+7c)}{4147200}. \tag{2.3.64}$$

Integrals of motion in WD_r CFT First, we write down the conserved currents in the CFT with WA_r-algebra symmetry. From the OPE analysis in the previous subsection, the conserved currents are summarized as

$$\begin{aligned} \hat{J}_2(v) &= \hat{T}(v), \\ \hat{J}_4(v) &= \hat{W}_4 + a_1 : \hat{T}\hat{T} : (v), \end{aligned} \tag{2.3.65}$$

where a_1 can be inferred from low-rank results ($r = 4, 5, 6, 7, 8$)

$$a_1 = -\frac{3(r-4)\left(2cr + c + 16r^2 - 10r\right)}{2(5c+22)(r-1)r(2r-1)}. \tag{2.3.66}$$

For the spin 6 conserved current in WD_4 algebra, it is given by

$$\hat{J}_6 = \hat{W}_6 + x_1 : T(: TT :) + x_2 : \partial T \partial T :, \tag{2.3.67}$$

where

$$x_1 = \frac{(11c+656)(52c+23)}{2646(2c-1)(7c+68)}, \quad x_2 = -\frac{(11c+656)\left(11c^2 - 654c - 2432\right)}{42336(2c-1)(7c+68)}. \tag{2.3.68}$$

Next, we evaluate the eigenvalue of the IoMs on the state $|\{\Delta_s\}\rangle$:

$$\mathbf{I}_s |\{\Delta_s\}\rangle = I_s |\{\Delta_s\}\rangle. \tag{2.3.69}$$

For $\mathbf{I}_s$ in (2.3.45), the eigenvalues are

$$I_2 = \Delta_1 - \frac{c}{24}, \tag{2.3.70}$$

$$I_4 = \Delta_4 + a_1 \left(\Delta_2^2 - \frac{c+2}{12}\Delta_2 + \frac{5c^2 + 22c}{2880} \right). \tag{2.3.71}$$

Finally, in WD_4 algebra, one can obtain

$$\begin{aligned} I_6 = \Delta_6 &+ \frac{(656+11c)(23+52c)}{2646(2c-1)(68+7c)}\Delta_2^3 - \frac{(c+4)(656+11c)(23+52c)}{21168(2c-1)(68+7c)}\Delta_2^2 \\ &+ \frac{(656+11c)(-96+364c+231c^2+26c^3)}{254016(2c-1)(68+7c)}\Delta_2 - \frac{c(656+11c)(60+13c)}{128024064}. \end{aligned} \tag{2.3.72}$$

Chapter 3

The ODE/IM correspondence from functional relations

As we mentioned at the beginning of the last chapter, the ODE/IM correspondence bridges the spectral analysis of certain ordinary differential equations (ODEs) and the T- and Q-operators in quantum integrable systems. In the preceding sections (2.1 and 2.2), we introduced the functional relations in quantum integrable models. Now, we turn our focus to the spectral analysis of ODEs, employing the WKB method. The connection coefficients between the asymptotic solutions of these ODEs near singularities define the Q-, T-, and Y-functions in quantum integrable models [82]. Their consistency conditions and analytical properties determine the functional relations among these functions.

The exact WKB periods correspond to the logarithms of the Y-functions in quantum integrable models. This connection has been used to reformulate the spectral problem of the Schrödinger equation in terms of solutions to the TBA equations [41]. The ODE/IM correspondence extends to higher-order ODEs and quantum integrable models associated with affine Lie algebras, where the Q-functions and T-Q relations have been extensively studied [45, 50–57].

In [83,84], the WKB analysis of higher-order ODE versions of the Schrödinger equation and their relation to the TBA system were investigated. Interestingly, the Schrödinger equation can also be expressed as a rank-two first-order linear differential system, allowing the WKB analysis of the rank-two system to be reduced to that of a second-order ODE. This linear system can be derived from the light-cone limit of the linear problem associated with the $A_1^{(1)}$ affine Toda field equation (Sinh-Gordon equation) modified by a conformal transformation. The linear problem associated with affine Toda field equations plays a central role in the massive ODE/IM correspondence [49, 55, 85, 86], where Baxter's T-Q relations are related to the connection coefficients between solutions at the origin and infinity.

To study the Y-system and TBA system through the ODE/IM correspondence, it is crucial to further develop the WKB analysis of linear problems arising from modified affine Toda field equations or related higher-order ODEs.

In this chapter, we begin with the second-order Schrödinger equation, which corresponds to the six-vertex model and the KdV equations. Baxter's T-Q relation, the Bethe ansatz equation, the quantum Wronskian, and the fusion hierarchy are derived on the ODE side in Sections 3.1, 3.3, and 3.2. Then, in Section 3.4, we generalize the correspondence to Toda field theory, introduce the linear problem arising from modified affine Toda field equations, and derive the BAEs from the linear problem using the ψ-system.

3.1 TQ relations in the ordinary differential equations

In order to correspond the KdV equations and the XXZ model, we shall start from the second-order Schrodinger equation:

$$(-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2})\Phi(x) = E\Phi(x) \tag{3.1.1}$$

with a complex variable x. Usually, a complete analytical solution to such a second-order differential equation is difficult to obtain. However, thanks to the development of the WKB method, the approximate solution as $|x| \to \infty$ is accessible. Due to the angular momentum term $l(l+1)/x^2$, it is natural to separate the solutions into two limits: $|x| \to \infty$ and $|x| \to 0$. However, the variable x now is a complex value which means the solution $\Phi(x)$ is a one-parameter function along the path of x in the complex plane. More specifically, the ODE now possesses two singular points:

- the origin, $x = 0$, is a regular singularity; solutions have straightforward series expansions in their vicinity which converge in their whole neighborhood and can be analytically continued simply.

- the infinity, $|x| \to \infty$, is an irregular singularity; in its neighborhood solutions possess asymptotic expansions that hold only in selected Stokes sectors, making analytic continuation a very subtle issue.

The solutions should be summarized into the following two types.

- Radial problem: The solution should converge along the path of x from the origin to some given sector of the infinity.

- Lateral problem: The solution should be convergent along the path of x in different Stokes sectors of the infinity. The path from the origin back to the origin turns out to be trivial so we ignore it here.

Lateral problem

For the lateral problem, the angular term $l(l+1)/x^2$ can be neglected, and it is straightforward to obtain the corresponding WKB solution:

$$\psi_\pm \sim p(x)^{-\frac{1}{4}} \exp\Big[\pm\tfrac{1}{M+1}\, e^{i\theta(1+M)}\,\rho^{1+M}\Big], \quad x = \rho\, e^{i\theta}, \tag{3.1.2}$$

where $p(x) = -x^{2M} + l(l+1)/x^2 - E$. We follow the convention in [61], taking the branch cut on the negative real axis and choosing the initial quantization contour on the imaginary axis.

For most values of θ, one of these solutions grows exponentially, while the other decays exponentially. However, when $\mathrm{Re}\big[e^{i\theta(1+M)}\big] = 0$, both solutions oscillate and neither dominates. The relevant values of θ are

$$\theta = \pm\frac{(2n+1)\pi}{2M+2}, \quad n \in \mathbb{Z}. \tag{3.1.3}$$

The rays defined by these θ values are called Stokes lines. The Stokes lines and the path of x are shown in the diagram below.

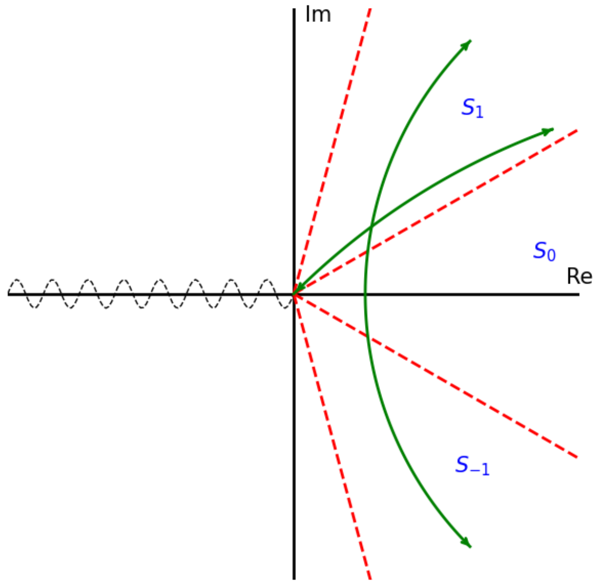


Figure 3.1.1: Stokes diagram for $M = 2.1$. The red lines are Stokes lines. The green line crossing sectors S_{-1} and S_1 indicates a lateral contour, and the green line emanating from the origin in sector S_1 is another lateral contour.

Next, we select the exponentially decaying solutions and simplify them under the large-$|x|$ approximation:

$$y(x, E, l) \;\sim\; \frac{1}{\sqrt{2i}}\, x^{-\frac{M}{2}} \;\exp\!\Big[-\tfrac{1}{M+1}\, x^{M+1}\Big]. \tag{3.1.4}$$

It is then possible to define the so-called Stokes sectors, lying between the Stokes lines, via

$$\mathcal{S}_k \;\equiv\; \Big\{ x \,\Big|\, \Big|\arg(x) \;-\; \frac{2\pi\, k}{2M+2}\Big| < \frac{\pi}{2M+2} \Big\}. \tag{3.1.5}$$

A whole family of solutions can be generated by Sibuya (or Symanzik) rotation. Define

$$y_k(x, E, l) \;\equiv\; \omega^{\frac{k}{2}}\, y\big(\omega^{-k} x,\; \omega^{2k} E,\; l\big), \quad \omega \;\equiv\; e^{\frac{2\pi i}{2M+2}}. \tag{3.1.6}$$

We note three remarkable properties here.

1. y is an analytical function of x and E except at $x = 0$ as we shall discuss later.

2. y_k is also the solution to Eq.(3.1.1) for all $k \in Z$ and the unique solution subdominant in $\mathcal{S}_k$ up to a constant.

3. Each pair y_k, y_{k+1} forms a basis of solutions to (3.1.1). So y_{-1} can be written as a linear combination of y_0 and y_1:

$$y_1(x, E, l) = C(E, l)y_0(x, E, l) + \tilde{C}y_{-1}(x, E, l). \tag{3.1.7}$$

Before going to the next step, we shall first introduce the two-dimensional Wronskians. The generic one will be given in the preceding parts. The Wronskian of two functions f and g is

$$W[f, g] \equiv fg' - f'g. \tag{3.1.8}$$

Since two solutions f and g to a second-order ODE without first-derivative term, their Wronskian is independent of x, and vanishes if and only if f and g are proportional. For further convenience, we denote $W_{k_1,k_2} \equiv y_{k_1}y'_{k_2} - y'_{k_1}y_{k_2}$. There are two useful properties which will be important in the future derivation:

$$W_{k_1+1,k_2+1}(E) = W_{k_1,k_2}(\omega^2 E), \qquad W_{0,1}(E) = 1. \tag{3.1.9}$$

They can be proved directly by substituting the solution into the Wronskian. Now go back to Eq.(3.1.7) and take Wronskians with first y_1 and then with y_0. It is easy to see that

$$C = \frac{W_{-1,1}}{W_{0,1}}, \qquad \tilde{C} = -\frac{W_{-1,0}}{W_{0,1}} = -1. \tag{3.1.10}$$

Then Eq.(3.1.7) becomes

$$C(E, l)y_0(x, E, l) = y_1(x, E, l) + y_{-1}(x, E, l) \tag{3.1.11}$$

Radial problem

To show the **TQ** relations on the ODE side, we should also consider the radial problem at $x = 0$, where the angular momentum term dominates. Unlike the lateral problem, the family of solutions is much simpler. At the origin, the solution to Eq.(3.1.1) behave as a linear combination of x^{l+1} and x^{-l}, a solution $\psi(x, E, l)$ can be defined by the requirement

$$\psi(x, E, l) \sim x^{l+1} + \mathcal{O}(x^{l+3}). \tag{3.1.12}$$

This solution is satisfied only if $\Re e(l) > -\frac{3}{2}$. A second solution can be obtained from the first by noting that Eq.(3.1.1) is invariant under the transformation $l \to -(l+1)$. Such a solution behaves as x^{-l} near the origin. Finally, we obtain two solutions

$$\begin{aligned} \psi_+(x,E) &\equiv \psi(x,E,l), \\ \psi_-(x,E) &\equiv \psi(x,E,-l-1). \end{aligned} \tag{3.1.13}$$

It can be proved that these two solutions are linear independent which means they also form into a basis. We now take the Wronskian of both sides of Eq.(3.1.7) with $\psi(x,E,l)$, then it becomes

$$C(E,l)W[y_0,\psi](E,l) = W[y_{-1},\psi](E,l) + W[y_1,\psi](E,l). \tag{3.1.14}$$

Set $W[y_0,\psi] = D(E,l)$ and take advantage of the property

$$W[y_k,\psi](E,l) = \omega^{(l+\frac{1}{2})k} W[y,\psi](\omega^{2k}E,l). \tag{3.1.15}$$

Finally, we obtain the relation

$$C(E,l)D(E,l) = \omega^{-(l+\frac{1}{2})} D(\omega^{-2}E,l) + \omega^{(l+\frac{1}{2})} D(\omega^{2}E,l). \tag{3.1.16}$$

This is nothing but the **TQ** relation on the ODE side.

The spectral interpretation

Finally, let us interpret the physical meaning of $C(E,l)$ and $D(E,l)$. Recall that both $C(E,l) = W_{-1,1}$ and $D(E,l) = W_{y_0,\psi}$ vanish if and only if the Wronskian vanishes, which means y_{-1} and y_1 are linearly dependent or y and ψ are linearly dependent. Therefore, $C(E,l)$ and $D(E,l)$ are the spectral determinants. The spectral determinant of an eigenvalue problem is a function that vanishes exactly at the eigenvalues of that problem: it generalizes to infinite dimensions the characteristic polynomial $\det(M - \lambda \mathrm{I})$ of a finite-dimensional matrix. In this sense, the vanishment of $C(E,l)(D(E,l))$ represents that the ODE has a solution decaying in the two sectors $\mathcal{S}_{-1}$ and $\mathcal{S}_1$ (at both $x \to \infty$ and $x = 0$) simultaneously, which is exactly the lateral(radial) eigenvalue problem.

3.2 The fusion hierarchy

Now that the **TQ** relation has been mapped onto a Stokes relation, it is natural to ask whether an analog of the fusion hierarchy can also be found in the spectral analysis, and it has been shown in [87].

Let us begin with the expansion of y_k in terms of any other basis, for example, $\{y_{k+r-1}, y_{k+r}\}$:

$$y_{k-1} = C_k^{(r)} y_{k+r-1} + \widetilde{C}_k^{(r)} y_{k+r} \tag{3.2.1}$$

Set the basis transformation from $\{y_{k+r-1}, y_{k+r}\}$ to $\{y_{k-1}, y_k\}$ to be

$$\begin{pmatrix} y_{k-1} \\ y_k \end{pmatrix} = \mathbf{C}_k^{(r)} \begin{pmatrix} y_{k+r-1} \\ y_{k+r} \end{pmatrix} \quad , \quad \mathbf{C}_k^{(r)} = \begin{pmatrix} C_k^{(r)} & \widetilde{C}_k^{(r)} \\ C_{k+1}^{(r-1)} & \widetilde{C}_{k+1}^{(r-1)} \end{pmatrix}, \tag{3.2.2}$$

where the matrix $\mathbf{C}_k^{(r)}$ should only depend on E and l. It is not difficult to see that

$$\begin{gathered} \mathbf{C}_k^{(r)}(E, l) = \mathbf{C}_{k-1}^{(r)}\left(\omega^2 E, l\right), \\ \mathbf{C}_k^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C}_k^{(1)} = \begin{pmatrix} C_k^{(1)} & -1 \\ 1 & 0 \end{pmatrix}, \end{gathered} \tag{3.2.3}$$

and the fundamental multiple rule

$$\mathbf{C}_k^{(r)} \mathbf{C}_{k+r}^{(n)} = \mathbf{C}_k^{(r+n)}. \tag{3.2.4}$$

Especially, the $r = 1$ case leads to

$$\begin{aligned} C_k^{(1)} C_{k+1}^{(n)} - C_{k+2}^{(n-1)} &= C_k^{(n+1)} \\ C_k^{(1)} \widetilde{C}_{k+1}^{(n)} - \widetilde{C}_{k+2}^{(n-1)} &= \widetilde{C}_k^{(n+1)}, \end{aligned} \tag{3.2.5}$$

and more generally $\widetilde{C}_k^{(n)} = -C_k^{(n-1)}$. Comparing two equations in Eq.(3.2.5), one can obtain

$$C(E) C^{(n)}\left(\omega^{n+1} E\right) = C^{(n-1)}\left(\omega^{n+2} E\right) + C^{(n+1)}\left(\omega^n E\right), \tag{3.2.6}$$

if we set

$$C^{(n)}(E) = C_0^{(n)}\left(\omega^{-n+1} E\right). \tag{3.2.7}$$

This matches the second fusion relation in Eq.(2.1.39). Since we have known $C^{(0)}(E) = 1 = t_0(E)$ and $C^{(1)}(E) = C(E) = T_{1/2}(vE)$, this establishes the basic equality

$$C^{(n)}(E) = T_{n/2}(vE). \tag{3.2.8}$$

It is also not difficult to find the first fusion relation in Eq.(2.1.39). From the Wronskians of Eq.(3.2.1), one can obtain

$$C_k^{(r)} = W_{k-1,k+r} \quad , \quad \widetilde{C}_k^{(r)} = -W_{k-1,k+r-1}, \tag{3.2.9}$$

which leads to $C_k^{(r)} = -C_{k+r+1}^{(-r-2)}$. Combining it with Eq.(3.2.4) at $r = -n$ results in

$$C^{(r-1)}\left(\omega^{-1}E\right)C^{(r-1)}(\omega E) - C^{(r)}(E)C^{(r-2)}(E) = 1, \tag{3.2.10}$$

namely the first fusion relation in Eq.(2.1.39).

Finally, let us discuss the truncation on the ODE side. When M is half-integer and $l(l+1)$ vanishes, all solutions of the ODE are single-valued functions of x, and the sectors $\mathcal{S}_{n+2M+2}$ coincides with $\mathcal{S}_n$. It leads to the proportional relation: $y_{n+2M+2}(x,E,l) = -y_n(x,E,l)$, where the minus comes from the asymptotic state. Then one can conclude

$$C^{(2M)}(E) = 1 \quad , \quad C^{(2M+1)}(E) = 0, \tag{3.2.11}$$

and give the truncation

$$C^{(r)}\left(\omega^{-1}E\right)C^{(r)}(\omega E) = 1 + \prod_{n=1}^{2M-1}\left(C^{(n)}(E)\right)^{G_{nr}} \tag{3.2.12}$$

3.3 The correspondence dictionary

Now we have introduced both integrable models and ordinary differential equations. From **TQ** relation (2.2.24) and CD relation (3.1.16), we can match the parameters on both sides. If we set the anisotropy η and the twist parameter ϕ as

$$\beta^2 = \frac{1}{M+1}, \qquad p = \frac{2l+1}{4M+4}, \tag{3.3.1}$$

or according to the relation (2.2.25), set the coupling constant β and the momentum parameter as

$$\eta = \frac{\pi}{2}\frac{M}{M+1} \quad , \quad \phi = \frac{(2l+1)\pi}{2M+2}. \tag{3.3.2}$$

Integrable continuum models	Integrable lattice models	Ordinary differential equations
Spectral parameter λ	Spectral parameter λ	Energy E
Coupling constant β	Anisotropy η	Degree of potential M
Momentum parameter p	Twist parameter ϕ	Angular momentum l
Transfer operator T	Transfer matrix T	Lateral spectral determinant C
Auxiliary operator Q	Auxiliary matrix Q	Radial spectral determinant D

then the correspondence can be summarized as

So far, we have just discussed the correspondence between the 2nd-order differential equation and the XXZ model or quantum KdV equation. The correspondence can be further generalized into other KdV hierarchies with the generalized Bethe ansatz equations and the ψ system.

3.4 Functional relations: the ψ system

The ψ system is a set of functional relations among unique defined solutions $\psi^{(a)}$ to a (pseudo-) ODE for $a = 1, \ldots, \mathrm{rank}(g)$. So far, we have just learned the correspondence between the Sine-Gordon model and the ODE (3.1.1) which is, however, trivial in the ψ system due to the lowest rank 1. So before we introduce the ψ system. Let us first generalize the ODE/IM correspondence to general affine Toda field theory. As an example, we take $A_r^{(1)}$ type as an simple example. For other types, someone could check [88].

3.4.1 The linear problem of $A_r^{(1)}$ affine Toda field equations

As we mentioned in the last of Section 2.3.2, one can add a massive term to the action of Toda field theory without breaking its integrability [31, 89, 90].

$$S[\phi] = \int \mathrm{d}^2 z \left[\frac{1}{2} \partial_z \phi \cdot \partial_{\bar{z}} \phi + \left(\frac{m^2}{\beta} \right) \left(\sum_{i=1}^{r} \exp\left(\beta \alpha_i \cdot \phi\right) + \exp\left(\beta \alpha_0 \cdot \phi\right) \right) \right]. \tag{3.4.1}$$

In the massive term, α_0 is the affine root. The equation of motion for the action (3.4.1) is given by

$$\partial_{\bar{z}} \partial_z \phi(z, \bar{z}) - \left(\frac{m^2}{\beta} \right) \left[\sum_{i=1}^{r} \alpha_i \exp\left(\beta \alpha_i \cdot \phi\right) + \alpha_0 \exp\left(\beta \alpha_0 \cdot \phi\right) \right] = 0. \tag{3.4.2}$$

Under the conformal transformation [49, 55, 85, 91]:

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z}), \quad \phi(z,\bar{z}) \to \hat{\phi}(z,\bar{z}) = \phi(w,\bar{w}) + \frac{\rho^\vee}{\beta}\log(\partial_z w \partial_{\bar{z}}\bar{w}), \tag{3.4.3}$$

the equation (3.4.2) written in terms of $(w,\bar{w})$ coordinates becomes[1]

$$\partial_{\bar{z}}\partial_z\phi(z,\bar{z}) - \left(\frac{m^2}{\beta}\right)\left[\sum_{i=1}^{r}\alpha_i \exp\left(\beta\alpha_i\cdot\phi\right) + p(z)\bar{p}(\bar{z})\alpha_0\exp\left(\beta\alpha_0\cdot\phi\right)\right] = 0 \tag{3.4.4}$$

with

$$p(z) = (\partial_z w)^h, \quad \bar{p}(\bar{z}) = (\partial_{\bar{z}}\bar{w})^h. \tag{3.4.5}$$

The parameter $\rho(\rho^\vee)$ in Eq.(3.4.3) is the Weyl(co-Weyl) vector and the Coxeter number h is given in the table C.4.1. Especially, $h = r+1$ for A_r Lie algebra.

As a classical integrable model, the modified affine Toda field equation (3.4.4) can be rewritten as the integrability condition for a pair of linear systems

$$\mathcal{L}\Psi = \bar{\mathcal{L}}\Psi = 0, \tag{3.4.6}$$

where $\mathcal{L}$ and $\bar{\mathcal{L}}$ denote the Lax operators

$$\begin{aligned}
\mathcal{L} &= \partial_z + \sum_{i=1}^{r}\beta\partial_z\phi_i(z)H_i + \lambda\left(\sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}\right), \\
\bar{\mathcal{L}} &= \partial_{\bar{z}} + \lambda^{-1}e^{-\beta\sum_{i=1}^{r}\phi_i H_i}\left(\sum_{i=1}^{r}E_{-\alpha_i} + \bar{p}(\bar{z})E_{-\alpha_0}\right)e^{\beta\sum_{i=1}^{r}\phi_i H_i}.
\end{aligned} \tag{3.4.7}$$

Here $E_{\pm\alpha_0}$, $E_{\pm\alpha_i}$ and $H_i = \alpha_i^\vee\cdot H$ $(i = 1,\ldots,r)$ are the Chevalley generators of affine Lie algebra $\hat{\mathfrak{g}}$, and λ is the spectral parameter. The fundamental representation for these generators can be found in Appendix C.5. The Lax operator $\mathcal{L}$ is nothing but the one that appears in the KdV hierarchies on a complex plane. The connection between these two integrable theories is based on the Drinfeld-Sokolov reduction [21]. More details can be found in Appendix A.

Let us focus on the holomorphic part of the linear problem

$$\mathcal{L}\Psi(z,\bar{z}) = 0. \tag{3.4.8}$$

[1] Here we have exchanged z and w

Without loss of generality, we choose the fundamental representation with the set of weight vectors $\epsilon_1, \cdots, \epsilon_{r+1}$ as we did in Eq.(2.3.19) in Toda field theory. The highest weight $\epsilon_1 = \omega_1$ satisfy $\epsilon_i - \epsilon_{i+1} = \alpha_i$ where the ω_1 is the fundamental weight and the α_i is the simple root of $A_r^{(1)}$. If we write (4.2.1) in explicit matrix form and act on the basis $\{\psi_i\}$, it becomes [55, 59]

$$\begin{pmatrix} \partial + \beta\epsilon_1 \cdot \partial\phi & m\lambda & 0 & \cdots & 0 \\ 0 & \partial + \beta\epsilon_2 \cdot \partial\phi & m\lambda & & \vdots \\ & & \ddots & & \\ \vdots & & & \partial + \beta\epsilon_{r+1} \cdot \partial\phi & m\lambda \\ m\lambda p(z) & & & 0 & \partial + \beta\epsilon_{r+1} \cdot \partial\phi \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_r \\ \psi_{r+1} \end{pmatrix} = 0. \tag{3.4.9}$$

The representations we used are summarized in C.5. Extract the first term ψ_1 and cancel other components, we can obtain a single differential equation on ψ_1

$$(\partial + \beta\epsilon_{r+1} \cdot \partial\phi) \cdots (\partial + \beta\epsilon_1 \cdot \partial\phi)\psi_1 = (-m\lambda)^h p(z)\psi_1. \tag{3.4.10}$$

The LHS is namely the Miura transformation of the scalar Lax operator in [21] and the Eq.(A.2.26) in Appendix A.

Now we are going to fix the singularity information of the ODEs. The steps are as follows. If one further takes the light-cone limit together with the conformal limit, Eq. (3.4.10) will become the one appearing in the ODE/IM correspondence [54, 55]. To see this, set the functions $p(z)$ and $\bar{p}(\bar{z})$ to be

$$p(z) = z^{hM} - s^{hM}, \quad \bar{p}(\bar{z}) = \bar{z}^{hM} - \bar{s}^{hM}, \tag{3.4.11}$$

where M is a positive real number with $M > 1/(h-1)$ and s is an arbitrary parameter. The function $p(z)$ determines the behavior of $\phi(z)$ at infinity [49, 55], while at the origin, we impose the boundary condition for $\phi(z)$ as [49]

$$\phi(z) = \frac{l}{\beta}\log(z) + \mathcal{O}(1), \tag{3.4.12}$$

with l an r-dimensional vector. We first take the light-cone limit $\bar{z} \to 0$ and the conformal limit $\lambda \to \infty$

$$z = \lambda^{-\frac{1}{1+M}} x, \quad s^{hM} = \lambda^{-\frac{hM}{1+M}} E \tag{3.4.13}$$

with x and E finite. In these limits, the linear problem reduces to a single holomorphic linear differential equation. Further rescaling x and E as

$$x \to \epsilon^{\frac{1}{1+M}} x, \quad E \to \epsilon^{\frac{hM}{1+M}}, \tag{3.4.14}$$

and keep E finite. Substitute these two terms into (3.4.10) and set z to 0, we can obtain

$$\epsilon^{r+1}(\partial_x + \beta\epsilon_{r+1} \cdot \frac{l}{x}) \cdots (\partial_x + \beta\epsilon_1 \cdot \frac{l}{x})\psi = (-1)^h (x^{hM} - E)\psi. \tag{3.4.15}$$

The constant ϵ acts like a Planck constant, playing a key role in Chapter 4, but we'll leave it out here for simplicity. Finally, we obtain the ordinary differential equation. For lateral problem, the WKB analysis shows the approximate solution y to be

$$y(x, E, l) \sim C x^{-\frac{rM}{2}} \exp\left(-\frac{x^{M+1}}{M+1}\right). \tag{3.4.16}$$

The unique subdominant solutions in the Stokes sectors

$$\mathcal{S}_k \equiv |\arg(x) - \frac{2\pi k}{hM + h}| < \frac{\pi}{hM + h}. \tag{3.4.17}$$

can be given by Symanzik rotation (3.1.6)

$$y_k(x, E, l) = w^{\frac{rk}{2}} \psi(\omega^{-k} x, \omega^{hMk} E, l), \tag{3.4.18}$$

where $\omega \equiv e^{\frac{2\pi i}{2M+2}}$, and the generic Coxter number h replaced 2 in $A_1^{(1)}$ algebra. On the other hand, a set of solutions to the radial problem is (up to the leading order)

$$\psi_i \sim x^{\nu_i}, \qquad i = 1, \dots r + 1 \tag{3.4.19}$$

where $\nu_i = i - 1 - \beta h_{i+1} \cdot l$. This can be easily verified by direct substitution. Similar to the lateral problem, we can also find the rotation of $\{\chi_i\}$'s

$$\psi_{i[k]}(x, E, l) = w^{-k\nu_i} \psi_i(x, E, l). \tag{3.4.20}$$

Since $\{\chi_i\}$'s are a basis of solution, it is possible to write

$$y(x, E, l) = Q_1^{(1)}(E, l)\psi_1(x, E, l) + Q_2^{(1)}(E, l)\psi_2(x, E, l) + \dots, \tag{3.4.21}$$

where the upper index (1) in Q denotes the eigenvalues of the ground state corresponding to the q function that appears in the BAE in Eq.(2.1.9) . So far, we have just shown the ODEs for $A_r^{(1)}$ linear problem. It can actually generalized into other Lie algebraic structures. For convenience, let us define the differential operator

$$D(a) \equiv \partial + \beta a \cdot \partial \phi, \tag{3.4.22}$$

then the corresponding (pseudo) ODEs with other affine Lie algebras can be given by

(3.4.23)

$A_r^{(1)}$	$D(\boldsymbol{\epsilon})\psi = (-m\lambda)^h\, p(z)\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_{r+1})$
$D_r^{(1)}$	$D\left(\boldsymbol{\epsilon}^\dagger\right)\partial^{-1} D(\boldsymbol{\epsilon})\psi = 2^{r-1}(m\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_r)$
$B_r^{(1)}$	$D\left(\boldsymbol{\epsilon}^\dagger\right)\partial D(\boldsymbol{\epsilon})\psi = 2^{r}(m\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_r)$
$A_{2r-1}^{(2)}$	$D\left(\boldsymbol{\epsilon}^\dagger\right) D(\boldsymbol{\epsilon})\psi = -2^{r-1}(m\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_r)$
$C_r^{(1)}$	$D\left(\boldsymbol{\epsilon}^\dagger\right) D(\boldsymbol{\epsilon})\psi = (m\lambda)^h\, p(z)\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_r)$
$D_{r+1}^{(2)}$	$D\left(\boldsymbol{\epsilon}^\dagger\right)\partial D(\boldsymbol{\epsilon})\psi = 2^{r+1}(m\lambda)^{2h}\, p(z)\partial^{-1}p(z)\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_r)$
$A_{2r}^{(2)}$	$D\left(\boldsymbol{\epsilon}^\dagger\right)\partial D(\boldsymbol{\epsilon})\psi = -2^{r}\sqrt{2}\,(m\lambda)^h\, p(z)\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_r)$
$G_2^{(1)}$	$D\left(\boldsymbol{\epsilon}^\dagger\right)\partial D(\boldsymbol{\epsilon})\psi = 8\,(m\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$	$\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$
$D_4^{(3)}$	$\big[D\left(\boldsymbol{\epsilon}^\dagger\right)\partial D(\boldsymbol{\epsilon}) + (\zeta+1)2\sqrt{3}\,(m\lambda)^4 D\left(\boldsymbol{\epsilon}^\dagger\right)p(z)$ $-\zeta 4\sqrt{3}\,(m\lambda)^4 D(-\epsilon_1)(\partial p(z) + p(z)\partial)D(\epsilon_1)$ $-(\zeta+1)2\sqrt{3}\,(m\lambda)^4 p(z)D(\boldsymbol{\epsilon})$ $+(\zeta-1)^2 12\,(m\lambda)^8 p(z)\partial^{-1}p(z)\big]\,\psi_1 = 0$	$\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$

where the Planck constant ϵ is still omitted and ζ is a twist parameter with $\zeta^3 = 1$. The derivation can be found in [55]. Next, we are going to obtain the generalized BAEs from these (pseudo)-ODEs. Let us still take the $A_r^{(1)}$ case for an example.

3.4.2 The ψ system and the generalized BAEs

First recall that the fundamental representation possess the highest wight ω_1. Now we try to generalize it. Define the fundamental representations $V^{(a)}(a = 1, \dots, r)$ with the highest weight ω and denote $\psi^{(a)}$ as the subdominant solution and $\chi_i^{(a)}$ as the basis around the origin. We also generalize $Q_1^{(1)}$ in (3.4.21) into $Q_1^{(a)}$. In fact, the ath fundamental representation $V^{(a)}$ is isomorphic to the a-anti-symmetric tensor product of $V^{(1)}$: $V^{(a)} \simeq \wedge^a V^{(1)}$. With respect to the subdominant solution, the Symanzik rotation should also be inserted to keep the same asymptotic behavior.

$$\psi^{(a)} = \wedge_{b=1}^{a}\ \psi^{(1)}_{[b-\frac{a+1}{2}]} \qquad a = 1, \dots, r. \tag{3.4.24}$$

Substitute Eq.(3.4.21) into this relation, one can obtain

$$Q_1^{(a)}(E,l) = \sum_{j_0,\dots,j_{a-1}=1}^{a} \epsilon_{j_0,\dots,j_{a-1}} \prod_{k=0}^{a-1} \omega^{-k\nu_{j_k}} Q_{j_k[\frac{a-1}{2}-k]}(E,l), \tag{3.4.25}$$

where $\epsilon_{j_0,\dots,j_{a-1}}$ is the totally anti-symmetric symbol normalized as $\epsilon_{01\cdots a-1}=1$. This is a general Wronskian for $Q^{(a)}$ operators [45].

There exists another functional relation used to derive the BAEs from the representation structure. For the representation $V^{(a)}$ of a simple-laced Lie algebra g, there is an inclusion map [52]

$$\iota: \quad V^{(a)} \wedge V^{(a)} \to \otimes_{b=1}^{r} (V^{(b)})^{2\delta_{ab}-C_{ab}}. \tag{3.4.26}$$

After inserting the Symanzik rotation, the ψ system for $A_r^{(1)}$ algebra takes the form

$$\iota(\psi_{-\frac{1}{2}}^{(a)} \wedge \psi_{\frac{1}{2}}^{(a)}) = \otimes_{b=1}^{r} (\psi^{(b)})^{2\delta_{ab}-C_{ab}}, \tag{3.4.27}$$

where $C_{ab} = 2\delta_{ab} - \delta_{a\pm1,b}$ is the Cartan matrix of A_r algebra. Substitute this identity into Eq.(3.4.21), we can obtain Q operator relations:

$$\prod_{b=1}^{r} [Q_1^{(b)}]^{2\delta_{ab}-C_{ab}} = \omega^{-\frac{1}{2}(\nu_1^{(a)}-\nu_2^{(a)})} Q_{1[-\frac{1}{2}]}^{(a)} Q_{2[\frac{1}{2}]}^{(a)} - \omega^{\frac{1}{2}(\nu_1^{(a)}-\nu_2^{(a)})} Q_{1[\frac{1}{2}]}^{(a)} Q_{2[-\frac{1}{2}]}^{(a)}. \tag{3.4.28}$$

Evaluating the $\frac{1}{2}$ and $-\frac{1}{2}$ Symanzik rotation of (3.4.28) at the zeros $E_i^{(a)}$ $(i=0,1,\dots)$ of $Q_1^{(a)}$. We obtain

$$\begin{aligned} Q_{1[\frac{1}{2}]}^{(a+1)} Q_{1[\frac{1}{2}]}^{(a-1)} &= -\omega^{\frac{1}{2}(\nu_1^{(a)}-\nu_2^{(a)})} Q_{1[1]}^{(a)} Q_2^{(a)}, \\ Q_{1[-\frac{1}{2}]}^{(a+1)} Q_{1[-\frac{1}{2}]}^{(a-1)} &= \omega^{-\frac{1}{2}(\nu_1^{(a)}-\nu_2^{(a)})} Q_{1[-1]}^{(a)} Q_2^{(a)}. \end{aligned} \tag{3.4.29}$$

Then divide these two equations, we find the A_r-type Bethe Ansatz equations

$$\omega^{(\nu_1^{(a)}-\nu_2^{(a)})} \frac{Q_{1[-\frac{1}{2}]}^{(a+1)} Q_{1[-\frac{1}{2}]}^{(a-1)} Q_{1[1]}^{(a)}}{Q_{1[\frac{1}{2}]}^{(a+1)} Q_{1[\frac{1}{2}]}^{(a-1)} Q_{1[-1]}^{(a)}} = -1. \tag{3.4.30}$$

To be more explicit, we make some simplification. Recall $\Omega = \omega^{hM}$ and the value of ν_i introduced in (3.4.19)

$$\nu_1 - \nu_2 = \beta h_2^{(a)} \cdot l - \beta h_1^{(a)} \cdot l - 1 = -\alpha_a \cdot l - 1 = -2[\omega_a \cdot (l+\rho^\vee)] \frac{C_{ab}}{2}. \tag{3.4.31}$$

So we define the components of twist parameter as

$$\phi_a \equiv -\frac{2}{hM}(\omega_a \cdot (l+\rho^\vee)), \tag{3.4.32}$$

then Eq.(3.4.30) becomes

$$\prod_{b=1}^{\mathrm{rank}(g)} \Omega^{\frac{C_{ab}}{2}\phi_b} \frac{Q^{(b)}_{C_{ab}}(E^{(a)}_i,\phi)}{Q^{(b)}_{-C_{ab}}(E^{(a)}_i,\phi)} = -1. \tag{3.4.33}$$

This is the generalized A_r-type BAEs. After setting $\mathfrak{g} = A_1$, we can obtain the familiar BAE in Eq.(2.1.36). So far, we have derived the ODE/IM correspondence for $A_r^{(1)}$ algebra, but the BAEs (3.4.33) is also valid for another simple laced Lie algebra $D_r^{(1)}$. For non-simple laced case, the BAEs are summarized below

- $\mathbf{C}_r^{(1)}$-type linear problem

$$\begin{aligned} &\prod_{b=1}^{r} \Omega^{C_{ab}\phi_b/2} \frac{Q^{(b)}_{[C_{ab}/2]}(E^{(a)}_i,\phi)}{Q^{(b)}_{[-C_{ab}/2]}(E^{(a)}_i,\phi)} = -1, \quad \text{for} \quad a = 1,\dots,r-1, \\ &\Omega^{\phi_r-\phi_{r-1}} \left.\frac{\left[Q^{(r-1)}_{[-1/2]}\right]^2 Q^{(r)}_{[1]}}{\left[Q^{(r-1)}_{[1/2]}\right]^2 Q^{(r)}_{[-1]}}\right|_{E^{(r)}_i} = -1. \end{aligned} \tag{3.4.34}$$

- $\mathbf{B}_r^{(1)}$-type linear problem

$$\begin{aligned} &\prod_{b=1}^{r} \Omega^{C_{ab}\phi_b/2} \frac{Q^{(b)}_{[C_{ab}/2]}(E^{(a)}_i,\phi)}{Q^{(b)}_{[-C_{ab}/2]}(E^{(a)}_i,\phi)} = -1, \quad \text{for} \quad a = 1,\dots,r-2,r, \\ &\Omega^{-\frac{1}{2}\phi_{r-2}+\phi_{r-1}-\phi_r} \left.\frac{Q^{[r-2)}_{[-1/2]} Q^{(r-1)}_{[1]} \left[Q^{(r)}_{[-1/2]}\right]^2}{Q^{(r-2)}_{[1/2]} Q^{(r-1)}_{[-1]} \left[Q^{(r)}_{[1/2]}\right]^2}\right|_{E^{(r-1)}_i} = -1. \end{aligned} \tag{3.4.35}$$

- $\mathbf{F}_4^{(1)}$-type linear problem

$$\begin{aligned} &\prod_{b=1}^{r} \Omega^{C_{ab}\phi_b/2} \frac{Q^{(b)}_{[C_{ab}/2]}(E^{(a)}_i,\phi)}{Q^{(b)}_{[-C_{ab}/2]}(E^{(a)}_i,\phi)} = -1, \quad \text{for} \quad a = 1,2,4, \\ &\Omega^{-\phi_2+\phi_3-\frac{1}{2}\phi_4} \left.\frac{\left[Q^{(2)}_{[-1/2]}\right]^2 Q^{(3)}_{[1]} Q^{(4)}_{[-1/2]}}{\left[Q^{(2)}_{[1/2]}\right]^2 Q^{(3)}_{[-1]} Q^{(4)}_{[1/2]}}\right|_{E^{(3)}_i} = -1. \end{aligned} \tag{3.4.36}$$

- $\mathbf{G}_2^{(1)}$-type linear problem

$$\Omega^{\phi_1-\frac{1}{2}\phi_2}\frac{Q^{(1)}_{[1]}Q^{(2)}_{[-1/2]}}{Q^{(1)}_{[-1]}Q^{(2)}_{[1/2]}}\Bigg|_{E_i^{(1)}}=-1,\quad \Omega^{-\frac{3}{2}\phi_1+\phi_2}\frac{\left[Q^{(1)}_{[-1/2]}\right]^3Q^{(2)}_{[1]}}{\left[Q^{(1)}_{[1/2]}\right]^3Q^{(2)}_{[-1]}}\Bigg|_{E_i^{(2)}}=-1. \tag{3.4.37}$$

Further details on the derivation can be found in [55]. By comparing the results with the BAEs derived from the integrable model side, it becomes apparent that the linear problem associated with the dual algebra $g^\vee$ leads to the $\hat{g}$-type BAEs, as outlined in [61]. This demonstrates that the ODE/IM correspondence inherently respects Langlands duality, which may not be immediately apparent when dealing with simple-laced cases, such as the A_r algebras.

3.4.3 The ODE/IM correspondence with numerical techniques

Before concluding this chapter, let us briefly discuss a numerical approach for efficiently verifying the ODE/IM correspondence. Since the functional **TQ** relations in the ground state and the CD spectral analysis share the same structure, it is natural that the Langlands dual Y-systems and Bethe ansatz equations can also be formulated on the ODE side. In the previous chapter, we demonstrated that the Y-system and the BAEs give rise to two types of integral equations: the TBA equations and the DDV equations. While these are often challenging to solve analytically, they are highly effective for numerical computations. Integral equations are generally easier to handle than differential ones, and spectral determinants can encode all eigenvalues simultaneously.

Moreover, the D-functions on the ODE side can be expressed as the inner product of the solution to the dual linear problem and the subdominant solution of the original linear problem. Using Cheng's algorithm [92], the zeros of these D-functions can be determined with remarkable efficiency. By comparing these zeros with the Q-functions obtained from the DDV equations, the ODE/IM correspondence can be numerically verified for simply-laced affine Lie algebras, including exceptional types [56].

Chapter 4

The Quantum/Classical Correspondence

So far, we have established the ODE/IM correspondence between the functional relations on the integrable model (IM) side and the spectral analysis on the ordinary differential equation (ODE) side. Let us briefly summarize the main results presented in the previous section. The narrative begins with the six-vertex model and the Schrödinger-type ordinary differential equation. This correspondence is demonstrated through various approaches, including functional relations ($\mathbf{T}$-systems, $\mathbf{Q}$-systems, and $\mathbf{Y}$-systems), Bethe ansatz equations, and integral equations (such as TBA and DDV equations). Importantly, regardless of the path taken to establish the correspondence, it consistently leads to an infinite set of IoMs.

It is natural to ask whether an alternative approach to the ODE/IM correspondence might exist—one that involves local IoMs and WKB integrals [25, 49, 58]. On another front, the correspondence has been extended to continuum models, such as KdV hierarchies, Toda field theories, and certain higher-order differential equations. These equations, intriguingly, can also be derived from the linear problem of modified affine Toda field equations. This reveals a profound connection between these (pseudo) ODEs and classical integrable models, linking the WKB integrals to classical conserved charges [59]. Ultimately, the WKB integrals bridge classical and quantum local IoMs, encapsulating the essence of the chapter's title: quantum/classical correspondence.

The structure of this chapter is as follows. We first establish the connections between (pseudo) ODEs and affine Toda field equations through the linear problem and the

Drinfeld-Sokolov reduction [21, 59]. As we will see, the Lax operators in the KdV hierarchies exhibit a structure analogous to those in affine Toda field equations, introduced earlier in Section 3.4.1.

In Section 4.1, we review the classical KdV equations and the Drinfeld-Sokolov reduction. Section 4.2 focuses on a detailed WKB analysis of the linear problems and their corresponding (pseudo) ODEs. In this section, we also introduce the WKB integrals for linear problems, evaluated along the Pochhammer contour for simple-laced affine Lie algebras after taking the conformal limit.

Section 4.3 presents the diagonalization method for linear problems, applying it to $A_1^{(1)}$, $A_2^{(1)}$, and then generalizing it to $A_r^{(1)}$. This method is further extended in Section 4.3.2 to affine Lie algebras of types $A_{2r-1}^{(2)}$, $B_r^{(1)}$, $D_{r+1}^{(2)}$, and $D_r^{(1)}$. In Section 4.4, we identify the diagonal elements as conserved densities of the KdV hierarchies.

Finally, we demonstrate the relationship between WKB integrals and the quantum IoMs in the ground state. In Section 4.5, we perform a WKB analysis of higher-order ODEs of types $A_r^{(1)}$ and $D_r^{(1)}$ [54, 55, 59], and compute their WKB integrals along the Pochhammer contour up to the 8th order for general rank r. In Sections 4.5.1 and 4.5.2, we compare the WKB integrals with the eigenvalues of the IoMs, thereby confirming the ODE/IM correspondence for linear problems associated with simple-laced affine Lie algebras of types $A_r^{(1)}$ and $D_r^{(1)}$.

4.1 The classical KdV equations and Lax formlism

The KdV equation is one of the most famous continuous integrable models describing the evolution of long, one-dimensional nonlinear waves in shallow water. On a cylinder, it is defined by the differential equation

$$u_t + 6uu_v + u_{vvv} = 0, \tag{4.1.1}$$

where the notation $u_v = \partial_v u$, $u(v,t)$ is a periodic function with $u(v+2\pi) = u(v)$ and satisfies the Poisson bracket algebra

$$\{u(v), u(w)\} = 2(u(v) + u(w))\delta'(v-w) + \delta'''(v-w), \tag{4.1.2}$$

It is the second Hamiltonian structure of the KdV equation provided we take one of the infinite sets of classical **integrals of motion**

$$
\begin{aligned}
I_2^{(cl)} &= \int_0^{2\pi} \frac{dv}{2\pi} u(v), \\
I_4^{(cl)} &= \int_0^{2\pi} \frac{dv}{2\pi} u^2(v), \\
I_6^{(cl)} &= \int_0^{2\pi} \frac{dv}{2\pi} \left[u^3(v) - \frac{(u'(v))^2}{2} \right], \\
&\cdots
\end{aligned}
\tag{4.1.3}
$$

They are namely the classical version of Eq.(2.2.5). As is introduced in Appendix A, one can rewrite the equation (4.1.1) into the Lax equation

$$
\frac{dL}{dt} = [M, L]. \tag{4.1.4}
$$

with the scalar Lax pair

$$
L = \partial_v^2 + u, \qquad M = \partial_v^3 + \frac{3}{2} + \frac{3}{4} u_v. \tag{4.1.5}
$$

Moreover, the KdV equations can be equivalently given by the Drinfeld Sokolov reduction with the Lie algebraic Lax operator

$$
\mathcal{L} = \partial_v - \partial_v \phi(v) H - \lambda(E + F), \tag{4.1.6}
$$

where E, F and H are the generators of A_1 Lie algebra with

$$
[H, E] = 2E, \qquad [H, F] = -2E, \qquad [E, F] = 2H, \tag{4.1.7}
$$

and $\phi(v)$ is a free scalar field on a cylinder with

$$
u(v) = -(\partial\phi)^2 - \partial^2\phi. \tag{4.1.8}
$$

Usually, $\phi(u)$ satisfies the quasi-periodic condition

$$
\phi(u + 2\pi) = \phi(u) + 2\pi i p \tag{4.1.9}
$$

to keep the $u(v)$ periodic. It is also the canonical variable with the Poisson brackets

$$
\begin{aligned}
\{\phi(v), \phi(w)\} &= \frac{1}{2}\epsilon(v - w); \\
\epsilon(v) = n \quad \text{for} \quad 2\pi n < v &< 2\pi(n+1); \quad n \in \mathbb{Z},
\end{aligned}
\tag{4.1.10}
$$

It is not difficult to see that the Lax operator here is the same as Eq.(3.4.7) introduced in the affine Toda field equation after setting $\hat{\mathfrak{g}} = A_1^{(1)}$ and taking a coordinate transformation from a cylinder to a complex plane or a Riemann sphere. Hence, it is also possible to consider the linear problem $\mathcal{L}\Psi = 0$. In the fundamental representation, we can find

$$\begin{pmatrix} \partial_v - \partial_v\phi(v) & -\lambda \\ -\lambda & \partial_v + \partial_v\phi(v) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{4.1.11}$$

After eliminating ψ_2, there is a second-ordinary differential equation

$$(\partial_v + \partial_v\phi(v))(\partial_v - \partial_v\phi(v))\psi_1 = \lambda^2\psi_1. \tag{4.1.12}$$

Substitute the Miura transformation (4.1.8), one can reproduce the linear problem of the scalar Lax operator

$$L\psi_1 = \left(\partial^2 + u(v)\right)\psi_1 = \lambda^2\psi_1. \tag{4.1.13}$$

It is also not difficult to solve the equation directly. The result is given by [18].

$$\Psi(v) = \pi_j\left[e^{\phi(v)H}\mathcal{P}\exp\left(\lambda\int_0^v dv'(e^{-2\phi(v')}E + e^{2\phi(v')}F)\right)\Psi_0\right], \tag{4.1.14}$$

where π_j means the matrix representation (dim$= 2j+1$) with spin j for the A_1 algebra, $\mathcal{P}$ means the "path ordered" exponential, and Ψ_0 is an arbitrary vector. The associated monodromy matrices can be calculated directly from Eq.(A.1.11).

$$M_j(\lambda) = \pi_j\left[e^{2\pi ipH}\mathcal{P}\exp\left(\lambda\int_0^{2\pi} dv'(e^{-2\phi(v')}E + e^{2\phi(v')}F)\right)\right], \tag{4.1.15}$$

where p is a constant. It is the classical version of Eq.(2.2.10). The classical transfer matrix is given by

$$T_j = \operatorname{Tr} M_j \tag{4.1.16}$$

with the involution condition

$$\{T_j(\lambda), T_{j'}(\mu)\} = 0. \tag{4.1.17}$$

The Poisson brackets follow the definition (4.1.10).

The higher-order KdV hierarchies

So far, we have only discussed the simplest case: the Lax operator with $A_1^{(1)}$ algebra. In the Drinfeld-Sokolov reduction, it is also possible to consider other affine Lie algebras $\hat{\mathfrak{g}}$. Here we first focus on the diagonal reduction on a complex plane, leading to the modified same linear problem in Eq.(3.4.7).

$$\begin{aligned}
\mathcal{L} &= \epsilon\partial_z + \epsilon\sum_{i=1}^{r}\beta\partial_z\phi_i(z)H_i + \left(\sum_{i=1}^{r}E_{\alpha_i} + E_{\alpha_0}\right), \\
\mathcal{L}_m &= \epsilon\partial_z + \epsilon\sum_{i=1}^{r}\beta\partial_z\phi_i(z)H_i + \left(\sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}\right),
\end{aligned} \tag{4.1.18}$$

where we have added the Planck constant ϵ which can be realized by similar rescaling as we did in Eq.(3.4.14). As mentioned in Section 3.4.1, we also explore the conformal limit of the linear problem and its corresponding (pseudo) ODEs, where the coordinate z is replaced with x. To clarify, $\mathcal{L}(\mathcal{L}_m)$ in Eq.(4.1.18) refers to the massive Lax operator. They are equivalent to some higher-order nonlinear equations with corresponding Lax operators. More details are given in Appendix A.

4.2 WKB analysis to the linear problems

In this section, we will continue the WKB analysis given in Section 3.4.1 and give more details. Recall in Section 3.4.1, we have seen that the linear problem

$$\mathcal{L}_m\Psi(z,\bar{z}) = 0. \tag{4.2.1}$$

leads to a set of (pseudo)-ODEs (3.4.23). Especially, for the simple-laced affine Lie algebras, the (pseudo)-ODEs are[1]

$$\left[\epsilon^{r+1}\left(\partial_z - \epsilon_1\cdot\partial_z\phi\right)\ldots\left(\partial_z - \epsilon_{r+1}\cdot\partial_z\phi\right) + (-1)^r p(z)\right]\psi(z,\epsilon) = 0 \tag{4.2.2}$$

for $A_r^{(1)}$ type, and

$$\begin{aligned}
&\epsilon^{2r-2}\left(\partial_z - \epsilon_1\cdot\partial_z\phi\right)\cdots\left(\partial_z - \epsilon_r\cdot\partial_z\phi\right)\partial_z^{-1}\left(\partial_z + \epsilon_r\cdot\partial_z\phi\right)\cdots\left(\partial_x + \epsilon_1\cdot\partial_z\phi\right)\psi(z,\epsilon) \\
&- 4\sqrt{p(x)}\partial_z\sqrt{p(z)}\psi(z,\epsilon) = 0
\end{aligned} \tag{4.2.3}$$

[1] Here we inverse the order of l_i.

for $D_r^{(1)}$ type. Here we focus on the holomorphic coordinate z and set $\bar{z}$ fixed. There are two approaches to the WKB analysis of the linear problem. Let us introduce both of them for $A_r^{(1)}$ type.

The first approach is the so-called abelianization method [93]. It starts from the $A_r^{(1)}$-type ordinary differential equation (4.2.2). To solve the ODE, let us apply the WKB ansatz by setting

$$\psi(z,\epsilon) = \exp\biggl(\frac{1}{\epsilon}\int^z dz\, P(z,\epsilon)\biggr) \tag{4.2.4}$$

with $P(z,\epsilon) = \sum_{i=0}^{\infty}\epsilon^i P_i(z)$. The equation satisfied by $P(z,\epsilon)$ is called the Riccati equation. For further convenience, denote $P(z,\epsilon)$ as the WKB solution in the massive case. One can calculate $P_i(z)$ recursively from the Riccati equation. The WKB analysis for $A_r^{(1)}$-type ODEs with $\phi_i = 0$ have been done in [83] (see also [93, 94]), where it has been shown there that $P_i(z)$ for odd i and $i = 2(r+1)k + r + 2$ $(k = 0, 1, \dots)$ are total derivatives. However, this is not true when $\phi_i \neq 0$, as seen in Section 4.3.

The second approach is applying the WKB method directly to the linear problem[2]. For simplicity, let us set $\phi_i = 0$. In the linear problem (4.2.1), we take the following WKB ansatz:

$$\Psi(z,\epsilon) = \bigl(\exp\biggl(\frac{1}{\epsilon}\int^z P_1(z',\epsilon)dz'\biggr), \dots, \exp\biggl(\frac{1}{\epsilon}\int^z P_{2r+2}(z',\epsilon)dz'\biggr)\bigr)^{\mathrm{T}}, \tag{4.2.5}$$

where $P_i(z,\epsilon) = \sum_{n=0}^{\infty}\epsilon^n P_{i,n}(z)$. In particular, $P_1(z,\epsilon) = P(z,\epsilon)$ with $\phi_i = 0$. After substituting the ansatz (4.2.5) into the linear problem (4.2.1), one find the following recursive relations

$$\begin{aligned} &P_{i+1,n} = -(\frac{P_i'}{P_i})_{n-1} + P_{i,n}, \quad 1 \le i \le r, \\ &\sum_{i=1}^{r+1}(\frac{P_i'}{P_i})_{n-1} - \delta_{1n}(\frac{p'(z)}{p(z)}) = 0. \end{aligned} \tag{4.2.6}$$

which are equivalent to the Riccati equation for the ODE (4.2.2) with $\phi_i = 0$ [83]

$$(\epsilon\partial_z + P(z,\epsilon))^r P(z,\epsilon) + (-1)^r p(z) = 0. \tag{4.2.7}$$

This method is applicable to any affine Lie algebra, but rather cumbersome to find the solutions. Especially it can be used to solve the $D_{r+1}^{(2)}$-type pseudo-differential equations. See [59] for more details.

[2]For the $\mathfrak{sl}_3$ Miura opers related to affine Gaudin models, the WKB analysis has bee studied in [95].

So far, we have only considered $A_r^{(1)}$ types. The first WKB analysis approach can also be applied to $A_{2r-1}^{(2)}$, $B_r^{(1)}$-type ordinary differential equations in Eq.(3.4.23). But it is still difficult to apply the same WKB analysis to $D_{r+1}^{(2)}$ and $D_r^{(1)}$-type differential equations for $\phi_i \neq 0$, where the obtained ones include the pseudo-differential operators. The second approach can be applied to linear problems with affine Lie algebras $A_{2r-1}^{(2)}$, $B_r^{(1)}$, and $D_{r+1}^{(2)}$ when $\phi_i = 0$ but it is still challenging to solve the general $D_r^{(1)}$ type with $\phi_i \neq 0$. Fortunately, it has been found that we can obtain the WKB solutions based on an embedding of $B_{r-1}^{(1)}$ into $D_r^{(1)}$ [59]. We will pay more attention to this prescription in Section 4.3.2. Before that, let us first compute the WKB integral along the so-called Pochhammer contour in the conformal limit.

4.2.1 The WKB integral in the conformal limit

The WKB integral in the conformal limit plays an important role in the ODE/IM correspondence as we will see. In general, the contour integrals of the massive WKB solution $P(z,\epsilon)$ in (4.2.4) are difficult to handle. However, due to the special structure of the solutions after lightcone limit and the conformal limit in Section 3.4.1, the linear problem (Lax operator) (4.1.18) becomes

$$\epsilon \partial_x \Psi + \epsilon \frac{1}{x}(l \cdot H)\Psi + \left(\sum_{i=1}^{r} E_{\alpha_i} + p(x) E_{\alpha_0} \right) \Psi = 0. \tag{4.2.8}$$

It is possible to choose a suitable contour to compute the WKB integral analytically [49].

The WKB analysis of the $A_r^{(1)}$ type

Let us first begins with the $A_r^{(1)}$ type. As we have shown, the ODE (the first approach) and the linear problem (the second approach) are equivalent. Here we choose the first approach. Consider the ODE (4.2.2), after the lightcone limit and the conformal limit, this ODE is the same as that appears in the ODE/IM correspondence [39, 61].

$$\left[\epsilon^{r+1} \left(\partial_x - \frac{l_1}{x} \right) \left(\partial_x - \frac{l_2}{x} \right) \dots \left(\partial_x - \frac{l_r}{x} \right) \left(\partial_x - \frac{l_{r+1}}{x} \right) + (-1)^r p(x) \right] \psi(x,\epsilon) = 0 \tag{4.2.9}$$

with $l_i = \epsilon_i \cdot l$. To find the WKB solution, it is convenient to expand the first term in Eq.(4.2.9) and rewrite it in the form

$$\left[(\epsilon\partial_x)^h + \sum_{i=2}^{h} \epsilon^i \frac{L_i}{x^i}(\epsilon\partial_x)^{h-i} + (-1)^r p(x)\right]\psi(x,\epsilon) = 0 \tag{4.2.10}$$

with the coefficients L_i some functions of l. In particular, they are expressed in terms of the the elementary symmetric polynomials of $(\epsilon_i, l+\rho)$:

$$s_k = \sum_{1\le i_1<i_2<\cdots<i_k\le r+1} (\epsilon_{i_1}, l+\rho)(\epsilon_{i_2}, l+\rho)\dots(\epsilon_{i_k}, l+\rho).$$

The first five coefficients are found to be

$$\begin{aligned}
L_2 &= s_2 + \frac{1}{24}r(r+1)(r+2),\\
L_3 &= -s_3 - (r-1)L_2,\\
L_4 &= s_4 + \frac{3}{2}(r-2)s_3 + \frac{1}{24}(r+24)(r-1)(r-2)s_2\\
&\quad + \frac{1}{5760}(5r+228)(r+2)(r+1)r(r-1)(r-2),\\
L_5 &= -s_5 - 2(r-3)s_4 - \frac{1}{24}(r-3)(r-2)(r+44)s_3\\
&\quad - \frac{1}{12}(r-3)(r-2)(r-1)(r+12)s_2\\
&\quad - \frac{1}{2880}(r-3)(r-2)(r-1)r(r+1)(r+2)(5r+108),
\end{aligned} \tag{4.2.11}$$

and the 6th, 7th, and 8th orders are listed in Appendix D.

We now study the WKB solution of (4.2.10). To distinguish from the massive WKB solution $P(z,\epsilon)$ in (4.2.4), we denote the WKB solution here as $S(x,\epsilon)$ with

$$\psi(x,\epsilon) = \exp\left(\frac{1}{\epsilon}\int^x dx\, S(x,\epsilon)\right), \tag{4.2.12}$$

$$S(x,\epsilon) = \sum_{i=0}^{\infty} \epsilon^i S_i(x). \tag{4.2.13}$$

Substituting Eq.(4.2.12) into Eq.(4.2.10), one obtains the Riccati equation for $S(x,\epsilon)$:

$$(\epsilon\partial_x + S)^r S + \sum_{i=2}^{r+1} \frac{L_i}{x^i}(\epsilon\partial_x + S)^{r-i} S + (-1)^r p(x) = 0, \tag{4.2.14}$$

where we define $(\epsilon\partial_x+S)^{-1}S = 1$ in the summation. Then we expand $S(x,\epsilon)$ as Eq.(4.2.13) and solve the equation (4.2.14) recursively. Here the first two terms of order ϵ^n $(n=0,1)$ are

$$\epsilon^0: \quad (-1)^r p(x) + S_0^{r+1} = 0, \quad \epsilon^1: \quad \frac{1}{2}(r+1)S_0^{r-1}\left(rS_0' + 2S_0S_1\right) = 0. \tag{4.2.15}$$

The higher-order terms can be determined similarly. Their solutions are given by

$$\begin{aligned} S_0(x) &= -p(x)^{\frac{1}{h}}, \\ S_1(x) &= \frac{h-1}{2h}\partial_x \log(p), \\ S_2(x) &= \frac{L_2}{h}p^{-\frac{1}{h}}x^{-2} - \frac{(h-1)(h+1)M(hM-1)}{12}p^{-\frac{1}{h}-1}x^{hM-2} \\ &\quad + \frac{(h-1)(h+1)(2h+1)M^2}{24}p^{-\frac{1}{h}-2}x^{2hM-2}, \\ &\cdots \end{aligned} \tag{4.2.16}$$

Note that $S_0(x)$ has $(r+1)$ solutions with phases given by $(r+1)$-th root of unity. Here we choose the minus sign. It has been shown that all the $(1+hk)$-th terms $(k \geq 0)$ in S_i becomes total derivatives [59, 83] and hence the corresponding WKB period integral vanishes.

Next, we consider the period integral of the coefficient $S_i(x)$ in the WKB expansion along the cycle C. Here we choose C as the Pochhammer contour which starts from $\infty \cdot e^{+i0}$ to $x=1$, goes around $x=1$ and finally ends with $\infty \cdot e^{-i0}$ as in Fig.4.2.1. We define the period integral of $S_i(x)$ as

$$Q_i = \oint_{\mathcal{C}} dx\ S_i(x). \tag{4.2.17}$$

For $r=1$, Q_i corresponds to the holomorphic part of the highest-weight quantum conserved charges for the quantum Sinh-Gordon model [49].

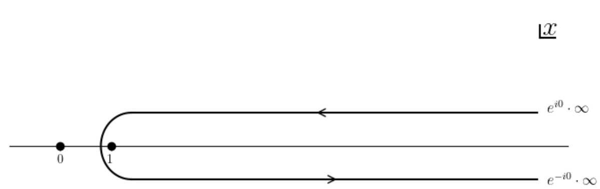


Figure 4.2.1: Integral contour C

Now we evaluate the integral (4.2.17) for $A_r^{(1)}$ with general r. Let us compute the first nontrivial term $Q_2(x)$. Since there appear integrants of the form $p(x)^a x^b$ frequently, it is convenient to introduce the symbol $J(a,b)$:

$$J(a,b) \equiv \int_C \left(x^{hM}-1\right)^a x^b dx = -\frac{e^{\pi i a} 2\pi i}{hM} \frac{\Gamma\left(-a-\frac{b+1}{hM}\right)}{\Gamma(-a)\Gamma\left(1-\frac{b+1}{hM}\right)}, \tag{4.2.18}$$

where Γ is the Gamma function. Using the formula $\Gamma(x+1)=x\Gamma(x)$, the integral satisfies the recurrence relation:

$$J(a+m, b+(hM)n) = e^{i\pi m} \frac{(\frac{b+1}{hM}+n)!(a+m+1)!(a+\frac{b+1}{hM}+1)!}{(a+m+\frac{b+1}{hM}+n+1)!(\frac{b+1}{hM})!(a+1)!} J(a,b,h;M). \tag{4.2.19}$$

After evaluating the integral of $S_2(x)$ in Eq.(4.2.16), there are three integrals. According to the recurrence relation (4.2.18), they are combined into the following form.

$$\begin{aligned} Q_2 =& \frac{L_2}{h} J(-\frac{1}{h}, -2) - \frac{(h^2-1)M(hM-1)}{12} J(-\frac{1}{h}-1, -2+hM) \\ &+ \frac{(h^2-1)(2h+1)M^2}{24} J(-\frac{1}{h}-2, -2+2hM) \\ =& \frac{1}{h} J(-\frac{1}{h}, -2)\Big(L_2 + \frac{1}{24}(h-1)h(hM-1)\Big). \end{aligned}$$

This construction rule can be applied to higher orders. For general $Q_i(x)$ case, there are $J(a+m, b+(hM)n)$ contained with two integers $0 \le m, n \le i$. Thanks to the recurrence relation and the construction rule above, it is possible to design an algorithm to compute general Q_i. Let us list the first five nontrivial terms here, and the 6th, 7th and 8th terms in Appendix D.

$$\begin{aligned} Q_2 &= \frac{1}{h} J_{1,2}\Big(L_2 + \frac{1}{24}(h-1)h(hM-1)\Big), \\ Q_3 &= \frac{1}{h} J_{2,3}\Big(-L_3 - (h-2)L_2\Big), \\ Q_4 &= \frac{1}{h} J_{3,4}\Big(L_4 + (h-3)\{-\frac{1}{2h}L_2^2 + \frac{3}{2}L_3 - \frac{1}{8}[h(M-4)+9]L_2 \\ &\quad + \frac{1}{1920}(h-1)[hM-3][hM-1][2h(M+1)-1]\}\Big), \\ Q_5 &= \frac{1}{h} J_{4,5}\Big(-L_5 + \frac{1}{3h}(h-4)\{3L_2L_3 + 3(h-2)L_2^2 - 6L_4 \\ &\quad + [h(M-3)+11]L_3 + (h-2)(hM+2)L_2\}\Big), \end{aligned} \tag{4.2.20}$$

where L_k vanishes when $k > h$ and $J(-n/h, m)$ are abbreviated to $J_{n,m}$ for convenience. We can check that $Q_{(1+hk)}$ ($k \in \mathbf{Z}$) becomes zero, which is consistent with the WKB analysis.

The WKB analysis of the $D_r^{(1)}$ type

Let us consider the $D_r^{(1)}$-type linear problem with the basis $\epsilon_1, \ldots, \epsilon_r, -\epsilon_r, \ldots, -\epsilon_1$, where ϵ_i are orthonormal basis with $\epsilon_i \cdot \epsilon_j = \delta_{ij}$. The simple roots are $\alpha_i = \epsilon_1 - \epsilon_{i+1}$ ($i = 1, \ldots, r-1$) and $\alpha_r = \epsilon_r + \epsilon_{r-1}$. The Weyl vector $\rho = \sum_{i=1}^{r}(r-i)\epsilon_i$.

The pseudo-ODE becomes

$$\epsilon^{2r-2}\left(\partial_x - \frac{l_1}{x}\right)\cdots\left(\partial_x - \frac{l_r}{x}\right)\partial_x^{-1}\left(\partial_x + \frac{l_r}{x}\right)\cdots\left(\partial_x + \frac{l_1}{x}\right)\psi = 4\sqrt{p(x)}\partial_x\sqrt{p(x)}\psi, \tag{4.2.21}$$

with $l_k = \epsilon_k \cdot l$ and $h = 2r-2$ in $D_r^{(1)}$ after the lightcone limit and conformal limit.

The ODE (4.2.21) is difficult to solve directly with the WKB method due to the existence of a pseudo-differential operator. As it will be shown in Section 4.3.2 and [21, 59, 96], the Riccati equations are constructed by two functions: $S(x,\epsilon)$ and $K(x,\epsilon)$ to determine the WKB solution of the linear problem. We can find the WKB series $S(x,\epsilon)$ based on an embedding of $B_{r-1}^{(1)}$ into $D_r^{(1)}$.

First, we set l_r to be zero, by which the pseudo-ODE (4.2.21) reduces to the $B_{r-1}^{(r)}$-type ODE:

$$\begin{aligned}\epsilon^{2r-2}&\left(\partial_x - \frac{l_1}{x}\right)\cdots\left(\partial_x - \frac{l_{r-1}}{x}\right)\partial_x\left(\partial_x + \frac{l_{r-1}}{x}\right)\cdots\left(\partial_x + \frac{l_1}{x}\right)\psi(x)\\ &- 4\sqrt{p(x)}\partial_x\sqrt{p(x)}\psi(x) = 0.\end{aligned} \tag{4.2.22}$$

The WKB expansion of this ODE can be studied as in the $A_r^{(1)}$ case. Expanding the differential operator in (4.2.22) such as

$$\left(\partial_x - \frac{l_1}{x}\right)\cdots\left(\partial_x - \frac{l_{r-1}}{x}\right)\partial_x\left(\partial_x + \frac{l_{r-1}}{x}\right)\cdots\left(\partial_x + \frac{l_1}{x}\right) = \partial_x^{2r-1} + \sum_{k=2}^{2r-1}\frac{L'_k}{x^k}\partial_x^{2r-1-k}, \tag{4.2.23}$$

and substituting (4.2.4) into (4.2.22), we obtain the Riccati equation:

$$(\epsilon\partial_x + S)^{2r-2}S + \sum_{i=2}^{2r-1}\frac{L'_i}{x^i}(\epsilon\partial_x + S)^{2r-2-i}S - 4\sqrt{p(z)}\partial_z\sqrt{p(z)} = 0. \tag{4.2.24}$$

The Riccati equation can be solved recursively as

$$
\begin{aligned}
S_0(x) &= 2^{\frac{2}{h}} p(x)^{\frac{1}{h}}, \\
S_1(x) &= -\frac{1}{2}\partial_x \log(p), \\
S_2(x) &= -4^{-1/h}\frac{L_2'}{h} p^{-\frac{1}{h}} x^{-2} - 2^{-\frac{2}{h}-3}\frac{(h+1)(h+2)(2h+1)M^2}{3} p^{-\frac{1}{h}-2} x^{-2+2hM} \\
&\quad + \frac{4^{-\frac{h+1}{h}}(h+1)(h+2)(hM-1)M}{3h} p^{-\frac{1}{h}-1} x^{-2+hM},
\end{aligned}
\tag{4.2.25}
$$

where $h = 2r-2$. The parameters L_k' in (4.2.23) are shown to be symmetric polynomials of $(\epsilon_1, l+\rho)^2, \ldots, (\epsilon_{r-1}, l+\rho)^2$. It is observed that the WKB expansion $S(x,\epsilon)$ for $D_r^{(1)}$ can be obtained by promoting L_k' to the symmetric polynomial of the same order by adding $(\epsilon_r, l+\rho)^2$. Explicitly, the first five L_k' for $D_r^{(1)}$ are given by

$$
\begin{aligned}
L_2' =& - s_1 + \frac{1}{12}(2r-1)r(2r-2), \\
L_3' =& - (2r-3)L_2', \\
L_4' =& s_2 - \frac{1}{6}(r-2)(r+11)(2r-3)s_1 \\
&+ \frac{1}{360}(r-2)(r-1)r(2r-3)(2r-1)(5r+109), \\
L_5' =& (10-4r)L_4' + 2(r-2)(2r-5)(2r-3)L_2', \\
L_6' =& - s_3 + \frac{1}{6}(r-3)(r+34)(2r-5)s_2 \\
&- \frac{1}{360}(r-3)(r-2)(2r-5)(2r-3)(5r^2+339r+1096)s_1 \\
&+ \frac{1}{45360}(r-3)(r-2)(r-1)r(2r-1)(2r-3)(2r-5)(35r^2+3549r+22306),
\end{aligned}
\tag{4.2.26}
$$

and s_k is defined by

$$
s_k = \sum_{1\le i_1 < i_2 < \cdots < i_k \le r} (\epsilon_{i_1}, l+\rho)^2 (\epsilon_{i_2}, l+\rho)^2 \ldots (\epsilon_{i_k}, l+\rho)^2.
$$

The other function $K(x,\epsilon)$ provides another WKB solution whose leading term is given by $2p^{-\frac{1}{2}} x^{-r} K_r$ where

$$
K_r := x^r \left(\partial_x - \frac{l_1}{x}\right) \cdots \left(\partial_x - \frac{l_r}{x}\right) \cdot 1 = (\epsilon_1, l+\rho) \cdots (\epsilon_r, l+\rho). \tag{4.2.27}
$$

Note that $s_1, \ldots, s_{r-1}, K_r$ characterize the Casimir invariants of D_r. After substituting these diagonal elements into Eq.(4.2.17), we can obtain the first five (Q_6 can be found in the Appendix D) and the extra WKB integrals:

$$
\begin{aligned}
Q_2 &= -\frac{2^{-\frac{2}{h}}}{h} J_{1,2} \left(L_2' + \frac{1}{24} h(h+2)(hM-1) \right), \\
Q_3 &= -\frac{2^{-\frac{4}{h}}}{h} J_{2,3} \left[L_3' + (2r-3) L_2' \right] = 0, \\
Q_4 &= -\frac{2^{-\frac{6}{h}}}{h} J_{3,4} \bigg[L_4' - \frac{h-3}{2h} L_2'^2 + \frac{3(h-2)}{2} L_3' \\
&\quad - \frac{1}{8} \left[(M-4)h^2 - (6M-9)h - 6 \right] L_2' \\
&\quad + \frac{1}{1920} h(h-6)(h+2)(Mh-3)(Mh-1)(2(M+1)h-1) \bigg], \\
Q_5 &= -\frac{2^{-\frac{8}{h}}}{h} J_{4,5} \bigg[L_5' + \frac{4}{h} L_2' L_3' + 2(h-3) L_4' - \frac{(h-4)(h-1)}{h} L_2'^2 \\
&\quad + \frac{1}{6} \left(40 - 28h + 6h^2 + (11-2h)hM \right) L_3' \\
&\quad - \frac{1}{6} (h-1) \left((2h-11)hM - 2(h+2) \right) L_2' \bigg], \\
Q_r' &= 2J(-\frac{1}{2}, -r)(\epsilon_1, l_1 + \rho) \cdots (\epsilon_1, l_r + \rho),
\end{aligned}
\tag{4.2.28}
$$

where L_k' also vanishes when $k > h$.

Above all, we have obtained the contour integrals of the WKB solutions in modified affine Toda field theories with simple-laced Lie algebras. Next, we will obtain these WKB solutions from another approach − the diagonalization, which also reveals the deep relation between $D_r^{(1)}$-type and $B_r^{(1)}$-type (pseudo) ODEs. The WKB analysis of the linear problem in this section begins with the solution Ψ. It inspires us that there may exist a method directly from the modified Lax operators (4.1.18). This is a central problem in the next section.

4.3 Diagonalization of integrable linear problems with WKB method

In this section, we reconsider the WKB analysis of the linear problem from the viewpoint of the classical integrable models. The Lax formalism allows us to construct an infinite number of conserved charges algebraically. For the Lax equations associated with affine Lie algebras, Drinfeld and Sokolov have diagonalized the Lax operator and systematically constructed the conserved charges [21]. In the diagonalized basis, the linear problem reduces to the set of independent first-order differential equations which can be solved easily.

In the following diagonalization procedure, we choose the specific fundamental representation instead of the general basis $\{\boldsymbol{\epsilon}\}$ for convenience. Look up Appendix C for more detail. According to the Lax operator (4.1.18), the $A_r^{(1)}$-type linear problem becomes

$$\left\{\epsilon\partial_z + \begin{pmatrix} \epsilon\partial_z\phi_1 & 1 & & & \\ & \epsilon(\partial_z\phi_2-\partial_z\phi_1) & 1 & & \\ & & \ddots & & \\ & & & \epsilon(\partial_z\phi_r-\partial_z\phi_{r-1}) & 1 \\ p(z) & & & & -\epsilon\partial_z\phi_r \end{pmatrix}\right\}\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_r\\ \psi_{r+1}\end{pmatrix} = 0, \tag{4.3.1}$$

where $\Psi(z,\bar{z}) = (\psi_1,\ldots,\psi_{r+1})^{\mathrm{T}}$ is expanded into an $(r+1)$-dimensional vector. Compared with the expansion in Eq.(3.4.9), we omit the mass parameter m for simplicity. It leads to $A_r^{(1)}$-type ODEs and other types can be similarly derived. In summary, they are given in the following table including $A_r^{(1)}$, $A_{2r-1}^{(2)}$, $B_r^{(1)}$, $D_{r+1}^{(2)}$, and $D_r^{(1)}$.

$$\begin{aligned}
A_r^{(1)}:\ & (-\epsilon)^{(r+1)}(\partial_z-\partial_z\phi_r)(\partial_z+\partial_z\phi_r-\partial_z\phi_{r-1}) \\
&\qquad\cdots(\partial_z+\partial_z\phi_2-\partial_z\phi_1)(\partial_z+\partial_z\phi_1)\psi - p(z)\psi = 0 \\
A_{2r-1}^{(2)}:\ & \epsilon^{(2r-1)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-\partial_z\phi_r+\partial_z\phi_{r-1})(\partial_z+\partial_z\phi_r-\partial_z\phi_{r-1}) \\
&\qquad\cdots(\partial_z+\partial_z\phi_1)\psi + 2\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0, \\
B_r^{(1)}:\ & \epsilon^{2r}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-2\partial_z\phi_r+\partial_z\phi_{r-1})\partial_z(\partial_z+2\partial_z\phi_r-\partial_z\phi_{r-1}) \\
&\qquad\cdots(\partial_z+\partial_z\phi_1)\psi - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0, \\
D_{r+1}^{(2)}:\ & \epsilon^{(2r+2)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-2\partial_z\phi_r+\partial_z\phi_{r-1})\partial_z(\partial_z+2\partial_z\phi_r-\partial_z\phi_{r-1}) \\
&\qquad\cdots(\partial_z+\partial_z\phi_1)\psi - 4p(z)\partial_z^{-1}p(z)\psi = 0, \\
D_r^{(1)}:\ & \epsilon^{(2r-2)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-\partial_z\phi_r-\partial_z\phi_{r-1}+\partial_z\phi_{r-2})\partial_z^{-1} \\
&\quad(\partial_z+\partial_z\phi_r+\partial_z\phi_{r-1}-\partial_z\phi_{r-2})\cdots(\partial_z+\partial_z\phi_1)\psi - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0.
\end{aligned} \tag{4.3.2}$$

We can also construct the ODEs for $C_r^{(1)}$ and $A_{2r}^{(2)}$ in the fundamental representations but they are in the same form as $A_r^{(1)}$ type.

4.3.1 Diagonalization of $A_r^{(1)}$-type linear problem

We first illustrate a diagonalization procedure for the $A_r^{(1)}$-type linear problem. Let us consider the following linear problem $\mathcal{L}\Psi = 0$ in the $(r+1)$-dimensional representation. Here $\mathcal{L}$ is a general form of Eq. (4.1.18)

$$\mathcal{L} = \partial_z + q(z,\lambda) + \lambda\Lambda, \tag{4.3.3}$$

where $q(z,\lambda)$ is a $(r+1)\times(r+1)$ lower triangular matrix, and $\Lambda = \sum_{i=0}^{r} E_{\alpha_i}$ which is invertible. If $q(z)$ depends on a time coordinate t and satisfies the Lax equation $d\mathcal{L}/dt = [\mathcal{A},\mathcal{L}]$ for a specific differential operator $\mathcal{A}$, one can define a classical integrable system with infinite conserved charges. There exists a formal series T such that the operator $\mathcal{L}_0 = T\mathcal{L}T^{-1}$ has the form [21]

$$\mathcal{L}_0 = \partial_z + \lambda\Lambda + \sum_{i=0}^{\infty} \lambda^{-i} I_i(z)\Lambda^{-i} \tag{4.3.4}$$

with

$$T(z,\lambda) = E + \sum_{i=1}^{\infty} \lambda^{-i} h_i(z)\Lambda^{-i}, \tag{4.3.5}$$

where E is the identity matrix, $I_i(z)$ are functions of z, and $h_i(z)$ are diagonal matrices whose entries are functions of z. The series $T(z,\lambda)$ is determined up to multiplication on the left by series of the form $d(z) = E + \sum_{i=1}^{\infty}\lambda^{-i}d_i(z)\Lambda^{-i}$, where $d_i(z)$ are functions of z. The form of Eq. (4.3.4) implies that $\mathcal{L}_0$ is a diagonalized Lax operator for the eigenvectors of Λ. $I_i(z)$ are argued to be the conserved charge densities of the integrable model, which will be discussed in Section 4.4. For the $(r+1)$-dimensional fundamental representation of A_r, the eigenvalues of Λ are $1,\omega,\ldots,\omega^r$ with $\omega = \exp\left(\frac{2\pi i}{r+1}\right)$. We can obtain $I_i(z)$ and $h_i(z)$ recursively by expanding $\mathcal{L}_0 T = T\mathcal{L}$ in λ^{-1} and comparing the coefficients of λ^{-i}. For $i=0$ and $i\geq 1$, we find

$$\begin{aligned} h_1^{\sigma}(z) + I_0(z)E &= q_0(z), \\ h_{i+1}(z) - h_{i+1}^{\sigma}(z) - I_i(z)E &= \partial_z h_i(z) + \sum_{j=0}^{i-1}\big(I_j(z)E - q_j(z)\big)h_{i-j}(z) - q_i(z), \end{aligned} \tag{4.3.6}$$

where $q(z) = \sum_{i=0}^{\infty} \lambda^{-i} q_i(z) \Lambda^{-i}$ with $q_i(z)$ diagonal matrices, and $h_i^\sigma = \Lambda h_i \Lambda^{-1}$. I_i is uniquely determined up to total derivatives due to the choice of $d(z)$ in $T(z, \lambda)$. From the relation $\mathcal{L}_0 = T\mathcal{L}T^{-1}$, we obtain

$$\mathrm{Tr}(\lambda^{-i} \sum_{i=0}^{\infty} (I_i(z) - q_i(z))\Lambda^{-i}) = -\mathrm{Tr}(\partial_z T T^{-1}) = -\partial_z \ln\det T, \qquad (4.3.7)$$

which varies up to $-\partial_z \ln\det d(z)$ with further multiplication on the left of T by $d(z)$. After the diagonalization of Λ, Eq.(4.3.4) can be written as

$$\begin{aligned}
\mathcal{L}_{\mathrm{diag}} &= \partial_z + \lambda\Lambda_{\mathrm{diag}} + \sum_{i=0}^{\infty} \lambda^{-i} I_i(z) \Lambda_{\mathrm{diag}}^{-i} \\
&= \partial_z + \lambda\Lambda_{\mathrm{diag}} + \begin{pmatrix} I(z, e^{\frac{2\pi i}{r+1}}\lambda) & & & & \\ & \ddots & & & \\ & & I(z, e^{\frac{2\pi i(r-1)}{r+1}}\lambda) & & \\ & & & I(z, e^{\frac{2\pi i r}{r+1}}\lambda) & \\ & & & & I(z,\lambda) \end{pmatrix},
\end{aligned} \qquad (4.3.8)$$

where $\Lambda_{\mathrm{diag}} = \mathbf{Diag}\{e^{\frac{2\pi i r}{r+1}}, e^{\frac{2\pi i(r-1)}{r+1}}, e^{\frac{2\pi i(r-2)}{r+1}}, \ldots, 1\}$ and $I(z,\lambda) = \sum_{i=0}^{\infty} I_i(z)\lambda^{-i}$. The order of eigenvalues in Λ_{diag} and $I(z, e^{\frac{2\pi i n}{r+1}}\lambda)$ in Eq. (4.3.8) is not fixed.

The gauge transformation and diagonalization of the Lax operator (4.3.3) based on $A_r^{(1)}$ can be generalized to an affine Lie algebra $\hat{\mathfrak{g}}$ and its arbitrary representation. In general, $q(z)$ is decomposed as $q(z) \in \oplus_{i=\infty}^{0} \mathfrak{g}^{-i}$, the negative canonical gradation (lower ladder operators) and $\Lambda = \sum_{i=0}^{r} E_{\alpha_i}$ admits zero eigenvalues. More detail can be found in Appendix A.2. After the gauge transformation by T, the Lax operator $\mathcal{L}$ becomes

$$\mathcal{L}_0 = T\mathcal{L}T^{-1} = \partial_z + \lambda\Lambda + H(z,\lambda) \qquad (4.3.9)$$

with

$$T(z,\lambda) = e^{U(z,\lambda)}, \quad U(z,\lambda) = \sum_{i=1}^{\infty} \lambda^{-i} U^i(z), \quad U^i(z) \in \mathfrak{g}^{-i} \qquad (4.3.10)$$

and $H(z,\lambda) \in \mathrm{Ker}\ \mathrm{ad}\Lambda$ satisfies $[H, \Lambda] = 0$. For the fundamental representation with the highest weight ω_1 for $\hat{\mathfrak{g}} = B_r^{(1)}, D_{r+1}^{(2)}, A_{2r-1}^{(2)}$, and $D_r^{(1)}$, $H(z,\lambda)$ is given by [21]

$$H(z,\lambda) = \sum_{i=0}^{\infty} \lambda^{-(2i+1)} I_i(z) \Lambda^{-(2i+1)}, \quad \text{for } B_r^{(1)},\ D_{r+1}^{(2)},\ A_{2r-1}^{(2)}, \qquad (4.3.11)$$

$$H(z,\lambda)=\sum_{i=0}^{\infty}\lambda^{-(2i+1)}I_i(z)\Lambda^{-(2i+1)}+\sum_{i=0}^{\infty}\lambda^{-(2i+1)}J_i(z)F,\quad \text{for } D_r^{(1)}. \tag{4.3.12}$$

From the eigenvalues of Λ, we can see $\Lambda^{-(2i+1)}=\Lambda^{-(2i+1)+kh}$ with sufficiently large k for $B_r^{(1)}$, $A_{2r-1}^{(2)}$, $D_r^{(1)}$ and $D_{r+1}^{(2)}$. In Eq.(4.3.12), there is an extra matrix F commuting with matrix Λ, which generates a new set of conserved densities in $D_r^{(1)}$ case corresponding to two zero eigenvalues in matrix Λ [96]. I_i, J_i, and U^i can be determined recursively. However, $U(z,\lambda)$ is not fixed uniquely because the form of $\mathcal{L}_0$ is preserved after the further gauge transformation $e^{\mathbf{ad}U'}$ ($U'\in$ Ker adΛ). Up to total derivatives, I_i and J_i are independent of different choices of $U(z)$. The terms proportional to Λ^{-2i} do not appear in Eq.(4.3.11) and Eq.(4.3.12) since their coefficients are total derivatives. Eq.(4.3.9) can be understood as diagonalization of $\mathcal{L}$ in terms of eigenvectors of matrix Λ.

We have seen that the Lax operator (4.3.3) is diagonalized after the gauge transformation, and the linear problem in the diagonalized form can be solved easily. Here we apply this approach to the linear problem $\mathcal{L}_m\Psi=0$ associated with the modified affine Toda field equation, which provides a way to find the WKB expansion to the solution. In the previous subsection, we have obtained diagonalized Lax operators for general affine Lie algebras. It is interesting to apply the diagonalization method to the linear problem for the modified affine Toda field equation.

Although the approach based on the recursive relation like Eq.(4.3.6) is enough to perform the diagonalization, it is rather complicated to find the concrete form of $T(z)$. We will investigate a direct approach to diagonalize the linear problem. Let us begin with the modified Lax operator in Eq.(4.1.18)

$$\mathcal{L}_m=\epsilon\partial_z+\epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i+\sum_{i=1}^{r}E_{\alpha_i}+p(z)E_{\alpha_0}. \tag{4.3.13}$$

One can view the Lax operator as a covariant derivative with corresponding connection $A(z)=\sum_{i=1}^{r}\epsilon\partial_z\phi_i(z)H_i+\sum_{i=1}^{r}E_{\alpha_i}+p(z)E_{\alpha_0}$. Then the gauge transformation is given by

$$\mathbf{Gau}_T[A(z)]=T^{-1}(z)A(z)T(z)+\epsilon T^{-1}(z)\partial_zT(z). \tag{4.3.14}$$

The next step is to find a matrix $T(z)$ to perform the diagonalization. In this paper, we diagonalize $A(z)$ row by row from bottom to top, which is regarded as a generalization of

Section 11.1 in [97]. The matrix is given by

$$T(z) = T_d T_{d-1} \dots T_3 T_2 T_1, \tag{4.3.15}$$

where d is the dimension of the representation and $T_i(z)$ are $d \times d$ matrices satisfying

$$T_i(z)_{ab} = \begin{cases} 1, & \text{if} \quad a = b, \\ g_{i,b}(z), & \text{if} \quad a = i, \quad b \neq i, \quad 1 \leq b \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

For each step of the gauge transformation $\mathbf{Gau}_{T_i}$, we fix $g_{i,b}(z)$ such that the connection $A'(z)$ satisfies

$$A'_{ij} = 0, \quad 1 \leq j \leq d, \quad j \neq i. \tag{4.3.16}$$

There are finally $d-1$ constraints in Eq. (4.3.16) to diagonalize the i-th row in $A(z)$ and fix the diagonal elements perturbatively. Except for the $D_r^{(1)}$ type, the $(d-1)$ constraints in a row are reduced to a single Riccati equation[3]. The final diagonalized connection $A_{\text{diag}}(z)$ is given by

$$A_{\text{diag}}(z) = \mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2} \dots \mathbf{Gau}_{T_{d-2}} \circ \mathbf{Gau}_{T_{d-1}} \circ \mathbf{Gau}_{T_d}[A(z)]. \tag{4.3.17}$$

Due to the traceless condition, $d-1$ Riccati equations are necessary to determine all the diagonal elements. The diagonal elements are uniquely determined up to total derivatives. One can further act $T_{\mathbf{diag}} = \mathbf{diag}\{\exp(t_1(z,\epsilon)), \dots, \exp(t_d(x,\epsilon))\}$ after $T(z)$, then the diagonal element $[A_{\mathbf{diag}}(z)]_{ii}$ will shift by $\partial_z t_i(z)$. In the following, we will explain the diagonalization procedure for lower-rank examples.

The diagonalization of $A_1^{(1)}$

The diagonalization of the $A_1^{(1)}$-type linear problem has been studied in [97]. The approach there can be viewed as a particular case of ours. The modified Lax operator (4.1.18) becomes

$$\mathcal{L}_m = \epsilon\partial_z + \epsilon\partial_z\phi(z)H + E_\alpha + p(z)E_{-\alpha} \tag{4.3.18}$$

[3]For $D_r^{(1)}$ type, the constraints are reduced to two Riccati equations. We shall discuss the details in Section 4.3.2.

with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.3.19}$$

We define a matrix: $T(z) = T_1 T_2$ of the gauge transformation $\mathcal{L} \to \mathcal{L}' \equiv T^{-1}\mathcal{L}T$ by

$$T_2(z) = \begin{pmatrix} 1 & 0 \\ g_{2,1}(z,\epsilon) & 1 \end{pmatrix}, \quad T_1(z) = \begin{pmatrix} 1 & g_{1,2}(z,\epsilon) \\ 0 & 1 \end{pmatrix}. \tag{4.3.20}$$

$T_2(z)$ is determined to diagonalize the second row

$$\mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} g_{2,1} + \phi' & 1 \\ -2\epsilon g_{2,1}\epsilon\phi' + \epsilon g_{2,1}' - g_{2,1}^2 + p & -g_{2,1} - \epsilon\phi' \end{pmatrix}. \tag{4.3.21}$$

It gives the condition for $g_{2,1}(z)$:

$$g_{2,1}^2(z,\epsilon) + 2\epsilon g_{2,1}(z,\epsilon)\phi'(z,\bar{z}) - \epsilon g_{2,1}'(z,\epsilon) - p(z) = 0. \tag{4.3.22}$$

The diagonal element $f(z,\epsilon) := -g_{2,1}(z,\epsilon) - \epsilon\phi'(z)$ satisfies

$$f^2(z,\epsilon) + \epsilon f'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0 \tag{4.3.23}$$

with $u_2(z) = \phi'(z)^2 - \phi''(z)$. After the diagonalization of the second row, the second gauge transformation $T_1(z)$ gives

$$\mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} -f(z,\epsilon) & 1 - 2g_{1,2}(z,\epsilon)f(z,\epsilon) \\ 0 & f(z,\epsilon) \end{pmatrix}. \tag{4.3.24}$$

We do not need to extract the diagonalization condition from the first row since $g_{1,2}$ is independent of the diagonal elements. Let us substitute $f(z,\epsilon) = \sum_{i=0}^{\infty} \epsilon^i f_i(z)$ into Eq. (4.3.22), then Eq.(4.3.23) can be solved perturbatively. The first four orders of diagonal elements $f(z,\epsilon)$ are listed below

$$\begin{aligned}
f_0(z) &= \sqrt{p(z)}, \\
f_1(z) &= -\frac{1}{2}\partial_z \ln f_0, \\
f_2(z) &= \frac{f_0''}{16f_0^2} + \frac{u_2(z)}{2f_0} + \partial_z\left(\frac{3f_0'}{16f_0^2}\right), \\
f_3(z) &= \partial_z\left(\frac{u_2(z)}{4f_0^2} - \frac{3f_0'^2}{16f_0^4} + \frac{f_0''}{8f_0^3}\right).
\end{aligned}$$

Besides, the diagonal elements are uniquely determined up to total derivative terms like Eq.(4.3.7). Therefore, the diagonal elements can be summarized as

$$A_{\text{diag}}(z) = \begin{pmatrix} -f(z,-\epsilon)+d(*) & 0 \\ 0 & f(z,\epsilon) \end{pmatrix}, \tag{4.3.25}$$

where $d(*)$ denotes total derivatives, and the form of the first diagonal element can be seen from

$$\begin{aligned}
-f_0(z) &= -\sqrt{p(z)}, \\
-f_1(z) &= f_1(z) + \partial_z \ln f_0, \\
-f_2(z) &= -f_2(z), \\
-f_3(z) &= f_3(z) - \partial_z\Big(\frac{u_2(z)}{2f_0^2} - \frac{3f_0'^2}{8f_0^4} + \frac{f_0''}{4f_0^3}\Big).
\end{aligned}$$

Next, we show the relations between $f(z,\epsilon)$ and the WKB solutions $P(z,\epsilon)$ in A_r-type ordinary differential equations (4.2.2). The $A_1^{(1)}$-type ordinary differential equation is given by

$$\big(\epsilon^2\partial_z^2 + \epsilon^2\partial_z^2\phi(z) - \epsilon^2(\partial_z\phi)^2 - p(z)\big)\psi(z,\epsilon) = 0. \tag{4.3.26}$$

Substituting the WKB ansatz (4.2.4), one can obtain the Riccati equation for $P(z,\epsilon)$

$$P^2(z,\epsilon) + \epsilon P'(z,\epsilon) - \epsilon^2\big(\phi'(z)^2 - \phi''(z)\big) - p(z) = 0. \tag{4.3.27}$$

The perturbative solution can be given by substituting the expansion $P(z,\epsilon) = \sum_{i=0}^{\infty}\epsilon^i P_i(z)$ to Eq.(4.3.27). The Riccati equation here is the same as Eq.(4.3.23). After setting $P_0(z) = \sqrt{p(z)}$, one can find the equality between $f_i(z)$ and $P_i(z)$.

Finally, let us also compare $f(z,\epsilon)$ with $I(z,\lambda)$ in Eq.(4.3.4). Due to the existence of ϵ in front of ∂_z in Eq.(4.1.18), $f_0(z)$ is actually the (-1)-th term corresponding to the $\lambda\Lambda_{\text{diag}}$ term. Besides this, $f_i(z)$ here corresponds to $I_{i-1}(z)$ in Eq.(4.3.4). Especially, if one takes the inverse conformal transformation, namely $p(z) = 1$, $S_0(z) = 1$ matches the result $\lambda\Lambda_{\text{diag}}$. Actually, $f(z,\epsilon)$ and $I(z,\lambda)$ turn out to be the same set of conserved densities, which will be the main part of Section 4.4.

The diagonalization of $A_2^{(1)}$

The equality $f_i(z) = P_i(z)$ can be generalized into general $A_r^{(1)}$ with $r \geq 2$. As a trial, let us further investigate the $A_2^{(1)}$ case. The diagonalization approach in Eq.(4.3.4) can be applied directly. The modified Lax operator is

$$\epsilon\mathcal{L}_m = \epsilon\partial_z + \sum_{i=1}^{2}\epsilon\partial_z\phi_i(z,\bar{z})H_i + \sum_{i=1}^{2}E_{\alpha_i} + p(z)E_{\alpha_0} \tag{4.3.28}$$

with the representation of $A_2^{(1)}$ in Appendix C.

We perform the gauge transformation with $T = T_3T_2T_1$, from which we find $A_{\rm diag}(z)$. The first gauge transformation by T_3 leads to

$$\mathbf{Gau}_{T_3}[A(z)] = \begin{pmatrix} \epsilon\phi_1' & 1 & 0 \\ g_{3,1} & g_{3,2}+\epsilon(\phi_2'-\phi_1') & 1 \\ \mathbf{Gau}_{T_3}[A(z)]_{3,1} & \mathbf{Gau}_{T_3}[A(z)]_{3,2} & -g_{3,2}-\epsilon\phi_2' \end{pmatrix}, \tag{4.3.29}$$

where

$$\begin{aligned} \mathbf{Gau}_{T_3}[A(z)]_{3,1} &= -g_{3,1}\left(g_{3,2}+\epsilon\left(\phi_1'+\phi_2'\right)\right)+\epsilon g_{3,1}'+p, \\ \mathbf{Gau}_{T_3}[A(z)]_{3,2} &= \epsilon g_{3,2}\left(\phi_1'-2\phi_2'\right)+\epsilon g_{3,2}'-g_{3,2}^2-g_{3,1}. \end{aligned} \tag{4.3.30}$$

Let us set the diagonal element $f(z,\epsilon) \equiv -g_{3,2}-\epsilon\phi_2'$. The diagonalization condition $\mathbf{Gau}_{T_3}[A(z)]_{3,1} = \mathbf{Gau}_{T_3}[A(z)]_{3,2} = 0$ solves $g_{3,1}$ and gives the equation for $g_{3,2}$ or $f(z,\epsilon)$.

After the second gauge transformation T_2, the connection becomes

$$\mathbf{Gau}_{T_2T_3}[A(z)] = \begin{pmatrix} \epsilon\phi_1'+g_{2,1} & 1 & g_{2,3} \\ \mathbf{Gau}_{T_2T_3}[A(z)]_{2,1} & g_{3,2}+\epsilon(\phi_2'-\phi_1')-g_{2,1} & \mathbf{Gau}_{T_2T_3}[A(z)]_{2,3} \\ 0 & 0 & f \end{pmatrix}, \tag{4.3.31}$$

where

$$\begin{aligned} \mathbf{Gau}_{T_2T_3}[A(z)]_{2,1} &= -2\epsilon g_{2,1}\phi_1'+\epsilon g_{2,1}\phi_2'+\epsilon g_{2,1}'-g_{2,1}^2+g_{3,2}g_{2,1}+g_{3,1}, \\ \mathbf{Gau}_{T_2T_3}[A(z)]_{2,3} &= -\epsilon g_{2,3}\phi_1'+2\epsilon g_{2,3}\phi_2'+\epsilon g_{2,3}'+2g_{3,2}g_{2,3}-g_{2,1}g_{2,3}+1. \end{aligned} \tag{4.3.32}$$

We impose the conditions $\mathbf{Gau}_{T_2T_3}[A(z)]_{2,1} = \mathbf{Gau}_{T_2T_3}[A(z)]_{2,3} = 0$ and set $h(z,\epsilon) \equiv g_{2,1}+\epsilon\phi_1'$. Eq. $\mathbf{Gau}_{T_2T_3}[A(z)]_{2,1} = 0$ leads to the equation for $h(z,\epsilon)$, while $g_{2,3}$ is determined by solving $\mathbf{Gau}_{T_2T_3}[A(z)]_{2,3} = 0$.

Gauge transformation by T_1 eliminates the off-diagonal elements in the first row. Finally, we obtain the diagonalized connection $A_{\rm diag}(z) = \mathbf{Gau}_{T_1T_2T_3}[A(z)]$. $A_{\rm Diag}(z)$ becomes of the form $\mathbf{diag}\{h(z,\epsilon), -f(z,\epsilon)-h(z,\epsilon), f(z,\epsilon)\}$ satisfy the Riccati equations:

$$\begin{aligned} &f^3+3\epsilon ff'-\epsilon^2u_2f+\epsilon^2f''-\epsilon^3u_3-p=0, \\ &h^2+fh+f^2-\epsilon h'+\epsilon f'-\epsilon^2u_2=0. \end{aligned} \tag{4.3.33}$$

with

$$
\begin{aligned}
u_2(z) &= \phi_1'(z)^2 - \phi_2'(z)\phi_1'(z) + \phi_2'(z)^2 - \phi_1''(z) - \phi_2''(z), \\
u_3(z) &= 2\phi_2'(z)\phi_2''(z) - \phi_1'(z)\phi_2''(z) - \phi_1'(z)\phi_2'(z)^2 + \phi_1'(z)^2\phi_2'(z) - \phi_2^{(3)}(z).
\end{aligned}
\tag{4.3.34}
$$

The second equation implies that $h(z,\epsilon)$ depends on $f(z,\epsilon)$, while the first equation can be obtained from that of $P(z,\epsilon)$ for the $A_2^{(1)}$ ordinary differential equation of $\psi(z,\epsilon) = \exp\left(\frac{1}{\epsilon}\int dz\, P(z,\epsilon)\right)$

$$
(-\epsilon)^3(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2 + \partial_z\phi_1)(\partial_z + \partial_z\phi_2)\psi + p(z)\psi = 0. \tag{4.3.35}
$$

Eq.(4.3.35) is the adjoint ODE of Eq.(4.2.2) with $r = 2$, where the adjoint means $\partial_z \to -\partial_z$ and $\phi_i \to \phi_{h-i}$. We can solve the coupled Riccati equations (4.3.33) perturbatively. Let us expand f and h as $f = \sum_{n=0}^{\infty} f_n\epsilon^n$ and $h = \sum_{n=0}^{\infty} h_n\epsilon^n$. The leading terms are f_0 and h_0, which can be obtained by solving $f_0^3 = p$ and $h_0^2 + f_0h_0 + f_0^2 = 0$. The result is

$$
f_0 = p^{\frac{1}{3}}, \quad h_0 = e^{\pm\frac{2\pi i}{3}} f_0. \tag{4.3.36}
$$

Here, we take the minus sign in the second equation then the higher-order terms are represented in terms of f_0:

$$
\begin{aligned}
f_1(z) &= -\frac{f_0'}{f_0}, & h_1(z) &= f_1(z) + 2\partial_z(\ln f_0), \\
f_2(z) &= \frac{f_0''}{6f_0^2} + \frac{u_2(z)}{3f_0} + \partial_z\left(\frac{f_0'}{2f_0^2}\right), & h_2(z) &= e^{\frac{2\pi i}{3}} f_2(z), \\
f_3(z) &= -\frac{u_3(z)}{3f_0^2} + \frac{f_0' u_2(z)}{3f_0^3} - \partial_z\left(-\frac{f_0'^2}{2f_0^4} + \frac{f_0''}{3f_0^3} - \frac{u_2(z)}{3S_0^2}\right), & h_3(z) &= e^{\frac{4\pi i}{3}}\left(f_3(z) - \partial_z\left(\frac{u_2}{3f_0^2}\right)\right).
\end{aligned}
$$

It is shown f_i and h_i, up to total derivatives, are connected by phase relations $e^{\frac{2\pi i n}{3}}$ ($n = 0, 1, 2$) which is expected from the structure in (4.3.4). The diagonal connection is summarized as

$$
A_{\text{diag}}(z) = \begin{pmatrix} e^{-\frac{i2\pi}{3}} f(z, e^{\frac{i2\pi}{3}}\epsilon) + d(*) & 0 & 0 \\ 0 & e^{-\frac{i4\pi}{3}} f(z, e^{\frac{i4\pi}{3}}\epsilon) + d(*) & 0 \\ 0 & 0 & f(z,\epsilon) \end{pmatrix}, \tag{4.3.37}
$$

where $d(*)$ denotes total derivatives. The order of the three diagonal elements is not fixed but determined by their lowest order in ϵ. For instance, the first two diagonal elements will be exchanged if one takes $h_0 = e^{\frac{2\pi i}{3}} f_0$ instead of $h_0 = e^{-\frac{2\pi i}{3}} f_0$.

After solving the three independent linear equations $(\epsilon\partial + A_{\rm diag})\Psi' = 0$, one obtains the solution of the linear problem by the gauge transformation $\Psi = T\Psi'$.

Finally, there is one comment on $u_i(z)$. The relations between $u_i(z)$ and $\partial_z\phi_i(z)$ are obtained by the Miura transformation. There exists another equivalent canonical Lax operator in terms of $u_i(z)$:

$$\epsilon\mathcal{L}_{\rm can} = \epsilon\partial_z + \sum_{i=1}^{r} \epsilon^{i+1} u_{1+i}(z) e_{i+1,1} + \sum_{i=1}^{r} E_{\alpha_i} + p(z)E_{\alpha_0} \tag{4.3.38}$$

with $(e_{i,j})_{a,b} = \delta_{i,a}\delta_{j,b}$. One can obtain the same Riccati equation after the same diagonalization method. In Section 4.4, we will discuss the canonical Lax operator in detail and utilize it to derive the WKB solutions from conserved densities. Before that, let us generalize the diagonalization to other affine Toda field theory types in the following subsection and Section 4.3.2.

Generalization to $A_r^{(1)}$

In the last two subsections, we only show the diagonal results of $A_1^{(1)}$ and $A_2^{(1)}$, but the method can be applied to general $A_r^{(1)}$ directly. So far, we have checked the results until $A_5^{(1)}$ case. Let us consider the linear problem for $A_r^{(1)}$ in the fundamental representation with dimension $h = r + 1$. After the step-by-step diagonalization procedure for the connection $A(z)$, the functions $g_{i,j}$ $(i > j)$ in the gauge transformations obey a set of non-linear equations, which reduce to the Riccati equations of the diagonal components in $A_{\rm diag}$. The bottom component of the diagonal element $A_{\rm diag}(z)$ obeys the Riccati equation of the ordinary differential equations

$$(-\epsilon)^h(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2 + \partial_z\phi_1)\cdots(\partial_z + \partial_z\phi_r)\psi(z,\epsilon) = p(z)\psi(z,\epsilon), \tag{4.3.39}$$

which is adjoint of the $A_r^{(1)}$-type in Eqs.(4.3.2). One can solve the set of Riccati equations perturbatively as we did in the $A_2^{(1)}$ case. It turns out that other diagonal elements in the connection can be summarized as the phase rotation of ϵ with one of the eigenvalues of Λ_0: $\{1, e^{\frac{2\pi i}{h}}, e^{\frac{4\pi i}{h}}, \dots, e^{\frac{2\pi i r}{h}}\}$ up to total derivatives. Then the diagonalized connection $A_{\mathbf{diag}}$ can be of the form

$$\mathbf{Diag}\{e^{-\frac{2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \dots, e^{-\frac{2\pi i r}{h}} f(z, e^{\frac{2\pi i r}{h}}\epsilon) + d(*), f(z,\epsilon)\}. \tag{4.3.40}$$

As we have mentioned in $A_2^{(1)}$ type, the phase in front of a diagonal element is still not fixed but determined by the lowest order in ϵ.

The traceless condition does not change up to total derivatives after the gauge transformation:

$$\sum_{n=0}^{h-1} e^{-\frac{2\pi i n}{h}} f(x, e^{\frac{2\pi i n}{h}}\epsilon) = d(*), \tag{4.3.41}$$

which implies that the $(1+hk)$-th term ($k \in \mathbf{Z}$) should be the total derivative and it agrees with the results in [83].

4.3.2 Diagonalization of other linear problems

Since our diagonalization approach is quite general, we can apply it to the linear problems associated with other classical affine Lie algebras and study their WKB solutions. Especially, the algebras $D_r^{(1)}$ and $D_{r+1}^{(2)}$ are interesting since they correspond to the ODEs including a pseudo-differential operator, while generalizations to $B_r^{(1)}$, $A_{2r-1}^{(2)}$, $A_{2r}^{(2)}$, and $C_r^{(1)}$ is straightforward. In this section, we will diagonalize $D_r^{(1)}$-, $B_r^{(1)}$-, $D_{r+1}^{(2)}$-, and $A_{2r-1}^{(2)}$-type linear problems for lower-rank cases explicitly. The diagonalization can be applied directly except for $D_r^{(1)}$ where there are two sets of extra conserved densities.

Let us return the modified Lax operator (4.1.18) for an affine Lie algebra $\hat{\mathfrak{g}}$. After a series of gauge transformations, one obtains the diagonalized connection $A_{\mathrm{diag}}(z)$. We denote $f(z,\epsilon)$ as the bottom component of diagonal elements of $A_{\mathrm{diag}}(z)$, which is expanded as

$$f(z,\epsilon) = \sum_{i=0}^{\infty} \epsilon^i f_i(z). \tag{4.3.42}$$

In the last section, we have seen that the diagonal element $f(z,\epsilon)$ for $A_r^{(1)}$ is related to the WKB solution $P(z,\epsilon)$ to the adjoint ODE (4.3.39). One of the goals of this section is to check the equality $f(z,\epsilon) = P(z,\epsilon)$ for other affine Lie algebras. For $B_r^{(1)}$ and $A_{2r-1}^{(2)}$ types, it is possible to find WKB solutions directly, and the equality can be checked. For $D_r^{(1)}$ and $D_{r+1}^{(2)}$ types, the diagonal elements may lead to the WKB solutions to the ODEs including a pseudo-differential operator, which will also be confirmed after taking specific limits. The adjoint ODEs of $D_r^{(1)}$, $B_r^{(1)}$, $D_{r+1}^{(2)}$, and $A_{2r-1}^{(2)}$-type differential equations (4.3.2) are defined by $\partial_z \to -\partial_z$ and reversing the order of the action of differentials.

The form of $\sum_{i=1}^{r}\partial_z\phi_i(z,\bar{z})H_i$ in Eq. (4.1.18) in our representation is a little bit tedious for other affine Lie algebras. We make the following simplification:

$$\boldsymbol{\phi}_i = \begin{cases}\phi_1, & i=1,\\ \phi_i-\phi_{i-1}, & 2\le i\le r-2,\end{cases}$$
$$\boldsymbol{\phi}_{r-1} = \begin{cases}\phi_{r-1}+\phi_r-\phi_{r-2}, & D_r^{(1)},\\ \phi_{r-1}-\phi_{r-2}, & \text{others},\end{cases}\quad \boldsymbol{\phi}_r = \begin{cases}2\phi_r-\phi_{r-1}, & B_r^{(1)},\ D_{r+1}^{(2)},\\ \phi_r-\phi_{r-1}, & \text{others}.\end{cases} \tag{4.3.43}$$

Define $D(\boldsymbol{\phi}_i)\equiv\partial_z+\partial_z\boldsymbol{\phi}_i$ so that $\partial_z+\sum_{i=1}^{r}\partial_z\phi_i(z,\bar{z})H_i$ in Eq. (**??**) and differential equations (4.1.18) can be rewritten in terms of $D(\boldsymbol{\phi}_i)$.

Diagonalization of $A_{2r-1}^{(2)}$

First, we begin with the WKB solution for the $A_{2r-1}^{(2)}$ linear problem in the $2r$-dimensional representation given in Appendix C. Λ in $2r$-dimensional representation is given by

$$\Lambda = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & & \ddots & & \\ 1 & & & & 0 & 1 \\ & 1 & & & & 0 \end{pmatrix}, \tag{4.3.44}$$

which has the eigenvalues: $\{0,1,e^{\frac{2\pi i}{h}},e^{\frac{4\pi i}{h}},\ldots,e^{\frac{2\pi i(2r-2)}{h}}\}$ with $h=2r-1$. The higher-order ordinary differential equation in Eq. (4.3.2) becomes

$$\epsilon^h D(-\boldsymbol{\phi}_1)\cdots D(-\boldsymbol{\phi}_r)D(\boldsymbol{\phi}_r)\cdots D(\boldsymbol{\phi}_1)\psi(z,\epsilon)+2\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi(z,\epsilon)=0. \tag{4.3.45}$$

We apply the diagonalization of $A(z)$ from the bottom row with the gauge transformation T_{2r}. This implies the diagonalization conditions for $g_{2r,i}$'s:

$$\mathbf{Gau}_{T_{2r}}[A(z)]_{2r,1}=\cdots=\mathbf{Gau}_{T_{2r}}[A(z)]_{2r,2r-1}=0. \tag{4.3.46}$$

$\mathbf{Gau}_{T_{2r}}[A(z)]_{2r,i}=0$ $(2\le i\le 2r-1)$ eliminate $g_{2r,i}$ in terms of $g_{2r,2r-1}$. Then $\mathbf{Gau}_{T_{2r}}[A(z)]_{2r,1}$ $=0$ determines the Riccati equation for $g_{2r,2r-1}$. Expanding $g_{2r,2r-1}=\sum_{i=0}^{\infty}(g_{2r,2r-1})_i\epsilon^i$, $(g_{2r,2r-1})_0$ satisfy

$$[(g_{2r,2r-1})_0(z)]^{2r}-2(g_{2r,2r-1})_0(z)p(z)=0. \tag{4.3.47}$$

Choosing a solution $(g_{2r,2r-1})_0 = (2p)^{1/h}$, we can solve $g_{2r,2r-1}$ perturbatively in ϵ. The bottom diagonal component is given by $f(z,\epsilon) = -g_{2r,2r-1} - \epsilon\boldsymbol{\phi}_1'$. The further diagonalization steps are similar and the final diagonal connection $A_{\text{diag}}(z)$ becomes

$$\mathbf{Diag}\{h(z,\epsilon) + d(*), e^{-\frac{2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i(h-1)}{h}} f(z, e^{\frac{2\pi i(h-1)}{h}}\epsilon) + d(*), f(z,\epsilon)\}, \tag{4.3.48}$$

where we fix $h(z,\epsilon)$ to be the top element determined by the traceless condition

$$\sum_{n=0}^{2r-2} e^{-\frac{2\pi i n}{h}} f(z, e^{\frac{2\pi i n}{h}}\epsilon) + h(z,\epsilon) = d(*), \tag{4.3.49}$$

and $f(z,\epsilon)$ obeys the Riccati equation for the adjoint of the ODE (4.3.45):

$$\epsilon^h D(-\boldsymbol{\phi}_1)\cdots D(-\boldsymbol{\phi}_r)D(\boldsymbol{\phi}_r)\cdots D(\boldsymbol{\phi}_1)\psi(z,\epsilon) - 2\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi(z,\epsilon) = 0 \tag{4.3.50}$$

with $\psi = \exp\big(\frac{1}{\epsilon}\int f(z,\epsilon)dz\big)$. The order of diagonal elements in Eq. (4.3.48) follows the convention in the $A_r^{(1)}$ type.

We now present the simplest example, the $A_3^{(2)}$ case. The $f(z,\epsilon)$ for the first five orders are listed below

$$\begin{aligned}
f_0(z) &= (2)^{\frac{1}{3}} p(z)^{\frac{1}{3}}, \\
f_1(z) &= -\frac{3}{2}\partial_z \ln f_0, \\
f_2(z) &= \frac{5f_0''}{3{f_0}^2} - \frac{5{f_0'}^2}{2{f_0}^2} + \frac{f_0'' u_2}{3f_0}, \\
f_3(z) &= -\partial_z\Big(\frac{u_2}{3f_0^2} - \frac{5\,(f_0')^{\,2}}{2f_0^4} + \frac{5f_0''}{3f_0^3}\Big), \\
f_4(z) &= \frac{109u_2\,(f_0')^{\,2}}{36f_0^5} - \frac{29f_0'u_2'}{18f_0^4} - \frac{17u_2f_0''}{18f_0^4} + \frac{955\,(f_0')^{\,2}f_0''}{12f_0^6} - \frac{955\,(f_0')^{\,4}}{16f_0^7} \\
&\quad - \frac{145{f_0}^{(3)}f_0'}{9f_0^5} - \frac{125\,(f_0'')^{\,2}}{12f_0^5} + \frac{29{f_0}^{(4)}}{18f_0^4} - \frac{2u_2'' + 3u_4}{9{f_0}^3},
\end{aligned} \tag{4.3.51}$$

with

$$\begin{aligned}
u_2(z) &= {\boldsymbol{\phi}_1'}^2 + {\boldsymbol{\phi}_2'}^2 - 3\boldsymbol{\phi}_1'' - \boldsymbol{\phi}_2'', \\
u_4(z) &= {\boldsymbol{\phi}_1'}^2\boldsymbol{\phi}_2'' + 2\boldsymbol{\phi}_2'\boldsymbol{\phi}_1'\boldsymbol{\phi}_2'' + {\boldsymbol{\phi}_2'}^2\boldsymbol{\phi}_1'' - {\boldsymbol{\phi}_2'}^2{\boldsymbol{\phi}_1'}^2 + {\boldsymbol{\phi}_1}^{(3)}\boldsymbol{\phi}_1' - {\boldsymbol{\phi}_2}^{(3)}\boldsymbol{\phi}_1' \\
&\quad - \boldsymbol{\phi}_1''\boldsymbol{\phi}_2'' - {\boldsymbol{\phi}_1}^{(4)},
\end{aligned} \tag{4.3.52}$$

and $h(z,\epsilon) = \sum_{i=0}^{\infty}\epsilon^i h_i(z)$ can be given by

$$h_i(z) = \begin{cases} -3f_i(z), & \text{if} \quad i = 1 + (2r-1)k,\ k \in \mathbf{Z}, \\ \ 0, & \text{otherwise.} \end{cases}$$

Diagonalization of $B_r^{(1)}$

Next, we study the $B_r^{(1)}$ type linear problem in the $(2r+1)$-dimensional representation. From the Appendix C, Λ is given by

$$\Lambda = \begin{pmatrix} 0 & 1 & & & & & & & & \\ & 0 & 1 & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 0 & 1 & & & & & \\ & & & & 0 & \sqrt{2} & & & & \\ & & & & & 0 & \sqrt{2} & & & \\ & & & & & & 0 & 1 & & \\ & & & & & & & \ddots & & \\ 1 & & & & & & & & 0 & 1 \\ & 1 & & & & & & & & 0 \end{pmatrix}, \tag{4.3.53}$$

whose eigenvalues are $2^{\frac{1}{r}}\{0, 1, e^{\frac{2\pi i}{h}}, e^{\frac{4\pi i}{h}}, \ldots, e^{\frac{2\pi i(2r-1)}{h}}\}$ with $h = 2r$. The higher-order ordinary differential equation (4.3.2) obtained from the linear problem is

$$\epsilon^h D(-\boldsymbol{\phi}_1)\cdots D(-\boldsymbol{\phi}_r)\partial_z D(\boldsymbol{\phi}_r)\cdots D(\boldsymbol{\phi}_1)\psi(z,\epsilon) - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi(z,\epsilon) = 0. \tag{4.3.54}$$

The diagonalization approach can be applied without modification. $\mathbf{Gau}_{T_{2r+1}}[A(z)]_{2r+1,i} = 0$ $(2 \le i \le 2r)$ eliminate $g_{2r+1,i}$ in terms of $g_{2r+1,2r}$. Then $\mathbf{Gau}_{T_{2r+2}}[A(z)]_{2r+1,1} = 0$ determines the Riccati equation for $g_{2r+1,2r}$. Expanding $g_{2r+1,2r} = \sum_{i=0}^{\infty}(g_{2r+1,2r})_i\epsilon^i$, $(g_{2r+1,2r})_0$ is shown to satisfy

$$[(g_{2r+1,2r})_0(z)]^{2r+1} - 4(g_{2r+1,2r})_0(z)p(z) = 0. \tag{4.3.55}$$

Here we will choose a solution $(g_{2r+1,2r})_0(z) = (4p(z))^{\frac{1}{h}}$. The bottom diagonal component is given by $f(z,\epsilon) = -g_{2r+1,2r} - \epsilon\boldsymbol{\phi}_1'$ which is the solution to the Riccati equation of the ODE (4.3.54) with $\psi = \exp\left(\frac{1}{\epsilon}\int f(z,\epsilon)dz\right)$. Note that the ODE (4.3.54) is self-adjoint. The diagonalized connection A_{diag} can be summarized as

$$\mathbf{Diag}\{d(*), e^{-\frac{2\pi i}{h}}f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i(2r-1)}{h}}f(z, e^{\frac{2\pi i(2r-1)}{h}}\epsilon) + d(*), f(z,\epsilon)\}, \tag{4.3.56}$$

where we choose the top element such that it corresponds to 0 eigenvalue of Λ. The form (4.3.56) has been checked for $r \le 4$. The traceless condition implies that $f_{1+2rk}(z)$ $(k \in \mathbf{Z})$ are total derivatives.

Here are the first five nonzero terms for $B_2^{(1)}$ case.

$$
\begin{aligned}
f_0(z) &= -\sqrt{2}p(z)^{\frac{1}{4}}, \\
f_1(z) &= -2\partial_z \ln f_0, \\
f_2(z) &= -\frac{u_2}{4f_0} - \frac{15\,(f_0')^{\,2}}{4f_0^3} + \frac{5f_0''}{2f_0^2}, \\
f_3(z) &= \partial_z\Big(\frac{u_2}{4f_0^2} + \frac{15\,(f_0')^{\,2}}{4f_0^4} - \frac{5f_0''}{2f_0^3}\Big), \\
f_4(z) &= -\frac{33u_2\,(f_0')^{\,2}}{16f_0^5} + \frac{9f_0'u_2'}{8f_0^4} + \frac{5u_2f_0''}{8f_0^4} + \frac{885\,(f_0')^{\,2}f_0''}{8f_0^6} \\
&\quad - \frac{2655\,(f_0')^{\,4}}{32f_0^7} - \frac{45{f_0}^{(3)}f_0'}{2f_0^5} - \frac{115\,(f_0'')^{\,2}}{8f_0^5} + \frac{9{f_0}^{(4)}}{4f_0^4} + \frac{u_2^2+8u_4}{32{f_0}^3},
\end{aligned}
\tag{4.3.57}
$$

with

$$
\begin{aligned}
u_2(z) &= -{\boldsymbol{\phi}_1'}^2 - {\boldsymbol{\phi}_2'}^2 + 4\boldsymbol{\phi}_1'' + 2\boldsymbol{\phi}_2'', \\
u_4(z) &= 2{\boldsymbol{\phi}_1'}^2\boldsymbol{\phi}_2'' + 2\boldsymbol{\phi}_2'\boldsymbol{\phi}_1'\boldsymbol{\phi}_2'' + 2{\boldsymbol{\phi}_2'}^2\boldsymbol{\phi}_1'' - {\boldsymbol{\phi}_2'}^2{\boldsymbol{\phi}_1'}^2 + 3{\boldsymbol{\phi}_1}^{(3)}\boldsymbol{\phi}_1' - 2{\boldsymbol{\phi}_2}^{(3)}\boldsymbol{\phi}_1' \\
&\quad + {\boldsymbol{\phi}_2}^{(3)}\boldsymbol{\phi}_2' + \boldsymbol{\phi}_2''^2 - 4\boldsymbol{\phi}_1''\boldsymbol{\phi}_2'' - 4{\boldsymbol{\phi}_1}^{(4)} - {\boldsymbol{\phi}_2}^{(4)}.
\end{aligned}
\tag{4.3.58}
$$

The results agree with the WKB solutions obtained by direct calculations.

Diagonalization of $D_{r+1}^{(2)}$ So far, we discussed the linear problem associated with the higher order ODE, including the differential operator ∂_z. For the D-type linear problem, we need to introduce the pseudo-differential operator ∂_z^{-1} to reduce the linear problem to the single ODE, where we cannot apply the standard WKB approach. In the following two subsections, we will apply the diagonalization method to this problem. Let us first consider the $D_{r+1}^{(2)}$-type linear problem in the $(2r+2)$-dimensional representation. The Λ is

$$
\Lambda = \begin{pmatrix}
0 & 1 & & & & & & & & & & \\
 & 0 & 1 & & & & & & & & & \\
 & & & \ddots & & & & & & & & \\
 & & & & 0 & \sqrt{2} & & & & & & \\
 & & & & & 0 & & \sqrt{2} & & & & \\
\sqrt{2} & & & & & & 0 & & & & & \\
 & & & & & & & 0 & 1 & & & \\
 & & & & & & & & & \ddots & & \\
 & & & & & & & & & & 0 & 1 \\
 & & & & & & \sqrt{2} & & & & & 0
\end{pmatrix}.
\tag{4.3.59}
$$

The differential equation from the linear problem is also self-adjoint:

$$\epsilon^h D(-\boldsymbol{\phi}_1)\cdots D(-\boldsymbol{\phi}_r)\partial_z D(\boldsymbol{\phi}_r)\cdots D(\boldsymbol{\phi}_1)\psi(z,\epsilon) - 4p(z)\partial_z^{-1}p(z)\psi(z,\epsilon) = 0, \tag{4.3.60}$$

with $h = 2r+2$. We can apply the diagonalization method to the linear problem. The first step of the bottom-row diagonalization requires the conditions:

$$\mathbf{Gau}_{T_{2r+2}}[A(z)]_{2r+2,1} = \cdots = \mathbf{Gau}_{T_{2r+2}}[A(z)]_{2r+2,2r+1} = 0. \tag{4.3.61}$$

$\mathbf{Gau}_{T_{2r+2}}[A(z)]_{2r+2,i} = 0$ $(2 \leq i \leq 2r+1)$ eliminate $g_{2r+2,i}$ in terms of $g_{2r+2,2r+1}$. Then $\mathbf{Gau}_{T_{2r+2}}[A(z)]_{2r+2,1} = 0$ determines the Riccati equation for $g_{2r+2,2r+1}$. Expanding $g_{2r+2,2r+1} = \sum_{i=0}^{\infty}(g_{2r+2,2r+1})_i\epsilon^i$, $(g_{2r+2,2r+1})_0$ satisfy

$$[(g_{2r+2,2r+1})_0(z)]^{2r+2} - 4p(z)^2 = 0. \tag{4.3.62}$$

Here we will choose a solution $(g_{2r+2,2r+1})_0(z) = (2p(z))^{\frac{1}{r+1}}$. The bottom diagonal component is given by $f(z,\epsilon) = -g_{2r+2,2r+1} - \epsilon\boldsymbol{\phi}_1'$. The further diagonalization steps are done in a similar way, and the final diagonal connection $A_{\mathrm{diag}}(z)$ is given by

$$\mathbf{Diag}\{e^{-\frac{2\pi i}{h}}f(z,e^{\frac{2\pi i}{h}}\epsilon) + d(*),\ldots,e^{-\frac{2\pi i(2r+1)}{h}}f(z,e^{\frac{2\pi i(2r+1)}{h}}\epsilon) + d(*), f(z,\epsilon)\}. \tag{4.3.63}$$

The traceless condition implies the $f_{1+(2r+2)k}(z)$ $(k \in \mathbf{Z})$ are total derivatives.

Let us discuss the simplest example $D_3^{(2)}$. The first five non-total derivative terms for $D_3^{(2)}$ are

$$\begin{aligned}
f_0(z) &= -\big(2p(z)\big)^{\frac{1}{3}},\\
f_1(z) &= -2\partial_z \ln f_0,\\
f_2(z) &= -\frac{u_2}{6f_0} - \frac{5\,(f_0')^{\,2}}{2f_0^3} + \frac{5f_0''}{3f_0^2},\\
f_3(z) &= 0,\\
f_4(z) &= \frac{61u_2\,(f_0')^{\,2}}{36f_0^5} - \frac{7f_0'u_2'}{9f_0^4} - \frac{11u_2f_0''}{18f_0^4} - \frac{475\,(f_0')^{\,2}f_0''}{6f_0^6}\\
&\quad + \frac{475\,(f_0')^{\,4}}{8f_0^7} + \frac{140{f_0}^{(3)}f_0'}{9f_0^5} + \frac{65\,(f_0'')^{\,2}}{6f_0^5} - \frac{14{f_0}^{(4)}}{9f_0^4} + \frac{3u_2^2 + 12u_4 + 22u_2''}{72S_0^4},
\end{aligned} \tag{4.3.64}$$

with

$$\begin{aligned}
u_2(z) &= -{\boldsymbol{\phi}_1'}^2 - {\boldsymbol{\phi}_2'}^2 + 4\boldsymbol{\phi}_1'' + 2\boldsymbol{\phi}_2'',\\
u_4(z) &= 2{\boldsymbol{\phi}_1'}^2\boldsymbol{\phi}_2'' + 2\boldsymbol{\phi}_2'\boldsymbol{\phi}_1'\boldsymbol{\phi}_2'' + 2{\boldsymbol{\phi}_2'}^2\boldsymbol{\phi}_1'' - {\boldsymbol{\phi}_2'}^2{\boldsymbol{\phi}_1'}^2 + 3{\boldsymbol{\phi}_1}^{(3)}\boldsymbol{\phi}_1' - 2{\boldsymbol{\phi}_2}^{(3)}\boldsymbol{\phi}_1'\\
&\quad + {\boldsymbol{\phi}_2}^{(3)}\boldsymbol{\phi}_2' + \boldsymbol{\phi}_2''^2 - 4\boldsymbol{\phi}_1''\boldsymbol{\phi}_2'' - 4{\boldsymbol{\phi}_1}^{(4)} - {\boldsymbol{\phi}_2}^{(4)}.
\end{aligned} \tag{4.3.65}$$

The perturbative results for $f(z,\epsilon)$ here agree with those for $P_1(z,\epsilon)$ in the case of $\phi_i(z) = 0$. See Appendix C.

Diagonalization of $D_r^{(1)}$

Finally, we study $D_r^{(1)}$ case. Here we consider the linear problem in the $2r$-dimensional vector representation of D_r. The matrix Λ in Eq.(4.3.3) is given by

$$\Lambda = \begin{pmatrix} 0 & 1 & & & & & & & & \\ & 0 & 1 & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 0 & 1 & 1 & & & & \\ & & & & 0 & & 1 & & & \\ & & & & & 0 & 1 & & & \\ & & & & & & 0 & 1 & & \\ & & & & & & & \ddots & & \\ 1 & & & & & & & & 0 & 1 \\ & 1 & & & & & & & & 0 \end{pmatrix}, \tag{4.3.66}$$

with the eigenvalues: $2^{\frac{1}{r-1}}\{0,0,1,e^{\frac{2\pi i}{h}},e^{\frac{4\pi i}{h}},\dots,e^{\frac{2\pi i(h-1)}{h}}\}$. Due to the existence of double zero eigenvalues in Λ, we introduce a matrix F commuting with Λ, which is given by

$$F = ie_{1,r} - ie_{1,r+1} - ie_{r,1} + ie_{r+1,1} + ie_{r,2r} - ie_{r+1,2r} - ie_{2r,r} + ie_{2r,r+1}. \tag{4.3.67}$$

The matrix F with eigenvalues $\{-2,2,0,0,0,\dots,0\}$ is nothing but the one that appears in Eq.(4.3.12). It leads to a new set of conserved densities. The corresponding ODE associated with the linear model is given by [55]. It takes the form of

$$\epsilon^h D(-\boldsymbol{\phi}_1)\cdots D(-\boldsymbol{\phi}_r)\partial_z^{-1} D(\boldsymbol{\phi}_r)\cdots D(\boldsymbol{\phi}_1)\psi(z,\epsilon) - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi(z,\epsilon) = 0, \tag{4.3.68}$$

with $h = 2r-2$. This equation includes the pseudo-differential operator.

We apply the same diagonalization steps for $A(z)$ from the bottom row with the following conditions for $g_{2r,i}$:

$$\mathbf{Gau}_{T_{2r}}[A(z)]_{2r,1} = \cdots = \mathbf{Gau}_{T_{2r}}[A(z)]_{2r,2r-1} = 0. \tag{4.3.69}$$

These equations, however, cannot be expressed locally in terms of $g_{2r,2r-1}$ in contrast to our previous examples. The most concise way to solve the constraints is to express $g_{2r,i}$

in terms of $g_{2r,2r-1}$ and $g_{2r,r+1}$, which give rise to the two Riccati equations:

$$\mathbf{Gau}_{T_{2r}}[A(z)]_{2r,1}=0,\quad \mathbf{Gau}_{T_{2r}}[A(z)]_{2r,r}=0. \tag{4.3.70}$$

Substituting the expansions $g_{2r,2r-1}=\sum_{i=0}^{\infty}(g_{2r,2r-1})_i\epsilon^i$ and $g_{2r,r+1}=\sum_{i=0}^{\infty}(g_{2r,r+1})_i\epsilon^i$ to (4.3.70), one can solve Eqs. (4.3.70) recursively in ϵ. Here $(g_{2r,2r-1})_0$ and $(g_{2r,r+1})_0$ satisfy

$$2(g_{2r,r+1})_0(g_{2r,2r-1})_0-[(g_{2r,2r-1})_0]^r=0,\quad -4p(z)(g_{2r,2r-1})_0+(g_{2r,r+1})_0[(g_{2r,2r-1})_0]^r=0. \tag{4.3.71}$$

Then $(g_{2r,2r-1})_0$ satisfies $(g_{2r,2r-1})_0=0$ or $4p(z)=[(g_{2r,2r-1})_0]^h$. Solving Eqs. (4.3.70), one obtains the bottom diagonal element $f(z,\epsilon)=-g_{2r,2r-1}-\epsilon\phi_1'$. The i-th row elements can be diagonalized similarly for $r+2\le i\le 2r$. For $i\le r+1$, the procedure can be reduced to solve one single Riccati equation for $g_{i,i-1}$.

We have performed the diagonalization procedure and obtained $A_{\rm diag}(z)$ for $r\le 4$, where we have chosen $(g_{2r,2r-1})_0=(4p(z))^{1/h}$. The diagonalized connection can be summarized as

$$\begin{aligned}&\mathbf{Diag}\{e^{\frac{-2\pi i}{h}}f(z,e^{\frac{2\pi i}{h}}\epsilon)+d(*),\ldots,e^{\frac{-2\pi i(r-1)}{h}}f(z,e^{\frac{2\pi i(r-1)}{h}}\epsilon)+d(*),\,e^{i\pi(r-1)}K(z,-\epsilon)+d(*),\\&K(z,\epsilon),\,e^{-\frac{2\pi ir}{h}}f(z,e^{\frac{2\pi ir}{h}}\epsilon)+d(*),\ldots,e^{-\frac{2\pi i(2r-1)}{h}}f(z,e^{\frac{2\pi i(2r-1)}{h}}\epsilon)+d(*),\,f(z,\epsilon)\}.\end{aligned} \tag{4.3.72}$$

The elements $K(z,\epsilon)$ (resp. $K(z,-\epsilon)$) corresponding to eigenvalues ± 2 of F are obtained from the $(r+1)$ (resp. r)-th diagonalization conditions

$$\begin{aligned}&\mathbf{Gau}_{T_{r+1}\ldots T_{2r}}[A(z)]_{r+1,i}=0,\ 1\le i\le 2r,\ i\ne r+1,\\&\mathbf{Gau}_{T_r\ldots T_{2r}}[A(z)]_{r,i}=0,\ 1\le i\le 2r,\ i\ne r.\end{aligned} \tag{4.3.73}$$

$\mathbf{Gau}_{T_{r+1}\ldots T_{2r}}[A(z)]_{r+1,1}=0$ (resp. $\mathbf{Gau}_{T_r\ldots T_{2r}}[A(z)]_{r,1}=0$) determines the Riccati equations for $g_{r+1,r}$ (resp. $g_{r.r-1}$). $K(z,\epsilon)$ (resp. $e^{-\frac{2\pi ir}{h}}K(z,-\epsilon)$) are namely the $r+1$ (resp. r)-th diagonal elements in terms of $g_{r+1,r}(g_{r.r-1})$ up to total derivatives. The expansion of $K(z,\epsilon)=\sum_{i=0}^{\infty}\epsilon^iK_i(z)$ corresponds to the $J_i(z)$ terms in Eq.(4.3.12). Note that the traceless condition of $A_{\rm diag}$ implies the $f_{1+(2r-2)k}(z)$ ($k\in\mathbf{Z}$) are total derivatives.

We now give the simplest example $D_3^{(1)}$ in the six-dimensional representation. This is equivalent to the $A_3^{(1)}$ in the same dimensional representation. The first five nonzero terms of $D_3^{(1)}$ are given by

$$\begin{aligned}
f_0(z) &= -\sqrt{2}p(z)^{\frac{1}{4}},\\
f_1(z) &= -2\partial_z \ln f_0,\\
f_2(z) &= -\frac{u_2}{4f_0} - \frac{15\,(f_0')^{\,2}}{4f_0^3} + \frac{5f_0''}{2f_0^2},\\
f_3(z) &= \partial_z\big(\frac{u_2}{4f_0^2} + \frac{15\,(f_0')^{\,2}}{4f_0^4} - \frac{5f_0''}{2f_0^3}\big),\\
f_4(z) &= -\frac{33u_2\,(f_0')^{\,2}}{16f_0^5} + \frac{9f_0'u_2'}{8f_0^4} + \frac{5u_2f_0''}{8f_0^4} + \frac{885\,(f_0')^{\,2}f_0''}{8f_0^6}\\
&\quad - \frac{2655\,(f_0')^{\,4}}{32f_0^7} - \frac{45{f_0}^{(3)}f_0'}{2f_0^5} - \frac{115\,(f_0'')^{\,2}}{8f_0^5} + \frac{9{f_0}^{(4)}}{4f_0^4} + \frac{u_2^2+8u_4}{32{f_0}^3},\\
K_3(z) &= \frac{v_3}{{f_0}^2},
\end{aligned} \tag{4.3.74}$$

with

$$\begin{aligned}
u_2(z) &= -{\phi_1'}^2 - {\phi_2'}^2 - {\phi_3'}^2 + 4\phi_1'' + 2\phi_2'',\\
u_4(z) &= 2{\phi_1'}^2\phi_2'' + 2\phi_2'\phi_1'\phi_2'' + 2\phi_3'\phi_1'\phi_3'' + 2{\phi_2'}^2\phi_1'' + 2{\phi_3'}^2\phi_1'' + 2\phi_2'\phi_3'\phi_3''\\
&\quad - {\phi_3'}^2{\phi_1'}^2 + 3{\phi_1}^{(3)}\phi_1' - 2{\phi_2}^{(3)}\phi_1' - {\phi_1'}^2{\phi_2'}^2 - {\phi_2'}^2{\phi_3'}^2 + {\phi_2}^{(3)}\phi_2'\\
&\quad + {\phi_3}^{(3)}\phi_3' + \phi_2''^2 - 4\phi_1''\phi_2'' - 4{\phi_1}^{(4)} - {\phi_2}^{(4)},\\
v_3(z) &= \phi_3'\phi_2'' + \phi_1'\phi_3'' + \phi_2'\phi_3'' - \phi_1'\phi_2'\phi_3' - {\phi_3}^{(3)},
\end{aligned} \tag{4.3.75}$$

where $K_3(z)$ vanishes when $\phi_3 = 0$. It can be understood from the fact that the pseudo-differential operator is canceled in Eq. (4.3.68), and the ODE is reduced to $B_{r-1}^{(1)}$ type for $\phi_3 = 0$. The diagonalized connection obtained here confirms this relation. One may notice the diagonal elements share the same form between $B_{r-1}^{(1)}$ and $D_r^{(1)}$ types and the difference only appears in the construction of u_i.

Remarks on $f(z,\epsilon)$

The diagonal element $f(z,\epsilon)$ in $A_{\mathrm{diag}}(z)$ for $D_r^{(1)}$, $B_r^{(1)}$, $D_{r+1}^{(2)}$ and $A_{2r-1}^{(2)}$ shares the following common properties.

First, during the diagonalization of the linear problem for an affine Lie algebra $\hat{\mathfrak{g}}$ except $A_r^{(1)}$, we observe that all of $f_{2i-1}(z)$ ($i \in \mathbb{N}$) are total derivatives. It is explained by Eq. (4.3.9) where $H(z,\epsilon)$ is expanded in terms of $\Lambda^{-(2i-1)}$ ($i \in \mathbb{N}$) and the coefficients in Λ^{-2i}

are total derivatives. However, this is not true for $A_r^{(1)}$ type, where Eq.(4.3.4) admits the even terms Λ^{-2i}, and $f_{2i-1}(z)$ are non-zero in general.

Second, we can consider the $\phi_i = 0$ case, where $D(\phi_i)$ becomes the differential operator ∂, and the ODE becomes simple. In this case, it is possible to study higher-order corrections for an affine Lie algebra with higher ranks. The diagonal elements $f_i(z)$ are uniquely determined up to total derivatives. For the practical computation of the WKB periods, where the total derivatives are not relevant, it is convenient to express them in a simple form. It is possible to make the following simplification:

$$f_i(f_0, f_0', f_0'', \cdots) = S_i(f_0, f_0'', \cdots) + d(*),$$

where S_i is independent of f_0'. All the f_0' terms are absorbed into total derivatives $d(*)$. We found that $S_i(z)$ can be fixed by $S_0 = f_0$. The details for $A_r^{(1)}$, $B_r^{(1)}$, $D_r^{(1)}$, $D_{r+1}^{(2)}$, and $A_{2r-1}^{(2)}$ are summarized in Appendix C.

4.4 Classical conserved densities and WKB solutions

In Sections 4.3 and 4.3.2, we have studied the WKB solutions of the linear problem for a classical affine Lie algebra $\hat{\mathfrak{g}}$ from diagonalizing the connection $A(z)$. The diagonalized connection $A_{\mathbf{diag}}(z)$ has a structure similar to the conserved densities which appear in the construction of the generalized KdV hierarchies associated with $\hat{\mathfrak{g}}$. We will check the correspondence between the coefficients in the WKB expansion and the conserved densities for the $A_r^{(1)}$-type linear problem, and we believe it also holds for other affine Lie algebras.

Continuity equations from the KdV hierarchies

First, let us review the generalized KdV hierarchies associated with the Lax operator $\mathcal{L}$ in Eq. (4.1.18) for an affine Lie algebra $A_r^{(1)}$ [21]. The adjoint ordinary differential equation obtained from the linear problem is given in Eq.(4.3.39), where the operator on the left-hand side is nothing but the scalar Lax operator for the modified KdV hierarchies.

$$L_{\text{scalar}} = (\partial_x - \partial_x\phi_1)\cdots(\partial_x + \partial_x\phi_2 - \partial_x\phi_1)(\partial_x + \partial_x\phi_r). \tag{4.4.1}$$

Here we have changed the coordinates $(z, \bar{z})$ into (x, t). The coordinates x and t are real in this setup. The operator (4.4.1) can be written in the canonical form:

$$L_{\mathrm{can}} = \partial_x^{r+1} - \sum_{i=0}^{r-1} u_{r+1-i}(x) \partial_x^i. \tag{4.4.2}$$

Here $u_i(x)$ is expressed in terms of ϕ_i's obtained by the Miura transformation. In the $A_1^{(1)}$ case, it leads to

$$u_2(x) = (\partial_x \phi_1)^2 - \partial_x^2 \phi_1, \tag{4.4.3}$$

which is the classical energy-momentum tensor in the sinh-Gordon model [18, 21]. It implies the deep relation between the KdV hierarchies and Toda field theories. The equivalence between $\mathcal{L}$ in Eq. (4.1.18) and the scalar L_{can} has been proved in [21] and can be found in Appendix A where we can see both the Lie algebraic and the scalar Lax operators satisfy the Lax equation. Let us first focus on the scalar one. The linear problem can be written as the equation $L\psi(x) = \lambda\psi(x)$ with λ the eigenvalue of L. Introducing time parameters t_i $(i = 1, 2, \dots)$ with $t_1 = t$, the integrable hierarchies are defined by the Lax equation

$$\partial_{t_i} L = [A_i, L], \tag{4.4.4}$$

with $A_i = (L^{\frac{i}{h}})_+$, where $(A)_+$ denotes the non-negative part in ∂_x of the differential operator A. Further acting ∂_{t_i} on $L\psi(x) = \lambda\psi(x)$, one can obtain $(L-\lambda)(\partial_{t_i}\psi - A_i\psi) = 0$ which implies, for some function $g(t_i)$,

$$\partial_{t_i}\psi(x) - A_i\psi(x) = g(t_i)\psi(x). \tag{4.4.5}$$

Substitute the WKB expansion $\psi(x, \epsilon) = \exp\left(\frac{1}{\epsilon}\int dx\, P(x, \epsilon)\right)$, it leads to

$$\partial_{t_i} P(x, \epsilon) - \partial_x a_i(x) = 0, \tag{4.4.6}$$

with $a_i(x, \epsilon) = \epsilon A_i \psi(x, \epsilon)/\psi(x, \epsilon)$. This equation implies $P(x, \epsilon)$ is the conserved density.

On the other hand, the integrable hierarchies defined by $\mathcal{L}$ are given by

$$\partial_t \mathcal{L} = [\mathcal{A}, \mathcal{L}], \tag{4.4.7}$$

where $\mathcal{A}(x) = \sum_{i=0}^{\infty} \mathcal{A}_i(x)(\lambda\Lambda)^{-i}$. After the diagonalization, the Lax operator $\mathcal{L}_{\mathrm{diag}}$ is the form of Eq. (4.3.8). One can obtain $\partial_t \mathcal{L}_{\mathbf{diag}} = [\mathcal{A}', \mathcal{L}_{\mathbf{diag}}]$ with $\mathcal{A}' = T^{-1}\mathcal{A}T$. This is nothing but the continuity equation

$$\partial_t f_i + \partial_x \mathcal{A}_i' = 0. \tag{4.4.8}$$

The functions $f_i(x)$ are also conserved densities. Since both $P(x,\epsilon)$ and the diagonal element $f(x,\epsilon)$ are conserved densities, it implies the equality $f(x,\epsilon) = P(x,\epsilon)$ up to total derivatives.

Conserved densities in $A_1^{(1)}$ and $A_2^{(1)}$ affine Toda field theories

We have seen the correspondence between the WKB expansions and the conserved densities for $A_r^{(1)}$ with $p(z) = 1$ case. Here we show the relations between WKB solutions and classical conserved densities with the conformal transformation. It is convenient to begin with the canonical Lax operator (4.3.38). Recall the appearance of $p(z)$: the conformal transformation Eq.(3.4.3) . It implies [49]

$$dw = \sqrt{p(z)}dz, \quad \hat{u}_2\big(w(z)\big) = \frac{1}{p(z)}\Big[u_2(z) + \frac{4pp'' - 5p'^2}{16p^2}\Big], \tag{4.4.9}$$

with $u_2(z) = \phi'(z)^2 - \phi''(z)$. Then the modified canonical Lax operator (4.3.38) for $A_1^{(1)}$ in the representation can be written as

$$\mathcal{L}_{\text{can}} = \epsilon\partial_w + e_{1,2} + \big(\epsilon^2 \hat{u}_2(w) + 1\big)e_{2,1}. \tag{4.4.10}$$

After the similar diagonalization procedure in Section 4.3.1, one can see the bottom diagonal element satisfies

$$\hat{f}(w,\epsilon)^2 + \epsilon\partial_w \hat{f}(w,\epsilon) - \epsilon^2 \hat{u}_2(w) - 1 = 0. \tag{4.4.11}$$

The expansion $\hat{f}(w,\epsilon) = \sum_{i=0}^{\infty} \epsilon^i f_i(w)$ can be calculated perturbatively. The first five terms are

$$\begin{aligned}
\hat{f}_0(w) &= 1, \\
\hat{f}_1(w) &= 0, \\
\hat{f}_2(w) &= \frac{\hat{u}_2(w)}{2}, \\
\hat{f}_3(w) &= -\frac{\partial_w \hat{u}_2(w)}{4}, \\
\hat{f}_4(w) &= \frac{\partial_w^2 \hat{u}_2(w) - \hat{u}_2^2(w)}{8},
\end{aligned} \tag{4.4.12}$$

which are the conserved densities up to total derivatives [18, 49, 97].[4] Substitute $\hat{u}(w)$ with Eq. (4.4.9) and multiply $\sqrt{p(z)}$, one can obtain the solution to Eq.(4.3.23) again. $\sqrt{p(z)}$ is from coordinate transformation in integral of motions: $\oint dw\ \hat{f}_i \to \oint dz\ \sqrt{p(z)} f_i$. In conclusion, Π_i defined below and conserved charges $\mathcal{Q}_i$ are related as follows:

$$\Pi_i \equiv \oint dz\ f_i(z) = \oint dz\ \sqrt{p(z)} \hat{f}_i = \oint dw\ \hat{f}_i \equiv \mathcal{Q}_i. \tag{4.4.13}$$

This implies that the WKB periods of the linear problem for an affine Toda field theory is a generating function of the classical conserved charges of the integrable hierarchies. It is known that for an affine Toda lattice, a finite-dimensional version of the present model, the integrals of motion are given by the period integral over the spectral curves. The contour integral $\sum_{i=0}^{\infty} \epsilon^i \Pi_i$ defined on the spectral curve can be viewed as the quantum Seiberg-Witten periods in $4d$ $\mathcal{N} = 2$ super Yang-Mills theories (SYMs). The equality here can be another evidence of the correspondence between SYMs and integrable models [34, 35, 98, 99]. See also [100] for the relation to the Gelfand-Dickii algebra.

Similar results can also be obtained from the $A_2^{(1)}$ affine Toda field equation. The generalized Miura transformation:

$$(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2 + \partial_z\phi_1)(\partial_z + \partial_z\phi_2) = \partial_z^3 - \sum_{i=0}^{2} u_{3-i}\partial_z^i \tag{4.4.14}$$

leads to the Miura transformation in Eq.(4.3.34). After the transformation [101]: $u_2(z) \to 2u_2(z)$, $u_3(z) \to u_3(z) + u_2'(z)$, $u_1(z)$ becomes the classical energy-momentum tensor and $u_3(z)$ becomes the additional classical local spin-3 field in $\mathcal{W}_3$ conformal field theory [42]. The conformal transformation leads to

$$p(z) = (\partial_z w)^3, \quad \hat{\phi}_i(w) = \phi_i(z) - \frac{1}{3}\log p(z). \tag{4.4.15}$$

The modified canonical Lax operator now becomes

$$\mathcal{L}_{\text{can}} = \epsilon\partial_w + \epsilon^2 u_2(z) e_{2,1} + (\epsilon^3 u_3(z) + 1)e_{3,1} + e_{1,2} + e_{2,3}. \tag{4.4.16}$$

The first diagonalization condition gives the Riccati equation of $\hat{f}(w, \epsilon)$:

$$\hat{f}^3 + 3\epsilon\hat{f}\hat{f}' - \epsilon^2\hat{u}_2\hat{f} + \epsilon^2\hat{f}'' - \epsilon^3\hat{u}_3 - 1 = 0, \tag{4.4.17}$$

[4] One may need $u_0 \to -u_0$ according to different convention in $L = \partial^2 \pm u$.

and the second diagonalization condition only leads to phase rotation. So the diagonal elements can be summarized by

$$\hat{A}(w) = \mathbf{Diag}\{e^{-\frac{2\pi}{3}}\hat{f}(w, e^{\frac{2\pi}{3}}\epsilon) + d(*), e^{-\frac{4\pi}{3}}\hat{f}(w, e^{\frac{4\pi}{3}}\epsilon) + d(*), \hat{f}(w,\epsilon)\}. \tag{4.4.18}$$

The first five perturbative solutions to Eq.(4.4.17) are nothing but the conserved densities up to total derivatives

$$\begin{aligned}
\hat{f}_0(w) &= 1,\\
\hat{f}_1(w) &= 0,\\
\hat{f}_2(w) &= \frac{\hat{u}_2(w)}{3},\\
\hat{f}_3(w) &= \frac{\hat{u}_3(w) - \partial_w \hat{u}_2(w)}{3},\\
\hat{f}_4(w) &= \frac{1}{9}\left(3\partial_w u_3(w) - \partial_w^2 u_2(w)\right),\\
\hat{f}_5(w) &= \frac{1}{9}\left(\hat{u}_3(w)\hat{u}_2(w) - 2\partial_w^2 \hat{u}_3(w) + \partial_w^3 \hat{u}_2(w)\right),
\end{aligned}$$

where $\hat{f}_4(w)$ is a total derivative. Substitute the classical conformal transformation Eq.(4.4.15) and multiply $\big(p(z)\big)^{\frac{1}{3}}$, one can obtain the same WKB solution to Eq.(4.3.35).

So far, we have shown the correspondence between WKB solutions and classical conserved densities. Although it is given in a fundamental representation (4.3.2), the conclusion is valid for general basis (3.4.23).

4.5 The ODE/IM correspondence in local integrals of motion

In this section, we will study the ODE/IM correspondence by comparing the eigenvalues of IoMs in $WA_r(WD_r)$ algebras with the contour integral of WKB solutions (the WKB integral for abbreviation) for $A_r^{(1)}(D_r^{(1)})$ modified affine Toda field equations in Section 4.2.1. Our result can be viewed as a generalization of Chapter 3 and [18, 42, 102], where the correspondence was studied for WA_1 and WA_2. Recall in Section 3.4 and the previous works [54, 56, 61], the ODE/IM correspondence was studied from the Non-Linear Integral Equations (NLIE) satisfied by the Q-functions and the connection coefficients of the solutions of the ODE (3.4.21). In particular, for the linear problem associated with an affine Lie algebra $\hat{\mathfrak{g}}$ with potential $p(z) = z^{hM} - E$, the effective central charge c_{eff} is evaluated as

$$c_{\text{eff}} = r - \frac{12}{h^2(M+1)}(l + \rho^\vee)^2. \tag{4.5.1}$$

Comparing with the formula $c_{\text{eff}} = c - 24\Delta_0$, where c is the central charge and Δ_0 is the conformal dimension of the vacuum state introduced in Section 2.3.3, we obtain the relation between the parameters in ODE and CFT:

$$c = r - 12\alpha_0^2\rho^2, \quad \Delta_0 = \frac{1}{2}\alpha_0 l \cdot (\alpha_0 l + \alpha_0 \rho) \tag{4.5.2}$$

with the correspondence

$$\alpha_0^2 = \frac{1}{h^2(M+1)}. \tag{4.5.3}$$

The effective central charge appears in non-unitary CFT with the normal central charge shifted by the vacuum energy Δ_0. It is also the eigenvalue of the IoM: $-\mathbf{I}_2/24$. Comparison between the higher IoMs and the integrals of the WKB solutions provides a further non-trivial check for the ODE/IM correspondence.

4.5.1 The ODE/IM correspondence for $A_r^{(1)}$-type ODEs

Let us first consider the ODE (4.2.9). The integrals of the WKB expansions Q_k ($k = 2, 3, \dots$) are given in Eq.(4.2.20). We compare these WKB integrals with the vacuum eigenvalue I_k of IoMs $\mathbf{I}_k$. Since we use the free field representation of W algebra, it is convenient to use momentum basis, where I_k can be expressed in terms of symmetric polynomials in p.

The WKB integral Q_2

Let us consider Q_2, which is expressed as

$$Q_2 = -h(M+1)J_{1,2}\left[-\frac{1}{h^2(M+1)}s_2 - \frac{r}{24}\right]. \tag{4.5.4}$$

Compared with

$$I_2 = -\sigma_2 - \frac{r}{24}, \tag{4.5.5}$$

it is found that

$$Q_2 = h(M+1)J_{1,2}\, I_2 \tag{4.5.6}$$

by identifying

$$s_2 = h^2(1+M)\sigma_2, \tag{4.5.7}$$

which implies

$$p = \frac{1}{h\sqrt{1+M}}(l+\rho). \tag{4.5.8}$$

in the momentum basis. Eq.(4.5.8) leads to the general formula

$$s_k = \left(h\sqrt{1+M}\right)^k \sigma_k \tag{4.5.9}$$

for $k \geq 2$.

The WKB integral Q_3

We will check the formula (4.5.9) for higher-order IoMs. After substituting L_3 from Eq.(4.2.11) into Eq.(4.2.20), the WKB integral Q_3 is given by

$$Q_3 = \frac{1}{h}J_{2,3}s_3. \tag{4.5.10}$$

Then from Eq.(4.5.9) and $I_3 = \sigma_3$ from Eq.(2.3.55), we find

$$Q_3 = h^2(1+M)^{\frac{3}{2}}J_{2,3}\, I_3. \tag{4.5.11}$$

Finally, particular attention should be given that s_k also vanishes when $k > h$. This condition is also applied to Q_k ($k > 3$) unless an explicit statement.

The WKB integral Q_4

The comparison of the fourth-order IoM is crucial since it includes a constant term. This relation leads to a new equation for the parameters. We note that the formula for I_4 and I_5 in Eq.(2.3.55) is only inferred from the low-rank computation ($r \leq 9$). So the correspondence predicts the IoMs for higher-rank WA_r algebra with $r \geq 10$. According to Eq.(4.2.11) and Eq.(4.2.20), we rewrite Q_4 in terms of s_i as

$$Q_4 = J_{3,4}\left[\frac{1}{h}s_4 - (h-3)\left(\frac{1}{2h^2}s_2^2 + \frac{1+M}{8h}s_2 - \frac{(1+M)^2}{1920}h^2(h-1)(\frac{2hM^2}{1+M} - 9)\right)\right]. \tag{4.5.12}$$

While I_4 in Eq.(2.3.55) in terms of σ_i becomes

$$I_4 = -\sigma_4 + \frac{r-2}{2(r+1)}\sigma_2^2 + \frac{r-2}{8(r+1)}\sigma_2 + \frac{(r-2)r(9-2(r+1)\alpha_0^2)}{1920(1+r)}.$$

After applying Eq.(4.5.9), we can see that the coefficients for σ_2^2 and σ_2 have already matched. The correspondence

$$Q_4 = h^3(1+M)^2 J_{3,4}\, I_4 \tag{4.5.13}$$

is satisfied when

$$\frac{1}{1920h}(h-3)(h-1)(9-2h\alpha_0^2) = \frac{1}{1920h}(h-3)(h-1)\left(9-\frac{2hM^2}{1+M}\right), \tag{4.5.14}$$

which is the same as the relation (4.5.3). Because $\alpha_0 = \beta - \beta^{-1}$, it is expressed as

$$M = \beta^{-2} - 1 \text{ or } M = \beta^2 - 1. \tag{4.5.15}$$

We will choose the first equality in the remaining part because of its appearance in [56], where the correspondence is shown from the Bethe ansatz equations.

Finally, based on the relations (2.3.25), (B.4.3) between Δ_i and σ_i, one can check the coefficient a_1 in Eq.(2.3.45) agrees with that we inferred in Eq.(2.3.46).

The WKB integral Q_5

Next, we will check the correspondence for the fifth-order IoMs. Substituting Eqs.(4.5.8) and (4.5.15) into Q_5 (4.2.20), one can obtain

$$Q_5 = J_{4,5}\left(\frac{1}{h}s_5 - \frac{h-4}{h^2}s_2 s_3 + \frac{1}{3}(h-4)(1+M)s_3\right). \tag{4.5.16}$$

From Eq.(4.5.8), one obtains

$$Q_5 = h^4\beta^{-5} J_{4,5}\, I_5, \tag{4.5.17}$$

where I_5 in Eq.(2.3.55) is given by

$$I_5 = \sigma_5 - \frac{r-3}{r+1}\sigma_2\sigma_3 - \frac{r-3}{3(r+1)}\sigma_3. \tag{4.5.18}$$

This relation also implies that the coefficient b_1 in Eq.(2.3.45) is valid for general r.

The WKB integral Q_6

Unlike the order $k \leq 5$ IoMs, we have only the formula of i_6 for $r \leq 3$ and do not find a general expression for general rank. On the other hand, for general rank r, the WKB integral Q_6 in terms of s_i is given by

$$Q_6 = J_{5,6}\left[\frac{1}{h}s_6 + (h-5)\left(\frac{1}{h}s_2s_4 + \frac{1}{2h}s_3^2 + \frac{2h-5}{6h^2}s_2^3 - \frac{5}{8}h(1+M)s_4 \right.\right.$$
$$\left.\left. + \frac{5}{48}(3h-7)(1+M)s_2^2 - h^3(1+M)^2c_6^{(1)}s_2 + h^5(1+M)^3c_6^{(2)}\right)\right]$$

with the coefficients $c_6^{(1)}$ and $c_6^{(2)}$

$$c_6^{(1)} = \frac{1}{1152h}[2h(5h-11)\frac{M^2}{(1+M)^2} - 91h + 205],$$
$$c_6^{(2)} = \frac{1}{580608h}(h-1)[8h^2(h^2+6h-19)\frac{M^4}{(1+M)^4} - 5(61h-139)(2h\frac{M^2}{(1+M)^2} - 5)].$$

After substituting the equation (4.5.8) and $M = \beta^{-2} - 1$, it becomes

$$Q_6 = -\frac{h^5}{\beta^6}J_{5,6}\left[-\sigma_6 - \frac{h-5}{h}\left(\sigma_2\sigma_4 + \frac{1}{2}\sigma_3^2 + \frac{2h-5}{6h}\sigma_2^3 - \frac{5}{8}\sigma_4 \right.\right.$$
$$\left.\left. + \frac{5(3h-7)}{48h}\sigma_2^2 - c_6^{(1)}\sigma_2 + c_6^{(2)}\right)\right]. \tag{4.5.19}$$

When $r = 2$ or 3, where I_6 is given by (2.3.59) and (2.3.60), we find the relation

$$Q_6 = -h^5\beta^{-6}J_{5,6}\, I_6$$

holds. Therefore, we conjecture that the vacuum eigenvalue of the IoM $\mathbf{I}_6$ is

$$I_6 = -\sigma_6 - \frac{h-5}{h}\left(\sigma_2\sigma_4 + \frac{1}{2}\sigma_3^2 + \frac{2h-5}{6h}\sigma_2^3 - \frac{5}{8}\sigma_4 + \frac{5(3h-7)}{48h}\sigma_2^2 - c_6^{(1)}\sigma_2 + c_6^{(2)}\right)\Bigg] \tag{4.5.20}$$

for general r.

Based on the above observations, the ODE/IM correspondence between the WKB integrals and the IoMs is expressed as

$$Q_k = \frac{(-h)^{k-1}}{\beta^k}J_{k-1,k}\, I_k. \tag{4.5.21}$$

Finally, we notice that such correspondence in WA_r algebra has also been studied up to the fifth order when $\alpha_0 = 0$ [43].

4.5.2 The ODE/IM correspondence for $D_r^{(1)}$-type pseudo ODEs

We now apply the same analysis to test the correspondence for WD_r algebra and $D_r^{(1)}$-type affine Toda field equation.

The WKB integral $\mathbf{Q}_2$

Let us begin with the WKB integral Q_2. Inserting Eq.(4.2.26) into Eq.(4.2.28), we can see

$$Q_2 = 2^{-\frac{2}{h}+1} J_{1,2} \left(\frac{1}{2h} s_1 - \frac{r}{24} h(1+M) \right).$$

Compared with the IoM from (2.3.70) in terms of σ_i

$$I_2 = \frac{1}{2}\sigma_1 - \frac{r}{24},$$

one can obtain the equality

$$Q_2 = 2^{-\frac{2}{h}+1} h(1+M) J_{1,2}\, I_2. \tag{4.5.22}$$

if the following condition is satisfied

$$s_k = \left(h^2(1+M)\right)^k \sigma_k \tag{4.5.23}$$

for $k = 1$. The relation (4.5.23) is satisfied if the momentum p satisfies

$$p = \frac{1}{h\sqrt{1+M}}(l+\rho). \tag{4.5.24}$$

In summary, we obtain the correspondence $Q_2 = 2^{-\frac{2}{h}+1} h(1+M) J_{1,2}\, I_2$, and the relation between M and β will be determined in Q_4

The WKB integral $\mathbf{Q}_4$

Since there are no third-order IoMs, we proceed to the fourth-order one. The WKB integral Q_4 in (4.2.28) in terms of momentum becomes

$$\begin{aligned} Q_4 = &- 2^{-\frac{6}{h}} J_{3,4} \\ &\times \left[\frac{1}{h} s_2 - \frac{h-3}{2h^2} s_1^2 + \frac{h-6}{8}(1+M)s_1 + \frac{h(h-6)}{1920}(h+2)(1+M)^2 \left(\frac{2hM^2}{1+M} - 9 \right) \right], \end{aligned}$$

and the IoM in Eq.(2.3.70) in terms of σ_i becomes

$$I_4 = \frac{1}{2}\sigma_2 - \frac{2r-5}{8(r+1)}\sigma_1^2 + \frac{r-4}{16(r-1)}\sigma_1 - \frac{r(r-4)(9-4(r-1)\alpha_0^2)}{1920(r-1)}. \tag{4.5.25}$$

After applying (4.5.23), one can see the correspondence

$$Q_4 = -2^{-\frac{6}{h}+1}h^3(1+M)^2 J_{3,4}\, I_4$$

is satisfied when

$$\frac{1}{1920h}(h-6)(h+2)(2ha^2-9) = \frac{1}{1920h}(h-6)(h+2)\left(\frac{2hM^2}{1+M}-9\right). \tag{4.5.26}$$

This equation implies that α_0 satisfies Eq.(4.5.3). M is given by Eq.(4.5.15), where we will choose the first equality as in the case of WA_r algebra.

Finally, if we rewrite Q_4 in terms of the highest-weight eigenvalues of primary W fields,

$$Q_4 = -2^{-\frac{6}{h}+1}\frac{h^3}{\beta^4}J_{3,4}\left(\Delta_4 + c_1\left(\Delta_2^2 - \frac{c+2}{12}\Delta_2 + \frac{5c^2+22c}{2880}\right)\right), \tag{4.5.27}$$

where the coefficient c_1 is nothing but the one we inferred in Eq.(2.3.66).

The WKB integral $\mathbf{Q}_6$

Since Q_5 is absent in the WKB expansion, we will present the sixth-order result, which is given by

$$\begin{aligned} Q_6 = 2^{\frac{10}{h}}J_{5,6}\Bigg[\frac{1}{h}s_3 - \frac{h-5}{h^2}\left(s_1 s_2 - \frac{2h-5}{6h}s_1^3 + \frac{1}{48}5h(3h-10)(1+M)s_1^2\right) \\ + \frac{5}{8}(h-6)(1+M)s_2 + h^3(1+M)^2 c_6^{(1)} s_1 + h^5(1+M)^3 c_6^{(2)}\Bigg] \end{aligned} \tag{4.5.28}$$

with

$$\begin{aligned} c_6^{(1)} &= \frac{1}{1152h^2}[(-200h+72h^2-10h^3)\frac{M^2}{(1+M)^2} + 91h^2 - 780h + 2300], \\ c_6^{(2)} &= \frac{1}{580608h^2}[-8h^2(h^4-50h^2+144h++472)\frac{M^4}{(1+M)^4} \\ &\qquad + (h+2)(61h^2-564h+1700)(10h\frac{M^2}{(1+M)^2}-25)]. \end{aligned}$$

After substituting the equation (4.5.24) and $M = \beta^{-2} - 1$, one can obtain

$$Q_6 = 2^{\frac{10}{h}+1} J_{5,6} \frac{h^5}{\beta^6} \left[\frac{1}{2}\sigma_3 - \frac{h-5}{2h} \left(\sigma_1\sigma_2 - \frac{2h-5}{6h}\sigma_1^3 + \frac{5(3h-10)}{48h}\sigma_1^2 \right) \right. \\ \left. + \frac{5}{16h}(h-6)\sigma_2 + \frac{1}{2}c_6^{(1)}\sigma_1 + \frac{1}{2}c_6^{(2)} \right]. \tag{4.5.29}$$

If we set rank $r = 4$ and apply the Miura transformation (B.4.13), one can obtain the correspondence

$$Q_6 = 2^{\frac{10}{h}+1} J_{5,6} \frac{h^5}{\beta^6}\ I_6.$$

We can also predict that I_6 with general rank is given by

$$I_6 = \frac{1}{2} \left[\sigma_3 - \frac{h-5}{h} \left(\sigma_1\sigma_2 - \frac{2h-5}{6h}\sigma_1^3 + \frac{5(3h-10)}{48h}\sigma_1^2 \right) + \frac{5}{8h}(h-6)\sigma_2 + c_6^{(1)}\sigma_1 + c_6^{(2)} \right]. \tag{4.5.30}$$

Based on the above observations, the ODE/IM correspondence between the WKB integrals and the IoMs is expressed as

$$Q_k = (-1)^{\frac{k}{2}-1} 2^{\frac{2k-2}{h}+1} \frac{h^{k-1}}{\beta^k} J_{k-1,k}\ I_k. \tag{4.5.31}$$

Finally, there is an extra WKB integral Q'_r in Eq.(4.2.28). It leads to the spin-r IoM by the current R_r (2.3.36). The correspondence is expressed as

$$Q'_r = 2J(-\frac{1}{2}, -r) \frac{h^r}{\beta^r}\ \tilde{\Delta}_r. \tag{4.5.32}$$

Above all, we have obtained the relation between the IoMs and the WKB integrals for WD_r algebras, which confirms the ODE/IM correspondence for the ODE with the pseudo-differential operator.

Chapter 5

Conclusions and Discussion

In this thesis, we delved deeply into the ODE/IM correspondence, especially uncovering its rich structures from the viewpoint of the integrals of motion based on two orginal research [58, 59]. Finally, let us summarize the thesis and give the furture directions. In Chapter 2, we began with the six-vertex model, exploring fundamental frameworks such as the **T**-, **Q**-, and **Y**-systems, along with Bethe ansatz equations. This foundation naturally extended to the continuous KdV equation, where we demonstrated how its **Q**-systems lead to the DDV equations. We then introduced integrable quantum field theories, including integrable conformal field theories like the Toda field theories and their massive extensions, the affine Toda field theories. The intricate connections between the KdV hierarchies and the Toda field theories were also explored. For Toda field theories with simple-laced Lie algebra structures, we analyzed their W-symmetries and derived local IoMs for WA_r and WD_r CFTs up to the sixth order using the quantum Drinfeld-Sokolov reduction.

Chapter 3 focused on the ODE/IM correspondence between the six-vertex model and Schrödinger-type ordinary differential equations. We introduced spectral analysis techniques using the WKB method and demonstrated how the spectral determinants of asymptotic ODE solutions correspond to the **T**- and **Q**-functions. This correspondence was extended to higher-order (pseudo) ODEs and affine Toda field theories, massive integrable models based on general affine Lie algebras. By examining the linear problems associated with classical affine Toda field equations, we derived generalized Bethe ansatz equations expressed through ψ-systems, further bridging the quantum/classical correspondence.

In Chapter 4, based on the previous research, we originally explored the ODE/IM correspondence through local IoMs and WKB integrals. By analyzing the WKB expansion of linear problems for modified affine Toda field equations, we diagonalized the connection via gauge transformations. This yielded diagonalization conditions framed as Riccati equations for WKB solutions to the adjoint ODE. Solving these equations revealed the bottom components of the diagonalized connection, with others emerging from $\mathbb{Z}_h$ phase rotations and traceless conditions [83]. For $A_1^{(1)}$ and $A_2^{(1)}$, we confirmed that the diagonal elements correspond to classical conserved densities after conformal transformation, suggesting the results generalize to other affine Lie algebras. We systematically calculated WKB period integrals along the Pochhammer contour, achieving results up to the 8th order for $A_r^{(1)}$ and the 6th order for $D_r^{(1)}$. These WKB integrals, predicted by the ODE/IM correspondence, matched the eigenvalues of quantum IoMs on the vacuum state in WA_r and WD_r algebras, subject to normalization factors. This alignment solidifies the link between WKB integrals and quantum IoMs, providing compelling evidence for the correspondence.

Crucially, WKB integrals connect both classical conserved charges and quantum IoMs in the ground state. This reinforces the proposed quantum/classical correspondence for conserved charges, as seen in prior works [18, 42, 103, 104], while offering a more efficient method to compute IoMs, particularly for higher-rank Lie algebras.

Next, we give some interesting furture plans. Despite these advances, significant open questions remain, pointing to various promising directions for further research. A natural next step is extending the correspondence to general affine Lie algebras. In particular, exploring the relationship between the linear problem associated with affine Toda field equations for non-simple-laced $\hat{\mathfrak{g}}$ and the quantum IoMs of Langlands dual $W\hat{\mathfrak{g}}^\vee$-algebras could provide a deeper understanding of the celebrated Langlands duality in mathematics. This investigation may also uncover new algebraic structures underpinning the correspondence.

Another intriguing direction involves modifying the potential term $p(x, E)$ in the associated linear problems, leading to two distinct generalizations. First, altering p allows the study of quantum IoMs beyond the vacuum state, with a focus on excited states [102]. This approach could unveil new patterns and symmetries, enriching our understanding of the spectral properties in integrable models. The second direction explores supersymmetric extensions of affine Toda field theories [105], particularly in linking SUSY structures with integrable systems. This line of inquiry may illuminate generalized versions of the ODE/IM correspondence, potentially extending its framework to include previously unexamined supersymmetric models.

Explicitly constructing IoMs for WA_r and WD_r CFTs remains a significant challenge. Advances in this area would greatly enhance the analysis of thermal correlators in CFTs with W-symmetry [80], offering crucial insights into the thermodynamic properties and providing a framework to test the generalized Eigenstate Thermalization Hypothesis. This hypothesis suggests that the expectation values of observables in individual eigenstates depend solely on the local IoMs of the system.

Finally, the SUSY/IM correspondence, briefly mentioned in the introduction, remains an intriguing area for further exploration. Completing the dictionary between integrable models with differential Lax operators, such as the KdV hierarchies and Toda field theories, could benefit from insights drawn from ODEs in quantum Seiberg-Witten curves in 4D $\mathcal{N} = 2$ supersymmetric gauge theories with a twisted omega background [106]. Through this lens, the ODE/IM correspondence may provide a path to fully realize this dictionary and unlock new perspectives in integrable systems and field theory.

Appendix A

The Drinfeld-Sokolov Reduction

The Drinfeld-Sokolov (DS) Reduction is a useful method for constructing abstract classical integrable models from Kac-Moody algebras [21]. The essential idea is to restrict the infinite-dimensional algebra using a Hamiltonian reduction process that imposes constraints compatible with integrability or Hamiltonian structures. The algebra with the constraints finally turn out to be the Gelfand-Dikii algebras [107].

The quantum DS reduction is realized by quantizing the free fields in the classical Miura transformation [24,25,76] where the Gelfand-Dikii algebras become the W-algebras in 2D CFT. Furthermore, in the QDS reduction, there exist "screening charges" as the centralizer of the W-algebras which plays an important role in computing the local integrals of motion.

Later, it was found that the quantum DS reduction is equivalent to the BRST procedure with ghost fields [108, 109]. The screening charges in the QDS reduction turn out to be equivalent to the cohomology charge in the BRST procedure.

In this chapter, we will briefly review the DS reduction. Let us first begin with an introduction to the classical integrability. Consider a dynamical Hamiltonian system with phase space M of dimension $2n$ with canonical coordinates p_i, q_i. To introduce an integrable system, recall the Poisson bracket in classical mechanics

$$\frac{dF}{dt} = \{H, F\}, \tag{A.0.1}$$

where the Poisson bracket is defined by

$$\{F, G\} \equiv \sum_i \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}. \tag{A.0.2}$$

From the definition, the quantity $H(p,q)$ is automatically conserved namely $\frac{dH}{dt} = 0$. In this thesis, we will introduce the definition of an integrable system through the Liouville theorem [97].

Definition A.0.1 *The system is integrable if it possesses* n *independent conserved quantities* $Q_i, i = 1, \ldots, n,\ \ \{H, Q_i\} = 0$ *in involution*

$$\{Q_i, Q_j\} = 0. \tag{A.0.3}$$

For an integrable system, the number of conserved quantities is equal to the degree of freedom of the system. Here, the structure of the Poisson bracket will be specified in the KdV hierarchies.

A.1 The scalar Lax equation

Lax formalism is a well-known formalism for classical integrable models. A Lax pair consists of two matrices L and M, which are functions on the phase space. And the equation of motion (EoM) (A.0.1) can be rewritten into the Lax pair by

$$\frac{dL}{dt} = [M, L]. \tag{A.1.1}$$

Here $[M, L]$ is the commutator of two matrices or operators whose entries are dynamical variables of the system. For example, the Korteweg-de Vries equation given by

$$u_t + 6uu_x + u_{xxx} = 0, \tag{A.1.2}$$

can be also represented by Lax pairs

$$L = \partial^2 + u, \qquad M = \partial^3 + \frac{3}{2} + \frac{3}{4}u_x, \tag{A.1.3}$$

were u is a function of x and t, and $\partial = \partial_x$. It is possible to generalize the Lax equation (A.1.1) into the Zakharov-Shabat construction [97].

$$\partial_{t_i} L = [M_i(\lambda), L], \quad i \in Z, \tag{A.1.4}$$

where t_i is not fixed to be time with the comutaivity $[\partial_{t_i}, \partial_{t_j}] = 0$ and M_i is determined by L and a spectral parameter λ. We will see the effect of this parameter in the KdV equation section. Since the detail is not necessary, we skip it and discuss the application. From $[\partial_{t_i}, \partial_{t_j}] = 0$ and $\partial_i \partial_j L = \partial_j \partial_i L$, we can obtain

$$[\partial_i M_j - \partial_j M_i, L] + [M_i, [L, M_j]] + [M_j, [M_i, L]] = 0. \tag{A.1.5}$$

Substitute the Jacobian identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, one can obtain

$$[\partial_i M_j - \partial_j M_i - [M_j, M_i], L] = 0, \tag{A.1.6}$$

which should hold for infinite number of M_i and this yields

$$\partial_i M_j - \partial_j M_i - [M_j, M_i] = 0. \tag{A.1.7}$$

This can be viewed as the zero-curvature condition. To be more obvious, let us consider the two-dimensional case

$$[\partial_x - U, \partial_t - V] = 0, \tag{A.1.8}$$

where U and V can be viewed as local connection along x and t direction. The zero curvature condition expresses the compatibility condition of the associated linear system

$$(\partial_x - U)\Phi = 0, \qquad (\partial_t - V)\Phi = 0. \tag{A.1.9}$$

Choosing a path from γ from the origin to the point (x, t), the wavefuntion can be written as

$$\Phi(x, t) = \mathcal{P} \exp\left[\int_\gamma (U dx + V dt)\right], \tag{A.1.10}$$

where $\mathcal{P}$ represents the path ordering along the path γ. We can choose a specific path with $x \in [0, 2\pi]$ and fix the time

$$T(t, \lambda) = \mathcal{P} \exp\left[\oint_\gamma U(x, t, \lambda) dx\right]. \tag{A.1.11}$$

where λ is the spectral parameter and T is known as the monodromy matrix which is equivalent to the Lax operator L in classical IM since

$$\partial_t T(t, \lambda) = [V(t, \lambda), T(\lambda, t)]. \tag{A.1.12}$$

Finally, the Lax pair is not fixed uniquely for a dynamical system. It can be matrices or operators. In fact, any pair that gives the corresponding equation of motion can be called a Lax pair. For example the monodromy matrix (A.1.11) can be the Lax operator. The differential operator used by Drinfel'd and Sokolov is another equivalent Lax operator which we will see later.

A.1.1 The construction of the Lax equation

Here we are going to introduce the Lax formalism mathematically and give its Hamiltonian structure.

Definition A.1.1 *The algebra of differential operators is the algebra generated by* $\boldsymbol{C}$ *valued functions of one variable* x *and the derivation symbol* ∂*, with the Leibnitz rule* $\partial.a = a.\partial + (\partial a)$

An element in this algebra can be written as

$$A = \sum_{i=0}^{N} a_i \partial^i, \tag{A.1.13}$$

where a_i are the functions of x. We extend the algebra of differential operators by introducing the integration symbol with the following rules.

$$\partial^{-1}\partial = \partial\partial^{-1} = 1, \tag{A.1.14}$$

$$\partial^{-1} a = \sum_{i=0}^{\infty} (-1)^i (\partial^i a) \partial^{-i-1}, \tag{A.1.15}$$

where the second rule is from the integration by parts. Here the pseudo-differential operator leads to a definite integral which cancel the differential operator, for instance, $\partial^{-1} x^n = x^{n+1}/(n+1)$. The algebra of pseudo-differential operators consists of the elements of the form

$$A = \sum_{i=-\infty}^{N} a_i \partial^i. \tag{A.1.16}$$

Let $\mathcal{P} = \{A = \sum_{i=-\infty}^{N} a_i \partial^i\}$ be the set of formal pseudo-differential operators in one variable. Then we separate it into two parts:

$$\mathcal{P}_+ = \{A = \sum_{i=0}^{N} a_i \partial^i\}, \qquad \mathcal{P}_- = \{A = \sum_{i-\infty}^{-1} a_i \partial^i\} \tag{A.1.17}$$

where the $+$ denotes the subalgebra of differential operators and the $-$ denotes the subalgebra of integral operators. For further application, we also define the residue as the coefficient of ∂^{-1} in A

$$\mathrm{Res}_\partial A \equiv a_{-1}(x). \tag{A.1.18}$$

Such a form can be used to define Adler trace

$$\langle A\rangle = \int dx\ \mathrm{Res}_\partial A = \int dx\ a_{-1}(x) \tag{A.1.19}$$

This trace helps us define the inner product on $\mathcal{P}$ by:

$$(A, B) = \langle AB\rangle = \int dx\ \ \mathrm{Res}_\partial(AB) \tag{A.1.20}$$

It is proved that $(A, B) = (B, A)$. Such an inner product is important when constructing the Possion braket which we will see later. We should mention that $\mathcal{P}_+$ and $\mathcal{P}_-$ are now in dual with the inner product (A.1.20) which can be seen from

$$\langle \partial^{-1-i} b_i, a_j\partial^j\rangle = \delta_{ij}\int dx\ \ (a_i b_j) \tag{A.1.21}$$

Now we can turn back to the Lax operator. Let L be a differential operator:

$$L = \partial^{n+1} - \sum_{i=0}^{n-1} u_i\partial^i. \tag{A.1.22}$$

Here we impose that the coefficient of ∂^n vanishes.

Proposition A.1.1 *L satisfies the Lax equations*

$$\partial_{t_i} L = [(L^{\frac{i}{n+1}})_+, L] \tag{A.1.23}$$

where the notation $\pm$ means only the differential/integral part survives. Such proposition gives one of the specific forms of M in the Lax equation (A.1.1). Take the KdV equation for an example, recall $L = \partial^2 - u,\ M = \partial^3 + \frac{3}{2} + \frac{3}{4}u_x$ which means $n = 1$. Now consider $i = 1$, $M^2 = L$, which can be solved that

$$M = (L)^{\frac{1}{2}} = \partial - \frac{1}{2}u\partial^{-1} + \frac{1}{4}(\partial u)\partial^{-2} + \dots \tag{A.1.24}$$

$M_+ = (L^{\frac{1}{2}})_+ = \partial$ and the Lax equation here becomes trivial

$$\partial_{t_1} L = [\partial, L] \tag{A.1.25}$$

which only means $t_1 = x$. It is not difficult to show that the second order is still trivial. And the familiar KdV equations appear in the third order where

$$M_+^3 = (L^{\frac{3}{2}})_+ = \partial^3 + \frac{3}{2} + \frac{3}{4}u_x \tag{A.1.26}$$

As we introduced in the beginning, the Hamiltonian structure and the integrals of motion play a fundamental role in an integrable model. In the KdV hierarchies, there are two equivalent Poisson brackets which is sometimes called the Bi-hamiltonian structure. For any functions f and g on $\mathcal{P}_+$, the Kostant-Kirillov bracket is defined by

$$\{f, g\}_1(L) = \langle L, [df, dg]\rangle \quad \forall L \in \mathcal{P}_+ \tag{A.1.27}$$

Recall that $\mathcal{P}_+$ and $\mathcal{P}_-$ are in dual. Similar with one form in differential manifold. Now we set $df \in \mathcal{P}_-$. Later we also denote $df_X = X, \quad X \in \mathcal{P}_-$ for convenience. Then two Poisson brackets can be given by

$$\{f_X, g_Y\}_1(L) = \langle L, [X, Y]\rangle, \tag{A.1.28}$$

$$\{f_X, g_Y\}_2(L) = \langle (LX)_+(LY)_-\rangle - \langle (XL)_+(YL)_- - \frac{1}{n+1}\int dx((\partial)^{-1}[L, X]_{-1})[L, Y]_{-1}. \tag{A.1.29}$$

These two Poisson brackets are connected by the classical Miura transformation, and the integrals of motion are given by the following proposition.

Proposition A.1.2 *The functions of L given by*

$$H_k(L) = (\frac{n+1}{n+k+1})\langle L^{\frac{k}{n+1}+1}\rangle \tag{A.1.30}$$

are the conserved quantities in involution of the generalized KdV hierarchy and the equations of motion can be given by

$$\frac{dL}{dt} = \{H_k, L\}_1 = [(L^{\frac{k}{n+1}})_+, L]. \tag{A.1.31}$$

Finally, we approach the Hamiltonian structure from the Lax equations.

In the example given above, the Lax equations (A.1.23) are also called the KdV hierarchies.

A.2 Lie algebraic Zakharov-Shabat system

So far, we have introduced the general Lax operator (A.1.22). However, the high order is a difficult task to solve. Drinfel'd and Sokolov gave a corresponding Lax operator with a

Lie algebra structure. The contents follows the original paper [21]. To be self-contained, we begin with the gradation of affine Lie algebras.

The untwist affine Lie algebra $\hat{\mathfrak{g}}$ is the tensor product of the simple Lie algebra g and the Laurent polynomial $C[\lambda, \lambda^{-1}]$. For the simple Lie algebra with rank r, we need r linearly independent simple roots to describe the whole root system. While in the affine case, due to the existence of a central element, we need to add one more root α_0 with ladder operator in Chevalley basis given by $e_0 = \lambda E_{-\theta}$ and $f_0 = \lambda^{-1} F_\theta$, where E and F are basis in Catan-Weyl basis, and θ is the highest root of simple Lie algebra g. More information on Lie algebras can be found in Appendix.C.

Now the whole Chevalley basis of the affine Lie algebra $\{e_i, f_i, h_i\}_{i=0,1,\ldots,r}$ can be given by

$$[h_i, h_j] = 0, \qquad [e_i, f_j] = \delta_{ij} h_j, \tag{A.2.1}$$

$$[h_i, e_j] = a_{ji} e_j, \qquad [h_i, f_j] = -a_{ji} f_j, \tag{A.2.2}$$

where $i = 0, 1, \ldots, r$ and a_{ij} is the Cartan matrix. The affine Lie algebra $\hat{g}$ is equipped with $\mathbb{Z}$ gradation. There are two typical gradations:

- The canonical/principal gradation

  $$\hat{\mathfrak{g}} = \bigoplus_{k\in\mathbb{Z}} \hat{\mathfrak{g}}^k, \quad [\hat{\mathfrak{g}}^k, \hat{\mathfrak{g}}^l] = \hat{\mathfrak{g}}^{k+l} \tag{A.2.3}$$

  is defined by assigning $+1(-1)$ to $e_a(f_a)$ and 0 to other generators.

- The standard/homogenous gradation is related to the Dynkin diagram of $\hat{\mathfrak{g}}$. To understand this gradation, we need first introduce the proposition below.

  Proposition A.2.1 *Let $S \subset \{0, 1, \ldots, r\}$ be a proper subset. Then the subalgebra generated by the elements $e_i, f_i, h_i, i \in S$, is semisimple. Especially when $S = \{1, \ldots, r\}$, it reduces to $\mathfrak{g}$.*

  The standard gradation with respect to the vertex c_0 in the Dynkin diagram of $\hat{\mathfrak{g}}$

  $$\hat{\mathfrak{g}} = \bigoplus_{k\in Z} \hat{\mathfrak{g}}_k, \quad [\hat{\mathfrak{g}}_k, \hat{\mathfrak{g}}_l] = \hat{\mathfrak{g}}_{k+l} \tag{A.2.4}$$

is defined by assigning degree $+1(-1)$ to $E_0(F_0)$ and 0 to other generators. It is obvious that $\hat{\mathfrak{g}}_0 = \mathfrak{g}$.

The Borel, Cartan and nilpotent subalgebras of $\hat{\mathfrak{g}}_0$ are defined by

$$\mathfrak{h} = \hat{\mathfrak{g}}_0 \cap \hat{\mathfrak{g}}^0, \qquad \mathfrak{n} = \hat{\mathfrak{g}}_0 \cap (\bigoplus_{k<0} \hat{\mathfrak{g}}^k), \tag{A.2.5}$$

$$\mathfrak{b} = \hat{\mathfrak{g}}_0 \cap (\bigoplus_{k\leq 0} \hat{\mathfrak{g}}^k) = \mathfrak{h} \bigoplus \mathfrak{n}. \tag{A.2.6}$$

For later convenience, it is better to explain the relations between Borel and nilpotent in detail. First, we divide the Borel subalgebra by the canonical gradation.

$$\mathfrak{b} = \bigoplus_{i\geq 0} \mathfrak{b}_i, \tag{A.2.7}$$

which means $\mathfrak{b}_i = \mathfrak{g}^{-i}$. Consider operator $I = \sum_{i=1}^r e_i$, *ad* I: acts from $\mathfrak{b}_i$ to $\mathfrak{b}_{i-1}$. We choose a vector subspace $V_i \subset \mathfrak{b}_i$ so that $\mathfrak{b}_i = [I, \mathfrak{b}_{i+1}] \bigoplus V_i$. Attention that the map *ad* I from $\mathfrak{b}_{-1}$ to $\mathfrak{b}_0$ is bijective so there is no V_0. In summary, we can obtain

$$\mathfrak{b} = V \bigoplus [I, \mathfrak{n}] \tag{A.2.8}$$

and $\dim V = \dim \mathfrak{b} - \dim \mathfrak{n} = \dim \mathfrak{h} = r$.

Now we turn back to the integrable models. Define the differential operator

$$\mathcal{L} = \partial + q + \Lambda \tag{A.2.9}$$

where $q \in C^\infty(\mathbf{R}, \mathfrak{b}), \Lambda = \sum_{i=0}^r e_i$. So far we have not fixed the algebra yet. It can be shown that there exists $S \in C^\infty(\mathbf{R}, \mathfrak{n})$ acting on $\mathcal{L}$ as gauge transformation

$$\mathcal{L}' = \text{ad } e^S(\mathcal{L}) = e^S(\mathcal{L})e^{-S}, \tag{A.2.10}$$

where $\mathcal{L}'$ also has the form (A.2.9). There are two important forms. The first one is canonical reduction $\mathcal{L}^{can}$

$$\mathcal{L}^{can} = \frac{d}{dx} + q^{can} + \Lambda, \tag{A.2.11}$$

where $q \in C^\infty(\mathbf{R}, V)$ is differential polynomials in q. The second one is diagonal reduction $\mathcal{L}^{diag}$

$$\mathcal{L}^{diag} = \frac{d}{dx} + q^{diag} + \Lambda, \tag{A.2.12}$$

where $q^{diag} \in C^\infty(\mathbf{R}, \mathfrak{h})$. In fact, such $\mathcal{L}$ is the Lax operator we are looking for. To show this, first consider a time evolution equation

$$\partial_t \mathcal{L} = [\mathcal{A}, \mathcal{L}]. \tag{A.2.13}$$

In analogy to what we have done in the scalar Lax formalism. It is possible to find that $\mathcal{A}$ is given by

$$\mathcal{A} = (e^{-adU}(u))^+, \quad U \in C^\infty(\mathbf{R}, \mathfrak{n}), \quad u \in \ker ad\Lambda. \tag{A.2.14}$$

Now the notation + denotes the gradation. It can be proved that the time evolution (A.2.13) preserves gauge equivalence and leads to the same equation for the class with gauge equivalence. Later it can be seen that the relation between canonical and diagonal case is the Miura transformation introduced in [21]. Especially, if we consider the canonical case, Eq.(A.2.13) becomes

$$\frac{\partial q^{can}}{\partial t} = F(q^{can}, \frac{\partial q^{can}}{\partial x}, \dots), \tag{A.2.15}$$

where F is a differential polynomial in q^{can}. In the diagonal case,

$$\frac{\partial q^{diag}}{\partial t} = -\partial A^0. \tag{A.2.16}$$

where $A_0 \in \mathfrak{g}^0$ is a differential polynomial in q^{diag}.

Proposition A.2.2 *If an operator $\mathcal{L}$ of the form* (A.2.9) *satisfies Eq.*(A.2.13)*, then any gauge equivalent equations (For instance, Eq.*(A.2.15) *or Eq.*(A.2.16) *is a coordinate realization of the generalized KdV equation.*

So far, we have not known how the $\mathcal{L}$ we obtain here relates the scalar Lax operator (A.1.22) we have introduced before. Recall the definition of pseudo-differential operators $\mathcal{P}$ we have given in Eq.(A.1.17). Now define the action $\mathcal{P}$ on vector η by

$$\mathcal{P} \cdot \eta = \sum_{i=0}^{n} a_i \mathcal{L}^i(\eta), \tag{A.2.17}$$

Recall that $\mathcal{L}$ belongs to $\mathcal{P}$ and rewrite the indices in L to match the rank of the algebra

$$L = \partial^r + \sum_{0}^{r-1} u_i \partial^i. \tag{A.2.18}$$

Set

$$L \cdot \psi = (\mathcal{L}^r + \sum_{i=0}^{r-1} u_i \mathcal{L}^i)\psi = \lambda\psi. \tag{A.2.19}$$

Note the gauge equivalent $\mathcal{L}$ corresponds to the same scalar Lax operator L. Here we construct the relations between scalar Lax operators and g-valued Lax operator $\mathcal{L}$.

A.2.1 The generalized KdV hierarchy

This method is a little bit abstract so we take the generalized KdV hierarchy for an example. In the generalized KdV hierarchy, the gauge fixed operators $\mathcal{L}^{can}$ and $\mathcal{L}^{diag}$ can be given in the following form

$$q^{can}(x,t) = U_1 e_{1,r+1} + U_2 e_{2,r+1} + \cdots + U_r e_{r,r+1}, \tag{A.2.20}$$

$$q^{diag}(x,t) = \sum_{a=1}^{r} v_a h_a = diag(q_1, \ldots, q_{r+1}), \tag{A.2.21}$$

or in matrix form

$$q^{can}(x,t) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & U_1 \\ 0 & 0 & 0 & \ldots & 0 & U_2 \\ 0 & 0 & 0 & \ldots & 0 & U_3 \\ \ldots & & & \ldots & & \ldots \\ 0 & 0 & \ldots & 0 & 0 & U_r \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}, \quad q^{diag}(x,t) = \begin{pmatrix} q_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & q_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & q_3 & \ldots & 0 & 0 \\ \ldots & & & \ldots & & \ldots \\ 0 & 0 & \ldots & 0 & q_r & 0 \\ 0 & 0 & \ldots & 0 & & q_{r+1} \end{pmatrix}. \tag{A.2.22}$$

Here we have taken the $(r+1)\times(r+1)$ matrix representation of $sl(r+1)^{(1)}$ algebra with Chevalley basis given by $e_a = e_{a+1,a}, f_a = e_{a,a+1}$, where $e_{a,b}$ denotes the matrix having unity at the (a,b) site and zeros elsewhere. And Λ now is

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \lambda \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & & & \ldots & & \ldots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}. \tag{A.2.23}$$

The representation here is reversed comparing the conventional one. This caters to the action from the left in (A.2.19). Take the standard basis $\{\psi_1, \ldots, \psi_{r+1}\}$, $\psi_1 = (1, 0, \ldots, 0)^T$. Noting

$$\Lambda^i \psi_1 = \psi_{1+i}, \quad (i < r), \tag{A.2.24}$$

$(q^{can})^2 = 0$ and $q^{can}\Lambda^r\psi_1 = \sum_1^r U_i\psi_1$, So in the canonical reduction, we find that $U_i = -u_i$ which relates the parameter between g-valued Lax operator and scalar Lax operator.

While in the diagonal reduction we have

$$(\mathcal{L} - q_r)\cdots(\mathcal{L} - q_2)(\mathcal{L} - q_1)\psi_1 = \lambda\psi_1. \tag{A.2.25}$$

Compare with (A.2.21), it implies the generalized Miura transformation.

$$L = (\partial - q_r)\cdots(\partial - q_2)(\partial - q_1). \tag{A.2.26}$$

So from the Lie algebraic approach we can find out the Miura transformation simply at least for A type algebra. As an example, we consider $A(1)$ algebra.

$$\partial^2 - u(x) = (\partial - \partial\phi(x))(\partial + \partial\phi(x)), \tag{A.2.27}$$

which means

$$u(x) = (\partial\phi)^2 - \partial^2\phi. \tag{A.2.28}$$

This is namely the Miura transformation.

A.2.2 The affine Toda field theory

In affine Toda field theory, the $\mathfrak{g}$-valued Lax operator is usually chosen to be diagonal reduction. For further convenience, we choose the complex coordinate,

$$\mathcal{L} = \partial_z + \partial_z\phi_i h_i + \Lambda, \tag{A.2.29}$$

where $\Lambda = \sum_{i=0}^r e_i$. Different from the generalized KdV equations, the linear differential operator $\mathcal{L}$ can be viewed as covariant derivative in the zero-curvature condition combining with

$$\bar{\mathcal{L}} = \partial_{\bar{z}} + e^{-\phi_i h_i}\bar{\Lambda}e^{\phi_i h_i} \tag{A.2.30}$$

with $\bar{\Lambda} = \sum_{i=0}^r f_i$. The zero curvature condition is written as

$$0 = [\mathcal{L}, \bar{\mathcal{L}}] = -\sum_{i=0}^r [\partial_{\bar{z}}\partial_z\phi_i h_i - \exp\left(\sum_{j=0}^r a_{ij}\phi_j\right)h_j] \tag{A.2.31}$$

and it yields the affine Toda field equation

$$\partial_{\bar{z}}\partial_z\phi - \sum_{i=0}^r \alpha_i \exp\left(\sum_{j=0}^r a_{ij}\phi_j\right) = 0, \tag{A.2.32}$$

where we set the coupling constant to be 1.

Appendix B

2D Conformal field theory

As we have explained in the continuum limit of the six-vertex model (Sec.2.1.4), the KdV equation (Sec.2.2) and Toda field theory (Sec.2.3), integrable models are deeply connected to conformal field theory. In this appendix, we will briefly introduce the related parts in the two-dimensional conformal field theory (2D CFT).

Two-dimensional conformal field theory (2D CFT) is a class of quantum field theories that are invariant under conformal transformations in two-dimensional spacetime. These theories are of profound importance in theoretical physics and mathematics due to their high degree of symmetry. There are several features for 2D CFT we are going to cover in this appendix: conformal symmetry, operator product expansion, Virasoro algebra, and operator/state correspondence.

B.1 2D Conformal transformation

Conformal symmetry refers to transformations that preserve angles and shapes locally, such as scaling and rotations. In two dimensions, this symmetry is much more powerful than in higher dimensions because the space of transformations becomes infinite-dimensional and it is this infinite symmetry that gives 2D CFT the exact solvability. Let us show this in more detail.

Definition B.1.1 *Consider differentiable map ϕ: where U and V are open subsets of manifold M and M' respectively, Denoting g and g' as the metric tensors of M and M' respectively, we can pull back g' by using the map ϕ. A conformal transformation is then* *defined by such a map that its pullback metric $\phi^* g'$ satisfying $\phi^* g' = \Lambda g$*

Denoting $x' = \phi(x)$ with $x \in U$, we can express the condition in a covariant form:

$$g'_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x), \tag{B.1.1}$$

where $\Lambda(x)$ is the so-called scale factor. We can find that any smooth transformation in one dimension is conformal. Here we consider $M = M'$ to be flat spaces with a constant metric $\eta_{\rho\sigma} = diag(-1, \ldots, +1, \ldots)$ Thus, the condition for a conformal transformation can be written as

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) \eta_{\mu\nu}. \tag{B.1.2}$$

So far, the transformation is in finite form namely $x' = f(x)$. If one considers the infinitesimal transformation:

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x). \tag{B.1.3}$$

Plunge this equation into (B.1.2), we can find the following restriction on the transformation (B.1.13) to be conformal:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial^{i}\epsilon_{i})\eta_{\mu\nu}. \tag{B.1.4}$$

From such infinitesimal conformal transformation, we can find out the generators of conformal symmetry. There are indeed four basic conformal transformations corresponding to their generators: translation, rotation, dilation, and special conformal transformation (SCT). An illustration in two dimensions is shown in the table below.

Table B.1.1: Finite conformal transformation and corresponding generators

Transformations		Generators
translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} \ = -i\partial_{\mu}$
dilation	$x'^{\mu} = \alpha x^{\mu}$	$D \ = -ix^{\mu}\partial_{\mu}$
rotation	$x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
SCT	$x'^{\mu} = \frac{x^{\mu} - (x\cdot x)b^{\mu}}{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}$	$K_{\mu} \ = -i(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu})$

B.1.1 Energy Momentum Tensor

In Noether's theorem, every symmetry of a system is always generated by a conserved current j^{μ} with the conservation law $\partial_{\mu} j^{\mu} = 0$. The one for the conformal symmetry is the energy-momentum (EM) tensor $T^{\mu\nu}$

$$T^{\mu\nu} = -2\frac{\delta S}{\delta g_{\mu\nu}}. \tag{B.1.5}$$

with the conservation law

$$\partial_\mu T^{\mu\nu} = 0. \tag{B.1.6}$$

For the conformal transformation (B.1.13), the corresponding current can be written as

$$j_\mu(x) = T_{\mu\nu}\epsilon^\nu(x). \tag{B.1.7}$$

The conserved current condition not only gives (B.1.6) but also implies

$$T_{\mu\nu}\eta^{\mu\nu} = T^\mu_\mu = 0, \tag{B.1.8}$$

which means that the EM tensor is traceless. This is the result of dilation symmetry.

In 2D CFT, it is convenient to write it on a complex plane. After coordinate transformation, the system is separated into two independent parts: the holomorphic (z) component and the anti-holomorphic component ($\bar{z}$). For further convenience, we denote

$$T(z) \equiv T_{zz}(z), \qquad \bar{T}(\bar{z}) = \bar{T}_{\bar{z}\bar{z}}(\bar{z}). \tag{B.1.9}$$

On a complex plane, the conserved equation (B.1.6) becomes

$$\partial_z \bar{T}(\bar{z}) = 0, \qquad \partial_{\bar{z}} T(z) = 0. \tag{B.1.10}$$

On the other hand, the current (B.1.7) becomes

$$j(z, \bar{z}) = \epsilon(z)T(z) + \epsilon(\bar{z})T(\bar{z}) \tag{B.1.11}$$

and the conserved charge (the Hamiltonian) is given by the spatial integral

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\, \epsilon(z)T(z). \tag{B.1.12}$$

Here we only consider the holomorphic component for simplicity. The conserved charge can be used as a generator of the associated symmetry [110]

$$\delta_\epsilon \Phi(w) = -[Q_\epsilon, \Phi(w)]. \tag{B.1.13}$$

Example: 2D CFT on an infinite cylinder

In a 2D CFT, it is also useful to define the theory on an infinite space-time cylinder, with time $(-\infty, \infty)$ along the flat direction of the cylinder, and space being compact with a coordinate $x \in (0, l)$, the points $(0, t)$ and (l, t) identical. The cylinder now can be described by a complex coordinate $\xi = t + ix$. We then map the cylinder onto a complex plane by a conformal transformation (In two dimensions, any holomorphic transformation turns out to be conformal).

$$z \to \xi(z), \qquad \text{with } \; z = e^{\frac{2\pi\xi}{l}}. \tag{B.1.14}$$

The most important feature for such transformation is that a time ordering product now becomes the radial ordering product

$$R(\mathcal{O}_1(z)\mathcal{O}_2(w)) = \begin{cases} \mathcal{O}_1(z)\mathcal{O}_2(w), & \text{for} |z| > |w| \\ \mathcal{O}_2(w)\mathcal{O}_1(z), & \text{for} |z| < |w| \end{cases} \tag{B.1.15}$$

which relates the operator product to commutation relations. We can see

$$\oint_w dz\, a(z)b(w) = \oint_{C_1} dz\, a(z)b(w) - \oint_{C_2} dz\, b(w)a(z) = [A, b(w)], \tag{B.1.16}$$

where the operator A is the contour integral of $a(z)$ at a fixed time

$$A = \oint dz\, a(z). \tag{B.1.17}$$

Here we take the contours C_1 and C_2 at fixed-radius $|w| + \epsilon$ and $|w| - \epsilon$ with small positive number ϵ. Then, with $B = \oint dz\, b(z)$, we can obtain

$$[A, B] = \oint_0 dw [A, b(w)] = \oint_0 dw \oint_w dz\, a(z)b(w). \tag{B.1.18}$$

To further compute the contour integral, we need to expand the interaction between fields $a(z)$ and $b(w)$ when they are approaching each other, the operator product expansion. But before that, let us first define an important field in 2D CFT.

B.1.2 Primary Fields

In 2D CFT, a primary field is a type of operator that plays a fundamental role in the structure of the theory. It is defined as an operator that transforms in the simplest way under conformal transformations, directly reflects the symmetries of the theory, and serves as the starting point for constructing more complex behaviors. To define a primary field, let us introduce the conformal dimension.

Definition B.1.2 *If a field $\phi(z, \bar{z})$ transforms under scaling $z \to \lambda z$ according to*

$$\phi(z, \bar{z}) \to \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda} \bar{z}), \tag{B.1.19}$$

it is said to have conformal dimensions $(h, \bar{h})$. Then a primary field is defined as follows.

Definition B.1.3 *If a field transforms under conformal transformation $z \to z'$ according to*

$$\phi(z, \bar{z}) \to (\frac{\partial z'}{\partial z})^h (\frac{\partial \bar{z}'}{\partial \bar{z}})^{\bar{h}} \phi(z', \bar{z}'), \tag{B.1.20}$$

it is called a primary field with conformal dimensions $(h, \bar{h})$.

B.1.3 Operator Product Expansion

The operator product expansion (OPE) determines how two fields behave when brought close. It rewrites the product of two fields into a sum of other fields, with coefficients that encode interaction information. This expansion simplifies the computation of the infinitesimal transformation (B.1.13) of an operator under certain symmetry. Recall that conformal transformation is generated by the EM tensor $T(z)$, whose OPE is given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots, \tag{B.1.21}$$

where c is the central charge. The term $\ldots$ denotes the regular terms. The primary field $\Phi(z)$ with conformal dimension Δ is defined by the OPE

$$T(z)\Phi(w) = \frac{\Delta \Phi(w)}{(z-w)^2} + \frac{\partial_w \Phi_(w)}{z-w} + \cdots. \tag{B.1.22}$$

It can be easily seen that the EM tensor is not a primary field. The symmetry algebra of a CFT can be extended by adding currents. In particular, the algebra generated by higher spin currents is called the W algebra. Let $W_s(z)$ denote the spin s current. Note that $T(z)$ is a spin 2 current: $W_2(z) = T(z)$.

Let $A(z)$ and $B(z)$ be two symmetry generators with conformal dimension Δ_A and Δ_B, respectively. The operator product expansion takes the form

$$A(z)B(w) = \sum_{1 \le k \le \Delta_A + \Delta_b} \frac{\{AB\}_k(w)}{(z-w)^k} + \,\mathring{\circ}\, AB \,\mathring{\circ}\, (w) + O(z-w). \tag{B.1.23}$$

Here $\mathring{\circ}\, AB \,\mathring{\circ}\, (w)$ is the normal ordered product on the complex plane defined by the radial ordered product:

$$\mathring{\circ}\, AB \,\mathring{\circ}\, (w) = \frac{1}{2\pi i} \oint dz \frac{R(A(z)B(w))}{z-w}. \tag{B.1.24}$$

B.1.4 Mode Expansion and Virasoro Algebra

The local conformal transformations are encoded by the generators in the so-called Virasoro algebra. A key feature of this algebra is the central charge that influences the behavior of the theory and helps classify different CFTs. To derive Virasoro algebra, let us first expand a field with conformal dimension $(h, \bar{h})$ as

$$\phi(z, \bar{z}) = \sum_{m \in Z} \sum_{n \in Z} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}. \tag{B.1.25}$$

We can also inverse the expansion (B.1.25) by the theorem of residues

$$\phi_{m,n} = \frac{1}{2\pi i} \oint dz\; z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z}\; \bar{z}^{n+\bar{h}-1} \phi(z, \bar{z}). \tag{B.1.26}$$

The conformal generators in Hilbert space and its corresponding algebra are constructed as follows. First expand the holomorphic EM tensor as

$$T(z) = \sum_{n \in Z} z^{-n-2} L_n \tag{B.1.27}$$

with L_n equals to

$$L_n = \frac{1}{2\pi i} \oint dz\; z^{n+1} T(z), \tag{B.1.28}$$

and the power -2 represents the spin index. Then, we expand the infinitesimal conformal change $\epsilon(z)$ as

$$\epsilon(z) = \sum_{n \in Z} z^{n+1} \epsilon_n, \tag{B.1.29}$$

which leads to the conserved charge (B.1.12) becomes

$$Q_\epsilon = \sum_{n\in Z} \epsilon_n L_n. \tag{B.1.30}$$

The commutator of L_n can be obtained by using (B.1.18) and TT operator product expansion (B.1.21)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}. \tag{B.1.31}$$

On the other hand, the anti-holomorphic part can be done similarly.

$$\begin{aligned}
&[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \\
&[L_n, \bar{L}_m] = 0.
\end{aligned} \tag{B.1.32}$$

This is the celebrated Virasoro algebra. For higher spin currents, the similar algebras can also be computed from the OPE among $T(z)$ and other W-currents. For more detail on W-algebra, see [28].

B.2 Example: Coulomb Gas Approach

So far, we have introduced the fundamental structure in 2d CFT. Now let us show the simplest example: the free boson and its modification: Coulomb Gas, which is deeply related to Toda field theory introduced in Section 2.3.

B.2.1 Free Theory of a Bosonic Field

Since the Liouville field theory and its generalization − Toda field theory are all 2d CFT modified from free boson. It is worthwhile to pay more attention to its formalism. The action for the free boson in 2D CFT is

$$S = \frac{1}{2}\int d^2z\ \partial\phi\bar{\partial}\phi, \tag{B.2.1}$$

and the corresponding EM tensor is

$$T(z) = -\frac{1}{2} :(\partial_z\phi)^2(z): \quad . \tag{B.2.2}$$

The correlator function of two primary fields ϕ can be given by the conformal symmetry

$$\langle\phi(z)\phi(z')\rangle = -\frac{1}{2}\ln(z-z'). \tag{B.2.3}$$

Since this correlator is also the Green function of a two-dimensional electrostatic problem, the formalism in this section is also known as the Coulomb gas approach. Plunge this into the EM tensor with Wick contraction, we can obtain the TT OPE [28]

$$T(z)T(w) = \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \tag{B.2.4}$$

As is evident from the propagator of $\phi(x)$, it is not a primary field but its derivative and its vertex operator

$$V_\Lambda(z) = :e^{i\Lambda\cdot\phi(z)}: \tag{B.2.5}$$

are primary fields and the Λ is a charge parameter as we will see later. The two-point correlation function of the vertex operators satisfying the neutrality condition[1] is

$$\langle V_\Lambda(z)V_{-\Lambda}(w)\rangle = \frac{1}{(z-w)^{\Lambda^2}}, \tag{B.2.6}$$

where we can see the conformal weight for the vertex operator is $\Delta_\Lambda = \Lambda^2/2$.

B.2.2 Modified Coulomb Gas

Consider a vertex operator with charge $-2\alpha_0$ and insert it into a correlation function, moving its position to infinity or to the north pole of the Riemann sphere. This procedure modifies the conformal dimensions of the vertex operators and the central charge. However, it spoils unitarity except for discrete values of the central charge and a finite set of vertex operators corresponding to minimal models. Such an effect can be realized by coupling the free boson to the scalar curvature R of the space manifold:

$$S = \int d^2z\sqrt{g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + i\alpha_0 R\varphi\right). \tag{B.2.7}$$

It makes anomalous the original $U(1)$ symmetry implemented infinitesimal transformation $\phi \to \phi + \eta$:

$$\delta S = i\alpha_0\int d^2x\sqrt{g}R = i8\pi\alpha_0(1-h), \tag{B.2.8}$$

[1] Otherwise the operator product expansion vanishes.

where h is the number of handles of the Riemann surface and we have $h = 0$ for a sphere. There is a new version of the EM tensor given by Noether's theorem and its analytic component reads

$$T(z) = -\frac{1}{2} \,\mathop{\circ}\limits^{\circ} (\partial\phi^2) \mathop{\circ}\limits^{\circ}\, (z) - i\alpha_0 \partial^2 \phi(z) \tag{B.2.9}$$

This is nothing but the EM tensor for a Liouville field theory with $\alpha_0 = \beta - 1/\beta$. Let us show how the conformal dimensions of the vertex operators and the central charge modify. The OPE between the new EM tensor with the vertex operator

$$T(z)V_\Lambda(w) = \frac{\Lambda\,(\Lambda + 2\alpha_0)}{2(z-w)^2} V_\Lambda(w) + \frac{1}{z-w} \partial V_\Lambda(w), \tag{B.2.10}$$

which implies the conformal dimension $h = (\Lambda^2 + 2\Lambda\alpha_0)/2$. The TT OPE leads to

$$T(z)T(w) = \frac{1 - 24\alpha_0^2}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}, \tag{B.2.11}$$

which implies the central charge $c = 1 - 24\alpha_0^2$.

B.3 Normal orderings on the cylinder

So far, we have introduced the fundamental parts of 2D conformal field theory related to the integrable models in this thesis. The readers can find more details on it in [28, 30]. To close this appendix, let us show explicitly how we obtain some important formulas for normal orderings and their zero modes on the cylinder from Eq.(2.3.44) in Section 2.3.4. After introducing the Bernoulli polynomial $\psi_n(x)$ of the second kind by

$$\frac{(z+1)^x}{\log(1+z)} = \sum_{n=0}^{\infty} \psi_n(x) z^{n-1}, \quad |z| < 1 \tag{B.3.1}$$

and the OPE

$$A_R(z)B_R(w) = \sum_{k=0}^{\Delta_A+\Delta_B} \frac{\{A_R B_R\}_k(w)}{(z-w)^k} + \{A_R B_R\}_0(w) + \cdots, \tag{B.3.2}$$

we expands the integrand in Eq.(2.3.44) at $z = w$. Following the procedure in [81], one obtains

$$:\hat{A}\hat{B}:(v) = \hat{A}_-(v)\hat{B}(v) + \hat{B}(v)\hat{A}_+(v) + \sum_{k=1}^{\Delta_A+\Delta_B} f_k(\Delta_A - 1)\{A_R B_R\}_k(w) w^{\Delta_A+\Delta_B-k}, \tag{B.3.3}$$

where

$$f_k(x) = \psi_k(x) - \frac{(x)_k}{k!}, \tag{B.3.4}$$

$$\hat{A}_+(v) = \sum_{n=0}^{\infty} \hat{A}_n e^{-nv}, \quad \hat{A}_-(v) = \sum_{n=1}^{\infty} \hat{A}_{-n} e^{nv}, \tag{B.3.5}$$

and $(a)_n = a(a-1)\cdots(a-n+1)$. Some values of $f_k(x)$ are seen in Table B.3.

k	1	2	3	4	5	6
$f_k(1)$	$\frac{1}{2}$	$\frac{5}{12}$	$-\frac{1}{24}$	$\frac{11}{720}$	$-\frac{11}{1440}$	$\frac{271}{60480}$
$f_k(2)$	$\frac{1}{2}$	$\frac{11}{12}$	$\frac{3}{8}$	$-\frac{19}{720}$	$\frac{11}{1440}$	$-\frac{191}{60480}$

Table B.3.1: The values of $f_k(x) = \psi_k(x) - \frac{(x)_k}{k!}$ for $x = 1$ and 2.

It is convenient to rewrite the first two terms in (B.3.3) in terms of the normal ordered product on the complex plane to compute multiple normal ordered products. It is found that

$$\begin{aligned} \hat{A}_-(v)\hat{B}(v) + \hat{B}(v)\hat{A}_+(v) = {} & w^{\Delta_A+\Delta_B}\{A_R B_R\}_0(w) \\ & + \sum_{n=1}^{h_A-1}\sum_{k=1}^{\Delta_A+\Delta_B} w^{\Delta_A+\Delta_B-k}\frac{(\Delta_A-n-1)_{k-1}}{(k-1)!}\{A_R B_R\}_k(w). \end{aligned} \tag{B.3.6}$$

Based on the technique above, let us show some important normal ordering in the local IoMs. The first one is

$$:\hat{T}\hat{T}:(v) = w^4\{TT\}_0(w) - \frac{c-10}{12}w^2 T(w) + \frac{3}{2}w^3\partial T(w) + \frac{22c+5c^2}{2880}. \tag{B.3.7}$$

Its zero mode is given by

$$(:\hat{T}\hat{T}:)_0 = 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{5c^2+22c}{2880}. \tag{B.3.8}$$

The formula above determines the normal ordered product of the energy-momentum tensors on the cylinder in terms of that on the complex plane. One can further calculate the normal ordered product of $\hat{T}$ and $:\hat{T}\hat{T}:$ from the OPE of $T_R(z)(\circ TT \circ)_R(w)$ as

$$:\hat{T}(:\hat{T}\hat{T}:):(v) = \hat{T}_- :\hat{T}\hat{T}:(v) + :\hat{T}\hat{T}:(v)\hat{T}_+ + \sum_{k=1}^{6} f_k(1)\{T_R(\circ TT \circ)_R\}_k(w), \tag{B.3.9}$$

where $T_R(w) = T(w) - \frac{c}{24w^2}$, and

$$:\partial\hat{T}\partial\hat{T}:(v) = \partial\hat{T}_{-}\partial\hat{T}:(v) + \partial\hat{T}:(v)\partial\hat{T}_{+} + \sum_{k=1}^{6} f_k(2)\{\partial T_R \partial T_R\}_k(w). \tag{B.3.10}$$

Their zero modes are

$$\begin{aligned}(:\hat{T}(:\hat{T}\hat{T}:):)_0 &= \sum_{n=1}^{\infty} \tilde{L}_{-n}(:\hat{T}\hat{T}:)_n + (:\hat{T}\hat{T}:)_{-n}\tilde{L}_n + L_0^3 - \frac{c+4}{8}L_0^2 \\ &\quad + \left(\frac{c^2}{192} + \frac{7c}{160} + \frac{1}{15}\right)L_0 + \left(-\frac{c^3}{13824} - \frac{11c^2}{11520} - \frac{47c}{15120}\right), \end{aligned} \tag{B.3.11}$$

$$(:\partial\hat{T}\partial\hat{T}:)_0 = -2\sum_{n=1}^{\infty} n^2 L_{-n}L_n + \frac{31c}{30240} - \frac{1}{60}L_0. \tag{B.3.12}$$

Next, we give the normal ordered product of T and spin s primary current W_s.

$$:\hat{T}\hat{W}_s:(v) = \hat{W}_{-}(v)\hat{T}(v) + \hat{T}(v)\hat{W}_{+}(v) + f_1(1)\{T_R W\}_1(w) + f_2(1)\{T_R W\}_2(w). \tag{B.3.13}$$

where $\{T_R W\}_1 = \partial W$ and $\{T_R W\}_2 = sW$. Its zero mode of $:\hat{T}\hat{W}:(u)$ is found to be

$$(:\hat{T}\hat{W}_s:)_0 = \sum_{n=1}^{\infty} L_{-n}(W_s)_n + \sum_{n=1}^{\infty}(W_s)_{-n}L_n + (W_s)_0 L_0 - \frac{2h+c}{24}(W_s)_0. \tag{B.3.14}$$

Finally, we show the normal ordering $:\hat{W}_3\hat{W}_3:(v)$ which appears in the 6th order IoM. It takes the form

$$:\hat{W}_3\hat{W}_3:(v) = \hat{W}_{-}(v)\hat{W}(v) + \hat{W}(v)\hat{W}_{+}(v) + \sum_{k=1}^{6} f_k(2)\{W_3 W_3\}_k(w), \tag{B.3.15}$$

The structure of the OPE $W_3(z)W_3(w)$ depends on the WA_r-algebra. We refer [?] and [?, ?] for low-rank results ($r = 2, 3$). From the normal ordered products of operators on the cylinder, one can calculate the zero modes, we give the results for WA_2 and WA_3 algebras. For the WA_2 algebra, the zero mode is

$$(:\hat{W}_3\hat{W}_3:)_0 = 2\sum_{n=-\infty}^{\infty}(W_3)_{-n}(W_3)_n + (W_3)_0^2 - \frac{191c}{181440} + \left(\frac{17}{360} - \frac{b^2}{30}\right)L_0 - \frac{b^2}{60}L_0^2. \tag{B.3.16}$$

Here $b^2 = \frac{16}{22+5c}$. For the WA_3 algebra, the zero mode is given by

$$
\begin{aligned}
(:\hat{W}_3\hat{W}_3:)_0 = 2\sum_{n=-\infty}^{\infty}(W_3)_{-n}(W_3)_n + (W_3)_0^2 - \frac{1}{3}(W_4)_0 - \frac{191c(c+7)}{1814400} \\
- \frac{4(c+7)}{15(22+5c)}(2\sum_{n=1}^{\infty}L_{-n}L_n + L_0^2 + 2L_0) + \frac{(c+7)(2102+85c)}{3600(22+5c)}L_0. \quad \text{(B.3.17)}
\end{aligned}
$$

where L_n and $(W_3)_n$ are the mode expansion of $T(z)$ and $W_3(z)$.

B.4 Construction of primary W-currents

Here we will explain an explicit construction of the primary W-currents. In general, the spin k current $\tilde{W}_k(z)$ obtained from the Miura transformation is not a primary field. But we can redefine these fields into primary fields by adding lower spin fields and their derivatives. For $\tilde{W}_k(z)$ the primary field is denoted by $W_k(z)$. Here we present our results in WA_r and WD_r algebras with lower ranks.

B.4.1 Primary W-currents in WA_r algebras

Computing the OPEs in lower rank WA_r algebras[2], it is possible to infer the primary W-currents:

$$
\begin{aligned}
&T(z) = \tilde{W}_2(z), \\
&W_3(z) = \tilde{W}_3(z) - \frac{h-2}{2}\alpha_0\partial_z T(z), \\
&W_4(z) = \tilde{W}_4(z) + (h-3)\left(x_1 \,\circ\!\!\!\circ\, TT \,\circ\!\!\!\circ\, (z) - \frac{1}{2}\alpha_0\partial_z\tilde{W}_3(z) - x_2\partial_z^2T(z)\right), \qquad \text{(B.4.1)} \\
&W_5(z) = \tilde{W}_5(z) - \frac{r-3}{2}\alpha_0\partial_z\tilde{W}_4(z) + y_1 \,\circ\!\!\!\circ\, TW_3 \,\circ\!\!\!\circ\, (z) + y_2\partial_z^2W_3(z) + y_3\alpha_0\partial_z^3T(z),
\end{aligned}
$$

[2] We used Mathematica package [?] for computation of OPE.

with

$$
\begin{aligned}
x_1 &= \frac{(h-2)((5h+7)c-2h+2)}{2(5c+22)h\,(h+1)\,(h-1)}, \quad x_2 = \frac{(h-2)(2c^2+(h+15)c-10(h-1))}{4(5c+22)h\,(h+1)\,(h-1)},\\
y_1 &= \frac{(h-4)(h-3)((7h+13)c-6h+6)}{(7c+114)h\,(h+1)\,(h-1)},\\
y_2 &= -\frac{3(h-4)(h-3)(c^2+(h+21)c-18(h-1))}{4(7c+114)h\,(h+1)\,(h-1)},\\
y_3 &= \frac{(h-4)(h-3)\,((32-7h)c^2+(7h^2-117h+620)c+6(h-1)(19h-92))}{24(7c+114)h\,(h+1)\,(h-1)}.
\end{aligned}
\tag{B.4.2}
$$

Correspondingly, the relations between conformal weights Δ_i of primary $W_k(z)$ and $\tilde{\Delta}_i$ can be derived from Eq.(B.4.1).

$$
\begin{aligned}
&\Delta_2 = \tilde{\Delta}_2,\\
&\Delta_3 = \tilde{\Delta}_3 + (h-2)\alpha_0\Delta_2,\\
&\Delta_4 = \tilde{\Delta}_4 + (h-3)\left(c_4^{(1)}\Delta_2^2 + \frac{3}{2}\alpha_0\tilde{\Delta}_3 + c_4^{(2)}\Delta_2\right),\\
&\Delta_5 = \tilde{\Delta}_5 + (h-4)\left(c_5^{(1)}\Delta_2\Delta_3 + 2\alpha_0\tilde{\Delta}_4 + c_5^{(2)}\Delta_3 - (h-3)(h-2)\alpha_0^2\Delta_2\right)
\end{aligned}
\tag{B.4.3}
$$

with the coefficients

$$
\begin{aligned}
c_4^{(1)} &= \frac{(h-2)[(5h+7)c-2h+2]}{2(5c+22)h\,(h+1)\,(h-1)}, \quad c_4^{(2)} = -\frac{(h-2)[6c^2-(7h-31)c-26(h-1)]}{2(5c+22)h\,(h+1)\,(h-1)},\\
c_5^{(1)} &= \frac{(h-3)[(7h+13)c-6h+6]}{(7c+114)h\,(h+1)\,(h-1)}, \quad c_5^{(2)} = -\frac{3(h-3)\,[3c^2-4(c+12)h+50c+48]}{(7c+114)h\,(h+1)\,(h-1)}.
\end{aligned}
$$

B.4.2 Primary W-currents in WD_r algebra

Here we first discuss a recursive construction of W-currents in WD_r algebra. We first compute the OPE of $V_r^{(r)}(z)$ using the relation

$$
R_r^{(r)}(z)V_r^{(r)}(w) = (-\tilde{p}_r R_{r-1}^{(r-1)} + \alpha_0\partial R_{r-1}^{(r-1)})(z)(-\tilde{p}_r R_{r-1}^{(r-1)} + \alpha_0\partial R_{r-1}^{(r-1)})(w). \tag{B.4.4}
$$

We then use the formula (2.3.27) for $r-1$ in the RHS of the OPE, which leads to the recursion relations for the constant A_r and the W-currents $W_{2k}^{(r)}$:

$$
A_r = A_{r-1} - (2r-1)(2r-2)a^2 A_{r-1}, \tag{B.4.5}
$$

$$W_2^{(r)} = W_2^{(r-1)} + \frac{1}{2}(\tilde{p}_r\tilde{p}_r) - (r-1)a\partial\tilde{p}_r, \tag{B.4.6}$$

$$\begin{aligned} W_4^{(r)} &= W_4^{(r-1)} + \frac{1}{4}(1-(2r-3)(2r-4)a^2)\Big\{(\partial^2\tilde{p}_r\tilde{p}_r) - \frac{1}{2}\partial^2(\tilde{p}_r\tilde{p}_r)\Big\} \\ &+ (\tilde{p}_r(\tilde{p}_r W_2^{(r-1)})) - 2(r-2)a(\partial\tilde{p}_r W_2^{(r-1)}) - a(\tilde{p}_r\partial W_2^{(r-1)}) \\ &+ \frac{1}{12}(r-1)a(1-(2r-3)(2r-4)a^2)\partial^3\tilde{p}_r + (r-2)a^2\partial^2 W_2^{(r-1)}. \end{aligned} \tag{B.4.7}$$

For $r=1$, we define

$$W_2^{(1)} = \frac{1}{2}(\tilde{p}_1\tilde{p}_1), \tag{B.4.8}$$

$$W_4^{(1)} = \frac{1}{4}(1-2a^2)\Big\{(\partial^2\tilde{p}_1\tilde{p}_1) - \frac{1}{2}\partial^2(\tilde{p}_1\tilde{p}_1)\Big\}. \tag{B.4.9}$$

Thus we obtain the non-primary W-currents $\tilde{W}_4$. The primary W_4 current is given by

$$W_4(z) = \tilde{W}_4(z) + a_1 :TT:(z) + a_2\partial_z^2 T(z) \tag{B.4.10}$$

with the coefficients

$$\begin{aligned} a_1 &= -\frac{5r+1}{22+5c}\left(1-2(r-2)(2r-3)\alpha_0^2\right), \\ a_2 &= \frac{1}{22+5c}\left(1-2(r-2)(2r-3)\alpha_0^2\right)\left(\frac{4r+5}{2} - r(r-1)(2r-1)\alpha_0^2\right). \end{aligned} \tag{B.4.11}$$

The relation between Δ_4 of the primary $W_4(z)$ and $\tilde{\Delta}_4$ is expressed as

$$\Delta_4 = \tilde{\Delta}_4 + a_1\Delta_2^2 + 2(a_1+3a_2)\Delta_2, \tag{B.4.12}$$

where a_1 and a_2 is given in (B.4.11).

For $W_6(z)$, so far, we can only give its value in WD_4 algebra. The primary spin 6 current W_6 is given by

$$\begin{aligned} W_6 &= \tilde{W}_6 - \frac{1}{3}:T\tilde{W}_4: - \frac{-13+1388\alpha_0^2-22560\alpha_0^4}{84(4-241\alpha_0^2+2352\alpha_0^4)} :T(:TT:): \\ &- \frac{9-2992\alpha_0^2+70688\alpha_0^4-288960\alpha_0^6}{168(4-241\alpha_0^2+2352\alpha_0^4)} :T\partial_z^2 T: \\ &- \frac{23-815\alpha_0^2+6302\alpha_0^4+15960\alpha_0^6}{84(4-241\alpha_0^2+2352\alpha_0^4)} :\partial T\partial T: \\ &+ \frac{1-4\alpha_0^2}{6}\partial^2\tilde{W}_4 - \frac{-2+425\alpha_0^2-10274\alpha_0^4+53496\alpha_0^6-40320\alpha_0^8}{144(4-241\alpha_0^2+2352\alpha_0^4)}\partial^4 T, \\ \Delta_6 &= \tilde{\Delta}_6 - \frac{1}{3}\Delta_2\tilde{\Delta}_4 + t_3\Delta_2^3 + t_4\tilde{\Delta}_4 + t_5\Delta_2^2 + t_6\Delta_2, \end{aligned} \tag{B.4.13}$$

with

$$
\begin{aligned}
t_3 &= \frac{13 - 1388\alpha_0^2 + 22560\alpha_0^4}{84(4 - 241\alpha_0^2 + 2352\alpha_0^4)}, \\
t_4 &= 2 - \frac{40\alpha_0^3}{3}, \\
t_5 &= \frac{-41 + 3908\alpha_0^2 - 101912\alpha_0^4 + 803040\alpha_0^6}{84(4 - 241\alpha_0^2 + 2352\alpha_0^4)}, \\
t_6 &= \frac{-1 - 30\alpha_0^2 + 6796\alpha_0^4 - 186480\alpha_0^6 + 1411200\alpha_0^8}{42(4 - 241\alpha_0^2 + 2352\alpha_0^4)}.
\end{aligned}
$$

The expansion of $\tilde{\Delta}_6$ in terms of momentum becomes

$$
\Delta_6 = \frac{1}{2}\sigma_3 + (-1 + \frac{11}{2}\alpha_0^2)\sigma_2 + (\frac{1}{2} - 8a^2 + \frac{61}{2}\alpha_0^4)\sigma_1 - 7\alpha_0^2 + 161\alpha_0^4 - \frac{1429\alpha_0^6}{2}.
$$

Appendix C

Simple Lie algebra and Kac-Moody algebra

C.1 Preliminary: $su(2)$ algebra

To warm up before the introduction to more general Lie algebras, we study the construction of representations for $su(2)$, the simplest non-abelian algebra given by

$$[J_a, J_b] = i\epsilon_{abc}J_c. \tag{C.1.1}$$

A familiar representation is provided by the Pauli matrices $J_a = \sigma_a/2$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.1.2}$$

This representation arises as the action of the 3D rotation group on spin 1/2 particles.

C.1.1 Cartan Weyl basis

We now combine the generators J_a into Cartan-Weyl form. The first step is to choose a maximal set of mutually commuting generators (this is the so-called Cartan subalgebra whose dimension is called the rank of the algebra). For $su(2)$, there is only one generator J_3 in this subalgebra.

Next, we take the linear combinations on remaining generators

$$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \tag{C.1.3}$$

such that they have simple commutation relations with the Cartan generator J_3

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_-. \tag{C.1.4}$$

$J_\pm$ are the ladder operators. To see this, it is natural to take a basis where the representative of J_3 is diagonal and label each vector in the basis by its J_3 eigenvalue, $|\mu\rangle$

$$J_3 |\mu\rangle = \mu |\mu\rangle . \tag{C.1.5}$$

Such charges μ are called weights of the representation. Starting with a state of weight $|\mu\rangle$, we can build the states $J_\pm |\mu\rangle$, which are eigenstates of J_3 with eigenvalues $\mu \pm 1$

$$J_3 J_\pm |\mu\rangle = ([J_3, J_\pm] + J_\pm J_3) |\mu\rangle = (\pm J_\pm + \mu J_\pm) |\mu\rangle = (\mu \pm 1) J_\pm |\mu\rangle . \tag{C.1.6}$$

Since the representation we are interested in are finite dimensional, we define $\mu = j$ as the highest weight. Recall what we have learnt in quantum mechanics, there is also a lowest weight which is just $\mu = -j$. Finally, in total we obtain a $2j + 1$ dimensional representation from j to $-j$.

Now turn back to Eq.(C.1.4), if we rewrite the commutators into

$$J_3 |J_+\rangle = J_+, \qquad J_3 |J_-\rangle = -J_-, \tag{C.1.7}$$

where the generators $J_\pm$ themselves become the basis, such a representation is called adjoint representation. The weights in the adjoint representation is also called the roots of the algebra. In $su(2)$ case, we have the roots $-1, 0, 1$ for J_-, J_3, J_+ respectively.

C.2 General simple Lie algebras

Definition C.2.1 *A Lie algebra can be specified by a set of generators $\{J_a\}$ and their commutation relations*

$$[J_a, J_b] = \sum_c i f_{abc} J_c. \tag{C.2.1}$$

The constants f_{abc} are the structure constants. And the number of generators is called the dimension d.

Definition C.2.2 *Simple Lie algebras are Lie algebras that contain no proper ideal (meaning no proper subset of generators* $\{L_a\}$ *such that* $[L_a, J_b] \in \{L_a\}$ *for any* J_b*). A direct sum of simple algebras is said to be semisimple.*

In this section, we will only consider the simple Lie algebras.

C.2.1 Cartan Weyl basis

Now we generalize the contents in the $su(2)$ algebra. Pick a maximal set of mutually commuting hermitian generators, which we call H_i, $i = 1\ldots, r$. The number of such generators is called the rank r of the group; they generate the Cartan subalgebra $U(1)^r$ of the Lie algebra.

The second step is to take linear combinations of the remaining operators so that they have easy commutators with the H_i:

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{C.2.2}$$

Now they are no longer Hermitian, rather $E_\alpha^\dagger = E_{-\alpha}$.

Using the Jacobi identity, it is also possible to show that

$$\begin{aligned} [E_\alpha, E_{-\alpha}] &= \frac{2}{|\alpha|^2}\sum_{i=1}^{r}\alpha_i H_i, \\ [E_\alpha, E_\beta] &= \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta}, & \text{if } \alpha+\beta \text{ is root} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{C.2.3}$$

where $N_{\alpha,\beta}$ are some constants and $|\alpha|^2 = \sum_i \alpha_i\alpha_i$ is the inner product. It is not difficult to notice that the number of roots is usually not equal to the rank which means the roots are not linearly independent. So we define a simple root as a positive root which cannot be written as a sum of positive roots with positive coefficients. The root is said to be positive if the first nonzero number in the sequence $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ is positive. Simple roots have nice properties, in particular, the set of simple roots of an algebra is linearly independent, and there are r simple roots to form a basis of root space.

C.2.2 Weight representation

To construct a representation, we choose a basis of the representation space where all matrices representing the Cartan generators are diagonal, and we label the vectors in the basis (eigenstates of the matrix representing H_i) by the corresponding eigenvalues.

$$H_i |\mu\rangle = \mu_i |\mu\rangle . \tag{C.2.4}$$

For weight vector $(\mu_1, \ldots, \mu_r)$, we define the inner product: $(\mu, \lambda) = \mu \cdot \lambda = \mu_i \lambda_i$.

As in $su(2)$ case, weights differ by roots.

$$H_i E_{\pm\alpha} |\mu\rangle = (\alpha_i E_{\pm\alpha} + E_{\pm\alpha} H_i) |\mu\rangle = (\mu_i \pm \alpha_i) E_{\pm\alpha} |\mu\rangle . \tag{C.2.5}$$

So $E_\alpha |\mu\rangle$, if nonzero, must be proportional to a state $|\mu + \alpha\rangle$. Representation of interest are the finite-dimensional ones. For any state $|\mu\rangle$ in a finite-dimensional representation, there are necessarily two positive integers p and q, such that

$$\begin{aligned} (E_\alpha)^{p+1} |\mu\rangle &\sim E_\alpha |\mu + p\alpha\rangle = 0, \\ (E_{-\alpha})^{q+1} |\mu\rangle &\sim E_{-\alpha} |\mu - q\alpha\rangle = 0. \end{aligned} \tag{C.2.6}$$

By means of the inner product, it is possible the project the algebra onto $su(2)$ subalgebra associated with the root α where $J_3 = \frac{\alpha \cdot H}{|\alpha|^2}$. Let the dimension of the later be $2j+1$; then from the state $|\mu\rangle$, the highest and lowest weights can be obtained from Eq.(C.2.6).

$$j = \frac{(\alpha, \mu)}{|\alpha|^2} + p, \qquad -j = \frac{(\alpha, \mu)}{|\alpha|^2} - p. \tag{C.2.7}$$

Eliminating j yields

$$2\frac{(\alpha, \mu)}{|\alpha|^2} = -(p - q). \tag{C.2.8}$$

This conclusion plays an important role in the construction of Cartan matrix and Dynkin diagram.

C.3 Dynkin diagrams

The scalar products of simple roots define the Cartan matrix

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_j^2}. \tag{C.3.1}$$

According to Eq.(C.2.8),

$$\frac{(\alpha_i, \alpha_j)}{|\alpha_j|^2} = -\frac{1}{2} m; \qquad \frac{(\alpha_i, \alpha_j)}{|\alpha_i|^2} = -\frac{1}{2} m'; \quad m, m' \in \mathbf{Z}. \tag{C.3.2}$$

We obtain a constraint on the relative angle of the roots

$$\cos^2 \theta_{\alpha_i,\alpha_j} = \frac{(\alpha_i, \alpha_j)^2}{|\alpha_i|^2|\alpha_j|^2} = \frac{mm'}{4}. \tag{C.3.3}$$

The angle is constrained to be 0, 30, 45, 60, 90, 120, 135, 150 or 180 degrees. On the other hand

$$(\alpha_i, \alpha_j) \leq 0, \quad i \neq j. \tag{C.3.4}$$

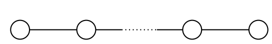
So only 0, 90, 120, 135, 150 or 180 degrees are posiible. In conclusion $i \neq j$, A_{ij} could only be -1, -2, or -3 and due to $A_{ij}A_{ji} < 4$, at least one of A_{ij} or A_{ji} should be -1.

The information of Cartan matrix is conveniently codified in a picture called the Dynkin diagram. The rules to obtain the simple root system from the diagram are as follows.

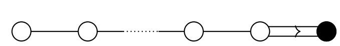
- Each node corresponds to a simple root.
- The number of lines joining two nodes gives us the angle between the two simple roots: no line means 90°, one line means 120°, two lines means 135°, three lines means 150°.
- Dark nodes correspond to shorter roots (The length of roots can be seen from the relative angle) and arrows point to shorter roots.

Due to different types of Dynkin diagram, simple Lie algebra can be classified as in the following figure.

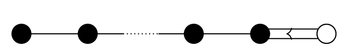
$\mathbf{A}_n$:



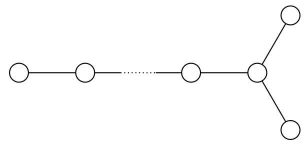
$\mathbf{B}_n$:



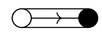
$\mathbf{C}_n$:



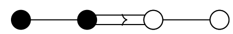
$\mathbf{D}_n$:



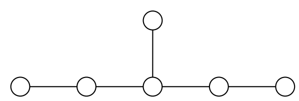
$\mathbf{G}_2$:



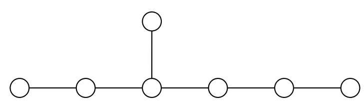
$\mathbf{F}_4$:



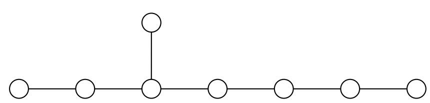
$\mathbf{E}_6$:



$\mathbf{E}_7$:



$\mathbf{E}_8$:



C.4 The Chevalley Basis

The Cartan matrix fixes completely the commutation relations of the algebra. In some reference, the Dynkin diagram can even be used as the definition of simple Lie algebra. The Chevalley basis is constructed to manifest the Cartan matrix in the commutation relations. Set

$$e_i = E_{\alpha_i}, \qquad f_i = E_{-\alpha_i}, \qquad h_i = \frac{2\alpha_i \cdot H}{|\alpha_i|^2}, \tag{C.4.1}$$

whose commutation relations are

$$\begin{aligned} &[h_i, h_j] = 0, \\ &[h_i, e_j] = A_{ij} e_j, \\ &[h_i, f_j] = -A_{ij} f_j, \\ &[e_i, f_j] = \delta_{ij} h_i. \end{aligned} \tag{C.4.2}$$

The remaining ladder operators are subject to the Serre relations

$$\begin{aligned}
&[\mathrm{ad}(e_i)]^{1-A_{ji}} e_j = 0, \\
&[\mathrm{ad}(f_i)]^{1-A_{ji}} f_j = 0,
\end{aligned} \tag{C.4.3}$$

where $\mathrm{ad}(e_i)e_j = [e_i, e_j]$.

C.4.1 Convention

Finally, we introduce some useful notation used in this thesis. It is convenient to introduce the co-roots as

$$\check{\alpha}_i = \frac{2\alpha_i}{|\alpha_i|^2}. \tag{C.4.4}$$

Then the fundamental weights $\{\omega_i\}$ can be defined by

$$(\omega_i, \check{\alpha}_j) = \delta_{ij}. \tag{C.4.5}$$

Similarly, the co-fundamental weights $\{\check{\omega}_i\}$ can be defined as

$$(\check{\omega}_i, \alpha_j) = \delta_{ij}. \tag{C.4.6}$$

There is so-called the Weyl (co-Weyl) vector given by the fundamental (co-fundamental) weights

$$\rho = \sum_{i=1}^{r} \omega_i, \quad \rho^\vee = \sum_{i=1}^{r} \check{\omega}_i \tag{C.4.7}$$

which we used in the conformal transformation (3.4.3) in Toda field theory. There is an important relation between the Weyl vector ρ and the dual Coxter number $h^\vee$ that is equal to h for a simply-laced Lie algebra $\mathfrak{g}$:

$$\rho^2 = \frac{h^\vee \dim(\mathfrak{g})}{12}. \tag{C.4.8}$$

Finally, the highest root $\theta = \sum m_i \alpha_i$ is defined to make the sum $\sum m_i$ maximized. The highest root and its co-root are usually written as

$$\theta = \sum_{i=1}^{r} a_i \alpha_i, \qquad \check{\theta} = \sum_{i=1}^{r} \check{a}_i \check{\alpha}_i, \tag{C.4.9}$$

where a_i ($\check{a}_i$) are called the marks (comarks) from which we can define another important parameter called the (dual) Coxter number

$$h = \sum_{i=1}^{r} a_i + 1, \qquad \check{h} = \sum_{i=1}^{r} \check{a}_i + 1. \tag{C.4.10}$$

The Coxeter numbers for the affine Lie algebras in this paper are listed in Table C.4.1.

$\hat{\mathfrak{g}}$	$A_r^{(1)}$	$D_r^{(1)}$	$C_r^{(1)}$	$B_r^{(1)}$	$A_{2r-1}^{(2)}$	$D_{r+1}^{(2)}$
h	$r+1$	$2r-2$	$2r-1$	$2r$	$2r-1$	$2r+2$

Table C.4.1: The Coxeter numbers of affine Lie algebras

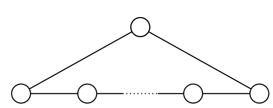
C.5 Affine Lie algebras

Now we generalize the simple Lie algebra g to the affine Lie algebras $\hat{g}$ (In this thesis, we only consider untwist case). The simplest way to understand it is from the extended Cartan matrix and the extended Dynkin diagram. The Dynkin diagram of $\hat{g}$ is obtained from that of g by the addition of an extra node representing an extra root α_0 besides simple roots. α_0 is linearly dependent with other simple roots $\alpha_0 = -\theta = \sum_{i=1}^{r} a_i \alpha_i$. The corresponding Cartan matrix now is generlized by adding additional entries

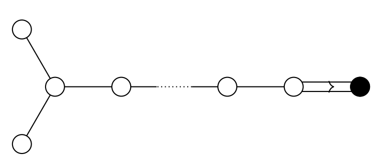
$$A_{0j} = (\alpha_0, \check{\alpha}_j) = -(\theta, \check{\alpha}_j) = -\sum_{i=1}^{r} a_i (\alpha_i, \check{\alpha}_j). \tag{C.5.1}$$

So the extra nodes in the extended Dynkin diagram linked to the α_i-nodes by $A_{0i}A_{i0}$ lines. The extended Dynkin diagram is given as follows

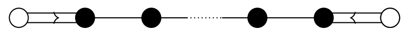
$\mathbf{A}_n^{(1)}$:



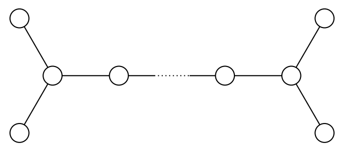
$\mathbf{B}_n^{(1)}$:



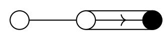
$\mathbf{C}_n^{(1)}$:



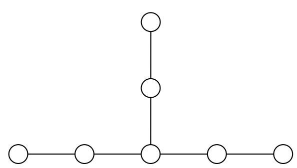
$\mathbf{D}_n^{(1)}$:



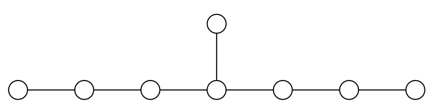
$\mathbf{G}_2^{(1)}$:



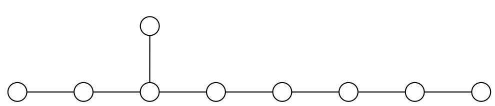
$\mathbf{E}_6^{(1)}$:



$\mathbf{E}_7^{(1)}$:



$\mathbf{E}_8^{(1)}$:



Finally we should mention that the zeroth (co)mark $\check{a}_0$ is defined to be 1. So from (C.5.1), we can obtain

$$\sum_{i=0}^{r} a_i A_{ij} = \sum_{i=0}^{r} A_{ij}\check{a}_j = 0. \tag{C.5.2}$$

And the dual Coxter number (C.4.10) reads

$$h = \sum_{i=0}^{r} \check{a}_i. \tag{C.5.3}$$

Finally, the affine extension of the Chevally basis can be given by

$$e_0 = \lambda E_{\alpha_0}, \qquad f_0 = \lambda^{-1} E_{-\alpha_0}, \qquad h_0 = \hat{k} - \theta \cdot H, \tag{C.5.4}$$

whose commutation relations are

$$\begin{aligned} [h_i, h_j] &= 0, \\ [h_i, e_j] &= A_{ij} e_j, \\ [h_i, f_j] &= -A_{ij} f_j, \\ [e_i, f_j] &= \delta_{ij} h_i \end{aligned} \tag{C.5.5}$$

with the Serre relations

$$\begin{aligned} [\mathrm{ad}(e_i)]^{1-A_{ji}} e_j &= 0, \\ [\mathrm{ad}(f_i)]^{1-A_{ji}} f_j &= 0, \end{aligned} \tag{C.5.6}$$

where λ is some variable and $\hat{k}$ is the central element commuting with all generators. Strictly speaking, the untwist affine Lie algebra can be viewed as follows: $\tilde{g} = g \otimes \mathbf{C}[\lambda, \lambda^{-1}]$.

The representations for affine Lie algebras

Finally, we show the representations for the generators of affine Lie algebras $\hat{\mathfrak{g}}$ used in this thesis. The notation $e_{i,j}$ below denotes the matrix with components $(e_{i,j})_{a,b} = \delta_{i,a}\delta_{j,b}$. α_i $(1 \leq i \leq r)$ denote the simple roots and α_0 denotes the affine root. We also set $E_{-\alpha_i} = E_{\alpha_i}^{\mathrm{T}}$.

$A_r^{(1)}$: The Coxeter number $h = r + 1$. The $(r+1)$-dimensional representation is $(r \geq 1)$

$$E_{\alpha_0} = e_{r+1,1}, \quad E_{\alpha_i} = e_{i,i+1}. \tag{C.5.7}$$

$B_r^{(1)}$: The Coxeter number $h = 2r$. The $(2r+1)$-dimensional representation is $(r \geq 2)$

$$\begin{aligned} E_{\alpha_0} = e_{2r,1} + e_{2r+1,2}, \quad E_{\alpha_i} &= e_{i,i+1} + e_{2r+1-i,2r+2-i}, \\ E_{\alpha_r} &= \sqrt{2}(e_{r,r+1} + e_{r+1,r+2}). \end{aligned} \tag{C.5.8}$$

$D_r^{(1)}$: The Coxeter number $h = 2r - 2$. The $2r$-dimensional representation is $(r \geq 3)$

$$E_{\alpha_0} = e_{2r-1,1} + e_{2r,2}, \quad E_{\alpha_i} = e_{i,i+1} + e_{2r-i,2r+1-i}, \quad E_{\alpha_r} = e_{r-1,r+1} + e_{r,r+2}. \tag{C.5.9}$$

$A^{(2)}_{2r-1}$**:** The Coxeter number $h = 2r - 1$. The $2r$-dimensional representation is $(r \geq 2)$

$$E_{\alpha_0} = e_{2r,2} + e_{2r-1,1}, \quad E_{\alpha_i} = e_{i,i+1} + e_{2r-i,2r+1-i}, \quad E_{\alpha_r} = e_{r,r+1}. \tag{C.5.10}$$

$D^{(2)}_{r+1}$**:** The Coxeter number $h = 2r+2$. The $(2r+2)$-dimensional representation is $(r \geq 2)$

$$\begin{gathered} E_{\alpha_0} = \sqrt{2}(e_{r+2,1} + e_{2r+2,r+2}), \quad E_{\alpha_i} = e_{i,i+1} + e_{2r+2-i,2r+3-i}, \\ E_{\alpha_r} = \sqrt{2}(e_{r,r+1} + e_{r+1,r+3}). \end{gathered} \tag{C.5.11}$$

$A^{(2)}_{2r}$**:** The Coxeter number $h = 2r$. The $(2r+1)$-dimensional representation is $(r \geq 1)$

$$E_{\alpha_0} = e_{2r+1,1}, \quad E_{\alpha_i} = e_{i,i+1} + e_{2r+1-i,2r+2-i}, \quad E_{\alpha_r} = \sqrt{2}(e_{r,r+1} + e_{r+1,r+2}. \tag{C.5.12}$$

$C^{(1)}_r$**:** The Coxeter number $h = 2r - 1$. The $2r$-dimensional representation is $(r \geq 1)$

$$E_{\alpha_0} = e_{2r,1}, \quad E_{\alpha_i} = e_{i,i+1} + e_{2r-i,2r+1-i}, \quad E_{\alpha_r} = e_{r,r+1}. \tag{C.5.13}$$

Appendix D

Higher-order WKB integrals

In this appendix, we present the results of higher-order WKB integrals Q_k with $k = 6, 7, 8$ for the ODE associated with $A_r^{(1)}$ algebras and Q_6 for $D_r^{(1)}$ algebras.

D.1 Higher-order WKB integrals in $A_r^{(1)}$-type ODEs

In Eq.(4.2.11), we have shown the formula for L_i up to $i = 5$ from the $A_r^{(1)}$-type ODE. Here we present the formula for L_6, L_7 and L_8:

$$\begin{aligned} L_6 = s_6 &+ \frac{5}{2}(r-4)s_5 + \frac{1}{24}(r-4)(r-3)(r+70)s_4 \\ &+ \frac{5}{48}(r-4)(r-3)(r-2)(r+20)s_3 \\ &+ \frac{1}{5760}(r-4)(r-3)(r-2)(r-1)(5r^2+698r+5760)s_2 \\ &+ \frac{1}{2903040}(r-4)(r-3)(r-2)(r-1)r(r+1)(r+2)(35r^2+7238r+103560), \end{aligned} \tag{D.1.1}$$

$$\begin{aligned} L_7 = -s_7 &- 3(r-5)s_6 - \frac{1}{24}(r-5)(r-4)(r+102)s_5 \\ &- \frac{1}{8}(r-5)(r-4)(r-3)(r+30)s_4 \\ &- \frac{1}{5760}(r-5)(r-4)(r-3)(r-2)(5r^2+1080r+13152)s_3 \\ &- \frac{1}{1920}(r-5)(r-4)(r-3)(r-2)(r-1)(5r^2+1490r+1920)s_2 \\ &- \frac{1}{967680}(r-5)(r-4)(r-3)(r-2)(r-1)r(r+1)(r+2)(35r^2+3038r+33000), \end{aligned} \tag{D.1.2}$$

$$
\begin{aligned}
L_8 = {} & s_8 + \frac{7}{2}(r-6)s_7 + \frac{1}{24}(r-6)(r-5)(r+140)s_6 \\
& + \frac{7}{48}(r-6)(r-5)(r-4)(r+42)s_5 \\
& + \frac{1}{5760}(r-6)(r-5)(r-4)(r-3)(5r^2+1398r+25984)s_4 \\
& + \frac{7}{11520}(r-6)(r-5)(r-4)(r-3)(r-2)(5r^2+418r+4032)s_3 \\
& + \frac{1}{2903040}(r-6)(r-5)(r-4)(r-3)(r-2)(r-1) \\
& \times (35r^3+14658r^2+539800r+2903040)s_2 \\
& + \frac{1}{1393459200}(r-6)(r-5)(r-4)(r-3)(r-2)(r-1)r(r+1)(r+2) \\
& \times (175r^3+97230r^2+5144984r+45632832).
\end{aligned}
\tag{D.1.3}
$$

In Eq.(4.2.20), we have shown the WKB integral up to the 5th order. Here we put the 6th, 7th and the 8th WKB integrals, which predict the corresponding vacuum eigenvalues of the IoMs:

$$
\begin{aligned}
Q_6 = {} & \frac{1}{h}J_{5,6}\Big(L_6 + (h-5)\Big[-\frac{1}{2h}L_3^2 - \frac{1}{h}L_2L_4 + \frac{2h-5}{6h^2}L_2^3 + \frac{5}{2}L_5 \\
& - \frac{1}{2h}(5h-13)L_2L_3 - \frac{5}{24}(3hM-8h+41)L_4 \\
& + \frac{1}{48h}(15h^2M-40h^2-35hM+205h-257)L_2^2 - \frac{15}{16}(h-3)(hM+3)L_3 \\
& - \frac{1}{1152}(10h^4M^3+10h^4M^2-22h^3M^3-113h^3M^2+288h^3M+192h^3 \\
& + 205h^2M^2-1750h^2M-730h^2+2410hM-235h+1895)L_2 \\
& + \frac{1}{580608}(h-1)h(hM-5)(hM-1)(8h^5M^3+8h^5M^2+48h^4M^3 \\
& + 96h^4M^2+48h^4M-152h^3M^3-474h^3M^2-474h^3M-152h^3+478h^2M^2 \\
& + 969h^2M+478h^2-309hM-309h+61)\Big]\Big),
\end{aligned}
$$

$$
\begin{aligned}
Q_7 = \frac{1}{h}J_{6,7}\Big(&-L_7+(h-6)\Big[\frac{1}{h}L_3L_4+\frac{1}{h}L_2L_5-\frac{h-3}{h^2}L_2^2L_3-3L_6+\frac{3(h-3)}{2h}L_3^2 \\
&-\frac{(h-3)(h-2)}{h^2}L_2^3+\frac{1}{2}((2M-5)h+33)L_5-\frac{(h-3)((2M-5)h+18)}{2h}L_2L_3 \\
&+\frac{(3h-10)}{h}L_2L_4+2(h-4)(hM+4)L_4-\frac{(h-3)(h-2)(hM+2)}{h}L_2^2 \\
&+\frac{1}{40}(h-3)\left(h^3M^3+h^3M^2-12h^2M^2+31h^2M+20h^2-195hM-42h-254\right)L_3 \\
&+\frac{1}{40}(h-3)(h-2)(hM+2)\left(h^2M^2+h^2M-14hM-11h-7\right)L_2\Big]\Big).
\end{aligned} \tag{D.1.4}
$$

In Section 4.2.1, we have given the prediction for i_6 in terms of the momentum parameter σ_i. It is also possible to rewrite i_6 in terms of Δ_i.

$$
i_6 = \Delta_6 + a_6^{(1)}\left\{\Delta_4\Delta_2+\frac{1}{2}\left[\Delta_3^2+a_6^{(2)}\left(\Delta_2^3+a_6^{(3)}\Delta_2^2+a_6^{(4)}\Delta_2+a_6^{(5)}\right)\right]\right\}
$$

with the tedious coefficients

$$
\begin{aligned}
a_6^{(2)} &= \frac{(-101+5h+29h^2-5h^3)c+2(-1+h)(-37-48h+25h^2)}{3(5c+22)h\left(h^2-1\right)}, \\
a_6^{(3)} &= \frac{(-5h^3+29h^2+5h-101)c^2+(5h^4-8h^3+69h^2+104h-626)c}{8\left(5h^3-29h^2-5h+101\right)c-16(-1+h)(-37-48h+25h^2)} \\
&\quad-\frac{2(h-1)\left(h^3-157h^2+282h+188\right)}{8\left(5h^3-29h^2-5h+101\right)c-16(-1+h)(-37-48h+25h^2)}, \\
a_6^{(4)} &= \frac{5\left(5h^3-29h^2-5h+101\right)c^3}{960(101-5h-29h^2+5h^3)c-1920(h-1)\left(25h^2-48h-37\right)} \\
&\quad-\frac{2\left(25h^4-165h^3+698h^2+650h-3662\right)c^2}{960(101-5h-29h^2+5h^3)c-1920(h-1)\left(25h^2-48h-37\right)} \\
&\quad+\frac{(25h^5-295h^4-1843h^3+6103h^2-6418h+10948)c}{960(101-5h-29h^2+5h^3)c-1920(h-1)\left(25h^2-48h-37\right)} \\
&\quad+\frac{2(h-1)\left(55h^4-398h^3-2147h^2+5922h+1248\right)}{960(101-5h-29h^2+5h^3)c-1920(h-1)\left(25h^2-48h-37\right)},
\end{aligned}
$$

$$
\begin{aligned}
a_6^{(5)} = &\frac{35\,(5h^3-29h^2-5h+101)\,c^4}{483840\,(5h^3-29h^2-5h+101)\,c-967680(h-1)\,(25h^2-48h-37)}\\
&-\frac{(525h^5-3815h^4+12621h^3+34311h^2-67134h-89284)c^3}{483840(h+1)\,(5h^3-29h^2-5h+101)\,c-967680(h^2-1)\,(25h^2-48h-37)}\\
&+\frac{(525h^6-5880h^5-12432h^4+30930h^3-87505h^2+154110h+318892)c^2}{483840(h+1)\,(5h^3-29h^2-5h+101)\,c-967680(h^2-1)\,(25h^2-48h-37)}\\
&-\frac{(h-1)\,(175h^6-3500h^5+14388h^4+133250h^3-189355h^2-369510h-13128)\,c}{483840(h+1)\,(5h^3-29h^2-5h+101)\,c-967680(h^2-1)\,(25h^2-48h-37)}\\
&-\frac{22(h-4)(h-3)(h-2)(h-1)^2\,(35h^2+112h+93)}{483840(h+1)\,(5h^3-29h^2-5h+101)\,c-967680(h^2-1)\,(25h^2-48h-37)}.
\end{aligned}
$$

Here $a_6^{(1)}$ is absent due to the difficulty in constructing Δ_6. But when $r=1$, 2, and 3, it leads to the result in Eq. (2.3.59), Eq.(2.3.60) and Eq.(2.3.61) after some normalization.

Next, we turn to Q_7 and Q_8. Based on Eqs.(D.1.2) and (D.1.4), one can obtain the WKB integral in terms of momentum parameter s_i

$$
\begin{aligned}
Q_7 = J_{6,7}\Big[\frac{1}{h}s_7 + (h-6)\Big(&-\frac{1}{h^2}(\sigma_2\sigma_5+\sigma_3\sigma_4)+\frac{h-3}{h^3}\sigma_2^2\sigma_3+\frac{h-3}{h}(1+M)\sigma_2\sigma_3\\
&-(1+M)\sigma_5-\frac{h(h-3)(1+M)\,(hM^2(1+M)-12)}{40}\sigma_3\Big)\Big].
\end{aligned}
$$

After the plunge of the ODE/IM correspondence (4.5.8), one can predict that the vacuum eigenvalue of the IoM i_7 is of the form

$$
i_7 = \sigma_7 - (h-6)\Big(\frac{1}{h}(\sigma_2\sigma_5+\sigma_3\sigma_4)-\frac{h-3}{h^2}(\sigma_2^2\sigma_3+\sigma_2\sigma_3)+\frac{1}{h}\sigma_5+\frac{(h-3)\,(h\alpha_0^2-12)}{40h^2}\sigma_3\Big).
$$

Finally, we can obtain Q_8 after following a similar step with Q_7.

$$
\begin{aligned}
Q_8 =& J_{7,8}\Big[\frac{1}{h}s_8 + (h-7)\Big(-\frac{1}{h^2}\left(s_6 s_2 + s_5 s_3 + \frac{1}{2}s_4^2 - \frac{2h-7}{2h}s_2^2 s_4 + \frac{(2h-7)(3h-7)}{24h^2}s_2^4\right) \\
& - \frac{35}{24}(1+M)s_6 + \frac{2h-7}{2h}(1+M)s_2 s_3^2 - \frac{7(2h-7)(5h-11)}{144h^2}(1+M)s_2^3 \\
& + \frac{7(5h-19)}{24h}(1+M)s_2 s_4 \\
& - \frac{7h(1+M)\left(14h^2M^2 - 54hM^2 - 211hM - 211h + 791M + 791\right)}{1920}s_4 \\
& + \frac{7(1+M)}{11520}(42h^3M^2 - 232h^2M^2 - 633h^2M - 633h^2 \\
& + 302hM^2 + 3524hM + 3524h - 4627M - 4627)s_2^2 \\
& + \frac{h^2(1+M)}{414720}(56h^5M^4 + 392h^4M^4 - 3416h^3M^4 - 9982h^3M^3 - 9982h^3M^2 \\
& + 4888h^2M^4 + 54624h^2M^3 + 96361h^2M^2 + 83474h^2M + 41737h^2 \\
& - 70322hM^3 - 299978hM^2 - 459312hM - 229656h \\
& + 297479M^2 + 594958M + 297479)s_2 \\
& + \frac{h^4(h-1)(1+M)}{199065600}(96h^7M^6 + 768h^6M^6 - 1584h^5M^6 - 8176h^5M^5 \\
& - 8176h^5M^4 - 18048h^4M^6 - 27244h^4M^5 - 27244h^4M^4 + 36048h^3M^6 \\
& + 343504h^3M^5 + 695436h^3M^4 + 703864h^3M^3 + 351932h^3M^2 - 526484h^2M^5 \\
& - 2474612h^2M^4 - 4512137h^2M^3 - 3795771h^2M^2 - 1847643h^2M - 615881h^2 \\
& + 2540356hM^4 + 8489936hM^3 + 12768028hM^2 + 10227672hM \\
& + 3409224h - 4445623M^3 - 13336869M^2 - 13336869M - 4445623)\Big)\Big].
\end{aligned}
$$

Substituting the relation (4.5.8), one can predict the vacuum eigenvalue of the IoM I_8 as

$$
\begin{aligned}
I_8 =&\sigma_8 + (h-7)\Big(-\frac{1}{h}\left(\sigma_6\sigma_2 + \sigma_5\sigma_3 + \frac{1}{2}\sigma_4^2 - \frac{2h-7}{2h}\sigma_2^2\sigma_4 + \frac{(2h-7)(3h-7)}{24h^2}\sigma_2^4\right) \\
&- \frac{35}{24h}\sigma_6 + \frac{2h-7}{2h^2}\sigma_2\sigma_3^2 - \frac{7(2h-7)(5h-11)}{144h^3}\sigma_2^3 + \frac{7(5h-19)}{24h^2}\sigma_2\sigma_4 \\
&- \frac{7\left(h\left(2(7h-27)\alpha_0^2 - 211\right) + 791\right)}{1920h^2}\sigma_4 \\
&+ \frac{7\left(h\left(2\alpha_0^2(h(21h-116)+151) - 633h + 3524\right) - 4627\right)}{11520h^3}\sigma_2^2 \\
&- \frac{1}{414720h^3}\big(56\alpha_0^4h^5 + 392\alpha_0^4h^4 - 14\alpha_0^2\left(244\alpha_0^2 + 713\right)h^3 \\
&- 14\left(5023\alpha_0^2 + 16404\right)h + \left(4888\alpha_0^4 + 54624\alpha_0^2 + 41737\right)h^2 + 297479\big)\sigma_2 \\
&+ \frac{h-1}{199065600h^3}\big(96\alpha_0^6h^7 + 768\alpha_0^6h^6 - 1584\alpha_0^6h^5 - 8176\alpha_0^4h^5 - 18048\alpha_0^6h^4 \\
&- 27244\alpha_0^4h^4 + 36048\alpha_0^6h^3 + 343504\alpha_0^4h^3 + 351932\alpha_0^2h^3 - 526484\alpha_0^4h^2 \\
&- 1948128\alpha_0^2h^2 - 615881h^2 + 2540356\alpha_0^2h + 3409224h - 4445623\big)\Big).
\end{aligned}
$$

The result agrees with the one in [102] when $h = 3$. Further Q_k can be calculated similarly.

D.2 Higher-order WKB integrals in $D_r^{(1)}$-type pseudo ODEs

Finally, we present Q_6 in terms of L_i' in $D_r^{(1)}$-type pseudo ODEs

$$
\begin{aligned}
Q_6 = -\frac{2^{-\frac{10}{h}}}{h} J_{5,6} \Big(& L_6' + (h-5)\Big[-\frac{1}{2h} L_3'^2 - \frac{1}{h} L_2' L_4' + \frac{2h-5}{6h^2} L_2'^3 \Big] \\
& + \frac{5}{2}(h-4) L_5' - \frac{1}{2h}(h-5)(5h-13) L_2' L_3' \\
& - \frac{5}{24}(3hM(h-6) - 8h^2 + 57h - 118) L_4' \\
& + \frac{h-5}{48h}(15h^2 M - 40h^2 - 50hM + 85h - 62) L_2'^2 \\
& - \frac{15}{16}(h-2)(3hM(h-6) + h - 22) L_3' \\
& - \frac{1}{1152}(10h^5 M^3 + 10h^5 M^2 - 72h^4 M^3 - 163h^4 M^2 + 200h^3 M^3 + +980h^3 M^2 \\
& + 288h^4 M - 2470h^3 M - 2300h^2 M^2 + 3960h^2 M - 3320hM \\
& + 192h^4 - 1690h^3 + 2695h^2 - 740h - 820) L_2' \\
& + \frac{1}{580608} h(h+2)(hM-5)(hM-1)(8h^6 M^3 - 16h^5 M^3 - 368h^4 M^3 \\
& + 8h^6 M^2 + 32h^5 M^2 + 48h^5 M - 1074h^4 M^2 + 1888h^3 M^3 + 5320h^3 M^2 \\
& - 5672h^2 M^2 - 858h^4 M + 5139h^3 M - 11652h^2 M + 3228hM \\
& - 152h^4 + 1694h^3 - 5939h^2 + 3268h - 572) \Big).
\end{aligned}
$$

The result in terms of s_i and the correspondence analysis has been given in (4.5.28).

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